Problem Set 1

Tyler Gordon
January 17, 2018

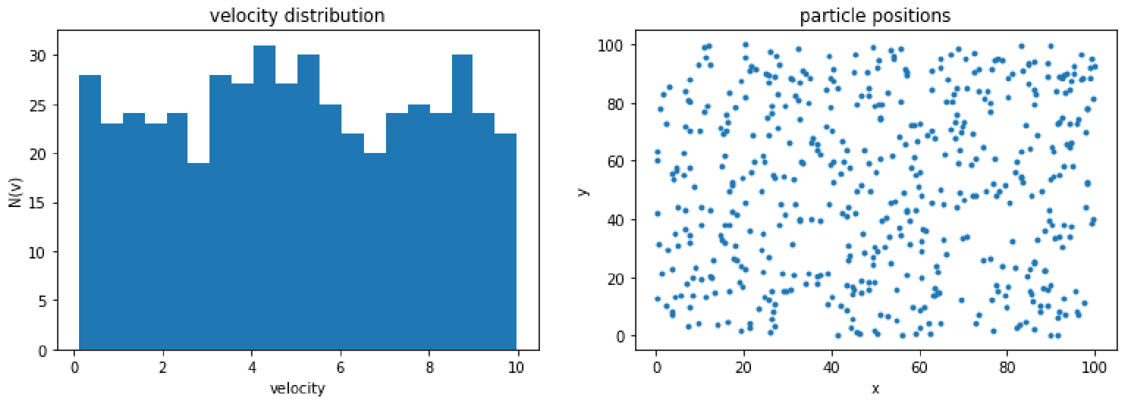
Problem 1

Part A

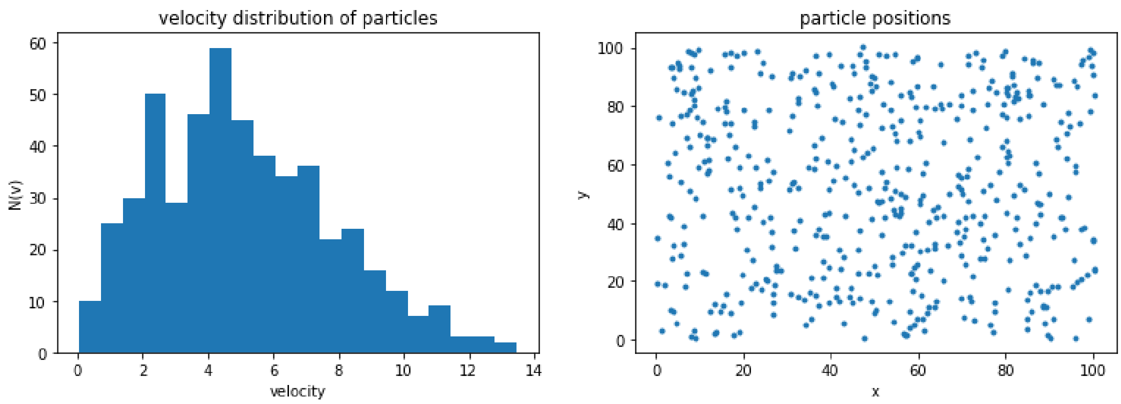
On github: http://github.com/tagordon/ASTR-507

Part B

Initial configuration:



Final configuration:



Problem 2

Part A

Deriving the Maxwellian velocity distribution:

$dN = f(p)dp_xdp_y$ gives the number of particles with momentum between p_x and $p_x + dp_x$, and p_y and $p_y + dp_y$. Starting with this expression, we have:

$$dN = f(p)dp \propto e^{-E/kT}dp_xdp_y = e^{-m(v_x^2+v_y^2)/2kT}m^2dv_xdv_y$$

rewriting this in polar coordinates, we get

$$dN \propto e^{-mv^2/2kT}m^2vdvd\theta$$

Integrating over all directions introduces a factor of 2π:

$$dN \propto 2\pi e^{-mv^2/kT}m^2vdv$$

We now must normalize such that the total number of particles is 1. By integrating dN over all velocities from zero to infinity, we find that the normalization constant must be $2m/kT$ from which we find the final distribution:

$$\frac{dN}{dv} = \frac{2m}{kT}ve^{-mv^2/kT}$$

Deriving the Maxwellian energy distribution:

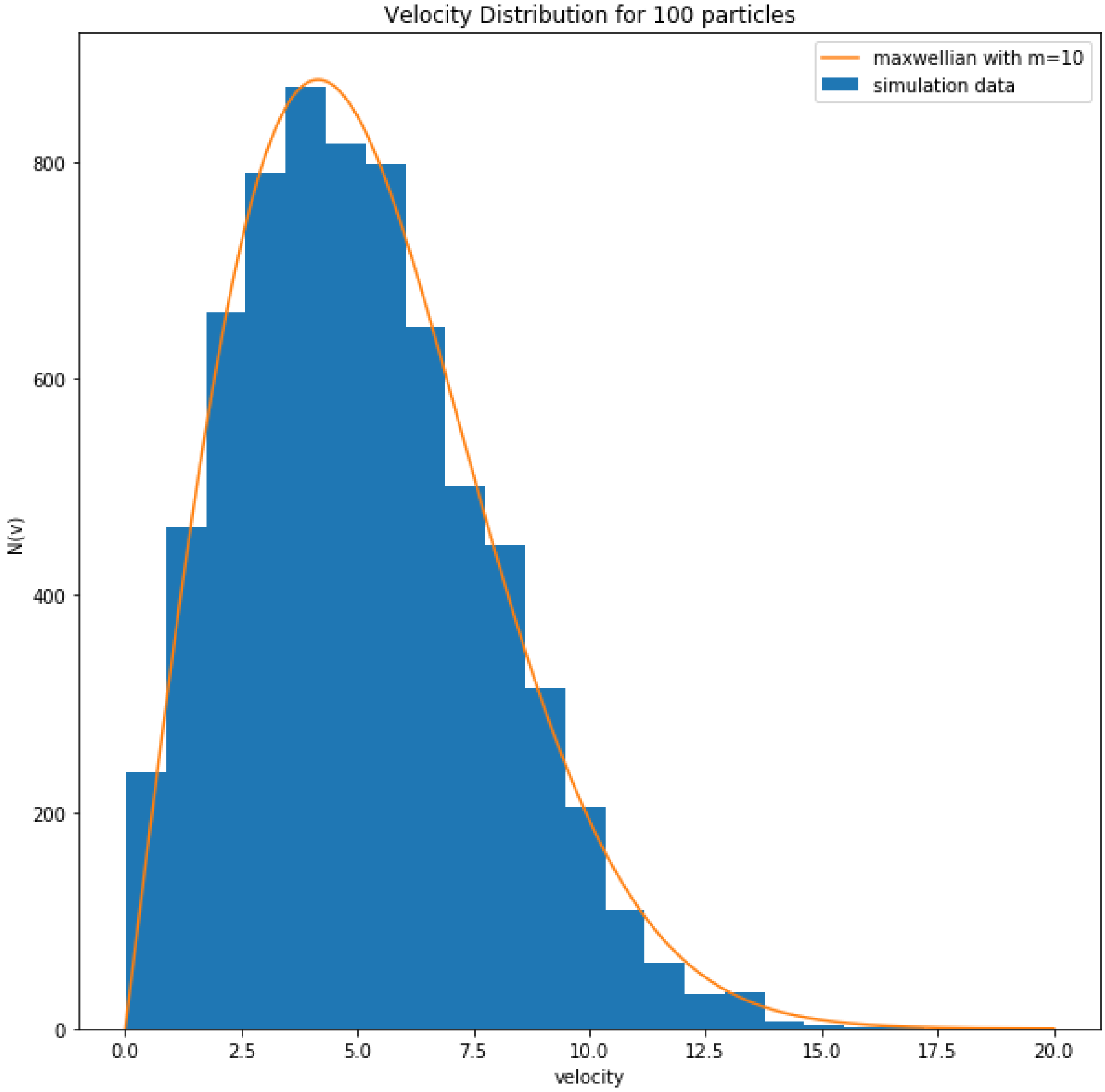
The number of particles with energy between E and $E + dE$ is given by $dN = f(p)dE$:

$$dN \propto e^{-E/kT}dE$$

Normalizing this, we find:

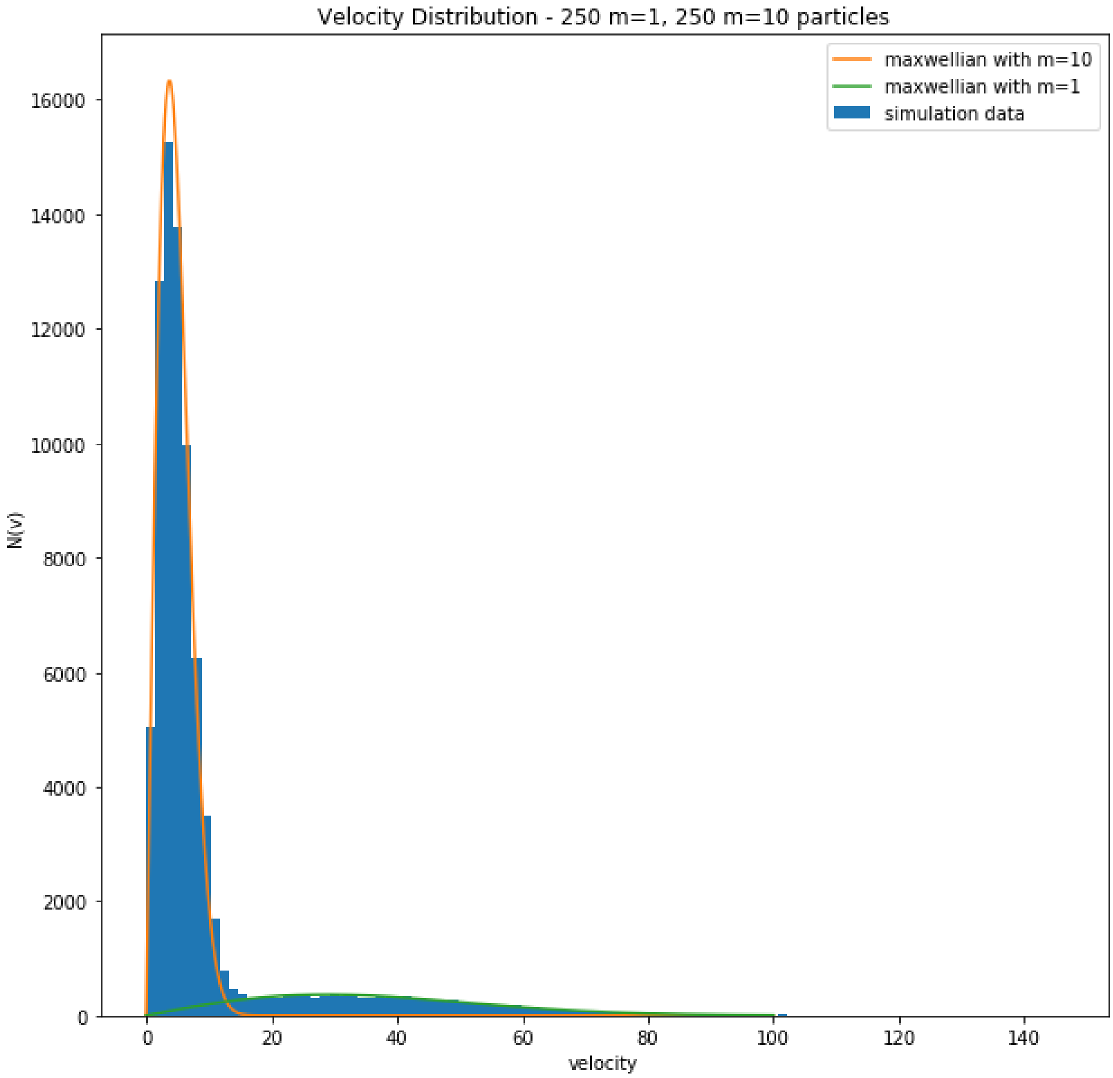
$$\frac{dN}{dE} = \frac{e^{-E/kT}}{kT}$$

Part B



Velocity Distribution for 100 particles
maxwellian with m=10
simulation data
N(v)
velocity
0
200
400
600
800
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

Part C



Each type of particle ($m = 1$ vs. $m = 10$) has a separate Maxwellian velocity distribution. This is the expected behavior - the more massive particles come from the same Maxwellian distribution in energy as do the less massive particles, but because of their mass, they have smaller velocities for the same energies. Thus in velocity, the two populations each have their own Maxwellian distribution.

