Question 1: Expression Manipulation

1A:

$$A_1 e^{-\alpha t} u(t) + A_2 e^{-\beta t} \sin(\omega_0 t + \theta) u(t)$$

1B:

$$\frac{A_1}{s\left(\frac{\alpha}{s}+1\right)} + \frac{A_2\left(\omega_0 \cos(\theta) + (\beta+s)\sin(\theta)\right)}{\omega_0^2 + (\beta+s)^2}$$

To calculate the laplace transform, we have:

$$X(s) = L\{x(t)\} = L\{A_1 e^{-\alpha t} u(t)\} + L\{A_2 e^{-\beta t} sin(\omega_0 t + \theta) u(t)\}$$

$$= A_1 \frac{1}{s+\alpha} + L\{A_2 e^{-\beta t} sin(\omega_0 t) cos(\theta) + A_2 e^{-\beta t} cos(\omega_0 t) sin(\theta)\}$$

$$= \frac{A_1}{s+\alpha} + \frac{A_2 \omega_0 cos(\theta)}{(s+\beta)^2 + \omega_0^2} + \frac{A_2 (\beta+s) sin(\theta)}{(s+\beta)^2 + \omega_0^2}$$

1C:

$$\frac{A_1\beta^2 + 2A_1\beta s + A_1\omega_0^2 + A_1 s^2 + A_2\alpha\beta\sin(\theta) + A_2\alpha\omega_0\cos(\theta) + A_2\alpha s\sin(\theta) + A_2\beta s\sin(\theta) + A_2\omega_0 s\cos(\theta) + A_2 s^2\sin(\theta)}{\alpha\beta^2 + 2\alpha\beta s + \alpha\omega_0^2 + \alpha s^2 + \beta^2 s + 2\beta s^2 + \omega_0^2 s + s^3}$$

1D:

$$\frac{A_1}{\alpha+s} + \frac{A_2\left(\sin(\theta) - \mathrm{j}\cos(\theta)\right)}{2\left(\beta - \mathrm{j}\omega_0 + s\right)} + \frac{A_2\left(\sin(\theta) + \mathrm{j}\cos(\theta)\right)}{2\left(\beta + \mathrm{j}\omega_0 + s\right)}$$

By using this notation, we can implement many algorithms for various computations for rational functions like computations for antiderivatives, Taylor series expansions and even inverse of Z-transform and laplace-transform.

1E:

$$\frac{(A_1 + A_2\sin(\theta))\left(s + \frac{2A_1\beta + A_2\alpha\sin(\theta) + A_2\beta\sin(\theta) + A_2\omega_0\cos(\theta)}{2A_1 + 2A_2\sin(\theta)} - \frac{\sqrt{-4(A_1+A_2\sin(\theta))\left(A_1\beta^2 + A_1\omega_0^2 + A_2\alpha\beta\sin(\theta) + A_2\alpha\omega_0\cos(\theta)\right) + (2A_1\beta + A_2\alpha\sin(\theta) + A_2\beta\sin(\theta) + A_2\omega_0\cos(\theta))^2}}{2(A_1+A_2\sin(\theta))}\right)\left(s + \frac{2A_1\beta + A_2\alpha\sin(\theta) + A_2\beta\sin(\theta) + A_2\omega_0\cos(\theta)}{2A_1 + 2A_2\sin(\theta)} + \frac{\sqrt{-4(A_1+A_2\sin(\theta))\left(A_1\beta^2 + A_1\omega_0^2 + A_2\alpha\beta\sin(\theta) + A_2\alpha\omega_0\cos(\theta)\right) + (2A_1\beta + A_2\alpha\sin(\theta) + A_2\beta\sin(\theta) + A_2\omega_0\cos(\theta))^2}}{2(A_1+A_2\sin(\theta))}\right)}{(\alpha + s)(\beta - j\omega_0 + s)(\beta + j\omega_0 + s)}$$

And for ROC we have:

$$ROC = ROC_1 \cap ROC_2 \rightarrow ROC\text{: } Re\{s\} > max(-\alpha, -\beta)$$

Question 2: Laplace Analysis/Transform

Laplace Analysis:

2-1A:

We implement this circuit(It's shown in notebook).

2-1B:

$$\frac{\frac{1}{C}\frac{1}{L}}{s^2 + \frac{s(CR_1R_2+L)}{CLR_2} + \frac{R_1+R_2}{CLR_2}}$$

2-1C:

ZPK notation of transfer function:

$$\frac{\frac{1}{C}\frac{1}{L}}{\left(s + \frac{CR_1R_2+L}{2CLR_2} - \frac{\sqrt{C^2R_1^2R_2^2 - 2CLR_1R_2 - 4CLR_2^2 + L^2}}{2CLR_2}\right)\left(s + \frac{CR_1R_2+L}{2CLR_2} + \frac{\sqrt{C^2R_1^2R_2^2 - 2CLR_1R_2 - 4CLR_2^2 + L^2}}{2CLR_2}\right)}$$

