

## Lecture 8 - The Lindblad equation

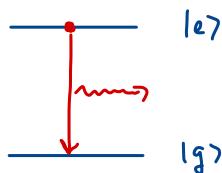
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Last time we gave arguments for a quantum master equation in the Lindblad form

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \sum_p (\hat{L}_p \hat{\rho} \hat{L}_p^\dagger - \frac{1}{2} \hat{L}_p^\dagger \hat{L}_p \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}_p^\dagger \hat{L}_p)$$

where  $\hat{\rho}$  is the reduced density matrix of a system,  $[\cdot, \cdot]$  is a commutator. The operators  $\hat{L}_p$  are often called Lindblad operators or jump operators, and describe effects of the environment. (I changed the notation from  $\hat{v}$  to  $\hat{L}$ ). We now want to understand this equation a little bit better.

### Spontaneous emission of a two-level system



An excited two-level atom in vacuum at zero temperature will eventually spontaneously decay to the ground state emitting a photon. The physics

behind this is vacuum fluctuations of the quantized electromagnetic field. We will not discuss details of this process. It is sufficient to say that we can use Fermi's golden rule to calculate the transition rate  $T_1$ , see for example section 19.5 in Ballentine, Quantum Mechanics.

Here we can simply use our physical intuition to identify the jump operator relevant for this process. There is only one, namely

$$\hat{L} = \sqrt{T_1} |g\rangle \langle e|$$

We can write the Hamiltonian of the atom as

$$\hat{H} = \frac{1}{2}\omega(|e\rangle \langle e| - |g\rangle \langle g|) = \frac{1}{2}\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\omega \sigma_z$$

using the Pauli matrix  $\sigma_z$ . We then have  $\hat{L} = \sqrt{T_1} \sigma_-$

$$\text{where } \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x - i\sigma_y) = \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right].$$

We also have

$$\sigma_+^+ = \sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The Lindblad equation takes the form

$$\frac{dg}{dt} = -i\frac{\omega}{2} [\sigma_z, g] + T \sigma_- g \sigma_+ - \frac{T}{2} (\sigma_+ \sigma_- g + g \sigma_+ \sigma_-)$$

We can write this out in components: denote

$$g = \begin{pmatrix} g_{ee} & g_{eg} \\ g_{ge} & g_{gg} \end{pmatrix} \quad \text{and} \quad \sigma_+ \sigma_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

so:

$$-i\frac{\omega}{2} [\sigma_z, g] = i\omega \begin{pmatrix} 0 & -g_{eg} \\ g_{ge} & 0 \end{pmatrix}$$

$$\sigma_- g \sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_{ee} & g_{eg} \\ g_{ge} & g_{gg} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ g_{ee} & g_{eg} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ee} \end{pmatrix}$$

$$\begin{aligned} \sigma_+ \sigma_- g + g \sigma_+ \sigma_- &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_{ee} & g_{eg} \\ g_{gg} & g_{gg} \end{pmatrix} + \begin{pmatrix} g_{ee} & g_{eg} \\ g_{ge} & g_{gg} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} g_{ee} & g_{eg} \\ g_{ge} & 0 \end{pmatrix} + \begin{pmatrix} g_{ee} & 0 \\ g_{ge} & 0 \end{pmatrix} = \begin{pmatrix} 2g_{ee} & g_{eg} \\ g_{ge} & 0 \end{pmatrix} \end{aligned}$$

Together, we get then:

$$\frac{dg_{ee}}{dt} = -T g_{ee}$$

$$\frac{dg_{gg}}{dt} = +T g_{ee}$$

$$\frac{dg_{eg}}{dt} = -i\omega g_{eg} - \frac{T}{2} g_{eg}$$

$$\frac{dg_{ge}}{dt} = +i\omega g_{ge} - \frac{T}{2} g_{ge}$$

That is, the diagonal elements decay with a rate  $T$ , and off-diagonal elements with a rate  $T/2$ . Or, in other words, excited state probability decays with rate  $T$  and atomic coherences with rate  $T/2$ .

