Framelet expansion in convolutional neural networks

Outline

1. Motivating example.
   - ❑ Original UNET vs Framelet based UNET
2. Convolutional neural networks (CNNs).
3. Frames and filter banks.
4. Framelet expansion –analysis and mathematical framework.
   - ❑ Analysis of frames
5. Framelet expansion in CNNs
6. Summary.

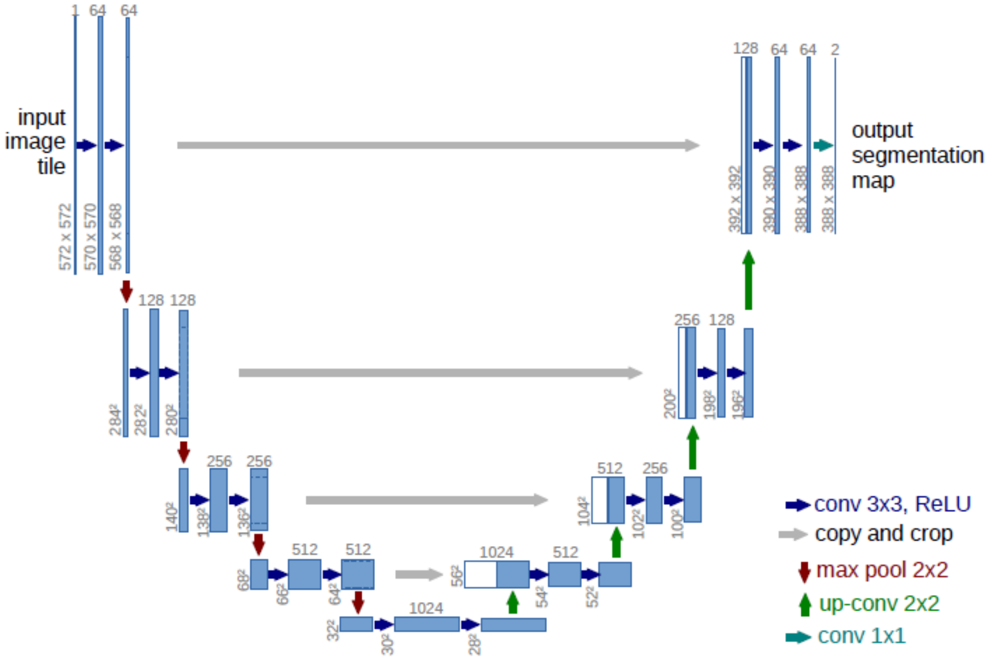
Ye et al 2018: Deep convolutional framelets: a general deep learning framework for inverse problems

Extra - Denoising example in 3D

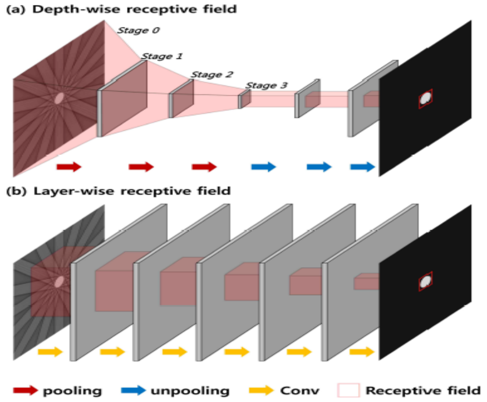
Motivating example

1. The original UNET, proposed by Ronneberger et al in 2015, has been used in many learning based inversion applications.
2. Trying to learn a optimal low rank approximation of the features in multiple levels (sparsest Fourier domain).
3. The low-rank approximation is done by pooling operations, which essentially is a low-pass filtering.
4. Pooling leads to an exponential increase in the receptive field, i.e. the CNN kernels becomes larger relative to its feature size.
5. High frequencies are retained by skip connections.
6. The problem is that skip connections and pooling leads to smoothing effects.

Ronneberger et al 2015: U-Net: Convolutional Networks for Biomedical Image Segmentation



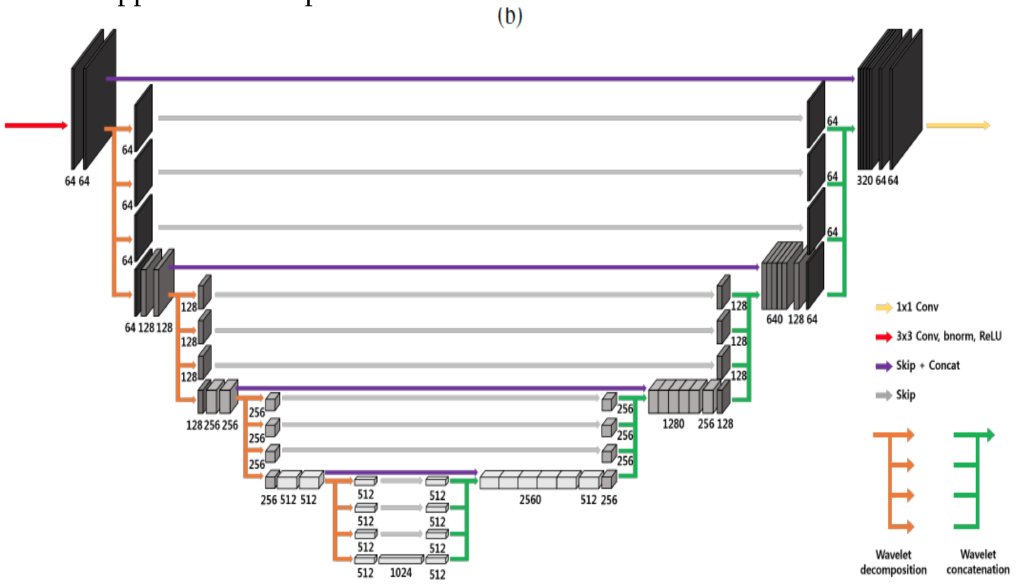
Ye et al 2018: Deep convolutional framelets: a general deep learning framework for inverse problems



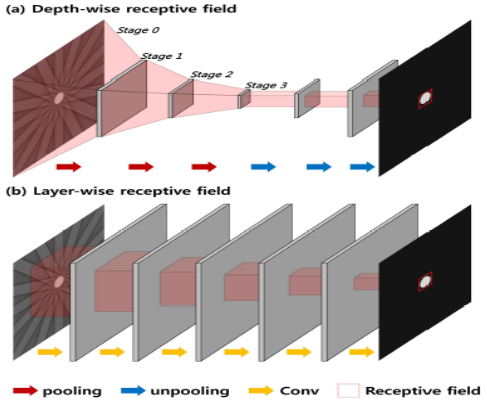
Motivating example

1. In 2018 Ye et al, conducted a mathematical analysis of the UNET architecture which lead to the theory of deep convolutional framelets.
2. This resulted in generalizing the pooling operation in CNNs with the use of framelet expansion.
3. Generalized pooling has several benefits over pooling, such as it meets perfect reconstruction condition and makes the architecture design more flexible towards choosing suitable basis functions for the data.

Han & Ye 2018: Framing U-Net via Deep Convolutional Framelets: Application to Sparse-view CT

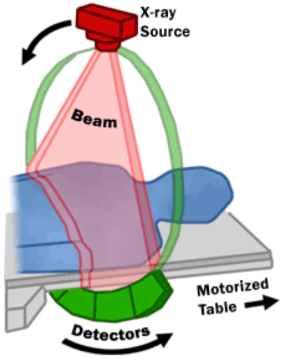


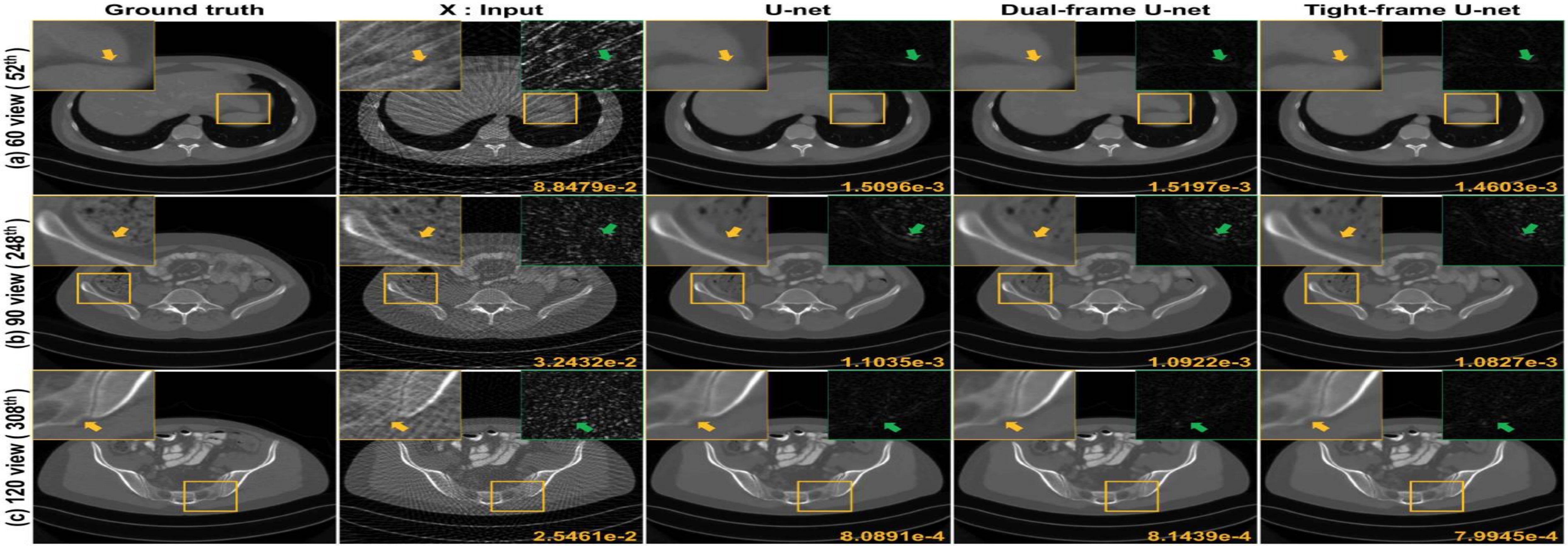
Ye et al 2018: Deep convolutional framelets: a general deep learning framework for inverse problems



Motivating example

https://en.wikipedia.org/wiki/CT_scan#/media/File:Drawing_of_CT_fan_beam_(left)_and_patient_in_a_CT_imaging_system.gif

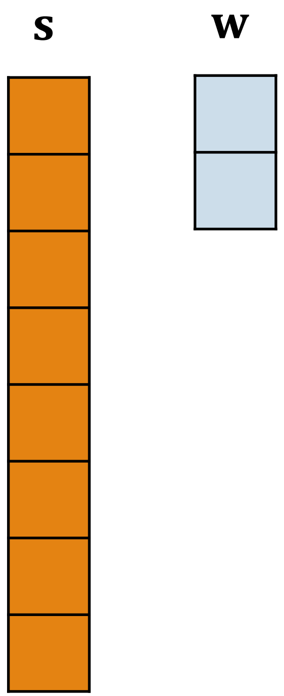




Han & Ye 2018: Framing U-Net via Deep Convolutional Framelets: Application to Sparse-view CT

Convolutional neural networks (CNNs)

- Sliding dot product
- Signal **s**
- Filter **w**

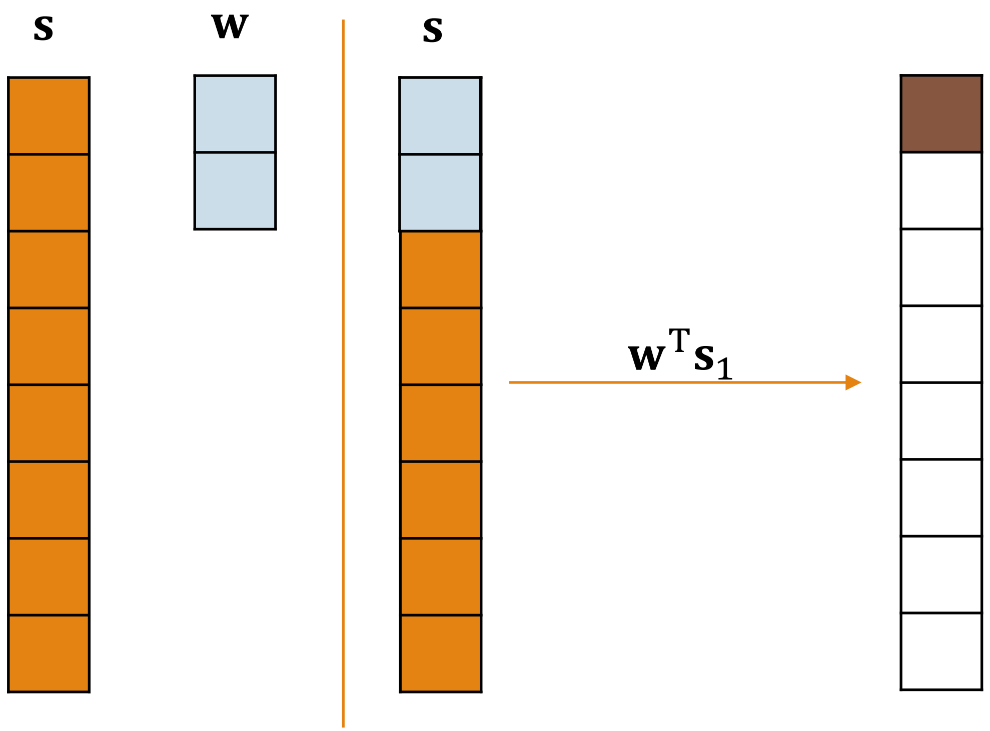


1D convolution between a filter **w** with length d, and a signal **s**

$$(\mathbf{w} * \mathbf{s}) = \sum_{u=-\mathrm{d}/2}^{\mathrm{d}/2} \mathrm{w}_u \mathrm{s}_{x-u} = \mathbf{W}\mathbf{s}$$

Convolutional neural networks (CNNs)

- Sliding dot product
- Signal **s**
- Filter **w**
- Stride = 1



$Single\ input, single\ output\ (SISO)$

$SISO$: $\mathbf{Ws} = \mathbf{f}$

Convolutional neural networks (CNNs)

- Sliding dot product
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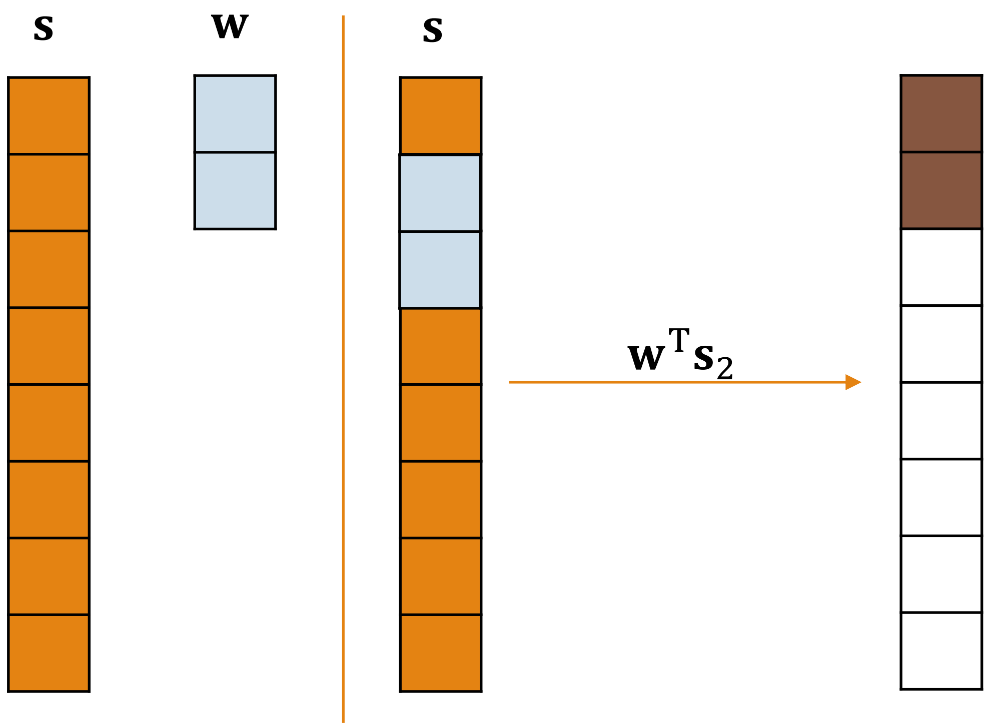

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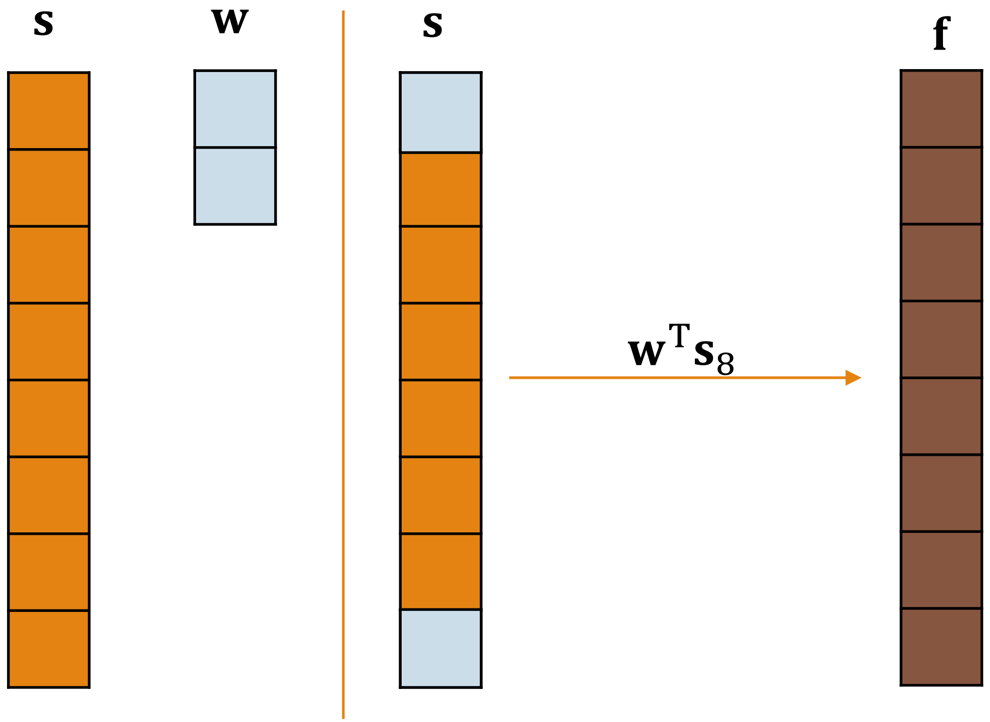
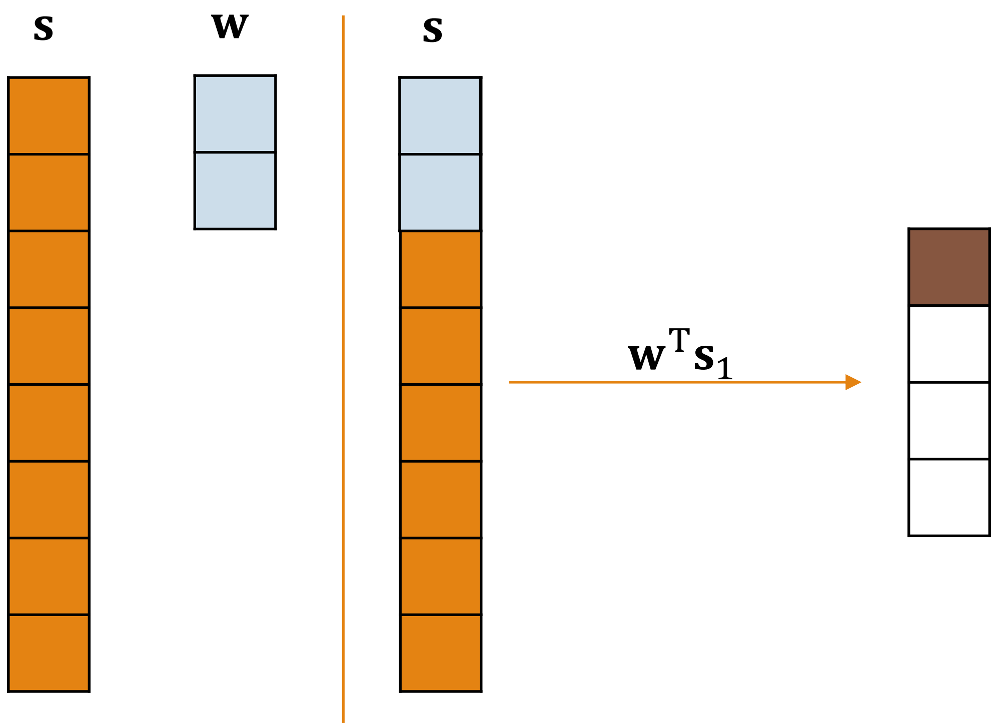


$Single\ input, single\ output\ (SISO)$

$SISO$: $\mathbf{Ws} = \mathbf{f}$

Convolutional neural networks (CNNs)

