Affine Springer Fibers

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1 Springer fibers

1.1 The setup

First we introduce a classical object, a springer fiber. In this section, we assume that k is an algebraically closed field and G is a connected semisimple algebraic group over k. Also suppose that $\mathrm{char}(k)$ is large compared to G and let r be a rank of G.

Fix a maximal torus T of G and a Borel subgroup of G contains T. Then we have $B = T \ltimes U$ with an unipotent radical U of B. We denote the Lie algebras of these groups by $\mathfrak{b}$, $\mathfrak{t}$, and $\mathfrak{n}$ respectively. Note that we have $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ and $\mathfrak{n}$ is a nilpotent radical $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ of $\mathfrak{b}$. Let $W = N_G(T)/T$ be a Weil group.

Let $\mathcal{B}$ be the set of all Borel subalgebras in $\mathfrak{g}$. Then $\mathcal{B}$ becomes a closed subvariety of a Grasmannian of dimmension $\dim \mathfrak{b}$ consists of solvable Lie subalgebras. Hence $\mathcal{B}$ is a projective variety. For a borel subalgebra $\mathfrak{b}$, its stabilizer $G_{\mathfrak{b}} = B$ under an adjoint action is a borel subgroup of B. Since all Borel subalgebras are conjugate to each other, we have a bijection

$$\begin{aligned} G/B &\longrightarrow \mathcal{B} \\ g &\mapsto \mathrm{Ad}(g)(\mathfrak{b}). \end{aligned}$$

Note that G/B is a smooth variety with a G-action, and indeed the above bijection is G-equivariant isomorphism of schemes.

Actually, $\mathcal{B}$ is a genearlized version of flag varieties. We show this when $G = SL_n$.

Definition 1.1.1. (Flag varieties) Let V be a vector space over k of dimension n and let $\mathcal{B}$ be a functor on k-algebras given by

$$X(R) = \{0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = V_R\}$$

where a rank of V_i is i. Then X is represented by a geometrically connected smooth projective variety, covered by affine open subsets with nonempty overlaps. We call scheme represents X as a (full) *flag variety*.

Lemma 1.1.2. *Let* $G = SL_n$. *Then a variety* $\mathcal{B}$ *is isomorphic to a flag varieties on* V *of dimension* n.

Proof. For a flag

$$F = \{V_0 \subseteq \cdots \subseteq V_n\},$$

define the Lie subalgebra $\mathfrak{b}_F = \{x \in \mathfrak{g} \mid x(V_i) \subseteq V_i \text{ for any } i\}$. By choosing the good basis, we may write the element $x \in \mathfrak{b}_F$ with

$$x = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix}$$

an upper triangular matrices in $\mathfrak{sl}_n$. Hence $\mathfrak{b}_F$ is a borel subalgebra of $\mathfrak{sl}_n$. The inverse is given by the Lie's theorem. □

It is important to study the relation between semisimple Lie algebra and the flag variety associated to it. We define the following variety to deal with this. Note that we may see $\mathfrak{g}$ as a scheme by $\mathfrak{g} = \mathrm{Spec}(\mathrm{Sym}\, \mathfrak{g}^*)$.

1.2 Grothendieck Alterations

Definition 1.2.1. (Grothendieck Alteration) A variety defined by

$$\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}$$

is called a *Grothendieck alteration*. It is a closed subscheme of $\mathfrak{g} \times \mathcal{B}$.

For a projection $\pi_{\mathcal{B}} : \tilde{\mathfrak{g}} \to \mathcal{B}$, a fiber of $\mathfrak{b} \in \mathcal{B}$ is given by $\{x \in \mathfrak{b}\}$, which is a vector space $\mathfrak{b}$. Indeed, $\tilde{\mathfrak{g}} \to \mathcal{B}$ is a G-equivariant vector bundle whose fibers are borel subalgebras of $\mathfrak{g}$. Here G acts on $\mathfrak{g}$ by $g \cdot (x, \mathfrak{b}) = (\mathrm{Ad}(g)x, \mathrm{Ad}(g)\mathfrak{b})$.

Define an action of B on $G \times \mathfrak{b}$ by

$$b \cdot (g, x) = (g \cdot b^{-1}, \mathrm{Ad}(b)x).$$

This action is free since the action on the first factor is free. Denote $G \times_B \mathfrak{b}$ for the orbit-space. Then the projection

$$\begin{aligned} G \times_B \mathfrak{b} &\longrightarrow G/B \\ (g, x) &\mapsto gB \end{aligned}$$

becomes a G-equivariant vector bundle whose fiber over gB is $\{g\} \times \mathfrak{b}$. Here G acts on $G \times_B \mathfrak{b}$ by $g \cdot (g', x) = (gg', x)$. Note that we have a G-equivariant isomorphism $\mathcal{B} \cong G/B$.

Lemma 1.2.2. *For* G*-equivariant vector bundles* $\pi_{\mathcal{B}} : \tilde{\mathfrak{g}} \to \mathcal{B}$ *and* $G \times_B \mathfrak{b} \to G/B$*, there exists a* G*-equivariant isomorphism*

$$\begin{aligned} G \times_B \mathfrak{b} &\xrightarrow{\cong} \tilde{\mathfrak{g}} \\ (g, x) &\mapsto (\mathrm{Ad}(g)x, \mathrm{Ad}(g)\mathfrak{b}) \end{aligned}$$

compatible with $G/B \cong \mathcal{B}$.

Proof. It is easily checked that the morphism we construct is G-equivariant and compatible with $G/B \cong \mathcal{B}$. To show that is is an isomorphism, it suffices to check it on fibers. For gB corresponding to $\mathrm{Ad}(g)\mathfrak{b}$, we have an isomorphism between their fibers

$$\begin{aligned} \{g\} \times \mathfrak{b} &\to \mathrm{Ad}(g)\mathfrak{b} \\ x &\mapsto \mathrm{Ad}(g)x, \end{aligned}$$

which is a conjugation. □

Now we analyze the other projection $\pi_{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \mathfrak{g}$.

Lemma 1.2.3. *The projection $\pi_{\mathfrak{g}}$ is proper.*

Proof. Since $\mathcal{B}$ is a projective variety, a projection $\mathfrak{g} \times \mathcal{B} \to \mathfrak{g}$ is proper. Since $\tilde{\mathfrak{g}}$ is a closed subscheme of $\mathfrak{g} \times \mathcal{B}$, the composition $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is proper. $\square$

The fiber of $x \in \mathfrak{g}$ along $\pi_{\mathfrak{g}}$ is given by $\mathcal{B}_x \subseteq \mathcal{B}$, consists of Borel subalgebras containing x. If $x = 0$, then we have $\pi_{\mathfrak{g}}^{-1}(x) = \mathcal{B}$.

Note that over an algebraically closed field, $x \in \mathfrak{g}$ is regular semisimple if the discriminant of characteristic polynomial of $\mathrm{ad} x$ is nonzero. Let $\mathfrak{g}^{rs}$ be a subset of $\mathfrak{g}$ consists of regular semisimple elements then it follows that it is a G-stable dense open subset. Let $\tilde{\mathfrak{g}}^{rs} = \pi_{\mathfrak{g}}^{-1}(\mathfrak{g}^{rs})$.

Lemma 1.2.4. *A projection $\pi_{\mathfrak{g}}|_{\tilde{\mathfrak{g}}^{rs}} : \tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is a W-torsor.*

Proof. For a regular semisimple element $x \in \mathfrak{g}^{rs}$, its centralizer is a Cartan subalgebra $\mathfrak{t}$. Then its fiber $\mathcal{B}_x$ is given by a set of Borel subalgebras containing $\mathfrak{t}$. Hence $N_G(T)$ acts on $\mathcal{B}_x$ since $g \in N_G(T)$ fixes $\mathfrak{t}$. This action is transitive since any Borel subalgebras are conjugate. Since a Borel subgroup is self-normalizing, a stabilizer of $\mathfrak{b} \in \mathcal{B}_x$ is given by $N_G(T) \cap B \subseteq G$. Note that we have $N_G(T) \cap B = N_B(T) = Z_B(T) = Z_G(T) \cap B$(See [Springer,6.3.6] for details). Since $Z_G(T) = T$, we are done. $\square$

Remark 1.2.5. Note that a projection $\tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is also a G-equivariant morphism, and the action is compatible that of W. A W-action does not change x in $(x, \mathfrak{b})$, and this is the key difference with G-action.

Define a morphism

$$\begin{aligned} \nu : \tilde{\mathfrak{g}} &\to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \\ (x, \mathfrak{b}') &\mapsto x \in \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}'] \cong \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]. \end{aligned}$$

To check the well-definedness, we need the following lemma.

Lemma 1.2.6. *For a Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$, we have a canonical isomorphism $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}']$.*

Proof. Note that every Borel subgroups are conjugate. Suppose $B' = gBg^{-1} = g'Bg'^{-1}$. Then we have $g^{-1}g' \in N_G(B) = B$. Since an unipotent radical $U \subseteq B$ is normal in B, it follows that $gUg^{-1} = g'Ug'^{-1}$. Hence we obtain an isomorphism $B/U \cong gBg^{-1}/gUg^{-1}$ which is independent of the choice of g. $\square$

Hence we can canonically identify all $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$'s, so we can define $\nu : \tilde{\mathfrak{g}} \to \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ as above. Indeed, it is a G-equivariant morphism. For $G \times_B \mathfrak{b}$, we have a canonical G-equivariant projection $G \times_B \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ sending (g,x) to $x \mod [\mathfrak{b},\mathfrak{b}]$. These morphisms are compatible with a G-equivariant isomorphism $G \times_B \mathfrak{b} \to \tilde{\mathfrak{g}}$ we constructed; i.e. The following diagram commutes :

$$\begin{tikzcd} G \times_B \mathfrak{b} \arrow[d, "\cong"] \arrow[dr] & \\ \tilde{\mathfrak{g}} \arrow[r, "\nu"] & \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]. \end{tikzcd}$$

Now let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. We choose a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{t}$. Then the composite $\mathfrak{t} \to \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ is an isomorphism of Cartan subalgebras. For another choice of $\mathfrak{b}'$ containing $\mathfrak{t}$, there exists a unique $g \in W(T)$ such that $\mathfrak{b}' = g\mathfrak{b}g^{-1}$. Under this isomorphism, we obtain a commutative diagram

$$\begin{tikzcd} \mathfrak{t} \arrow[r, "\cong"] \arrow[dr, "\cong"] & \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \arrow[d, "\cong"] \\ & \mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}'] \end{tikzcd}$$

Example 1.2.7. Let $G = SL_n$, then we may identify $\mathcal{B}$ with a flag variety on a vector space V of dimension n. Then we have

$$\tilde{\mathfrak{g}} = \{(x,F) \mid xV_i \subseteq V_i\}.$$

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{sl}_n$ consists of diagonal matrices whose traces are zero. Then we may identify $\mathfrak{t}$ with $\mathbb{A}^n_k$. For a flag F and $x \in \mathfrak{b}_F$, we have a linear map $x : V_i/V_{i-1} \to V_i/V_{i-1}$ between 1-dimensional vector spaces for every i. By taking an eigenvalue of this linear map for each i, we have n eigenvalues and these are diagonal entries of $\mathfrak{b}_F$. Hence $\nu : \tilde{\mathfrak{g}} \to \mathfrak{t}$ is identified with

$$\nu : (x,F) \mapsto (\lambda_1, \cdots, \lambda_n)$$

satisfying $\sum_i \lambda_i = 0$.

Since a Weyl group of SL_n is given by S_n, thus for $x \in \mathfrak{sl}_n^{rs}$, its fiber $\mathcal{B}_x$ has a freely transitive S_n-action.

1.3 Chevalley Restriction Theorem

Let $k[\mathfrak{g}] = \mathrm{Sym}\,\mathfrak{g}^*$, then it is a set of polynomial functions on $\mathfrak{g}$. Note that G acts on $\mathfrak{g}$ by adjoint representation, so we can extend it to $k[\mathfrak{g}]$. We denote

$k[\mathfrak{g}]^G$ as a set of G-invariant polynomials. For a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$, we have a restriction map $k[\mathfrak{g}] \to k[\mathfrak{t}]$. For a Weyl group W of $\mathfrak{t}$, we have the following restiction $k[\mathfrak{g}]^G \to k[\mathfrak{t}]^W$.

Theorem 1.3.1. *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Cartan algebra. Then the restriction map*

$$k[\mathfrak{g}]^G \to k[\mathfrak{t}]^W$$

is an isomorphism.

Proof. First we show the injectivity. Note that $\mathfrak{g}$ is a reduced scheme, thus polynomials on it are determined by its values on points. For $f \in k[\mathfrak{g}]^G$, suppose its restriction on $\mathfrak{h}$ is zero. Note that every regular semisimple element $x \in \mathfrak{g}^{rs}$ is conjugate to some element of $\mathfrak{h}$. Since f is invariant under a conjugation, f becomes zero at $\mathfrak{g}^{rs}$. Since $\mathfrak{g}^{rs}$ is dense in $\mathfrak{g}$ and a zero locus is a closed subset, it follows that $f = 0$.

Showing the surjectivity is rather difficult. Let $f \in k[\mathfrak{t}]^W$ and $\mathfrak{b}$ a Borel subalgebra contaning $\mathfrak{h}$. Then we obtain an isomorphism $\mathfrak{t} \cong \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$. We may write $f : \mathfrak{t} \to \mathbb{A}^1_k$, then $f \in k[\mathfrak{t}]^W$ means that the composite

$$\mathfrak{t} \xrightarrow{\text{conj.}} \mathfrak{t} \xrightarrow{f} \mathbb{A}^1_k$$

admits with f. Under the isomorphism $\mathfrak{t} \cong \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, we may rewrite f with $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \to \mathbb{A}^1_k$, and this is independent of the choice of $\mathfrak{b}$ since f is W-invariant. Now we replace $\mathfrak{t}$ with $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, then we have a morphism $\nu : \tilde{\mathfrak{g}} \to \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, and $f \circ \nu$ gives a polynomial defined on $\tilde{\mathfrak{g}}$. By replacing ν with a projection $G \times_B \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ sending (g, x) to x, we can obtain that $f \circ \nu$ is G-invariant.

By 1.2.4, $\tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is W-torsor. Since f is W-invariant, we have that

$$f \circ \nu|_{\tilde{\mathfrak{g}}^{rs}} : \tilde{\mathfrak{g}}^{rs} \to \mathbb{A}^1_k$$

is W-invariant morphism over $\tilde{\mathfrak{g}}^{rs}$. The Galois descent for $\tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ gives the corresponding morphism $f' : \mathfrak{g}^{rs} \to \mathbb{A}^1_k$. Indeed, since $\tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is G-equivariant and $f \circ \nu|_{\tilde{\mathfrak{g}}^{rs}}$ is G-equivariant, the obtained morphism $f' : \mathfrak{g}^{rs} \to \mathbb{A}^1_k$ is also G-equivariant.

It remains to show that the restriction of f' equals to f. It suffices to show that they admit on a dense open subset $\mathfrak{t}^r$ of $\mathfrak{t}$. For $(x, \mathfrak{b})$ with $x \in \mathfrak{t}^r$, we have

$$f'(x) = f \circ \nu(x, \mathfrak{b}) = f(x + [\mathfrak{b},\mathfrak{b}]) = f(x).$$

□

Suppose a finite group G is acting on an affine scheme $X = \operatorname{Spec} A$. Then $A^G \subseteq A$ is an integral extension, and an affine GIT quotient $\operatorname{Spec} A \to \operatorname{Spec} A^G$ gives an homeomorphism $|\operatorname{Spec} A|/G \cong |\operatorname{Spec} A^G|$. We usually denote $X/\!/G = \operatorname{Spec} A^G$. Note that this does not hold for general group scheme. For example, consider a $\mathbb{G}_m$-action on $\mathbb{A}^2_k$ by $t \cdot (x, y) = (tx, t^{-1}y)$. So in particular, we have a GIT quotient $\mathfrak{t}/\!/W = \operatorname{Spec} k[\mathfrak{t}]^W$ whose topology admits with a quotient topology. From the morphism

$$k[\mathfrak{t}]^W \cong k[\mathfrak{g}]^G \subseteq k[\mathfrak{g}],$$

we obtain the corresponding morphism

$$\mathfrak{g} \to \mathfrak{t}/\!/W.$$

Indeed, we have a diagram

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\nu"] \arrow[d, "\pi_{\mathfrak{g}}"] & \mathfrak{t} \arrow[d, "\pi"] \\ \mathfrak{g} \arrow[r, "\rho"] & \mathfrak{t}/\!/W. \end{tikzcd}$$

Lemma 1.3.2. *(1) The above diagram commutes.*

(2) We have a cartesian diagram

$$\begin{tikzcd} \tilde{\mathfrak{g}}^{rs} \arrow[r, "\nu"] \arrow[d, "\pi_{\mathfrak{g}}"] & \mathfrak{t} \arrow[d] \\ \mathfrak{g}^{rs} \arrow[r] & \mathfrak{t}/\!/W. \end{tikzcd}$$

Proof. (1) It suffices to show that a diagram

$$\begin{tikzcd} \mathcal{O}(\tilde{\mathfrak{g}}) & k[\mathfrak{t}] \arrow[l, "\nu^*"'] \\ k[\mathfrak{g}] \arrow[u, "\pi_{\mathfrak{g}}^*"] & k[\mathfrak{t}]^W. \arrow[l, "\rho^*"'] \arrow[u, "\pi^*"'] \end{tikzcd}$$

commutes. For $f \in k[\mathfrak{t}]^W$, with the notation in the proof of 1.3.1, we have

$$\pi_{\mathfrak{g}}^* \rho^* f = \pi_{\mathfrak{g}}^* f' = f \circ \nu|_{\tilde{\mathfrak{g}}^{rs}} = \nu^* \pi^* f.$$

(2) It suffices to check on global sections. As stated in the proof, a global section of $\mathfrak{g}^{rs}$ corresponds to a global section of $\tilde{\mathfrak{g}}^{rs}$ which is W-invariant. $\square$

Example 1.3.3. Let $G = SL_n$. For $\mathbb{A}^n_k = \operatorname{Spec} k[\lambda_1, \cdots, \lambda_n]$, let $H = V(\lambda_1 + \cdots + \lambda_n)$ be a hyper space of $\mathbb{A}^n_k$, Then we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{sl}}_n & \xrightarrow{(\lambda_1,\cdots,\lambda_n)} & H \\ \downarrow{\scriptstyle \pi_{\mathfrak{g}}} & & \downarrow \\ \mathfrak{sl}_n & \longrightarrow & H/\!/S_n. \end{array}$$

Note that $k[\lambda_1, \cdots, \lambda_n]^{S_n} \cong k[s_1, \cdots, s_n]$, thus we can identify $H/\!/S_n$ with $\mathbb{A}^{n-1}_k$. Then the lower horizontal morphism is identified with $\mathfrak{sl}_n \to \mathbb{A}^{n-1}_k$, sending $x \in \mathfrak{sl}_n$ to its $(n-i)$-th coefficients of characteristic polynomial.

Lemma 1.3.4. *Let $\mathfrak{b}$ be a Borel subalgebra with nilradical $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. For any G-invariant polynomial $f \in k[\mathfrak{g}]^G$ and any $x \in \mathfrak{b}$, the restriction $f|_{x+\mathfrak{n}}$ is constant with the vallue $f(x)$. Equivalently, for any $f \in k[\mathfrak{g}]^G$, we have $\mathfrak{n} \subseteq V(f - f(x)) \subseteq \mathfrak{g}$.*

Proof. Let $y = x + n$ for some $n \in \mathfrak{n}$. Then we have liftings $\tilde{x} = (x, \mathfrak{b})$ and $\tilde{y} = (y, \mathfrak{b})$ in $\tilde{\mathfrak{g}}$. In $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$, we have $\nu(\tilde{x}) = \nu(\tilde{y})$. The commutativity gives the claim. $\square$

1.4 Nilpotent cones

We call $x \in \mathfrak{g}$ is ad-nilpotent(resp. ad-semisimple) if $\operatorname{ad}x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent(resp. semisimple). Equivalently, the characteristic polynomial of $\operatorname{ad}x$ is t^n. In general, it is different from that x is nilpotent. For example, $I \in \mathfrak{gl}_n$ is ad-nilpotent but not nilpotent. We can see easily that nilpotency implies ad-nilpotency.

Lemma 1.4.1. *Let $\mathfrak{g} = \mathfrak{gl}(V)$ with a finite dimensional vector space V over a field k. If $x \in \mathfrak{g}$ is nilpotent, then it is ad-nilpotent. Similarly, if k is algebraically closed and x is semisimple, then it is ad-semisimple.*

Proof. First suppose that x is nilpotent. Consider linear maps $L, R : \mathfrak{g} \to \mathfrak{g}$ defined by $Ly = xy$ and $Ry = yx$. Then L and R are commutes since $(LR)y = (RL)y = xyx$. Note that $\operatorname{ad}x = L - R$ and $L^n = R^n = 0$. Then we have

$$\operatorname{ad}x^{2n} = (L-R)^{2n} = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} L^{2n-i} R^i = 0.$$

Now suppose that x is semisimple. Then we have a basis $\{v_1, \cdots, v_n\}$ of V diagonalizes x with eigenvalues $\lambda_1, \cdots, \lambda_n$ respectively. Let e_{ij} be a basis of $\mathfrak{gl}(V)$ defined by $e_{ij}(v_k) = \delta_{ik}v_j$. Then we have

$$\mathrm{ad}x(e_{ij})(v_k) = (xe_{ij} - e_{ij}x)v_k = x\delta_{ik}v_j - e_{ij}a_k v_k = a_j\delta_{ik}v_j - a_k\delta_{ik}v_j.$$

Since $a_k\delta_{ik} = a_i\delta_{ik}$, we obtain

$$\mathrm{ad}x(e_{ij})(v_k) = (a_j - a_i)\delta_{ik}v_j = (a_j - a_i)e_{ij}(v_k).$$

Thus e_{ij} is an eigenvector of $\mathrm{ad}x$ with an eigenvalue $a_j - a_i$. $\square$

When $\mathfrak{g}$ is a semisimple Lie algebra, any element $x \in \mathfrak{g}$ can be decomposed into $x = x_s + x_n$ where x_s is ad-semisimple and x_n is ad-nilpotent with $[x_s, x_n] = 0$. Then for a finite dimensional representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$, we have that $\varphi(x) = \varphi(x_s) + \varphi(x_n) = \varphi(x)_s + \varphi(x)_n$ where $\varphi(x)_s$ and $\varphi(x)_n$ are Jordan-Chevalley decomposition in $\mathfrak{gl}(V)$. In particular, for $\mathfrak{g} = \mathfrak{sl}_n$, $x \in \mathfrak{g}$ is ad-nilpotent(resp. ad-semisimple) if and only if x is nilpotent(resp. semisimple).

Now we define a nilpotent cone.

Definition 1.4.2. (Nilpotent cones) Let $\mathcal{N}$ be a set of ad-nilpotent elements of $\mathfrak{g}$. Since these are points whose characteristic polynomial of adjunct is t^n, it follows that $\mathcal{N}$ is a closed subscheme of $\mathfrak{g}$. We call $\mathcal{N}$ as a *nilpotent cone* of $\mathfrak{g}$. Since a characteristic polynomial is G-stable, $\mathcal{N}$ is a G-stable subscheme of $\mathfrak{g}$. It is also $\mathbb{G}_m$-stable with respect to a scalar multiplication. This is why $\mathcal{N}$ is called a cone.

For a Grothendieck alteration $\pi_{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \mathfrak{g}$, consider a fiber product

$$\tilde{\mathcal{N}} = \pi_{\mathfrak{g}}^{-1}(\mathcal{N}) = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}.$$

Note that we have a commutative diagram

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[r] \arrow[d] & \tilde{\mathfrak{g}} \arrow[r, "\nu"] \arrow[d, "\pi_{\mathfrak{g}}"] & \mathfrak{t} \arrow[d] \\
\mathcal{N} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{t}//W.
\end{tikzcd}$$

where the left square is cartesian. We call $\tilde{N} \to N$ as a Springer resolution.

For a Borel subalgebra $\mathfrak{b} \in \mathcal{B}$, its fiber over $\tilde{\mathcal{N}} \to \mathcal{B}$ consists of ad-nilpotent elements of $\mathfrak{b}$. Note that we have a decomposition $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ where

$\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, and $x \in \mathfrak{b}$ is ad-nilpotent if and only if its Cartan component is zero. Hence $x \in \mathfrak{b}$ is ad-nilpotent if and only if it is contained in $\mathfrak{n}$. This shows that $\tilde{N}$ is a vector bundle over $\mathcal{B}$ with fiber $\mathfrak{n}$. Since any nilpotent element of $\mathfrak{g}$ is conjugate to some element of $\mathfrak{n}$, we have a G-equivariant isomorphism

$$\tilde{\mathcal{N}} \cong G \times_B \mathfrak{n}.$$

In particular, $\tilde{\mathcal{N}}$ is a smooth variety.

Furthermore, a Springer resolution $\tilde{\mathcal{N}} \to \mathcal{N}$ is surjective since every unipotent subgroup is contained in a Borel subgroup over an algebraically closed field.

Lemma 1.4.3. *Let $T^*\mathcal{B}$ be a cotangent bundle of $\mathcal{B}$. Then we have a G-equivariant isomorphism*

$$\tilde{\mathcal{N}} \cong T^*\mathcal{B}$$

of vector bundles of $\mathcal{B}$.