Problem 3

Part A

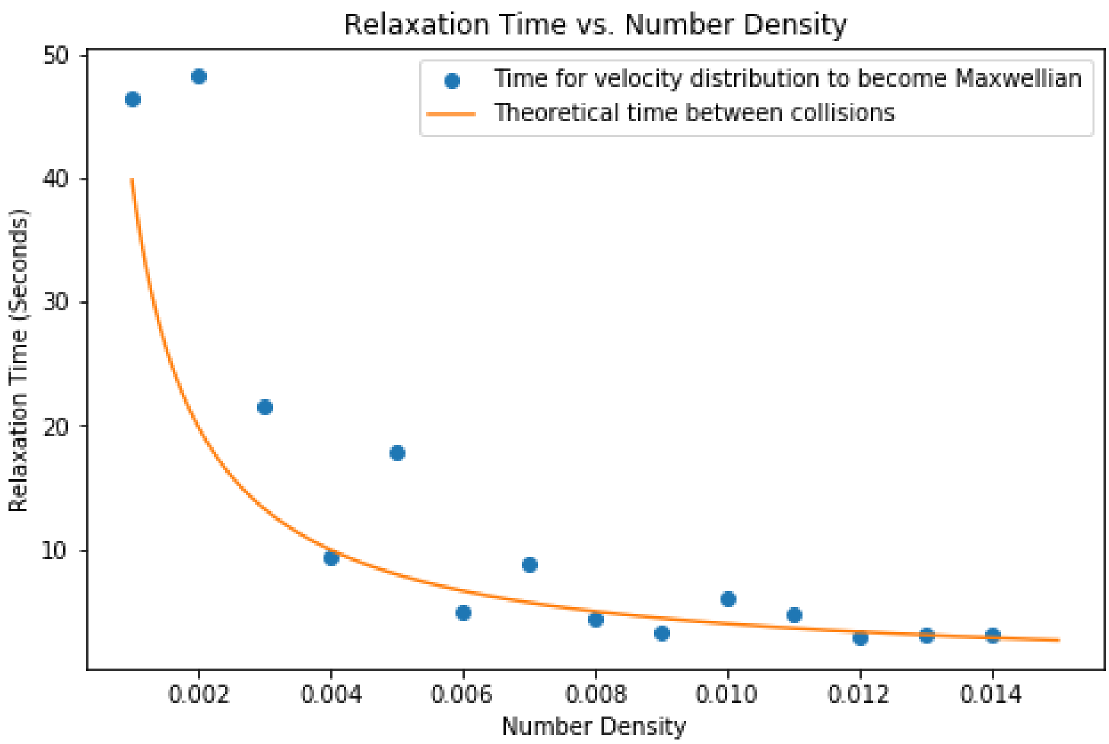
A *very* rough criterion could be arrived at by computing the difference between the population in two widely different energy bins and checking that it is approximately the difference expected from the Maxwellian distribution, to within some specified error. i.e. how long does it take for the energies to be distributed approximately according to the Maxwellian distribution? To make this criterion more robust to the small number of particles in my simulations, I can split the velocities in half at some cutoff, and consider the number of particles with velocities greater than v_{cut} versus the number of particles with velocities less than v_{cut}.