- Sliding dot product
- Signal **s**
- Filter **w**
- Stride = 2



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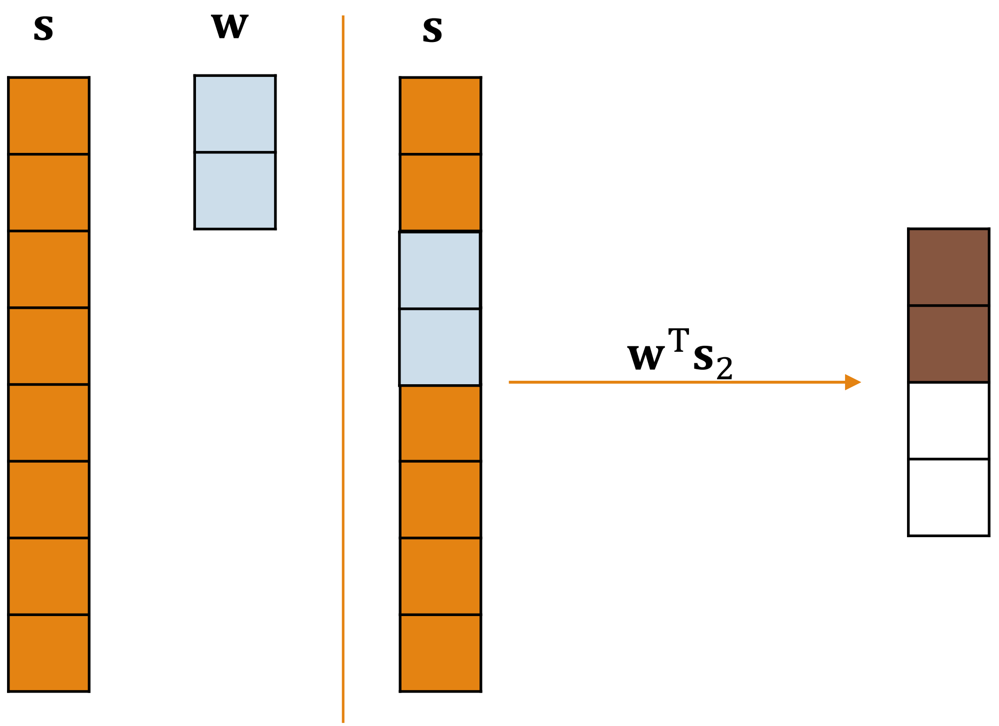

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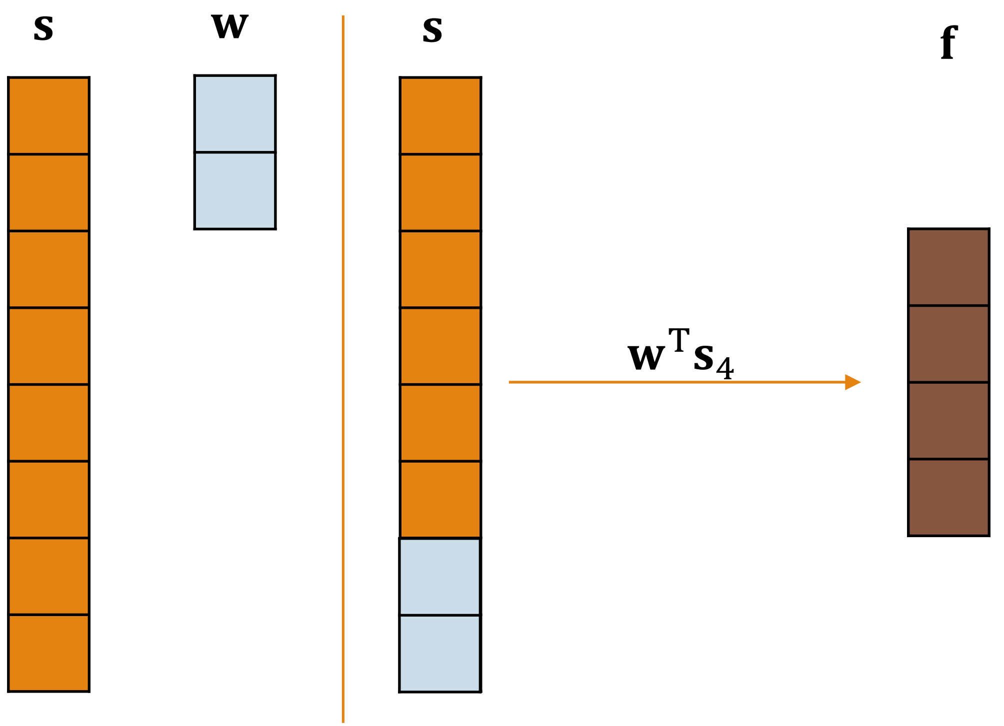
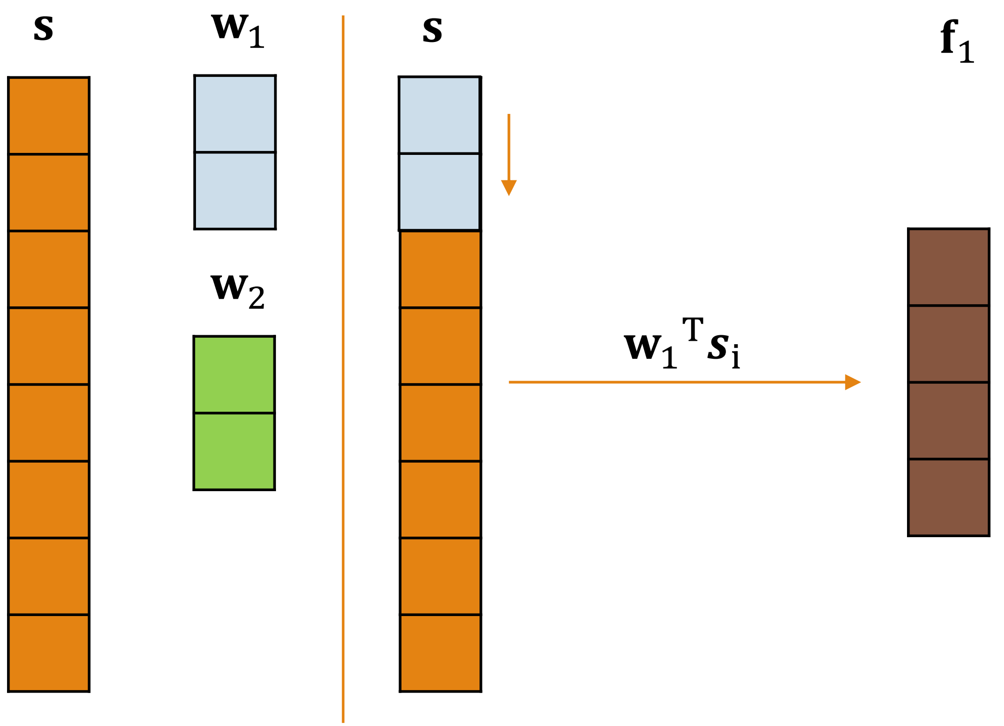


$Single\ input, single\ output\ (SISO)$

$SISO$: $\mathbf{Ws} = \mathbf{f}$

Convolutional neural networks (CNNs)

- Sliding dot product
- Signal $\mathbf{s}$
- Filter $\mathbf{w}_1$ and $\mathbf{w}_2$
- Stride = 2



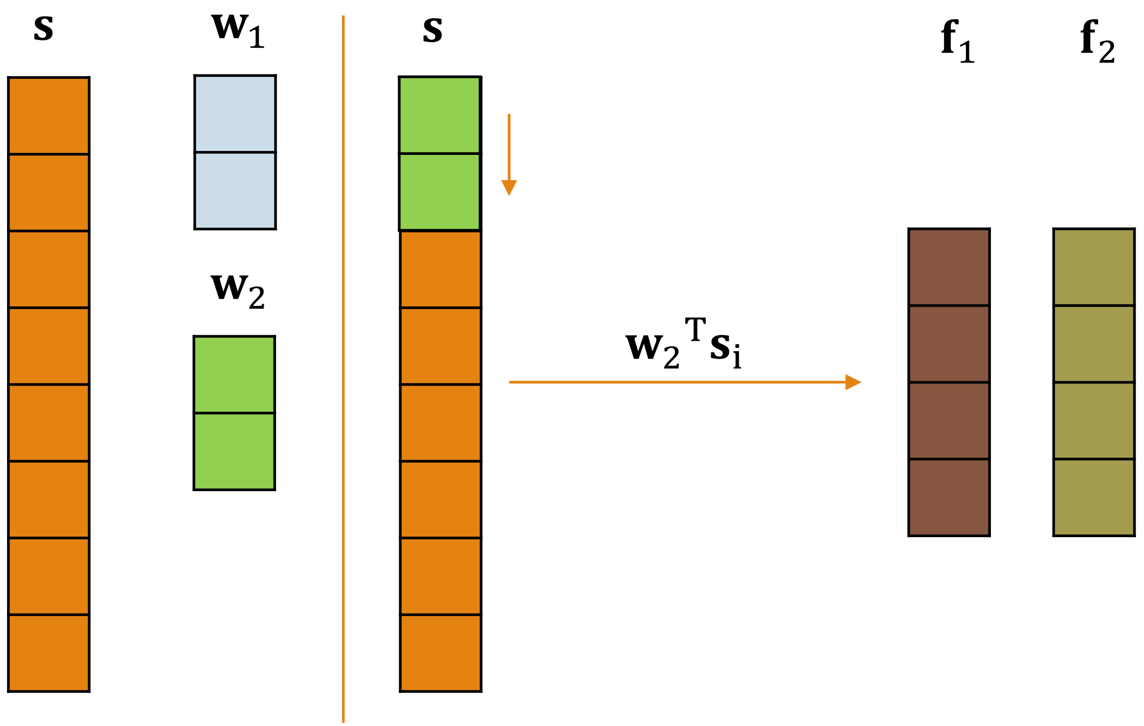
$Single\ input, multi\ output\ (SIMO)$

$SISO$: $\mathbf{Ws} = \mathbf{f}$

$SIMO$: $[\mathbf{W}_1\ \mathbf{W}_2]\boldsymbol{I}\mathbf{s} = [\mathbf{f}_1\ \ \mathbf{f}_2]$

Convolutional neural networks (CNNs)

- Sliding dot product
- Signal $\mathbf{s}$
- Filter $\mathbf{w}_1$ and $\mathbf{w}_2$
- Stride = 2



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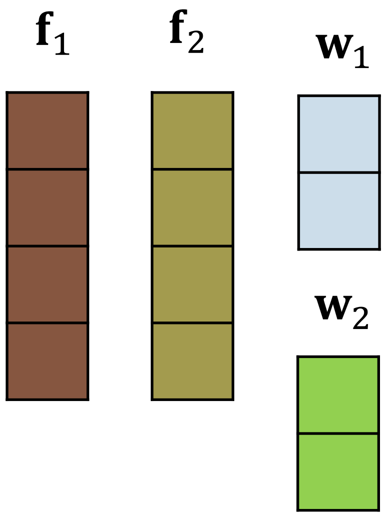
Convolutional neural networks (CNNs)

- Sliding dot product
- Feature $\mathbf{f}_1, \mathbf{f}_2$
- Filter $\mathbf{w}_1$ and $\mathbf{w}_2$
- Stride = 1



$Multi\ input, single\ output\ (MISO)$

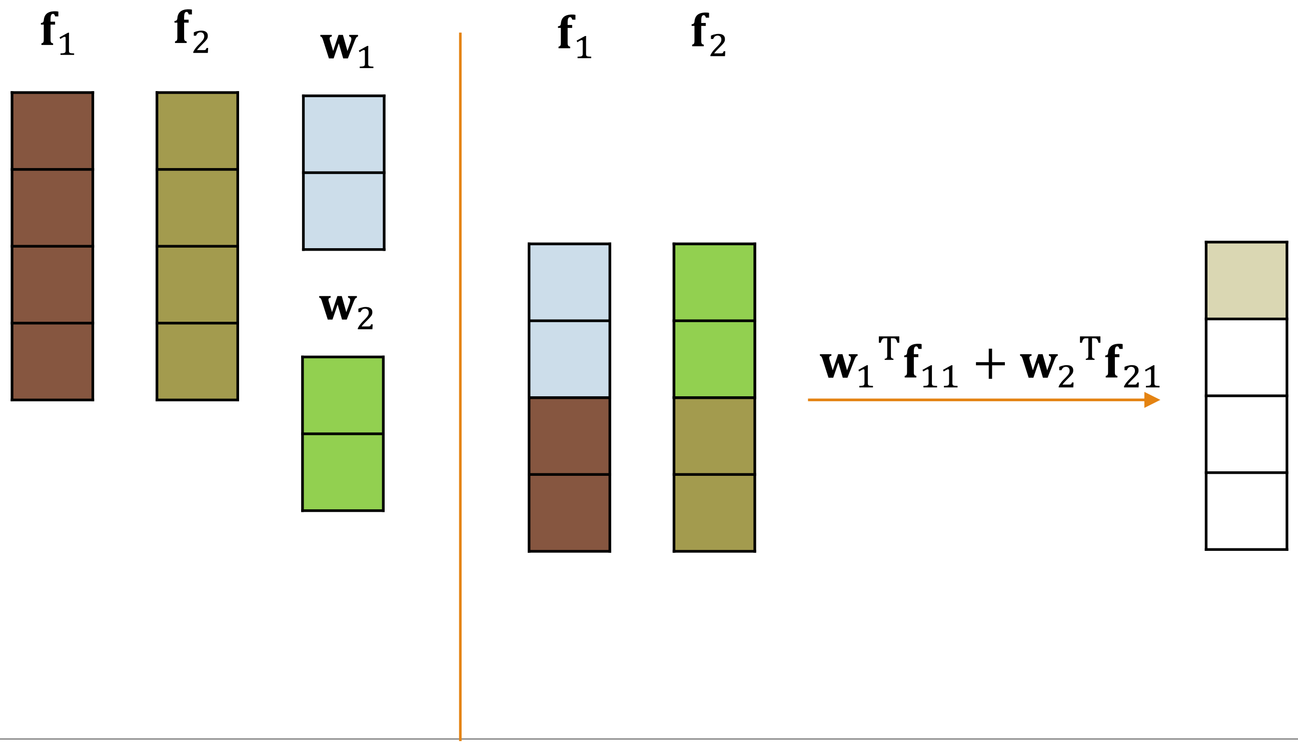
$SISO$: $\mathbf{Ws} = \mathbf{f}$

$SIMO$: $[\mathbf{W}_1\ \mathbf{W}_2]\boldsymbol{I}\mathbf{s} = [\mathbf{f}_1\ \ \mathbf{f}_2]$

$MISO$: $[\mathbf{W}_1\ \mathbf{W}_2]\begin{bmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{bmatrix} = \mathbf{f}_3$

Convolutional neural networks (CNNs)

- Sliding dot product
- Feature $\mathbf{f}_1, \mathbf{f}_2$
- Filter $\mathbf{w}_1$ and $\mathbf{w}_2$
- Stride = 1



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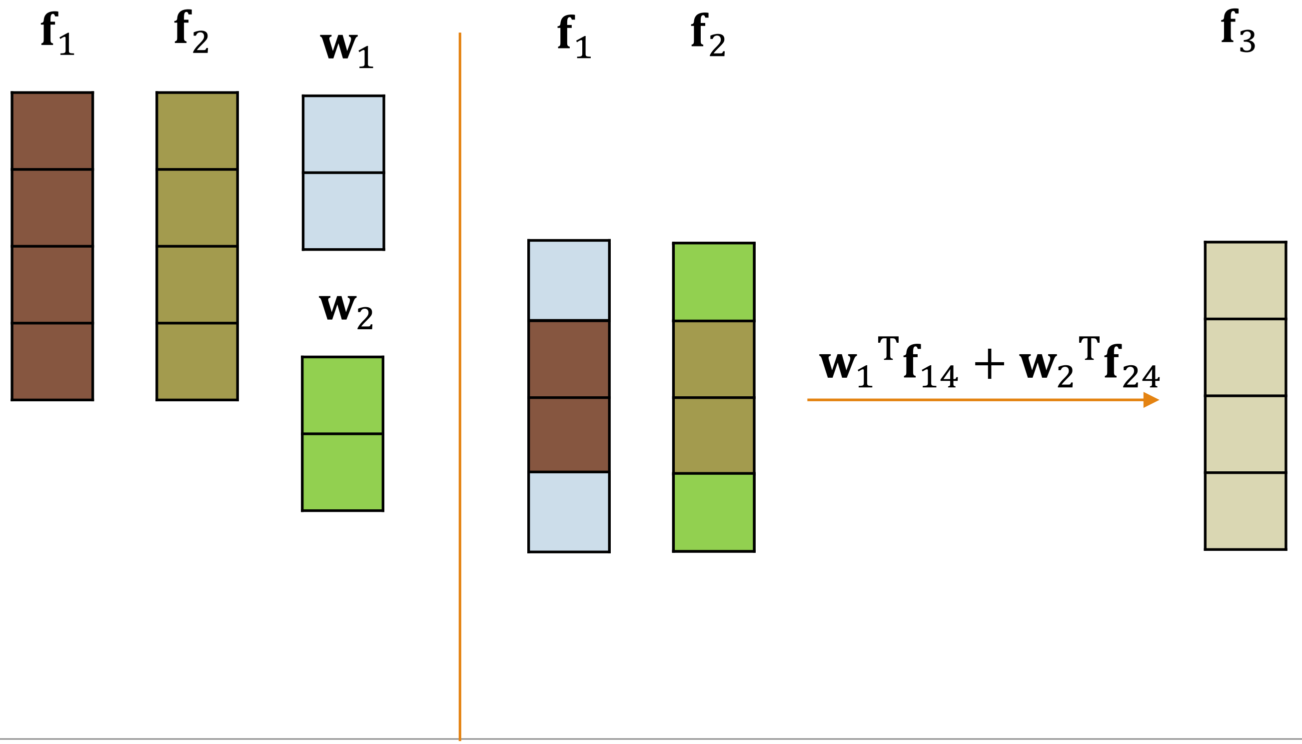

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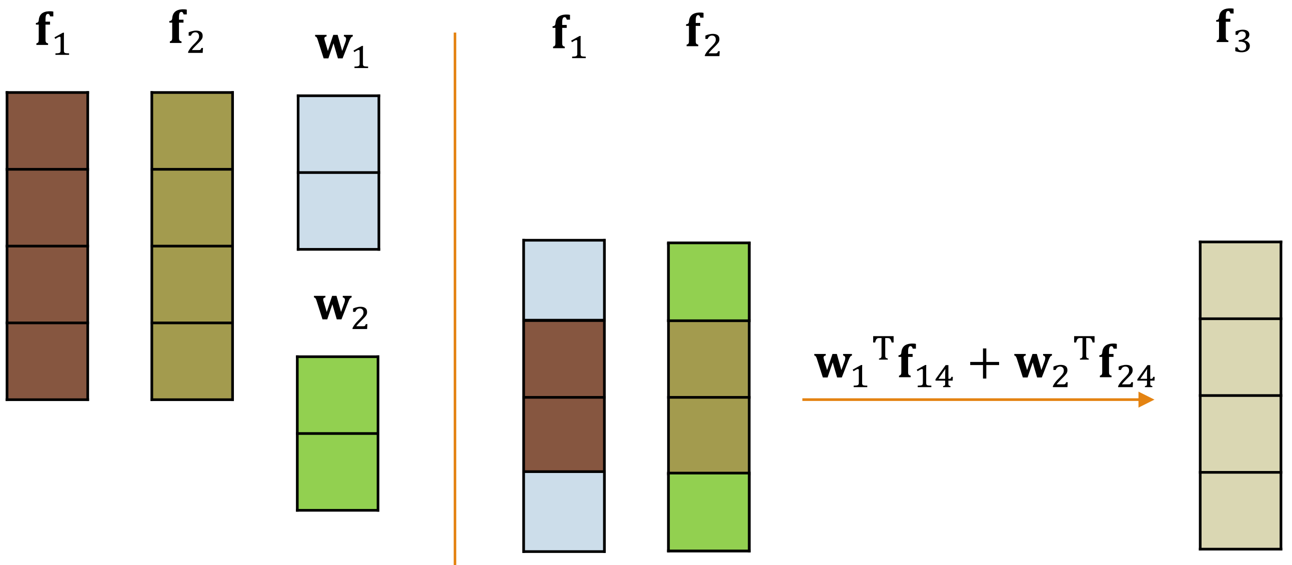
Convolutional neural networks (CNNs)

- Sliding dot product
- Feature $\mathbf{f}_1, \mathbf{f}_2$
- Filter $\mathbf{w}_1$ and $\mathbf{w}_2$
- Stride = 1



