In the name of God

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HW #3, Convex Optimization, Dr. Babazadeh

Problem 1

a) The domain of the relaxed version is the continuous range $[0,1]$. Suppose the original problem reached a minimum value of p^*. This value is reached on the points $\{0,1\}$. These two points are also in the domain of the relaxed version, so if the relaxed version were to opt a larger value than p^*, it could have simply chosen the points $\{0,1\}$ & reached the same minimum value of p^*, so the result of the LP relaxation is a lower bound for the Boolean LP.

b) $$\mathcal{L}(x,\nu,\eta) = c^T x + \nu^T(Ax-b) + \underbrace{\sum_{i=1}^{n} \eta_i x_i (1-x_i)}_{Q}$$

$$Q = x^T \operatorname{diag}(\eta)(d-x) \quad \text{where} \quad \operatorname{diag}(\eta) = \begin{bmatrix} \eta_1 & & 0 \\ & \eta_2 & \\ 0 & & \ddots \; \eta_n \end{bmatrix} = D, \quad d = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= x^T \underbrace{Dd}_{\eta} - x^T D x = x^T \eta - x^T D x$$

$$\Rightarrow \mathcal{L}(x,\nu,\eta) = \underbrace{(c^T + \nu^T A)}_{B} x - \nu^T b + x^T \eta - x^T D x$$

$$\Rightarrow \nabla \mathcal{L} = B^T + \eta - 2Dx \overset{\nabla = 0}{\Longrightarrow} x^* = \frac{1}{2} D^{-1}(B^T + \eta)$$

$$\Rightarrow g(\nu,\eta) = \frac{1}{4} B D^{-1} B^T + \frac{1}{2} B D^{-1} \eta + \frac{1}{4} \eta^T \underbrace{\underbrace{D^{-1}\eta}_{d}}_{\sum_i \eta_i} - \nu^T b$$

given $\eta_i > 0$. [illegible] $BD^{-1}B^T$ can tend to $-\infty$

$$\Rightarrow g(\nu,\eta) = \begin{cases} \frac{1}{4}(c^T + \nu^T A) D^{-1} (c + A^T \nu + 2\eta) + \sum \eta_i - \nu^T b & \eta_i > 0 \\ -\infty & \text{o.w} \end{cases}$$

$\Rightarrow$ dual problem: maximize $g(\nu,\eta)$

c)

$$\mathcal{L}(\lambda,\mu,\vartheta) = c^T x + \underbrace{\sum \lambda_i (x_i - 1)}_{\lambda^T (x-d)} + \underbrace{\sum \mu_i (-x_i)}_{-\mu^T x} + \vartheta^T (Ax-b) \qquad d = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= (c^T + \lambda^T - \mu^T + \vartheta^T A)x - \lambda^T d - \vartheta^T b$$

$$\nabla \mathcal{L} = 0 \Rightarrow c^T + \lambda^T - \mu^T + \vartheta^T A = 0$$

$$\Rightarrow g(\lambda,\mu,\vartheta) = \begin{cases} -\lambda^T d - \vartheta^T b & , c^T + \lambda^T - \mu^T + \vartheta^T A = 0 \\ -\infty & \text{o.w} \end{cases}$$

$\Rightarrow$ dual problem:

$$\begin{aligned} &\text{maximize} && -\lambda^T d - \vartheta^T b \\ &\text{subject to} && c^T + \lambda^T - \mu^T + \vartheta^T A = 0 \\ & && \lambda_i \geq 0, \mu_i \geq 0 \end{aligned}$$

let's simplify the dual problem of part b

$$\overset{z_i > 0}{\Rightarrow} g(z, \vartheta, \eta) = \frac{1}{4} \underbrace{B D^{-1} (B + 2\eta)}_{\sum \left(\frac{B_i^2}{z_i} + 2B_i\right)} + \eta^T d - \vartheta^T b \qquad \text{where } B_i = \vartheta^T a_i + c_i \qquad A = [a_1 | a_2 | \cdots | a_n]$$

d) An example could perhaps be the decision of which power generators to keep ON & which to turn OFF.