Part B

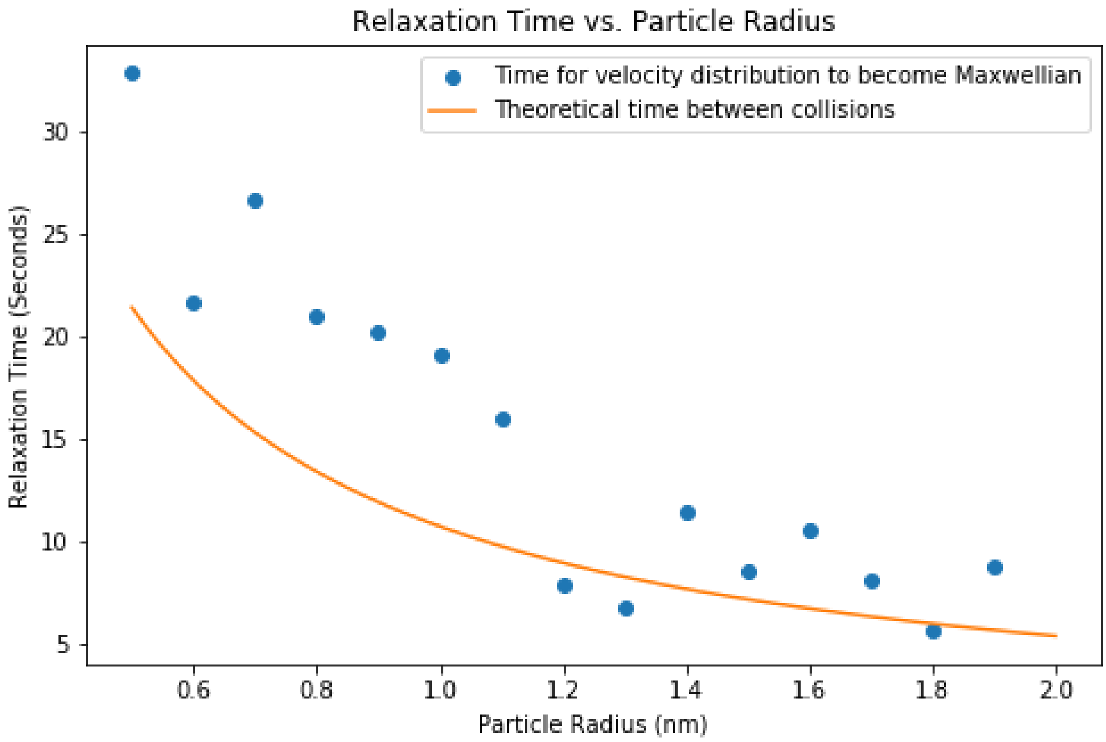
Since the particles need to collide to "communicate" their velocities to each other, the time to relax should scale with the time between collisions. The mean free path in three dimensions is given by $l = 1/\sigma n$ where σ is the cross section of the particles and n is the number density of particles. The time between collisions for a particle with velocity v is thus $\tau = 1/\sigma n v$. In two dimensions, the cross section is actually the linear extent of the particle, or twice its radius.

Part C

If we choose to split the 2-D Maxwellian at $v = \sqrt{2 \ln 2} v_{\max}$ where $v_{\max}$ is the maximum of the velocity distribution at $v_{\max} = \sqrt{m/2kT}$ then we get the same population on each side of the velocity cut. We can use this expectation to test for the system being relaxed. I'll arbitrarily consider the system to be relaxed when the populations on either side of this cut are within ten percent of each other.



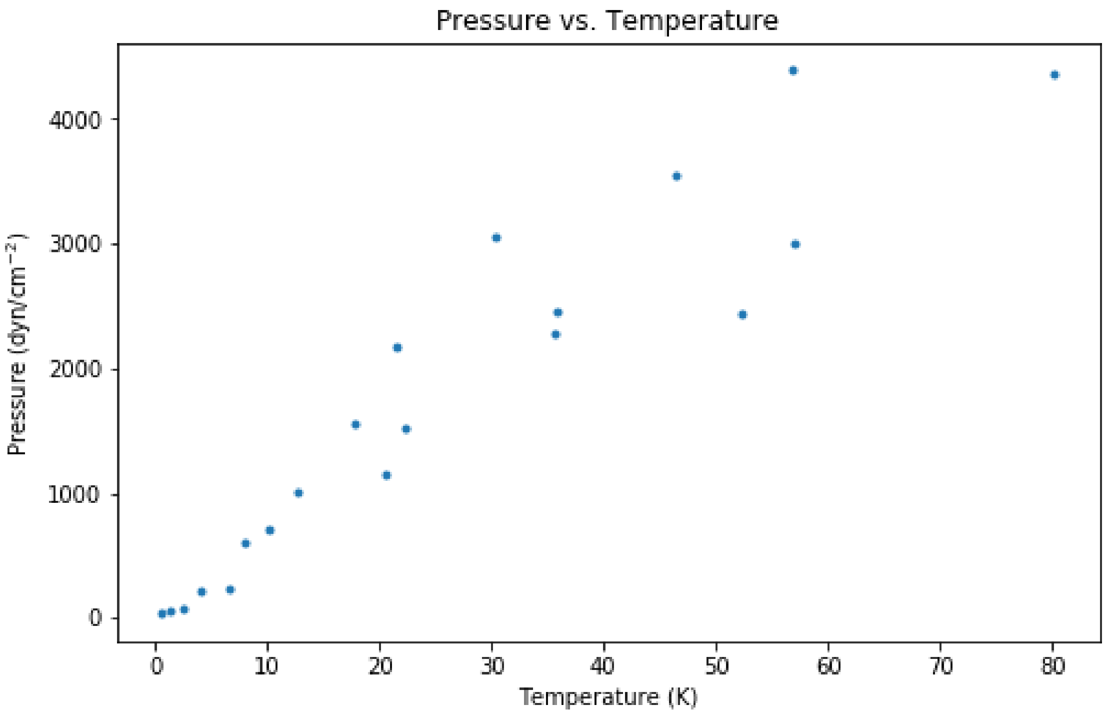
This plot demonstrates that my numerical criterion for relaxation scales with number density the same as my theoretical criterion: the average time between collisions. In fact, the two criteria correspond rather closely, showing that they not only scale the same, but that the velocity distribution approaches a Maxwellian on the same timescale as the average time between collisions for the system.

