Model predictive control for horizontal bipedal locomotion

Tania Aguirre
ENS-Paris Saclay
Saclay, France
tania.aguirre_lordoguin@ens-paris-saclay.fr

Imen Mahdi
Télécom Paris
Palaiseau, France
imen.mahdi@telecom-paris.fr

1 INTRODUCTION

Model Predictive Control (MPC) aims to solve a sequence of optimal control problems online over a limited horizon of steps. However, its success highly depends on the choice of the optimal control problem to be solved online, ensuring stability for future steps.

The ZMP Preview Control scheme has been widely used to generate dynamically stable motion for humanoid robot locomotion. However, previous implementations lack robustness to eventual perturbations that may occur during movement.

This work addresses these limitations by considering the ZMP scheme as the optimal control problem to be solved within the MPC framework for a humanoid robot. The work can be accessed in https://github.com/taguirre19/robotics-project.git

2 THEORETICAL PROBLEM FORMULATION

This section reviews the model predictive control (MPC) problem formulation for a humanoid walking trajectory explored in the paper [1].

2.1 Assumptions

The following assumptions are made in this study:

- The robot's center of mass remains at a constant height, indicating no vertical movement. The height of h_{CoM} is consistently stable.
- The robot's center of pressure must be situated only on the convex hull of its feet's contact points with the ground.
- We disregard any inertia effects caused by the rotation of the robot's body.

This results in the following formula:

$$z = x - \frac{h_{CoM}}{g}\ddot{x} \tag{1}$$

Here, z represents the position of the Center of Pressure on the ground, x is the horizontal position of the Center of Mass, $\ddot{x}$ refers to its horizontal acceleration, h_{CoM} denotes the altitude of the Center of Mass(CoM), and g is the gravitational constant. This discussion concentrates solely on one horizontal movement. The forward motion is handled in the same manner as we will observe later, given that the two motions are independent.

2.2 Goal

The objective is to create a dynamically feasible CoM trajectory while maintaining fixed positions of the feet. Refer to Figure 1.

If we assume that the jerk $\dddot{x}$ remains constant over discrete time intervals of length T, we can formulate the following equation for

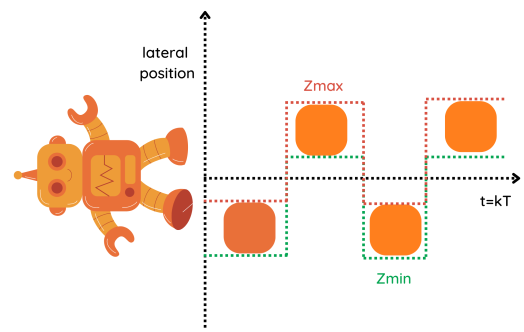


Figure 1: Visualization of the robot steps and the z^{min} and z^{max} lateral positions over time.

each kT:

$$\hat{x}_k = \begin{bmatrix} x(kT) \\ \dot{x}(kT) \\ \ddot{x}(kT) \end{bmatrix}, \ \dddot{x}_k = \dddot{x}(kT), \ z_k = z(kT), \tag{2}$$

Where the vector $\hat{x}_k$ represents the state vector containing the center of mass position, velocity, and acceleration at time interval k. Its corresponding CoP is denoted as z_k. By integrating over the time interval T, the following recursive equation is obtained:

$$\hat{x}_{k+1} = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_k \tag{3}$$

The equation can be expressed as:

$$z_k = \left[\ 1, \ 0, \ -\frac{h_{CoM}}{g} \ \right] \hat{x}_k \tag{4}$$

To constrain the CoP to lie on the convex hull of the feet's contact points with the ground, the following inequality constraint can be written:

$$z_k^{min} \leq z_k \leq z_k^{max} \tag{5}$$

With the minimum and maximum values of the CoP on the ground being z_k^{min} and z_k^{max} respectively, refer to figure 1 for a more graphical description of the variables.

In other words, the robot is capable of exerting force exclusively on its feet when on the ground surface. If the terrain is not convex, the robot can apply force within the convex hull.

Therefore, a set of jerks $\dddot{x}_k$ complying with the inequality constraints must be identified for each time step k in order to generate a CoM trajectory. Minimizing jerk is also vital for achieving a smooth motion.