Proof. Note that we have $T_e(G/B) \cong \mathfrak{g}/\mathfrak{b}$. Consider a trivial bundle $G/B \times \mathfrak{g}$ of G/B and a G-equivariant morphism

$$\begin{aligned} G/B \times \mathfrak{g} &\to T(G/B) \\ (gB, x) &\mapsto \mathrm{Ad}g(\overline{x}) \end{aligned}$$

of vector bundles over G/B. Observe that is is surjective(which can be checked on fibers). Here B acts on $G/B \times \mathfrak{g}$ by $b \cdot (gB, x) = (gB, \mathrm{Ad}b(x))$. Let $\underline{b}$ be the kernel of this morphism, then its fiber over gB consists of x with $x \in \mathfrak{b}$ and $\mathrm{Ad}bx - x$ with $b \in B$ and $x \in \mathfrak{g}$. It follows that $G \times_B \mathfrak{g}/\mathfrak{b} \cong T(G/B)$. Then dual gives that $G \times_B (\mathfrak{g}/\mathfrak{b})^* \cong T^*(G/B)$. Note that we have

$$(\mathfrak{g}/\mathfrak{b})^* \cong \mathfrak{b}^\perp = \{\varphi \in \mathfrak{g}^* \mid \varphi|_{\mathfrak{b}} = 0\}.$$

Under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ given by the Killing form, $\mathfrak{b}^\perp$ corresponds to $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Hence we have $\tilde{\mathcal{N}} \cong G \times_B \mathfrak{n} \cong T^*(G/B)$. □

Example 1.4.4. Let $\mathfrak{g} = \mathfrak{sl}_2$. For

$$x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g},$$

we have $p_A = t^2$ if and only if $\det x = -a^2 - bc = 0$. Hence a nilpotent cone is given by

$$\mathcal{N} = V(a^2 + bc) \subseteq \mathbb{A}^3_k,$$

which is a usual cylinder cone. Note that $\mathcal{N}$ has a singularity at $(0, 0, 0)$. We have $\mathcal{B} \cong \mathbb{P}^1_{\mathbb{C}}$, so $\tilde{\mathcal{N}} \cong T^*\mathcal{B} \cong T^*\mathbb{P}^1_{\mathbb{C}}$.

We can identify the ad-nilpotent elements of $\mathfrak{g}$ in the following way.

Lemma 1.4.5. *An element $x \in \mathfrak{g}$ is ad-nilpotent if and only if it is contained in the fiber of $0 \in \mathfrak{g}/\!/G$ along a projection $\mathfrak{g} \to \mathfrak{g}/\!/G$. Equivalently, $\mathcal{N}$ is a fiber of $\rho : \mathfrak{g} \to \mathfrak{t}/\!/W$ over 0.*

Proof. Suppose $x \in \mathfrak{g}$ is ad-nilpotent; i.e. $x \in \mathcal{N}$. We can identify $\mathfrak{g}/\!/G$ with $\mathfrak{t}/\!/W$ by the theorem 1.3.1. For $\tilde{x} = (x, \mathfrak{b}) \in \pi_{\mathcal{N}}^{-1}(x)$, we have $x \in [\mathfrak{b}, \mathfrak{b}]$. Using the notation of 1.3.2, we have that $\nu(\tilde{x}) = 0$. Then it follows that $\pi \circ \nu(\tilde{x}) = \rho \circ \pi_{\mathfrak{g}}(\tilde{x}) = \rho(x) = 0$.

Conversely, suppose $\rho(x) = 0$ for $x \in \mathfrak{g}$. Choose $\tilde{x} = (x, \mathfrak{b}) \in \pi_{\mathfrak{g}}^{-1}(x)$. Again, we have $\pi \circ \nu(\tilde{x}) = \rho \circ \pi_{\mathfrak{g}}(\tilde{x}) = \rho(x) = 0$. This shows that $\nu(\tilde{x}) \in \pi^{-1}(0)$. Since W is a finite group, every element of $\mathfrak{t}$ is stable. It follows that $\pi^{-1}(0) = 0$. Hence we obtain $\nu(\tilde{x}) = 0$, so $(x, \mathfrak{b}) \in \nu^{-1}(0)$. This shows that $x \in \mathfrak{n}$, thus x is ad-nilpotent. □

$$
\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[rr] \arrow[dd] \arrow[dr] & & 0 \arrow[dr] \arrow[dd, dashed] & \\
& \tilde{\mathfrak{g}} \arrow[rr, "\nu"] \arrow[dd, "\pi_{\mathfrak{g}}"] & & \mathfrak{t} \arrow[dd, "\pi"] \\
\mathcal{N} \arrow[rr, dashed] \arrow[dr] & & 0 \arrow[dr, dashed] & \\
& \mathfrak{g} \arrow[rr, "\rho"] & & \mathfrak{g}/\!/G \cong \mathfrak{t}/\!/W
\end{tikzcd}
$$

Lemma 1.4.6. *A nilpotent cone $\mathcal{N}$ is an irreducible of dimension* $2\mathrm{dim}\mathfrak{n}$.

Proof. Observe that $T^*\mathcal{B}$ is smooth and connected, it is irreducible. Since $\mu : T^*\mathcal{B} \to \mathcal{N}$ is surjective, it follows that $\mathcal{N}$ is irreducible. Also we have $\dim \mathcal{N} \leq \dim T^*\mathcal{B} = 2\dim\mathcal{B} = 2(\dim\mathfrak{g} - \dim\mathfrak{b}) = 2\dim\mathfrak{n}$(Note the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$).

A nilpotent cone $\mathcal{N}$ is given by a fiber of $\mathfrak{g} \to \mathfrak{t}/\!/W$. Since W is a finite group, we have $\dim \mathfrak{t}/\!/W = \dim \mathfrak{t}$. Hence we have $\dim \mathcal{N} \geq \dim \mathfrak{g} - \dim \mathfrak{t}/\!/W = \dim \mathfrak{g} - \dim \mathfrak{t} = 2\dim \mathfrak{n}$. □

Example 1.4.7. Let $G = SL_{2,\mathbb{C}}$, then a nilpotent cone $V(a^2 + bc)$ is of dimension 2 and $\mathfrak{n}$ is of dimension 1.

Lemma 1.4.8. *The number of G-orbits in $\mathcal{N}$ is finite.*

Example 1.4.9. Let $G = SL_n(\mathbb{C})$, then the lemma is clear since in a Jordan canonical form, nilpotent matrices are given by Jordan blocks with zero eigenvalues.

Lemma 1.4.10. *The set $\mathcal{N}^r$ of regular nilpotent element is a dense open G-orbit in $\mathcal{N}$.*

Proof. By 1.4.8, we have a finitely many G-orbits in $\mathcal{N}$. It follows that $\mathcal{N}$ is a union of closure of orbits, then since $\mathcal{N}$ is irreducible, there exists an orbit $G \cdot x \subseteq \mathcal{N}$ such that $\overline{G \cdot x} = \mathcal{N}$. By [Brion,1.11], $G \cdot x$ is a dense open subset of $\overline{G \cdot x} = \mathcal{N}$ and $\dim G \cdot x = \dim G - \dim G_x$ where G_x is a stabilizer subgroup of G. Since the action is given by a conjugation, we have that $G_x = Z_G(x)$. Now we have $\dim Z_G(x) = \dim G - \dim G \cdot x = \dim \mathfrak{g} - \dim \mathcal{N} = \dim \mathfrak{g} - 2 \dim \mathfrak{n} = \operatorname{rk} \mathfrak{g}$. It follows that x is regular, hence every elements in $G \cdot x$ is regular. Also [Brion,1.11] gives that the other orbits have dimension strictly less than the dimension of $G \cdot x$. Again by the dimension formula, an element in the other orbit is not a regular element. $\square$

Example 1.4.11. Let $G = SL_n(\mathbb{C})$. In the Jordan normal form, we have only one regular nilpotent element x with a unique Jordan block with eigenvalue 0. Then its centralizer is the linear sum of matrices $x, x^2, \cdots, x^{n-1}$. The dimension of the centralizer is $n - 1 = \operatorname{rk} \mathfrak{sl}_n(\mathbb{C})$.

Let B be a Borel subgroup containing a maximal (split) torus T. Then we may write $B = T \ltimes U$ with a unipotent radical U. Since U is a normal subgroup of B, a Borel subgroup B acts on U by a conjugation. Hence B acts on $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

Now let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ be a root space decompostion. Note that $\mathfrak{g}$ is semisimple. For a projection $\mathfrak{n} \to \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, choose simple positive roots $e_1, \cdots, e_l$ with $l = \operatorname{rk} \mathfrak{g}$ where $\overline{e}_1, \cdots, \overline{e}_n$ forms a basis of $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. Consider $\{\sum \lambda_i \cdot \overline{e}_i, \lambda_i \in k^\times\} \subseteq \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, then it is an open subscheme since it is given by a finite intersection of open subscheme where the coefficients of $\overline{e}_i$ is nonzero. Consider a fiber $\mathfrak{n}^r = \{x \in \mathfrak{n} \mid \overline{x} = \sum \lambda_i \overline{e}_i, \lambda_i \in k^\times\}$ under a projection, then $\mathfrak{n}^r$ is an open subscheme of $\mathfrak{n}$.

Lemma 1.4.12. *$\mathfrak{n}^r$ is a single B-orbit consisting of regular nilpotent elements in $\mathfrak{g}$.*

Proof. The image of $\mathfrak{n}^r$ in $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is a single open dense T-orbit in $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. Hence $\mathfrak{n}^r$ contains a regular nilpotent element x. Since $\operatorname{Ad} g \cdot x - x \in [\mathfrak{n}, \mathfrak{n}]$ for any $g \in U$, $x + [\mathfrak{n}, \mathfrak{n}]$ is stable under U-action. Since x is regular, we have $\dim Z_G(x) = \operatorname{rk}(\mathfrak{g})$. It follows that

$$\dim U \cdot x = \dim U - \dim Z_G(x) = \dim \mathfrak{n} - \operatorname{rk}(\mathfrak{g}) = \dim [\mathfrak{n}, \mathfrak{n}].$$

Since $U \cdot x$ is a locally closed subvariety of $\mathfrak{n}$ and is of the same dimension with $x + [\mathfrak{n}, \mathfrak{n}]$, the orbit $U \cdot x$ is open in $x + [\mathfrak{n}, \mathfrak{n}]$. Furthermore, since U is unipotent, the orbit $U \cdot x$ is closed. Hence we have $U \cdot x = x + [\mathfrak{n}, \mathfrak{n}]$. Now we obtain

$$B \cdot x = T \cdot (x + [\mathfrak{n}, \mathfrak{n}]) = \mathfrak{n}^r.$$

Since x is regular, $\mathfrak{n}^r$ is a single B-orbit consisting of regular nilpotent elements. $\square$

Corollary 1.4.13. *The element* $e = e_1 + \cdots + e_l$ *is a regular nilpotent element in* $\mathfrak{g}$.

Proof. It is clear that $e = e_1 + \cdots + e_l \in \mathfrak{n}^r$. The preceding lemma gives the result. $\square$

Lemma 1.4.14. *Any regular nilpotent element is contained in a unique Borel algebra. Equivalently, a fiber of a regular nilpotent element along a Springer resolution* $\tilde{\mathcal{N}} \to \mathcal{N}$ *consists of a single element.*

Proof. By 1.4.3 and 1.4.6, we have that

$$\dim \tilde{\mathcal{N}} = \dim T^*\mathcal{B} = 2 \dim G/B = 2(\dim \mathfrak{g} - \dim \mathfrak{b}) = 2 \dim \mathfrak{n} = \dim \mathcal{N}.$$

Hence a Springer resolution $\tilde{\mathcal{N}} \to \mathcal{N}$ is a surjection between irreducible varieties of the same dimension. Thus there exists a dense open subset of $\mathcal{N}$ where fibers are of dimension 0. Take a regular nilpotent element x contained in this dense open subset. Then $\mathcal{B}_x$ is a discrete. We may choose a Borel subgroup B and a maximal (split) torus T where $x \in [\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$. Choose positive simple roots $\alpha_1, \cdots, \alpha_l$ determined by B and T and $e_1, \cdots, e_n$ be a generator of corresponding root spaces. Let $t \in \mathfrak{t}$ be an element such that $\alpha_i(t) = 1$ for any i. Note that an element $h \in \mathfrak{t}$ is regular if and only if $\alpha(h) \neq 0$ for all $\alpha \in \Phi$. This follows from that if $x \in \mathfrak{g}_\alpha$, then $[h, x] = \alpha(h) \cdot x$. Hence h is regular and $[h, e_i] = \alpha(s)e_i = e_i$. By 1.4.13, we may suppose $x = e = e_1 + \cdots e_n$. Then we have $[h, e] = e$.

Consider a one-parameter subgroup $\mathbb{G}_m \to G$ such that $\mathrm{Lie}\mathbb{G}_m = kh$. For any $t \in k^\times$, we have $t \cdot e = \exp(t)e$. Hence $\mathbb{G}_m$ stabilizes the fiber $\mu^{-1}(e)$. Since the fiber is discrete, dimension of orbit is zero thus $\mathbb{G}^m$ fixes every point. It follows that $h \in \mathfrak{b}'$ for any $\mathfrak{b}' \in \mu^{-1}(e)$. Since h is regular, $\mathfrak{h} \subseteq \mathfrak{b}'$ for every such $\mathfrak{b}'$. But then $\mu^{-1}(e) \subseteq W \cdot \mathfrak{b}$, thus there exists only one point in this orbit containing the nilpotent e. $\square$

2 The Affine Grassmannian and the Affine Flag Manifold

2.1 Ind-Schemes

Definition 2.1.1. Let S be a scheme. A S-*space* X is a sheaf $X : (\mathrm{Sch}\,/S)^{\mathrm{op}} \to \mathrm{Set}$ with respect to fqpc-topology on Sch_S. An *ind-scheme* X over S is a filtered colimit $X = \varinjlim X_i$ in the category of S-spaces of where each X_i is represented by a S-scheme. We call X is represented by (X_i).

A morphism between ind-schemes is given by a morphism of sheaves.

Note that we have fully-faithful functors

$$\mathrm{Sch}\,/S \hookrightarrow \text{ind-}\mathrm{Sch}\,/S \hookrightarrow \mathrm{Sp}/S$$

where Sp/S is a category of S-spaces.

In this context, we only deal with ind-schemes X where the index category has a cofinal subcategory identified with a set $\mathbb{N}$ of natural numbers.

An ind-scheme X is called as of *ind-finite type*(resp. *ind*-P) if we can write $X = \varinjlim X_n$ where each X_n is of finite type(resp. P). Note that X can be written as a colimit of schemes in various ways.

Lemma 2.1.2. *Let T be a quasicompact S-scheme and $X = \varinjlim X_i$ an ind-scheme over S. Then a natural morphism*

$$\varinjlim \mathrm{Hom}_S(T, X_i). \to \mathrm{Hom}_S(T, X)$$

is an isomorphism.

In particular, for $S = \operatorname{Spec} k$ and a k-algebra R, we have $X(R) = \varinjlim X_i(R)$.

Proof. By A.0.4, X is a Zariski-sheafification of filtered colimit of presheaves. Let $\mathcal{F} = \varinjlim \mathrm{Hom}_S(-, X_i)$ is a filtered colimit of presheaves. Then we have

$$X(T) = \varinjlim_{\mathfrak{U}} \check{H}^0(\mathfrak{U}, \mathcal{F})$$

where $\mathfrak{U}$ is a Zariski open cover of T. Since any open cover $\mathfrak{U}$ of T can be refined by an finite open cover, it suffices to show

$$\mathcal{F}(T) \cong \check{H}^0(\mathfrak{U}, \mathcal{F})$$

with a finite open cover $\mathfrak{U}$ of T. Since filtered colimits commute with finite limits, we are done. $\square$

Remark 2.1.3. This kind of argument is widely used. For example, in the similar manner we can prove that étale cohomology is compatible with filtered colimits. But when in that situation, we need assume T is quasicompact and quasiseparated. Here quasiseparatedness is necessary since the sheafification is done in two steps. However, Zariski-sheafification is done in one step, so we don't need quasisepratedness.

Note that for $X = \varinjlim_i X_i$ and $Y = \varinjlim_j Y_j$, a morphism between filtered system $f_i : (X_i) \to (Y_j)$ induces a unique morphism $f : X \to Y$.

In some cases, we also have the converse.

Lemma 2.1.4. *Let* $X = \varinjlim_i X_i$ *and* $Y = \varinjlim_j Y_j$ *are ind-schemes over* S *where any* X_i *and* Y_j *are quasicompact. Then a morphism* $f : X \to Y$ *is induced from some compatible morphisms* $f_i : X_i \to Y_{i_j}$.

Proof. By 2.1.2, we have

$$\begin{aligned}\mathrm{Hom}_S(X, Y) &\cong \mathrm{Hom}_S(\varinjlim_i X_i, \varinjlim_j Y_j)\\ &\cong \varprojlim_i \mathrm{Hom}_S(X_i, \varinjlim_j Y_j)\\ &\cong \varprojlim_i \varinjlim_j \mathrm{Hom}_S(X_i, Y_j).\end{aligned}$$

□

Lemma 2.1.5. *Let* $X = \varinjlim_i X_i$, $Y = \varinjlim_i Y_i$, $Z = \varinjlim_i Z_i$ *be ind-scheme over* S *and* $f_i : X_i \to Z_i$, $g_i : Y_i \to Z_i$ *are compatible morphisms. For an induced morphisms* $f : X \to Z$ *and* $g : Y \to Z$*, a fiber product is given by* $X \times_Z Y \cong \varinjlim_i X_i \times_{Z_i} Y_i$.

Proof. Filtered colimits commute with finite limites. □

For now, we assume $S = \operatorname{Spec} k$ with a field k.

Example 2.1.6. We want to identify $k[\![t]\!]$ with k-points of some k-scheme. A power series $\sum_{i \geq 0} a_i t^i$ is given by its countably many coefficients, so it can be identified with a k-point of $\mathbb{A}^\infty = \prod_{i \geq 0} \mathbb{A}^1 = \operatorname{Spec} k[x_0, x_1, \cdots]$. In the similar manner, we can identify the set of doubly infinite power series $\sum_{i=-\infty}^{\infty} a_i t^i$ as k-pointes of infinite dimensional affine space.

Now consider a set of Laurent series $k(\!(t)\!)$. Clearly it is contained in a set of k-points of $\prod_{i=-\infty}^{\infty} \mathbb{A}^1$, but the condition that all but finitely many coefficients of negative indices are zero cannot be given as polynomial equations. Hence we cannot express $k(\!(t)\!)$ as a closed subscheme of doubly infinite product. Instead, we have

$$k((t)) = \bigcup_{i \leq 0} t^i k[[t]].$$

For $i \leq 0$, $X_i = \prod_{j \geq i} \mathbb{A}^1$. Then for any $i' \leq i$, we have a closed immersion $X_i \hookrightarrow X_{i'}$. Set $X = \varinjlim_{i \leq 0} X_i$, then 2.1.2 gives that

$$X(k) = k((t)).$$

2.2 The affine Grassmannian of GL_n

Let k be a field and R a k-algebra. We denote by $R[[t]]$ the ring of formal power series over R and by $R((t))$ the ring of Laurent series over R. Note that $R((t))$ is a localization of $R[[t]]$ with respect to t.

Definition 2.2.1. An R-family of lattices in $k((t))^n$ is a finitely generated projective $R[[t]]$-submodule Λ of $R((t))^n$ such that $\Lambda \otimes_{R[[t]]} R((t))^n = R((t))^n$.

We call $\Lambda_0 = R[[t]]^n$ a standard lattice.

Definition 2.2.2. The affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}$ is the presheaf sending a k-algbra R to the set of R-families of lattices in $k((t))^n$.

For simplicity, we denote $\mathrm{Gr}_{\mathrm{GL}_n}$ by Gr in this subsection.

Theorem 2.2.3. *The affine Grassmannian* Gr *is represented by an ind-projective scheme.*

Proof. For an R-family of lattice $\Lambda \subseteq R((t))^n$, we claim that there exists some positive integer N such that

$$t^N R[[t]]^n \subseteq \Lambda \subseteq t^{-N} R[[t]]^n.$$

Since Λ is finite projective $R[[t]]$-module of rank n, we have $(g_1, \cdots, g_s) \in R[[t]]$ which generates the unit ideal and Λ_{g_i} is free $R[[t]]_{g_i}$-module of rank n. Since the units of $R[[t]]$ are the power series whose constant terms are units, $f_i := g_i(0)$ generates the unit ideal in R. For any i such that $f_i \neq 0$, we have that g_i is a unit in $R_{f_i}[[t]]$. For such i's, it follows that

$$\Lambda \otimes_{R[[t]]} R_{f_i}[[t]] = \Lambda \otimes_{R[[t]]} R[[t]]_{g_i} \otimes_{R[[t]]_{g_i}} R_{f_i}[[t]]$$

is free of rank n over $R_{f_i}[[t]]$. Hence Λ is free Zariski-locally.

Thus we obtain a faithfully-flat $R \to R'$ such that

$$\Lambda_{R'} := \Lambda \otimes_{R[\![t]\!]} R'[\![t]\!]$$

is free of rank n. Then we may write $\Lambda_{R'} = g\Lambda^0$ where $\Lambda^0 = R'[\![t]\!]$ is a standard torus of $R'(\!(t)\!)$ and $g \in GL_n(R'(\!(t)\!))$. By considering the valuation of each entries of g, we have a positive integer N such that

$$t^N R'[\![t]\!]^n \subseteq \Lambda_{R'} \subseteq t^{-N} R'[\![t]\!]^n.$$

Note that all three modules are contained in $R'(\!(t)\!)^n$, thus we can intersect them with $R(\!(t)\!)^n \subseteq R'(\!(t)\!)^n$. It is clear that $t^{\pm N} R'[\![t]\!] \cap R(\!(t)\!)^n = t^{\pm} R[\![t]\!]$. Since $\Lambda \otimes_{R[\![t]\!]} R(\!(t)\!) = R(\!(t)\!)^n$ and Λ is a flat $R[\![t]\!]$-module, we obtain

$$\begin{aligned} \Lambda_{R'} \cap R(\!(t)\!)^n &= (\Lambda \otimes_{R[\![t]\!]} R'[\![t]\!]) \cap (\Lambda \otimes_{R[\![t]\!]} R(\!(t)\!)) \\ &= \Lambda \otimes_{R[\![t]\!]} (R'[\![t]\!] \cap R(\!(t)\!)) \\ &= \Lambda. \end{aligned}$$

Thus by intersecting with $R(\!(t)\!)^n$ in $R'(\!(t)\!)^n$, we obtain

$$t^N R[\![t]\!]^n \subseteq \Lambda \subseteq t^{-N} R[\![t]\!]^n.$$

Actually, the converse also holds : See the lemma 2.11 of [Gortz].

Now let $\mathrm{Gr}^{(N)}$ be the subfunctor of Gr consisting of lattices satisfying the above condition. Then we obtain

$$\mathrm{Gr} = \bigcup_{N \geq 1} \mathrm{Gr}^{(N)}.$$

Thus is suffices to show that each $\mathrm{Gr}^{(N)}$ is projective k-scheme.

To show this, first define a presheaf $\mathrm{Gr}^{(N),f}$ given by

$$\mathrm{Gr}^{(N),f}(R) := \{R[\![t]\!]\text{-quotient modules of } \frac{t^{-N}R[\![t]\!]^n}{t^N R[\![t]\!]^n} \text{ that are projective } R\text{-modules}\}.$$

We denote $\mathrm{Gr}(m)$ by the usual Grassmannian variety parametrizing finite dimensional subspace of k^m. It is a disjoint union of $\mathrm{Gr}(r, m)$, parametrizing subspaces of dimension r.

Lemma 2.2.4. *The presheaf* $\mathrm{Gr}^{(N),f}$ *is represented by a closed subscheme of* $\mathrm{Gr}(2nN)$.

Note that a R-point of a Grassmannian $\mathrm{Gr}(r, m)$ is given by

$$0 \to M \to R^m \to N \to 0$$

where N is a locally free R-module of rank r, which is equivalent to say that N is finite projective R-module of rank r. Note that M is also a locall free R-module of finite rank $m - r$.

To prove this lemma, we identify $t^{-N}R[\![t]\!]^n/t^N R[\![t]\!]^n$ with R^{2nN}. Since $N \in \mathrm{Gr}^{(N),f}(R)$ is a quotient $R[\![t]\!]$-module, it follows that M is Φ-invariant where Φ is a multiplication by t. Hence a R-point of $\mathrm{Gr}^{(N),f}$ can be identified with a projective submodule $M \subseteq R^{2nN}$ with a projective cokernel, which is an Φ-invariant submodule. Since Φ is a nilpotent operator on R^{2nN}, any Φ-invariant projective submodule is given by the kernel of some Φ^i. It follows that $\mathrm{Gr}^{(N),f}$ is a zero locus of polynomials, hence it is a closed subscheme of Gr^{2nN}.

Next we need the lemma.

