In the name of God

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HW #1, Convex Optimization, Dr. Babazadeh

Problem 1

a) $A \succeq 0 \Rightarrow \lambda_i \geq 0$

$\left.\begin{array}{l} \\ trace(A) = 0 = \sum \lambda_i \end{array}\right\} \Rightarrow \lambda_i = 0 \Rightarrow$ Since all the eigen values are zero, & $A \succeq 0$, the matrix A can only be the zero matrix

c) $A \succeq 0$. First it is easy to show that for a PSD matrix, all elements on the diagonal are positive: if $a_{ii} \leq 0$ the $e_i^T A e_i = a_{ii} \leq 0$

e_i ↓ column vector with only the i th element non zero & equal to 1.

So now we must show that the largest value is on the diagonal. suppose a_{ij} where $i \neq j$ is the largest element & also suppose that $a_{ij} \leq a_{ii}$ & $a_{ij} \leq a_{jj}$. It is easy to see that for $x = e_i - e_j$ we have $x^T A x = a_{ii} + a_{jj} - 2a_{ij} \leq 0$. ▨

d) Using what we proved in part c, we know that all diagonal enteries are non-negative thus since both A & B are PSD, if $A + B = 0$, then all diagonal entries must be zero. We also proved that the largest entery is on the diagonal & since all diagonal enteries are 0 this means that all other element are non positive $\begin{array}{l} a_{ij} \leq 0 \\ b_{ij} \leq 0 \end{array}$ but $a_{ij} + b_{ij} = 0$

$\Rightarrow a_{ij} = b_{ij} = 0$

$A = B = 0$ ▨

b) A counter example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$eig(A) = 0, 1$

$eig(B) = 0, 1$ $\Rightarrow A \succeq 0, B \succeq 0$ & $AB = 0$.

Problem 2

$$\left.\begin{array}{l}\text{dom}(f) \text{ is convex}\\ f \text{ is convex}\end{array}\right\} \iff (\nabla f(x) - \nabla f(y))^T (x-y) \geq 0 \quad \forall x,y$$

Proving $\Rightarrow$:

The first order condition imply the inequality below for a convex function on a convex domain :

$$f(y) \geq f(x) + \nabla f(x)^T (y-x)$$

$$\Rightarrow \nabla f(x)^T (y-x) \leq f(y) - f(x) \Rightarrow \nabla f(x)^T (x-y) \geq f(x) - f(y) \quad (I)$$

$$\hookrightarrow \nabla f(y)^T (x-y) \leq f(x) - f(y) \quad (II)$$

$$(I) - (II) \Rightarrow (\nabla f(x) - \nabla f(y))^T (x-y) \geq f(x) - f(y) - (f(x) - f(y)) = 0$$

proving $\Leftarrow$:

$$g(\theta) = f(y + \theta(x-y)) \qquad 0 < \theta < 1$$

$$g'(\theta) = \nabla f^T (y + \theta(x-y)) (x-y)$$

$$g'(1) - g'(0) = (\nabla f(x) - \nabla f(y))^T (x-y) \geq 0$$

g & g' are both increasing functions $\Rightarrow g$ is convex

$\Rightarrow f$ is convex

Problem 3

a) This set is convex. the case $\beta \geq 1$ is the half plane to the side of a. For $\beta < 1$ the locus is a hyperball & all that lie within. The simple case of R^2 is proven here. The higher dimension proof is similar

$$\|x-a\| \leq \beta\|x-b\| \Rightarrow \sqrt{(x-a_x)^2+(y-a_y)^2} \leq \beta\sqrt{(x-b_x)^2+(y-b_y)^2}$$

for simplicity, suppose $a = \begin{bmatrix}0\\0\end{bmatrix}$

$$\Rightarrow (1-\beta^2)x^2 + (1-\beta^2)y^2 + 2\beta^2 b_x x + 2\beta^2 b_y y - \beta^2(b_x^2+b_y^2) \leq 0$$

$$x^2+y^2+\frac{2\beta^2 b_x}{1-\beta^2}x + \frac{2\beta^2 b_y}{1-\beta^2}y - \frac{\beta^2}{1-\beta^2}(b_x^2+b_y^2) \leq 0$$

$$\Rightarrow \left(x+\frac{\beta^2 b_x}{1-\beta^2}\right)^2 + \left(y+\frac{\beta^2 b_y}{1-\beta^2}\right)^2 \underbrace{- \frac{\beta^4 b_x^2}{(1-\beta^2)^2} - \frac{\beta^4 b_y^2}{(1-\beta^2)^2} - \frac{\beta^2(b_x^2+b_y^2)}{1-\beta^2}}_{-R^2} \leq 0$$

$$\Rightarrow \frac{\left(x+\frac{\beta^2 b_x}{1-\beta^2}\right)^2}{R^2} + \frac{\left(y+\frac{\beta^2 b_y}{1-\beta^2}\right)^2}{R^2} \leq 1$$

which is the equation of a circle in R^2

b) Consider the function $f(x) = x_1 x_2 \cdots x_n$, $x_i > 0$

the 1-super levelset is $f(x) \geq 1$, which is, the set in the question.

taking the logarithm of both sides yields $\log f(x) \geq 0$

$\log(f(x)) = \sum \log(x_i)$ which is the sum of concave functions & is concave $\Rightarrow \log f(x) \geq 0$ is the 0 superlevel set of a concave function $\Rightarrow$ It is convex $\Rightarrow S$ is convex

Problem 4

- If the epigraph of a function is a half space ($A^T x \geq b$) then so is the hypo of this function ($A^T x \leq b$) thus the function is both concave & convex thus it is affine
- If $f(\alpha x) = \alpha f(x)$ for $\alpha \geq 0$, the epigraph is a convex cone
  a convex cone is parameterized by $\{\alpha x + \beta y \in S \;\; \forall \alpha, \beta \geq 0, \; x, y \in S\}$ & thus all points satisfy the equation.
- A polyhedron is the intersection of several half spaces so as in part a we would expect it to be some sort of an affine function.
  A piecewise affine function has the mentioned property

Problem 5

$$f(x) = \sqrt[n]{\lambda_1(x) \cdots \lambda_n(x)} = \sqrt[n]{\det(X)}$$

$g(t) = f(X + tV)$ $\qquad$ f is convex iff g is convex

$$= \det(X + tV)^{1/n} \qquad X \succeq 0 \Rightarrow X = X^{0.5} X^{0.5}$$

$$= \det(X^{0.5} X^{0.5} + X^{0.5} X^{-0.5} + V X^{-0.5} X^{0.5})^{1/n}$$

$$= \{\det(X) \det(I + X^{-0.5} t V X^{-0.5})\}^{1/n}$$

$$= \det(X)^{1/n} \det(I + t X^{-0.5} V X^{-0.5})^{1/n}$$

$$= \det(X)^{1/n} \underbrace{\left(\prod_{i=1}^{n} (1 + t\lambda_i)\right)^{1/n}}_{\text{we have shown in the class that the geometric mean is concave}}$$

$\Rightarrow g(t)$ is concave $\Longleftrightarrow f(X)$ is concave