In other words, if we have a specific location where the robot should position its feet, we can calculate the ideal position for its

CoM to ensure stability. This translates to solving the subsequent problem:

$$\underset{\dddot{x}_k, \dddot{x}_{k+1} \ldots,}{\text{minimize}} \sum_{i=k}^{\infty} \frac{1}{2} Q \left(z_{i+1} - z_{i+1}^{ref} \right)^2 + \frac{1}{2} R \dddot{x}_i^2 \tag{6}$$

where R/Q allows to balance the minimization of the jerks $\dddot{x}_i$ with the tracking references positions z_i^{ref}.

2.3 Quadratic Problem over finite intervals

In this section, we aim to solve the problem over a finite interval rather than the original QP of equation (2.2). Our goal is to minimize jerk over an interval of N samples, which can be achieved by minimizing the vector $\dddot{X}_k = [\dddot{x}_k, \ldots, \dddot{x}_{k+N}]^T$. Thus, we must solve the following optimization problem:

$$\begin{aligned} \underset{\dddot{X}_k}{\text{minimize}} \quad & \dddot{X}_k^{\,2} \\ \text{s.t.} \quad & Z_{k+1}^{min} \leq z_k \leq Z_{k+1}^{max} \end{aligned} \tag{7}$$

Due to the recursive equation of $\hat{x}_k$, we can rephrase the problem with its N relations, yielding a correlation between the CoP and the jerk, as shown in the formula below:

$$Z_{k+1} = P_x \hat{x}_k + P_u \dddot{X}_k \tag{8}$$

We have observed an error in the definition of the matrix P_u in the article. The authors define it as follows:

$$P_u = \begin{bmatrix} (\frac{T^3}{6} - T\frac{h_{\text{CoM}}}{g}) & 0 & 0 \\ \vdots & \ddots & \vdots \\ (1+3N+3N^2)\frac{T^3}{6} - T\frac{h_{\text{CoM}}}{g} & \ldots & (\frac{T^3}{6} - T\frac{h_{\text{CoM}}}{g}) \end{bmatrix} \tag{9}$$

If we examine the dimensions, we can deduce that the Z_{k+1} vector has a size of N, whereas the P_u matrix measures (N+1) x (N+1). This is because the last row ranges from $(1+3N+3N^2)\frac{T^3}{6} - T\frac{h_{CoM}}{g}$ to $\frac{T^3}{6} - T\frac{h_{CoM}}{g}$.

The correct formulation is explained in the appendix A, in which we obtained the following expression for P_u:

$$P_u = \begin{bmatrix} (\frac{T^3}{6} - T\frac{h_{\text{CoM}}}{g}) & 0 & 0 \\ \vdots & \ddots & \vdots \\ (1+3(N-1)+3(N-1)^2)\frac{T^3}{6} - T\frac{h_{\text{CoM}}}{g} & \ldots & (\frac{T^3}{6} - T\frac{h_{\text{CoM}}}{g}) \end{bmatrix} \tag{10}$$

While the matrix P_x is described in the following way:

$$P_x = \begin{bmatrix} 1 & T & (T^2 - \frac{h_{\text{CoM}}}{g}) \\ \vdots & \vdots & \vdots \\ 1 & NT & (\frac{N^2T^2}{2} - \frac{h_{\text{CoM}}}{g}) \end{bmatrix} \tag{11}$$

2.4 MPC Formulation as a QP with inequality constraints

We can now write our MPC problem as the following optimization problem:

$$\begin{aligned} \underset{\dddot{X}_k}{\text{minimize}} \quad & \dddot{X}_k^T \dddot{X}_k \\ \text{s.t.} \quad & Z_{k+1}^{min} \leq Z_{k+1} \leq Z_{k+1}^{max} \end{aligned} \tag{12}$$

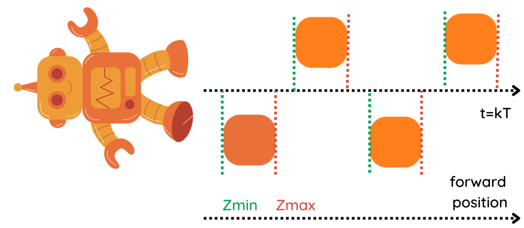


Figure 2: Visualization of the robot steps and the z^{min} and z^{max} forward positions over time.

