Problem Set 2
Convex Optimization

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Problem 1. Consider the function $f:\ \mathbf{S}^n \times \mathbf{R}^n \to \mathbf{R}$ defined by,

$$f(X, y) = y^T X^{-1} y.$$

(a) Let $\mathrm{dom}\, f = \{(X, y) | X = X^T \succ 0\}$. Show that the function f is convex.
`Hint:` You can use the fact that for $X = X^T \succ 0$ and $z \geq 0$, the constraint $y^T X^{-1} y \leq z$ is equivalent to the LMI $\begin{bmatrix} X & y \\ y^T & z \end{bmatrix} \succeq 0$.

(b) Let $\mathrm{dom}\, f = \{(X, y) | X + X^T \succ 0\}$., Is f convex? If so, prove it. If not, give a (simple) counterexample.

Problem 2. Let $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ be concave and convex functions, respectively. Show that if $f_1(x) \leq f_2(x)\ \forall x \in \mathbb{R}^n$, then there exists an affine function $f_a(x)$ such that,

$$f_1(x) \leq f_a(x) \leq f_2(x)\ \forall x \in \mathbb{R}^n$$

Problem 3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex with $\mathrm{dom}\, f = \mathbb{R}^n$, and bounded above on $\mathbb{R}^n$. In this case, f is constant. Can you explain why?

Problem 4. In general the product or ratio of two convex functions is not convex. However, there are some results that apply to functions on $\mathbb{R}$. Prove the following.

(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.

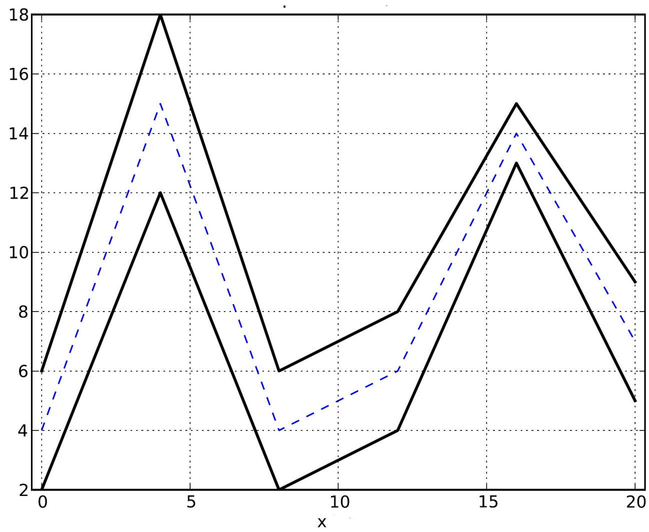
(b) If f, g are concave, positive, with one nondecreasing and the other nonincreasing, then fg is concave.

(c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then $\dfrac{f}{g}$ is convex.

Matlab Assignment

Problem 5. Consider the problem of traveling from the point $(x_0, y_0) = (0, 4)$ to the point $(x_6, y_6) = (24, 4)$ by going through 5 parallel gates located at fixed positions (x_i, y_i) with width c_i reported in the following Table.

(a) Use optimization modeling to find the path which minimizes the total length of the path.



i	x_i	y_i	c_i
0	0	4	N/A
1	4	15	3
2	8	4	2
3	12	6	2
4	16	14	1
5	20	7	2
6	24	4	N/A

(b) Use the function "`quadprog`" in Matlab to solve the problem with the data provided in the following Table.

Problem 6. ***(Extra point)*** In radiation treatment, radiation is delivered to a patient, with the goal of killing or damaging the cells in a tumor, while carrying out minimal damage to other tissue. The radiation is delivered in beams, each of which has a known pattern; the level of each beam can be adjusted. (In most cases multiple beams are delivered at the same time, in one shot, with the treatment organized as a sequence of shots.) We let b_j denote the level of beam j, for $j = 1, ..., n$. These must satisfy $0 \leq b_j \leq B^{max}$, where B_{max} is the maximum possible beam level. The exposure area is divided into m voxels, labeled $i = 1, \ldots, m$. The dose d_i delivered to voxel i is linear in the beam levels, i.e., $d_i = \sum_{j=1}^{n} A_{ij} b_j$ where $A \in \mathbb{R}_+^{m \times n}$ (known) matrix that characterizes the beam patterns. We now describe a simple radiation treatment planning problem.
A (known) subset of the voxels, $\tau \subset \{1, \ldots, m\}$, corresponds to the tumor or target region. We require that a minimum radiation dose D^{target} be administered to each tumor voxel, i.e., $d_i \geq D^{target}$ for $i \in \tau$. For all other voxels, we would like to have $d_i \leq D^{other}$, where D^{other} is a desired maximum dose for non-target voxels. This is generally not feasible, so instead we settle for minimizing the penalty,

$$E = \sum_{i \notin \tau} \left(d_i - D^{other}\right)_+ .$$

where $()_+$ denotes the nonnegative part of its argument (i.e., $(z)_+ = \max\{0, z\}$). We can interpret E as the total nontarget excess dose.

(a) Show that the treatment planning problem is a linear program. The optimization variable is $b \in \mathbb{R}^n$; the problem data are $B^{max}, A, T, D^{target}$, and D^{other}

(b) Solve the problem instance with data generated by the file `treatment-planning-data.m` using the matlab function "`linprog`". Here we have split the matrix A into `Atarget`, which contains the rows corresponding to the target voxels, and `Aother`, which contains the rows corresponding to other voxels. Plot the dose histogram for the target voxels, and also for the other voxels in Matlab (You can use the Matlab function "`hist`" to plot histograms.) Make a brief comment on what you see. Remark: The beam pattern matrix in this problem instance is randomly generated, but similar results would be obtained with realistic data.