$Multi\ input, multi\ output\ (MIMO)$

$SISO\text{:}\ \mathbf{Ws} = \mathbf{f}$

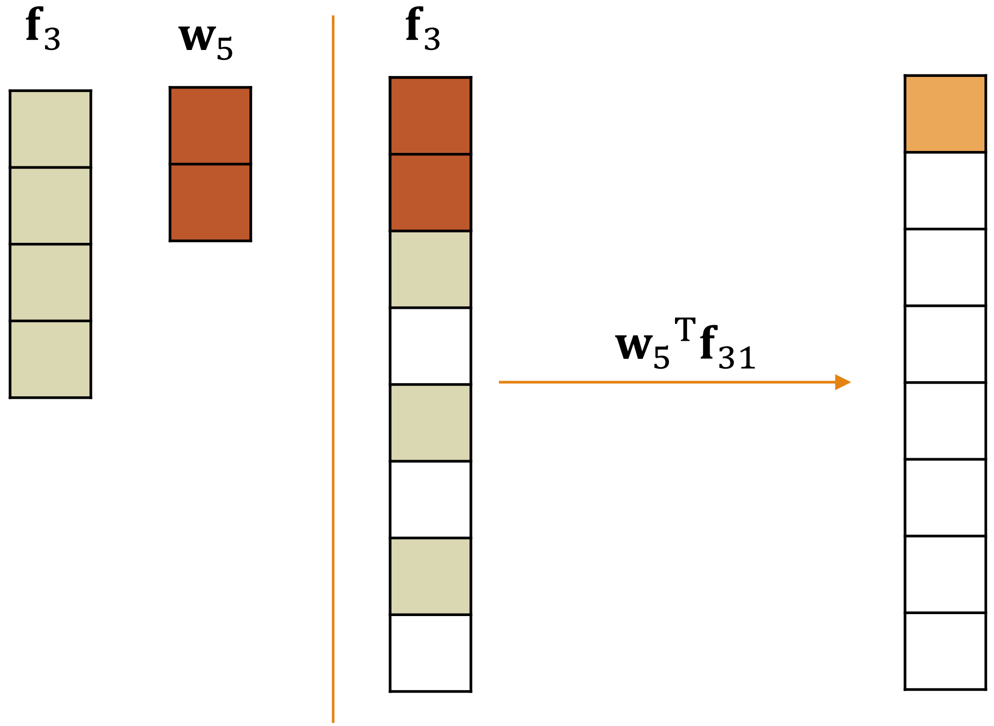
$MIMO\text{:} \begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_2 \\ \mathbf{W}_3 & \mathbf{W}_4 \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_3 \\ \mathbf{f}_4 \end{bmatrix}$

$SIMO\text{:} [\mathbf{W}_1\ \mathbf{W}_2]\boldsymbol{Is} = [\mathbf{f}_1\ \ \mathbf{f}_2]$

$MISO\text{:} [\mathbf{W}_1\ \mathbf{W}_2] \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \mathbf{f}_3$

Convolutional neural networks (CNNs)

- Sliding dot product
- Feature $\mathbf{f}_3$
- Filter $\mathbf{w}_5$
- Stride = 2 transposed



$SISO\ Transposed\ convolution$

$SISO$: $\mathbf{Ws} = \mathbf{f}$

$MIMO$: $\begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_2 \\ \mathbf{W}_3 & \mathbf{W}_4 \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_3 \\ \mathbf{f}_4 \end{bmatrix}$

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- Sliding dot product
- Feature $\mathbf{f}_3$
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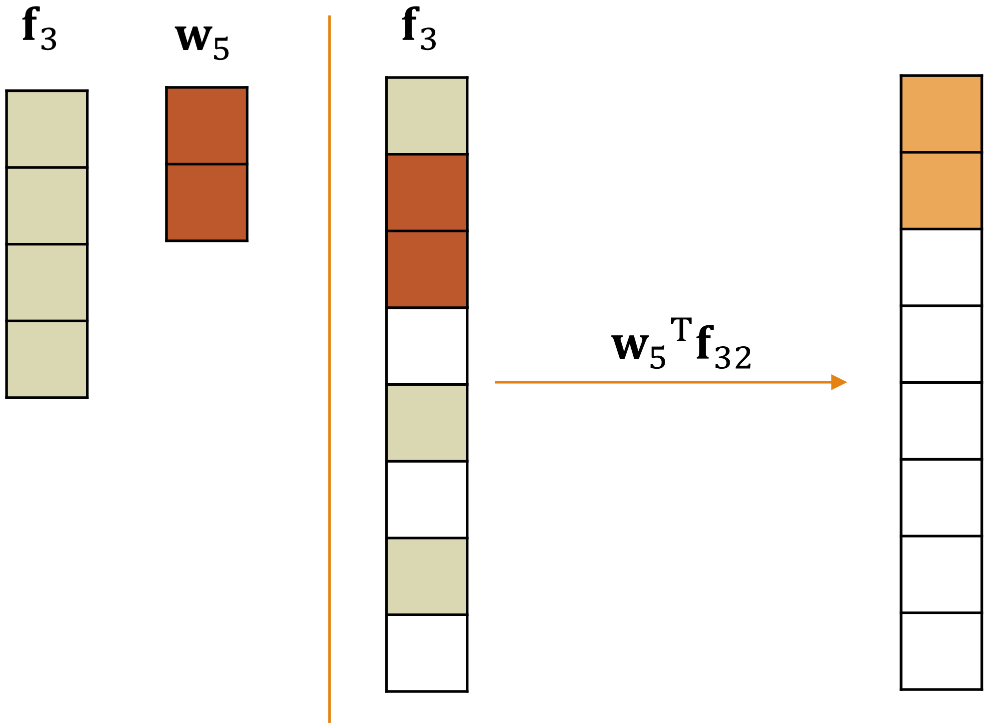

SISO Transposed convolution

$SISO$: $\mathbf{W}\mathbf{s} = \mathbf{f}$

$MIMO$: $\begin{bmatrix}\mathbf{W}_1 & \mathbf{W}_2\\ \mathbf{W}_3 & \mathbf{W}_4\end{bmatrix}\begin{bmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{bmatrix} = \begin{bmatrix}\mathbf{f}_3\\ \mathbf{f}_4\end{bmatrix}$

$SIMO$: $[\mathbf{W}_1\ \mathbf{W}_2]\boldsymbol{Is} = [\mathbf{f}_1\ \ \mathbf{f}_2]$

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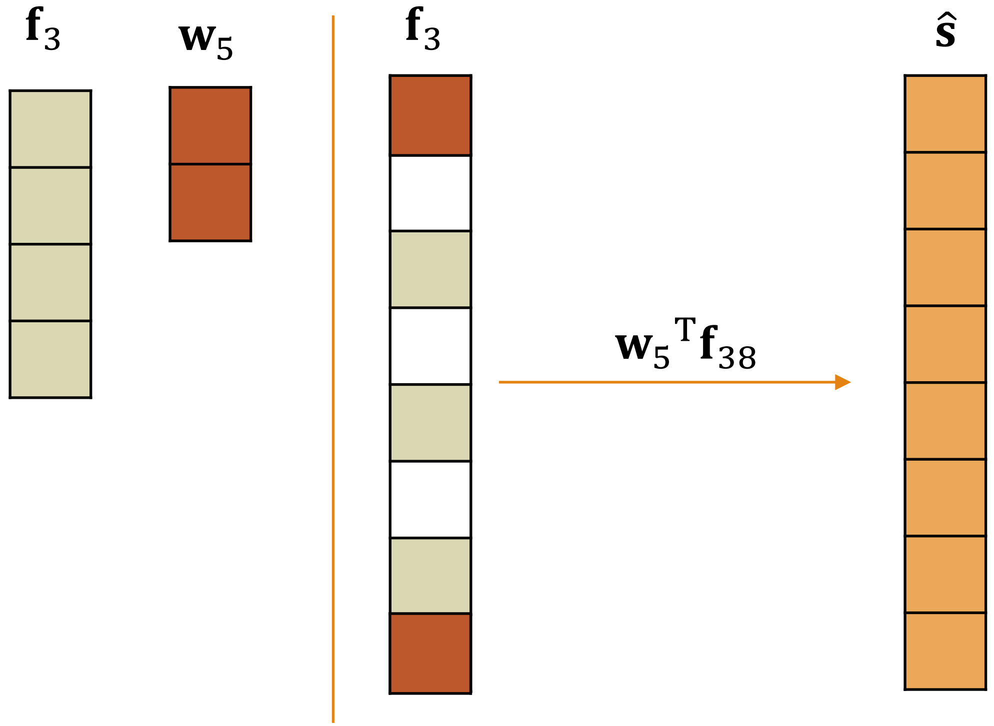

$SISO\ Transposed\ convolution$

$SISO\colon \mathbf{Ws} = \mathbf{f}$

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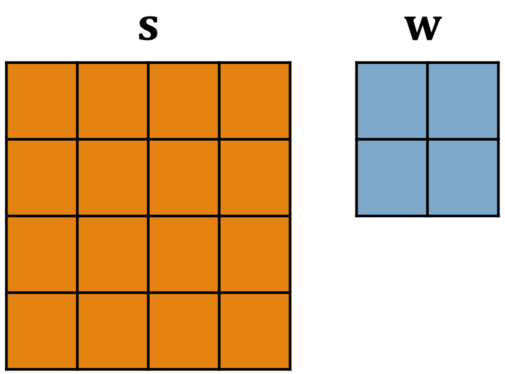
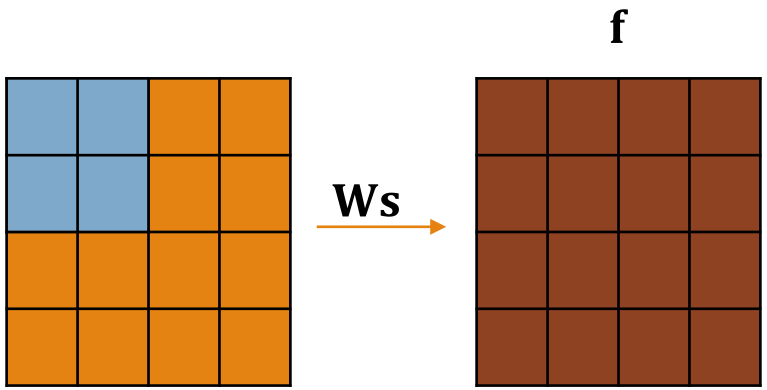
Convolutional neural networks (CNNs) 2D

- Sliding dot product
- Feature **f**
- Filter **w**
- Stride = 1





2D convolution between a filter **w** with length d, and a signal **s**

$SISO$: $\mathbf{Ws} = \mathbf{f}$

$MIMO$: $\begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_2 \\ \mathbf{W}_3 & \mathbf{W}_4 \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_3 \\ \mathbf{f}_4 \end{bmatrix}$

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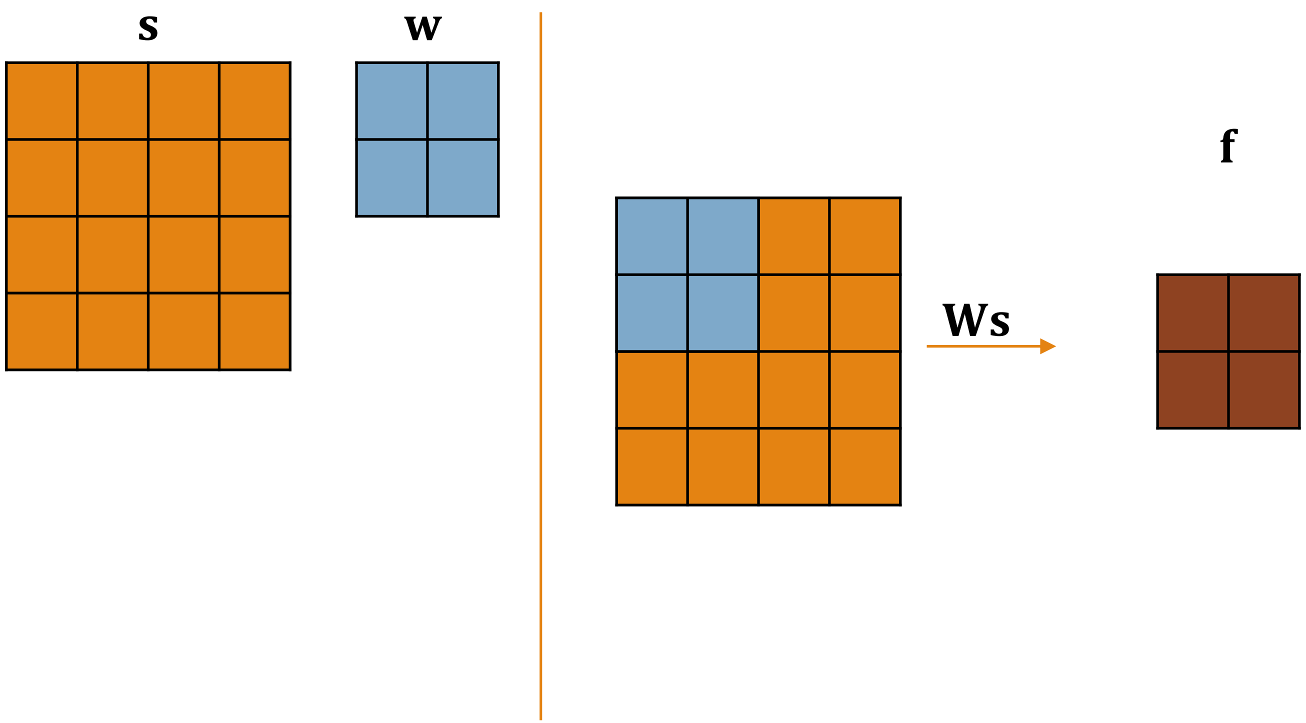
Convolutional neural networks (CNNs) 2D

- Sliding dot product
- Feature **f**
- Filter **w**
- Stride = 2



2D strided convolution between a filter **w** with length d, and a signal **s**

$$SISO\text{: } \mathbf{Ws} = \mathbf{f}$$

$$MIMO\text{: } \begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_2 \\ \mathbf{W}_3 & \mathbf{W}_4 \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_3 \\ \mathbf{f}_4 \end{bmatrix}$$

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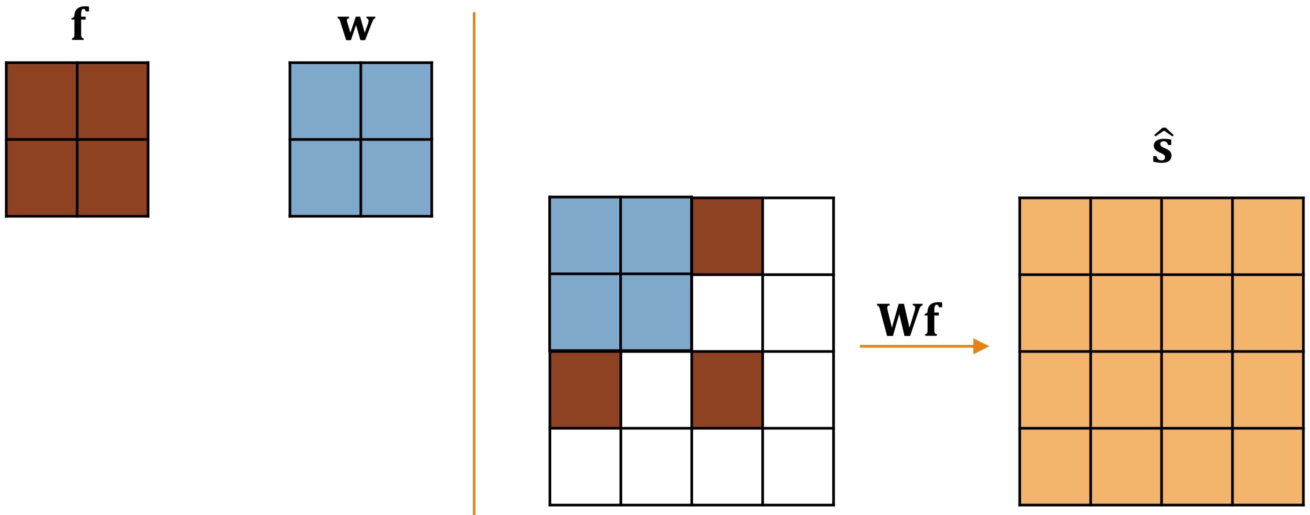
Convolutional neural networks (CNNs) 2D

- Sliding dot product
- Feature **f**
- Filter **w**
- Stride = 2 transposed



2D transposed convolution between a filter **w** with length d, and a signal **s**

$SISO$: $\mathbf{Ws} = \mathbf{f}$

$MIMO$: $\begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_2 \\ \mathbf{W}_3 & \mathbf{W}_4 \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_3 \\ \mathbf{f}_4 \end{bmatrix}$

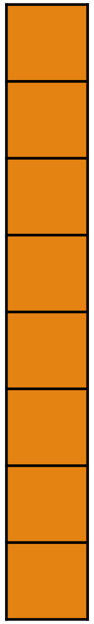
$SIMO$: $[\mathbf{W}_1\ \mathbf{W}_2]\boldsymbol{Is} = [\mathbf{f}_1\ \ \mathbf{f}_2]$

$MISO$: $[\mathbf{W}_1\ \mathbf{W}_2] \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \mathbf{f}_3$

Convolutional neural networks (CNNs)

- The neural network
- Layer $l \in \{1, \dots, 3\}$
- Filters $\mathbf{W}^{[i]}, i \in l$
- Features $\mathbf{f}$
- Non-linear function $\boldsymbol{\sigma}(\cdot)$
- Activations $\mathbf{a}^{[i-1]}$

$\mathbf{s} = \mathbf{a}^{[0]}$



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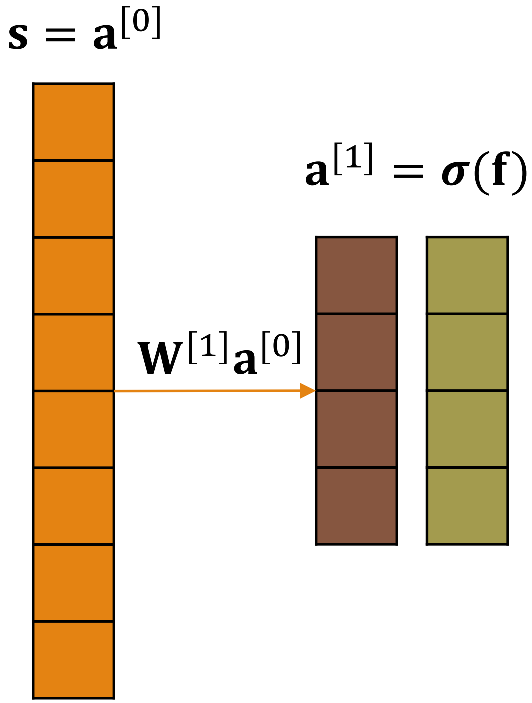

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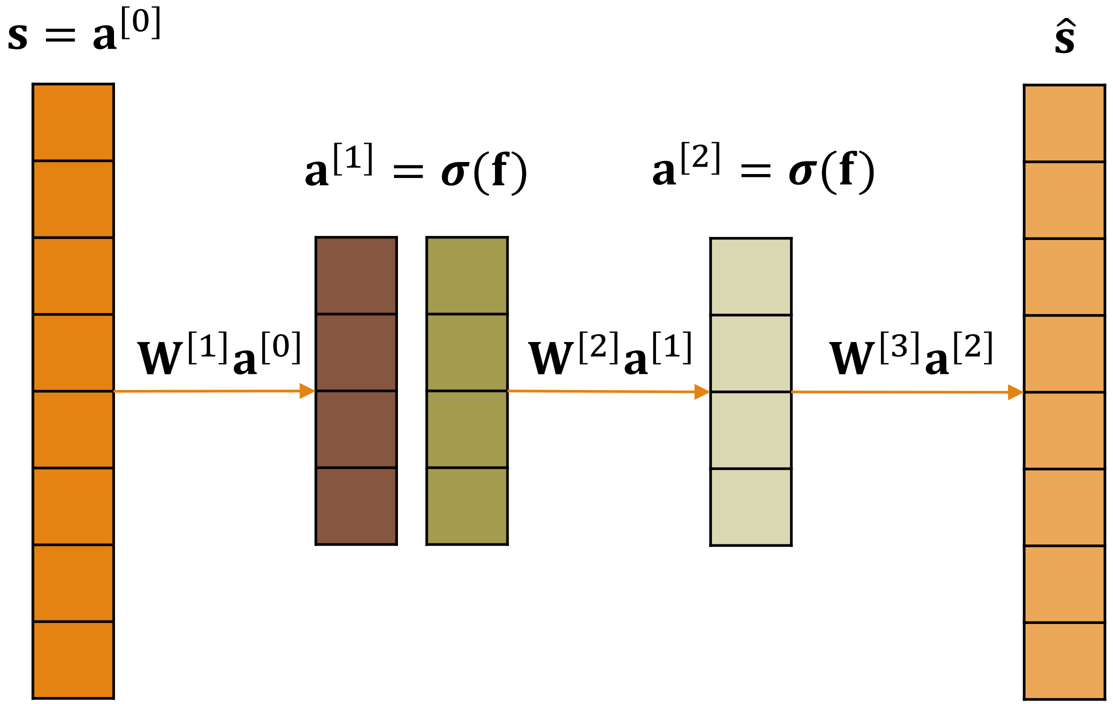
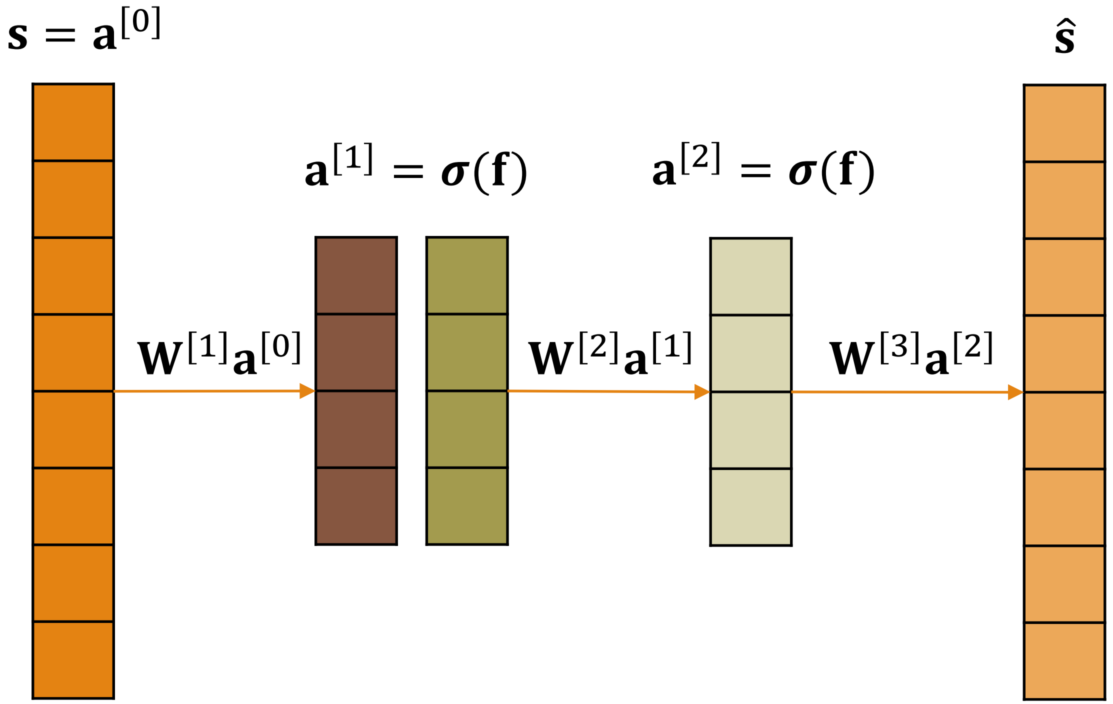