Lemma 2.2.5. *A presheaf* $\mathrm{Gr}^{(N)}$ *is identified with* $\mathrm{Gr}^{(N),f}$ *by the map*

$$\Lambda \mapsto Q = t^{-N}R[\![t]\!]^n/\Lambda.$$

First we check that this map is well-defined. Since Λ is projective $R[\![t]\!]$-module, it is a direct summand of a free $R[\![t]\!]$-module. It follows that $\Lambda/t\Lambda$ is a projective R-module. We have

$$\begin{aligned} R(\!(t)\!)^n/\Lambda &\cong ((R[\![t]\!]_t)^n \otimes_{R[\![t]\!]^n} \Lambda)/\Lambda \\ &\cong \Lambda_t/\Lambda \\ &\cong \bigoplus_{k \geq 0} t^{-k-1}\Lambda/t^{-k}\Lambda. \end{aligned}$$

Since $\Lambda/t\Lambda$ is projective, so is $R(\!(t)\!)^n/\Lambda$. Then Q is also projective since is admits a short exact sequence

$$0 \to Q \to R(\!(t)\!)^n/\Lambda \to R(\!(t)\!)^n/t^{-N}R[\![t]\!]^n \to 0.$$

This shows that we can identify $\mathrm{Gr}^{(N)}$ as a subsheaf of $\mathrm{Gr}^{(N),f}$.

It remains to show the surjectivity on R-points. By the construction, an R-point of $\mathrm{Gr}^{(N),f}$ is given by a $R[\![t]\!]$-quotient map $t^{-N}R[\![t]\!]^n/t^N R[\![t]\!]^n \to Q$. Consider the $R[\![t]\!]$-module

$$\Lambda := \ker(t^{-N}R[\![t]\!]^n \to t^{-N}R[\![t]\!]^n/t^N R[\![t]\!]^n \to Q).$$

It suffices to show that Λ is a finite projective $R[\![t]\!]$-module. Since the localization $R(\!(t)\!)$ is a flat $R[\![t]\!]$-module, the functor $-\otimes_{R[\![t]\!]^n} R(\!(t)\!)^n$ is exact. In particular,

$$0 = t^{-N}R[\![t]\!]^n/t^N R[\![t]\!]^n \otimes_{R[\![t]\!]^n} R(\!(t)\!)^n \to Q \otimes_{R[\![t]\!]^n} R(\!(t)\!)^n$$

is surjective. Hence $Q \otimes_{R[\![t]\!]^n} R(\!(t)\!)^n = 0$. Then $\Lambda \otimes_{R[\![t]\!]^n} R(\!(t)\!)^n$ is the kernel of $R(\!(t)\!)^n \to 0$, thus isomorphic to $R(\!(t)\!)^n$.

We may write $R = \varinjlim R_i$ where R_i are finitely generated k-algebras. By 2.2.4, $\mathrm{Gr}^{(N),f}$ is locally of finite type. Then we have(cf. [Stacks,01ZC])

$$\mathrm{Gr}^{(N),f}(\varinjlim R_i) = \varinjlim \mathrm{Gr}^{(N),f}(R_i).$$

Hence we may assume that R is finite k-module. Then $R[t]$ is Noetheirian, so [Stacks,00MB] gives that $R[t] \to R[\![t]\!]$ is flat. Also we have $t^{-N}R[t]^n/t^N/R[t]^n \cong t^{-N}R[\![t]\!]^n/t^N R[\![t]\!]^n$. We defined an $R[t]$-module

$$\Lambda_f := \ker(t^{-N}R[t]^n \to t^{-N}R[t]^n/t^N R[t]^n \to Q).$$

Since $R[t] \to R[\![t]\!]$ is flat, it follows that $\Lambda \cong \Lambda_f \otimes_{R[t]} R[\![t]\!]$. Hence it suffices to show that Λ_f is finite projective $R[t]$-module. Since $R[t]$ is Noetherian, Λ_f is finite $R[t]$-module. It remains to show that Λ_f is $R[t]$-projective. Since $R[t]$ is Noetherian and Λ_f is a finite $R[t]$-module, it suffices to show that Λ_f is a flat $R[t]$-module.

As R-module, Λ_f is the kernel of projective R-modules. Hence Λ_f is R-projective, so in particular R-flat. To show that Λ_f is $R[t]$-flat, it suffices to show $(\Lambda_f)_\mathfrak{q}$ is $(R[t])_\mathfrak{q}$-flat for every prime ideal $\mathfrak{q} \subseteq R[t]$. We will use [Stacks,00MH]. Let $\mathfrak{p} = \mathfrak{q} \cap R$ be a prime ideal of R and K a residue field of $\mathfrak{p}$. Since $R \to R[t]$ is flat, so is $R_\mathfrak{p} \to (R[t])_\mathfrak{q}$. Also $(\Lambda_f)_\mathfrak{p}$ is $R_\mathfrak{p}$-flat, thus $(\Lambda_f)_\mathfrak{q} \cong (\Lambda_f)_\mathfrak{p}$ is $R_\mathfrak{p}$-flat. It remains to show $(\Lambda_f)_\mathfrak{q}/\mathfrak{p}(\Lambda_f)_\mathfrak{q}$ is a free $(R[t])_\mathfrak{q}/\mathfrak{p}(R[t])_\mathfrak{q}$-module. It suffices to show $(\Lambda_f/\mathfrak{p}\Lambda_f)_\mathfrak{p} \cong \Lambda_f \otimes_{R[t]} K[t]$ is a free $(R[t]/\mathfrak{p}R[t])_\mathfrak{p} \cong K[t]$-module. Note that

$$\Lambda_f \otimes_{R[t]} K[t] = \ker(t^{-N}K[t]^n \to Q \otimes_{R[t]} K[t]),$$

thus $\Lambda_f \otimes_{R[t]} K[t]$ is a torsion-free $K[t]$-module. Since $K[t]$ is a dedekind domain, $\Lambda_f \otimes_{R[t]} K[t]$ is a flat $K[t]$-module. Then since it is finitely generated flat module over a Noetherian ring, it is projective module. Since $K[t]$ is a P.I.D, it is indeed a free $K[t]$-module. $\square$

Remark 2.2.6. The idea of replacing $R[\![t]\!]$ with $R[t]$ will be used again later.

2.3 Affine Grassmannians of general groups

Definition 2.3.1. (The disc) We denote

$$D = \operatorname{Spec} k[\![t]\!], \quad D^* = \operatorname{Spec} k(\!(t)\!)$$

and call them *the disc* and *the punctured disc*. For a k-algebra R, we write

$$D_R := D \hat{\times} \operatorname{Spec} R = \operatorname{Spec} R[\![t]\!], \quad D_R^* := D^* \hat{\times} \operatorname{Spec} R = \operatorname{Spec} R(\!(t)\!).$$

Note that D_R^* is a distinguished open subset of D_R.

Remark 2.3.2. Aware that $R[\![t]\!] \otimes_R R'$ is not isomorphic to $R'[\![t]\!]$ in general. For example, let $R = \mathbb{C}$ and $R' = \mathbb{C}[t]$.

Definition 2.3.3. (Affine Grassmannians of general groups) Let G be a smooth affine k-group. We definte the affine Grassmannian Gr_G of G as

$$\mathrm{Gr}_G(R) = \{(\mathcal{E}, \beta) \mid \mathcal{E} \text{ is a G-torsor on } D_R, \text{ and } \beta : \mathcal{E}|_{D_R^*} \cong G|_{D_R^*} \text{ is a trivialization}\}.$$

Here we mean torsor in fppf-topology. Indeed, this definition also makes sense for a smooth affine group scheme $\underline{G}$ over $k[\![t]\!]$. Note that each $\mathrm{Gr}_G(R)$ has the trivial element $(\mathcal{E}, \mathrm{id}_{D_R^*})$.

Remark 2.3.4. (i) Note that $(\mathcal{E}, \beta)$ the definition is different with a G-torsor on D_R trivialized at D_R^*. Here we specified the trivialization isomorphism β. For example, $tk[\![t]\!]$ and $k[\![t]\!]$ are isomorphic as $k[\![t]\!]$-module and tensor product to $k(\!(t)\!)$ becomes $k(\!(t)\!)$. But they are not the same in $\mathrm{Gr}_{GL_1}(k)$ since the trivialization isomorphisms are different.

(ii) The above definition and the following theorem also holds for the ring of integers of any local field.

Note that an element in $\mathrm{Gr}_G(R)$ can be written as

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "\pi"] \\ D_R^* \arrow[ur, "\beta"] \arrow[r] & D_R \end{tikzcd}$$

since a torsor with a global section is trivial. Actually this β corresponds to β^{-1}, but we write it with β for convenience.

Example 2.3.5. Let $G = GL_n$, then GL_n-torsors corresponds to locally free sheaves of rank n(in Zariski topology). Hence a GL_n-torsor on D_R trivialized at D_R^* corresponds to a locally free sheaf of rank n on $\operatorname{Spec} R[\![t]\!]$, which is a finitely generated projective $R[\![t]\!]$-module Λ of rank n whose base change to $R(\!(t)\!)$ is isomorphic to $R(\!(t)\!)^n$. Also under this identification, we can see Λ as an $R[\![t]\!]$-submodule of $R(\!(t)\!)^n$. Hence this is equivalent the definition of 2.2.1.

Example 2.3.6. Let $G = SL_n$, then we have an exact sequence

$$1 \to SL_n \to GL_n \to \mathbb{G}_m \to 1$$

of sheaves. Then we have the following exact sequence

$$1 \to H^1(D_R, SL_n) \to H^1(D_R, GL_n) \to H^1(D_R, \mathbb{G}_m)$$

of pointed sets. Hence we can identify SL_n-torsors with a locally free sheaf $\mathcal{E}$ of rank n whose determinant line bundle $\wedge^n\mathcal{E}$ is trivial. Let Λ be the $R[\![t]\!]$-submodule of $R(\!(t)\!)^n$ corresponding to $(\mathcal{E}, \beta) \in \mathrm{Gr}_{SL_n}$. Then we have $\wedge^n\Lambda = R[\![t]\!]$. Hence we obtain

$$\mathrm{Gr}_{SL_n}(R) = \{\Lambda : R\text{-family of lattices in } k(\!(t)\!)^n \text{ such that } \wedge^n \Lambda = R[\![t]\!]\}.$$

Such lattices are often called by unimodular lattices.

Example 2.3.7. Let $G = \mathbb{G}_a$. Note that a $\mathbb{G}_a$-torsor of a scheme X is a short exact sequence of quasicoherent sheaves

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X \to 0.$$

Thus $\mathrm{Gr}_{\mathbb{G}_a}(R)$ is the set of short exact sequences of $R[\![t]\!]$-modules

$$0 \to R[\![t]\!] \to \Lambda \to R[\![t]\!] \to 0$$

which splits after tensored with $R(\!(t)\!)$ over $R[\![t]\!]$.

Theorem 2.3.8. *The presheaf* $\mathrm{Gr}_{\underline{G}}$ *is represented by an ind-scheme finite type over* k*. If* $\underline{G}$ *is reductive, then* $\mathrm{Gr}_{\underline{G}}$ *is ind-projective.*

This theorem is proved by the following two propositions.

Remark 2.3.9. The above definition and the theorem makes sense for any local field and a ring of integer.

Proposition 2.3.10. *Let $\underline{G}$ be a smooth affine group scheme over* D*. Then there exists a faithful* $k[\![t]\!]$*-linear representation* $\rho : \underline{G} \to GL_n$ *such that* $GL_n / \underline{G}$ *is quasi-affine. Furthermore, if* $\underline{G}$ *is reductive, one can choose* ρ *such that* $GL_n / \underline{G}$ *is affine over* D*.*

Proof. See [Pappas,Proposition 1.3]. ∎

Suppose a group homomorphism $\varphi : G \to G'$ is given. Then we have a functor from G-torsors to G'-torsor by sending a G-torsor $\mathcal{E}$ to a contracted product $\mathcal{E} \times^{G} G'$ given by a sheaf associated to a presheaf $\mathcal{E} \times G'/\sim$ where $(s, g') \sim (s\varphi(g^{-1}), \varphi(g)g')$. Then for a linear representation $\underline{G} \to GL_n$, we obtain the induced morphism $\underline{G} \to GL_n$ defined by

$$\begin{tikzcd} & \mathcal{E}' \arrow[d, "\pi"] \\ D_R^* \arrow[ur, "\beta'"] \arrow[r] & D_R \end{tikzcd} \mapsto \begin{tikzcd} & \mathcal{E}' \times_{\underline{G}} GL_n \arrow[d, "\pi"] \\ D_R^* \arrow[ur, "\beta"] \arrow[r] & D_R \end{tikzcd}$$

where β is given by a pair β with a section $D_R \to GL_n$ corresponding to the identity.

Proposition 2.3.11. *Let* $\rho : \underline{G} \to GL_n$ *be a linear representation such that* $GL_n/\underline{G}$ *is quasi-affine. Then the induced morphism* $f_\rho : Gr_{\underline{G}} \to Gr_{GL_n}$ *is a locally closed embedding. Furthermore, if* $GL_n/\underline{G}$ *is affine, it is a closed embedding.*

Proof. Let $\operatorname{Spec} R \to Gr_{GL_n}$ be an R-point of Gr_{GL_n} given by $(\mathcal{E}, \beta)$. We need to show that a cartesian diagram

$$\begin{tikzcd} \mathcal{F} := \operatorname{Spec} R \times_{Gr_{GL_n}} \underline{G} \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "(\mathcal{E},\beta)"] \\ Gr_{\underline{G}} \arrow[r] & Gr_{GL_n} \end{tikzcd}$$

gives that $\operatorname{Spec} R \times_{Gr_{GL_n}} \underline{G}$ is a represented by locally closed embedding of $\operatorname{Spec} R$. Let R' be an R-algebra, then R' point of $\operatorname{Spec} R \times_{Gr_{GL_n}} \underline{G}$ is given by a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} R' \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \operatorname{Spec} R \times_{\mathrm{Gr}_{\mathrm{GL}_n}} \underline{G} \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "{(\mathcal{E},\beta)}"] \\
& \mathrm{Gr}_{\underline{G}} \arrow[r] & \mathrm{Gr}_{\mathrm{GL}_n}.
\end{tikzcd}$$

Equivalently, this says that R'-point corresponds to a G-torsor $\mathcal{E}'$ on $D_{R'}$ with a section β' on $D^*_{R'}$

$$\begin{tikzcd}
& \mathcal{E}' \arrow[d] \\
D^*_{R'} \arrow[ur, "\beta'"] \arrow[r] & D_{R'}
\end{tikzcd}$$

admit the commuative diagram

$$\begin{tikzcd}
& \mathcal{E}' \times^{\underline{G}} \mathrm{GL}_n \arrow[d, dashed, "\tilde{\pi}'"] \arrow[dr, "\cong"] & \\
& & \mathcal{E}_{R'} \arrow[dl, "\pi_{R'}"] \\
D^*_{R'} \arrow[uur, "\tilde{\beta}'"] \arrow[urr, "\beta_{R'}"] \arrow[r] & D_{R'} &
\end{tikzcd}$$

where the isomorphism is a GL_n-equivariant one. Since $\mathcal{E}' \times^{\underline{G}} \mathrm{GL}_n$ is already GL_n-torsor, any morphism $\mathcal{E}' \times^{\underline{G}} \mathrm{GL}_n \to \mathcal{E}$ admitting the diagram becomes an isomorphism. By the adjunction, it equivalent to give a $\underline{G}$-equivariant α admitting

$$\begin{tikzcd}
& \mathcal{E}' \arrow[dd, dashed, "\pi'"] \arrow[dr, "\alpha"] & \\
& & \mathcal{E}_{R'} \arrow[dl, "\pi_{R'}"] \\
D^*_{R'} \arrow[uur, "\beta'"] \arrow[urr, "\beta_{R'}"] \arrow[r] & D_{R'} &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
& \mathcal{E}' \arrow[dd, dashed, "\pi'"] \arrow[dr, "\alpha"] & \\
\underline{G} \arrow[ur, "\beta'"] \arrow[rr, "\beta_{R'}"] \arrow[d] & & \mathcal{E}_{R'} \arrow[dl, "\pi_{R'}"] \\
D^*_{R'} \arrow[r] & D_{R'} &
\end{tikzcd}$$

where $\pi_{R'}$ and $\beta_{R'}$ are given by retriction GL_n-actions to $\underline{G}$-actions. Note that a $D_{R'}$-point of $\overline{\pi} : [\mathcal{E}/\underline{G}] \to D_R$ is given by

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "\alpha"] \arrow[dr] & \mathcal{E}_{R'} \arrow[d] \\ & D_{R'} \end{tikzcd}$$

where $\mathcal{E}$ is a $\underline{G}$-torsor on $D_{R'}$ and α is a $\underline{G}$-equivariant morpism. By 2.3.10, $GL_n/\underline{G}$ is a (quasi-)affine scheme over D_R. Since $\mathcal{E}/\underline{G}$ is fpqc-locally isomorphic to $GL_n/\underline{G}$, it is also represented by a quasi-affine scheme over D_R. Also we have a natural map $[\mathcal{E}/\underline{G}] \to \mathcal{E}/\underline{G}$, and it is an isomorphism if and only if the stabilizers are trivial. But this holds since $\underline{G}$ acts freely on $\mathcal{E}$, hence we can identify $[\mathcal{E}/\underline{G}]$ with $\mathcal{E}/\underline{G}$. Hence $D_{R'}$-points of $[\mathcal{E}/\underline{G}]$ correspond to sections of $\overline{\pi}_{R'} : \mathcal{E}_{R'}/\underline{G}_{R'} \to D_{R'}$. The given β gives the induced section $\overline{\beta_{R'}}$ of $\overline{\pi}_{R'}$ over $D^*_{R'}$ as

$$\begin{tikzcd} \underline{G} \arrow[r] \arrow[dr] \arrow[rr, bend left, "\overline{\beta_{R'}}"] & GL_n \arrow[r] \arrow[d] & \mathcal{E}_{R'} \arrow[d] \\ & D^*_{R'} \arrow[r] & D_{R'} \end{tikzcd}$$

Thus $\mathcal{F}$ is the presheaf over $\operatorname{Spec} R$ where $\mathcal{F}(R')$ is the set of sections β' of $\overline{\pi}$ over $D_{R'}$ such that $\beta'|_{D^*_{R'}} = \overline{\beta}|_{D^*_{R'}}$.

Let W be an affine space over D_R where $\mathcal{E}/\underline{G} \subseteq W$ is a quasicompact open subscheme. To get the result, we need the following lemma :

Lemma 2.3.12. *Let $p : V \to D_R$ be an affine scheme of finite presentation and s a section of p over D^*_R. Then the presheaf $\mathcal{F}$ on $\operatorname{Spec} R$ where $\mathcal{F}(R')$ is the set of sections s' of p over $D_{R'}$ satisfying $s'|D^*_{R'} = s|_{D^*_{R'}}$ is represented by a closed subscheme of $\operatorname{Spec} R$.*

Proof. Let $V \subseteq \mathbb{A}^N_{D_R}$ is a closed embedding in to some affine space. Then we may write

$$s = (s_1(t), \cdots, s_N(t)), \quad s_i(t) = \sum s_{ij} t^j \in R((t))$$

where $s_i(t)$'s satisfy some equations. Then R'-point of $\mathcal{F}$ is given by

$$s' = (s'_1(t), \cdots, s'_N(t)), \quad s'_i(t) \in R'[[t]]$$

where $s_i'(t)$'s satisfy the same equations and $s_i'(t) = s_i(t)$ in $R'((t))$. This shows that $\mathcal{F}$ is represented by $\operatorname{Spec} R/I$ where I is the ideal generated by $\{s_{ij} \mid 1 \le i \le N, j < 0\}$. □

Let $U = \mathcal{E}/\underline{G} \subseteq W$, then a section $\overline{\beta}$ of U over D_R^* can be seen as a section of W over D_R^*. Applying the lemma with $V = W$ and $s = \overline{\beta} : D_R^* \to W$, we obtain a closed subscheme $\operatorname{Spec} A \subseteq \operatorname{Spec} R$ represents the presheaf $\mathcal{F}'$ constructed similarly with $W \to D_R$. Then for $R' = A$, the set of A-points of this presheaf has the natural point $\mathrm{id} : \operatorname{Spec} A \to \operatorname{Spec} A$ and the corresponding section $\beta_A : D_A \to W_{D_A}$ whose restriction to D_A^* equals to $\overline{\beta}|_{D_A^*}$. Now the R'-points of $\mathcal{F}$ is given by a base change of β_A. Base change along the closed embedding $\operatorname{Spec} A \to D_A$ defined by $t = 0$ gives the section $s_0 : \operatorname{Spec} A \to W \times_{D_R} \operatorname{Spec} A$. Then $\mathcal{F}$ is represented by the open subscheme $s_0^{-1}(\mathcal{E}/\underline{G} \times_{D_R} \operatorname{Spec} A)$ of $\operatorname{Spec} A$. □

Now let $\mathrm{Gr}_{GL_n} = \varinjlim X_i$ where X_i are projective and transtition maps $X_i \to X_j$ are closed immersions. Since $\mathrm{Gr}_{\underline{G}} \to \mathrm{Gr}_{GL_n}$ is a (locally) closed embedding, we have a cartesian diagram

$$\begin{tikzcd} X_i' \arrow[r, hook] \arrow[d] & X_i \arrow[d] \\ \mathrm{Gr}_{\underline{G}} \arrow[r] & \mathrm{Gr}_{GL_n} \end{tikzcd}$$

where $X_i' \hookrightarrow X_i$ is a (locally) closed embedding for each i. Hence X_i' is of finite type, and further projective if G is reductive. Since filtered colimits commute with finite limits, we have a carteisan diagram

$$\begin{tikzcd} \varinjlim X_i' \arrow[r, hook] \arrow[d] & \varinjlim X_i \arrow[d] \\ \mathrm{Gr}_{\underline{G}} \arrow[r] & \mathrm{Gr}_{GL_n}. \end{tikzcd}$$

Since the right vertical morphism is an isomorphism , so is the left vertical one.

2.4 Loop Groups

In this section we give another characterization of an affine Grassmannian.

Definition 2.4.1. Let X be a presheaf over $\mathcal{O} = k[\![t]\!]$. The space of n-jets $L^n X$ is the presheaf defined by

$$L^n X(R) = X(R[t]/t^n)$$

for any k-algebra R. The *positive loop space* L^+X is the presheaf defined by

$$L^+X(R) = X(R[\![t]\!]) = X(D_R)$$

for any k-algebra R. The *loop space* LX is the presheaf defined by

$$LX(R) = X(R(\!(t)\!)) = X(D_R^*)$$

for any k-algebra R. If X is a presheaf on k, then we denote $L^+(X \otimes_k \mathcal{O})$(resp. $LX(X \otimes_k \mathcal{O})$) by L^+X(resp. LX).

Proposition 2.4.2. *(1) Let X be an affine scheme of finite type over $\mathcal{O} = k[\![t]\!]$. Then $L^n X$ is represented by an affine scheme of finite type over k, and*

$$L^+X \cong \varprojlim L^+X$$

holds. In particular, L^+X is represented by an affine scheme over k.

(2) Let X be an affine scheme of finite type over $F = k(\!(t)\!)$. Then LX is represented by an ind-affine scheme over k.

Proposition 2.4.3. *For a scheme X over a commutative ring $\mathcal{O}$, $L^n X$ and L^+X are both schemes. If X is (locally) of finite type, then so is $L^n X$(but not L^+X). If X is affine, then so is $L^n X$ and L^+X.*

Proof. First suppose that $X = \mathbb{A}^1$. We have

$$L^n X(R) = \mathrm{Hom}(\mathrm{Spec}\, R[t]/(t^n), \mathbb{A}^1) = R[t]/(t^n) \cong \prod_{i=1}^{n} R.$$

This shows that $L^n X$ is represented by an affine scheme $\mathbb{A}^n$.

Now suppose that X is an affine scheme. We may write

$$X = \mathrm{Spec}\, \mathcal{O}[x_1, x_2, \cdots, x_i, \cdots]/(f_1, f_2, \cdots, f_r, \cdots).$$

Then X admits a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_i \mathbb{A}^1 \arrow[d, "f_r"] \\ \mathrm{Spec}\, \mathcal{O} \arrow[r, "0"] & \prod_r \mathbb{A}^1, \end{tikzcd}$$

thus X is a limit of a diagram consists of copies of $\mathbb{A}^1$. If X is of finite type, then it is a finite limit of copies of $\mathbb{A}^1$. Hence we may write $X = \lim_\alpha \mathbb{A}^1_\alpha$, then we obtain

$$\begin{aligned} L^n X(R) = X(R[t]/(t^n)) &= \mathrm{Hom}(\mathrm{Spec}\, R[t]/(t^n), \lim_\alpha \mathbb{A}^1_\alpha) \\ &= \lim_\alpha \mathrm{Hom}(\mathrm{Spec}\, R, L^n \mathbb{A}^1) \\ &= \mathrm{Hom}(\mathrm{Spec}\, R, L^n \lim_\alpha L^n \mathbb{A}^1). \end{aligned}$$

Since each $L^n\mathbb{A}^1$ is affine, such limit $\lim_\alpha L^n\mathbb{A}^1$ exits as an affine scheme so we obtain $X = \lim_\alpha L^n\mathbb{A}^1$.

Now let X be a general scheme and U an affine open subset of X. It can be easily checked that

$$\begin{tikzcd} U(R[t]/(t^n)) \arrow[r] \arrow[d] & X(R[t]/(t^n)) \arrow[d] \\ U(R) \arrow[r] & X(R) \end{tikzcd}$$

is cartesian for any commutative $\mathcal{O}$-algebra R thus so is

$$\begin{tikzcd} L^n U \arrow[r] \arrow[d] & L^n X \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

Now let U_i an affine open cover of X, then we obtain that $L^n U_i$ is a Zariski open cover of $L^n X$ by the above argument. Since each $L^n U_i$ is representable by a scheme, so is $L^n X$.

