Directed Graphical Models: Bayesian Networks

Probabilistic Graphical Models

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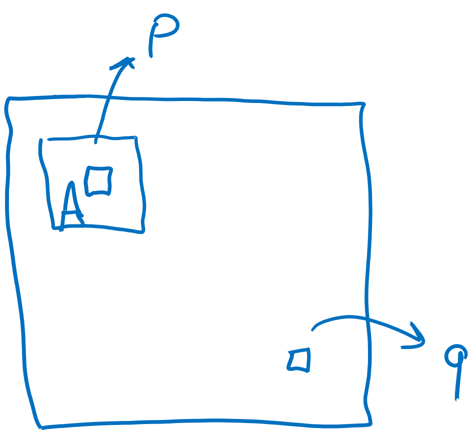
Slides from Dr. Soleymani's PGM course, Sharif University of Technology

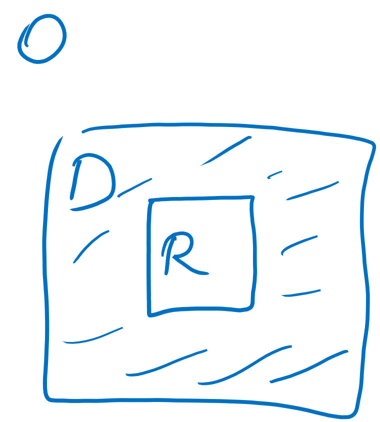
Basics

- Multivariate distributions with large number of variables

- Independency assumptions are useful
  - Independence and conditional independence relationships simplify representation and alleviate inference complexities



$$\boxed{p \perp q \mid A}$$



$$R \perp O \mid D$$

Conditional and marginal independence

- X and Y are **conditionally independent** given Z if:

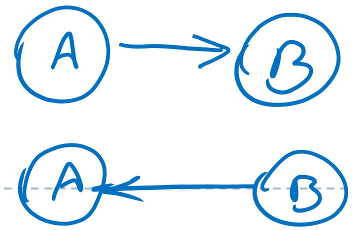
$X \perp Y | Z$

$$\boxed{P(X,Y|Z) = P(X|Z)P(Y|Z)} \longleftrightarrow \begin{array}{l} P(X|Y,Z) = P(X|Z) \\ P(Y|X,Z) = P(Y|Z) \end{array}$$

- X and Y are **marginal independent** if:

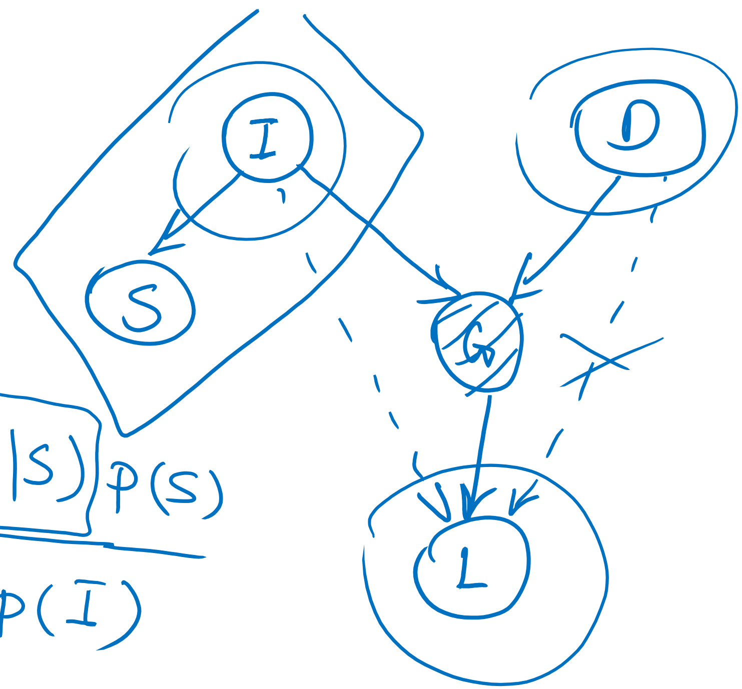
$\boxed{X \perp Y | \emptyset}$

$$\boxed{P(X,Y) = P(X)P(Y)} \longleftrightarrow \begin{array}{l} P(X|Y) = P(X) \\ P(Y|X) = P(Y) \end{array}$$