Note that

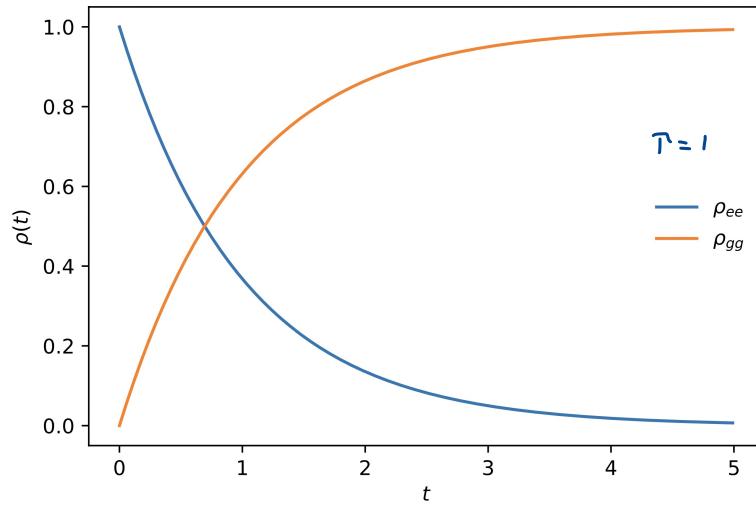
$$\frac{d}{dt} \text{tr } g = \frac{d}{dt} (\rho_{ee} + \rho_{gg}) = -T\rho_{ee} + T\rho_{ee} = 0$$

as needed for  $g$  to remain a density matrix at all times.

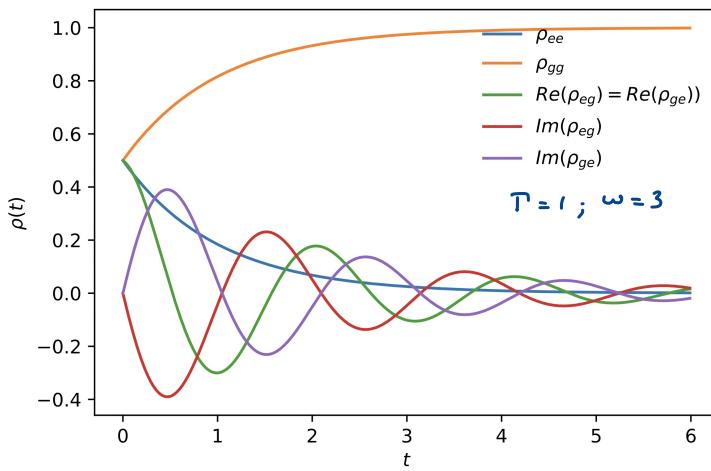
These equations are simple enough that we can solve them:

$$\begin{aligned}\rho_{ee}(t) &= \rho_{ee}(0) e^{-Tt} \\ \rho_{gg}(t) &= \rho_{gg}(0) + T \int_0^t \rho_{ee}(0) e^{-Tt'} dt' = \rho_{gg}(0) + \rho_{ee}(0)(1 - e^{-Tt}) \\ \rho_{eg}(t) &= \rho_{eg}(0) e^{-i\omega t - \frac{T}{2}t} \\ \rho_{ge}(t) &= \rho_{ge}(0) e^{+i\omega t - \frac{T}{2}t}\end{aligned}$$

If we prepare the system in the state  $|e\rangle$ , then  $\rho(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and only the diagonal elements are relevant, and follow a simple exponential decay towards the steady state  $\rho(\infty) = |g\rangle\langle g| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ .



If instead we prepare the system in the superposition  $|t\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$ , then  $\rho(0) = \frac{1}{2}(|e\rangle + |g\rangle)(\langle e| + \langle g|) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ . The diagonal elements behave similarly as before, apart from starting from different values; the off-diagonal elements oscillate with frequency  $\omega$  and decay with the rate  $T/2$  towards zero.



### Atom relaxing in a thermal cavity

When the atom is in a cavity in a presence of a thermal field there will be stimulated emission as well as absorption of photons. These leads to two changes: first the rate of emission will depend on the number of photons present  $n$ . That is  $T \rightarrow T(n+1)$ . This form could again be derived from the quantum theory of radiation using Fermi's golden rule, but we can also see that if the stimulated transition rate is proportional to the number of photons present the rate should be of the form  $T_{st} \cdot n + T_{sp}$ . That  $T_{st} = T_{sp} = T$  requires a bit more arguing, but let's assume that for the time being. The jump operator for emission then becomes

$$L_- = \sqrt{T(n+1)} \sigma_-$$

The second change we need is that we need to introduce a second jump operator to capture absorptions. This will be proportional to  $|\langle e | g \rangle| = \sigma_+$ . This will happen with rate  $T_{as}$  and since the Fermi golden rule is proportional to the absolute value  $|\langle e | \hat{H}_{int} | g \rangle|^2 = |\langle g | \hat{H}_{int} | e \rangle|^2$ , with  $\hat{H}_{int}$  the Hamiltonian describing the interaction between the

electromagnetic field and the atom, we have  $T_{as} = T$ .