Observe that $L^{n+1}X \to L^n X$ is an affine morphism. Thus $L^+X = \varprojlim_{n\geq 1} L^n X$, L^+X is represented by a scheme, and furthermore if $L^n X$ is affine, then so is L^+X. □

Proof. First let $X = \mathbb{A}^m_{\mathcal{O}}$. Then we have

$$X(R[t]/(t^n)) = \mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathcal{O}[x_1, \cdots, x_m], R[t]/(t^n)) \cong \prod_{i=1}^{r} R[t]/t^n \cong \prod_{i=1}^{m} R^{n+1},$$

where the last isomorphism is a bijection of sets. It follows that

$$X(R[t]/t^n) \cong X^{n+1}(R) \cong \prod_{i=1}^{m} R^{n+1}.$$

Hence L^nX is represented by X^{n+1}, which is affine and of finite type.

Now let $X = \operatorname{Spec} \mathcal{O}[x_1, \dots, x_m]/(f_1, \cdots, f_r)$. We have

$$X(R[t]/t^n) = \operatorname{Hom}_{\mathcal{O}\text{-alg}}(\mathcal{O}[x_1, \cdots, x_m]/(f_1, \cdots, f_r), R[t]/t^n),$$

thus $\varphi \in X(R[t]/t^n)$ is given by $\varphi(x_i) = \sum_{j=0}^{n-1} a_{ij} t^j$ satisfying that

$$f_l(\sum_{j=0}^{n-1} a_{1j} t^j, \cdots, \sum_{j=0}^{n-1} a_{mj} t^j) = 0$$

for each $l = 1, \cdots, r$. Note that we identify $R[t]/t^n$ with R^n as a set by sending $\sum a_j t^j$ to $(a_1, \cdots, a_{n-1})$. Under this isomorphism, we can identify φ as a homomorphism

$$\mathcal{O}[x_{ij}]_{1 \le i \le m, 0 \le j \le n-1} \to R$$

satisfying the equations given by the above ones. This shows that L^+X is represented by an affine scheme of finite type over k.

Since limits are commute, we have

$$L^+X(R) = X(R[\![t]\!]) = X(\varprojlim R[t]/t^n) = \varprojlim X(R[t]/t^n) = \varprojlim L^nX$$

for any k-algebra R. This shows that $L^+X \cong \varprojlim L^nX$. Since each L^nX is affine, the transition maps $L^{n+1}X \to L^nX$ are all affine. It follows that L^+X is represented by affine schemes.

Note that we have a morphism $L^+X \to L^+X$ given by

$$X(R[\![t]\!]) \xrightarrow{-\cdot t} X(R[\![t]\!]).$$

This gives a closed embedding $L^+X \to L^+X$, and we have a sequence

$$L^+X \xrightarrow{-\cdot t} L^+X \xrightarrow{-\cdot t} L^+X \xrightarrow{-\cdot t} \cdots.$$

Denote the i-th L^+X with Y_i and consider $Y = \varinjlim Y_i$. Since each Y_i is affine, 2.1.2 gives that

$$Y(R) = \varinjlim Y_i(R) = \varinjlim_{i \ge 0} \frac{1}{t^i} R[\![t]\!] = LX(R).$$

for any k-algebra R. This shows that LX is ind-affine scheme. $\square$

Let $\underline{X}$ be an affine scheme of finite over $\mathcal{O}$ and $X = \underline{X} \otimes_{\mathcal{O}} F$. Since each transition morphism of LX is a closed immersion, $L^+\underline{X} \subseteq LX$ is a closed immersion.

Example 2.4.4. Let $X = \mathbb{A}^1_k$, then $L^+X = \mathbb{A}^\infty_{i\geq 0}$ and $LX = \varinjlim_{i\leq 0} \mathbb{A}^\infty_{j\geq i}$.

Proposition 2.4.5. *The affine Grassmannian* $\mathrm{Gr}_{\underline{G}}$ *can be identified with a fqpc quotient* $[L\underline{G}/L^+\underline{G}]$.

Proof. To show this, we first prove the following lemma :

Lemma 2.4.6. *Every* $\underline{G}$*-torsor on* D_R *can be trivialized over* $D_{R'}$ *for some étale covering* $\operatorname{Spec} R' \to \operatorname{Spec} R$.

Proof. Note that the structure morphism of group scheme admits a section, so it is surjective. Since the surjectivity is fpqc-local, every structure morphism of $\underline{G}$-torsor is surjective. Since $\underline{G}$ is smooth , every $\underline{G}$-torsor $\mathcal{E}$ is smooth and surjective. Then $\mathcal{E} \otimes_{R[\![t]\!]} R \to \operatorname{Spec} R$ is also smooth and surjective. Thus it admits an étale-local section, and by the quasicompactness of $\operatorname{Spec} R$ we may assume that we have a section over some étale covering $\operatorname{Spec} R' \to \operatorname{Spec} R$. Since $R'[\![t]\!]$ is a henselian ring and $\mathcal{E}_{D_{R'}}$ is smooth over $D_{R'}$, this section can be lifted to a section over $D_{R'}$. In particular, $\mathcal{E}_{D_{R'}}$ is a trivial torsor. ∎

Now consider a morphism of presheaves

$$\begin{aligned} LG/L^+G &\longrightarrow \mathrm{Gr}_{\underline{G}} \\ \beta &\mapsto (G, \beta). \end{aligned}$$

We show that this is a sheafification morphism. By [Stacks,00WK], it suffices to show that for any $\operatorname{Spec} R$ and $(\mathcal{E}, \beta) \in \mathrm{Gr}_G(R)$, there exists a fpqc cover $\operatorname{Spec} R'$ and $\beta' \in LG/L^+G(R')$ whose image under the above morphism is $(\mathcal{E}, \beta)|_{D_{R'}}$ and satisfying $\mathrm{pr}_0^*\beta' = \mathrm{pr}_1^*\beta'$.

Note that the given morphism $LG/L^+G \to \mathrm{Gr}_{\underline{G}}$ is injective. By the above lemma, $(\mathcal{E}, \beta) \in \mathrm{Gr}_G$ is trivalized over some $D_{R'}$ where $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an étale cover, which is also a fpqc cover. Consider the commutative diagram

$$\begin{tikzcd} & LG/LG^+(R') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & LG/LG^+(R' \otimes_R R') \arrow[d] \\ \mathrm{Gr}_G(R) \arrow[r] & \mathrm{Gr}_G(R') \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Gr}_G(R' \otimes_R R'), \end{tikzcd}$$

then we have a $\beta' \in LG/LG^+(R')$ such that its image in $\mathrm{Gr}_G(R')$ coincides with $(\mathcal{E}, \beta)_{D_{R'}}$. Such β' is unique by the injectivity of $LG/LG^+ \to \mathrm{Gr}_G$. It remains to show $\mathrm{pr}_0^*\beta' = \mathrm{pr}_1^*\beta'$. Both have the same image in $\mathrm{Gr}_G(R' \otimes_R R')$ along the vertical morphism, and the injectivity gives the result. $\square$

Remark 2.4.7. For a field k, note that every locally free sheaf on $k[\![t]\!]$ is trivial since any open neighborhood of $[(t)]$ is $\operatorname{Spec} k[\![t]\!]$ itself. Then it follows that $LG(k)/LG^+(k) = \mathrm{Gr}_{GL_n}(k)$.

Proposition 2.4.8. *The affine Grassmannian* Gr_G *is formally smooth.*

Proof. First we prove that LG and L^+G are formally smooth. For a square-zero ideal $I \subseteq R$, we claim that $L^+G(R) \to L^+G(R/I)$ is surjective. Equivalently, we need to show that $G(R[\![t]\!]) \to G(R/I[\![t]\!])$ is surjective. Observe that the kernel of a surjection $R[\![t]\!] \to R/I[\![t]\!]$ is given by $I[\![t]\!]$ and $I[\![t]\!]$ is a square-zero ideal of $R[\![t]\!]$. Since G is smooth, we are done.

In the similar manner, we can show that LG is formally smooth. Then it follows that $\mathrm{Gr}_G = LG/L^+G$ is also formally smooth. $\square$

Indeed, we have a generalization of the theorem.

Theorem 2.4.9. *Let* P *be an smooth affine group scheme over* E*. Consider the fpqc-sheaf associated to a presheaf*

$$\mathcal{F}_P : R \mapsto LP(R)/L^+P(R) = P(R(\!(t)\!))/P(R[\![t]\!])$$

over k*. Then* $\mathcal{F}_P$ *is represented by an ind-*k*-scheme of ind-finite type over* k*. Furthermore, a quotient map* $LP \to \mathcal{F}_P$ *admits a section étale-locally, which is equivalent to say that* $\operatorname{Spec} R \times_{\mathcal{F}_P} LP \cong \operatorname{Spec} R \times_k L^P$ *is a* L^+P*-equivariant isomorpihsm for any strictly henselian* R*.*

Proof. See the theorem 1.4 of [PR]. $\square$

Remark 2.4.10. For $P = GL_n$, we can obtain the previous theorem.

2.5 Beauville-Laszlo's Theorem

Let X be a reduced connected curve over k and $x \in X$ a smooth closed point. We usually assume that X is smooth. Let G be an affine group scheme over X. We denote X_R^* by a base change of $X^* = X \setminus x$ by R.

Definition 2.5.1. Let $\mathrm{Gr}_{G,x}$ be the presheaf defined by

$$\mathrm{Gr}_{G,x}(R) = \{(\mathcal{E}, \beta) \mid \mathcal{E} \text{ is a } G\text{-torsor on } X_R, \text{ and } \beta : \mathcal{E}|_{X_R^*} \cong G|_{X_R^*} \text{ is a trivialization}\}.$$

We often call this as a *global Affine Grassmannian*.

Remark 2.5.2. In this situation, the originally defined Affine Grassmannian is called "local".

Since $x \in X$ is smooth, the formal completion of a stalk at x is given by $\widehat{\mathcal{O}_{X,x}} \cong k[\![t]\!]$. x

2.6 Schubert varieties

Let G be a connected split reductive group over a field $k = \overline{k}$ with a split maximal torus T.

2.6.1 Coweight lattices

Let $\Phi \subseteq \mathbb{X}^\bullet(T)$ be a set of roots in the weight lattice. Fir a Borel subgroup B containing T. Then we have a subset of positive roots $\Phi^+ \subseteq \Phi$. For a coweight lattice $\mathbb{X}_\bullet(T)$, we call $\lambda \in \mathbb{X}_\bullet(T)$ is *dominant* if $\langle \lambda, \alpha \rangle \geq 0$ for any $\alpha \in \Phi^+$. We denote the set of dominant coweights by $\mathbb{X}_\bullet(T)^+$. For a Weyl group W, note that we have a bijection $\mathbb{X}_\bullet(T)/W \cong \mathbb{X}_\bullet(T)^+$. There is a partial order $\leq$ on $\mathbb{X}_\bullet(T)$, where $\lambda \leq \mu$ if $\mu - \lambda$ is a non-negative integral sum of simple coroots. We say $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$. We usually denote by $2\rho \in \mathbb{X}^\bullet(T)$ the sum of all positive roots, and let $\rho = \frac{1}{2}(2\rho) \in \mathbb{X}^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.6.2 The relative position

For $\mu \in \mathbb{X}_\bullet(T)$, a morphism $\mu : \mathbb{G}_m \to T$ gives a group homomorphism $F^\times \to T(F)$ and we denote the image of $t \in F^\times$ by t^μ. The Cartan decomposition gives

$$G(F) = \bigsqcup_{\mu \in \mathbb{X}_\bullet(T)^+} G(\mathcal{O}) t^\mu G(\mathcal{O}).$$

Note that the double coset $G(\mathcal{O}) t^\mu G(\mathcal{O})$ does not depend on the choice of a uniformlizer t nor the choice of T. It follows that

$$G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}) \cong \mathbb{X}_\bullet(T)/W \cong \mathbb{X}_\bullet(T)^+.$$

Now for G-torsors $\mathcal{E}_1$ and $\mathcal{E}_2$ over D where both split over D^*, let

$$\beta : \mathcal{E}_1|_{D^*} \cong \mathcal{E}_2|_{D^*}$$

be an G-equivariant isomorphism over D^*. Then we have $\beta \in G(F)$, and if we choose $\mathcal{E}_1$ and $\mathcal{E}_2$ up to G-equivariant isomorphism on D, then we obtain a well-defined element $\mathrm{Inv}(\beta) \in G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}$. We call $\mathrm{Inv}(\beta)$ by the relative position of β.

Remark 2.6.1. Let K be a not necessarily algebraically closed field extension of k. For G-torsors $\mathcal{E}_1$, $\mathcal{E}_2$ over D_K and $\beta : \mathcal{E}_1|_{D_K^*} \cong \mathcal{E}_2|_{D_K^*}$, we can define $\mathrm{Inv}(\beta) \in \mathbb{X}_\bullet(T)$ as follows. For an algebraic closure $\overline{K}/K$, then the above construction gives

$$\mathrm{Inv}(\beta_{\overline{K}}) \in G(\overline{K}[\![t]\!])\backslash G(\overline{K}(\!(t)\!))/G(\overline{K}[\![t]\!]) \cong G(\mathcal{O})\backslash G(F)/G(\mathcal{O})$$

and this is independent of the choice of $\overline{K}$. Hence $\mathrm{Inv}(\beta) := \mathrm{Inv}(\beta_{\overline{K}})$ is well-defined.

Now consider a k-algebra R, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be G-torsors over D_R. For a G-equivariant isomorphism

$$\beta : \mathcal{E}^1|_{D_R^*} \cong \mathcal{E}^2|_{D_R^*}$$

we can obtain

$$\beta_{k(x)} : \mathcal{E}^1|_{D_{k(x)}^*} \cong \mathcal{E}^2|_{D_{k(x)^*}}$$

for any $x \in \operatorname{Spec} R$. We define $\mathrm{Inv}_x(\beta) = \mathrm{Inv}(\beta_{k(x)}) \in \mathbb{X}_\bullet(T)^+$.

Proposition 2.6.2. *Let* $X = \operatorname{Spec} R$ *for a* k*-algebra* R*. For a given dominant coweight* $\mu \in \mathbb{X}_\bullet(T)$*, the set of points*

$$X_{\leq \mu} := \{x \in X \mid \mathrm{Inv}_x(\beta) \leq \beta\}$$

is Zariski-closed in X*.*

Proof. See the proposition 2.1.4 of [Zhu] ☐

Note that an R-point of Gr_G is given by $(\mathcal{E}, \beta)$ where $\mathcal{E}$ is a G-torsor over D_R and $\beta : \mathcal{E}|_{D_R^*} \cong \mathcal{E}^0|_{D_R^*}$. Hence we have $\mathrm{Inv}_x(\beta) \in \mathbb{X}_\bullet(T)^+$ for any $x \in \operatorname{Spec} R$.

Definition 2.6.3. Let $\mu \in \mathbb{X}_\bullet(T)^+$. For an affine Grassmannian Gr_G, we define

$$\mathrm{Gr}_{\leq\mu}(R) := \{(\mathcal{E}, \beta) \in \mathrm{Gr}_G(R) \mid \mathrm{Inv}_x(\beta) \leq \mu \text{ for all } x \in \operatorname{Spec} R\},$$

a subfunctor $\mathrm{Gr}_{\leq\mu} \hookrightarrow \mathrm{Gr}_G$ with a reduced scheme structure. We call such functor a *Schubert variety*.

Lemma 2.6.4. *A subfunctor* $\mathrm{Gr}_{\leq\mu} \hookrightarrow \mathrm{Gr}_G$ *is a closed immersion. In particular,* $\mathrm{Gr}_{\leq\mu}$ *is a inj-projective scheme over* k*. Furthermore, it is contained in* $\mathrm{GL}_n^(N)$ *for some embedding* $G \hookrightarrow \mathrm{GL}_n$ *and for* $N \gg 0$*, thus it is projective scheme over* k*.*

Proof. It suffices to show that

$$\begin{tikzcd} \operatorname{Spec} R/I \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "(\mathcal{E},\beta)"] \\ \mathrm{Gr}_{\leq\mu} \arrow[r] & \mathrm{Gr}_G, \end{tikzcd}$$

is a cartesian diagram for some ideal $I \subseteq R$. For $\operatorname{Spec} R' \to \operatorname{Spec} R$, the R'-point of the fiber product is given by $(\mathcal{E}, \beta)$ where $\mathcal{E}$ is a G-torsor over D and $\beta : \mathcal{E}|_{D^*_R} \cong \mathcal{E}^0|_{D^*_R}$, such that $\beta_{R'} : \mathcal{E}|_{D^*_{R'}} \cong \mathcal{E}^0|_{D^*_{R'}}$ satisfying $\mathrm{Inv}_x(\beta) \leq \mu$ for any $x \in \operatorname{Spec} R'$. By the proposition 2.6.2, we obtain that such $\operatorname{Spec} R \to \operatorname{Spec} R'$ factor thorough some $\operatorname{Spec} R/I \to \operatorname{Spec} R$ and a set of R'-point of the fiber product corresponds to $\mathrm{Gr}_{\leq\mu}(R/I)$. □

Definition 2.6.5. We define

$$\mathrm{Gr}_\mu := \{(\mathcal{E}, \beta) \in \mathrm{Gr}_G \mid \mathrm{Inv}(\beta) = \mu\} = \mathrm{Gr}_{\leq\mu} \setminus \bigcup_{\lambda<\mu} \mathrm{Gr}_\lambda.$$

By the lemma 2.6.4, it is an open subvariety of $\mathrm{Gr}_{\leq\mu}$ and we call it as a *Schubert cell*.

Proposition 2.6.6. *(1) Gr_μ is a single LG^+-orbit and is a smooth quasi-projective scheme over k of dimension $(2\rho, \mu)$.*

(2) The schematic-closure of Gr_μ is $\mathrm{Gr}_{\leq\mu}$. In particular, $\mathrm{Gr}_{\leq\mu}$ is a projective variety.

Example 2.6.7. Let $G = GL_n$ and $\mu = (r, 0, \cdots, 0)$ under the identification $\mathbb{X}_\bullet(T) \cong \mathbb{Z}^n$. Positive roots are $e_i - e_j$ with $i < j$, thus dominant coweights are given by

$$\mathbb{X}_\bullet(T)^+ = \{(a_1, \cdots, a_n) \mid a_i \geq a_j \text{ for any } i < j\}.$$

Suppose $\lambda \in \mathbb{X}_\bullet(T)$ and $\lambda \leq \mu$, then λ becomes a n-partition of r including zeros. Note that we have a lattice description of $\mathrm{Gr}_{GL_n}(R)$. Since $\mathrm{Gr}_{GL_n}(k) \cong G(F)/G(\mathcal{O}) \cong G(\mathcal{O})\mathbb{X}_\bullet(T)^+$, a lattice $\Lambda = g\Lambda^0$ can be rewritten as

$$G(\mathcal{O}) \begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n} \end{pmatrix} \Lambda^0$$

with $a_i \in \mathbb{Z}$ satisfying $a_i \geq a_j$ for any $i > j$. Now if $\Lambda \in \mathrm{Gr}_{\leq\lambda}$, then $(a_1, \cdots, a_n)$ becomes a n-partition of r. Then

$$\begin{pmatrix} t^{a_n} & & \\ & \ddots & \\ & & t^{a_n} \end{pmatrix} \Lambda_0$$

are the lattices with $\Lambda \subseteq \Lambda^0$ and $\mathrm{rk}(\Lambda/\Lambda^0) = r$. Since $G(\mathcal{O})$ preserves a relative rank of lattices, we obtain that

$$\mathrm{Gr}_{\leq\mu}(R) = \{\Lambda \subseteq \Lambda^0 \mid \mathrm{rk}(\Lambda/\Lambda_0) = r\}.$$

For a coweight $\mu \in \mathbb{X}_\bullet(T)$, we obtain a parabolic subgroup $P_\lambda \subseteq G$ generated by a torus and root subgroups U_α with $\langle\alpha, \mu\rangle \leq 0$.

Example 2.6.8. Let $G = \mathrm{GL}_n$ and $\Lambda^0 = k[t]^n$. For a dominant coweight $\mu = (a_1, \cdots, a_n)$, the map $P_\mu : \mathrm{Gr}_{\leq\mu} \to G/P_\mu$ maps a lattice Λ to the decreasing filtration of $V = \Lambda^0/t\Lambda^0 \cong k^n$ defined by

$$\mathrm{Fil}^i V := \Lambda \cap \Lambda^0/$$

Lemma 2.6.9. *If* μ *is minuscule, then* $\mathrm{Gr}_{\leq\mu} \cong \mathrm{Gr}_\mu \cong G/P_\mu$.

2.7 Parahoric subgroups

Let G be a connected reductive group over $F = k((t))$ and let $\mathcal{B} = \mathcal{B}(G(F))$ be a Bruhat-Tits building of $G(K)$. For $K^\flat = \overline{k}((t))$, a facet $\underline{a}$ of $\mathcal{B}$ corresponds to a facet $\underline{a}^\flat$ of the building $\mathcal{B}(G(K^\flat))$ fixed under the action of $\Gamma = \mathrm{Gal}(K^\flat/K) = \mathrm{Gal}(\overline{k}/k)$. Note that a parahoric subgroup of $G(K^\flat)$ is given by the stabilizer of a facet $\underline{a}^\flat$. Then we obtain a connected smooth affine group scheme $P^\flat$ over $\mathcal{O} = \overline{k}[\![t]\!]$ whose generic fiber is $G^\flat$ and integral points is the given parabolic subgroup of $G(K^\flat)$. By a descent argument, we obtain P over $\mathcal{O}$ whose integral points is a parahoric subgroup of $G(K)$. Note that such P exists uniquely. Hence we can call such group scheme P as a parahoric subgroup of G. Note that we call a minimal parahoric subgroup as a Iwahori subgroup

Example 2.7.1. Let $G = \mathrm{SL}_3$ over F. Consider a Iwahoric subgroup

$$P = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

of $G(F)$ with $\mathfrak{p} = (t) \subseteq \mathcal{O}$. Then we may let

$$P = \operatorname{Spec} \frac{R[x_{ij}, y_{21}, y_{31}, y_{32}]_{1 \le i,j \le 3}}{(\det(x_{ij}) - 1, x_{21} - ty_{21}, x_{31} - ty_{31}, x_{32} - ty_{32})}.$$

Then we can obtain a subgroup sheaf $L^+P \subseteq LG$.

The subgroup $L^+G \subseteq LG$ is an example of a *parahoric subgroup*.

Let $B \subseteq G$ be a Borel subgroup and $\mathbf{I}$ a base change of B under a natural morphism $L^+G \to G$. Then $\mathbf{I}$ is an example of *Iwahori* subgroup. In general, Iwahori subgroups are conjugates of $\mathbf{I}$ in LG. Parahoric subgroups are connected subschemes of LG containing an Iwahori subgroup with finite codimension.

Note that Iwahori subgroups and Parahoric subgroups are analogous to Borel subgroups and Parabolic subgroups respectively. Like as the conjugacy classes of parabolic subgroups of G corresponds to Dynkin Diagrams of G, LG-conjugacy classes of parahoric subgroups of LG corresponds to proper subsets of vertices of the extended Dynkin diagram of G.

Example 2.7.2. The maximal parahoric subgroup of $GL_n(F)$ are the stabilizers of O_F-lattices in F^n. In particular, $GL_n(\mathcal{O}_F)$ is maximal parahoric. The Iwahori subgroups are conjugated to the subgroup $\mathbf{I}$ of matrices in $GL_n(\mathcal{O}_F)$ whose base change to k is a subgroup of upper triangular matrices of $GL_n(k)$. The parahoric subgroups are all groups conjugate to subgroups between $\mathbf{I}$ and $GL_n(\mathcal{O}_F)$ whose reduces to parabolic subgroups of $GL_n(k)$ over k.

Definition 2.7.3. (Affine flag varieties) Let P be a parahoric subgroup of G and consider a subgroup $L^+\mathbf{P} \subseteq LG$. The *affine partial flag variety* $\mathrm{Fl}_{\mathbf{P}}$ is the fpqc-sheaf associated to the presheaf

$$R \mapsto LG(R)/L^+P(R).$$

When $\mathbf{P}$ is an Iwahori subgroup $\mathbf{I}$, we denote $\mathrm{Fl}_{\mathbf{P}}$ by Fl_G or Fl and call it the *affine flag variety*.

Remark 2.7.4. By the varies of the theorem 2.4.9, LG/L^+P is represented by a strict ind-proj-scheme over k. The affine Grassmannian Gr_G is a special case when $\mathbf{P} = L^+G$, which is a maximal parahoric subgroup.

Example 2.7.5. (SL_n) Let $G = SL_n$. We denote the set of k-family of lattices in $F^n = k((t))^n$ by $\mathfrak{L}_n$. For two lattices $\Lambda_1, \Lambda_2 \in \mathfrak{L}_n$, define the *relative length* as

$$[\Lambda_1 : \Lambda_2] := \dim_k(\Lambda_1/\Lambda_1 \cap \Lambda_2) - \dim_k(\Lambda_2/\Lambda_1 \cap \Lambda_2).$$

For a nonempty subset $J \subseteq \mathbb{Z}/n\mathbb{Z}$, let $\tilde{J}$ be the preimage of J under the projection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. A *periodic* J-*chain* of lattices is a function $\Lambda : \tilde{J} \to \mathfrak{L}_n$ assigns each $i \in \tilde{J}$ to a lattice Λ_i satisfying

- $[\Lambda_i : k[\![t]\!]^n] = i$ for any $i \in \tilde{J}$;
- $\Lambda_i \subseteq \Lambda_j$ for any $i < j$;
- $\Lambda_i = t\Lambda_{i+n}$ for any $i \in \tilde{J}$.