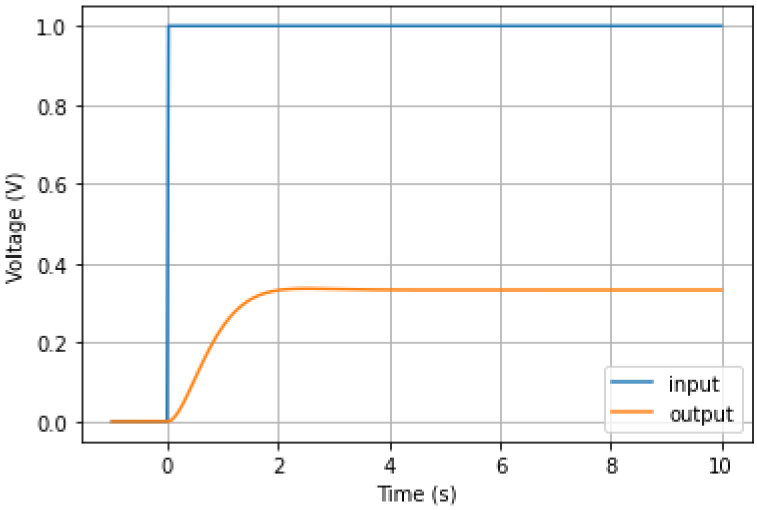
and ZPK notation of Laplace transform of step response:

$$\frac{\frac{1}{C}\frac{1}{L}}{s\left(s + \frac{CR_1R_2+L}{2CLR_2} - \frac{\sqrt{C^2R_1^2R_2^2 - 2CLR_1R_2 - 4CLR_2^2 + L^2}}{2CLR_2}\right)\left(s + \frac{CR_1R_2+L}{2CLR_2} + \frac{\sqrt{C^2R_1^2R_2^2 - 2CLR_1R_2 - 4CLR_2^2 + L^2}}{2CLR_2}\right)}$$

If we want to exclude sin/cos terms from the unit response, as you can see from above that the term $C^2R_1^2R_2^2 - 2CLR_1R_2 - 4CLR_2^2 + L^2$ should be positive. This is because if it's negative, then we would have a negative term under radical which will result in imaginary part and if we multiply two parentheses, we will have a term in a form of $(s + \alpha)^2 + \beta^2$, in which beta is a positive number and by separating fractions from each other and take an inverse Laplace transform, we'll see that sin/cos terms will be appeared in step response(we know that Laplace transform of sin/cos has a fraction form, which has a form of $(s + \alpha)^2 + \beta^2$ in its denominator). So the condition should be $C^2R_1^2R_2^2 - 2CLR_1R_2 - 4CLR_2^2 + L^2 > 0$.

2-1D:

$$\frac{\left(13e^{\frac{9t}{5}} - 3\sqrt{39}\sin\left(\frac{\sqrt{39}t}{5}\right) - 13\cos\left(\frac{\sqrt{39}t}{5}\right)\right)e^{-\frac{9t}{5}}u(t)}{39}$$



If we want to calculate the step response, we'll first calculate impedances and then use them to calculate $V_c(t)$. Since input is step, we have: $V_c(0^-) = 0$, $I_L(0^-) = 0$, so we have:

$Z_c(s) = \frac{1}{sc} = \frac{4}{s}$, $Z_L(s) = sL = 2.5s$, $Z_{3-0}(s) = \frac{Z_c * R_2}{Z_c + R_2} = \frac{4}{s+2}$

$\rightarrow Z_T(s) = \frac{4}{s+2} + 2.5s + 4 \rightarrow V_c(s) = \frac{Z_{3-0}}{Z_T} * \frac{1}{s} = \frac{4}{2.5s^3+9s^2+12s}$

$\rightarrow V_c(t) = L^{-1}\{V_c(s)\} = \frac{1}{3}u(t) - \frac{3\sqrt{39}}{39}sin(\frac{\sqrt{39}t}{5})e^{-\frac{9t}{5}}u(t) - \frac{1}{3}cos(\frac{\sqrt{39}t}{5})e^{-\frac{9t}{5}}u(t)$

2-1E:

To calculate the new transfer function, we can simply divide the previous transfer function by its impedance to get the new transfer function. So we will have:

$$\frac{s}{4CL\left(s^2 + \frac{s(CR_1R_2+L)}{CLR_2} + \frac{R_1+R_2}{CLR_2}\right)}$$

ZPK notation of new transfer function:

$$\frac{s\frac{1}{4CL}}{\left(s + \frac{CR_1R_2+L}{2CLR_2} - \frac{\sqrt{C^2R_1^2R_2^2-2CLR_1R_2-4CLR_2^2+L^2}}{2CLR_2}\right)\left(s + \frac{CR_1R_2+L}{2CLR_2} + \frac{\sqrt{C^2R_1^2R_2^2-2CLR_1R_2-4CLR_2^2+L^2}}{2CLR_2}\right)}$$