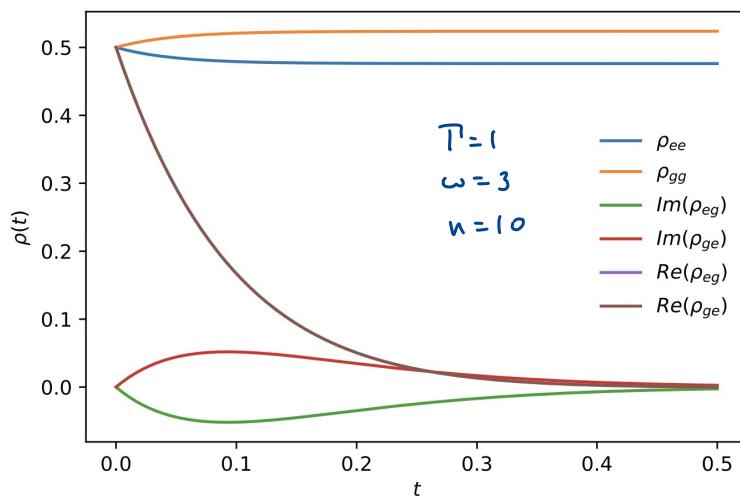
The jump operator for absorption is then

$$L_+ = \sqrt{Tn} \sigma_+$$

But what is  $n$ ? In principle we could try to keep track of the probability distribution of having  $n$  photons, but since we are assuming the environment is large and the fluctuations in the number of photons is therefore small, we can simply replace  $n$  by  $\langle n \rangle$  the expectation value of the number of photons in a thermal state. We then get.

$$\begin{aligned} \frac{d\varrho}{dt} = & -i \frac{\omega}{2} [\sigma_z, \varrho] + T(n+1) \left[ \sigma_- g \sigma_+ - \frac{1}{2} (\sigma_+ \sigma_- g + g \sigma_+ \sigma_-) \right] \\ & + Tn \left[ \sigma_+ g \sigma_- - \frac{1}{2} (\sigma_- \sigma_+ g + g \sigma_- \sigma_+) \right] \end{aligned}$$

We can solve this as before, or by numerical integration. Here is an example result starting from  $\varrho(0) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ .



We can look at some limits to make sure this makes sense.

At  $t \rightarrow \infty$  we expect a steady state with  $\frac{d\varrho}{dt} = 0$

Using that

$$\sigma_- g \sigma_+ = \begin{pmatrix} 0 & 0 \\ 0 & g_{ee} \end{pmatrix} \text{ and } \sigma_+ g \sigma_- + g \sigma_+ \sigma_- = \begin{pmatrix} 2g_{ee} & g_{eg} \\ g_{ge} & 0 \end{pmatrix}$$

from before, and

$$\sigma_+ g \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_{ee} & g_{eg} \\ g_{ge} & g_{jj} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} g_{ge} & g_{jj} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} g_{jj} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- \sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ so}$$

$$\sigma_- \sigma_+ g + g \sigma_- \sigma_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_{ee} & g_{eg} \\ g_{ge} & g_{jj} \end{pmatrix} + \begin{pmatrix} g_{ee} & g_{eg} \\ g_{ge} & g_{jj} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & g_{eg} \\ g_{ge} & 2g_{jj} \end{pmatrix}$$

$$\frac{dg}{dt} = 0 \text{ implies}$$

$$T(n+1) \left[ \sigma_- g \sigma_+ - \frac{1}{2} (\sigma_+ \sigma_- g + g \sigma_- \sigma_+) \right]_{aa} = - T_n \left[ \sigma_+ g \sigma_- - \frac{1}{2} (\sigma_- \sigma_+ g + g \sigma_- \sigma_+) \right]_{aa}$$

with the diagonal elements

$$T(n+1) \left( 0 - \frac{1}{2} \cdot 2 g_{ee} \right) = - T_n \left( g_{jj} - \frac{1}{2} \cdot 0 \right)$$