Note that each k-family of lattices is a k-point of $\mathrm{Gr}_{\mathrm{GL}_n}$ and LG acts on $\mathrm{Gr}_{\mathrm{GL}_n}$. Hence we can define a stabilizer of k-family of lattices in $k(\!(t)\!)^n$.

Let $\mathfrak{L}_J$ be the set of periodic J-chains of lattices. For each $\{\Lambda_i\}_{i\in\tilde{J}} \in \mathfrak{L}_J$, let $\mathbf{P}_{\{\Lambda_i\}_{i\in\tilde{J}}} \subseteq LG$ be a simultaneous stabilizer of all Λ_i's. Then $\mathbf{P}_{\{\Lambda_i\}_{i\in\tilde{J}}}$ is a parahoric subgroup of LG and we call it a parahoric subgroup of type J. Indeed, every parahoric subgroup of LG is given by this form with a unique $J \subseteq \mathbb{Z}/n\mathbb{Z}$ and a unique periodic J-chains of lattices. Thus we have a bijection between $\bigsqcup_J \mathcal{L}_J$ and a set of parahoric subgroups of LG. For example, let $J = \{0\}$ and $\Lambda_i = t^{-i/n}k[\![t]\!]^n$. Note that $\tilde{J} = n\mathbb{Z}$, and we have $\mathbf{P}_{\{\Lambda_i\}} = L^+G$.

For a parahoric subgroup $\mathbf{P}$, we have a pro-unipotent radical $\mathbf{P}^+ \subseteq \mathbf{P}$ and an exact sequence

$$1 \to \mathbf{P}^+ \to \mathbf{P} \to L_{\mathbf{P}} \to 1$$

of group schemes. We call $L_{\mathbf{P}}$ as a *Levi quotient* of $\mathbf{P}$.

For parahoric subgroups $\mathbf{P} \subseteq \mathbf{Q}$, note that $\mathbf{Q}^+ \subseteq \mathbf{P}^+$ and $\mathbf{P}/\mathbf{Q}^+$ is a parabolic subgroup of a Levi quotient $L_{\mathbf{Q}} = \mathbf{Q}/\mathbf{Q}^+$.

Now for $\mathbf{P} \subseteq \mathbf{Q}$, we have a natural projection

$$\mathrm{Fl}_{\mathbf{P}} \to \mathrm{FL}_{\mathbf{Q}}$$

then the fiber is given by $\mathbf{Q}/\mathbf{P} = L_{\mathbf{Q}}/(\mathbf{P}/\mathbf{Q}^+)$. This is the partial flag variety corresponds to a parabolic subgoup $\mathbf{P}/\mathbf{Q}^+$ of $L_{\mathbf{Q}}$. In particular, there is a natural projection

$$\mathrm{FL}_G \to \mathrm{Gr}_G$$

whose fibers are isomorphic to the (full-)flag variety $\mathcal{B}$.

2.8 Affine Springer fibers

Let G be a connected reductive group over an algebraically closed field k and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. For a k-algebra R, we denote $\mathfrak{g} \otimes_k R$ by

$\mathfrak{g}(R)$. Note that $\mathrm{Lie}(G_R) \cong \mathfrak{g}(R)$. For $F = k((t))$, let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element in $\mathfrak{g}(\overline{F})$. For $g \in LG(R) = G(R((t)))$, we have an adjoint representation $\mathrm{Ad}(g^{-1}) \in GL(\mathfrak{g}(R((t))))$. Then for $\gamma_R \in \mathfrak{g}(R((t)))$, we have $\mathrm{Ad}(g^{-1})\gamma \in \mathfrak{g}(R((t)))$.

Note that any element in $[LG/LG^+](R)$ is represented by $g \in LG(R')$ with an étale cover $\operatorname{Spec} R' \to \operatorname{Spec} R$ and a morphism $\alpha : \operatorname{Spec} R' \otimes_R R' \to LG^+ \times LG$ satisfying $g \circ pr_0 = s \circ \alpha$ and $g \circ pr_1 = t \circ \alpha$ where $s, t : LG^+ \times LG \to LG$ are given by the multiplication and the projection on the second factor respectively. We denote this element by $[g]$.

Definition 2.8.1. (Affine Springer fibers) Define a subfunctor of Gr_G given by

$$\tilde{\mathcal{X}}_\gamma(R) := \{[g] \in \mathrm{Gr}_G(R) \mid \mathrm{Ad}(g^{-1})\gamma \in \mathfrak{g}(R'[[t]])\}.$$

Then $\tilde{\mathcal{X}}_\gamma$ is a closed sub-ind-scheme of Gr_G.

An *affine Springer fiber* of γ is a reduced-ind-scheme given by $\mathcal{X}_\gamma = \tilde{\mathcal{X}}_\gamma^{\mathrm{red}}$.

Remark 2.8.2. Consider the Grothendieck alteration $\tilde{\mathfrak{g}} \to \mathfrak{g}$, then the Springer fiber of $x \in \mathfrak{g}$ is defined by

$$\{\text{borel algebras } \mathfrak{b} \text{ containing } x\}.$$

Note that we have $G/B \cong \mathcal{B}$ by sending $[g] \mapsto \mathrm{Ad}(g)\mathfrak{b}$ for a fixed borel algebra $\mathfrak{b}$. Hence the Springer fiber can be rewritten with

$$\{[g] \in G/B \mid \mathrm{Ad}(g^{-1})x \in \mathfrak{b}\},$$

which is analogous to the definition of the affine Springer fiber.

Remark 2.8.3. We can define an affine Springer fiber for an arbitrary $\gamma \in \mathfrak{g}(F)$, but then it does not satisfy the good geometric properties. For example, if γ is regular semisimple, then $\tilde{\mathcal{X}}_\gamma(F)$ is of finite dimensional. See [3.7,Görtz] for details.

Example 2.8.4. Let T be a split maximal k-torus of G. Consider $\gamma \in \mathfrak{t}(\mathcal{O}_F)$ such that $\overline{\gamma} \in \mathfrak{t}$ is regular semisimple.

Example 2.8.5. Let $G = GL_n$. We can identify Gr_G as a moduli-space of lattices in F^n. Let $\gamma \in \mathfrak{gl}_n(F)$ be a regular semisimple element. Any lattice Λ in $R((t))$ is characterized by $\Lambda'_R = g\Lambda^0$ for some étale cover $\operatorname{Spec} R' \to \operatorname{Spec} R$, $[g] \in GL_n(R'((t)))/GL_n(R'[[t]])$, and a standard lattice Λ^0 of $R'((t))$. Note that a stabilizer of Λ^0 is given by $GL_n(R'[[t]])$. For $\mathrm{Ad}(g^{-1})\gamma = g^{-1}\gamma g$, we have $g^{-1}\gamma g \in \mathfrak{g}(R'[[t]])$ if and only if $g^{-1}\gamma g\Lambda^0 \subseteq \Lambda^0$. This is equivalent to say $\gamma\Lambda_{R'} \subseteq \Lambda_{R'}$.

Example 2.8.6. Let $G = SL_n$. We can identify Gr_G as a moduli-space of lattices in $k((t))^n$ whose determinant bundle is trivial. Equivalently, of lattices such that $[\Lambda : \Lambda^0] = 0$. In the similar manner, we can show that the affine Springer fiber $\tilde{\mathcal{X}}_\gamma(R)$ is given by lattices Λ satisfying $\gamma\Lambda \subseteq \Lambda$.

Example 2.8.7. Let $G = SL_2$ and $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$. Note that γ is a regular semisimple element. Since $\mathcal{X}_\gamma = \tilde{\mathcal{X}}_\gamma^{\text{red}}$ is uniquely characterized by a subfunctor $\mathcal{X}_\gamma \subseteq \tilde{\mathcal{X}}_\gamma$ satisfying $\mathcal{X}_\gamma(R) = \tilde{\mathcal{X}}_\gamma^{\text{red}}(R)$ for any reduced K-algebra, it suffices to compute $\mathcal{X}_\gamma(R)$ for reduced k-algebras R. By the previous description, we need to classify a lattice Λ of $R((t))^2$ satisfying that $\gamma\Lambda \subseteq \Lambda$. The problem is that Λ may not be a free $R[[t]]$-module. First suppose that Λ is a free $R[[t]]$-module. Then we may let $\Lambda = g\Lambda^0$ for some $g \in G(R((t)))$. Consider the Iwasawa decomposition

$$G(R((t))) = U(R((t)))M(R((t)))G(R[[t]]).$$

where U is a unipotent subgroup consists of upper triangular matrix, and M is a standard torus. Since R is reduced, a unit of $R((t))$ is a Laurent series whose initial coefficient is unit. Hence we have $(R((t)))^\times \cong t^{\mathbb{Z}} \times (R[[t]])^\times$, so in the Iwasawa decomposition we may choose representative as

$$G(R((t))) = \bigsqcup_{n \in \mathbb{Z}} U(R((t))) \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} G(R[[t]]).$$

Thus for any $[g] \in Gr_G(R)$ can be represented by

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} = \begin{pmatrix} t^n & xt^{-n} \\ 0 & t^{-n} \end{pmatrix}$$

for some $x \in R((t))$. Since

$$\begin{pmatrix} t^n & xt^{-n} \\ 0 & t^{-n} \end{pmatrix}^{-1} \begin{pmatrix} t^n & yt^{-n} \\ 0 & t^{-n} \end{pmatrix} = \begin{pmatrix} 1 & t^{-2n}(x-y) \\ 0 & 1 \end{pmatrix},$$

the element x has the well-defined image in $R((t))/t^{2n}R[[t]]$ with unique $n \in \mathbb{Z}$. Now we have

$$\mathrm{Ad}(g^{-1})\gamma = g^{-1}\gamma g = \begin{pmatrix} t & 2t^{1-2n}x \\ 0 & -t \end{pmatrix} \in \mathfrak{g}(R[[t]])$$

if and only if $x \in t^{2n-1}R[[t]]$. Since x is well-defined module $t^{2n}R[[t]]$, we have $x \in t^{2n-1}R[[t]]/t^{2n}R[[t]]$. This is equivalent to say that $xt^{-n} \in t^{n-1}R[[t]]/t^nR[[t]]$. Since $\Lambda = g\Lambda^0$ has a basis $\{(t^n, 0), (xt^{-n}, t^{-n})\}$, it is equivalent to say that

$$t^nR[[t]] \oplus t^{-n+1}R[[t]] \subseteq \Lambda \subseteq t^{n-1}R[[t]] \oplus t^{-n}R[[t]].$$

Now we consider the case Λ is not free. Note that Λ is finite projective $R[\![t]\!]$-module of rank 2 satisfying $\Lambda \otimes_{R[\![t]\!]} R(\!(t)\!) = R(\!(t)\!)^2$. Then we have a fpqc-covering(or étale) $\operatorname{Spec} R' \to \operatorname{Spec} R$ such that $\Lambda_{R'}$ is free of rank 2. Then we can classify $\Lambda_{R'}$ as

$$t^n R'[\![t]\!] \oplus t^{-n+1} R'[\![t]\!] \subseteq \Lambda_{R'} \subseteq t^{n-1} R'[\![t]\!] \oplus t^{-n} R'[\![t]\!].$$

for some unique $n \in \mathbb{Z}$. By the similar argument used in the proof of 2.2.3, we obtain

$$t^n R[\![t]\!] \oplus t^{-n+1} R[\![t]\!] \subseteq \Lambda \subseteq t^{n-1} R[\![t]\!] \oplus t^{-n} R[\![t]\!]$$

by intersecting with $R(\!(t)\!)^2$.

For $n \in \mathbb{Z}$, consider the subscheme C_n of Gr_G classifying the lattices $\Lambda \subseteq R(\!(t)\!)^2$ satisfying the above inequality. Again by the similar argument used in the proof of 2.2.3, we have

$$C_n(R) = \left\{ R[\![t]\!]\text{-quotient of } \frac{t^{n-1}R[\![t]\!] \oplus t^{-n}[\![t]\!]}{t^n R[\![t]\!] \oplus t^{-n+1}[\![t]\!]}, \text{projective as } R\text{-module} \right\}$$

and since the above module is isomorphic to $R \oplus R$ and t acts trivially, this is equivalent to choose a quotient of R-module $R \oplus R \to P$ where P is a projective. From the condition $[\Lambda : \Lambda^0] = 0$, we have that P is a projective of rank 1. This is equivalent to $\mathbb{P}^1(R)$, so we obtain $C_n \cong \mathbb{P}^1$. We conclude that $\mathcal{X}_\gamma = \bigcup_{n \in \mathbb{Z}} C_n$. Note that C_n and C_{n+1} intersects at one point $t^{-n}R[\![t]\!] \oplus t^n R[\![t]\!]$ and otherwise components are disjoint. Hence $\mathcal{X}_\gamma$ is an infinite chain of $\mathbb{P}^1$'s.

Example 2.8.8. Let $G = SL_2$ and $\gamma = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ for nonzero $a \in k$, In the similar way, for any reduced k-algebra R we have

$$\mathrm{Ad}(g^{-1})\gamma = \begin{pmatrix} at^n & axt^{-2n} \\ 0 & -a \end{pmatrix} \in \mathfrak{g}(R[\![t]\!])$$

if and only if $x \in t^{2n}R[\![t]\!]$. But we chose x modulo $t^{2n}R[\![t]\!]$, thus we have $x = 0$. This shows that

$$g = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}$$

for some integer n. Note that this equals to the $t^\lambda := \lambda(t)$ for a cocharater $\lambda : \mathbb{G}_m \to T$. Thus we have $\mathcal{X}_\gamma = \mathbb{X}_*(T) \cong \mathbb{Z}$.

Example 2.8.9. Let $G = SL_2$ and $\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}$. In the similar way, we have

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} = \begin{pmatrix} t^n & xt^{-n} \\ 0 & t^{-n} \end{pmatrix}$$

and

$$\mathrm{Ad}(g^{-1})\gamma = g^{-1}\gamma g = \begin{pmatrix} -xt & t^{-2n+2} - (xt)^2 t^{-2n-1} \\ t^{2n+1} & xt \end{pmatrix} \in \mathfrak{g}(R[\![t]\!])$$

if and only if $xt \in R[\![t]\!]$, $2n+1 \geq 0$, $-2n+2 \geq 0$, and $2\mathrm{val}(x) + (-2n+1) \geq 0$. If $xt \notin tR[\![t]\!]$, we can obtain the contradiction easily. The remain possibilities are $xt \in tR[\![t]\!]$ with $n = 0$, or $xt \in t^2R[\![t]\!]$ with $n = 1$. Then $\Lambda = g\Lambda^0$ corresponds to

$$tR[\![t]\!] \oplus R[\![t]\!] \subseteq \Lambda \subseteq R[\![t]\!] \oplus t^{-1}R[\![t]\!]$$

satisfying $[\Lambda : \Lambda^0] = 0$, which is characterized with $\mathbb{P}^1(R)$. Thus we have $\mathcal{X}_\gamma = \mathbb{P}^1$.

Remark 2.8.10. (Invariance under conjugation) Suppose $\gamma, \gamma' \in \mathfrak{g}(F)$ satisfies that $\gamma = \mathrm{Ad}(g^{-1})\gamma'$ for some $g \in G(F)$. Then $\mathrm{Ad}(g'^{-1})\gamma = \mathrm{Ad}((gg')^{-1})\gamma'$, thus the left translation by g on Gr_G restricts to the isomorphism $\tilde{\mathcal{X}}_\gamma \cong \tilde{\mathcal{X}}_{\gamma'}$, and $\mathcal{X}_\gamma \cong \mathcal{X}_{\gamma'}$ as well. Thus $\mathcal{X}_\gamma$ is invariant under $G(F)$-conjugation up to isomorphisms. Recall the map $\chi : \mathfrak{g} \to \mathfrak{c} := \mathfrak{g}/\!/G \cong \mathfrak{t}/\!/W$. Note that $\gamma \in \mathfrak{g}(F)$ is regular semisimple if and only if $\chi(\gamma) \in \mathfrak{c}^{rs}(F)$. For $a \in \mathfrak{c}^{rs}(F)$, the fiber $\chi^{-1}(a)$ consists of a single $G(F)$-conjugacy class since k is an algebraically closed field. Hence the isomorphism type of $\mathcal{X}_\gamma$ only depends on $a = \chi(\gamma)$.

An affine springer fiber $\mathcal{X}_\gamma$ may be empty for some γ. Indeed, $\mathcal{X}_\gamma$ is nonempty if and only if $a = \chi(\gamma) \in \mathfrak{c}(\mathcal{O}_F)$. If $[g] \in \mathcal{X}_\gamma$, then we have $\mathrm{Ad}(g^{-1})\gamma \in \mathfrak{g}(\mathcal{O}_F)$ thus $\chi(\gamma) = \chi(\mathrm{Ad}(g^{-1})\gamma) \in \mathfrak{c}(\mathcal{O}_F)$. Conversely, with the identification $\mathfrak{c}^{reg} = e + \mathfrak{g}^f$ for a principal $\mathfrak{sl}_2$-triple (e, f, h) of $\mathfrak{g}$, we obtain a Kostant section $\epsilon : \mathfrak{c}^{reg} \to \mathfrak{g}$ of χ. Then for $a \in \mathfrak{c}^{rs} \cap \mathfrak{c}(\mathcal{O}_F)$, we have $\epsilon(a) \in \mathfrak{g}(\mathcal{O}_F)$ thus the unit coset in Gr_G is contained in $\mathcal{X}_{\epsilon(a)}$. Since we have $\mathcal{X}_\gamma \cong \mathcal{X}_{\epsilon(\chi(\gamma))}$, if $\chi(\gamma) \in \mathfrak{c}(\mathcal{O}_F)$ then we have $\mathcal{X}_\gamma$ is nonempty.

Example 2.8.11. Let $G = SL_n$ and consider a regular semisimple element

$$\gamma = \begin{pmatrix} t^{-1} & 0 \\ 0 & -t^{-1} \end{pmatrix} \in \mathfrak{g}(F).$$

Then the similar computation shows that

$$\mathrm{Ad}(g^{-1})\gamma = \begin{pmatrix} t^{-1} & 2t^{-1-2n}x \\ 0 & -t^{-1} \end{pmatrix}$$

thus $\mathcal{X}_\gamma$ is empty. Note that a characteristic polynomial of γ has coefficients not contained in $\mathfrak{g}(\mathcal{O}_F)$, thus $\chi(\gamma)$ is not contained in $\mathfrak{c}(\mathcal{O}_F)$. Note that $R((t))^\times$

We generalize the definition of affine springer fibers for affine flag varieties. Let $\mathbf{P}$ be a parahoric subgroup of G. Note that $\mathbf{P}$ is defined over $\mathcal{O}_F$, and $\mathrm{Lie}(\mathbf{P}) \subseteq \mathfrak{g}$ becomes a $\mathcal{O}_F$-lattice. Let $\gamma \in \mathrm{Lie}(\mathbf{P})$ be a regular semisimple element.

Definition 2.8.12. Define a subfunctor of $\mathrm{Fl}_\mathbf{P}$ by

$$\tilde{\mathcal{X}}_{\mathbf{P},\gamma}(R) = \{[g] \in \mathrm{Fl}_\mathbf{P}(R) \mid \mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}(\mathbf{P})\hat{\otimes}_k R\}$$

The reduced ind-scheme $\mathcal{X}_{\mathbf{P},\gamma} := \tilde{\mathcal{X}}^{\mathrm{red}}_{\mathbf{P},\gamma}$ is called the *affine Springer fiber* of γ of type $\mathbf{P}$.

2.9 Centralizer Action

For a regular semisimple $\gamma \in \mathfrak{g}(F)$, let G_γ be a centralizer of γ in G_F, which is the linear algebraic group over F obtained from G by base change. Note that since k is algebraically closed, G_F is reductive as well. Hence a Cartan subgroup G_γ is indeed a maximal torus of G_F over F. Note that G_γ may not be split in general. Consider a loop group LG_γ defined by $R \mapsto G_\gamma(R[\![t]\!])$ for any k-algebra R.

We will show that LG_γ acts on the ind-scheme $\tilde{\mathcal{X}}_\gamma$. Let $h \in LG_\gamma(R) = G_\gamma(R((t)))$ and $[g] \in \mathrm{Gr}_G(R)$. Then we have $\mathrm{Ad}(g^{-1})\gamma \in \mathfrak{g}(R[\![t]\!])$, thus we obtain

$$\mathrm{Ad}(hg)^{-1}\gamma = \mathrm{Ad}(g^{-1})\mathrm{Ad}(h^{-1})\gamma = \mathrm{Ad}(g^{-1})\gamma \in \mathfrak{g}(R[\![t]\!]).$$

This shows that $[hg] \in \mathrm{Gr}_G(R)$, thus we have defined an action of LG_γ on $\tilde{\mathcal{X}}_\gamma$. This induces an action of LG_γ on an affine srpinger fiber $\mathcal{X}_\gamma$.

For a regular semisimple $\gamma \in \mathfrak{g}(F)$, let $\mathbb{X}_*(G_\gamma) := \mathrm{Hom}_F(\mathbb{G}_m, G_\gamma)$ be a K-rational cocharacter lattice of a maximal torus G_γ of G_F. For $\lambda \in \mathbb{X}_*(G_\gamma)$, its F-points gives a group homomorphism

$$\lambda : k((t))^\times \to G_\gamma(F)$$

and we let $\lambda(t) \in G_\gamma(F)$ be the image of $t \in k((t))^\times$ under λ. Since a k-algebra map $k[x, x^{-1}] \to k((t))$ sending x to t is injective, the corresponding $\operatorname{Spec} k((t)) \to \mathbb{G}_m$ is dominant. Hence we have an injective homomorphism

$$\mathbb{X}_*(G_\gamma) \hookrightarrow G_\gamma(F)$$

and we denote the image by Λ_γ. From a F-rational subgroup $\Lambda_\gamma \subseteq G_\gamma(F)$, we obtain a F-subgroup of G_γ whose F-points are Λ_γ by taking a Zariski-closure. We denote this subgroup by Λ_γ as well. Later, we will see that Λ_γ acts on $\mathcal{X}_\mathfrak{g}$ freely. Indeed, we can show that the quotient $\Lambda_\gamma \backslash (LG_\gamma)^{\mathrm{red}}$ is an affine scheme which is a finitely many disjoint union(product?) of $\mathbb{G}_m^a \times \mathbb{A}^{\mathbb{N}}$. Soon we will see this by some examples.

2.9.1 Symmetry on affine Springer fibers

Ngô has found that the LG_γ-action on $\tilde{\mathcal{X}}_\gamma$ factor through a canonical finite-dimensional quotient. Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. For $a = \chi(\gamma) \in \mathfrak{c}^{\mathrm{red}}(F)$, we assume that $a \in \mathfrak{c}^{\mathrm{red}}(\mathcal{O}_F)$ to avoid the situation that $\mathcal{X}_\gamma$ is empty.

There exists a smooth affine group scheme J over $\mathfrak{c}$ called the *regular centralizer group scheme* characterized by the property that the pullback χ^*J over $\mathfrak{g}$ maps into the *universal centralizer group scheme* I over $\mathfrak{g}$, and the restriction of such map to $\mathfrak{g}^{\mathrm{reg}}$ becomes an isomorphism. Let J_a be the pullback of J along $a : \mathcal{O}_F \to \mathfrak{c}$. Then J_a is an smooth affine group scheme over $\mathcal{O}_F$ whose fiber over F is the torus G_γ. In particular, J_a is an integral model of G_γ. Then we can define the affine Grassmannian $P_a := LG_\gamma / L^+J_a$ which is called the *local Picard group*, and the reduced group scheme P_a^{red} is finite dimensional and locally of finite type. Ngô proved that the action of LG_γ(resp. $(LG_\gamma)^{\mathrm{red}}$ on $\tilde{\mathcal{X}}_\gamma$(resp. $\mathcal{X}_\gamma$) factor thorough P_a, and it does not factor though further quotient.

If we choose $\gamma \in \mathfrak{t}^{\mathrm{red}}(\mathcal{O}_F)$, then the centralizer can be defined as a subgroup of $G_{\mathcal{O}_F}$. If we denote such we may let J_a

Example 2.9.1. Suppose $\gamma \in \mathfrak{t}^{\mathrm{red}}(\mathcal{O}_F)$ satisfies $\gamma \in \overline{\mathfrak{t}^{\mathrm{res}}}$. Since the maximal torus G_γ splits, we have $LG_\gamma = LT$. It is known that LT acts on $\tilde{\mathcal{X}}_\gamma$ transitively with a stabilizer LT^+. Hence $\tilde{\mathcal{X}}_\gamma$ is a $LT/L^+T = \mathrm{Gr}_T$-torsor. Here J_a is just the split maximal torus T, thus $P_a = \mathrm{Gr}_T$ in this case.

Note that $(LT)^{\mathrm{red}} \cong \mathbb{X}_*(T) \times LT^+$, thus we obtain $X_\gamma \cong \mathbb{X}_*(T)$.