For simplification the inequalities constraints are rewritten in the following way:

$$\begin{aligned} \underset{\dddot{X}_k}{\text{minimize}} \quad & \dddot{X}_k^T \dddot{X}_k \\ \text{s.t.} \quad & P_x \hat{x}_k + P_u \dddot{X}_k \leq Z_{k+1}^{max} \\ & -P_x \hat{x}_k - P_u \dddot{X}_k \leq -Z_{k+1}^{min} \end{aligned} \tag{13}$$

Which is equivalent to:

$$\begin{aligned} \underset{\dddot{X}_k}{\text{minimize}} \quad & \dddot{X}_k^T \dddot{X}_k \\ \text{s.t.} \quad & P_u \dddot{X}_k \leq (Z_{k+1}^{max} - P_x \hat{x}_k) \\ & -P_u \dddot{X}_k \leq (-Z_{k+1}^{min} + P_x \hat{x}_k) \end{aligned} \tag{14}$$

And thus we can write the problem as a QP with inequality constraints:

$$\begin{aligned} \underset{\dddot{X}_k}{\text{minimize}} \quad & \dddot{X}_k^T \dddot{X}_k \\ \text{s.t.} \quad & A \dddot{X}_k \leq b \end{aligned} \tag{15}$$

with $A = \begin{bmatrix} P_u \\ -P_u \end{bmatrix}$ and $b = \begin{bmatrix} Z_{k+1}^{max} - P_x \hat{x}_k \\ -Z_{k+1}^{min} + P_x \hat{x}_k \end{bmatrix}$

2.5 Forward Movement

To generalize the problem to include both lateral and forward movements, it is possible to divide the problem into two separate ones. Each problem is a QP with inequality constraints as presented previously. The singular variation is in the values for z_min and z_max.

Therefore, we will construct the new problem for y:

$$\begin{aligned} \underset{\dddot{Y}_k}{\text{minimize}} \quad & \dddot{Y}_k^T \dddot{Y}_k \\ \text{s.t.} \quad & A \dddot{Y}_k \leq b \end{aligned} \tag{16}$$

with $A = \begin{bmatrix} P_u \\ -P_u \end{bmatrix}$ and $b = \begin{bmatrix} Z_{k+1}^{max} - P_y \hat{y}_k \\ -Z_{k+1}^{min} + P_y \hat{y}_k \end{bmatrix}$

Figure 2 shows the difference of z^{min} and z^{max} positions over time for the forward movement.

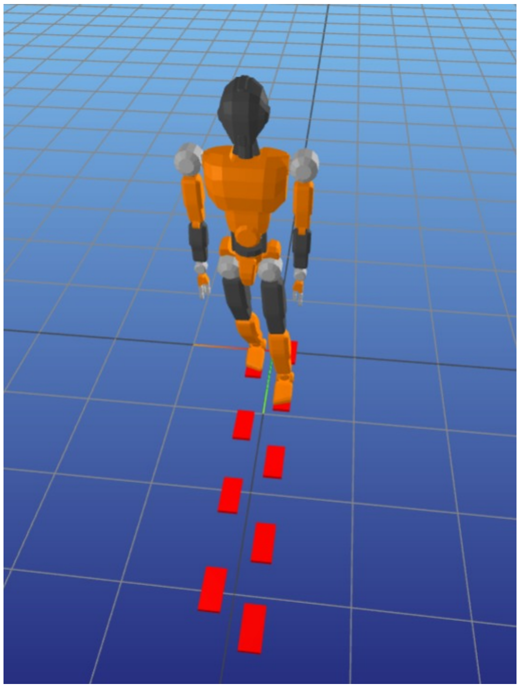


Figure 3: Robot Simulation

3 RESULTS

As part of the project, a simulation using the pink library was included. The accompanying figure displays a sample of the robot with expected footsteps represented by a rectangular block.

The results demonstrate that the generated CoM position enables stability, and the position of the CoP remains within the constraints of the feet boundaries for both movements: forward and lateral. Figure 4 presents the lateral trajectory first, then the forward trajectory, and finally the movement with the ultimate position accounting for a feet's square shape. This visual representation provides a comprehensive overview of our approach's success in this scenario.

In order to evaluate the resilience of the proposed method to external disturbances, we conducted simulations involving the application of forces to the robot at different magnitudes and time intervals. This examination aims to provide insights into the system's robustness, to see if our model is capable of adapting to perturbations while maintaining stable and effective performance.

Figure 5 describes the experimental results of applying n instantaneous forces at a given time with two different magnitudes. With small forces, our system is destabilized for a short period of time, but later is able to continue with the fixed points. However, as expected, when a significant perturbation is applied, the system is not able to return to its original position and continue walking. This observation is consistent with the graphs and with the simulation, and highlights the sensitivity of the system to significant external perturbations.