$$T(n+1) \left( g_{ee} - \frac{1}{2} \cdot 0 \right) = - T_n \left( 0 - \frac{1}{2} \cdot 2 g_{jj} \right)$$

$$\text{or} \quad \frac{n+1}{n} = \frac{g_{jj}}{g_{ee}} = \frac{g_{jj}}{1-g_{jj}} \quad \text{or} \quad (n+1)(1-g_{jj}) = n g_{jj}$$

so

$$g_{jj} = \frac{n+1}{2n+1}$$

$$\text{and} \quad g_{ee} = 1 - g_{jj} = \frac{n}{2n+1}$$

which for  $n=10$  gives  $g_{jj} = \frac{11}{21}$  and  $g_{ee} = \frac{10}{21}$ , which is consistent with the plot.

## Quantum trajectory (Monte Carlo) approach

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Since the density matrix is of size  $N^2$ , solving the Lindblad equation for large  $N$  can be demanding. There is another approach that instead calculate trajectories, which are wave functions  $|\Psi_i(t)\rangle$  that are stochastically generated. This is a bit like when we solved the Langevin equation to get paths  $v(t)$  and then averaged over many paths to get the probabilities  $P(v)$ .

I will describe one algorithm here. This can be shown to be equivalent to solving the Lindblad equation. I will not derive this rigorously, but just motivate it using the Langevin-Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle = \left[ \hat{H} - \frac{i}{2} \sum_p \hat{L}_p^\dagger \hat{L}_p + i \sum_p \eta_p(t) \hat{L}_p \right] |\Psi(t)\rangle$$

Now, the noise terms  $\eta_p(t) \hat{L}_p$  cause quantum jumps, where the wave function is not continuous, while the first two terms give a continuous damped time evolution with the non-hermitian operator

$$\hat{J} = \hat{H} - \frac{i}{2} \sum_p \hat{L}_p^\dagger \hat{L}_p$$

If we time evolve over a short time  $\tau$ , then to first order in  $\tau$ , the wave function will have decayed by

$$|\Psi(t+\tau)\rangle = (1 - i\tau \hat{J} + \dots) |\Psi(t)\rangle$$

that is

$$\langle \Psi(t+\tau) | \Psi(t+\tau) \rangle = \langle \Psi(t) | \Psi(t) \rangle - \tau \langle \Psi(t) | \sum_p \hat{L}_p^\dagger \hat{L}_p | \Psi(t) \rangle + \dots$$

We can then say that the probability for no quantum jump having occurred is

$$P_0 = 1 - \tau \langle \Psi(t) | \sum_p \hat{L}_p^\dagger \hat{L}_p | \Psi(t) \rangle.$$

The probability of quantum jump  $\hat{L}_p$  having happened must then be

$$P_p = \langle \psi(t) | \hat{L}_p^\dagger \hat{L}_p | \psi(t) \rangle \tau$$

We then have

$$P_0 + \sum_p P_p = 1$$

So, we can then construct the following stochastic algorithm to evolve the wavefunction.

- start with wave function  $|\psi_a(t)\rangle$ . When starting the algorithm  $|\psi_a(0)\rangle$  is given by the initial conditions. If  $t > 0$  it is the wave function from the last step below:
- calculate  $P_0$  and  $P_p$ . Draw a random number  $r \in [0, 1]$ .

If  $r < P_0$ :

$$\text{evolve } |\psi_a(t+\tau)\rangle = (\mathcal{U} - i \hat{J} \cdot \vec{\epsilon} + \dots) |\psi_a(t)\rangle$$

Else  $r \geq P_0$ :

Pick  $v$  as the smallest index such that  $\sum_{n=0}^v P_n \geq r$ .

$$\text{Then } |\psi_a(t+\tau)\rangle = \hat{L}_v |\psi_a(t)\rangle$$

- Renormalize  $|\psi_a(t+\tau)\rangle \rightarrow \frac{|\psi_a(t+\tau)\rangle}{\sqrt{\langle \psi_a(t+\tau) | \psi_a(t+\tau) \rangle}}$