2.9.2 The split case

Suppose $\gamma \in \mathfrak{t}^{rs}$ in contained in a split maximal torus. Then the split maximal torus equals to G_γ. For example, we may choose $\gamma \in \mathfrak{t}^{rs}(F)$ is induced from $\overline{\gamma} \in \mathfrak{t}^{rs}$ where $\mathfrak{t}$ is a Lie algebra of a (split)maximal torus T of G over k. Since centralizers are compatible with base changes, we have $G_\gamma = T \otimes_k F$ thus $LG_\gamma = LT$. Note that T splits since k is algebraically closed, thus we may let $T \cong \mathbb{G}_m^n$. Then we have

$$\begin{aligned} LT(R) &= \mathrm{Hom}_k(\mathrm{Spec}\, R(\!(t)\!), \mathbb{G}_m^n) \\ &\cong \prod_n \mathrm{Hom}_k(\mathrm{Spec}\, R(\!(t)\!), \mathbb{G}_m) \\ &\cong \mathbb{X}_*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m(R(\!(t)\!)) \end{aligned}$$

thus $LG_\gamma = LT \cong \mathbb{X}_*(T) \otimes_{\mathbb{Z}} L\mathbb{G}_m$. Note that we have an isomorhpism $\mathbb{X}_*(T) \cong \mathbb{Z}^n$. Explicitly, the isomorphism is given by

$$\begin{aligned} \mathbb{X}_*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m(R(\!(t)\!)) &\longrightarrow LT(R) \\ (\mathbb{G}_m \xrightarrow{\lambda} T) \otimes (\mathrm{Spec}\, R(\!(t)\!) \xrightarrow{r} \mathbb{G}_m) &\mapsto (\mathrm{Spec}\, R(\!(t)\!) \xrightarrow{\lambda \circ r} T) \end{aligned}$$

For a k-algebra R, note that $L\mathbb{G}_m(R) = R(\!(t)\!)^\times$ consists of $\sum_i a_i t^i \in R(\!(t)\!)$ where initial finitely many coefficients are nilpotent and the first non-nilpotent coefficient is invertible in R. Thus for a reduced R, $\sum_i a_i t^i \in R(\!(t)\!)$ is invertible if and only if the leading coefficient is invertible. Since $\sum_{i \geq 0} a_i t^i \in R[\![t]\!]$ is a unit if and only if $a_i \in R^\times$, it follows that that $R(\!(t)\!)^\times = t^{\mathbb{Z}} \cdot R[\![t]\!]^\times$ for a reduced k-algebra R. Hence we obtain $(L\mathbb{G}_m)^{red} \cong \mathbb{Z} \times L^+\mathbb{G}_m$ and $L^+\mathbb{G}_m \cong \mathbb{G}_m \times \mathbb{A}^{\mathbb{N}}$ with $\mathbb{A}^{\mathbb{N}} = \mathrm{Spec}\, k[x_1, x_2, \cdots]$. Hence for $\gamma \in \mathfrak{t}^{rs}$, we have $(LG_\gamma)^{red} \cong \mathbb{X}_*(T) \times L^+T$ where L^+T is an affine scheme of infinite type. Let $\Lambda_\gamma = \mathbb{X}_*(T)$ be a lattice in $(LG_\gamma)^{red}$, then we have $\Lambda_\gamma \backslash (LG_\gamma)^{red} \cong L^+T \cong \prod_i L^+\mathbb{G}_m$.

Example 2.9.2. Recall that examples 2.8.7 and 2.8.8 are both split cases. Hence both have $LG_\gamma \cong LT \cong L\mathbb{G}_m$ and $L(G_\gamma)^{red} \cong \mathbb{X}_*(T) \times L^+T$. Since $J_a = T$, we expect that L^+T acts trivially on $\mathcal{X}_\gamma$ for both. It is clear since $L^+T(R) = \mathbb{G}_m(R[\![t]\!]) = R[\![t]\!]^\times$ fixing lattices in both cases. Hence it remains to see the action of $\mathbb{X}_*(T)$ on $\mathcal{X}_\gamma$. In the second example, $\mathcal{X}_\gamma \cong \mathbb{X}_*(T)$ thus acts freely transitively. In the first example, we have $\mathcal{X}_\gamma$ is a union of $\mathbb{Z}$-copies of $\mathbb{P}^1$ where the only adjacent $\mathbb{P}^1$'s intersect in the one point. By the construction of 2.8.7, we can see that $n \in \mathbb{Z} \cong \mathbb{X}_*(T)$ moves i-th $\mathbb{P}^1$ to the $i+n$ one. Note that this action is not transitive and free.

2.9.3 The non-split case

Example 2.9.3. (The case $G = GL(V)$) Let V be a n-dimensional vector space over k. We further assume that $\mathrm{char}(k) > n$. Consider the characteristic polynomial

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in F[x]$$

of a regular semisimple element $\gamma \in \mathfrak{g}(F)$ acts on $V(F)$. It follows that $P(x)$ is separable. We decompose $P(x) = P_1(x) \cdots P_m(x)$ into irreducible factors $P_i(x)$. Then we have

$$F[x]/(P(x)) \cong \prod_i F[x]/(P_i(x)) \cong \prod_i F_i$$

where $F_i = F[x]/(P_i(x))$ is a separable finite field extension of F satisfying that $\sum_{1 \leq i \leq m} [F_i : F] = n$. We may identify $V(F)$ with $F[x]/(P(x))$, where γ corresponds to a left multiplication by x.

Since the residue field k is algebraically closed, any finite separable extension of F is totally ramified. Since $\mathrm{char}(k) > n$, each F_i/F is totally tamely ramified thus can be written with $k(\!(t^{1/e_i})\!)$ for $e_i = [F_i : F]$.

For an F-algebra A, a group $G(A)$ acts on $V(A) \cong \prod_i A[x]/(P_i(x))$. Consider an element $g \in G_\gamma(A)$ for an F-algebra A. Then g commutes with x, for any $v \in V(F)$ we have $P_i(x)(g \cdot v) = g \cdot P_i(x)(v)$. In particular, g preserves the decomposition $V(A) = \prod A[x]/(P_i(x))$. Hence we can restrict the action of g to $A[x]/(P_i(x))$ for each i. With an ordered basis $\{1, x, \cdots, x^{e_i - 1}\}$, we have $g \cdot x^j = x^j \cdot (g \cdot 1)$ thus the action of g is completely determined by $g(x) := g \cdot 1$. Since g is invertible, we have $g(x) \in (A[x]/(P_i(x)))^\times$.

Conversely, it is clear that $g(x) \in (A[x]/P_i(x))^\times$ commutes with x. This shows that

$$G_\gamma(A) \cong \prod_i (A[x]/(P_i(x)))^\times = \prod_i \mathbb{G}_m(A \otimes_F F_i)$$

for any F-algebra A, thus we obtain

$$G_\gamma \cong \prod_i \mathrm{Res}_F^{F_i} \mathbb{G}_m.$$

Hence we have

$$\mathbb{X}_*(G_\gamma) \cong \mathrm{Hom}_F(\mathbb{G}_m, \prod_i \mathrm{Res}_F^{F_i} \mathbb{G}_m) \cong \prod_i \mathrm{Hom}_{F_i}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}^m$$

and the injection $\mathbb{X}_*(G_\gamma) \hookrightarrow G_\gamma(F)$ is given by

$$\mathbb{X}_*(G_\gamma) \cong \mathbb{Z}^m \longrightarrow \prod_i F_i^\times \cong G_\gamma(F)$$
$$(d_1, \cdots, d_m) \mapsto (t^{d_1}, \cdots, t^{d_m}).$$

Consider a loop group LG_γ defined by $LG_\gamma(R) = G_\gamma(F\hat{\otimes}R)$ for a k-algebra R. For a reduced k-algebra R, we have

$$\begin{aligned} LG_\gamma(R) &\cong \prod_i \mathbb{G}_m(F_i \otimes_F R((t))) \\ &\cong \prod_i \mathbb{G}_m(k((t^{1/e_i})) \otimes_{k((t))} R((t))) \\ &\cong \prod_i \mathbb{G}_m(R((t^{1/e_i}))) \\ &\cong \prod_i L_{F_i}\mathbb{G}_m(R) \end{aligned}$$

where $L_{F_i}\mathbb{G}_m$ is isomorphic to $L\mathbb{G}_m$ by identifying t^{1/e_i} with t_i. By the computation in the split case, we obtain

$$(LG_\gamma)^{\mathrm{red}} \cong \prod_i t_i^{\mathbb{Z}} \times L^+_{F_i}\mathbb{G}_m \cong \prod_i t^{\frac{1}{e_i}\mathbb{Z}} \times L^+\mathbb{G}_m$$

with a sublattice $\Lambda_\gamma \cong \prod_i t_i^{e_i\mathbb{Z}} \cong \prod_i t^{\mathbb{Z}}$. In particular, the quotient is given by

$$\Lambda_\gamma \backslash (LG_\gamma)^{\mathrm{red}} \cong \prod_i t_i^{\mathbb{Z}/(e_i)} \times L^+_{F_i}\mathbb{G}_m$$

and contains the group scheme $\prod L^+_{F_i}\mathbb{G}_m$ with finite index. Note that we have an exact sequence

$$0 \to \mathcal{O}_{F_i}^\times / t^{\mathbb{Z}} \to \mathbb{Z}/e_i\mathbb{Z} \to 0$$

for each i. Alternatively, we may take a lattice $\tilde{\Lambda}_\gamma = (t_1^{\mathbb{Z}}, \cdots, t_m^{\mathbb{Z}}) \subseteq G_\gamma(F)$ for $t_i = t^{1/e_i}$. This lattice will be used to compute the orbital integral.

Example 2.9.4. (The case $G = SL_2$ Let $G = SL_2$ and $\gamma = \begin{pmatrix} 0 & t^n \\ 1 & 0 \end{pmatrix}$ Note that γ is regular semisimple. for some odd positive integer n. Then the easy computation gives that for a F-algebra A, we have

$$G_\gamma(A) = \begin{pmatrix} a & t^n b \\ b & a \end{pmatrix}$$

with $a, b \in A$ and $a^2 - t^n b^2 = 1$. This shows that $G_\gamma = \operatorname{Spec} F[x,y]/(x^2 - t^n y^2 - 1)$. For $A = R((t))$ with a reduced k-algebra R, the equation gives that $a, b \in R[\![t]\!]$, thus $G_\gamma(R((t))) = G_\gamma(R[\![t]\!])$. The torus G_γ is non-split, and splits over a quadratic field extension $F(t^{1/2})/F$. Since G_γ is of dimension 1 and non-split, the lattice $\Lambda_\gamma = \operatorname{Hom}_F(\mathbb{G}_m, G_\gamma) = 0$ is trivial. For a reduced k-algebra R, consider $LG_\gamma(R) = G_\gamma(F\hat{\otimes}R)$. Note that if we let $R = \varprojlim_{m\in\mathbb{Z}} k[s,t]/(s^2 - t, s^m)$, then we obtain $F\hat{\otimes}R \cong k((t^{1/2}))$. Hence the equation $a^2 + t^n b^2 = 1$ may have a solution in $F\hat{\otimes}R$ for some R. If we denote the solutions by $a = \sum_{i\geq 0} a_i t^i$ and $b = \sum_{j\geq 0} b_j t^j$, then the equation gives $a_0 = \pm 1$ and each a_i for $i \geq 1$ is determined by the Taylor expansion of $(1 + t^n b^2)^{1/2}$. Thus we have

$$(LG_\gamma)^{\mathrm{red}} \cong L^+G_\gamma \cong \{\pm 1\} \times \mathbb{A}^{\mathbb{N}}$$

where the last isomorphism is given by $(a, b) \mapsto (a_0, b_0, b_1, \cdots)$.

Example 2.9.5. The example 2.8.9 shows the non-split case.

2.10 Geometric properties of affine Springer fibers

2.10.1 Non-reducedness

The ind-scheme $\tilde{\mathcal{X}}_\gamma$ is never reduced if G is nontrivial and γ is regular semisimple in $\mathfrak{g}(F)$. For example, choose $\gamma \in \mathfrak{t}^{\mathrm{res}}$ as an example 2.8.8. Then $\tilde{\mathcal{X}}_\gamma$ is a Gr_T-torsor, and since $LT(R)$ may have nilpotent leading coeffiecients for nonreduced k-algebsra R $\mathrm{Gr}_T(R)$ may be different from $(\mathrm{Gr}_T)^{\mathrm{red}} \cong \mathbb{X}_*(T)(R)$. Hence Gr_T is not reduced, thus so is $\tilde{\mathcal{X}}_\gamma$.

2.10.2 Main theorem

Theorem 2.10.1. *For a regular semisimple element. $\gamma \in \mathfrak{g}(F)$, the followings are hold :*

(1) There exists a closed subscheme $Z \subseteq \mathcal{X}_\gamma$, which is projective over k and satisfying
$$X_\gamma = \bigcup_{l\in\Lambda_\gamma} l \cdot Z.$$

(2) The ind-scheme $\mathcal{X}_\gamma$ is represented by a scheme locally finite type over k.

(3) The action of a lattice Λ_γ on $\mathcal{X}_\gamma$ is free, and the fppf-sheaf quotient $\Lambda_\gamma\backslash\mathcal{X}_\gamma$ is representable by a proper algebraic space over k.

Proof. (1) We first consider the case when γ is contained in a split maximal torus. By $G(F)$-conjugation, we may assume that $\gamma \in \mathfrak{t}(F)$ and $\mathbb{X}_*(T) \cong \Lambda_\gamma$. Choose a Borel subgroup B contains T and denote the unipotent radical of B by N. Then we have an Iwasawa decomposition

$$LG = LN \cdot \Lambda_\gamma \cdot LG^+$$

thus

$$\mathrm{Gr} = LN \cdot \Lambda_\gamma \cdot LG^+/LG^+ = \bigsqcup_{\lambda \in \mathbb{X}_*(T)} LN \cdot t^\lambda LG^+/LG^+.$$

Now let $X := \mathcal{X}_\gamma \cap LN \cdot LG^+/LG^+ \subseteq \mathcal{X}_\gamma$. By the above decomposition, it suffices to show that X is contained in some Schubert variety $\mathrm{Gr}_{\leq\mu}$, for then its schematic-closure Z gives the projective variety we want. Note that $\{l \cdot X \mid l \in \Lambda_\gamma\}$ are disjoint and cover $\mathcal{X}_\gamma$.

Choose positive roots with respect to B and a pinning, then for any $u \in N(F)$ we may write

$$u = \prod_{\alpha > 0} x_\alpha(c_\alpha)$$

for root groups $x_\alpha : \mathbb{G}_a \cong N$ and $c_\alpha \in F$. We claim that there exists a lower bound of valuations of c_α for any $u \in N(F)$ satisfying $[u] \in X$. $\square$

Example 2.10.2. In the example 2.8.7, we have $Z = \mathbb{P}^1$ and $\mathcal{X}_\gamma = \bigcup_{\mathbb{Z}} \mathbb{P}^1$ over k.

Example 2.10.3. Let $G = GL_n$

3 Orbital Integrals

3.1 Integration on a p-adic group

Let F be a nonarcimedean local field; i.e. F is either a finite extension of $\mathbb{Q}_p$ or a finite extension of $\mathbb{F}_p((t))$. We may assume that the valuation $\mathrm{val} : F^\times \to \mathbb{Z}$ is surjective by normalization. Let $\mathcal{O}_F$ and k be the ring of integers and the residue field respectively. Note that k is a finite field, unlike the previous section. We assume that k is large enough respect to the groups we deal with.

3.1.1 Haar measure and Integration

Let G be a locally profinite group, then it has a right invariant Haar measure μ_G unique up to scalar multiplication. Note that a group is locally profinite if and only if every open neighborhood of the identity contains a compact open subgroup. For a compact open subgroup $K_0 \subseteq G$, we may assume that K_0 has a volume 1.

In this case, we define a function $f : G \to \mathbb{C}$ is *smooth* if it is locally constant. We denote the complex vecor space of smooth functions(resp. smooth compactly supported functions) by $C^\infty(G)$(resp. $C_c^\infty(G)$). It turns out that $C_c^\infty(G)$ an algebra under a convolution $*$ and it is called as the *Hecke algebra of* G. For a compact open subset $K \subseteq G$, let $C_c^\infty(G/\!/K)$ be the subalgebra of functions which are left and right K-invariant.

Lemma 3.1.1. *Let G be a locally profinite group. For any $f \in C_c^\infty(G)$, there exists a compact open subgroup $K \subseteq G$ such that f is left and right K-invariant; i.e. $f \in C_c^\infty(G/\!/K)$.*

Proof. Let $g \in \mathrm{supp}(f)$, then $f^{-1}(f(g))$ is an open subgroup of G. Note that $\mathrm{supp}(f)$ is both open and closed, thus compact by the assumption. Then $g^{-1} \cdot f^{-1}(f(g))$ is an open neighborhood of the identiy, so it contains a compact open subgroup $K(g) \subseteq G$. Now f is constant on $gK(g)$, and

$$\{gK(g) \mid g \in \mathrm{supp}(f)\}$$

is an open cover of $\mathrm{supp}(f)$. Since $\mathrm{supp}(f)$ is compact, we obtain a finite subcover

$$\{g_iK(g_i) \mid g_i \in \mathrm{supp}(f), 1 \leq i \leq n\}.$$

Let

$$K_r := \bigcap_{i=1}^{n} K(g_i),$$

then it is a compact open subgroup as well. By the construction, f is right K_r-invariant function. In the similar manner, we can construct K_l where f is left K_l-invariant. Then $K := K_l \cap K_r$ gives a compact open subgroup of G such that $f \in C_c^\infty(G/\!/K)$. $\square$

Remark 3.1.2. Let $H \subseteq G$ be a closed subgroup and f a left H-invariant locally constant function on G. For a locally constant function $\bar{f}$ on $H\backslash G$ defined by $\bar{f}(Hg) = f(g)$, suppose that $\mathrm{supp}(\bar{f}) \subseteq \mathrm{supp}(f)$ is compact(Hence the condition is weaker than the one of the above lemma). Since $G \to H\backslash G$ is an open map, $\{H\backslash gK(g) \mid g \in \mathrm{supp}(f)\}$ is an open cover of $\mathrm{supp}(\bar{f})$ as well thus we may choose finite subcover of $\operatorname{supp}(\bar{f})$. Then the preimage $\{g_i K(g_i)\}$ gives the finite open cover of $\operatorname{supp}(f)$, thus we can find K similarly.

Lemma 3.1.3. *Suppose $f \in C_c^\infty(G)$ is H-left and K-right invariant for some subgroups $H, K \subseteq G$. Then we may write*

$$f = \sum_{[g] \in H\backslash G/K} f(g) \cdot 1_{HgK} = \sum_{i=1}^{n} f(g_i) \cdot 1_{Hg_iK}$$

for some finitely many $g_i \in G$. Here 1_{HgK} denotes the characteristic function of HgK.

Proof. Since f is constant on HgK, thus first equality holds. Note that

$$\{HgK \mid g \in \operatorname{supp}(f)\}$$

is an open cover of $\operatorname{supp}(f)$, thus admits a finite open cover, say

$$\{Hg_iK \mid g_i \in \operatorname{supp}(f), 1 \le i \le n\}.$$

Since double-cosets are mutually disjoint, the cosets HgK contained in $\operatorname{supp}(f)$ equals to the one of Hg_iK. Hence we may write

$$f = \sum_{i=1}^{n} f(g_i) \cdot 1_{Hg_iK}.$$

□

The following theorem holds for a locally compact Hausdorff group G.

Theorem 3.1.4. *Let $H \subseteq G$ be a closed subgroup. There exists a right G-invariant Radon measure μ on H/G if and only if $\delta_G|_H = \delta_H$. In this case, μ is unique up to a scalar multiplication, and is characterized as follows. For given right Haar measures μ_G and μ_H, one has that*

$$\int_G f(g)dg = \int_{H\backslash G} \left(\int_H f(hg)dh \right) d\mu(Hg).$$

for $f \in C_c(G)$.

We will denote μ in the theorem by $\frac{dg}{dh}$.

3.1.2 Integration on Linear Algebraic Groups

Let G be a linear algebraic group over F. Then $G(F)$ becomes a locally profinite group. If we choose an integral model $\mathcal{G}$ over $\mathcal{O}_F$, then we usually let $K_0 = \mathcal{G}(F)$. Let $H \leq G$ be a linear algebraic subgroup over F. It follows that $H(F) \leq G(F)$ is a closed subgroup. Note that if G is reductive then $G(F)$ is unimodular. Further suppose that G is reductive and H is either reductive or $H(F)$ is compact. Then we obtain dg/dh by the theorem 3.1.4. Consider a left $H(F)$-invariant locally constant function f on $G(F)$ whose support is compact module $H(F)$; i.e. For a locally constant function $\overline{f}$ on $H(F)\backslash G(F)$ defined by $\overline{f}(Hx) = f(x)$, we have $\operatorname{supp}(\overline{f}) \subseteq H(F)\backslash G(F)$ is compact. The integral

$$\int_{H(F)\backslash G(F)} \overline{f}\,\frac{dg}{dh}$$

can be calculated as follows.

First we claim that there exists a compact open subgroup $K \subseteq K_0$ of finite index such that f is right K-invariant. Let K be a compact open subgroup of $G(F)$ such that f is right K-invariant. If we replace K with the intersection $K \cap K_0$, then since it is a compact open subgroup of a profinite group, it has a finite index with respect to K_0.

Now f is left $H(F)$-invariant and right K-invariant. Then $\{H(F)gK \mid g \in \operatorname{supp}(f)\}$ forms an open cover of $\operatorname{supp}(f)$, thus so does its quotient $\{H(F)gK \mid Hg \in \operatorname{supp}(\overline{f})\}$ of $\operatorname{supp}(\overline{f})$. Hence we may write

$$\overline{f} = \sum_{[g]\in H(F)\backslash G(F)/K} \overline{f}(H(F)g)\cdot \overline{1}_{H(F)gK} = \sum_{[g]\in H(F)\backslash G(F)/K} f(g)\cdot \overline{1}_{H(F)gK}$$

where $\overline{1}_{H(F)gK}$ is a characteristic function defined on $H(F)\backslash G(F)$. Note that

$$\overline{1}_{HgK}(Hx) = \begin{cases} 1 & \text{if } gKx^{-1}\cap H(F) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned}\int_{h\in H(F)} 1_{gK}(hx)dh &= \int_{h\in H(F)} 1_{gKx^{-1}}(h)dh \\ &= \operatorname{vol}(gKx^{-1}\cap H(F), dh)\cdot \overline{1}_{H(F)gK}(H(F)x)\end{aligned}$$

for $H(F)x \in H(F)\backslash G(F)$ and 1_{gK} on $G(F)$, thus the theorem 3.1.4 shows that

$$\begin{aligned}
&\int_{H(F)\backslash G(F)} \overline{1}_{H(F)gK}(H(F)x)\frac{dg}{dh} \\
=&\int_{H(F)\backslash G(F)} \left(\frac{1}{\mathrm{vol}(gKx^{-1}\cap H, dh)}\int_{h\in H} 1_{gK}(hx)dh\right) \\
=&\int_{x\in G} \frac{1}{\mathrm{vol}(gKx^{-1}\cap H, dh)} 1_{gK}(x)dg \\
=&\frac{\mathrm{vol}(gK, dg)}{\mathrm{vol}(gKg^{-1}\cap H, dh)} \\
=&\frac{\mathrm{vol}(K, dg)}{\mathrm{vol}(gKg^{-1}\cap H, dh)}
\end{aligned}$$

From $[K : K_0] = \mathrm{vol}(K)/\mathrm{vol}(K_0)$, we conclude that

$$\begin{aligned}
\int_{H(F)\backslash G(F)} \overline{f}\frac{dg}{dh} &= \sum_{[g]\in H(F)\backslash G/K} \int_{H(F)\backslash G(F)} f(g)\cdot \overline{1}_{H(F)gK}\frac{dg}{dh} \\
&= \sum_{[g]\in H(F)\backslash G/K} \frac{\mathrm{vol}(K, dg)}{\mathrm{vol}(H(F)\cap gKg^{-1}, dh)}\cdot f(g) \\
&= \frac{1}{[K_0 : K]} \sum_{[g]\in H(F)\backslash G/K} \frac{f(g)}{\mathrm{vol}(H(F)\cap gKg^{-1})}.
\end{aligned}$$

In particular, we have

$$\int_{G(F)} f dg = \frac{1}{[K_0 : K]} \sum_{[g]\in G/K} f(g).$$

when $H = 1$.

3.2 Orbital Integrals

3.2.1 Definition of orbital integrals

We follow the setup of 3.1.2. Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element and $\varphi \in \mathcal{S}(\mathfrak{g}(F))$. For a map $G(F) \to \mathfrak{g}(F)$ defined by $g \mapsto \mathrm{Ad}(g^{-1})\gamma$, the composition gives $f : g \mapsto \varphi \circ \mathrm{Ad}(g^{-1})\gamma$, a smooth function on $G(F)$. Then f is locally constant, left invariant under the centralizer $G_\gamma(F) \leq G(F)$, and compactly supported modulo $G_\gamma(F)$. Fix Haar measure μ_G and μ_{G_γ} on G and G_γ respectively. As in 3.1.2, we define an *orbital integral* on $\mathfrak{g}(F)$ by

$$O_\gamma(\varphi) := \int_{G_\gamma(F)\backslash G(F)} \varphi(\mathrm{Ad}(g^{-1})\gamma)\frac{\mu_G}{\mu_{G_\gamma}}$$

For the rest section, we restrict to ourselves to the specific situation. Let G be a split reductive group over F. Then we have a Chevally model, which is a reductive group scheme over $\mathbb{Z}$ with a maximal torus. This gives an integral model of G over $\mathcal{O}_F$ by base changing. In this way, we may let G be a reductive group scheme over $\mathcal{O}_F$. Let $K_0 = G(\mathcal{O}_F)$ and μ_G a Haar measure normalized to K_0. We have a canonical lattice $\mathfrak{g}(O_F)$ in $\mathfrak{g}(F)$, and our most important case is $\varphi = 1_{\mathfrak{g}(\mathcal{O}_F)}$.