Convolutional neural networks (CNNs)

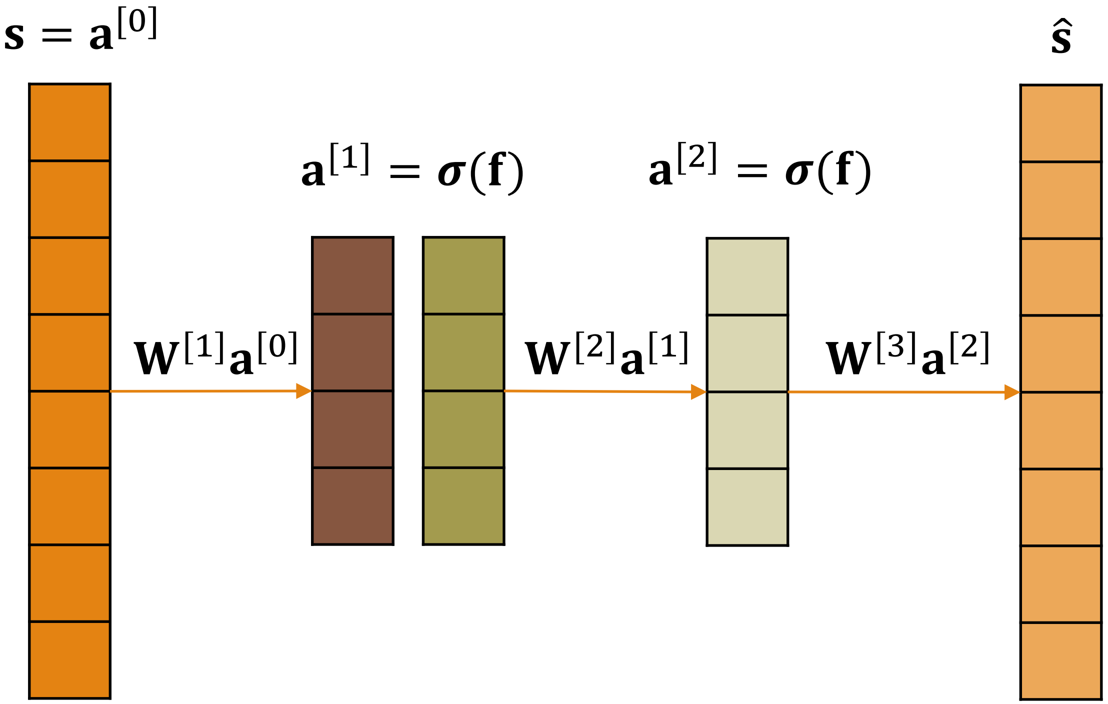
- The neural network
- Layer $l \in \{1, \ldots, 3\}$
- Filters $\mathbf{W}^{[i]}, i \in l$
- Features $\mathbf{f}$
- Non-linear function $\boldsymbol{\sigma}(\cdot)$
- Activations $\mathbf{a}^{[i-1]}$
- Bias $\mathbf{b}$



$$\mathbf{a}^{[i]} = \boldsymbol{\sigma}\left(\mathbf{W}^{[i]}\mathbf{a}^{[i-1]} + \mathbf{b}^{[i]}\right), \text{for } i = 1, \ldots n, \mathbf{s} = \mathbf{a}^{[0]}$$

Convolutional neural networks (CNNs)

- The neural network
- Layer $l \in \{1, \dots, 3\}$
- Filters $\mathbf{W}^{[i]}, i \in l$
- Features $\mathbf{f}$
- Non-linear function $\boldsymbol{\sigma}(\cdot)$
- Activations $\mathbf{a}^{[i-1]}$
- Bias $\mathbf{b}$



The Filters $\mathbf{W}^{[i]}$ and biases $\mathbf{b}^{[i]}$ are initialized to small values and needs to be trained (updated) through optimization.

$$\mathbf{a}^{[i]} = \boldsymbol{\sigma}\left(\mathbf{W}^{[i]}\mathbf{a}^{[i-1]} + \mathbf{b}^{[i]}\right), \text{for } i = 1, \dots n, \mathbf{s} = \mathbf{a}^{[0]}$$

$$C(\mathbf{W}, \mathbf{b}) = \frac{1}{2}\sum_{j=1}^{K}\left\|\mathbf{s}^{[j]} - \hat{\mathbf{s}}^{[j]}\right\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{W}\|_2^2 \quad \left\{\begin{array}{l} \mathbf{W}^{[l]} = \mathbf{W}^{[l]} + \eta\nabla_{\mathbf{W}^{[l]}}C(\mathbf{s}, \hat{\mathbf{s}}; \mathbf{W}, \mathbf{b}) \\ \mathbf{b}^{[l]} = \mathbf{b}^{[l]} + \eta\nabla_{\mathbf{b}^{[l]}}C(\mathbf{s}, \hat{\mathbf{s}}; \mathbf{W}, \mathbf{b}) \end{array}\right.$$

3. Frames and filter banks

- A filter is a linear operator which is computed as the inner product of an input signal with all translates of a fixed function.
- In CNNs these filters are commonly referred to as convolutional kernels.
- Filter banks consists of a group of filters, and is considered a special type of frame where each frame element are required to be translates of each other, i.e. Haar filters, Gabor filters etc.
- Filters and filter banks are therefore related to frames which generalizes the concept of filter banks and consists of equally spaced translates of a fixed set of functions, i.e. finite frames.

3. Frames and filter banks

- Simple 2D Haar filter (frame) example.

$$\boldsymbol{\phi} \in \{\boldsymbol{\phi}_i\}_{i=1}^{4}$$

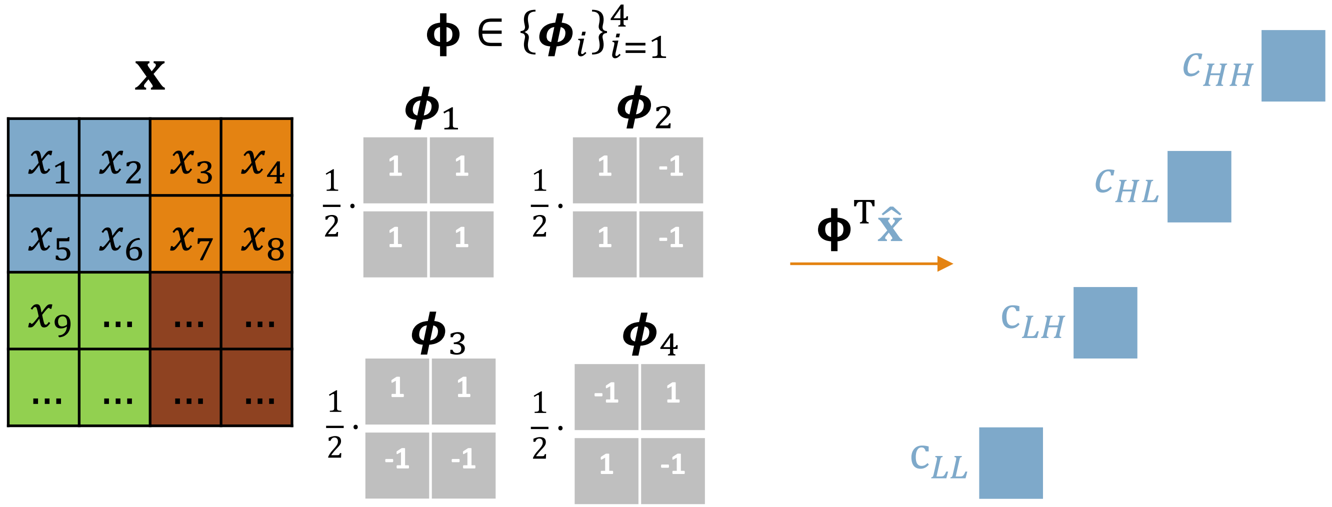
$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\boldsymbol{\phi}_1 = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \boldsymbol{\phi}_2 = \frac{1}{2} \cdot \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\boldsymbol{\phi}_3 = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad \boldsymbol{\phi}_4 = \frac{1}{2} \cdot \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

3. Frames and filter banks

- Simple 2D Haar filter (frame) example.



$$\boldsymbol{\phi}^{\mathrm{T}} = \begin{bmatrix} - & \boldsymbol{\phi}_1^{\mathrm{T}} & - \\ - & \vdots & - \\ - & \boldsymbol{\phi}_4^{\mathrm{T}} & - \end{bmatrix}$$

$$\boldsymbol{\phi}^{\mathrm{T}}\hat{\mathbf{x}} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} c_{LL} \\ c_{LH} \\ c_{HL} \\ c_{HH} \end{bmatrix}$$

3. Frames and filter banks

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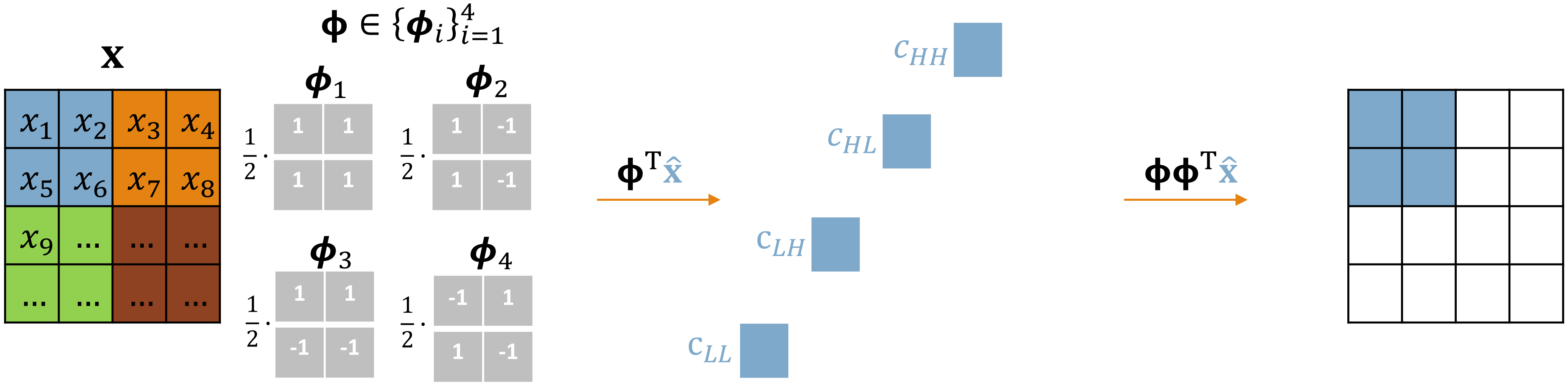


$$\boldsymbol{\Phi}^{\mathrm{T}} = \begin{bmatrix} - & \boldsymbol{\phi}_1^{\mathrm{T}} & - \\ - & \vdots & - \\ - & \boldsymbol{\phi}_4^{\mathrm{T}} & - \end{bmatrix}$$

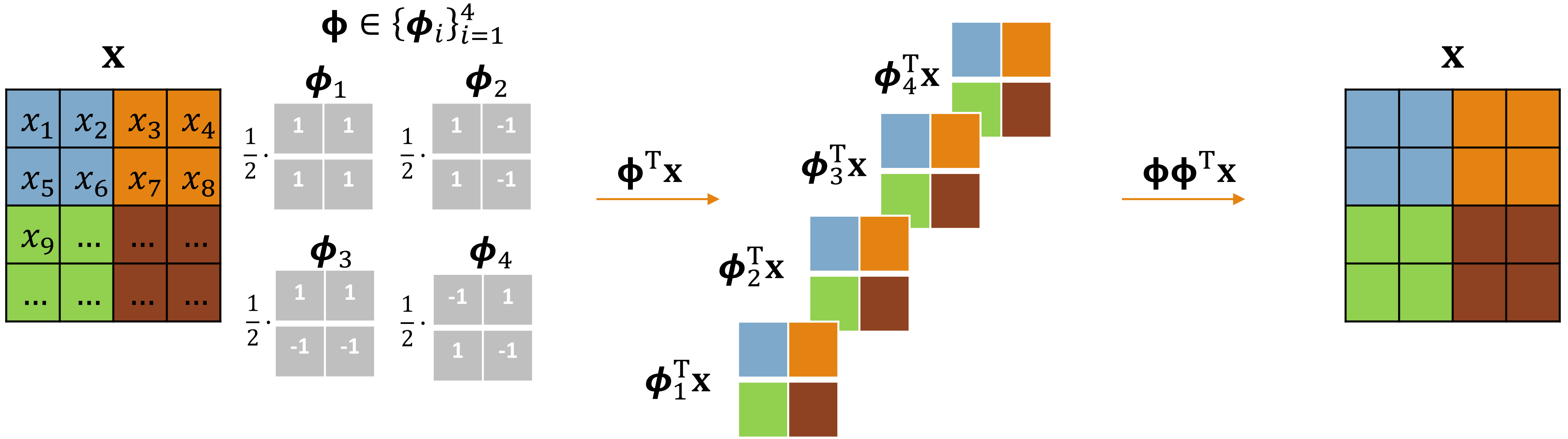
$$\boldsymbol{\Phi}^{\mathrm{T}}\hat{\mathbf{x}} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} c_{LL} \\ c_{LH} \\ c_{HL} \\ c_{HH} \end{bmatrix}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} | & & | \\ \boldsymbol{\phi}_1 & \cdots & \boldsymbol{\phi}_4 \\ | & & | \end{bmatrix}$$

$$\boldsymbol{\Phi}c = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} c_{LL} \\ c_{LH} \\ c_{HL} \\ c_{HH} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{bmatrix}$$

3. Frames and filter banks

- Simple 2D Haar filter (frame) example.



$$\boldsymbol{\Phi}^{\mathrm{T}} = \begin{bmatrix} - & \boldsymbol{\phi}_1^{\mathrm{T}} & - \\ - & \vdots & - \\ - & \boldsymbol{\phi}_4^{\mathrm{T}} & - \end{bmatrix} \qquad \boldsymbol{\Phi}^{\mathrm{T}}\hat{\mathbf{x}} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} c_{LL} \\ c_{LH} \\ c_{HL} \\ c_{HH} \end{bmatrix}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} | & | & | \\ \boldsymbol{\phi}_1 & \cdots & \boldsymbol{\phi}_4 \\ | & | & | \end{bmatrix} \qquad \boldsymbol{\Phi}c = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} c_{LL} \\ c_{LH} \\ c_{HL} \\ c_{HH} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{bmatrix}$$

3. Frames and filter banks

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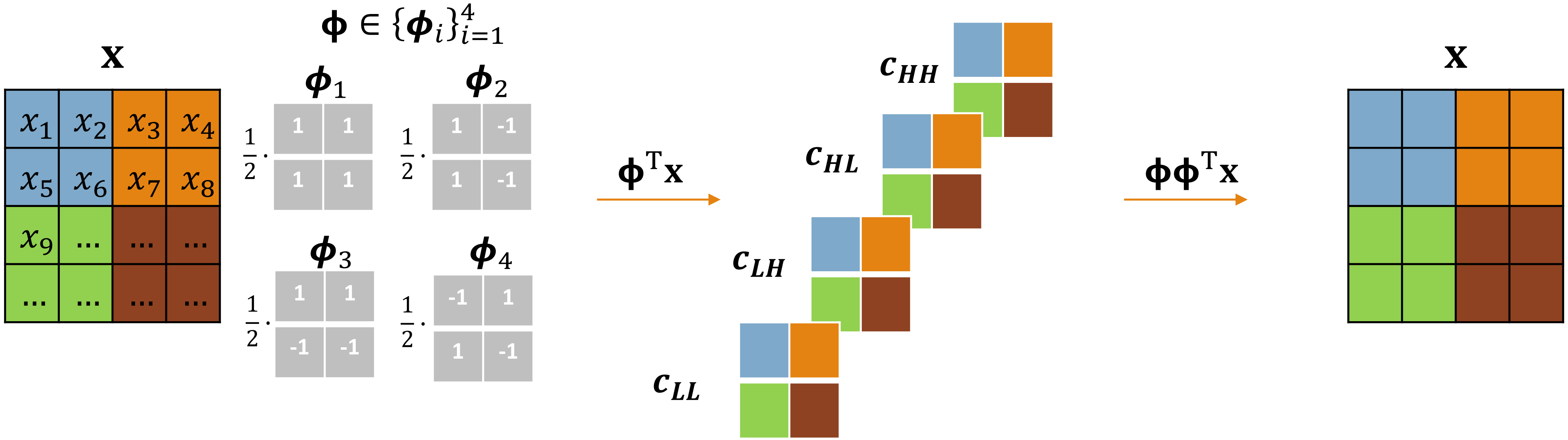

$$\boldsymbol{\phi}^{\mathrm{T}} = \begin{bmatrix} - & \boldsymbol{\phi}_1^{\mathrm{T}} & - \\ - & \vdots & - \\ - & \boldsymbol{\phi}_4^{\mathrm{T}} & - \end{bmatrix} \qquad \boldsymbol{\phi}^{\mathrm{T}}\hat{\mathbf{x}} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} c_{LL} \\ c_{LH} \\ c_{HL} \\ c_{HH} \end{bmatrix}$$

$$\boldsymbol{\phi} = \begin{bmatrix} | & | & | \\ \boldsymbol{\phi}_1 & \cdots & \boldsymbol{\phi}_4 \\ | & | & | \end{bmatrix} \qquad \boldsymbol{\phi}c = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} c_{LL} \\ c_{LH} \\ c_{HL} \\ c_{HH} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{bmatrix}$$

3. Frames and filter banks

- The family of vectors $\{\boldsymbol{\phi}_i\}_{i=1}^{M}$ in the Hilbert space H^n is called a frame if they satisfy

$$\alpha\|\mathbf{x}\|^2 \leq \sum_{i=1}^{M} |\langle \mathbf{x}, \boldsymbol{\phi}_i \rangle|^2 \leq \beta\|\mathbf{x}\|^2 \, , \mathbf{x} \in H^n$$

$Frame$: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{F}\mathbf{x}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

3. Frames and filter banks

- The family of vectors $\{\boldsymbol{\phi}_i\}_{i=1}^{M}$ in the Hilbert space H^n is called a frame if they satisfy

$$\alpha\|\mathbf{x}\|^2 \leq \sum_{i=1}^{M} |\langle \mathbf{x}, \boldsymbol{\phi}_i \rangle|^2 \leq \beta\|\mathbf{x}\|^2 \,, \mathbf{x} \in H^n$$

- Typically we associate frames with three operators 1. the frame operator, 2. the analysis operator and 3. the synthesis operator.

$$1.\ \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F} \quad 2.\ \boldsymbol{\phi}^{\mathrm{T}} = \begin{bmatrix} - & \boldsymbol{\phi}_1^{\mathrm{T}} & - \\ - & \vdots & - \\ - & \boldsymbol{\phi}_M^{\mathbf{T}} & - \end{bmatrix} \quad 3.\ \widetilde{\boldsymbol{\phi}} = \begin{bmatrix} | & | & | \\ \widetilde{\boldsymbol{\phi}}_1 & \cdots & \widetilde{\boldsymbol{\phi}}_M \\ | & | & | \end{bmatrix}$$

$Frame$: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{F}\mathbf{x}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator$: $\widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis$: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis$: $\mathbf{x} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$

3. Frames and filter banks

- The constants $0 < \alpha \leq \beta < \infty$ are known as frame bounds, and if $\alpha = \beta$ then the frame is known as a tight frame.