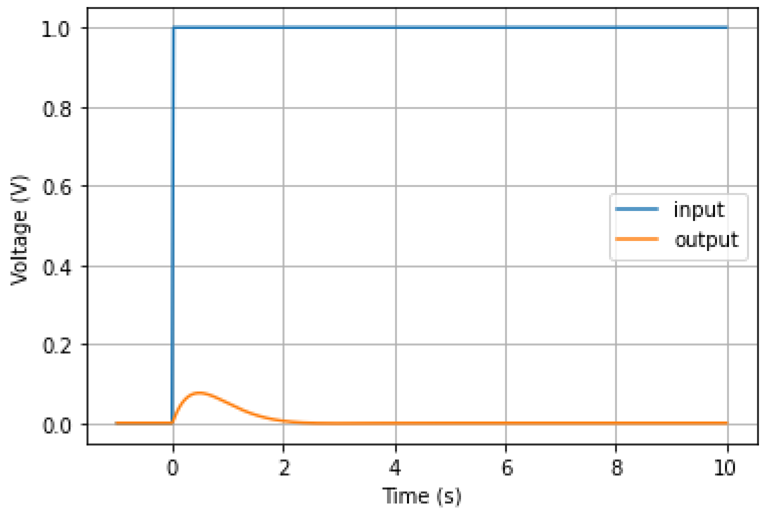
and ZPK notation of Laplace transform of step response:

$$\frac{\frac{1}{4}\frac{1}{C}\frac{1}{L}}{\left(s + \frac{CR_1R_2+L}{2CLR_2} - \frac{\sqrt{C^2R_1^2R_2^2-2CLR_1R_2-4CLR_2^2+L^2}}{2CLR_2}\right)\left(s + \frac{CR_1R_2+L}{2CLR_2} + \frac{\sqrt{C^2R_1^2R_2^2-2CLR_1R_2-4CLR_2^2+L^2}}{2CLR_2}\right)}$$

Again, if we want to exclude sin/cos terms from the unit response, we should have $C^2{R_1}^2{R_2}^2 - 2CLR_1R_2 - 4CL{R_2}^2 + L^2 > 0$ (The explanation is exactly same as above).

And finally, for step response, we have:

$$\frac{2\sqrt{39}e^{-\frac{9t}{5}}\sin\left(\frac{\sqrt{39}t}{5}\right)u(t)}{39}$$



If we want to calculate the step response, we can simply divide the previous step response by its impedance, so we have:

$$I_c(s) = \frac{V_c(s)}{Z_c(s)} = \frac{\frac{4}{2.5s^3+9s^2+12s}}{\frac{4}{s}} = \frac{1}{2.5s^2+9s+12} \rightarrow I_c(t) = L^{-1}\{I_c(s)\} = \frac{2\sqrt{39}e^{-\frac{9t}{5}}sin(\frac{\sqrt{39}t}{5})}{39}u(t)$$

Laplace Transform:

2-2A:

$$\frac{f_0^2}{2s} - 2f_0 + s + \frac{s\left(\frac{f_0^2}{2} - 6f_0 - 2\right)}{s^2+4} + 3 + \frac{2\left(\frac{3f_0^2}{2} + 2f_0 - 6\right)}{s^2+4}$$

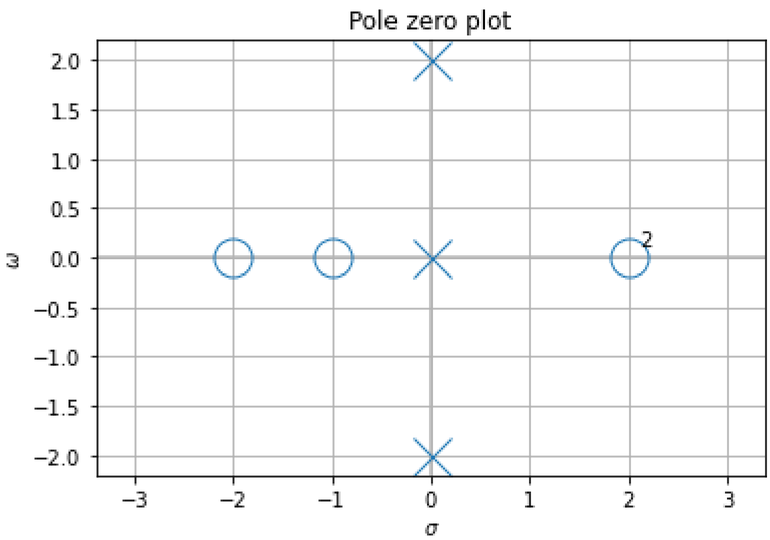
For calculating Laplace transform, we have:

$$X(s) = L\{x(t)\} = (3 - 2f_0)L\{\delta_0\} + (\frac{3f_0^2}{2} + 2f_0 - 6)L\{sin(2t)u(t)\}$$
$$+ (\frac{f_0^2}{2} - 6f_0 - 2)L\{cos(2t)u(t)\} + \frac{f_0^2}{2}L\{u(t)\} + L\{\delta^1(t)\}$$
$$\rightarrow X(s) = (3 - 2f_0) + (\frac{3f_0^2}{2} + 2f_0 - 6)\frac{2}{s^2+4}$$
$$+ (\frac{f_0^2}{2} - 6f_0 - 2)\frac{s}{s^2+4} + \frac{f_0^2}{2}\frac{1}{s} + s$$

2-2B:

$$(4\sin(2t) - 12\cos(2t) + 2)u(t) - \delta(t) + \delta^{(1)}(t)$$

2-2C:

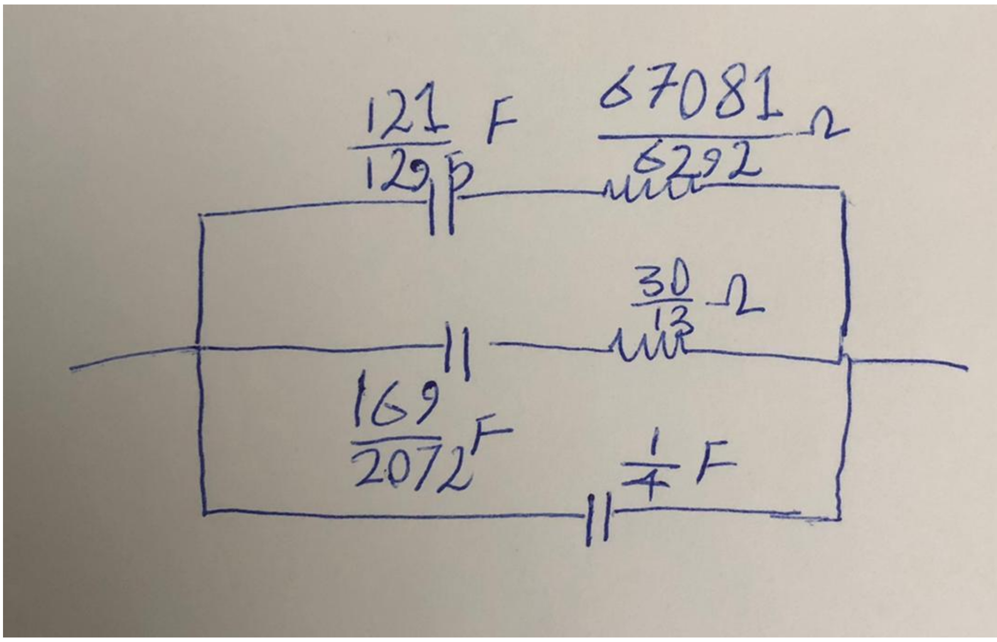


Question 3: Networks

3A:

$$(((C(\tfrac{121}{1295}) + R(\tfrac{67081}{6292}))|C(\tfrac{169}{2072})) + R(\tfrac{30}{13}))|C(\tfrac{1}{4})$$

The circuit:



To calculate the impedance of this circuit, we have:

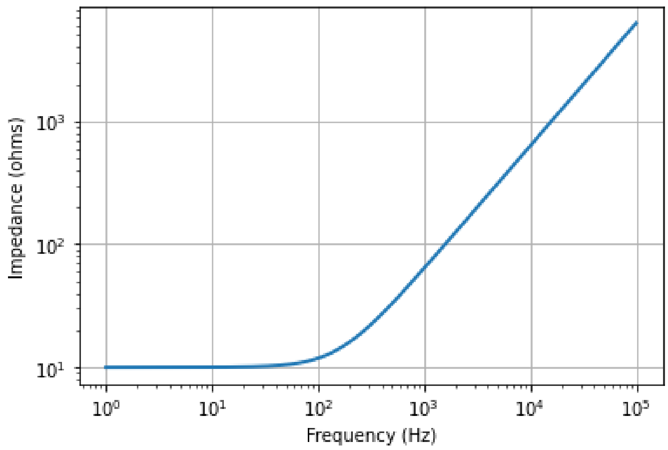
$$Z_{C_1}(s) = \frac{1}{s\frac{121}{1295}} = \frac{1295}{121s},\ Z_{R_1}(s) = \frac{67081}{6292} \rightarrow Z_1(s) = \frac{67340+67081s}{6292s}$$

$$Z_{C_2}(s) = \frac{1}{s\frac{169}{2072}} = \frac{2072}{169s},\ Z_{R_2}(s) = \frac{30}{13} \rightarrow Z_2(s) = \frac{2072+390s}{169s}$$

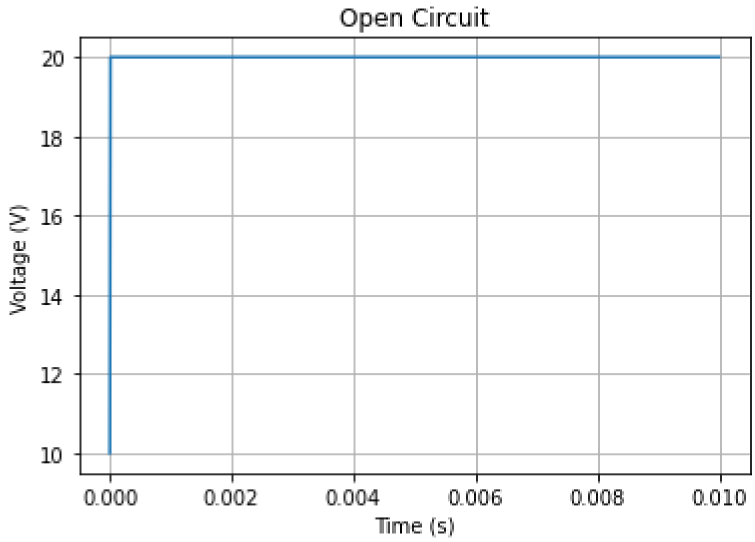
$$Z_3(s) = \frac{1}{\frac{1}{4}s} = \frac{4}{s} \rightarrow Z_T(s) = Z_1|Z_2|Z_3 \rightarrow Z_T(s) = \frac{60s^2+448s+320}{15s^3+138s^2+136s}$$

As you can see, the impedance calculated is the same as the first impedance given.