Since $\gamma \in \mathfrak{g}(F)$ is regular semisimple, a centralizer G_γ is a F-torus. For a maximal unramified extension $F^{un} = F\hat{\otimes}_k\bar{k}$, we have an injective homomorphism

$$\mathbb{X}_*(G_\gamma \otimes_F F^{un}) \hookrightarrow G_\gamma(F^{un})$$

by sending λ to $\lambda(t)$ where t is an uniformlizer of F, and F^{un} as well. We again denote the image by Λ_γ. The morphism is $\mathrm{Gal}(\bar{k}/k)$-equivariant, hence Λ_γ admits a galois action. Then we have the corresponding étale group scheme over k, again denoted by Λ_k. Note that $\Lambda_k(k)$ is a galois invariant of Λ_k if we regard the latter as a plain group. Then $\Lambda_\gamma(k) \subseteq G_\gamma(F)$ is a discrete cocompact subgroup. Here cocompact means that the quotient is compact.

Example 3.2.1. Let $G = GL_n$ and $\gamma \in \mathfrak{g}(F)$ a regular semisimple element, which might be not diagonalizable over F. Since γ is regular semisimple, the characteristic polynomial $P(x) = x^n + a_1x^{n-1} + \cdots + a_n \in F[x]$ of γ is separable. Then we have $F[x]/(P(x)) \cong \times_{i=1}^m F_i$ where each F_i is a field and $G_\gamma(F) \cong \prod_{i=1}^m F_i^\times$. Then a lattice Λ_γ is given by $t \mapsto (t^{d_1}, \cdots, t^{d_m})$ for $(d_1, \cdots, d_m) \in \mathbb{Z}^m$ Note that $F_i = l_i((t^{1/e_i}))$ where e_i is a ramification index. Hence we have an exact sequence

$$0 \to \mathcal{O}_{F_i}^\times \to F_i^\times/t^{\mathbb{Z}} \to \mathbb{Z}/e_i\mathbb{Z} \to 0.$$

This shows that the quotient $\Lambda_\gamma\backslash G_\gamma(F) \cong \prod_i F_i^\times/t^{\mathbb{Z}}$ is a compact.

3.2.2 Orbital integrals in terms of counting

For a regular semisimple element $\gamma \in \mathfrak{g}(F)$, consider

$$X_\gamma := \{[g] \in G(F)/G(\mathcal{O}_F) \mid \mathrm{Ad}(g^{-1})\gamma \in \mathfrak{g}(\mathcal{O}_F)\}$$

a subset of $G(F)/G(\mathcal{O}_F)$. This is a set-theoretical version of affine springer fiber. Then $G_\gamma(F)$ acts on X_γ by left translation.

Fact 3.2.2. For any free abelian group $L \subseteq G_\gamma(F)$, it acts on X_γ freely. More generally, for any discrete cocompact $G_\gamma(F)$, the action groupoid $L\backslash X_\gamma$ is finitary; i.e. it has finitely many isomorphism classes, and the automorphism group of each object is finite.

For a finitary groupoid Y, we define its cardinality by

$$\#Y := \sum_{y \in \mathrm{ob}(Y)/\cong} \frac{1}{\mathrm{Aut}(y)}.$$

Lemma 3.2.3. *Let γ be a regular semisimple element in $\mathfrak{g}(F)$. For a discrete cocompact subgroup $L \subseteq G_\gamma(F)$, we have*

$$O_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \frac{1}{\mathrm{vol}(G_\gamma(F)/L, \mu_{G_\gamma})} \#(L\backslash X_\gamma)$$

Proof. Let $H = G_\gamma(F)$ and $K = G(\mathcal{O}_F)$. Since $f = 1_{\mathfrak{g}(\mathcal{O}_F)}(\mathrm{Ad}(g^{-1})\gamma)$ is left H-invariant and right K-invariant, the discussion of 3.1.2 gives that

$$\begin{aligned} O_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) &= \int_{G_\gamma(F)\backslash G(F)} 1_{\mathfrak{g}(\mathcal{O}_F)}(\mathrm{Ad}(g^{-1})\gamma) \frac{\mu_G}{\mu_{G_\gamma}} \\ &= \sum_{[g] \in G_\gamma(F)\backslash G(F)/G(\mathcal{O}_F)} \frac{1_{\mathfrak{g}(\mathcal{O}_F)}(\mathrm{Ad}(g^{-1})\gamma)}{\mathrm{vol}(\mathrm{stab}_{G_\gamma(F)}(g), \mu_{G_\gamma})} \\ &= \sum_{G_\gamma(F)\backslash X_\gamma} \frac{1}{\mathrm{vol}(\mathrm{stab}_{G_\gamma(F)}(g), \mu_{G_\gamma})} \end{aligned}$$

Since the Haar measure on L is a counting measure, we have

$$\begin{aligned}
\mathrm{vol}(\mathrm{stab}_{G_\gamma(F)}(g), \mu_{G_\gamma}) &= \int_{G_\gamma(F)} 1_{\mathrm{stab}_{G_\gamma(F)}(g)}(h)\ \mu_{G_\gamma} \\
&= \int_{L\backslash G_\gamma(F)} \left(\int_L 1_{\mathrm{stab}_{G_\gamma(F)}(g)}(lh)\ \mu_L \right) \frac{\mu_{G_\gamma}}{\mu_L} \\
&= \int_{L\backslash G_\gamma(F)} \left(\sum_L 1_{\mathrm{stab}_{G_\gamma(F)}(g)}(lh) \right) \frac{\mu_{G_\gamma}}{\mu_L} \\
&= \int_{L\backslash G_\gamma(F)} \left(\#\{l \in L \mid l \in h^{-1}\mathrm{stab}_{G_\gamma(F)}(g)\} \right) \frac{\mu_{G_\gamma}}{\mu_L} \\
&= \int_{L\backslash G_\gamma(F)} \left(\#\{l \in L \mid l \in \mathrm{stab}_{G_\gamma(F)}(g)\} \right) \frac{\mu_{G_\gamma}}{\mu_L} \\
&= \#\{l \in L \mid l \in \mathrm{stab}_{G_\gamma(F)}(g)\} \int_{L\backslash G_\gamma(F)} \frac{\mu_{G_\gamma}}{\mu_L} \\
&= \#(\mathrm{stab}_{G_\gamma(F)}(g) \cap L) \cdot \mathrm{vol}(L\backslash G_\gamma(F), \mu_{G_\gamma(F)/\mu_L}).
\end{aligned}$$

It follows that

$\square$

Example 3.2.4. Let $G = GL_n$, then if k is algebraically clsoed then the example 2.9.3 gives that $G_\gamma(F) = \prod_i F_i^\times$ with a lattice $\tilde{\Lambda}_\gamma = \prod_i t_i^{\mathbb{Z}}$. When $k \neq \bar{k}$, then each F_i is given by a tower of unramified extension $l((t))$ of F and a totally tamely ramified extension $F_i = l((t^{1/e_i}))$ over $l((t))$. In this case, let $t_i = t^{1/e_i}$ be a uniformlizer of F_i where e_i is a ramification index. Then again we have a lattice $L_0 \subseteq G_\gamma(F)$ satisfying $G_\gamma(F)/L_0 \cong \prod_i \mathcal{O}_{F_i}^\times$. By normalizing the Haar measure on $G_\gamma(F)$ to make $\prod_i \mathcal{O}_{F_i}^\times$ of volume 1, then so is the volume of $G_\gamma(F)/L_0$. By the lemma 3.2.3, we have

$$O_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \#(L_0\backslash X_\gamma).$$

3.3 Relation with affine springer fibers

In this section, we restrict to the case F is a local function field; i.e. $F = k((t))$ for some finite field $k = \mathbb{F}_q$. The constructions of affine Grassmannians and affine springer fibers make sense even when k is a finite field, so we have ind-schemes Gr_G and $\mathcal{X}_\gamma$ defined over k.

Lemma 3.3.1. *The set of k-rational points $\mathcal{X}_\gamma(k)$ equals to X_γ, both as a subset of $\mathrm{Gr}_G(k)$.*

Question 3.3.2. Why the lemma holds? Can we say that the $H^1(k, LG^+(\overline{k}))$ is trivial for general G?

If we base change k to $\overline{k}$, then from the theorem 2.10.2 we know that $\Lambda_\gamma \backslash \mathcal{X}_{\gamma,\overline{k}}$ is a proper algebraic space over $\overline{k}$. Actually, the proof shows that the algebraic space is defined over k, which is denoted by $\Lambda_\gamma \backslash \mathcal{X}_\gamma$. Here we view Λ_γ as an étale group scheme over k, whose $\overline{k}$-point is the plain group used to be denoted Λ_γ.

3.3.1 k-points of a quotient

Let k be a perfect field. Suppose X is a scheme over k and A is an algebraic group over k acting on X. Consider a quotient stack $\mathcal{Y} = [A \backslash X]$, then $\mathcal{Y}(k)$ consists of a left A-torsor P over k and a A-equivariant morphism $P \to X$. Note that the isomorphism class of left A-torsors is classified by the first cohomology $H^1(k, A) := H^1(\mathrm{Gal}(\overline{k}/k), A(\overline{k}))$. A group action is given by $A \to \underline{\mathrm{Aut}}(X)$, we have the induced a map of pointed set $H^1(k, A) \to H^1(k, \underline{\mathrm{Aut}}(X))$. For a class $\xi \in H^1(k, A)$ corresponding to a left A-torsor P, the image under this map corresponds to a contracted product $P \times^A X := A \backslash P \times X$ where the action of A on $P \times X$ is a diagonal action. We denote the contracted product by A_ξ. Then we have $X_\xi(k) \cong \mathrm{Hom}_{\text{G-equiv.}}(A, X_\xi) \cong \mathrm{Hom}_{\text{A-equiv.}}(P_\xi, X)$. Now $[A \backslash X](k)$ is decomposed in to groupoids indexed by $H^1(k, A)$. Suppose further that A acts on X freely. Then the stabilizer are all trivial, equivalent to say that any automorphism of an object in a groupoid is trivial; a groupoid becomes a setoid. Each connected component of $[A \backslash X](k)$ consists of $A(k)$ many objects since there exists $A(k)$ many $A(k)$-equivariant morphism $P_\xi \to X$. Since a groupoid is a setoid, a k-point of a fppf sheaf quotient $A \backslash X$ is given by

$$\coprod_{\xi \in H^1(k,A)} A(k) \backslash X_\xi(k).$$

3.3.2 The case of GL_n

In the setup of 3.2.4, we have a lattice $L_0 = \prod_i t_i$ in $G_\gamma(F)$. By base changing to $\overline{k}$, we obtain a lattice $\widetilde{\Lambda}_\gamma$ of LG_γ as in the example 2.9.3 by choosing the same t_i above. Note that $F_i \hat{\otimes}_k \overline{k}$ may split into a product of fields, but the image of $t_i \otimes 1$ along each projection becomes the uniformizer of each factor. Then using the galois action on $\widetilde{\Lambda}_\gamma$ we may view it as an étale group scheme over k, and it satisfies $\widetilde{\Lambda}_\gamma(k) = L_0$. There is an analog of the theorem

2.10.2 if we replace Λ_γ with $\widetilde{\Lambda}_\gamma$, thus $\widetilde{\Lambda}_\gamma\backslash\mathcal{X}_\gamma$ is a proper algebraic space over k with admitting a surjective map from a projective k-scheme.

Now in GL_n case, we can compute the orbital integral by the trace formula.

Theorem 3.3.3. *Let $G = \mathrm{GL}_n$ and $\gamma \in \mathfrak{g}(F)$ a regular semisimple element. We normalize the Haar measure on $G_\gamma(F)$ to make the maximal compact subgroup of volume 1. Then we have*

$$O_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \#(\widetilde{\Lambda}\backslash\mathcal{X}_\gamma)(k) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_k, H^i(\widetilde{\Lambda}\backslash X_{\gamma,\bar{k}}, \mathbb{Q}_l)).$$

Proof. The second equality comes from the Grothendieck-Lefschetz trace formula. By the computation in the example 3.2.4, it suffices to show that $\widetilde{\Lambda}_\gamma(k)\backslash\mathcal{X}_\gamma(k) = (\widetilde{\Lambda}\backslash\mathcal{X}_\gamma)(k)$. It suffices to show that $H^1(k, \widetilde{\Lambda}_\gamma) := H^1(\mathrm{Gal}(\bar{k}/k), \widetilde{\Lambda}(\bar{k}))$ is trivial. By the construction, we have

$$\widetilde{\Lambda}_\gamma \cong \bigoplus_{i=1}^m \mathrm{Ind}_{\mathrm{Gal}(\bar{k}/k_j)}^{\mathrm{Gal}(\bar{k}/k)} \mathbb{Z}$$

as a $\mathrm{Gal}(\bar{k}/k)$-module where k_i is a residue field of F_i. By the Shapiro lemma, we obtain

$$H^1(k, \widetilde{\Lambda}) = \bigoplus_{i=1}^m H^1(k_i, \mathbb{Z})$$

which is trivial since the kernel of a continuous homomorphism $\mathrm{Gal}(\bar{k}/k_i) \to \mathbb{Z}$ is open thus the image is a finite subgroup of $\mathbb{Z}$, which is trivial. □

3.4 Stable Orbital Integrals

For a general reductive group, a formula of 3.3.3 does not hold since we cannot find $\widetilde{\Lambda}_\gamma$ with vanishing cohomology to make $\widetilde{\Lambda}_\gamma(k)\backslash\mathcal{X}_\gamma(k)$ equal to $(\widetilde{\Lambda}_\gamma\backslash\mathcal{X}_\gamma)(k)$. Thus we need to consider the twisted forms of $\mathcal{X}_\gamma$ all together.

3.4.1 Stable conjugacy

Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. We say that $\gamma' \in \mathfrak{g}(F)$ is *stably conjugate* to γ if there exists $g \in G(\bar{F})$ such that $\mathrm{Ad}(g)(\gamma) = \gamma'$; i.e. they in the same $G(\bar{F})$-orbit. This is equivalent to say that $\chi(\gamma) = \chi(\gamma') \in \mathfrak{c}(F)$. For γ' stably conjugate to γ, choose $g \in G(\bar{F})$ such that $\gamma' = g\gamma g^{-1}$. Since we choose γ, γ' in $\mathfrak{g}(F)$, they are fixed by $\Gamma = \mathrm{Gal}(\bar{F}/F)$. Hence we have $\gamma' = \sigma(g)\gamma\sigma(g)^{-1}$ for all $g \in \Gamma$. It follows that the map $\sigma \mapsto g^{-1}\sigma(g)$

defines a 1-cocycle of Γ in G_γ whose image in G becomes a coboundary. One can easily shows that the different choice of g gives the cohomologous 1-cocycle. Hence we obtain an element

$$\mathrm{inv}(\gamma, \gamma') \in \ker(H^1(F, G_\gamma) \to H^1(F, G)).$$

If γ' is $G(F)$-conjugate to γ, then we may choose $g \in G(F)$ thus $g^{-1}\sigma(g) = g^{-1}g = 1$ holds. Thus the map is defined on the $G(F)$-conjugacy classes of γ, so we have a well-defined pointed map

$$\{\gamma' \in \mathfrak{g}(F) \text{ stably conjugate to } \gamma\}/G(f) \to \ker(H^1(F, G_\gamma) \to H^1(F, G)).$$

We claim that this map is bijective. If $g^{-1}\sigma(g) = g'^{-1}\sigma(g)$ for any $\sigma \in \Gamma$, then gg'^{-1} is fixed by any $\sigma \in \Gamma$ thus contained in $G_(F)$. This shows that the map is injective. Suppose $f \in \ker(H^1(F, G_\gamma) \to H^1(F, G))$, then we have $g \in G(\overline{F})$ such that $f(\sigma) = g^{-1}\sigma(g)$ for any $\sigma \in \Gamma$. By letting $\gamma' = g\gamma g^{-1}$, we've showed that the map is surjective.

Example 3.4.1. (The GL_n case) Let $G = \mathrm{GL}_n$, then for a regular semisimple element $\gamma \in \mathfrak{g}(F)$ we have $G_\gamma = \prod_{i=1}^m \mathrm{Res}_F^{F_i} \mathbb{G}_m$. It follows that

$$H^1(F, G_\gamma) \cong \bigoplus_{i=1}^m H^1(F, \mathrm{Res}_F^{F_i} \mathbb{G}_m) \cong \bigoplus_{i=1}^m H^1(F_i, \mathbb{G}_m) = 1.$$

This shows that every γ' stably conjugate to γ is indeed $G(F)$-conjugate to γ.

Example 3.4.2. (SL_2) Let $a \in k^\times \setminus (k^\times)^2$. Consider two regular seimsimple elements

$$\gamma = \begin{pmatrix} 0 & at \\ t & 0 \end{pmatrix}, \quad \gamma' = \begin{pmatrix} 0 & at^2 \\ 1 & 0 \end{pmatrix}.$$

Since they have the same characteristic polynomial, they are stably conjugate to each other. Explicitly, let $g = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}$ then we have $\gamma' = g\gamma g^{-1}$. One may check that they are not $G(F)$-conjugate.

Definition 3.4.3. Let $\varphi \in \mathcal{S}(\mathfrak{g}(F))$ and $\gamma \in \mathfrak{g}(F)$ a regular semisipmle element. We define the *stable orbital integral* of φ with respect to γ by

$$\mathrm{SO}_\gamma(\varphi) = \sum_{\gamma'} O_{\gamma'}(\varphi)$$

where the index γ' represents the $G(F)$-orbit in the set of elements in $\mathfrak{g}(F)$ stably conjugate to γ.

3.5 Examples in SL_2

Let $G = SL_2$ and suppose $\mathrm{char}(k) > 2$. We will see that the computation of stable integral orbital by the sum of orbital integrals and by the cohomology are equal in by some examples. In the following examples, the centralizer G_γ becomes a norm-one torus over F which splits over a quadratic extension E/F(may be ramified). The norm-one torus G_γ admits an exact sequence

$$1 \to G_\gamma \to \mathrm{Res}_F^E \mathbb{G}_m \xrightarrow{\det} \mathbb{G}_m \to 1$$

of abelian groups, so we obtain an exact sequence

$$E^\times \xrightarrow{\mathrm{Nm}} F^\times \to H^1(F, G_\gamma) \to 1.$$

thus $F^\times/\mathrm{Nm}(E^\times) \cong H^1(F, G_\gamma)$. By the local class field theory, we have $F^\times/\mathrm{Nm}(E^\times) \cong \mathrm{Gal}(E/F)$. Then the discussion in 3.4.1 gives that γ has 2 stable conjugacy classes up to ordinal conjugacy classes. Hence to compute the stable orbital integral directly, it suffices to choose two representatives.

3.5.1 Unramified case : Orbital integrals

Example 3.5.1. Let

$$\gamma = \begin{pmatrix} 0 & at \\ t & 0 \end{pmatrix}$$

with a square-free $a \in k^\times$. Then we have

$$G_\gamma(R) = \begin{pmatrix} x & ay \\ y & x \end{pmatrix}$$

for a F-algebra R thus

$$G_\gamma = \mathrm{Spec}\, F[x, y]/(x^2 - ay^2 - 1)$$

is a norm 1 torus. Since it is anisotropic, we obtain that $G_\gamma(F)$ is compact. Hence we may choose the Haar measure on $G_\gamma(F)$ whose total volume is 1.

Let $\mathcal{X}_\gamma$ be an affine springer fiber of γ, which is a scheme over k. Then the lemma 3.2.3 gives that

$$\mathcal{O}_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \#X_\gamma = \#\mathcal{X}_\gamma(k).$$

Let $k' = k(\sqrt{a})$ and $E = k' \hat{\otimes}_k F = k'(\!(t)\!)$. By choosing the proper eigenbasis, we obtain

$$g = \begin{pmatrix} \frac{\sqrt{a}}{2} & -1 \\ \frac{1}{2} & \frac{1}{\sqrt{a}} \end{pmatrix} \in SL_2(E)$$

such that

$$g^{-1}\gamma g = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = \gamma_0$$

Thus $\mathcal{X}_{\gamma,k'}$ is isomorphic to an affine springer fiber $\mathcal{X}_{\gamma_0}$ over k'. In particular, the example 2.8.7 shows that $\mathcal{X}_{\gamma,k'}$ is a union of $\mathbb{Z}$-copies of $\mathbb{P}^1_{k'}$. We denote each $\mathbb{P}^1_{k'}$ by C_n so that $t \in \Lambda_\gamma$ sends C_n to C_{n+1}. The lattice $\Lambda_\gamma \subseteq G_\gamma(F^{\mathrm{un}})$ in contained in $G_\gamma(E) = E^\times = k'((t))^\times$, which is generated by t. Let $x_{n+1/2} := C_n \cap C_{n+1}$, which is a k'-point of $\mathcal{X}_\gamma$. We claim that σ acts on $\mathcal{X}_\gamma(k')$ by $\sigma(C_n) = C_{1-n}$. Since σ is a group action, it must send $C_n \cong \mathbb{P}^1_{k'}$ to the other copy of $\mathbb{P}^1$. Since G_γ is a twisted form of $\mathbb{G}_m$ splits over a quadratic field extension E/F, the galois descent gives that a nontrivial $\sigma \in \mathrm{Gal}(E/F) = \mathrm{Gal}(k'/k)$ acts on $G_\gamma(E)$ by an inversion, thus so on Λ_γ. Since σ-action is compatible with Λ_γ-action, we obtain

$$\sigma(t) \cdot \sigma(C_n) = \sigma(t \cdot C_n),$$

which is rewritten as

$$t^{-1} \cdot \sigma(C_n) = \sigma(C_{n+1}).$$

Hence it suffices to know the $\sigma(C_0) = C_m$. Note that C_0 and C_1 of $\mathcal{X}_{\gamma_0}(k')$ is given by

$$\mathcal{O}_E \oplus t\mathcal{O}_E \subseteq \Lambda \subseteq t^{-1}\mathcal{O}_E \oplus \mathcal{O}_E$$

and

$$t\mathcal{O}_E \oplus \mathcal{O}_E \subseteq \Lambda \subseteq \mathcal{O}_E \oplus t^{-1}\mathcal{O}_E$$

respectively. Since we have $\gamma = g\gamma_0 g^{-1}$, an isomorphism $\mathcal{X}_{\gamma_0}(k') \to \mathcal{X}_\gamma(k')$ is given by the left translation of g. In particular, C_0 of $\mathcal{X}_\gamma(k')$ is given by

$$g(\mathcal{O}_E \oplus t\mathcal{O}_E) \subseteq \Lambda \subseteq g(t^{-1}\mathcal{O}_E \oplus \mathcal{O}_E).$$

Note that

$$g\begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{a}}{2} - t \\ \frac{1}{2} - \frac{1}{\sqrt{a}}t \end{pmatrix}, \quad g\begin{pmatrix} t^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{a}}{2}t^{-1} - 1 \\ \frac{1}{2}t^{-1} - \frac{1}{\sqrt{a}} \end{pmatrix}.$$

Since we have $\sigma(t) = t^{-1}$, it follows that

$$\sigma\left(g\begin{pmatrix} 1 \\ t \end{pmatrix}\right) = g\begin{pmatrix} 1 \\ t^{-1} \end{pmatrix}, \quad \sigma\left(g\begin{pmatrix} t^{-1} \\ 1 \end{pmatrix}\right) = g\begin{pmatrix} t \\ 1 \end{pmatrix}.$$

This shows that $\sigma(C_0) = C_1$. It follows that

$$\sigma(C_n) = C_{1-n}, \quad \sigma(x_{n+1/2}) = x_{-n+1/2}$$

for any $n \in \mathbb{Z}$. In particular, $x_{1/2}$ is the only point fixed by σ thus defined over k.