In Figure 6, the experiment focused on maintaining a perturbation of low magnitude while altering the time at which it is applied. The first image depicts a scenario where the perturbation is successfully corrected, keeping the system within the valid interval. In the second image, the perturbation briefly leads the system outside the valid data range, but later, it manages to readjust and continue walking. The final result showcases a case where, despite applying the same force, the system cannot continue walking. This simple analysis demonstrates a very intuitive argument: in a position where the system is not stable, a small force can lead to total destabilization.

4 CONCLUSIONS

The online generation of a trajectory within the proposed framework effectively compensates for minor perturbations applied to the robot. However, the robot's adaptability relies on both the timing and magnitude of the external force. The pre-fixed trajectory becomes a limiting factor in extreme scenarios where the CoP reaches the step boundaries or when the force exceeds manageable limits. To enhance flexibility, exploring frameworks that allow for dynamic adjustment of the feet trajectory by incorporating additional terms directly into the objective function can be explored.

REFERENCES

[1] Pierre-brice Wieber. 2006. Trajectory Free Linear Model Predictive Control for Stable Walking in the Presence of Strong Perturbations. In *2006 6th IEEE-RAS International Conference on Humanoid Robots*. 137–142. https://doi.org/10.1109/ICHR.2006.321375

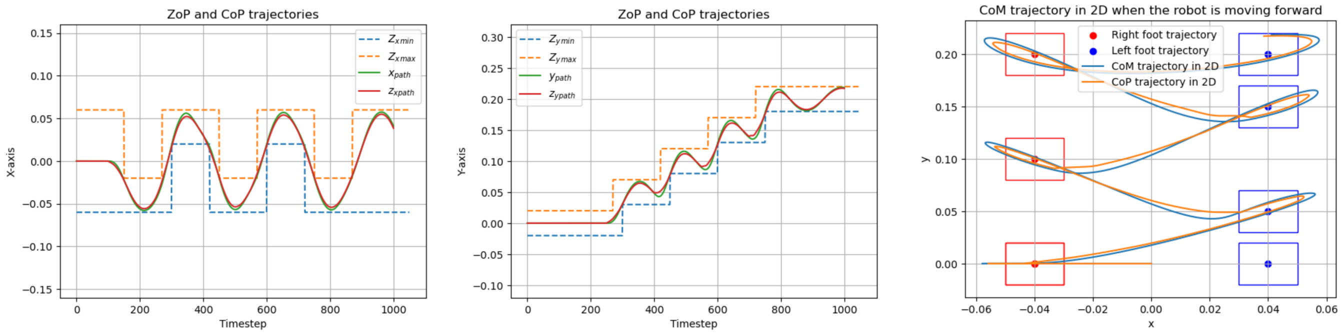
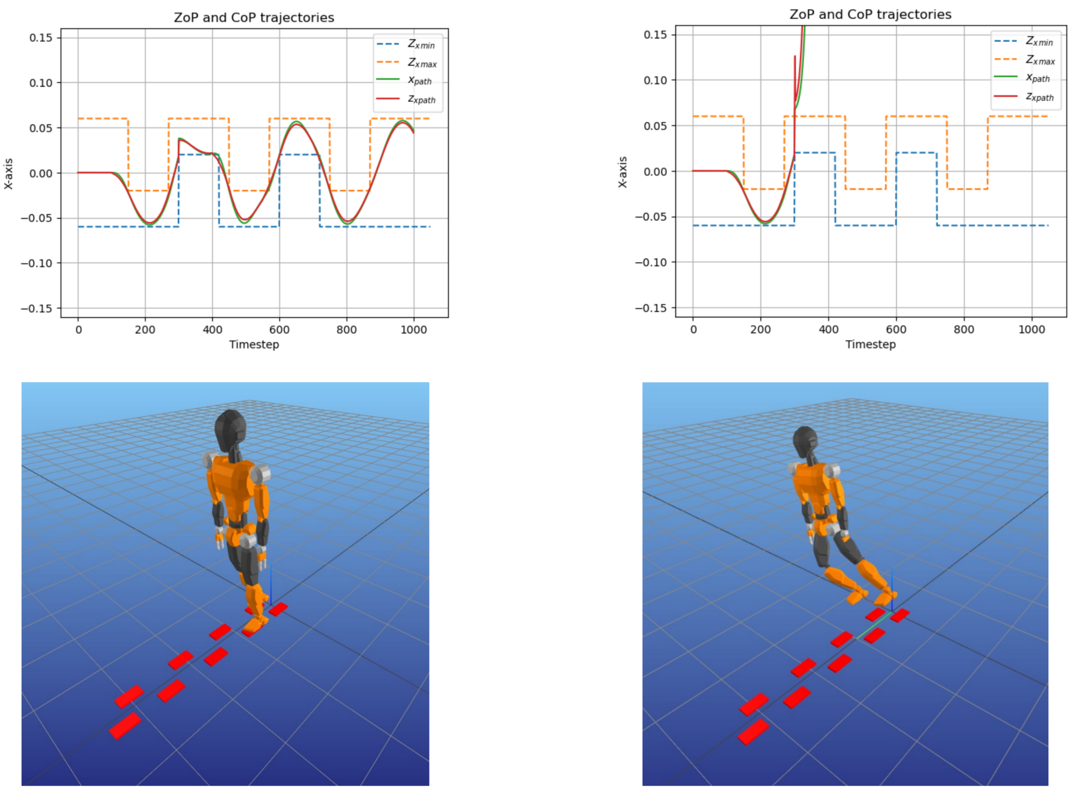
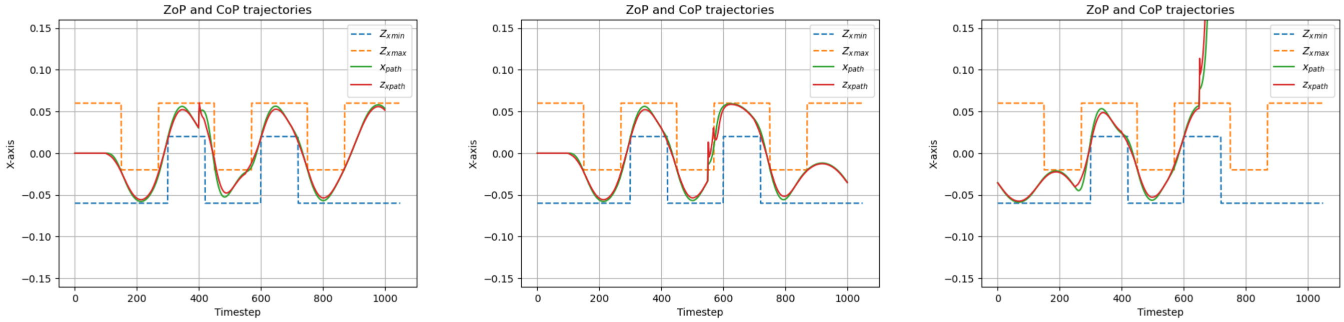