$\mathit{Frame}\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$\mathit{Frame\ operator}\colon \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$\mathit{Analysis}\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$\mathit{Synthesis}\colon \mathbf{x} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$

$\mathit{Tight\ frame}\colon\ \alpha = \beta$

3. Frames and filter banks

- The constants $0 < \alpha \leq \beta < \infty$ are known as frame bounds, and if $\alpha = \beta$ then the frame is known as a tight frame.
- The frame bounds are represented by the eigenvalues of $\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}$. This can be derived from the singular value decomposition:

$$\left.\begin{array}{l} \boldsymbol{\phi} = \mathbf{U\Sigma V^T} \\ \\ \boldsymbol{\phi}^{\mathrm{T}} = \mathbf{V\Sigma U}^{\mathrm{T}} \end{array}\right\} \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{U\Sigma V^T V\Sigma U}^{\mathrm{T}} = \mathbf{U\Sigma^2 U}^{\mathrm{T}} = \mathbf{U}\begin{bmatrix} \sigma_{\max}^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{\min}^2 \end{bmatrix}\mathbf{U^T}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$ $\qquad Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

3. Frames and filter banks

- If the lower bound α is non-zero, then recovery of the original signal can be done from the analysis $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$ by using the synthesis operator $\tilde{\boldsymbol{\phi}}\mathbf{c} = \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{x}$, where $\tilde{\boldsymbol{\phi}}$ satisfies the frame condition $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$.

$Frame$: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{F}\mathbf{x}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Bounds$: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ operator$: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Frame\ condition$: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Analysis$: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis$: $\mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame$: $\alpha = \beta$

3. Frames and filter banks

- If the lower bound α is non-zero, then recovery of the original signal can be done from the analysis $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$ by using the synthesis operator $\widetilde{\boldsymbol{\phi}}\mathbf{c} = \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{x}$, where $\widetilde{\boldsymbol{\phi}}$ satisfies the frame condition $\widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$.

- The synthesis operator is explicitly derived from the pseudo-inverse:

$$\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c} \rightarrow \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \boldsymbol{\phi}\mathbf{c} \rightarrow \mathbf{x} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}\mathbf{c} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\colon \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\colon \widetilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

3. Frames and filter banks

- If the lower bound α is non-zero, then recovery of the original signal can be done from the analysis $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$ by using the synthesis operator $\tilde{\boldsymbol{\phi}}\mathbf{c} = \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{x}$, where $\tilde{\boldsymbol{\phi}}$ satisfies the frame condition $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$.

- The synthesis operator is explicitly derived from the pseudo-inverse:

$$\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c} \rightarrow \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \boldsymbol{\phi}\mathbf{c} \rightarrow \mathbf{x} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}\mathbf{c} = \tilde{\boldsymbol{\phi}}\mathbf{c}$$

- The synthesis operator $\tilde{\boldsymbol{\phi}}$ also goes by the name dual frame or adjoint.

$Frame:\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis: \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame:\ \alpha = \beta$

$Bounds:\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame: \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

3. Frames and filter banks

- In the presence of noise: $\acute{\mathbf{x}} = \mathbf{x} + \mathbf{e} \rightarrow \acute{\mathbf{c}} = \boldsymbol{\phi}^{\mathrm{T}}\acute{\mathbf{x}} = \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} + \boldsymbol{\phi}^{\mathrm{T}}\mathbf{e} = \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} + \acute{\mathbf{e}}$, then

$$\acute{\mathbf{x}} = \widetilde{\boldsymbol{\phi}}\acute{\mathbf{c}} = \widetilde{\boldsymbol{\phi}}\left(\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} + \acute{\mathbf{e}}\right) = \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} + \widetilde{\boldsymbol{\phi}}\acute{\mathbf{e}}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{F}\mathbf{x}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ operator\colon \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Frame\ condition\colon \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Dual\ frame\colon \widetilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$Synthesis\colon \mathbf{x} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

3. Frames and filter banks

- In the presence of noise: $\acute{\mathbf{x}} = \mathbf{x} + \mathbf{e} \rightarrow \acute{\mathbf{c}} = \boldsymbol{\phi}^{\mathrm{T}}\acute{\mathbf{x}} = \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} + \boldsymbol{\phi}^{\mathrm{T}}\mathbf{e} = \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} + \acute{\mathbf{e}}$, then

$$\acute{\mathbf{x}} = \widetilde{\boldsymbol{\phi}}\acute{\mathbf{c}} = \widetilde{\boldsymbol{\phi}}\left(\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} + \acute{\mathbf{e}}\right) = \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} + \widetilde{\boldsymbol{\phi}}\acute{\mathbf{e}}$$

- Noise amplification is given by the condition number (CN)

$$\frac{\left\|\widetilde{\boldsymbol{\phi}}\acute{\mathbf{e}}\right\|^2}{\|\acute{\mathbf{e}}\|^2} = \frac{\sigma_{\max}^2}{\sigma_{\min}^2} = \frac{\beta}{\alpha} = \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) \rightarrow \text{Tight frame has the minimum noise amplification}$$

$Frame\!: \ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{F}\mathbf{x}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator\!: \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\!: \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\!: \mathbf{x} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\!: \ \alpha = \beta$

$Bounds\!: \ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\!: \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\!: \widetilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$CN\!: \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

4. Framelet expansion – analysis and mathematical framework

- The framelet expansion in CNNs is explained through **non-local** and **local basis**, **Hankel matrix** formulation of convolution, and the **SVD**.

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{F}\mathbf{x}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

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4. Framelet expansion – analysis and mathematical framework

- The framelet expansion in CNNs is explained through **non-local** and **local basis**, **Hankel matrix** formulation of convolution, and the **SVD**.
- First, let $\mathbf{x} = \left[\mathrm{x}[1], \ldots, \mathrm{x}[n]\right]^{\mathrm{T}}$ be a 1D signal with length n and let d represent a matrix pencil parameter (also filter length), then the wrap-around Hankel matrix is defined as

$$\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = \begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] & \cdots & \mathrm{x}[d] \\ \mathrm{x}[2] & \mathrm{x}[3] & \cdots & \mathrm{x}[d+1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{x}[n] & \mathrm{x}[1] & \cdots & \mathrm{x}[d-1] \end{bmatrix} \in \mathbb{R}^{n \times d}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

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$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- The SVD of the wrap-around Hankel matrix can be written as the following factorization

  $\mathcal{H}_{\mathrm{d}}(\mathbf{x}) = \mathbf{U\Sigma V}^{\mathrm{T}}$

- Where the column vectors of $\mathbf{U}$ and row vectors of $\mathbf{V}^{\mathrm{T}}$ denotes the left and right singular vectors, respectively, and the diagonal elements of $\mathbf{\Sigma}$ denotes the singular values.

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

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4. Framelet expansion – analysis and mathematical framework

- The SVD of the wrap-around Hankel matrix can be written as the following factorization

$\mathcal{H}_{\mathrm{d}}(\mathbf{x}) = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$

$$\underbrace{\begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] & \cdots & \mathrm{x}[d] \\ \mathrm{x}[2] & \mathrm{x}[3] & \cdots & \mathrm{x}[d+1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{x}[n] & \mathrm{x}[1] & \cdots & \mathrm{x}[d-1] \end{bmatrix}}_{n \times d} = \underbrace{\begin{bmatrix} | & \cdots & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_n \\ | & \cdots & | \end{bmatrix}}_{n \times n} \underbrace{\begin{bmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{nd} \end{bmatrix}}_{n \times d} \underbrace{\begin{bmatrix} - & \mathbf{v}_1^{\mathrm{T}} & - \\ \vdots & \vdots & \vdots \\ - & \mathbf{v}_d^{\mathrm{T}} & - \end{bmatrix}}_{d \times d}$$

$\mathit{Frame}\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{F}\mathbf{x}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$\mathit{Frame\ operator}\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$\mathit{Analysis}\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

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$\mathit{CN}\colon \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$\mathit{Hankel}\colon \mathcal{H}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- The Truncated-SVD of the wrap-around Hankel matrix, where $\mathrm{n} > k$ (low rank approximation)

$$\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = \mathbf{U\Sigma V}^{\mathrm{T}}$$

$$\underbrace{\begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] & \cdots & \mathrm{x}[d] \\ \mathrm{x}[2] & \mathrm{x}[3] & \cdots & \mathrm{x}[d+1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{x}[n] & \mathrm{x}[1] & \cdots & \mathrm{x}[d-1] \end{bmatrix}}_{n \times d} = \underbrace{\begin{bmatrix} | & \cdots & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_k \\ | & \cdots & | \end{bmatrix}}_{n \times k} \underbrace{\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{kk} \end{bmatrix}}_{k \times k} \underbrace{\begin{bmatrix} - & \mathbf{v}_1^{\mathrm{T}} & - \\ \vdots & \vdots & \vdots \\ - & \mathbf{v}_k^{\mathrm{T}} & - \end{bmatrix}}_{k \times d}$$

$Frame$: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator$: $\widetilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis$: $\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis$: $\mathbf{x} = \widetilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame$: $\alpha = \beta$

$Bounds$: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition$: $\widetilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame$: $\widetilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

CN: $\kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel$: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- If we now compute the left side with $\mathbf{U}^{\mathrm{T}}$ and $\mathbf{V}$, i.e. $\mathbf{U}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\mathbf{V} = \mathbf{U}^{\mathrm{T}}\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}\mathbf{V}$, we equivalently have

$$\boldsymbol{\Sigma} = \mathbf{U}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\mathbf{V}$$

$$\underbrace{\begin{bmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{nd} \end{bmatrix}}_{n \times d} = \underbrace{\begin{bmatrix} - & \mathbf{u}_1^{\mathrm{T}} & - \\ \vdots & \vdots & \vdots \\ - & \mathbf{u}_n^{\mathrm{T}} & - \end{bmatrix}}_{n \times n} \underbrace{\begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] & \cdots & \mathrm{x}[d] \\ \mathrm{x}[2] & \mathrm{x}[3] & \cdots & \mathrm{x}[d+1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{x}[n] & \mathrm{x}[1] & \cdots & \mathrm{x}[d-1] \end{bmatrix}}_{n \times d} \underbrace{\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_d \\ | & & | \end{bmatrix}}_{d \times d}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\colon \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

$CN\colon \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- If we now compute the left side with $\mathbf{U}^{\mathrm{T}}$ and $\mathbf{V}$, i.e. $\mathbf{U}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\mathbf{V} = \mathbf{U}^{\mathrm{T}}\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}\mathbf{V}$, we equivalently have

$$\boldsymbol{\Sigma} = \mathbf{U}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\mathbf{V} \rightarrow \sigma_{ij} = \mathbf{u}_i^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\mathbf{v}_j \;,1 \le i,j \le r$$

$$\underbrace{\begin{bmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{nd} \end{bmatrix}}_{n\times d} = \underbrace{\begin{bmatrix} - & \mathbf{u}_1^{\mathrm{T}} & - \\ \vdots & \vdots & \vdots \\ - & \mathbf{u}_n^{\mathrm{T}} & - \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] & \cdots & \mathrm{x}[d] \\ \mathrm{x}[2] & \mathrm{x}[3] & \cdots & \mathrm{x}[d+1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{x}[n] & \mathrm{x}[1] & \cdots & \mathrm{x}[d-1] \end{bmatrix}}_{n\times d} \underbrace{\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_d \\ | & & | \end{bmatrix}}_{d\times d}$$

Frame: $\alpha\|\mathbf{x}\|^2 \le \langle \mathbf{F}\mathbf{x},\mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

Frame operator: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

Analysis: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

Synthesis: $\mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

Tight frame: $\alpha = \beta$

Bounds: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

Frame condition: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

Dual frame: $\tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

CN: $\kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

Hankel: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- If we now compute the left side with $\mathbf{U}^{\mathrm{T}}$ and $\mathbf{V}$, i.e. $\mathbf{U}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\mathbf{V} = \mathbf{U}^{\mathrm{T}}\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}\mathbf{V}$, we equivalently have

$$\boldsymbol{\Sigma} = \mathbf{U}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\mathbf{V} \to \sigma_{ij} = \mathbf{u}_i^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\mathbf{v}_j \;, 1 \le i,j \le r \qquad \mathbf{u}_i^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) \to \text{nonlocal}, \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\mathbf{v}_j \to \text{local}$$

$$\underbrace{\begin{bmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{nd} \end{bmatrix}}_{n \times d} = \underbrace{\begin{bmatrix} - & \mathbf{u}_1^{\mathrm{T}} & - \\ \vdots & \vdots & \vdots \\ - & \mathbf{u}_n^{\mathrm{T}} & - \end{bmatrix}}_{n \times n} \underbrace{\begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] & \cdots & \mathrm{x}[d] \\ \mathrm{x}[2] & \mathrm{x}[3] & \cdots & \mathrm{x}[d+1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{x}[n] & \mathrm{x}[1] & \cdots & \mathrm{x}[d-1] \end{bmatrix}}_{n \times d} \underbrace{\begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_d \\ | & | & | \end{bmatrix}}_{d \times d}$$

Frame: $\alpha\|\mathbf{x}\|^2 \le \langle \mathbf{F}\mathbf{x}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

Frame operator: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

Analysis: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

Synthesis: $\mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

Tight frame: $\alpha = \beta$

Bounds: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

Frame condition: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

Dual frame: $\tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

CN: $\kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

Hankel: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

Basis: $nonlocal\{\mathbf{u}_i\}_{i=1}^n, local\ \{\mathbf{v}_i\}_{i=1}^d$

4. Framelet expansion – analysis and mathematical framework

- However, this type of decomposition holds for arbitrary basis matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\varphi}$

$$\boldsymbol{C} = \boldsymbol{\Phi}^{\mathrm{T}}\mathcal{H}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi} \rightarrow c_{ij} = \boldsymbol{\phi}_i^{\mathrm{T}}\mathcal{H}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}_j \ , i = 1, \ldots, n \text{ and j} = 1, \ldots, \mathrm{d}$$

$$\boldsymbol{C} = \underbrace{\begin{bmatrix} - & \boldsymbol{\phi}_1^{\mathrm{T}} & - \\ \vdots & \vdots & \vdots \\ - & \boldsymbol{\phi}_n^{\mathrm{T}} & - \end{bmatrix}}_{n \times n} \underbrace{\begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] & \cdots & \mathrm{x}[d] \\ \mathrm{x}[2] & \mathrm{x}[3] & \cdots & \mathrm{x}[d+1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{x}[n] & \mathrm{x}[1] & \cdots & \mathrm{x}[d-1] \end{bmatrix}}_{n \times d} \underbrace{\begin{bmatrix} | & | & | \\ \boldsymbol{\varphi}_1 & \cdots & \boldsymbol{\varphi}_d \\ | & | & | \end{bmatrix}}_{d \times d}$$

$Frame: \ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator: \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis: \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame: \ \alpha = \beta$

$Bounds: \ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition: \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame: \tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

$CN: \kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel: \mathcal{H}_{\mathrm{d}}(\mathbf{x})$

$Basis: nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^n, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^d$

4. Framelet expansion – analysis and mathematical framework

- The decomposition can be seen as an encoder-decoder of the wrap-around Hankel matrix

Encoder: $\boldsymbol{C} = \boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}$

Decoder: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}^{\mathrm{T}}$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Basis\colon nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Frame\ condition\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Dual\ frame\colon \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$CN\colon \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Tight\ frame\colon\ \alpha = \beta$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- The decomposition can be seen as an encoder-decoder of the wrap-around Hankel matrix

Encoder: $\boldsymbol{C} = \boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}$

Decoder: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}^{\mathrm{T}}$

- In the theory of convolutional framelet expansion the local basis $\{\boldsymbol{\varphi}_i\}_{i=1}^{d}$ are learned and the nonlocal $\{\boldsymbol{\phi}_i\}_{i=1}^{n}$ are chosen as a suitable basis for the data, that meets the frame condition.

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \le \langle\mathbf{Fx},\mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Basis\colon nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Frame\ condition\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Dual\ frame\colon \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$CN\colon \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Tight\ frame\colon\ \alpha = \beta$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- The decomposition can be seen as an encoder-decoder of the wrap-around Hankel matrix

Encoder: $\boldsymbol{C} = \boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi}$

Decoder: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = \tilde{\boldsymbol{\phi}} \boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi} \tilde{\boldsymbol{\varphi}}^{\mathrm{T}}$

$\tilde{\boldsymbol{\phi}} \boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}, \boldsymbol{\varphi} \tilde{\boldsymbol{\varphi}}^{\mathrm{T}} = \boldsymbol{I} \rightarrow Perfect\ reconstruction$

$Frame:\ \alpha \|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta \|\mathbf{x}\|^2$

$Bounds:\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Basis: nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Frame\ operator: \tilde{\boldsymbol{\phi}} \boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Frame\ condition: \tilde{\boldsymbol{\phi}} \boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Analysis: \boldsymbol{\phi}^{\mathrm{T}} \mathbf{x} = \mathbf{c}$

$Dual\ frame: \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi} \boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}} \boldsymbol{\phi}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\phi}} \mathbf{c}$

$CN: \kappa\left(\boldsymbol{\phi} \boldsymbol{\phi}^{\mathrm{T}}\right) = \beta / \alpha$

$Tight\ frame:\ \alpha = \beta$

$Hankel: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- Example in 1D using basis from the normalized Haar matrices $\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{H}_4$ and $\boldsymbol{\varphi} = \boldsymbol{H}_2$

$Frame$: $\alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x} \rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator$: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis$: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis$: $\mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame$: $\alpha = \beta$

$Bounds$: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition$: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame$: $\tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{\mathbf{-1}}\boldsymbol{\phi}$

CN: $\kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel$: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis$: $nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^n, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^d$

4. Framelet expansion – analysis and mathematical framework

- Example in 1D using basis from the normalized Haar matrices $\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{H}_4$ and $\boldsymbol{\varphi} = \boldsymbol{H}_2$

$$\boldsymbol{C} = \underbrace{\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{\boldsymbol{\phi}^{\mathrm{T}}} \underbrace{\begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] \\ \mathrm{x}[2] & \mathrm{x}[3] \\ \mathrm{x}[3] & \mathrm{x}[4] \\ \mathrm{x}[4] & \mathrm{x}[1] \end{bmatrix}}_{\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{\boldsymbol{\varphi}}$$

$Frame:\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x} \rangle \le \beta\|\mathbf{x}\|^2$

$Bounds:\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Basis: nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{n}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{d}$

$Frame\ operator: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Frame\ condition: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Analysis: \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Dual\ frame: \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$CN: \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Tight\ frame:\ \alpha = \beta$

$Hankel: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- Example in 1D using basis from the normalized Haar matrices $\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{H}_4$ and $\boldsymbol{\varphi} = \boldsymbol{H}_2$

$$\boldsymbol{C} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \overbrace{\begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] \\ \mathrm{x}[2] & \mathrm{x}[3] \\ \mathrm{x}[3] & \mathrm{x}[4] \\ \mathrm{x}[4] & \mathrm{x}[1] \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}^{\text{SIMO}}$$

$Frame:\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis: \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame:\ \alpha = \beta$

$Bounds:\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame: \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

$CN: \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis: nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{n}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{d}$

4. Framelet expansion – analysis and mathematical framework

- Example in 1D using basis from the normalized Haar matrices $\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{H}_4$ and $\boldsymbol{\varphi} = \boldsymbol{H}_2$

$$\boldsymbol{C} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} | & | \\ \mathbf{f}_1 & \mathbf{f}_2 \\ | & | \end{bmatrix}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Basis\colon nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{n}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{d}$

$Frame\ operator\colon \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Frame\ condition\colon \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Dual\ frame\colon \widetilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$Synthesis\colon \mathbf{x} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$

$CN\colon \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Tight\ frame\colon\ \alpha = \beta$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- Example in 1D using basis from the normalized Haar matrices $\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{H}_4$ and $\boldsymbol{\varphi} = \boldsymbol{H}_2$

$$\boldsymbol{C} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} | & | \\ \mathbf{f}_1 & \mathbf{f}_2 \\ | & | \end{bmatrix} \qquad \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = \tilde{\boldsymbol{\phi}} \boldsymbol{C} \tilde{\boldsymbol{\varphi}}^{\mathrm{T}} \rightarrow \tilde{\boldsymbol{\phi}} \boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}, \boldsymbol{\varphi} \tilde{\boldsymbol{\varphi}}^{\mathrm{T}} = \boldsymbol{I}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x} \rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}} \boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}} \mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\phi}} \mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\colon \tilde{\boldsymbol{\phi}} \boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\colon \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}} \boldsymbol{\phi}$

$CN\colon \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis\colon nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{n}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{d}$

4. Framelet expansion – analysis and mathematical framework

- Low rank example from the normalized Haar matrices $\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{H}_4$ (truncated) and $\boldsymbol{\varphi} = \boldsymbol{H}_2$

$Frame$: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Bounds$: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Basis$: $nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{n}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{d}$

$Frame\ operator$: $\widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Frame\ condition$: $\widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Analysis$: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Dual\ frame$: $\widetilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$Synthesis$: $\mathbf{x} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$

CN: $\kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Tight\ frame$: $\alpha = \beta$

$Hankel$: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

4. Framelet expansion – analysis and mathematical framework

- Low rank example from the normalized Haar matrices $\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{H}_4$ (truncated) and $\boldsymbol{\varphi} = \boldsymbol{H}_2$

$$\boldsymbol{C} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} \mathrm{x}[1] & \mathrm{x}[2] \\ \mathrm{x}[2] & \mathrm{x}[3] \\ \mathrm{x}[3] & \mathrm{x}[4] \\ \mathrm{x}[4] & \mathrm{x}[1] \end{bmatrix} \begin{bmatrix} \sqrt{1/2} & \sqrt{1/2} \\ \sqrt{1/2} & -\sqrt{1/2} \end{bmatrix}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x} \rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\colon \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$CN\colon \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis\colon nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{n}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{d}$

4. Framelet expansion – analysis and mathematical framework

- Low rank example from the normalized Haar matrices $\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{H}_4$ (truncated) and $\boldsymbol{\varphi} = \boldsymbol{H}_2$

$$\widehat{\boldsymbol{\mathcal{H}}}_{\mathrm{d}}(\mathbf{x}) = \widetilde{\boldsymbol{\phi}}\boldsymbol{C}\widetilde{\boldsymbol{\varphi}}^{\mathrm{T}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \sqrt{1/2} & \sqrt{1/2} \\ \sqrt{1/2} & -\sqrt{1/2} \end{bmatrix}$$

$Frame$: $\alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x} \rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator$: $\widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis$: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis$: $\mathbf{x} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame$: $\alpha = \beta$

$Bounds$: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition$: $\widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame$: $\widetilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

CN: $\kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel$: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis$: $nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{n}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{d}$

4. Framelet expansion – analysis and mathematical framework

- Low rank example from the normalized Haar matrices $\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{H}_4$ (truncated) and $\boldsymbol{\varphi} = \boldsymbol{H}_2$

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$$\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}} = \underbrace{\begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}}_{\alpha = 0}$$

- This type of low-rank approximation is similar to pooling/unpooling in CNNs

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$Frame:\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator: \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis: \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis: \mathbf{x} = \widetilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame:\ \alpha = \beta$

$Bounds:\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition: \widetilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame: \widetilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

$CN: \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis: nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{n}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{d}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with pooling to retain high frequencies (UNET)

$$\boldsymbol{C}_{aug} = \boldsymbol{\Phi}_{aug}^{\mathrm{T}} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi}, \qquad \boldsymbol{\Phi}_{aug}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}_{low}^{\mathrm{T}} \end{bmatrix}$$

$Frame:\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{F}\mathbf{x}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator: \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis: \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame:\ \alpha = \beta$

$Bounds:\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition: \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame: \tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\Phi}$

$CN: \kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis: nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis: \boldsymbol{\varphi}$

$Fixed\ basis: \boldsymbol{\Phi}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with pooling to retain high frequencies (UNET)

$$\boldsymbol{C}_{aug} = \boldsymbol{\phi}_{aug}^{\mathrm{T}} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}, \qquad \boldsymbol{\phi}_{aug}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\phi}_{low}^{\mathrm{T}} \end{bmatrix}$$

$$\boldsymbol{C}_{aug} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\phi}_{low}^{\mathrm{T}} \end{bmatrix} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi} = \Big[\underbrace{\boldsymbol{I}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}}_{\text{all pass}} \quad \underbrace{\boldsymbol{\phi}_{low}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}}_{\text{pooling}}\Big] = [\mathbf{f} \quad \boldsymbol{\phi}_{low}^{\mathrm{T}}\mathbf{f}]$$

$Frame:\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis: \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame:\ \alpha = \beta$

$Bounds:\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame: \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

$CN: \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis: nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis: \boldsymbol{\varphi}$

$Fixed\ basis: \boldsymbol{\phi}$

4. Framelet expansion – analysis and mathematical framework

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$$\boldsymbol{C}_{aug} = \boldsymbol{\Phi}_{aug}^{\mathrm{T}} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}, \qquad \boldsymbol{\Phi}_{aug}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}_{low}^{\mathrm{T}} \end{bmatrix}$$

$$\boldsymbol{C}_{aug} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}_{low}^{\mathrm{T}} \end{bmatrix} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi} = \big[\overbrace{\boldsymbol{I}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}}^{\text{all pass}} \quad \overbrace{\boldsymbol{\Phi}_{low}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}}^{\text{pooling}}\big] = [\mathbf{f} \quad \boldsymbol{\Phi}_{low}^{\mathrm{T}}\mathbf{f}]$$

$$\hat{\mathbf{f}} = \boldsymbol{\Phi}_{low}\boldsymbol{\Phi}_{low}^{\mathrm{T}}\mathbf{f} \rightarrow \overbrace{[\mathbf{f} \quad \hat{\mathbf{f}}]}^{\text{concatenate}}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\colon \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\colon \tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

$CN\colon \kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis\colon nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis\colon \boldsymbol{\varphi}$

$Fixed\ basis\colon \boldsymbol{\Phi}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with pooling to retain high frequencies (UNET)

$$\boldsymbol{C}_{aug} = \boldsymbol{\Phi}_{aug}^{\mathrm{T}} \mathcal{H}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi}, \qquad \boldsymbol{\Phi}_{aug}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}_{low}^{\mathrm{T}} \end{bmatrix}$$

$$\boldsymbol{C}_{aug} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}_{low}^{\mathrm{T}} \end{bmatrix} \mathcal{H}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi} = [\underbrace{\boldsymbol{I} \mathcal{H}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi}}_{\text{all pass}} \quad \underbrace{\boldsymbol{\Phi}_{low}^{\mathrm{T}} \mathcal{H}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi}}_{\text{pooling}}] = [\mathbf{f} \quad \boldsymbol{\Phi}_{low}^{\mathrm{T}} \mathbf{f}]$$

$$\hat{\mathbf{f}} = \tilde{\boldsymbol{\Phi}}_{low} \boldsymbol{\Phi}_{low}^{\mathrm{T}} \mathbf{f} \rightarrow \underbrace{[\mathbf{f} \quad \hat{\mathbf{f}}]}_{\text{concatenate}}$$

$$\mathcal{H}_{\mathrm{d}}(\mathbf{x}) = [\mathbf{f} \quad \hat{\mathbf{f}}] \begin{bmatrix} \tilde{\boldsymbol{\varphi}}_1^{\mathrm{T}} \\ \tilde{\boldsymbol{\varphi}}_2^{\mathrm{T}} \end{bmatrix} = \mathbf{f} \tilde{\boldsymbol{\varphi}}_1^{\mathrm{T}} + \hat{\mathbf{f}} \tilde{\boldsymbol{\varphi}}_2^{\mathrm{T}} = \mathcal{H}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi} \tilde{\boldsymbol{\varphi}}_1^{\mathrm{T}} + \boldsymbol{\Phi}_{low} \boldsymbol{\Phi}_{low}^{\mathrm{T}} \mathcal{H}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi} \tilde{\boldsymbol{\varphi}}_2^{\mathrm{T}}$$

$Frame\!: \ \alpha \|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x} \rangle \le \beta \|\mathbf{x}\|^2$

$Frame\ operator\!: \tilde{\boldsymbol{\Phi}} \boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\!: \boldsymbol{\Phi}^{\mathrm{T}} \mathbf{x} = \mathbf{c}$

$Synthesis\!: \mathbf{x} = \tilde{\boldsymbol{\Phi}} \mathbf{c}$

$Tight\ frame\!: \ \alpha = \beta$

$Bounds\!: \ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\!: \tilde{\boldsymbol{\Phi}} \boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\!: \tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi} \boldsymbol{\Phi}^{\mathrm{T}} \right)^{-1} \boldsymbol{\Phi}$

$CN\!: \kappa\left(\boldsymbol{\Phi} \boldsymbol{\Phi}^{\mathrm{T}} \right) = \beta / \alpha$

$Hankel\!: \mathcal{H}_{\mathrm{d}}(\mathbf{x})$

$Basis\!: nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis\!: \boldsymbol{\varphi}$

$Fixed\ basis\!: \boldsymbol{\Phi}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with pooling to retain high frequencies (UNET)

$$\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}_1^{\mathrm{T}} + \boldsymbol{\phi}_{low}\boldsymbol{\phi}_{low}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}_2^{\mathrm{T}}$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\colon \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$CN\colon \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis\colon nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis\colon \boldsymbol{\varphi}$

$Fixed\ basis\colon \boldsymbol{\phi}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with pooling to retain high frequencies (UNET)

$$\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = \overbrace{\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}_1^{\mathrm{T}} + \boldsymbol{\Phi}_{low}\boldsymbol{\Phi}_{low}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}_2^{\mathrm{T}}}^{\text{Both terms has the low frequency component}}$$

- This does not guarantee perfect reconstruction due to both terms has the low frequency component, and the low frequencies are therefore overly emphasized. This is assumed to be the main source of smoothing in the UNET architecture.

$Frame:\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{F}\mathbf{x}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator: \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis: \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame:\ \alpha = \beta$

$Bounds:\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition: \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame: \tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

$CN: \kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis: nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis: \boldsymbol{\varphi}$

$Fixed\ basis: \boldsymbol{\Phi}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with framelet expansion

$$\boldsymbol{C}_{aug} = \boldsymbol{\phi}_{aug}^{\mathrm{T}} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi}, \qquad \boldsymbol{\phi}_{aug}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\phi}^{\mathrm{T}} \end{bmatrix}$$

$Frame\text{: } \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator\text{: } \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\text{: } \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\text{: } \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\text{: } \alpha = \beta$

$Bounds\text{: } \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\text{: } \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\text{: } \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$CN\text{: } \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\text{: } \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis\text{: } nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis\text{: } \boldsymbol{\varphi}$

$Fixed\ basis\text{: } \boldsymbol{\phi}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with framelet expansion

$$\boldsymbol{C}_{aug} = \boldsymbol{\Phi}_{aug}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}, \qquad \boldsymbol{\Phi}_{aug}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}^{\mathrm{T}} \end{bmatrix}$$

$$\boldsymbol{C}_{aug} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}^{\mathrm{T}} \end{bmatrix}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi} = [\boldsymbol{I}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi} \quad \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}] = [\mathbf{f} \quad \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{f}]$$

$Frame\colon\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

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$Frame\ condition\colon \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\colon \tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

$CN\colon \kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis\colon nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis\colon \boldsymbol{\varphi}$

$Fixed\ basis\colon \boldsymbol{\Phi}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with framelet expansion

$$\boldsymbol{C}_{aug} = \boldsymbol{\Phi}_{aug}^{\mathrm{T}} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi}, \qquad \boldsymbol{\Phi}_{aug}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}^{\mathrm{T}} \end{bmatrix}$$

$$\boldsymbol{C}_{aug} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}^{\mathrm{T}} \end{bmatrix} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) \boldsymbol{\varphi} = [\boldsymbol{I}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi} \quad \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}] = [\mathbf{f} \quad \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{f}]$$

$$\acute{\mathbf{f}} = \tilde{\boldsymbol{\Phi}} \boldsymbol{\Phi}^{\mathrm{T}} \mathbf{f} \rightarrow [\overbrace{\mathbf{f} \quad \acute{\mathbf{f}}}^{\text{concatenate}}]$$

$Frame:\ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{F}\mathbf{x}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator: \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis: \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame:\ \alpha = \beta$

$Bounds:\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition: \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame: \tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

$CN: \kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis: nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis: \boldsymbol{\varphi}$

$Fixed\ basis: \boldsymbol{\Phi}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with framelet expansion

$$\boldsymbol{C}_{aug} = \boldsymbol{\Phi}_{aug}^{\mathrm{T}} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}, \qquad \boldsymbol{\Phi}_{aug}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}^{\mathrm{T}} \end{bmatrix}$$

$$\boldsymbol{C}_{aug} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Phi}^{\mathrm{T}} \end{bmatrix} \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi} = [\boldsymbol{I}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi} \quad \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}] = [\mathbf{f} \quad \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{f}]$$

$$\acute{\mathbf{f}} = \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{f} \rightarrow \overbrace{[\mathbf{f} \quad \acute{\mathbf{f}}]}^{\text{concatenate}}$$

$$\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = [\mathbf{f} \quad \acute{\mathbf{f}}] \begin{bmatrix} \tilde{\boldsymbol{\varphi}}_1^{\mathrm{T}} \\ \tilde{\boldsymbol{\varphi}}_2^{\mathrm{T}} \end{bmatrix} = \mathbf{f}\tilde{\boldsymbol{\varphi}}_1^{\mathrm{T}} + \acute{\mathbf{f}}\tilde{\boldsymbol{\varphi}}_2^{\mathrm{T}} = \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}_1^{\mathrm{T}} + \tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}_2^{\mathrm{T}}$$

$Frame$: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator$: $\tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis$: $\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis$: $\mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame$: $\alpha = \beta$

$Bounds$: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition$: $\tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame$: $\tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

CN: $\kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel$: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis$: $nonlocal \{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis$: $\boldsymbol{\varphi}$

$Fixed\ basis$: $\boldsymbol{\Phi}$

4. Framelet expansion – analysis and mathematical framework

- The introduction of skip connections with framelet expansion

$$\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x}) = \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}_1^{\mathrm{T}} + \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})\boldsymbol{\varphi}\tilde{\boldsymbol{\varphi}}_2^{\mathrm{T}}$$

- The framelet expansion meets the frame condition and therefore meets the perfect reconstruction condition.

$Frame\!: \ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator\!: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\!: \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\!: \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\!: \ \alpha = \beta$

$Bounds\!: \ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\!: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\!: \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-\mathbf{1}}\boldsymbol{\phi}$

$CN\!: \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\!: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

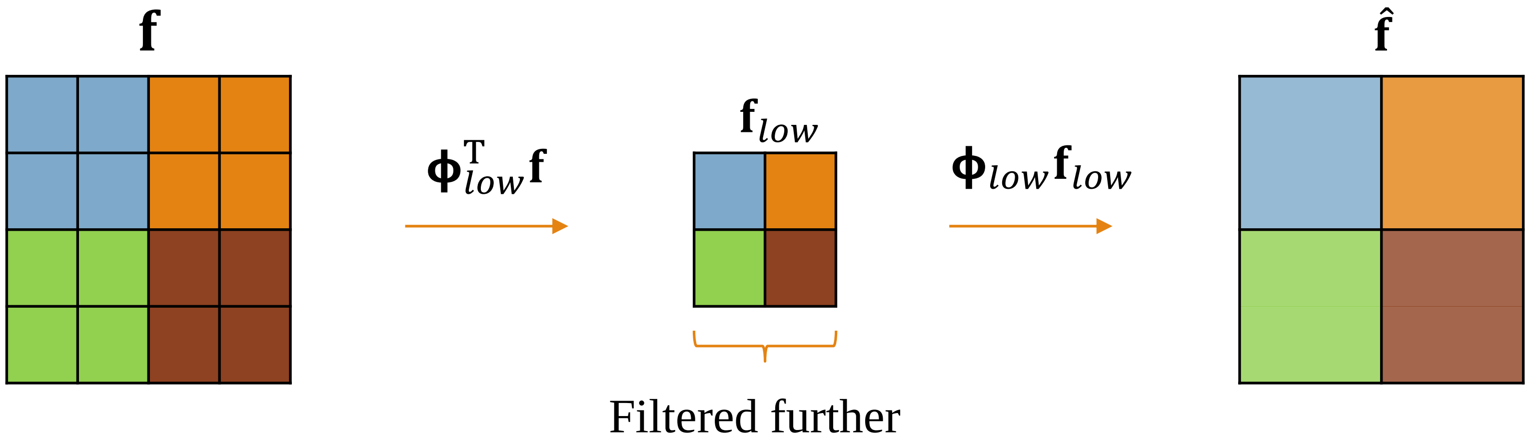
$Basis\!: nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis\!: \boldsymbol{\varphi}$

$Fixed\ basis\!: \boldsymbol{\phi}$

4. Framelet expansion in CNNs

- Pooling/unpooling of an arbitrary feauture within the network



Frame: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

Frame operator: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

Analysis: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

Synthesis: $\mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

Tight frame: $\alpha = \beta$

Bounds: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

Frame condition: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

Dual frame: $\tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

CN: $\kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

Hankel: $\mathcal{H}_{\mathrm{d}}(\mathbf{x})$

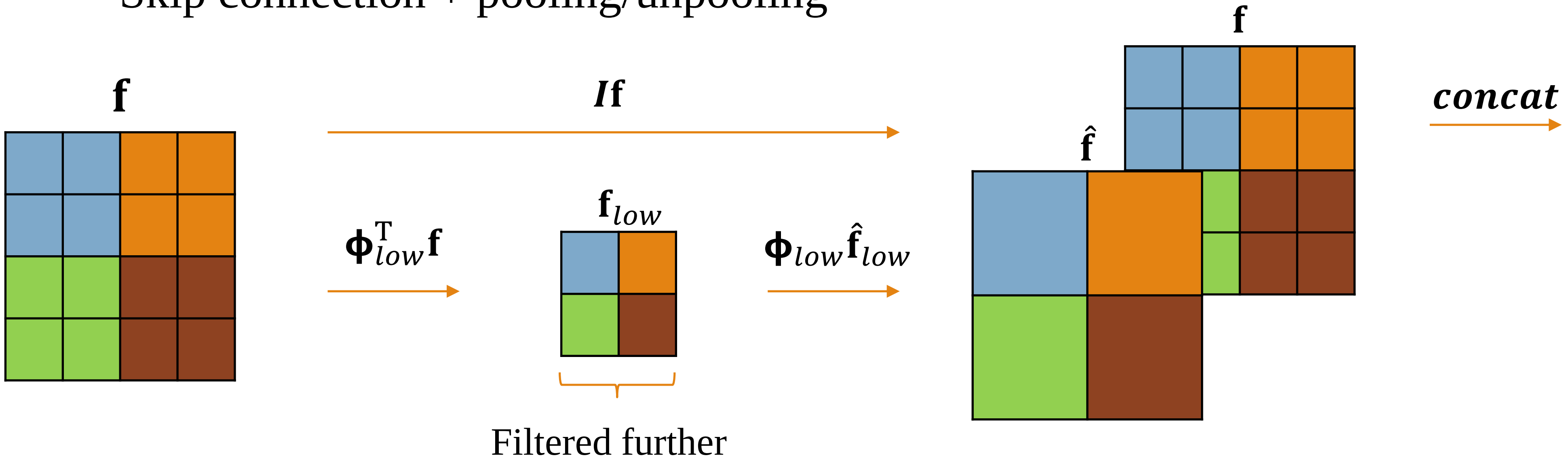
Basis: $nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

Learned basis: $\boldsymbol{\varphi}$

Fixed basis: $\boldsymbol{\phi}$

4. Framelet expansion in CNNs

- Skip connection + pooling/unpooling



$Frame\text{: } \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator\text{: } \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\text{: } \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\text{: } \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\text{: } \alpha = \beta$

$Bounds\text{: } \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\text{: } \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\text{: } \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

$CN\text{: } \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\text{: } \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

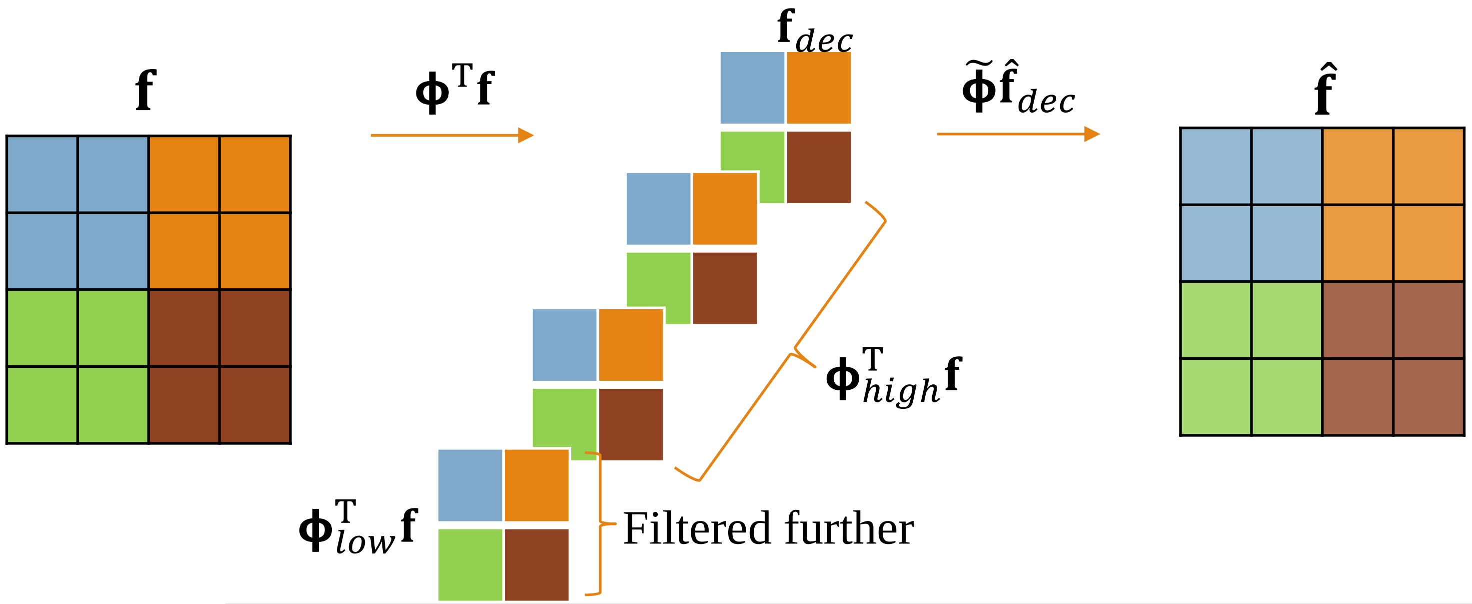
$Basis\text{: } nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis\text{: } \boldsymbol{\varphi}$

$Fixed\ basis\text{: } \boldsymbol{\phi}$

4. Framelet expansion in CNNs

- Generalized pooling/unpooling (framelet expansion with Haar filters)



Frame: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

Frame operator: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

Analysis: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

Synthesis: $\mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

Tight frame: $\alpha = \beta$

Bounds: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

Frame condition: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

Dual frame: $\tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

CN: $\kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

Hankel: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

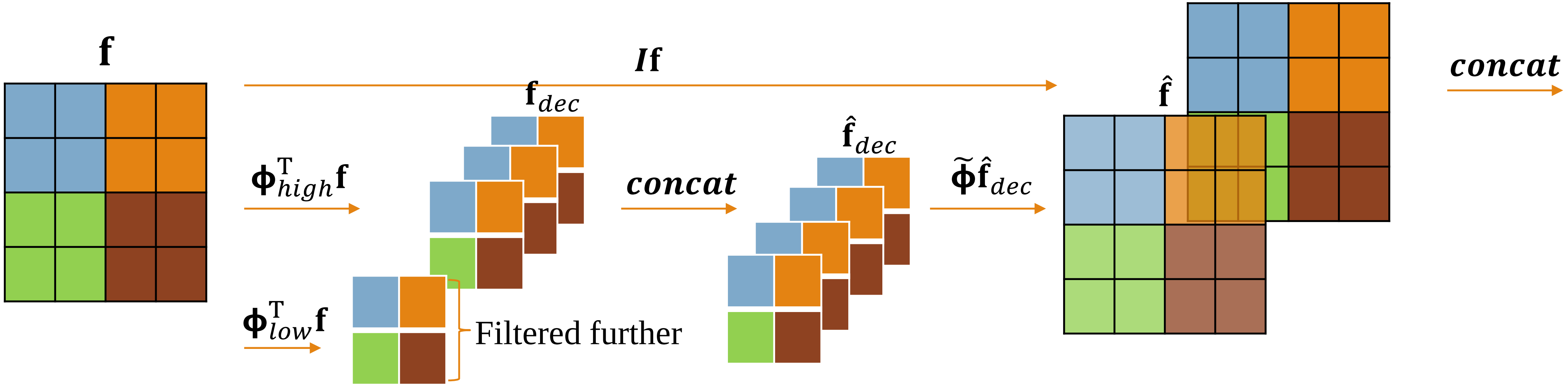
Basis: $nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

Learned basis: $\boldsymbol{\varphi}$

Fixed basis: $\boldsymbol{\phi}$

4. Framelet expansion in CNNs

- Skip connection with generalized pooling/unpooling



Frame: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

Frame operator: $\tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

Analysis: $\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

Synthesis: $\mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

Tight frame: $\alpha = \beta$

Bounds: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

Frame condition: $\tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

Dual frame: $\tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

CN: $\kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

Hankel: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

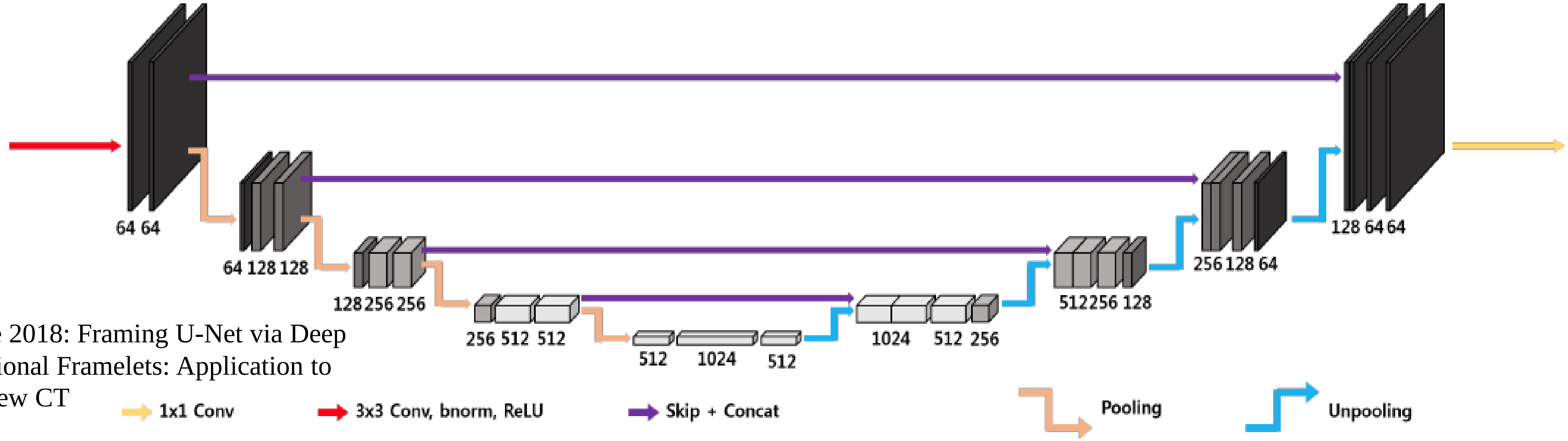
Basis: *nonlocal* $\{\boldsymbol{\phi}_i\}_{i=1}^{d}$, *local* $\{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

Learned basis: $\boldsymbol{\varphi}$

Fixed basis: $\boldsymbol{\Phi}$

4. Framelet expansion in CNNs

- UNET architecture with pooling



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$Frame: \ \alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x} \rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis: \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis: \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame: \ \alpha = \beta$

$Bounds: \ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition: \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame: \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

$CN: \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel: \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

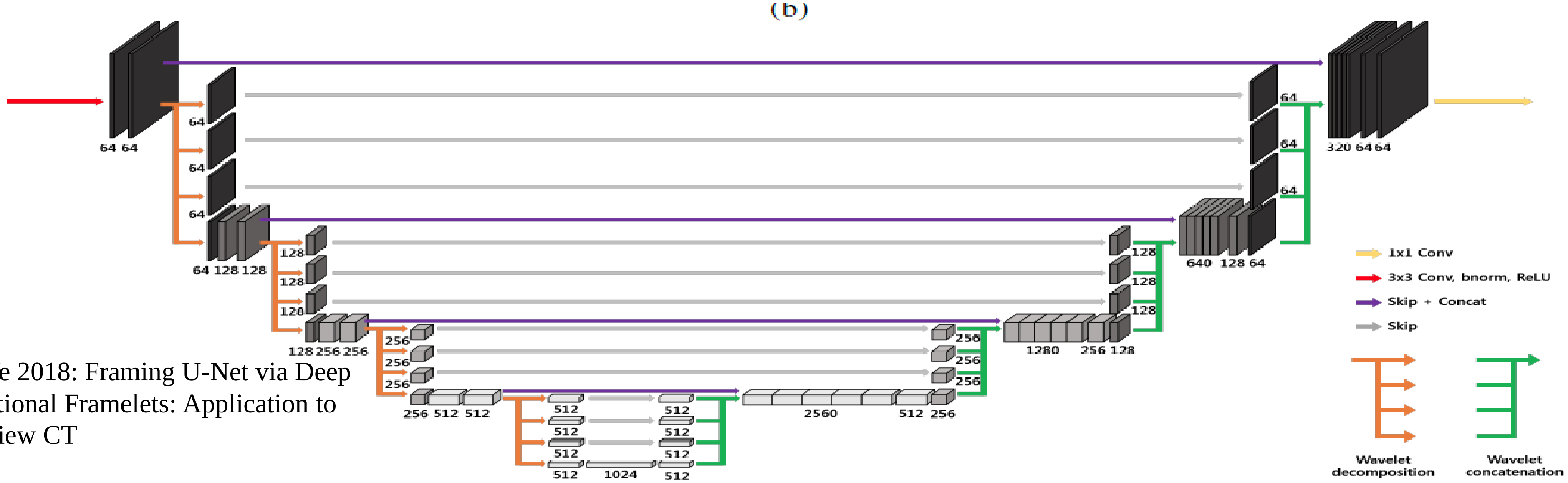
$Basis: nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis: \boldsymbol{\varphi}$

$Fixed\ basis: \boldsymbol{\phi}$

4. Framelet expansion in CNNs

- UNET architecture with generalized pooling



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$Frame\colon\ \alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis\colon \boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis\colon \mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

$Tight\ frame\colon\ \alpha = \beta$

$Bounds\colon\ \alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition\colon \tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame\colon \tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

$CN\colon \kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel\colon \boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis\colon nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

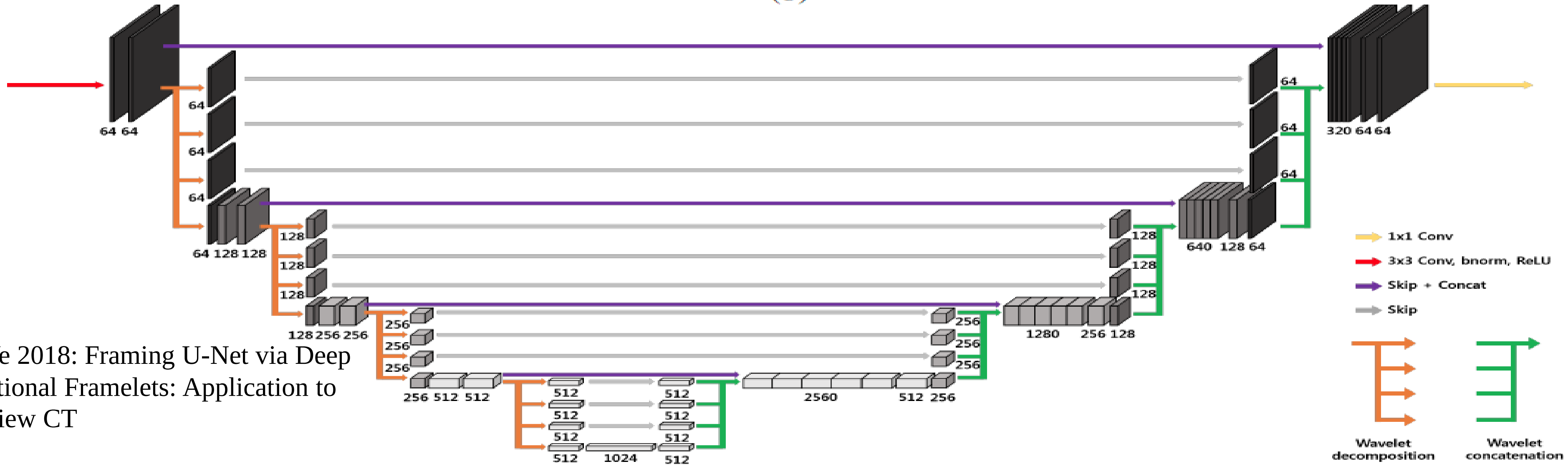
$Learned\ basis\colon \boldsymbol{\varphi}$

$Fixed\ basis\colon \boldsymbol{\phi}$

4. Framelet expansion in CNNs

- Why only the low-rank approximation further into the next level?

(b)



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$Frame$: $\alpha\|\mathbf{x}\|^2 \leq \langle \mathbf{Fx}, \mathbf{x}\rangle \leq \beta\|\mathbf{x}\|^2$

$Frame\ operator$: $\tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis$: $\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis$: $\mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame$: $\alpha = \beta$

$Bounds$: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition$: $\tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame$: $\tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

CN: $\kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel$: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

$Basis$: $nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

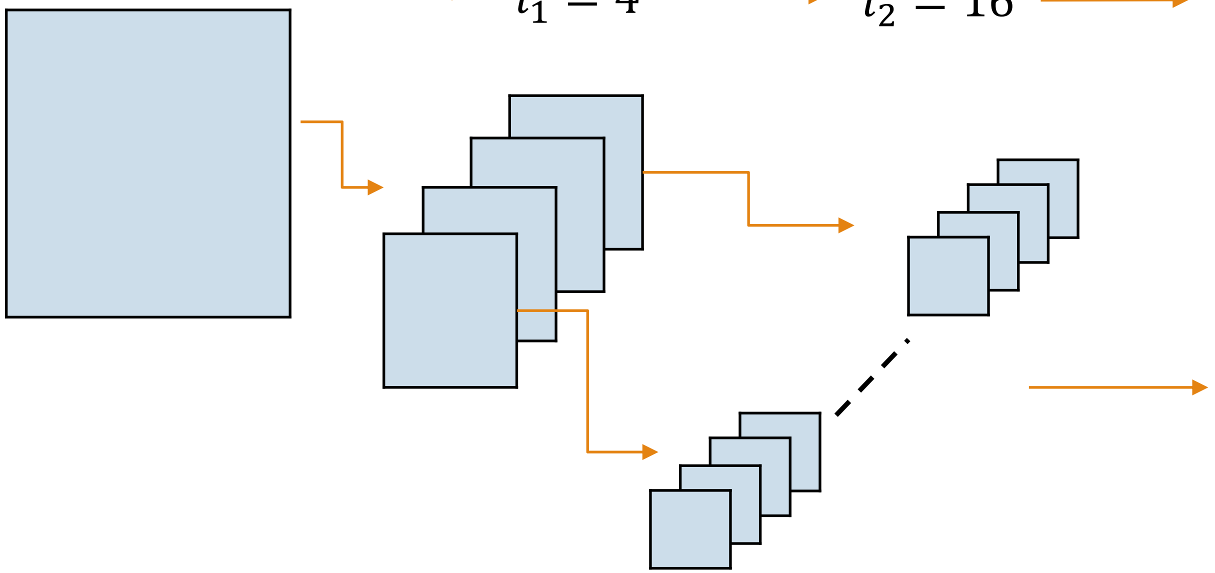
$Learned\ basis$: $\boldsymbol{\varphi}$

$Fixed\ basis$: $\boldsymbol{\Phi}$

4. Framelet expansion in CNNs

- Exponential increase in number of features, i.e. number of filters also increase exponentially.

$l_1 = 4 \longrightarrow l_2 = 16 \longrightarrow l_N = 4^N$



Frame: $\alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

Frame operator: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \mathbf{F}$

Analysis: $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

Synthesis: $\mathbf{x} = \tilde{\boldsymbol{\phi}}\mathbf{c}$

Tight frame: $\alpha = \beta$

Bounds: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

Frame condition: $\tilde{\boldsymbol{\phi}}\boldsymbol{\phi}^{\mathrm{T}} = \boldsymbol{I}$

Dual frame: $\tilde{\boldsymbol{\phi}} = \left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\phi}$

CN: $\kappa\left(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\right) = \beta/\alpha$

Hankel: $\mathcal{H}_{\mathrm{d}}(\mathbf{x})$

Basis: *nonlocal* $\{\boldsymbol{\phi}_i\}_{i=1}^{d}$, *local* $\{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

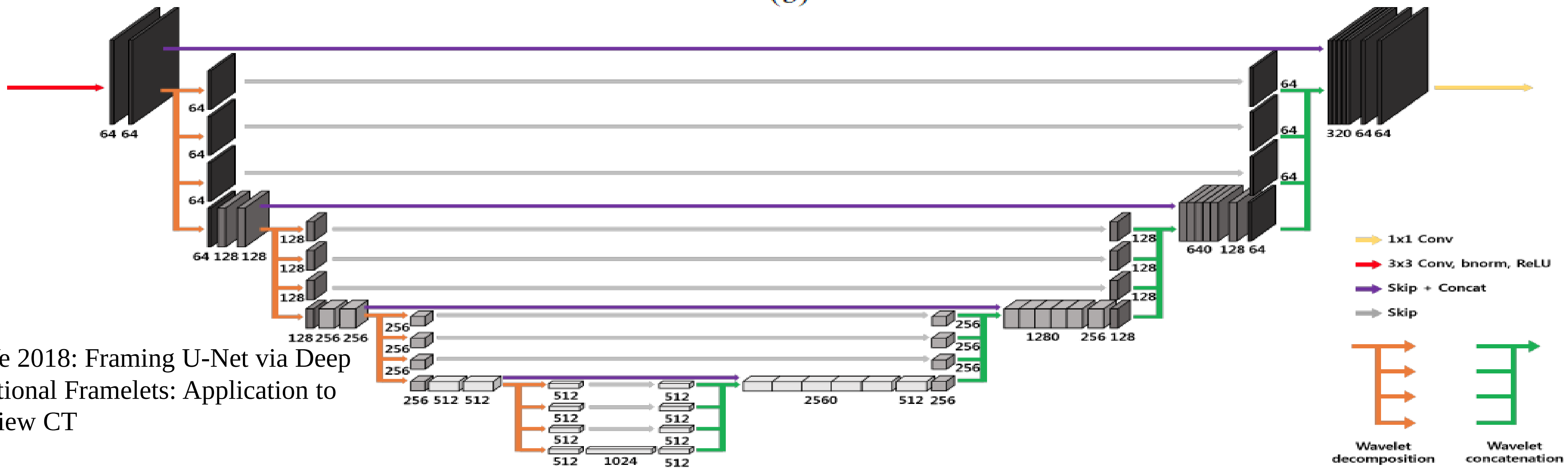
Learned basis: $\boldsymbol{\varphi}$

Fixed basis: $\boldsymbol{\phi}$

4. Framelet expansion in CNNs

- Why only the low-rank approximation further into the next level? <u>Computationally infeasible</u>.

(b)



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$Frame$: $\alpha\|\mathbf{x}\|^2 \le \langle \mathbf{Fx}, \mathbf{x}\rangle \le \beta\|\mathbf{x}\|^2$

$Frame\ operator$: $\tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{F}$

$Analysis$: $\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{x} = \mathbf{c}$

$Synthesis$: $\mathbf{x} = \tilde{\boldsymbol{\Phi}}\mathbf{c}$

$Tight\ frame$: $\alpha = \beta$

$Bounds$: $\alpha = \sigma_{\min}^2, \beta = \sigma_{\max}^2$

$Frame\ condition$: $\tilde{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$

$Dual\ frame$: $\tilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right)^{-1}\boldsymbol{\Phi}$

CN: $\kappa\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\right) = \beta/\alpha$

$Hankel$: $\boldsymbol{\mathcal{H}}_{\mathrm{d}}(\mathbf{x})$

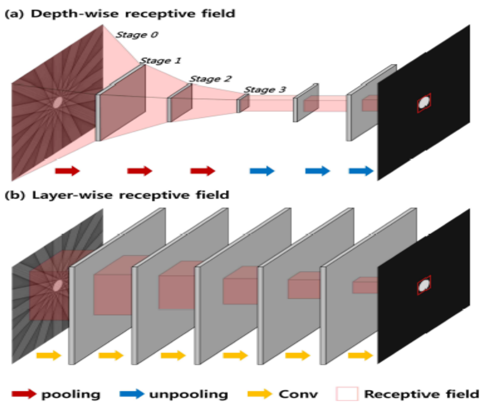
$Basis$: $nonlocal\{\boldsymbol{\phi}_i\}_{i=1}^{d}, local\ \{\boldsymbol{\varphi}_i\}_{i=1}^{n}$

$Learned\ basis$: $\boldsymbol{\varphi}$

$Fixed\ basis$: $\boldsymbol{\Phi}$

5. Summary

- Pooling of features in CNNs can be seen as computing a low-rank approximation of the input.
- Pooling is important due to the increase in the receptive field which increases the relative size of the filters.
- Introducing skip connections with pooling allows for learning a low-rank approximation in multiple levels and at the same time retaining high frequencies.
- As seen, skip connection with pooling overly emphasizes the low-frequency component.
- In the theory of framelet expansion, skip connections are combined with generalized pooling which allows for both learning an optimal low-rank approximation in multiple levels and at the same time meets the perfect reconstruction condition.
- Multiple research opportunities - designing good frames and architectures for data processing and inversion.



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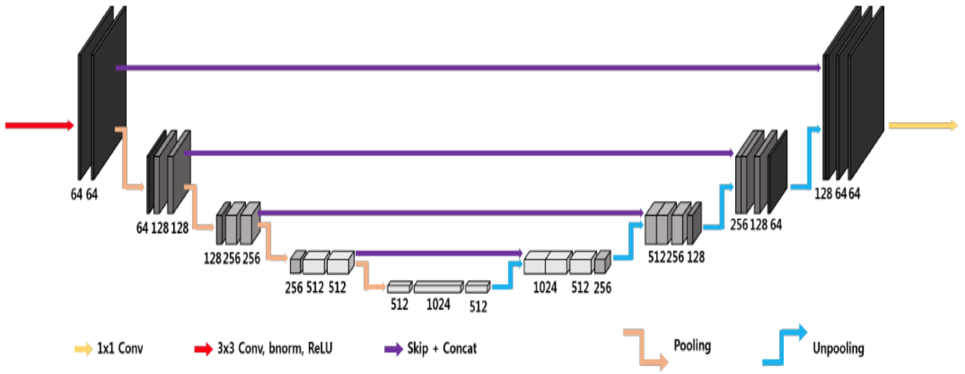
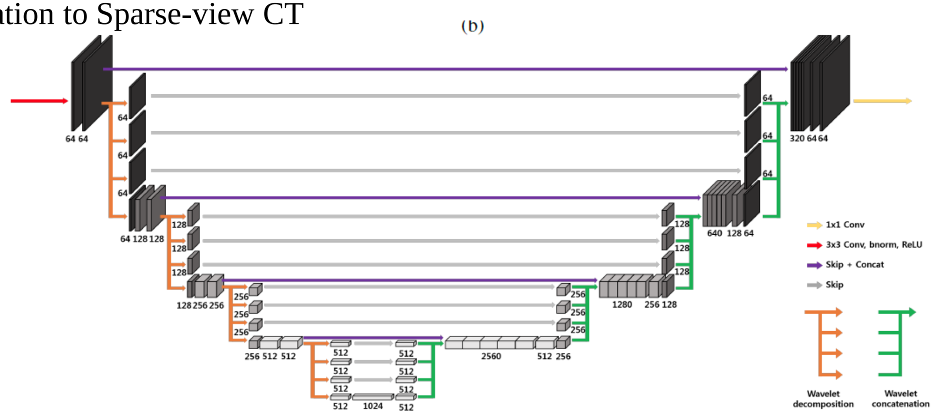
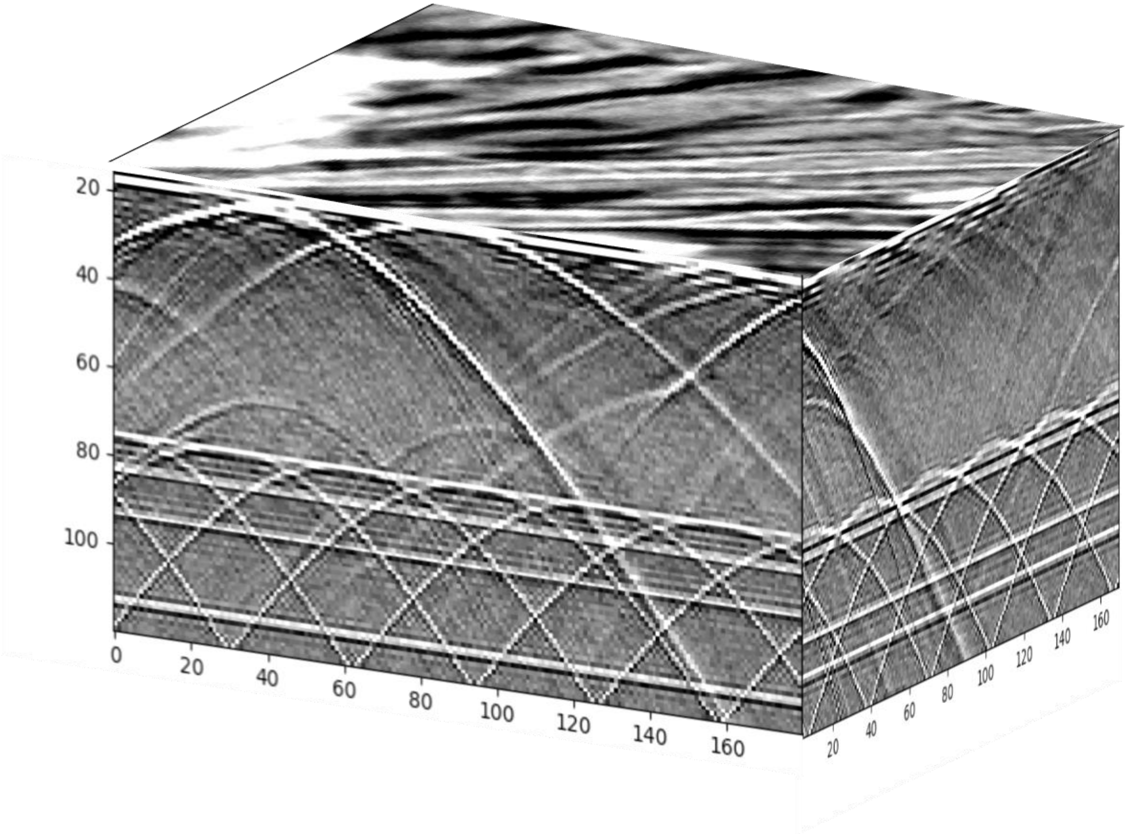




Extra - Denoising example in 3D

- Synthetic seismic data cube of size 180x120x180 (xline-6.25m, time-2ms, inline-6.25m)
- 3D CNN.
- 3D Haar like frames.



Extra - Denoising example in 3D

- Synthetic seismic data cube of size 180x120x180 (xline, time, inline)
- 3D CNN.
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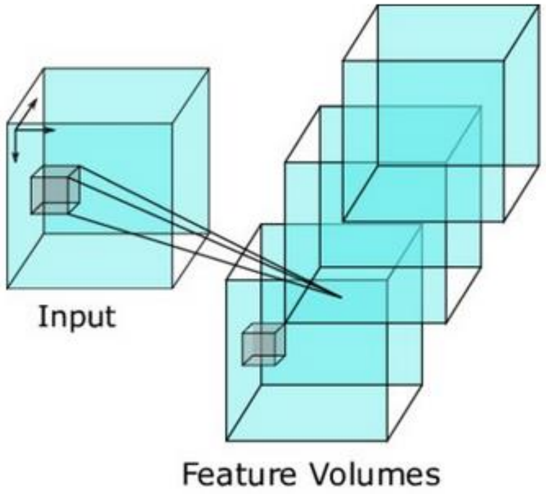

https://www.researchgate.net/publication/330912338_ECNN_Activity_Recognition_Using_Ensembled_Convolutional_Neural_Networks/figures?lo=1

Extra - Denoising example in 3D

- Synthetic seismic data cube of size 180x120x180 (xline, time, inline)
- 3D CNN.
- 3D Haar like frames.

$$\boldsymbol{\Phi}^{\mathrm{T}} = \frac{1}{\sqrt{8}} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}}_{\{\boldsymbol{\phi}_i\}_{i=1}^{8} \in \mathbb{R}^{2\times2\times2} \quad \{\tilde{\boldsymbol{\phi}}_i\}_{i=1}^{8} \in \mathbb{R}^{2\times2\times2}}$$

Extra - Denoising example in 3D

- Synthetic seismic data cube of size 180x120x180 (xline, time, inline)
- 3D CNN.
- 3D Haar like frames.
- Strided convolutions for analysis.
- Strided transposed convolutions for synthesis.

```
def wave_dec(x,analysis):

    shape = [i.value for i in x.get_shape()]
    num_features = shape[4]


    for i in range(num_features):

        xx = tf.nn.conv3d(x[:,:,:,:,i:(i+1)],analysis,strides=[1,2,2,2,1],padding='SAME')

        if i==0:
            low_pass  = xx[:,:,:,:,0:1]
            high_pass = xx[:,:,:,:,1:]

        else:
            low_pass  = tf.concat((low_pass,xx[:,:,:,:,0:1]),axis=4)
            high_pass = tf.concat((high_pass,xx[:,:,:,:,1:]),axis=4)

    return low_pass, high_pass


def wave_rec(x_low,x_high,synthesis):

    shape = [i.value for i in x_low.get_shape()]
    num_low_pass = shape[4]

    x_concat = []
    for i in range(num_low_pass):
        x = tf.concat((x_low[:,:,:,:,i:(i+1)],x_high[:,:,:,:,i*7:((i*7)+7)]),axis=4)
        x = _upsampling(x,synthesis,num_ch=1,stride=2)
        x_concat.append(x)
    x_concat = tf.concat(x_concat,axis=4)

    return x_concat
```

Extra - Denoising example in 3D

- Synthetic seismic data cube of size 180x120x180 (xline, time, inline)
- 3D CNN.
- 3D Haar like frames.
- Strided convolutions for analysis.
- Strided transposed convolutions for synthesis.
- Objective function $O = \frac{1}{2}\|\boldsymbol{d} - \widehat{\boldsymbol{d}}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{W}\|_2^2$

$$\boldsymbol{d} = \boldsymbol{d}_{truth} + \boldsymbol{e}, \boldsymbol{e} \sim \boldsymbol{N}(\mathbf{0}, \mathbf{0.01})$$

```
def wave_dec(x,analysis):

    shape = [i.value for i in x.get_shape()]
    num_features = shape[4]


    for i in range(num_features):

        xx = tf.nn.conv3d(x[:,:,:,:,i:(i+1)],analysis,strides=[1,2,2,2,1],padding='SAME')

        if i==0:
            low_pass  = xx[:,:,:,:,0:1]
            high_pass = xx[:,:,:,:,1:]

        else:
            low_pass  = tf.concat((low_pass,xx[:,:,:,:,0:1]),axis=4)
            high_pass = tf.concat((high_pass,xx[:,:,:,:,1:]),axis=4)

    return low_pass, high_pass


def wave_rec(x_low,x_high,synthesis):

    shape = [i.value for i in x_low.get_shape()]
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    x_concat = []
    for i in range(num_low_pass):
        x = tf.concat((x_low[:,:,:,:,i:(i+1)],x_high[:,:,:,:,i*7:((i*7)+7)]),axis=4)
        x = _upsampling(x,synthesis,num_ch=1,stride=2)
        x_concat.append(x)
    x_concat = tf.concat(x_concat,axis=4)

    return x_concat
```

Extra - Denoising example in 3D

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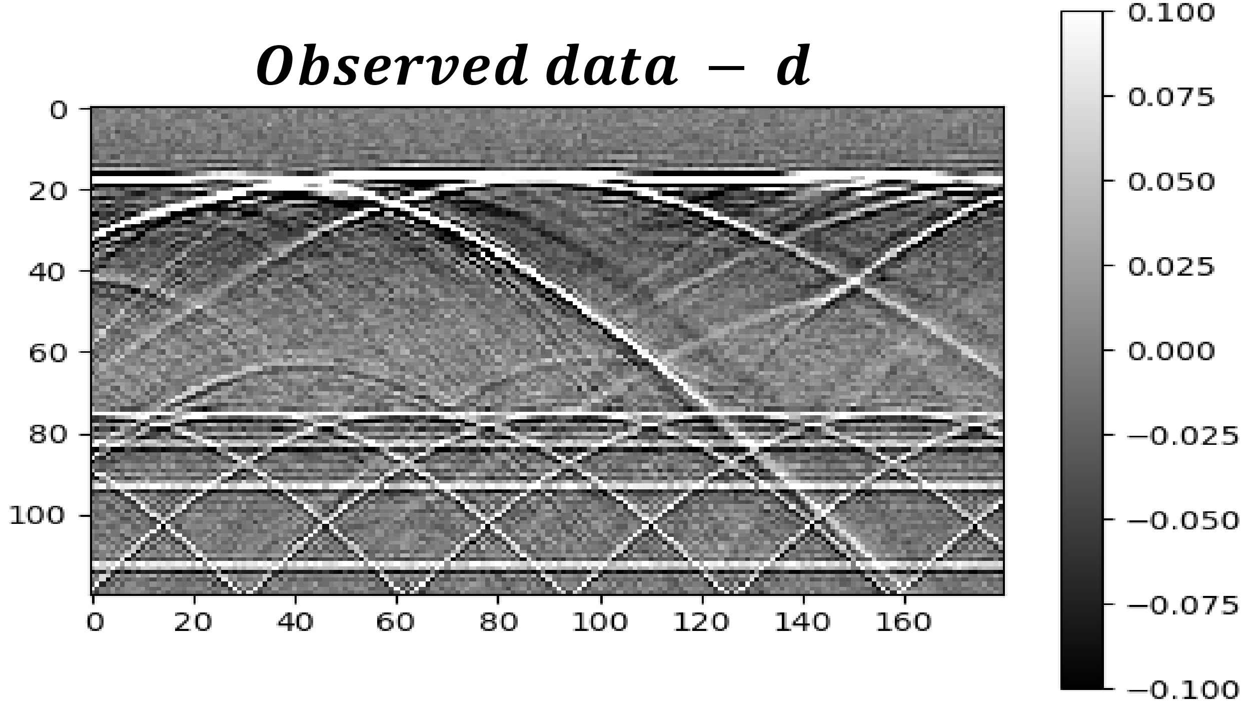
$$\boldsymbol{d} = \boldsymbol{d}_{truth} + \boldsymbol{e}, \boldsymbol{e} \sim \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{0.01})$$

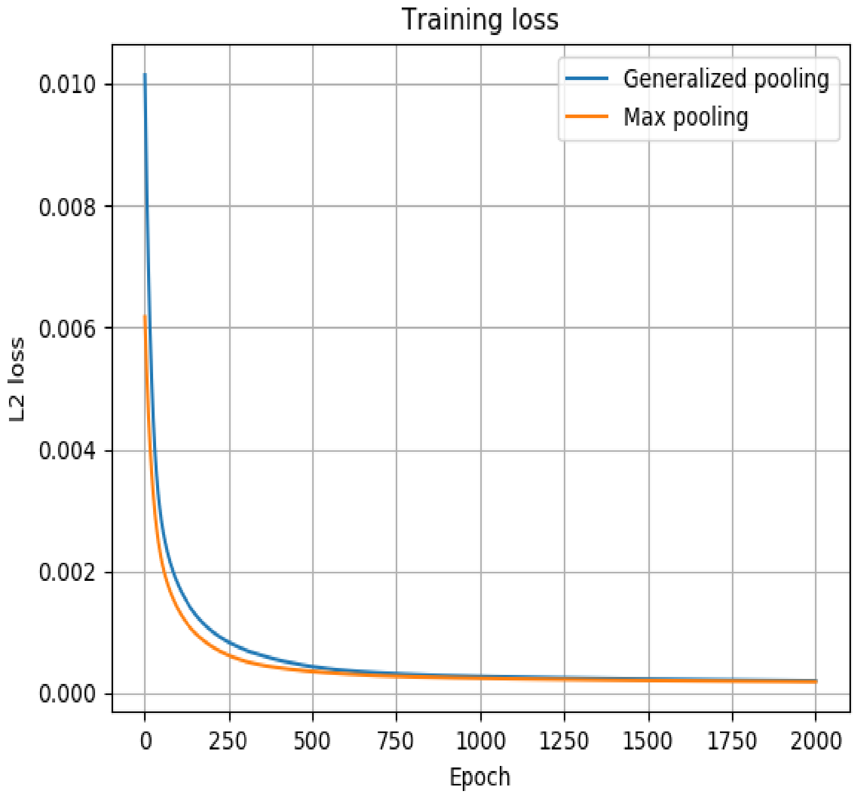
Architecture

```
[5,5,5,1,8],  # regular conv
pooling/generalized pooling
[3,3,3,8,16], # regular conv
[3,3,3,16,8], # regular conv
unpooling/generalized unpooling
[3,3,3,16,8], # concatenate conv
[3,3,3,8,1],  # reconstruction
```

Extra - Denoising example in 3D

- Setup: epochs – 2000 , learning rate – 0.0001, λ – 0.001.
- Computation time 422 sec for pooling/unpooling, 495 sec for generalized pooling





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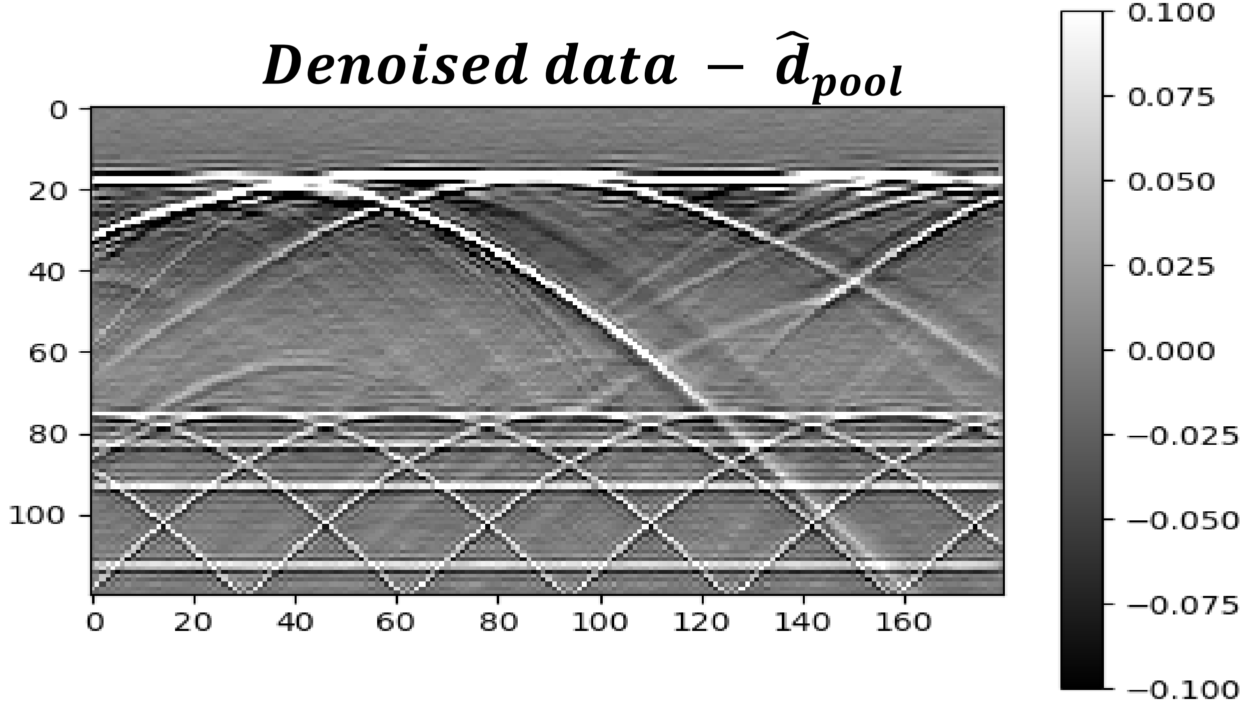

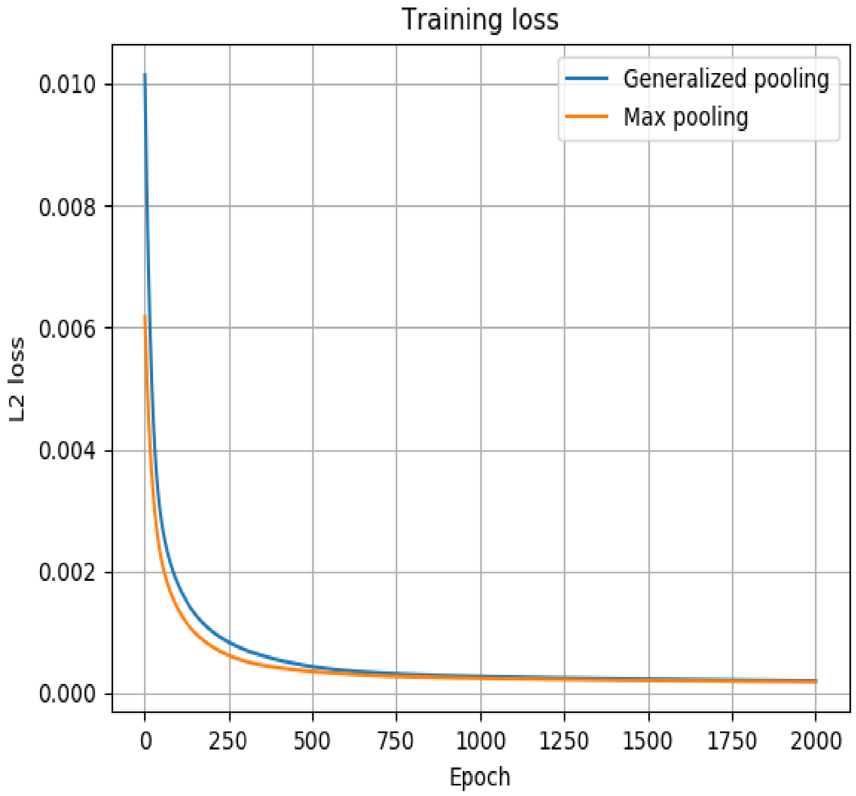

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