Since $\mathcal{X}_\gamma(k)$ consists of the $\mathrm{Gal}(k'/k)$-invariant elements of $\mathcal{X}_\gamma(k')$, we have $\mathcal{X}_\gamma(k) = x_{1/2}$. This implies that

$$O_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \#\mathcal{X}_\gamma(k) = 1.$$

For the other representative, let

$$\gamma' = \begin{pmatrix} 0 & at^2 \\ 1 & 0 \end{pmatrix}$$

with $a \in k^\times$ is square-free. Then we have

$$G_{\gamma'}(R) = \begin{pmatrix} x & at^2y \\ y & x \end{pmatrix}$$

for F-algebra R thus

$$G_{\gamma'} = \mathrm{Spec} F[x, y]/(x^2 - at^2y^2 - 1)$$

is a norm 1 torus. We obtain

$$g = \begin{pmatrix} \frac{\sqrt{a}}{2}t & -1 \\ \frac{1}{2} & \frac{1}{\sqrt{a}}t^{-1} \end{pmatrix}$$

such that

$$g^{-1}\gamma g = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = \gamma_0.$$

We proceed almost in the same way with the previous example. The only difference is that the galois action on $\mathcal{X}_{\gamma'}(k')$. Note that

$$g\begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{a}}{2}t - t \\ \frac{1}{2} - \frac{1}{\sqrt{a}} \end{pmatrix}, \quad g\begin{pmatrix} t^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{a}}{2} - 1 \\ \frac{1}{2}t^{-1} - \frac{1}{\sqrt{a}}t^{-1} \end{pmatrix}.$$

Since

$$\sigma\left(g\begin{pmatrix} 1 \\ t \end{pmatrix}\right) = \begin{pmatrix} \frac{\sqrt{a}}{2}t^{-1} - t^{-1} \\ \frac{1}{2} - \frac{1}{\sqrt{a}} \end{pmatrix}$$

gives the same lattice with

$$g\begin{pmatrix} t^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{a}}{2} - 1 \\ \frac{1}{2}t^{-1} - \frac{1}{\sqrt{a}}t^{-1} \end{pmatrix},$$

we have that $\sigma(C'_0) = C'_0$. It follows that

$$\sigma(C'_n) = C'_{-n}, \quad \sigma(x_{n+1/2}) = x_{-n-1/2}.$$

In particular, C'_0 is the only one defined over k. Since a (cohomological) Brauer group of a finite field is trivial, indeed $C'_0 \cong \mathbb{P}^1_k$ over k. This shows that $C'_0(k) = \mathbb{P}^1(k) = q+1$. Thus we obtain

$$O_{\gamma'}(1_{\mathfrak{g}(\mathcal{O}_F)}) = q+1.$$

Hence we obtain that

$$SO_{\gamma'}(1_{\mathfrak{g}(\mathcal{O}_F)}) = 1 + (q+1) = q+2.$$

3.5.2 Unramified case : Cohomology

For $\gamma = \begin{pmatrix} 0 & at \\ t & 0 \end{pmatrix}$ and $\gamma' = \begin{pmatrix} 0 & at^2 \\ 1 & 0 \end{pmatrix}$, we expect that

We have shown that both $\mathcal{X}_\gamma$ and $\mathcal{X}_{\gamma'}$ are an infinite chains of $\mathbb{P}^1$ over k'. Consider quotients $\Lambda_\gamma \backslash \mathcal{X}_\gamma$ and $\Lambda_{\gamma'} \backslash \mathcal{X}_{\gamma'}$ over k. Then over k', both are isomorphic to a nodal rational curve obtained from $\mathbb{P}^1$ by attaching two k-points into a nodal point y. Consider the case of γ'. Since a nodal point y is fixed by a galois action σ, it is defined over k. Note that none of x_j is defined over k since none of them is fixed by σ. Quotient by $\Lambda_{\gamma'}$ does not affect to the points $C'_0(k)$, thus $(\Lambda_{\gamma'} \backslash G_{\gamma'})(k)$ consists of $C'_0(k)$ and the nodal point y. In particular, we have $\#(\Lambda_{\gamma'} \backslash G_\gamma)(k) = q+1+1 = q+2$. Similarly, we can obtain the same result for γ. The connectedness of a quotient group scheme $\Lambda_{\gamma'} \backslash LG_{\gamma'}$ can be checked after base change to $\bar{k}$. Since $G_{\gamma'}$ is a norm-one torus, we have that $\Lambda_{\gamma',\bar{k}} \backslash G_{\gamma',\bar{k}}$ is isomorphic to $t^{\mathbb{Z}} \backslash (L\mathbb{G}_m)^{\mathrm{red}} \cong L^+\mathbb{G}_m$ over $\bar{k}$. This shows that $\Lambda_{\gamma'} \backslash LG_{\gamma'}$ is connected, thus the stable part of the cohomology of $\Lambda_{\gamma'} \backslash LG_{\gamma'}$ is indeed $H^*(\Lambda_{\gamma'} \backslash \mathcal{X}_{\gamma'})$. Then the Grothendieck-Lefschetz fixed point theorem gives that $(\Lambda_{\gamma'} \backslash \mathcal{X}_{\gamma'})(k) = \sum_{i=0}^2 H^i(\Lambda_{\gamma'} \backslash \mathcal{X}_{\gamma'}, \mathbb{Q}_l) = q+2$. We conclude that

$$SO_{\gamma'}(1_{\mathfrak{g}(\mathcal{O}_F)}) = \sum_{i=0}^{2} H^i(\Lambda_{\gamma'} \backslash \mathcal{X}_{\gamma'}, \mathbb{Q}_l) = q+2.$$

3.5.3 Ramified case : Orbital integrals

Let

$$\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}, \quad \gamma' = \begin{pmatrix} 0 & at^2 \\ a^{-1}t & 0 \end{pmatrix}.$$

Then we have

$$G_\gamma(R) = \begin{pmatrix} x & ty \\ y & x \end{pmatrix}, \quad G_{\gamma'}(R) = \begin{pmatrix} x & a^2 ty \\ y & x \end{pmatrix}$$

respectively, thus both are norm 1 tori. Let $E = k((t^{1/2}))$ be a ramified quadratic extension of F. We may choose the Haar measure on compact subgroups $G_\gamma(F)$ and $G_\gamma(F)$ with total volume 1. By the example 2.8.9, we know that both $\mathcal{X}_{\gamma,\bar{k}}$ and $\mathcal{X}_{\gamma',\bar{k}}$ are isomorphic to $\mathbb{P}^1_{\bar{k}}$. Since k is a finite field, its (cohomological) Brauer group is trivial. In particular, both $\mathcal{X}_\gamma$ and $\mathcal{X}_{\gamma'}$ are isomorphic to $\mathbb{P}^1_k$ over k. The lemma 3.2.3 gives that

$$\mathcal{O}_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \#\mathcal{X}_\gamma(k) = q+1$$

and the same for γ'. Hence we obtain

$$SO_\gamma(1_\mathfrak{g}(\mathcal{O}_F)) = 2(q+1).$$

3.5.4 Ramified case : Cohomology

By 2.9.4, we know that Λ_γ is trivial and $(LG_\gamma)^{\mathrm{red}} \cong \pm 1 \times \mathbb{A}^{\mathbb{N}}$. Since the component group acts on cohomology trivially, the stable part is the whole $H^*(\mathcal{X}_\gamma)$. A norm-one torus $G_\gamma(F)$ is itself compact, and in 2.9.4 we showed that $G_\gamma(F) = G_\gamma(\mathcal{O}_F)$. Hence let

$$\mathcal{G}_\gamma = \mathrm{Spec}\mathcal{O}_F[x,y]/(x^2 - ty^2 - 1),$$

then its special fiber is given by

$$\mathcal{G}_{\gamma,k} = \mathrm{Spec}k[x,y]/(x^2-1).$$

The neutral component of a special fiber is given by $\mathrm{Spec}k[x,y]/(x-1)$, and a nonneutral component is a closed subscheme

$$\mathrm{Spec}k[x,y]/(x+1) \hookrightarrow \mathrm{Spec}k[[t]][x,y]/(x^2-ty^2-1).$$

Then an open subscheme $\mathcal{G}_\gamma^0$ is the complement of nonneutral component, and its $\mathcal{O}_F$ point is given by

$$K_\gamma = \left\{ \begin{pmatrix} a & tb \\ b & a \end{pmatrix} \in \mathcal{G}_\gamma(\mathcal{O}_F) \mid \bar{a} = a_0 = 1 \right\}.$$

This is of index 2 in $G_\gamma(F)$, thus $\mathrm{vol}(K_\gamma) = 1/2$. Since $\Lambda_\gamma \backslash \mathcal{X}_\gamma = \mathcal{X}_\gamma \cong \mathbb{P}^1$, we have $\mathcal{X}_\gamma(k) = q+1$. It follows that

$$\frac{1}{\mathrm{vol}(K_\gamma, \mu_{G_\gamma(F)})} \sum_{i=0}^{2} (-1)^i \mathrm{Tr}(\mathrm{Frob}, H^i(\mathcal{X}_\gamma, \mathbb{Q}_l) = 2\mathcal{X}_\gamma(k) = 2(q+1).$$

4 Hitchin Fibration

4.1 The Hitchin moduli stack

4.1.1 The setup

Let X be a smooth projective connected curve over $k = \overline{k}$. For a reductive group G over k, we denote by Bun_G the moduli stack of étale (right) G-torsors over X. For a k-algebra R, the R-points of Bun_G is the groupoid of G-torsors over X_R. Then Bun_G is a smooth algebraic stack locally of finite type over k. Note that since G is smooth, every fppf G-torsor splits over some étale cover.

Example 4.1.1. Let $G = GL_n$, then every G-torsor over X corresponds to a vector bundle of rank n over X. Hence we have a canonical isomorphism of stacks $\mathrm{Bun}_{GL_n} \cong \mathrm{Bun}_n$ where Bun_n is the stack of vector bundles of rank n.

4.1.2 Associated bundle

Let V be a G-representation over k. Then a right G-torsor $\mathcal{E}$ over X gives an associated bundle

$$\mathcal{E} \times^G V := G \backslash \mathcal{E} \times_X V_X$$

where the G-action is given by $g \cdot (e, v) = (e \cdot g^{-1}, g \cdot v)$.

For a reductive group G over k, G acts on $\mathfrak{g}$ by the adjoint action. Then we have an associated bundle

$$\mathrm{Ad}(G) := \mathcal{E} \times^G \mathfrak{g}.$$

Example 4.1.2. Let $G = GL_n$ and $\mathcal{E}$ a GL_n-torsor. Then for a vector bundle $\mathcal{V}$ of rank n over X, the associated bundle $\mathcal{E} \times^G \mathcal{V}$ gives the equivalence of the moduli stack $\mathrm{Bun}_G \cong \mathrm{Bun}_n$.

4.1.3 G-Higgs bundles

Definition 4.1.3. Let $\mathcal{L}$ be a line bundle. An *$\mathcal{L}$-twisted Higgs bundle* is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a étale (right) G-torsor over X and φ is a global section of a vector bundle $\mathrm{Ad}(G) \otimes \mathcal{L}$. We call φ as a *$\mathcal{L}$-twisted Higgs field* on $\mathcal{E}$.

Example 4.1.4. Let $G = GL(V)$, then we have an isomorphism

$$\begin{aligned} \mathcal{E} \times^G \underline{\mathrm{End}}(V) &\longrightarrow \underline{\mathrm{End}}(\mathcal{E} \times^G V) \\ (e, f) &\mapsto ((e, v) \mapsto (e, f(v)) \end{aligned}$$

for a right G-torsor $\mathcal{E}$ over X. We denote $\mathcal{E} \times^G V$ by $\mathcal{V}$, then for a line bundle $\mathcal{L}$ a $\mathcal{L}$-twisted Higgs field is given by a $\mathcal{O}_X$-linear map

$$\mathcal{O}_X \to \underline{\mathrm{End}}(\mathcal{V}) \otimes \mathcal{L}$$

corresponding to

$$\mathcal{V} \to \mathcal{V} \otimes \mathcal{L}.$$

Example 4.1.5. Let $G = SL_n$, then $\mathcal{L}$-twisted G-Higgs bundle is give by $(\mathcal{V}, \iota, \varphi)$ is given by a vector bundle $\mathcal{V}$ of rank n over X, a nonvanishing section $\iota : \wedge^n \mathcal{V} \cong \mathcal{O}_X$, and $\mathcal{O}_X$-linear map $\varphi : \mathcal{V} \to \mathcal{V} \otimes \mathcal{L}$ such that $\mathrm{Tr}(\varphi) = 0 \in H^0(X, \mathcal{L})$. Note that the trace can be defined in coordinate-free way as follows: $\varphi : \mathcal{V} \to \mathcal{V} \otimes \mathcal{L}$ corresponds to $\mathcal{O}_X \to \mathcal{V}^\vee \otimes \mathcal{V} \otimes \mathcal{L}$, then take the evaluation map.

We have a moduli stack $\mathcal{M}_{G,\mathcal{L}}$ of $\mathcal{L}$-twisted Higgs G-torsors over X. For a k-algebra R, the R-points of $\mathcal{M}_{G,\mathcal{L}}$ is the groupoid of $\mathcal{L}_R$-twisted G-Higgs bundles on X_R. The forgetful functor $\mathcal{M}_{G,\mathcal{L}} \to \mathcal{M}_G$ is representable, thus $\mathcal{M}_{G,\mathcal{L}}$ is an algebraic stack over k as well.

4.2 Hitchin fibration

4.2.1 The GL_n-case

Let $G = GL_n$ and $(\mathcal{V}, \varphi)$ a $\mathcal{L}$-twisted G-Higgs bundle over X. For an $\mathcal{O}_X$-map

$$\varphi : \mathcal{V} \to \mathcal{V} \otimes \mathcal{L},$$

we obtain

$$\wedge^i \varphi : \wedge^i \mathcal{V} \to \wedge^i \mathcal{V} \otimes \mathcal{L}^{\otimes i}$$

for each $1 \leq i \leq n$. We define

$$a_i(\varphi) := \mathrm{Tr}(\varphi) \in H^0(X, \mathcal{L}^{\otimes i})$$

similar to the coefficient of characteristic polynomial of φ, then we obtain a morphism

$$f : \mathcal{M}_{n,\mathcal{L}} \to \mathcal{A}_{n,\mathcal{L}} := \prod_{i=1}^{n} H^0(X, \mathcal{L}^i).$$

sending $(\mathcal{V}, \varphi)$ to $(a_0, \cdots, a_n)$. We call such f as *Hithchin fibration* in $G = GL_n$.

4.2.2 The general case

Recall that G is a connected reductive group over an algebraically closed field $k = \overline{k}$. Let r be the rank of G and choose a split maximal torus T of G. The Chevalley's theorem gives $\mathfrak{g}//G \cong \mathfrak{t}//W$, where $\mathfrak{t}//W \cong \mathrm{Sym}(\mathfrak{t}^*)^W$ has r-variables. Then we may choose a homogeneous generators $(f_1, \cdots, f_r)$ of $\mathfrak{g}//G$ of degree $d_1 \leq \cdots \leq d_r$ respectively. Each f_i can be written as a symmetric multilinear function $f_i : \mathfrak{g}^{\otimes d_i} \to k$ invariant under G, thus for any G-torsor $\mathcal{E}$ we obtain a map of the associated bundle

$$f_i : \mathrm{Ad}(\mathcal{E})^{\otimes d_i} \to \mathcal{O}_X.$$

Note that G acts trivially on k, so we have $\mathcal{E} \times^G k = \mathcal{O}_X$. For a line bundle $\mathcal{L}$ over X, we obtain

$$f_i^{\mathcal{L}} : (\mathrm{Ad}(\mathcal{E}) \otimes \mathcal{L})^{\otimes d_i} \to \mathcal{L}^{\otimes d_i}$$

by tensoring with $\mathcal{L}^{\otimes d_i}$. For a $\mathcal{L}$-twisted G-Higgs field $\varphi \in H^0(X, \mathrm{Ad}(\mathcal{E}) \otimes \mathcal{L})$, the evaluation of φ^{d_i} along $f_i^{\mathcal{L}}$ defines

$$a_i(\varphi) \in H^0(X, \mathcal{L}^{d_i})$$

and the *Hitchin fibration* for $\mathcal{M}_{G,\mathcal{L}}$

$$f_{G,\mathcal{L}} : \mathcal{M}_{G,\mathcal{L}} \to A_{G,\mathcal{L}} := \prod_{i=1}^{r} H^0(X, \mathcal{L}^{\otimes d_i}).$$

Note that $H^0(X, \mathcal{L}^{\otimes d_i})$ is a presheaf over k whose R-points is $\mathcal{L}_R^{\otimes d_i}(X_R)$. If $\mathcal{L} = \mathcal{O}_X$, then the R-points of $\mathcal{L}$ is given by $X(R)$, which is a degree 0 part of the coordinate ring of a projective curve X_R. Hence as a k-sheaf, $\mathcal{O}_X$ is represented by the spectrum of degree 0 part of the coordinate ring of X. In particular, it is an affine space over k. This shows that $\mathrm{Res}_k^X(G_a)$ is represented by an affine space. Since $\mathcal{L}$ is a twisted form of $\mathcal{O}_X$, Then $\mathcal{L}^{d_i}$ is represented by d_i-copies of $\mathcal{L}$, so is an affine space. Now

We call $\mathcal{A}_{G,\mathcal{L}}$ as a *Hitchin base*.

Here we give more intrinsic description of the Hitchin base. Consider $\mathfrak{c} = \mathfrak{g}//G$, then $\mathfrak{c}$ has a G_m-action. This is equivalent to say that $\mathrm{Sym}(\mathfrak{g}^*)^G$ has a $\mathbb{Z}$-grading. Now consider a cartesian square

$$\begin{array}{ccc} \mathfrak{c} \times^{\mathbb{G}_m} \mathrm{Tot}(\mathcal{L}) & \longrightarrow & [\mathfrak{c}//\mathbb{G}_m] \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad \mathcal{L} \quad} & B\mathbb{G}_m, \end{array}$$

of stacks where $\mathcal{L}$ is a line bundle $\mathcal{L}$ and $\mathfrak{c} \times^{\mathbb{G}_m} \mathrm{Tot}(\mathcal{L})$ is quotient by the relation $(c, x) \simeq (\lambda \cdot c, \lambda^{-1} \cdot x)$. For the total space $\mathrm{Tot}(\mathcal{L})$, consider $\mathrm{Tot}^{\times}(\mathcal{L})$ the complement of zero section of $\mathcal{L}$. We denote $\mathfrak{c}_{X,\mathcal{L}} := \mathfrak{c} \times^{\mathbb{G}_m} \mathrm{Tot}^{\times}(\mathcal{L})$. Then a Hitchin base $\mathcal{A}_{G,\mathcal{L}}$ is a moduli space of a section of $\mathfrak{c}_{X,\mathcal{L}} \to X$.

Let $a \in \mathcal{A}_{G,\mathcal{L}}$, then we have the corresponding $\mathbb{G}_m$-equivariant map

$$a : \mathrm{Tot}(\mathcal{L}) \to \mathfrak{c},$$

then it gives a $\mathbb{Z}$-graded $\mathcal{O}_X$-algebra map

$$a : \mathrm{Sym}(\mathfrak{g}^*)^G \to \mathrm{Sym}(\mathcal{L}^{\vee}).$$

Recall that $\mathrm{Sym}(\mathfrak{g}^*)^G$ is generated by homogenous elements f_i of degree d_i respectively, thus there image under a is contained in $H^0(X, \mathcal{L}^{\vee})$. This shows that

$$a \in \prod_{i=1}^{r} H^0(X, \mathcal{L}^{\otimes d_i}).$$

4.2.3 The generically regular semisimple locus

Let η be a generic point of X. Trivializing $\mathcal{L}$ at η and restricting each a_i to η. Then we have $a_i \in F$ where $F = k(X)$ is a function field of X. We say $a \in \mathcal{A}_{n,\mathcal{L}}$ is *generically regular semisimple* if a characteristic polynomial $P_a(y) := y^n + a_1 y^{n-1} + \cdots + a_n \in F[y]$ is separable in $P_a[y]$. Let $\mathcal{A}^{\heartsuit}_{G,\mathcal{L}}$ be a generically regular semisimple locus, then it becomes an open subscheme of $\mathcal{A}^{\heartsuit}_{G,\mathcal{L}}$.

For general G, we have showed that $a \in \mathcal{A}_{G,\mathcal{L}}$ corresponds to $a : X \to \mathfrak{c}_{X,\mathcal{L}}$. Then a is generically regular semisipmle if a sends a generic point η to an open substack $[\mathfrak{c}^{rs}/\mathbb{G}_m]$. This defines an open subscheme $\mathcal{A}^{\heartsuit}_{G,\mathcal{L}} \subseteq \mathcal{A}_{G,\mathcal{L}}$.

4.2.4 Geometric properties

If $\deg \mathcal{L} > 2g - 2$, the stack $\mathcal{M}|_{\mathcal{A}^{\heartsuit}_{G,\mathcal{L}}}$ is smooth. In this case, the restriction $f_{G,\mathcal{L}}$ is flat over $\mathcal{A}^{\heartsuit}_{G,\mathcal{L}}$.qo

4.3 Hitchin Fibers

4.3.1 The case of GL_n and the spectral curve

Let $(a_1, \cdots, a_n) \in \mathcal{A}_{n,\mathcal{L}}$. We will define a curve Y_a equipped with a finite flat morphism $Y_a \to X$ of degree n. Let $\Sigma = \mathrm{Tot}(\mathcal{L})$ be a total space and

$\pi : \Sigma \to X$ a projection. Consider a $\mathcal{O}_X$-module map

$$\iota_a : \mathcal{L}^{\otimes -n} \to \pi_* \mathcal{O}_\Sigma = \mathcal{O} \oplus \mathcal{L}^{\otimes -1} \oplus \mathcal{L}^{\otimes -2} \oplus \cdots$$

given by $((-1)^n a_n, (-1)^{n-1} a_{n-1}, \cdots, -a_1, 1, 0, \cdots)$. Then we have the corresponding

$$\pi^* \mathcal{L}^{\otimes -n} \to \mathcal{O}_\Sigma$$

and we denote the image by $\mathcal{J}_a$. Then $\mathcal{J}_a$ is an ideal sheaf on Σ, and we define the *spectral curve*

$$Y_a = \mathrm{Spec}\mathcal{O}_\Sigma / \mathcal{J}_a = \mathrm{Spec}(\mathcal{O} \oplus \mathcal{L}^{\otimes -1} \oplus \mathcal{L}^{\otimes -2} \oplus \cdots)/\mathcal{J}_a.$$

For an open subset $U \subseteq X$ where $\mathcal{L}$ trivializes, the restriction $Y_a|_U$ is a closed subscheme of $U \times \mathbb{A}^1$ defined by the equation $y^n - a_1 y^{n-1} + a_2 y^{n-2} + \cdots + (-1)^n a_n = 0$ with a coordinate y of $\mathbb{A}^1$. Hence $\pi_a : Y_a \to X$ is finite flat of degree n, and the fiber of π_a are the roots of the characteristic polynomial.

If $a \in \mathcal{A}^{\heartsuit}_{G,\mathcal{L}}$, then Y_a is reduced thus smooth on a Zariski dense open subset. There is a moduli stack $\overline{\mathrm{Pic}}(Y_a)$ classifying torsion-free coherent $\mathcal{O}_{Y_a}$-modules that are generically of rank 1.

Example 4.3.1. Let $X = \mathbb{P}^1_k$ and $\mathcal{L} = \mathcal{O}_X$. Then we have $a_i \in H^0(\mathbb{P}^1, \mathcal{O}_X^i) = k$. So the characteristic polynomial $P_a(y)$ is contained in $k[y]$

Proposition 4.3.2. *Let $a \in \mathcal{A}^{\heartsuit}_{G,\mathcal{L}}$ and $\mathcal{M}_a$ the fiber of $f : \mathcal{M}_{n,\mathcal{L}} \to \mathcal{A}_{n,\mathcal{L}}$ over a. Then there is a canonical isomorpihsm of stacks*

$$\overline{\mathrm{Pic}}(Y_a) \cong \mathcal{M}_a.$$

4.4 Relation with affine Springer fibers

Theorem 4.4.1. *(Product formula) For $a \in \mathcal{A}^{\heartsuit}_{G,\mathcal{L}}$, there is a canonical isomorphism of stacks*

$$\mathcal{P}_a \times^{\prod_{x \in X \setminus U_a} \mathcal{P}_{a_x}} \prod_{x \in X \setminus U_a} \mathcal{X}_{a_x} \to \mathcal{M}_a.$$

A Appendix : fpqc-sheaves

Let S be a base scheme.

Theorem A.0.1. *Let $\mathcal{F}$ be a presheaf on Sch_S. Then $\mathcal{F}$ is a fpqc-sheaf on Sch_S if and only if is is a Zariski-sheaf on Sch_S and for any faithfully flat morphism of affine S-schemes $Y \longrightarrow X$, the diagram*

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(Y) \rightrightarrows \mathcal{F}(Y \times_X Y)$$

is an equalizer.

Proof. See lemma 2.60 if [Vistoli]. □

Lemma A.0.2. *Let $\mathcal{F}$ be a presheaf on Sch_S. Suppose $\mathcal{F}$ satisfies*

(1) For any faithfully flat morphism of affine S-schemes $Y \longrightarrow X$, the diagram

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(Y) \rightrightarrows \mathcal{F}(Y \times_X Y)$$

is an equalizer,

(2) For any affine S-schemes $Y_1, \cdots, Y_n$, we have

$$\mathcal{F}(Y_1 \sqcup \cdots \sqcup Y_n) \cong \mathcal{F}(Y_1) \times \cdots \times \mathcal{F}(Y_n).$$

Then the Zariski-sheafification $\mathcal{F}^\#$ of $\mathcal{F}$ is an fpqc-sheaf. In particular, the Zariski-sheafification coincides with the fpqc-sheafification. Furthermore, a natural morphism $\mathcal{F} \longrightarrow \mathcal{F}^\#$ restricts to an isomorphism on the category of affine schemes.

Proof. See the proposition 50 of [Kreidl]. I will add it later. □

Corollary A.0.3. *Let $\mathcal{F}$ be a presheaf in A.0.2. Then the restriction of $\mathcal{F}$ on the category of affine schemes is a fpqc-sheaf.*

Lemma A.0.4. *Let $\mathcal{F}$ be a presheaf on Sch_S given by a filtered colimit $\mathcal{F} = \varinjlim X_i$ in the category of presheaves where X_i are represented by S-schemes. Then a Zariski-shefifiaction of $\mathcal{F}$ is an fpqc-sheaf. Furthermore, if we restrict $\mathcal{F}$ to the category of affine scheme, then $\mathcal{F}$ itself is already a fpqc-sheaf.*

Proof. It suffices to show $\mathcal{F}$ satisfies the two conditions in A.0.2.

□

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