Problem Set 1
Convex Optimization

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Spring Semester 1397–98
Deadline: Monday–19 Farvardin

Problem 1. Let A and B be $n \times n$ matrices with real entries. Prove or disprove each of the following statements.

(a) If $A \succeq 0$ and $\text{trace}(A) = 0$, then $A = 0$.

(b) If $A \succeq 0$, $B \succeq 0$, and $AB = 0$, then $A = 0$ or $B = 0$.

(c) Suppose $A \succeq 0$. Then the largest entry in absolute value of A must be on the diagonal.

(d) If $A \succeq 0$, $B \succeq 0$, and $A + B = 0$, then $A = B = 0$.

Problem 2. Let f be twice differentiable, with $\text{dom}(f)$ convex. Show that f is convex if and only if,

$$(\nabla f(x) - \nabla f(y))^T (x - y) \geq 0 \quad \forall x, y.$$

Problem 3. Which of the following sets are convex?

a) $\left\{x \;\middle|\; \|x - a\|_2 \leq \beta \|x - b\|_2,\ a \neq b,\ 0 \leq \beta \leq 1\right\}$

b) $\left\{x \;\middle|\; x_1 x_2 \ldots x_n \geq 1,\ x_i > 0,\ i = 1, \ldots n\right\}$

Problem 4. Explain when is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

Problem 5. Let $\lambda_1(X) \geq \lambda_2(X) \geq \ldots \geq \lambda_n(X)$ be the eigenvalues of $X \in \mathbf{S}^n$. We have already shown in class that $\lambda_1(X)$ is convex and $\lambda_n(X)$ is a concave function on $\mathbf{S}^n$. In this problem, Show that $\sqrt[n]{\lambda_1(X)\lambda_2(X)\ldots\lambda_n(X)}$ is concave on $\mathbf{S}^n_{++}$

Figure 1: Haji Firouz in Problem 8

Matlab Assignment

Problem 6. Let Q_1 and Q_2 be arbitrary $n \times n$ symmetric matrices ($n > 2$).

(a) Use Matlab to draw the set $\mathcal{S} = \left\{ \left(x^T Q_1 x, x^T Q_2 x \right) \mid \|x\|_2 = 1 \right\}$ and investigate its convexity.

(b) Extra point: Can you show your conclusion in part (a) theoretically?

Problem 7. Let the set $\mathcal{S}$ be described by,

$$\mathcal{S} = \left\{ (y_1, y_2) | 2x_1^4 + x_2^4 + y_1 x_1 x_2^3 + y_2 x_1^3 x_2 \geq 0 \quad \forall (x_1, x_2) \in \mathbb{R}^2 \right\}.$$

(a) Use Matlab to draw the set $\mathcal{S}$ and investigate its convexity.

(b) Extra point: Can you show your conclusion in part (a) theoretically?

Problem 8. Let A be a real $m \times n$ matrix with a singular value decomposition given by $A = U \Sigma V^T$ (as discussed in class). For a positive integer $k \leq \min\{m, n\}$, we let A_k denote an $m \times n$ matrix which is an "approximation" of the matrix A obtained from its top k singular values and singular vectors, i.e.,

$$A_k = U_k \Sigma_k V_k^T,$$

where U_k has the first k columns of U, V_k has the first k columns of V , and Σ_k is the upper left $k \times k$ block of Σ.

(a) To provide a good approximation for A, consider the cost function $\|A - X\|_2$ where X is restricted to be an $m \times n$ matrix with $\text{rank}(X) \leq k$. Show that A_k is the minimizer of the cost function $\|A - X\|_2$.

(b) Download the file HajiFirouz.jpg. Read this file in Matlab by typing:

```
A=imread('HajiFirouz.jpg');
A=im2double(A) ;
A=rgb2gray(A) ;
```

The result is a 400×672 matrix A, with each entry representing a single pixel in the picture with a number between 0 and 1.

For different values of k, use Matlab to compute A_k, construct a compressed image with A_k (You can used the command `imwrite`), and report the value of $\|A - A_k\|_2$.

(c) Based on your experiments on part (b), provide a good compressed image for HajiFirouz and explain your interpretations.