Convex Optimization: Final Project

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Portfolio Optimization

In this project, you will learn how to choose an appropriate set of assets (e.g., stocks) based on the information about their returns. The information usually comes from historical records of asset returns as well as domain knowledge. We will describe it using the classical Markowitz portfolio optimization problem framework (described on page 155 of your textbook [1]), and study how to improve the classical portfolio optimization.

Step 1: Choosing a portfolio

There are n assets, say, stocks. You can buy real-valued amounts x_i of each stock i at the beginning of some time period, and you sell all of your holdings at the end of the time period. The stocks may lose or gain value. Let p_i be the relative price change of stock i during the time period. If stock i changes price from u to v, then $p_i = (v-u)/u$. For example, if you buy $x_i = 20$ of stock i and its value doubles over the time period ($v = 2u$), then $p_i = 1$, and your actual return from stock i is $p_i x_i = 20$. Your return from all your investments is $p^T x$. In this step, assume $x \geq 0$ and $\sum_{i=1}^n x_i = B$ where B is the budget. Usually we assume a unit budget $B = 1$, and in this case, x_i represents the proportion of investment on stock i.

Since p is the return rates of n stocks during your holding period in the future, it is an unknown vector. Consider p as a Gaussian random vector with mean $\bar{p}$ and covariance Σ, where $\bar{p}$ and Σ may be estimated from historical data or from expert knowledge. The return $r = p^T x$ of a portfolio x is a linear combination of Gaussian variables in p and thus, $r = p^T x$ is a Gaussian variable with mean $p^T x$ and variance $x^T \Sigma x$. Note that $\bar{p}$ and Σ are your predictions about the mean and covariance of asset returns (in the future).

Choosing x to maximize your expected return while minimizing the "risk" (i.e., variance of the return) can be described by the following QP:

$$\begin{aligned} \text{Maximize} \quad & \bar{p}^T x - \eta x^T \Sigma x \\ \text{Subject to} \quad & \sum_{i=1}^{n} x_i = 1, \quad x \geq 0. \end{aligned} \tag{1}$$

Note that the parameter η determines the tradeoff between the two objectives: maximizing the expected return $\bar{p}x$ and minimizing the risk $x^T \Sigma x$.

(a) Download the file `data.zip` from CW. Numerical values of $\bar{p}$ and Σ are provided in `pbar.txt` and `sigma.txt`, respectively. Use CVX to solve optimization problem (1) for at least 30 values of η, within the range 0.01 and 1000. Use a log-scale to choose the values of η.

(b) Make a plot with the standard deviation of the return (square root of $x^T \Sigma x$) on the x-axis and the mean return ($\bar{p}$x) on the y-axis.

(c) Using the portfolios x you just calculated, make a plot with values of η on the x-axis and allocation for each stock on the y-axis. Interpret your results.

Step 2: Short positions

In this step, short positions, or short selling in portfolio is introduced. Suppose that we now allow $x_i < 0$ in optimization problem (1). If you choose a portfolio with $x_i < 0$, you will sell stock i without really having it (e.g., via borrowing this stock from a broker), and then you will buy this stock at the end of the holding period (in order to return to the broker). In this sense, by a short position on x_i, you are betting that the stock i will lose value during your hold period. At the end of the period, you make a profit of $p_i x_i$ (since $x_i < 0$ and you bet $p_i < 0$). To allow short positions, we will introduce new variables x_l, x_s for long position and short position, respectively, and replace the constraints in (1) by the new constraints,

$$\begin{gathered} x = x_l - x_s, \quad x_l \geq 0, \quad x_s \geq 0, \\ \sum_{i=1}^{n} x_l = 1, \quad \sum_{i=1}^{n} x_s \leq \gamma \sum_{i=1}^{n} x_l \end{gathered} \tag{2}$$

The last constraint limits our short position to be at most a fraction ($0 \leq \gamma \leq 1$) of long position. **Note:** being able to control short selling by γ is the main reason that the two new vectors x_l and x_s are included, rather than simply removing the nonnegative constraints on x. In the rest of this project, you may set $\gamma = 0.5$. We also require that the sum of x_l (rather than x) is one, since now x_l is the real money we need to spend for buying at the beginning of the holding period.

(a) Modify the convex program (1) to allow for short positions. Find numerical values for $\bar{p}$ and Σ in `pbar-short.txt` and `sigma-short.txt`, respectively. The values of $\bar{p}$ and Σ in this step are the same as step 1, but with an addition of a new stock. This new stock has an expected return $\bar{p}_5 = 0$. Use the convex program in step 1 and the modified convex program in step 2 (with short selling) to find the optimal portfolios for different choices of η.

(b) Make plots of standard deviation of the return vs. mean return. Include the plots for both methods (without short positions and with short positions) in the same figure.

(c) Explain how the optimization with short positions benefits from having this new zero-expected-return stock.

Step 3: Portfolio Optimization with real data

Find an international exchange rates data-set in `stock-dataset.mat`. It includes the daily return rates of 12 different currencies. The data-set contains a period of about 3 years. The file contains two matrices, a 1017×12 matrix "past-returns", and a 90×12 matrix "future-returns". **The objective is to use the historical information in the past 1017 days to construct a portfolio, hold the portfolio during the future 90 days, and see how this portfolio performs in the holding period.**

(a) Compute the sample mean (a 12×1 vector) and sample covariance (a 12×12 matrix) of returns using the historical returns (the matrix "past-returns"). `Hint: You may use the Matlab functions mean and cov.` Use them as estimated $\bar{p}$ and Σ to obtain optimal portfolio x, for different values of η, with and without short positions.

(b) For each portfolio x you obtained, calculate the cumulative total return of the portfolio in the holding period (90 days) by using the matrix "future-returns".

(c) Plot the cumulative total return versus η for portfolios (i) without short selling and (ii) with short selling in the same figure.

(d) For each curve in the figure, locate the value of η which leads to the best cumulative total return. Consider this point as your best performance with (or without) short positions. Is the best portfolio with short selling still outperforms the best portfolio without short selling? Explain your answer.

Step 4: Improve your estimate

In step 3, you calculated the sample mean and sample covariance of historical returns, and used them directly as estimated $\bar{p}$ and Σ in portfolio optimization. In this step, you will improve your estimate Σ a little bit. In this step, the sample covariance is denoted by S. You are not supposed to directly set $\Sigma = S$. Instead, you should estimate the 12×12 covariance matrix Σ as follows.
Use Θ to denote Σ^{-1}. Each entry (i, j) in an inverse Gaussian covariance Σ^{-1} represents the conditional dependence of variables i and j. When the entry is zero, we know variables i and j are conditionally independent given other variables in the system. Given the 12×12 sample covariance S (which you computed using Matlab function cov on the past-returns matrix), the regularized estimate of $\Theta = \Sigma^{-1}$ is the following convex optimization problem:

$$\begin{aligned} \text{Maximize} \quad & \log\det\Theta - \operatorname{trace}(S\Theta) - \lambda \sum_{i \neq j} |\Theta_{ij}| \\ \text{Subject to} \quad & \Theta \succeq 0. \end{aligned} \tag{3}$$

In this step, let the regularization parameter λ to be 10^{-5}. For simplicity, only consider portfolios *without short positions.*

(a) Use CVX to solve the optimization problem (3) and subsequently, estimate Σ from sample covariance. Report the new covariance matrix Σ.

(b) Use estimated Σ in part (a) to construct optimal portfolio x (only without short positions).

(c) For each portfolio x you obtained in part (b), calculate the cumulative total return of the portfolio in the holding period (90 days) by using the matrix "future-returns".

(d) Plot the cumulative total return versus η for portfolios constructed using (i) $\Sigma = S$, and (ii) new estimated Σ from part (a), in the same figure.

(e) Which curve achieves the best portfolio return? Explain your answer.

(f) Can you explain how optimization problem (3) helps in estimating $\Theta = \Sigma^{-1}$?

Step 5: Portfolio optimization with loss risk constraints

Consider an additional loss risk constraint of the form

$$\textbf{prob}\,(r \le 0) \le \beta \tag{4}$$

where the return $r = p^T x$ is a Gaussian variable with mean $p^T x$ and variance $x^T \Sigma x$, as described in step 1. It is shown in page 158 of your textbook [1] that for $\beta < 0.5$ this constraint can be equivalently described as a second-order cone constraint.

(a) Use the procedure described in page 158 of your textbook [1] to show that the portfolio optimization (1) with additional loss risk constraint (4) ($\beta < 0.5$) can be expressed as an SOCP.

(b) Use the estimated Σ in step 4 and the SOCP form derived in part (a) to solve the portfolio optimization with loss risk constraints,

$$\begin{aligned} \text{Maximize} \quad & \bar{p}^T x - \eta x^T \Sigma x \\ \text{Subject to} \quad & \sum_{i=1}^{n} x_i = 1, \quad x \ge 0, \\ & \textbf{prob}\,(r \le 0) \le \beta. \end{aligned} \tag{5}$$

for at least 10 values of β, within the range 10^{-4} and 10^{-1}. `Hint: You may need the Matlab functions erfc and erfcinv.`

(c) Using the portfolios x you just calculated, plot allocation for each stock versus maximum probability β. Explain the results.

(d) calculate the cumulative total return of the portfolio in the holding period (90 days) by using the matrix "future returns". Plot the cumulative total return versus maximum probability β for portfolios constructed in part (b).

(e) Based on all the experiments in this project, you need to report your final suggestion for the best portfolio! Report your suggestion for $x \in \mathbf{R}^{12}$ and explain your answer.

Step 6: maximum risk for the chosen portfolio

Assume that based on some strategies, the portfolio x is determined, but only partial information is available about the covariance matrix. (see page 172 of your textbook [1]). Assume that your final suggestion for the best portfolio in step 5 is selected as the portfolio x. The diagonal entries of Σ are known. However, You find out that only the sign of the off-diagonal entries of Σ (based on the past data) are reliable! That is, you do not know the correlation between two individual return rates, but you know that they are non-negatively or non-positively correlated.

(a) Under the described conditions, compute the worst-case variance of the portfolio return, defined by,

$$\sigma_{wc} = \left\{ \sup_{\Sigma} x^T \Sigma x, \quad \Sigma \in \mathcal{C}, \quad \Sigma \succeq 0 \right\}, \tag{6}$$

where $\mathcal{C}$ is the prior information about Σ, described in step 6.

(b) Can you revise your choice of portfolio x such that the worst-case variance of the portfolio return is minimized? Explain your answer.

References

[1] Boyd, Stephen, and Lieven Vandenberghe. Convex optimization. Cambridge university press, 2004.