Problem 2.

a) The problem's is equivalent is :

Minimize t (or equivalently $\frac{1}{2}t\,P_0 t + q_0^T t + r_0$ where $P_0 = 0$, $r_0 = 0$, $q_0 = 1$)

Subject to $\frac{1}{2}x^T P_i x + q_i^T x + r \leq t$

$Ax \leq b$

$P_i \succeq 0$ $\quad i = 1, \dots, k$

which is the standard QCQP form.

b) $\mathcal{E} = \{P \mid P_0 - \gamma I \preceq P \preceq P_0 + \gamma I\}$ where $P_0 \succeq 0$

let's consider $P \preceq P_0 + \gamma I$, we can not multiply the matrix inequality but we know that the definition of $A \preceq B$ is $x^T A x \leq x^T B x$. So we have.

$$x^T P x \leq x^T (P_0 + \gamma I) x \implies \frac{1}{2}x^T P x + q_r^T x + r \leq \frac{1}{2}x^T P_0 x + \frac{\gamma}{2}x^T x + q_r^T x + r$$

$$\implies \sup_{P \in \mathcal{E}} \frac{1}{2}x^T P x + q_r^T x + r = \frac{1}{2}x^T P_0 x + \frac{\gamma}{2}x^T x + q_r^T x + r$$

$$\implies \quad \text{Minimize} \quad \frac{1}{2}x^T P_0 x + \frac{\gamma}{2}x^T x + q_r^T x + r$$

Subject to $Ax \leq b$

$P_0 \succeq 0$

Problem 3

The lagrangian is:

$$\mathcal{L}(x,\nu,\mu) = \sum_k x_k \log\frac{x_k}{y_k} + \nu^T(Ax-b) + \mu(\sum x - 1)$$

taking the derivative with respect to x_l we get:

$$\frac{\partial \mathcal{L}}{\partial x_l} = \log\frac{x_l}{y_l} + 1 + \underbrace{\nu^T a_l}_{=a_l^T\nu} + \mu = 0 \Rightarrow \log\frac{x_l}{y_l} = -(1+a_l^T\nu+\mu)$$

$$x_l = y_l e^{-(1+a_l^T\nu+\mu)}$$

$$\Rightarrow g(\nu,\mu) = -\sum y_k e^{-(1+a_k^T\nu+\mu)}(1+a_k^T\nu+\mu) + \mu\sum y_k e^{-(1+a_k^T\nu+\mu)} + \sum \underbrace{\nu^T a_k}_{a_k^T\nu} y_k e^{-(1+a_k^T\nu)}$$

$$-\nu^T b - \mu$$

$$= \sum y_k e^{-(1+a_k^T\nu+\mu)}(\cancel{\mu} + \cancel{\nu^T a_k} - 1 - \cancel{a_k^T\nu} - \cancel{\mu}) - \nu^T b - \mu$$

$$= -\nu^T b - \mu - \sum y_k e^{-(1+a_k^T\nu+\mu)}$$

$\Rightarrow$ dual problem: maximize $g(\nu,\mu)$

The form that the question has provided has no μ in it. so let's solve the above problem with respect to μ. since the problem is unconstrained, we take the derivate:

$$\frac{\partial g}{\partial \mu} = -1 + \sum y_k e^{-(1+a_k^T\nu+\mu)} = 0 \Rightarrow \sum y_k e^{-(1+a_k^T\nu+\mu)} = 1$$

$$\hookrightarrow \mu = \log\sum y_k e^{-a_k^T\nu} - 1$$

$\Rightarrow$ dual problem: maximize $-\nu^T b - \log\sum y_k e^{-a_k^T\nu}$

$$\nu \rightsquigarrow -z$$

$$\underline{b^T z - \log\sum y_k e^{a_k^T z}}$$