ASTR 507 Problem Set 2

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January 29, 2018

Problem 1

The heat capacity at constant pressure is defined as

$$C_p = \left(\frac{dU}{dT}\right)_P$$

And the heat capacity at constant volume is:

$$C_v = \left(\frac{dU}{dT}\right)_V$$

Starting with the first law of thermodynamics, $dU = dQ - PdV$, we note that the heat capacity at constant volume can also be written

$$C_v = \frac{dQ}{dT}$$

Setting this fact aside for now, we can differentiate the internal energy of the fluid to find the heat capacity at constant volume:

$$\left(\frac{dU}{dT}\right)_V = \frac{d}{dT}\left(\frac{f}{2}NkT\right) = \frac{f}{2}Nk$$

where f is the number of degrees of freedom in the energy of the molecule in question. Now for C_p, we have, from the first law,

$$C_p = \frac{dQ}{dT} - \frac{PdV}{dT}$$

The first term is just C_v. The second term can be found from the ideal gas law: $PdV = d(NkT) = Nk$ and therefore

$$C_p = C_v - Nk$$

Combining these relationships, we arrive at $C_v - C_p = Nk$, and

$$\frac{C_p}{C_v} = \frac{C_v - Nk}{C_v} = 1 - \frac{Nk}{C_v} = 1 - \frac{2}{f} = \gamma$$

Problem 2

We can find the speed of the fastest electron by solving the equation $nVB(v) = 1$ to find the lowest velocity above which we would expect to find only a single electron. Doing this numerically for the Boltzmann distribution with the given parameters ($m = m_e$, $T = 10^6$ K, $n = 0.01$ cm^{-3}, and $V = 40$ pc^3), I find $v_{\max} \sim 6 \times 10^7$ m/s. This is only about a factor of 5 less than the velocity of a cosmic ray, but that difference is still significant. The probability of finding a particle in this distribution with velocity $v \sim c$ is essentially zero. Of course, the Maxwellian velocity distribution is also non-relativistic. A properly relativistic velocity distribution should drop off even more quickly at velocities near c. Otherwise a sufficiently high temperature could drive electrons to faster than light speed. Thus cosmic rays cannot have a thermal origin.

Problem 3

Part A

The mean free path at the escape density is:

$$l = 1/n_{\rm esc}\sigma$$

Setting this equation to $H = kT/(mg)$, we find:

$$n_{\rm esc} = \frac{mg}{kT\sigma}$$

Part B

Integrate:

$$\phi(v)dv = \frac{f(v)}{4\pi}dv \int_0^{2\pi} d\phi \int_0^{\pi/2} v\cos\theta\sin\theta d\theta$$

The integral over $d\phi$ yields a factor of 2π, and the second integral is evaluated as

$$\int_0^{\pi/2} v\cos\theta\sin\theta d\theta = \frac{v}{2}\sin^2\theta\Big|_0^{\pi/2} = \frac{v}{2}$$

So the complete integral evaluates to

$$\phi(v)dv = \frac{f(v)}{4}vdv$$

Part C

To simplify things, let's consider the integral

$$\phi = \frac{A}{4}\int_{v_{\rm esc}}^{\infty} v^3 e^{-bv^2}$$

At the end of the evaluation we can replace A with the proper set of constants for the Maxwellian velocity distribution, and b with $b = m/2kT$. Integrating by parts, we have

$$\int_{v_{esc}}^{\infty} v^3 e^{-bv^2} = -\frac{v^2}{2b}e^{-bv^2}\Big|_{v_{esc}}^{\infty} + \frac{1}{b}\int_{v_{esc}}^{\infty} ve^{-bv^2} = \left(-\frac{v^2}{2b} + \frac{1}{2b^2}\right)e^{-bv^2}\Big|_{v_{esc}}^{\infty}$$

Evaluating between the two limits and multiplying the $A/4$ factor back onto the result, we have

$$\phi = \frac{A}{4}\left(\frac{v_{esc}^2}{2b} + \frac{1}{2b^2}\right)e^{-bv_{esc}^2}$$

Substituting in the constants for A and b, we have

$$\phi = \frac{\pi m^{3/2} n}{(2\pi kT)^{3/2}}\left(\frac{kTv_{esc}^2}{m} + \frac{2(kT)^2}{m^2}\right)e^{-mv_{esc}^2/2kT}$$

Finally, in terms of v_s and λ_{esc}, we have

$$\phi = \frac{nv_s}{2\sqrt{\pi}}(\lambda_{esc}+1)e^{-\lambda_{esc}}$$

Part D

Over 1 Gyr the Earth would lose 4.2×10^{42} molecular hydrogen molecules, nearly half of its current reservoir of hydrogen.

Part E

Over this same period of time the it is unlikely that even a single oxygen molecule will escape the atmosphere through this mechanism. I calculate that about 3×10^{-54} molecules escape over 1 Gyr. Thus, over time, the ratio of oxygen to hydrogen molecules should increase as hydrogen escapes at oxygen stays put.

Part E

For deuterium-hydrogen molecules, I calculate that 4×10^{39} molecules escape over 1 Gyr, compared to a thousand times more molecules for hydrogen alone. As a result, the D/H ratio should increase over time.

My calculations for parts D-E are on my github: