University of Oslo

MAT-INF3360

Introduction to Partial Differential Equations

Mandatory exercise 2

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Contents

1 (a)

Considering the problem

$$u_t = \epsilon u_{xx} + u \tag{1}$$

$$u(0,t) = u(1,t) = 0, u(x,0) = f(x) \tag{2}$$

where $f(x) = \sin(\pi x)$

First we use seperation of variables to find the solution $u(x,t)$

$$u = X(x)T(t), u_{xx} = \epsilon X''(x)T(t), u_t = X(x)T'(t) \tag{3}$$

Inserting the equations in (3) into (1) we get

$$\frac{T'(t)}{T(t)} = \frac{\epsilon X''(x) + X(x)}{X(x)} = -\lambda \tag{4}$$

where λ is a common constant (or eigenvalues). We see that (4) can separated into two ODE's as follows

$$T'(t) + \lambda T(t) = 0, X''(x) + \left(\frac{1+\lambda}{\epsilon}\right)X(x) = 0 \tag{5}$$

The general solution to second equation in (5) is given as

$$X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x) \tag{6}$$

It follows from the boundary conditions in (2) that

$$X(0) = X(1) = 0 \tag{7}$$

Let

$$\beta = \sqrt{\frac{1+\lambda}{\epsilon}} \tag{8}$$

$$X(0) = c_1 = 0 \tag{9}$$

and

$$X(1) = \sin(\beta) = 0 \tag{10}$$

which is satisfied by

$$\beta_k = k\pi, k = 1, 2, \ldots \tag{11}$$

From (8), the eigenvalues are therefore given as

$$\lambda_k = (k\pi)^2\epsilon - 1 \tag{12}$$

Considering the first equation in (5), which has the general solution

$$T(t) = e^{-\lambda_k t} = e^{(1-(k\pi)^2\epsilon)t} \tag{13}$$

It follows from (3), (6) and (13) that solution to (1) is given by

$$u(x,t) = e^{(1-(k\pi)^2\epsilon)t}\sin(k\pi x) \tag{14}$$

From (14) we see that $u(x,t) \to 0$ as $t \to \infty$ if $(1-(k\pi)^2\epsilon) < 0$ for $k \geq 1$. For $k = 1$ we get

$$1 < \pi^2\epsilon \implies \epsilon > \frac{1}{\pi^2} \tag{15}$$

2 (b)

Let

$$f(x) = \sum_{k=1}^{N} c_k \sin(k\pi x) \tag{16}$$

where c_k are the Fourier coefficients. The Fourier coefficients are given by

$$c_k = 2\int_0^1 f(x)\sin(m\pi x)dx = 2\int_0^1 \sin(k\pi x)\sin(m\pi x)dx \tag{17}$$

From the inner product in **lemma 2.9**

$$< \sin(k\pi x), \sin(m\pi x) >= \begin{cases} 0 & \text{if } k \neq \text{m} \\ 1/2 & \text{if } k = \text{m} \end{cases}$$

Which gives

$$c_k = 1 \tag{18}$$

For all k. The solution to the problem in (1) can be written as

$$u(x,t) = \sum_{k=1}^{N} c_k e^{(1-(k\pi)^2\epsilon)t}\sin(k\pi x) = \sum_{k=1}^{N} e^{(1-(k\pi)^2\epsilon)t}\sin(k\pi x) \tag{19}$$

From this, the same argument as derived in (a) can be stated. We see from the derivation of $u(x,t)$ using separation of variables that (13) does not depend on the initial function $f(x)$, so it is reasonable to expect this to be true for a more general function $f(x)$.

3 (c)

To show that condition (4) and (5) implies stability, we start by considering the analytical solution. Consider

$$u_t = \epsilon u_{xx} + u \tag{20}$$

We insert a particular solution on the form

$$u_k(x,t) = T_k(t)e^{ik\pi x} \tag{21}$$

Where i is the complex number $\sqrt{-1}$. By inserting (21) into (20) we get

$$T_k'(t)e^{ik\pi x} = -\epsilon T(t)(k\pi)^2 e^{ik\pi x} + T_k(t)e^{ik\pi x} \tag{22}$$

which gives

$$T(t) = e^{(1-(k\pi)^2\epsilon)t} = e^t e^{-(k\pi)^2\epsilon t} \tag{23}$$

For all k we have

$$|T(t)| \leq e^t \tag{24}$$

Now we use the numerical approximation on the form

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \epsilon \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{(\Delta x)^2} + v_j^m \tag{25}$$

where we insert a particular solution on the form

$$v_j^m = (a_k)^m e^{ik\pi x_j} \tag{26}$$

By inserting (26) into (25) we get

$$\frac{(a_k)^{m+1} - (a_k)^m}{\Delta t} e^{ik\pi x_j} = \epsilon \frac{e^{ik\pi x_{j-1}} - 2e^{ik\pi x_j} + e^{ik\pi x_{j+1}}}{(\Delta x)^2} (a_k)^m + (a_k)^m e^{ik\pi x_j} \tag{27}$$

By using $x_j = j\Delta x$ and $(a_k)^{m+1} = (a_k)^m a_k$ we get

$$\frac{a_k - 1}{\Delta t} = \epsilon \frac{e^{-ik\pi\Delta x} - 2 + e^{ik\pi\Delta x}}{(\Delta x)^2} + 1 \tag{28}$$

which gives for a_k

$$a_k = 1 + \Delta t - \frac{\epsilon 4\Delta t}{(\Delta x)^2} \sin^2(\frac{k\pi\Delta x}{2}) \tag{29}$$

Here, the relations $e^{-ik\pi\Delta x} = \cos(k\pi\Delta x) - i\sin(k\pi\Delta x)$ and $\cos(k\pi\Delta x) - 1 = -2\sin(k\pi\Delta x/2)$ has been used to go from (28) to (29). Based on the analytical solution we require that the numerical approximation satisfies

$$|(a_k)^m| \leq e^{t_m} \tag{30}$$

From (29) we get

$$|(a_k)^m| \leq |a_k|^m \iff |(a_k)^m| \leq \left(1 + \Delta t - \frac{\epsilon 4\Delta t}{(\Delta x)^2} \sin^2(\frac{k\pi\Delta x}{2})\right)^m \leq (1 + \Delta t)^m = e^{t_m} \tag{31}$$

which is satisfied if

$$|1 - \frac{\epsilon 4\Delta t}{(\Delta x)^2} \sin^2\left(\frac{k\pi\Delta x}{2}\right)| \leq 1 \tag{32}$$

which implies

$$\frac{\epsilon\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \tag{33}$$

since $0 \leq 4\sin^2\left(\frac{k\pi\Delta x}{2}\right) \leq 4$. Since $T(t)$ given in (24) satisfies

$$|T(t)| \leq 1 \tag{34}$$

For all $k \geq 1$ for $\epsilon > \frac{1}{\pi^2}$ we require the numerical approximation to satisfy

$$|(a_k)^m| \leq 1 \tag{35}$$

for all $k \geq 1$. For $k = 1$ we get

$$\frac{\epsilon 4\Delta t}{(\Delta x)^2} \sin^2(\frac{\pi\Delta x}{2}) \geq \Delta t \tag{36}$$

and

$$\epsilon \geq \frac{(\Delta x)^2}{4\sin^2((\pi\Delta x)/2)} \tag{37}$$

Taylor expansion on $\sin((\pi\Delta x)/2)$ gives

$$\epsilon \geq \frac{(\Delta x)^2}{4(\frac{(\pi\Delta x)}{2} - \frac{(\pi\Delta x)^3}{48})^2} \tag{38}$$

From (38) we see that we can multiply out $\pi\Delta x$ of each bracket and multiply 2 in to the brackets. From this we get

$$\epsilon \geq \frac{1}{\pi^2(1-\frac{(\pi\Delta x)^2}{24})^2} \tag{39}$$

4 (d)

Considering

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \epsilon \frac{v_{j-1}^{m+1} - 2v_j^{m+1} + v_{j+1}^{m+1}}{(\Delta x)^2} + v_j^{m+1} \tag{40}$$

By inserting (26) into (40) we get

$$\frac{(a_k)^m(a_k) - (a_k)^m}{\Delta t} e^{ik\pi x_j} = \epsilon \frac{e^{ik\pi x_{j-1}} - 2e^{ik\pi x_j} + e^{ik\pi x_{j+1}}}{(\Delta x)^2}(a_k)^m(a_k) + (a_k)^m(a_k)e^{ik\pi x_j} \tag{41}$$

Let $r = \Delta t/(\Delta x)^2$ and $x_j = j\Delta x$ we get

$$(a_k) - 1 = \epsilon r(e^{-ik\pi\Delta x} - 2 + e^{ik\pi\Delta x})(a_k) + \Delta t(a_k) \tag{42}$$

$$(a_k)(1 - \epsilon r(e^{-ik\pi\Delta x} - 2 + e^{ik\pi\Delta x}) - \Delta t) = 1 \tag{43}$$

which gives

$$a_k = \frac{1}{1 - \epsilon r 4\sin^2(k\pi\Delta x/2) - \Delta t} \tag{44}$$

The result from the numerical approximation in (44) should also satisfy (35). From this we get

$$1 \geq \frac{1}{1 - \epsilon r 4\sin^2(k\pi\Delta x/2) - \Delta t} \implies \epsilon r 4\sin^2(k\pi\Delta x/2) \geq \Delta t \tag{45}$$

which we observe is the same as in (37). This shows that condition 5 implies stability for this scheme.

5 (e)

Considering

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \epsilon \frac{v_{j-1}^{m+1} - 2v_j^{m+1} + v_{j+1}^{m+1}}{(\Delta x)^2} + v_j^{m+1} \tag{46}$$

we start with writing the first part of the right side of the equation using the matrix A given in (4.40) in the book .This gives

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \epsilon A v^{m+1} + v_j^{m+1} \tag{47}$$

where v^{m+1} is the vector $v^{m+1} = (v_1^{m+1}, v_2^{m+1}, ..., v_n^{m+1})^T$. We can write v_j^{m+1} as follows

$$v_j^{m+1} = I v^{m+1} = v^{m+1} \tag{48}$$

which gives us

$$I v^{m+1} - v^m = \epsilon \Delta t A v^{m+1} + \Delta t I v^{m+1} \tag{49}$$

Rearrange to

$$I v^{m+1} - \epsilon \Delta t A v^{m+1} - \Delta t I v^{m+1} = v^m \tag{50}$$

which gives

$$\big((1-\Delta t)I - \epsilon \Delta t A\big)v^{m+1} = v^m \tag{51}$$

6 (f)

```
%MAT-INF3360 Oblig 2 Implementation of explicit finite difference scheme
%Initial parameters
clear all,close all; clc
dt = 0.000125;
dx = 0.05;
r = dt/(dx)^2;
eps = 0.5;
%Initialization function
xmax = 1;
x = 0:dx:(xmax-dx*xmax);
f = zeros(1,length(x));
for i = 1:length(x)
        if x(i) <= 1/2
            f(i) = 2*x(i);
        else if x(i) > 1/2
            f(i) = 2*(1 - x(i));
            end
        end
end
plot(x,f)
%Time vector
T = 0:dt:1;
N = length(T)-1;
n = length(f);
v = zeros(N,n);
% Initialization and boundary conditions
v(1,:) = f;
v(:,1) = 0;
v(:,end) = 0;

%scheme
%v(i+1,j) = eps*dt/dx^2*v(i,j-1) + eps*dt/dx^2*v(i,j+1) + (1 + dt - 2*eps*dt/dx^2)*v(i,j)
C1 = eps*r;
C2 = (1 + dt - 2*eps*r);
for i = 1:N
    for j = 2:n-1
    v(i+1,j) = C1*v(i,j-1) + C1*v(i,j+1) + C2*v(i,j);
    end
end
```

Figure 1: Explicit Finite Difference Shceme

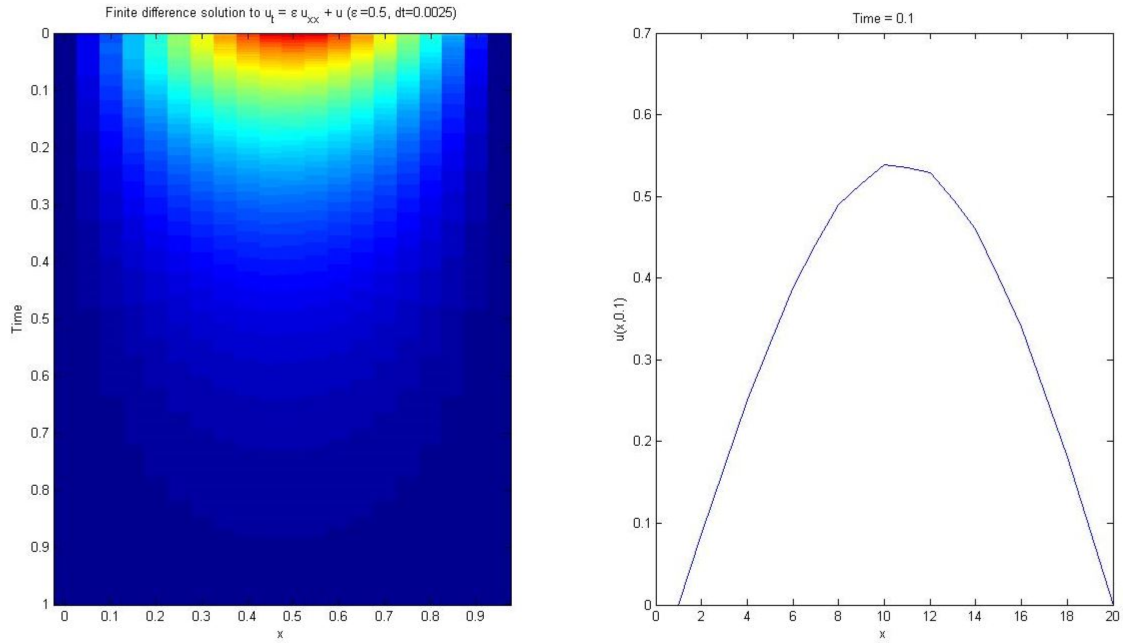

Figure 2: Result from the implementation of the explicit finite difference scheme. In this case stability criterion (4) is met with a $\Delta t = 0.0025$.

We see that the solution $u(x, t)$ gets smoother by decreasing the time step Δt, as shown in Figure 2 where criterion (4) is met exact compared to Figure 3 where the time step is much lower. We also see that the solution is critically unstable when criterion (4) is not met, as seen in Figure 4, where the Δt was chosen to give a number to not meet the criterion.

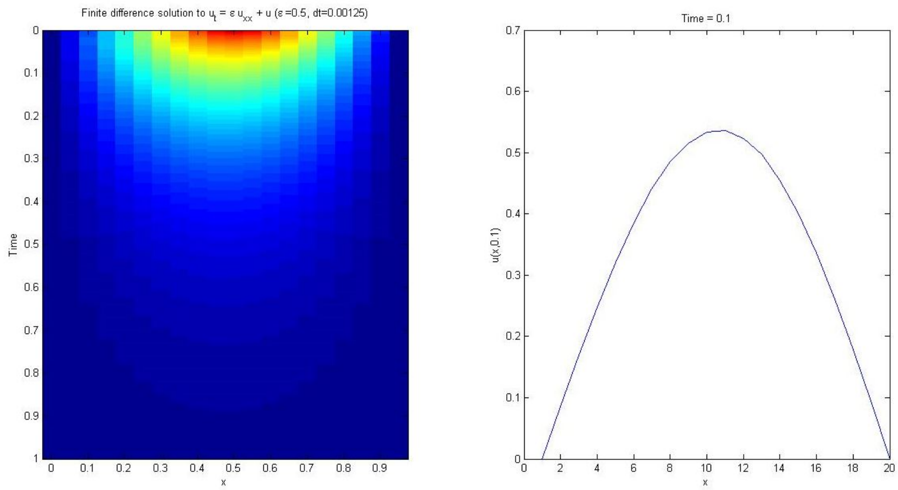

Figure 3: Result from the implementation of the explicit finite difference scheme. In this case stability criterion (4) is met with a $\Delta t = 0.00125$.

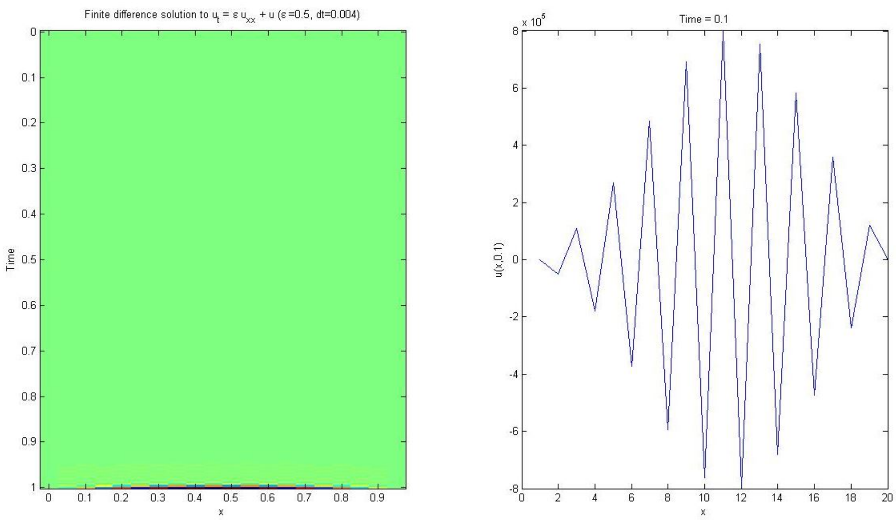

Figure 4: Result from the implementation of the explicit finite difference scheme. In this case stability criterion (4) is not met with a $\Delta t = 0.004$.

7 (g)

```
%MAT-INF3360 Oblig 2 Implementation of implicit finite difference scheme
%Initial parameters
clear all,close all; clc
dt = 0.0025;
dx = 0.05;
r = dt/(dx)^2;
eps = 0.05;


%Initialization function
xmax = 1;
x = 0:dx:(xmax-dx*xmax);
f = zeros(1,length(x));
for i = 1:length(x)
          if x(i) <= 1/2
               f(i) = 2*x(i);
          else if x(i) > 1/2
               f(i) = 2*(1 - x(i));
               end
          end
end
plot(x,f)
%Time vector
T = 0:dt:1;
N = length(T)-1;
n = length(f);
v = zeros(N,n);
% Initialiation and initial conditions
v(1,:) = f;
v(:,1) = 0;

v(:,end) = 0;
%Implicit scheme v^m+1 = B^-1v^m
I = eye(n,n);
toep = [2 -1 zeros(1,n-2)];
A = (1/dx^2)*toeplitz(toep);
Bin = inv(((1 - dt)*I + eps*dt*A));

for i = 1:N;
    v(i+1,:) = Bin*v(i,:)';
    v(:,1) = 0;
    v(:,end) = 0;
end
```

Figure 5: Implicit Finite Difference Shceme

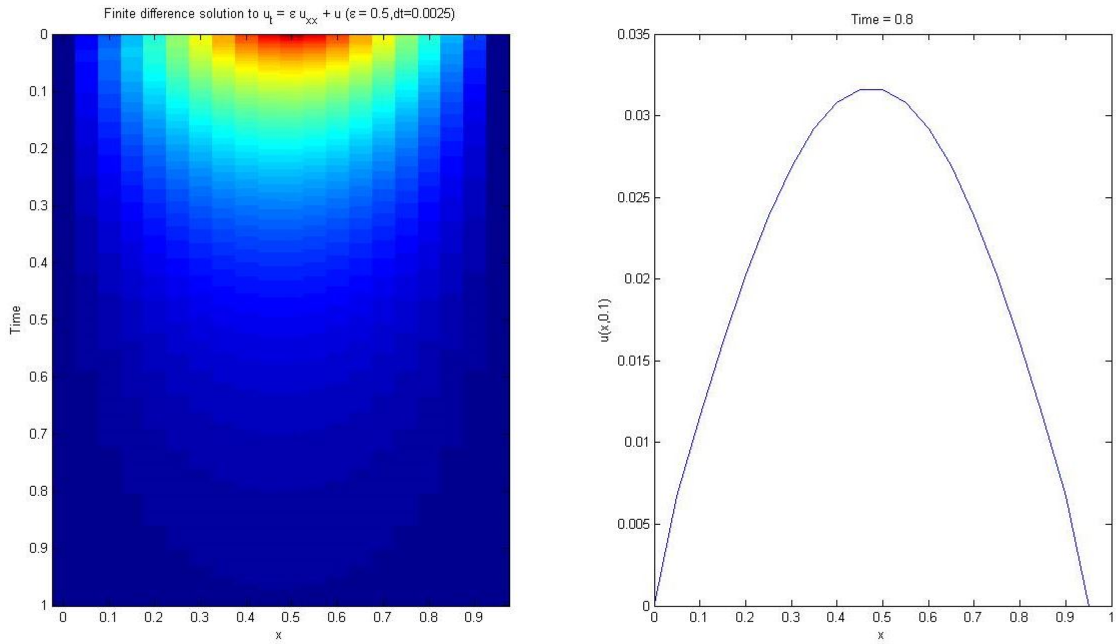

Figure 6: Result from the implementation of the implicit finite difference scheme. In this case criterion (5) is met with a $\epsilon = 0.5$.

For this exercise the parameters was set to $\Delta x = 0.05$ and $\Delta t = 0.0025$, equal to the previous exercise in Figure 2. We see that the solution $u(x, t)$ shown in Figure 6 is more or less the same as in Figure 2, only in this case plotted at a larger time. When the stability criterion is not met in this scheme, the solution does not go as critical as shown in Figure 4, as we can see in Figure 7. However, it is still unstable as we can see the solution increases in magnitude with time compared to Figure 6.

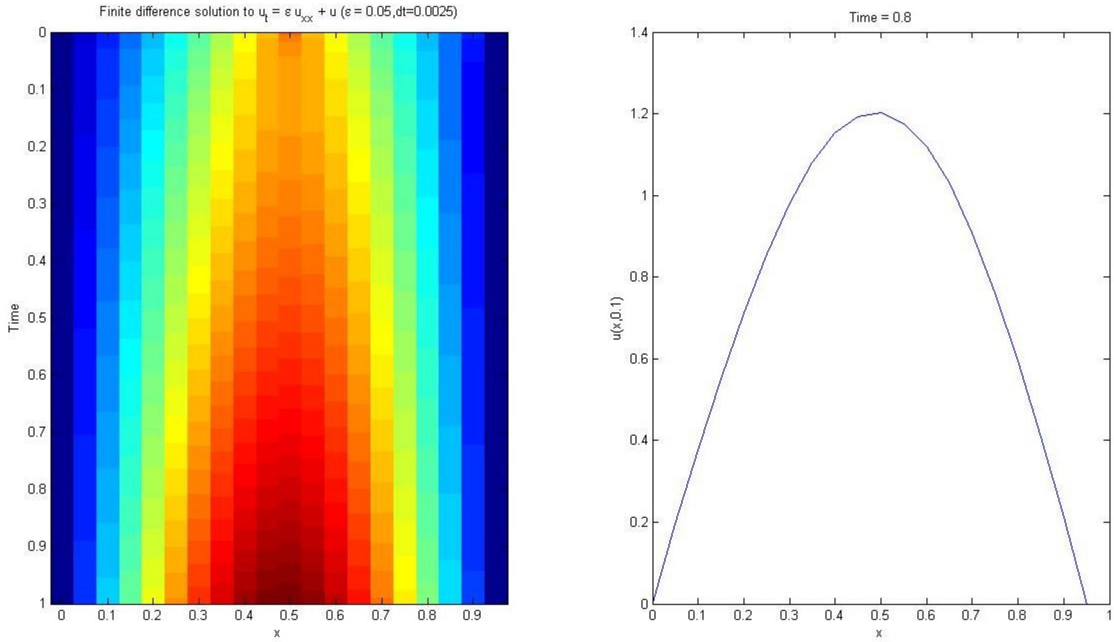

Figure 7: Result from the implementation of the implicit finite difference scheme. In this case criterion (5) is not met with a $\epsilon = 0.05$.