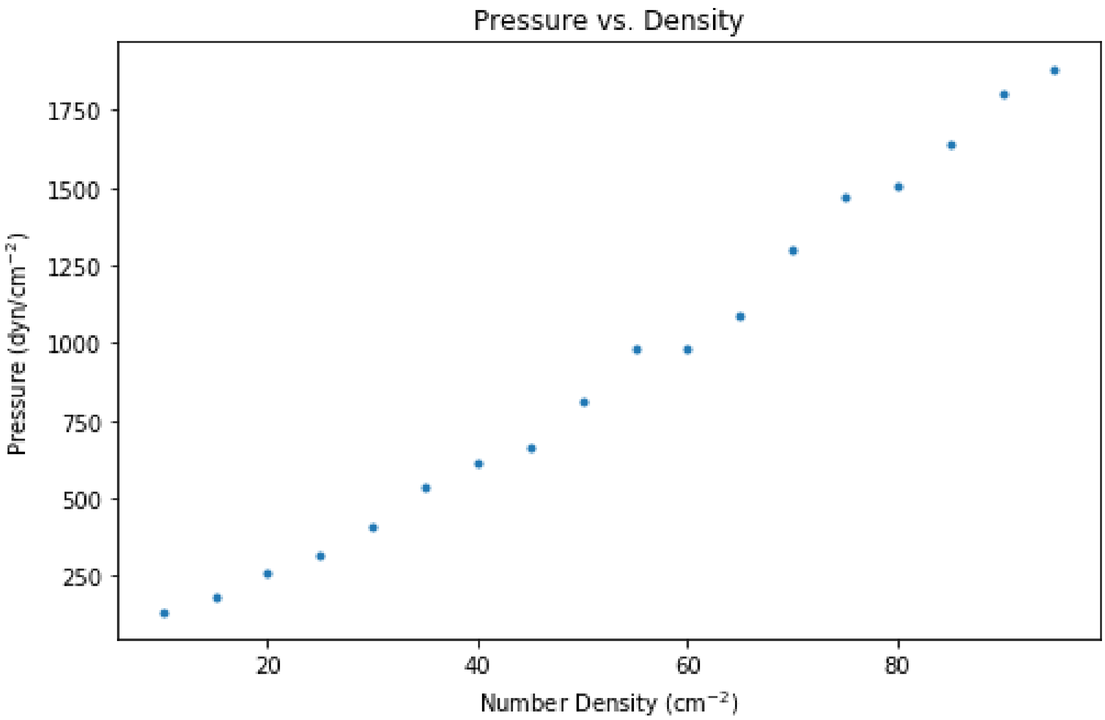


This plot demonstrates that my numerical relaxation criterion scales with particle radius in the same way as my theoretical criterion. The correlation between the two is not as clear as in the relaxation vs. number density plot, but the timescales appear to be similar.

Part D

The first plot shows the linear relationship between pressure and temperature which we expect from the ideal gas law. The second shows the linear relationship between pressure and number density.





The temperature plot shows a large scatter at higher temperatures. I think that this is because I computed temperature from the maximum of the velocity distribution using $v_{\max} = \sqrt{2kT/m}$. For more energetic simulations, there is a large scatter in velocities, and the corresponding computed temperature is also more uncertain.