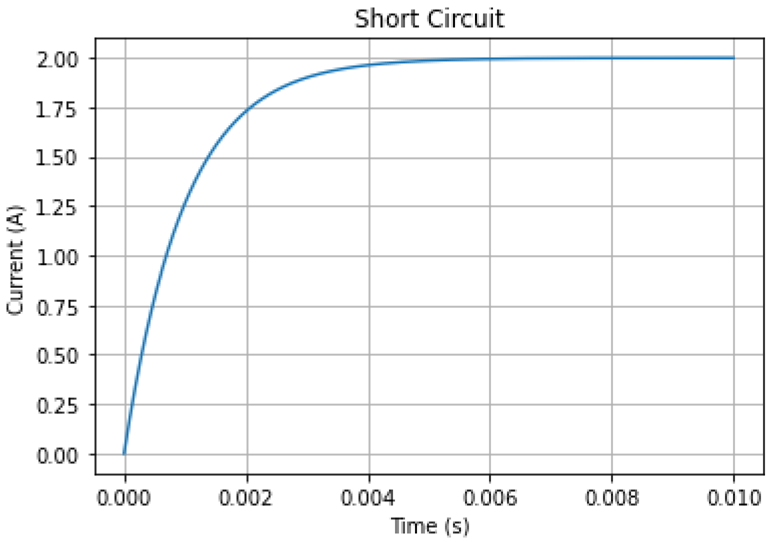
3B:



3C:



Open Circuit
Voltage (V)
20
18
16
14
12
10
0.000
0.002
0.004
0.006
0.008
0.010
Time (s)

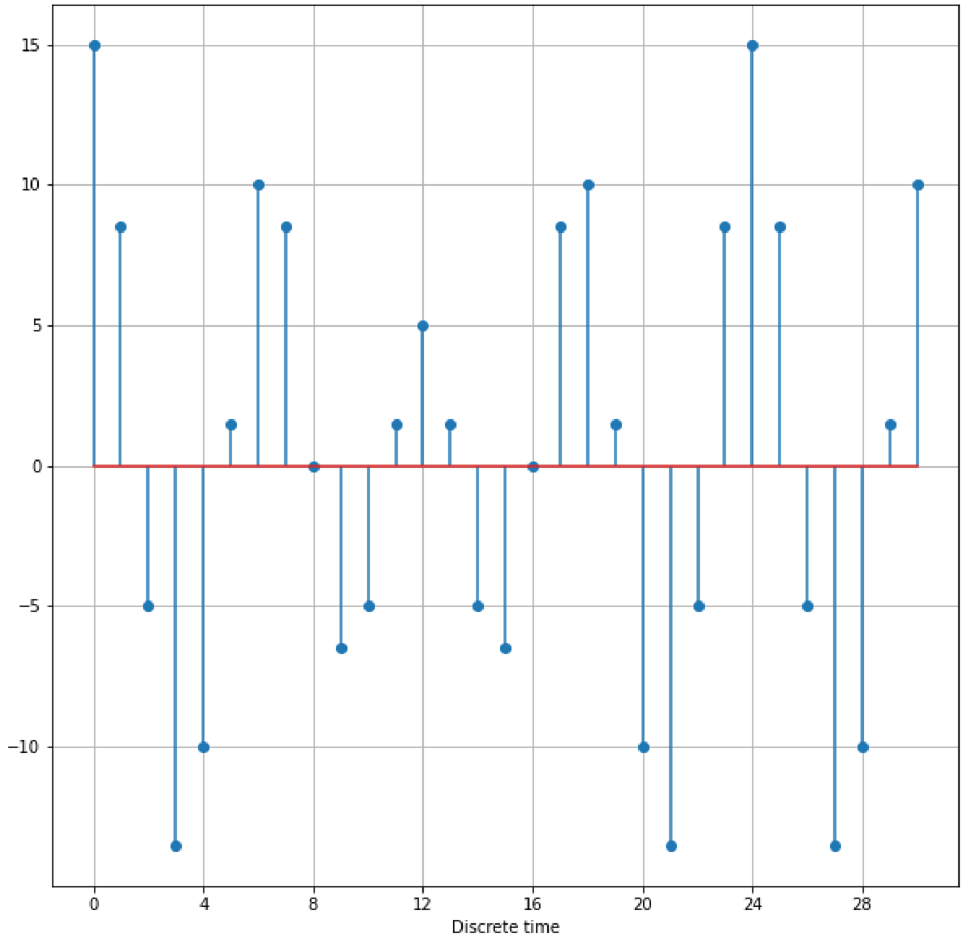


Short Circuit
Current (A)
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.000
0.002
0.004
0.006
0.008
0.010
Time (s)