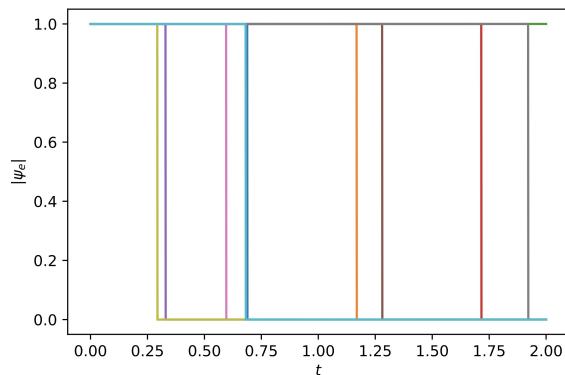
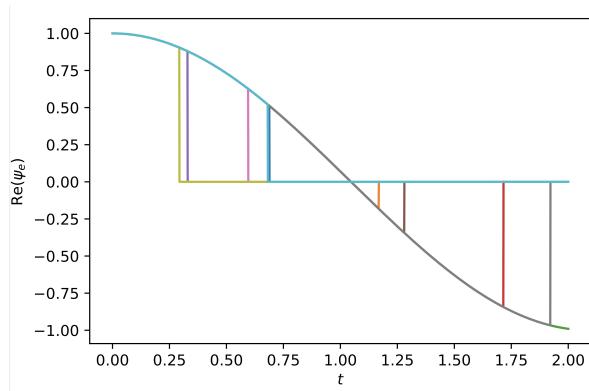
- Repeat.

This will give one trajectory  $|\psi_a(t)\rangle$ . Repeating this  $M$  times for different realizations of the random number generation gives the ensemble  $\{|\psi_a(t)\rangle \mid a=1, \dots, M\}$ . The corresponding density matrix is then

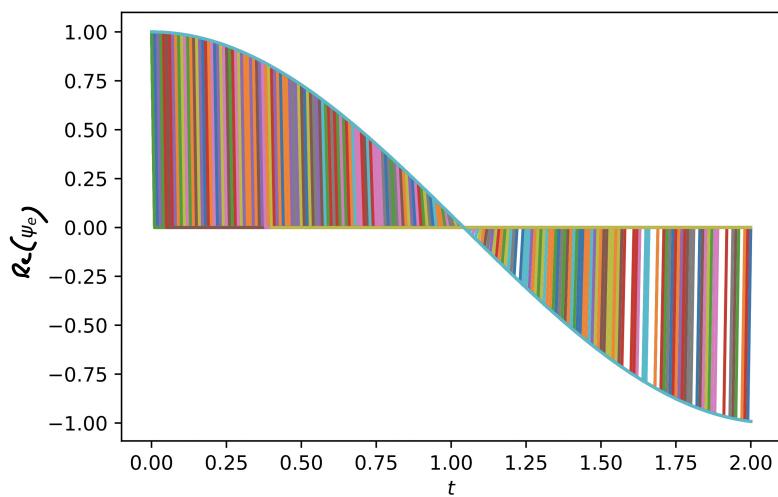
$$\bar{\rho}(t) = \frac{1}{M} \sum_a |\psi_a(t)\rangle \langle \psi_a(t)|$$

Let's do this calculation for the simple decay of a two-level system.

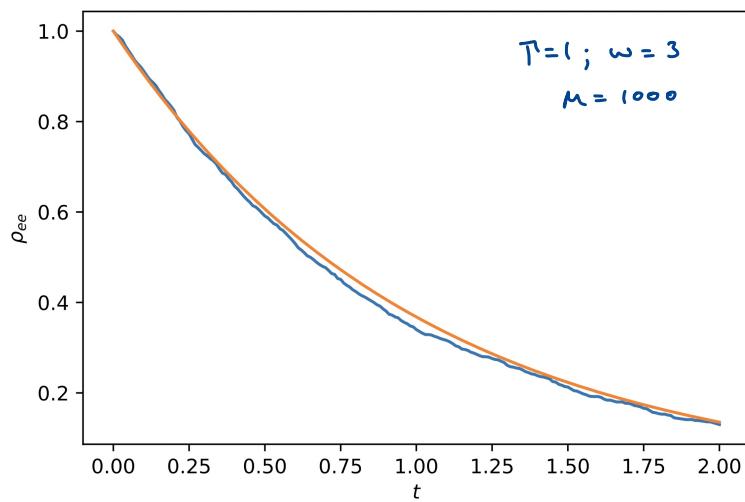
Here are some examples of quantum trajectories.



and here with many more:



Averaging gives the density matrix and the decay of the Sec



Agreement can be improved with more trajectories.