Figure 4: Plot of X(CoM) and z(CoP) obtained after solving numerically the QP problem.



(a) A force f applied at t=300 **(b) A force f'>f applied at t=300**

Figure 5: Impact of a simulated force applied to the robot at different magnitudes.



(a) A force f applied at t=400 **(b) A force f applied at t=550** **(c) A force f applied at t=600**

Figure 6: Impact of a simulated force applied to the robot at different timesteps.

A DERIVATION OF THE RECURSIVE EQUATION OF $\hat{x}_k$

We start by the case of $N = 1$

$$\hat{x}_{k+1} = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_k$$

For $N = 2$:

$$\begin{aligned}
\hat{x}_{k+2} &= \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_{k+1} + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+1} \\
&= \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}^2 \hat{x}_k + \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_k + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+1} \\
&= \begin{bmatrix} 1 & 2T & 2^2T^2 \\ 0 & 1 & 2T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \begin{bmatrix} (1+3+3)\frac{T^3}{6} \\ 3\frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_k + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+1} \\
&= \begin{bmatrix} 1 & 2T & 2^2T^2 \\ 0 & 1 & 2T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \sum_{p=0}^{1} \begin{bmatrix} (1+3p+3p^2)\frac{T^3}{6} \\ (2p+1)\frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{(k+2)-(p+1)}
\end{aligned}$$

Before we proof the general case, we can easily prove by recursion that for any $N \in \mathbb{N}$, we have:

$$\begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}^N = \begin{bmatrix} 1 & NT & N^2T^2 \\ 0 & 1 & NT \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