Question 4: Discrete time signals

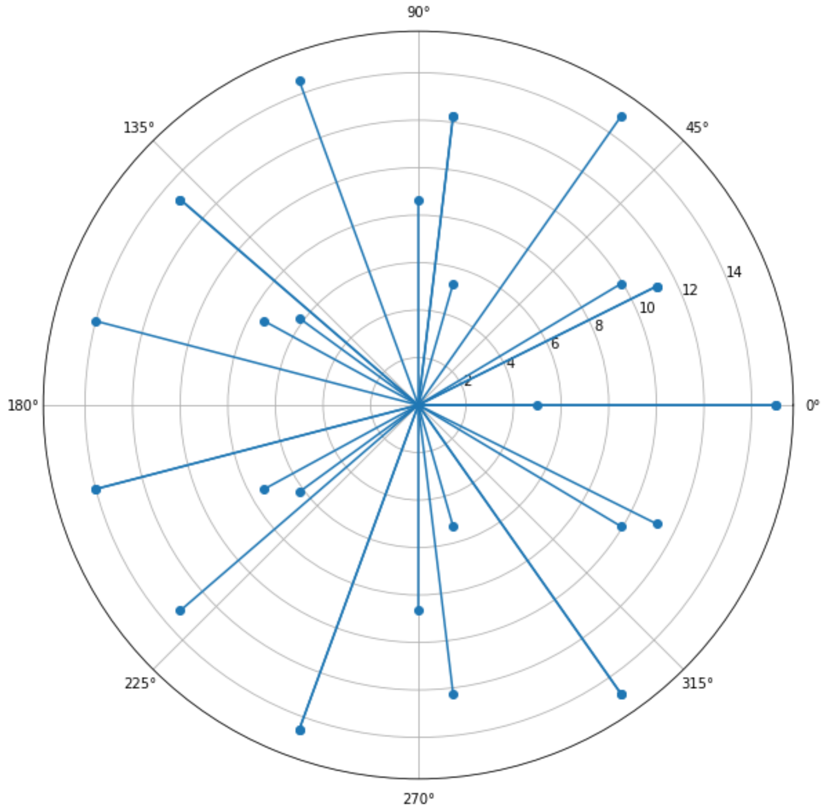
Discrete analysis:

4-1A:



As you can see from the plot, the period is 24.

4-1B:



If we check the polar plot, we will find that for n = 0, 1, 2, we have maximum magnitude and for n = 18, we have maximum phase, which is near 2π(I plot this plot for different intervals to find the values above).

Sequences:

4-2A:

Circular convolution, also known as **cyclic convolution**, is a special case of **periodic convolution**, which is the convolution of two periodic functions that have the same period. Periodic convolution arises, for example, in the context of the discrete-time Fourier transform (DTFT). In particular, the DTFT of the product of two discrete sequences is the periodic convolution of the DTFTs of the individual sequences. And each DTFT is a periodic summation of a continuous Fourier transform function (see DTFT § Definition). Although DTFTs are usually continuous functions of frequency, the concepts of periodic and circular convolution are also directly applicable to discrete sequences of data. In that context, circular convolution plays an important role in maximizing the efficiency of a certain kind of common filtering operation.

4-2B & 4-2C:

$$\left\{-2-3*j, -1+2j, 6+j, 7-6j, 2-7j, 3-2j\right\}$$

4-2D:

1	-j	-1	j	1	-j	: x_1
0	1	2	3	4	5	: x_2

$x_2 \xrightarrow{\text{flip } x_2}$

1	-j	-1	j	1	-j	x_1
5	4	3	2	1	0	x_2'

$x_4[0]$: 5 4 3 2 1

1	-j	-1	j	1	-j
0	5	4	3	2	1

$\rightarrow 0-5j-4+3j+2-j = -3j-2$

$x_4[1]$. 5 4 3 2

1	-j	-1	j	1	-j
1	0	5	4	3	2

$\rightarrow 1-5+4j+3-2j = -1+2j$

$x_4[2]$. we don't show the extra parts

1	-j	-1	j	1	-j
2	1	0	5	4	3

$\rightarrow 2-j+5j+4-3j = j+6$

$x_4[3]$.

1	-j	-1	j	1	-j
3	2	1	0	5	4

$\rightarrow 3-2j-1+5-4j = -6j+7$

$x_4[4]$.