Now let's assume that the recursive equation is true for N:

$$\hat{x}_{k+N} = \begin{bmatrix} 1 & NT & \frac{N^2T^2}{2} \\ 0 & 1 & NT \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \sum_{p=0}^{N-1} \begin{bmatrix} (1+3p+3p^2)\frac{T^3}{6} \\ (2p+1)\frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+N-p-1} \tag{18}$$

We can now prove that it is also true for $N + 1$:

$$\begin{aligned}
\hat{x}_{k+N+1} &= \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_{k+N} + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+N} \\
&= \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}^N \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \sum_{p=0}^{N-1} \begin{bmatrix} (1+3p+3p^2)\frac{T^3}{6} \\ (2p+1)\frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+N-p-1} + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+N+1} \\
&= \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}^{N+1} \hat{x}_k + \sum_{p=0}^{N-1} \begin{bmatrix} (1+3p+3p^2)\frac{T^3}{6} + 3(2p+1)\frac{T^3}{6} + \frac{T^3}{6} \\ (2p+1)\frac{T^2}{2} + \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+N-p-1} + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+N} \\
&= \begin{bmatrix} 1 & (N+1)T & (N+1)^2\frac{T^2}{2} \\ 0 & 1 & (N+1)T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \sum_{p=0}^{N-1} \begin{bmatrix} (1+3(p+1)+3(p+1)^2)\frac{T^3}{6} \\ (2(p+1)+1)\frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{(k+N+1)-(p+1)} + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+N} \\
&= \begin{bmatrix} 1 & (N+1)T & (N+1)^2\frac{T^2}{2} \\ 0 & 1 & (N+1)T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \sum_{p=1}^{N} \begin{bmatrix} (1+3p+3p^2)\frac{T^3}{6} \\ (2p+1)\frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{(k+N+1)-(p+1)} + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{k+(N+1)-(0+1)} \\
&= \begin{bmatrix} 1 & (N+1)T & (N+1)^2\frac{T^2}{2} \\ 0 & 1 & (N+1)T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \sum_{p=0}^{N} \begin{bmatrix} (1+3p+3p^2)\frac{T^3}{6} \\ (2p+1)\frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_{(k+N+1)-(p+1)}
\end{aligned}$$

Therefore we have proved that the recursive equation 18 is true for any $N \in \mathbb{N}$.
Now we can rewrite the equation of z_{k+N} as:

$$\begin{aligned}
z_{k+N} &= \left[\ 1,\ 0,\ -\frac{h_{CoM}}{g}\right]\hat{x}_k \\
&= \left[\ 1,\ 0,\ -\frac{h_{CoM}}{g}\right]\begin{bmatrix}1 & NT & N^2\frac{T^2}{2} \\ 0 & 1 & NT \\ 0 & 0 & 1\end{bmatrix}\hat{x}_k + \left[\ 1,\ 0,\ -\frac{h_{CoM}}{g}\right]\sum_{p=0}^{N-1}\begin{bmatrix}(1+3p+3p^2)\frac{T^3}{6} \\ (2p+1)\frac{T^2}{2} \\ T\end{bmatrix}\dddot{x}_{k+N-p-1} \\
&= \left[\ 1,\ NT,\ N^2\frac{T^2}{2} - \frac{h_{CoM}}{g}\right]\hat{x}_k + \sum_{p=0}^{N-1}\left((1+3p+3p^3)\frac{T^3}{6} - T\frac{h_{CoM}}{g}\right)\dddot{x}_{k+N-p-1}
\end{aligned}$$

We can now write the following equation:

$$Z_{k+1} = \begin{bmatrix}1 & T & T^2 - \frac{h_{CoM}}{g} \\ \vdots & \vdots & \vdots \\ 1 & NT & N^2T^2/2 - \frac{h_{CoM}}{g}\end{bmatrix}\hat{x}_k + \begin{bmatrix}\frac{T^3}{6} - T\frac{h_{CoM}}{g} & 0 & 0 \\ \vdots & \ddots & \vdots \\ (1+3(N-1)+3(N-1)^2)\frac{T^3}{6} - T\frac{h_{CoM}}{g} & \dots & \frac{T^3}{6} - T\frac{h_{CoM}}{g}\end{bmatrix} \times \begin{bmatrix}\dddot{x}_k \\ \vdots \\ \dddot{x}_{k+N-1}\end{bmatrix}$$