1	-j	-1	j	1	-j
4	3	2	1	0	5

$\rightarrow 4-3j-2+j-5j = 2-7j$

$x_4[5]$,

1	-j	-1	j	1	-j
5	4	3	2	1	0

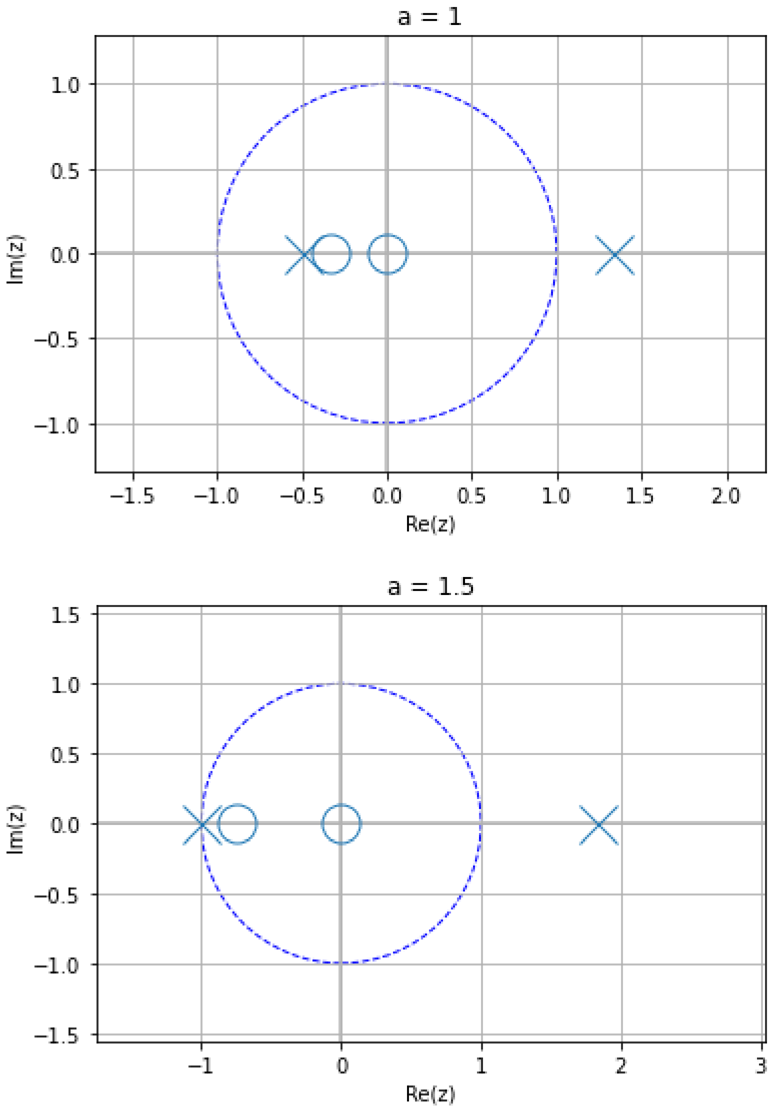
$\rightarrow 5-4j-3+2j+1 = 3-2j$

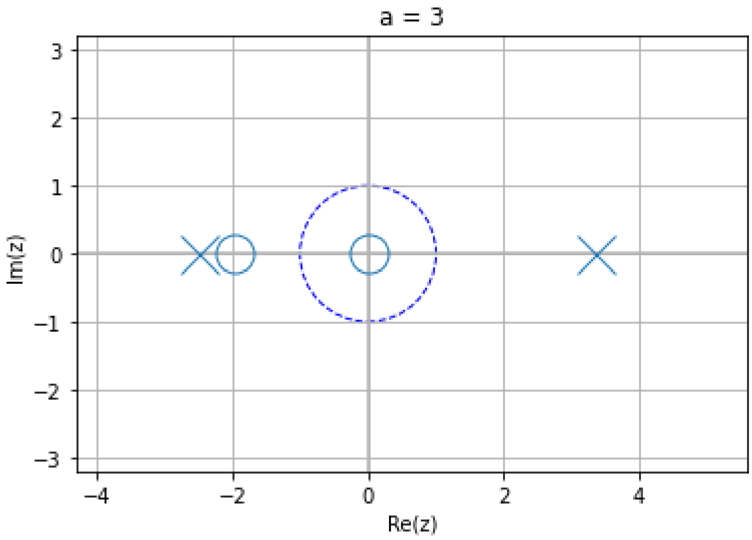
$\rightarrow x_4 = \{-2-3j, -1+2j, j+6, 7-6j, 2-7j, 3-2j\}$

As you can see, the calculated convolution is exactly the same as what lcapy found.

Discrete Transform:

4-3A:





As you can see, as a increases, we have two poles that get further and further(which is logical, because if we get a Z-transform from the function given, we'll see that we have an a in the denominator, which means by increasing a, the abs of poles increase too). This is the same for one of the zeros too, but the other one is fixed and doesn't change as a changes.

4-3B:

$$\frac{az}{(a-z)^2}$$

For Z-transform we have(using derivation property of Z):

$$Z\{n * a^n u[n]\} = -zd\frac{Z\{a^n u[n]\}}{dz} = \frac{az}{(a-z)^2}$$

Inverse of Z:

$$a^n nu[n]$$

DFT:

$$\frac{-Na^N + \left(Naa^N - aa^N + a\right)e^{-\frac{2j\pi k}{N}}}{\left(-ae^{-\frac{2j\pi k}{N}} + 1\right)^2}$$

Again if we use the DFT of famous function, we'll see that the result is the same as above.

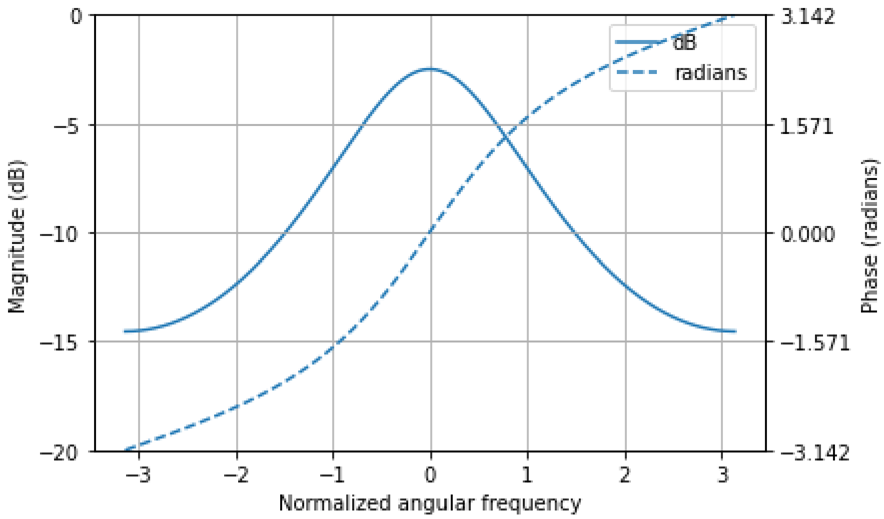
DTFT:

$$\frac{ae^{2j\pi\Delta_t f}}{a^2 - 2ae^{2j\pi\Delta_t f} + e^{4j\pi\Delta_t f}}$$

$$\frac{ae^{2j\pi F}}{a^2 - 2ae^{2j\pi F} + e^{4j\pi F}}$$

$$\frac{ae^{j\Omega}}{a^2 - 2ae^{j\Omega} + e^{2j\Omega}}$$

Plot:



Question 5: Difference Equation

5A:

$$y(n) = ax(n) + bx(n-1) - cy(n-1) - dy(n-2)$$

5B:

$$\frac{z\,(az+b)}{cz+d+z^2}$$

5C:

$$\frac{2^{-n}\left(\left(-c-\sqrt{c^2-4d}\right)^n\left(ac+a\sqrt{c^2-4d}-2b\right)+\left(-c+\sqrt{c^2-4d}\right)^n\left(-ac+a\sqrt{c^2-4d}+2b\right)\right)}{2\sqrt{c^2-4d}} \quad \text{for } n \geq 0$$

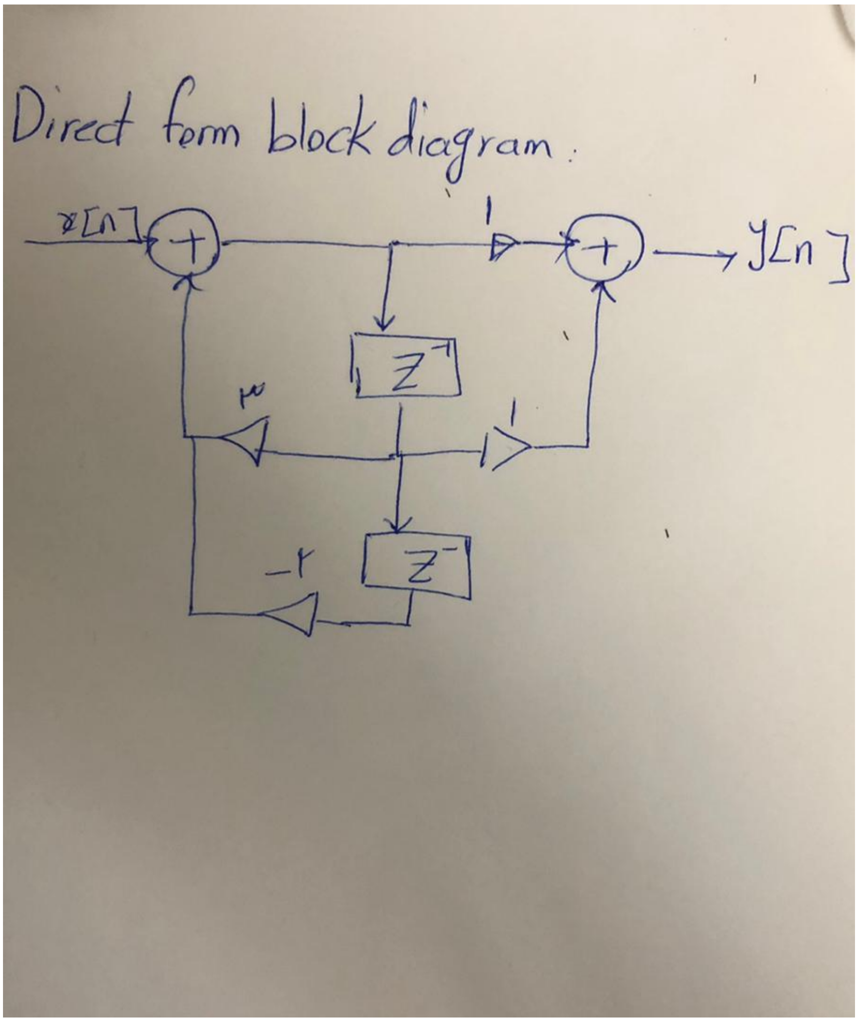
5D:

$$y(n) = x(n) + x(n-1) - 2y(n-2) + 3y(n-1)$$

5E:

$$y(n) = x(n) + x(n-1) - 2y(n-2) + 3y(n-1)$$

Block Diagram:



5F:

$$\{\underline{6}, 22, 66\}$$

Theory:

$$y[n] = x[n] + x[n-1] + 3y[n-1] - 2y[n-2]$$

$$\rightarrow y[n] - 3y[n-1] + 2y[n-2] = x[n] + x[n-1]$$

$$\xrightarrow{Z} H(z) = \frac{1 + z^{-1}}{1 - 3z^{-1} + 2z^{-2}} = \frac{z^2 + z}{z^2 - 3z + 2}$$

$$x[n] = 2^n \times u[n] \longrightarrow X(z) = \frac{1}{1 - 2z^{-1}}$$

$$\left. \begin{array}{l} H(z) \\ X(z) \end{array} \right\} \rightarrow Y(z) = \frac{z^2(z+1)}{(z-2)(z^2 - 3z + 2)} \xrightarrow{Z} y[n] = 2^n(3n-1) + 2, \quad n \geq 0$$

$$\left\{ \begin{array}{l} y[1] = 6 \\ y[2] = 22 \\ y[3] = 66 \end{array} \right.$$