Example

- Random variables:
  - D — Course difficulty
  - L — Quality of recommendation letter
  - I — Intelligence
  - G — Grade
  - S — SAT score

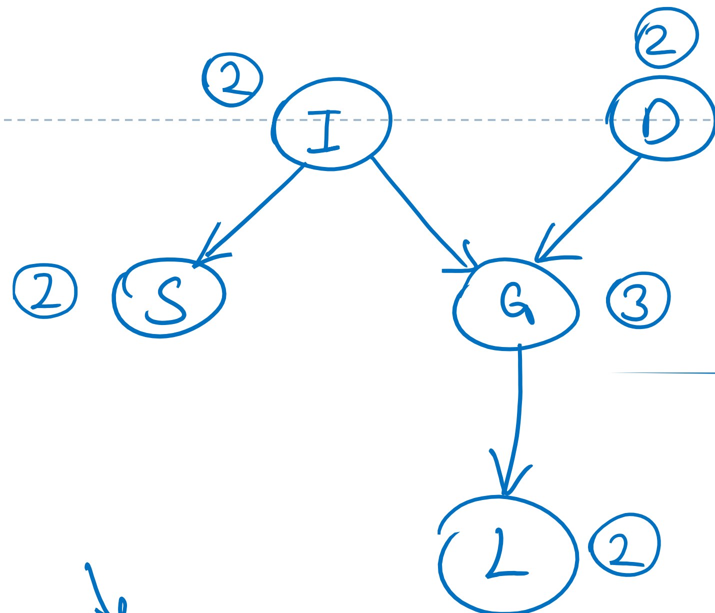


$L \perp I \mid G$

$$\boxed{P(S \mid I)} = \frac{\boxed{P(I \mid S)}\, P(S)}{P(I)}$$

نرمال

$P(S \mid I=0)$ $P(S \mid I=1)$



$$\boxed{P(X_1,\dots,X_n)=\prod_{i=1}^{n} P(X_i \mid Pa(X_i))}$$

$$\boxed{P(I,S,D,G,L)} = \underbrace{P(I)}_{1}\,\underbrace{P(D)}_{1}\,\underbrace{P(S \mid I)}_{2}\,\overbrace{\underbrace{P(G \mid I,D)}_{}}^{8}\,\underbrace{P(L \mid G)}_{3}$$

2

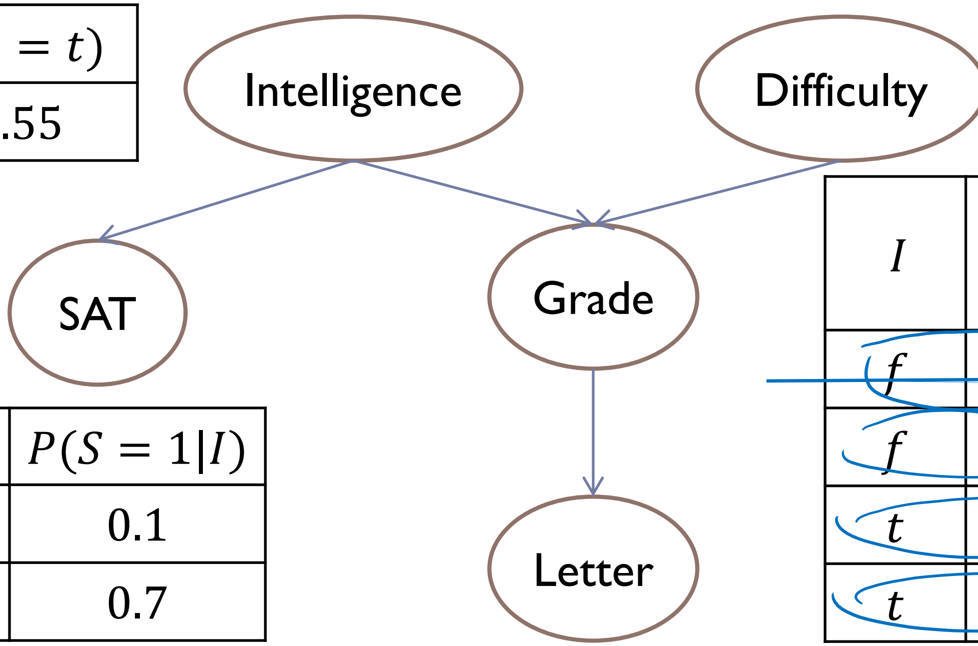
I S D G L	Prob
	○
	○
	○
	⋮
	○

48

15

Example

$P(I = t)$
0.55



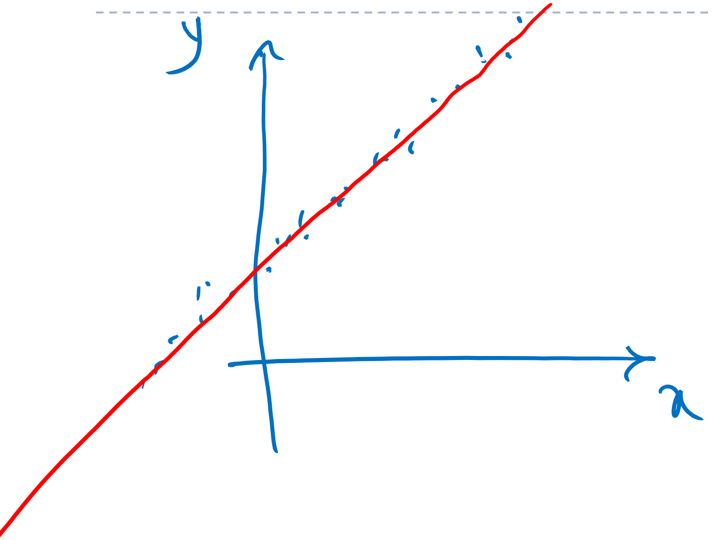
$P(D = t)$
0.65

I	$P(S = 1 \mid I)$
f	0.1
t	0.7

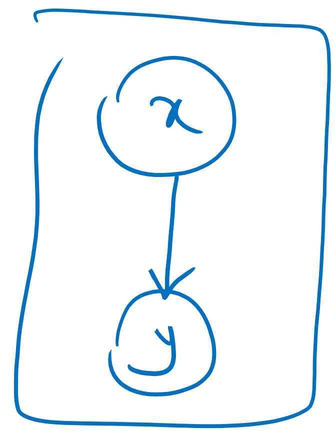
I	D	$P(G \mid I, D)$		
		$G = 1$	$G = 2$	$G = 3$
f	f	0.3	0.4	0.3
f	t	0.05	0.25	0.7
t	f	0.9	0.08	0.02
t	t	0.5	0.3	0.2

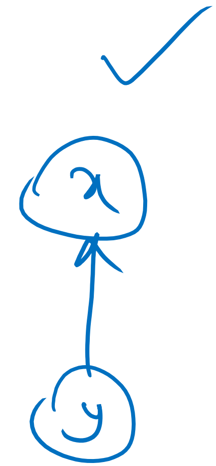
G	$P(L = t \mid G)$
1	0.9
2	0.5
3	0.05

Continuous example: Linear regression



$$\boxed{y = ax + b}$$



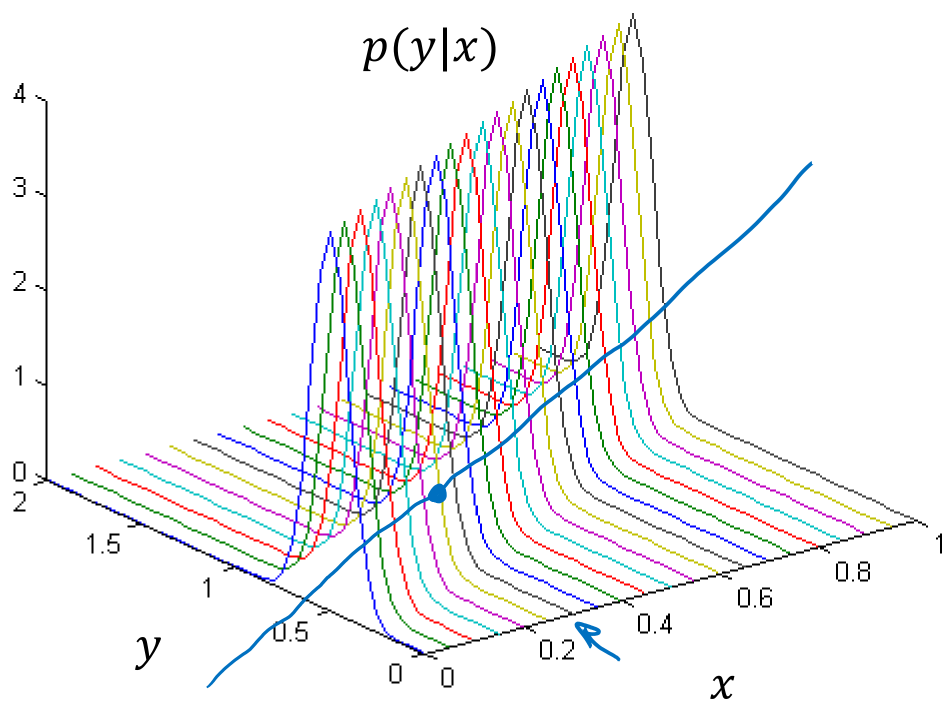


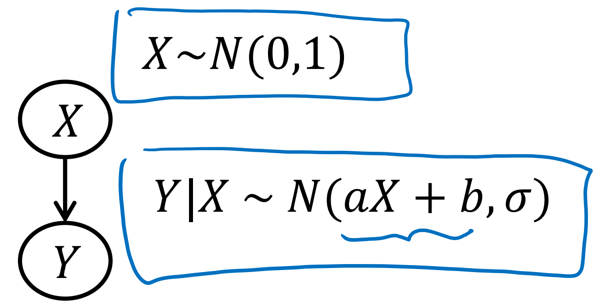
$$P(x,y) = \underbrace{P(x)}\,\underbrace{P(y|x)}$$

$$= P(y)\,P(x|y)$$

Continuous variables example

- Linear Gaussian





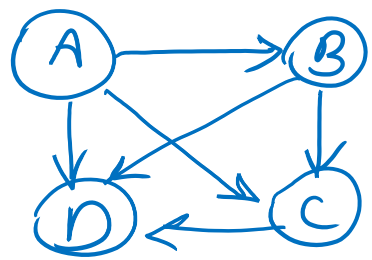
Missing edges

$X_1, X_2, \ldots, X_n$

(A, B, C, D)

- Chain Rule

$$P(A,B,C,D) = P(A)\,P(B|A)\,P(C|A,B)\,P(D|A,B,C)$$



- Missing edges imply conditional independencies.

- The more the sparse DAG, the more conditional independencies.

Compact representation

- A BN for a Boolean variables with k Boolean parents

$$\boxed{X_1, X_2, \dots, X_n}$$

$$\boxed{k \ll n}$$

$$P(X_1, \dots, X_n) = \prod_{i=1}^{n} \underbrace{P(X_i \mid Pa(X_i))}_{2^k}$$

$$\boxed{n \times 2^k}$$

$$\boxed{2^n - 1}$$

$I(P) = \{A \perp B \mid C,\ A \perp D \mid C,\ \dots\}$

Factorization & independence

- Let G be a graph over $X_1, \dots, X_n$, distribution P **factorizes** over G if:

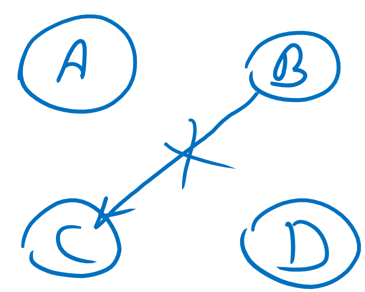
$$\boxed{P(X_1, \dots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid Pa(X_i)\right)}$$

P

G

$\boxed{I(P)}$

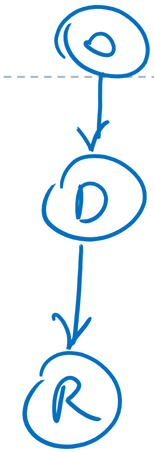
$\boxed{I(G)}$

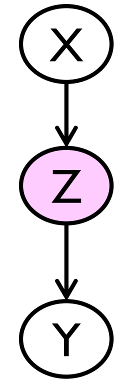


Basic structures

$O \perp R \mid D$

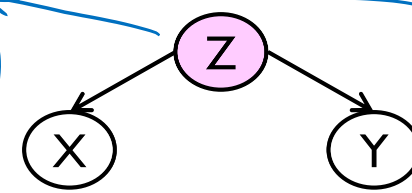
- $X \perp Y \mid Z$





- $X \perp Y \mid Z$

باهوش

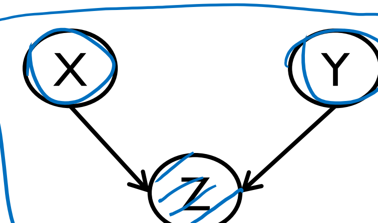


نمره فیزیک

نمره ریاضی

- $X \perp Y$

V-structure

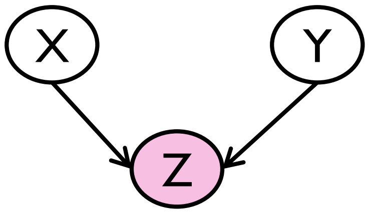


Explaining away

$$\boxed{Z = X + Y}$$

Explaining away

- When we condition on Z are X and Y are independent?



$$P(X,Y,Z) = P(X)P(Y)P(Z|X,Y)$$

- X and Y are marginally independent but given Z they are conditionally dependent

- This is called explaining away

- Two coins example

D-separation

I(P) I(G)

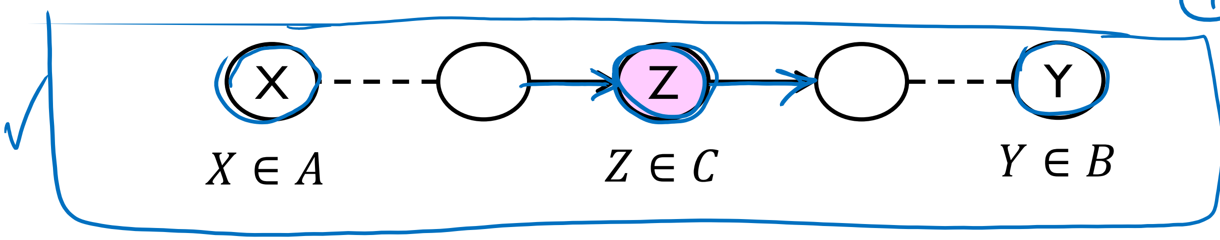
- Let A, B, C denote three disjoint sets of nodes, A is **d-separated** from B by C then $\boldsymbol{A} \perp \boldsymbol{B}|\boldsymbol{C}$

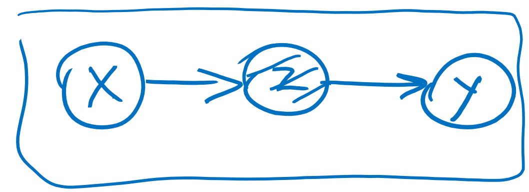
- A is **d-separated** from B by C if <u>all undirected paths between A and B are **blocked** by C</u>

Undirected path blocking

- Head-to-tail at a node $Z \in C$

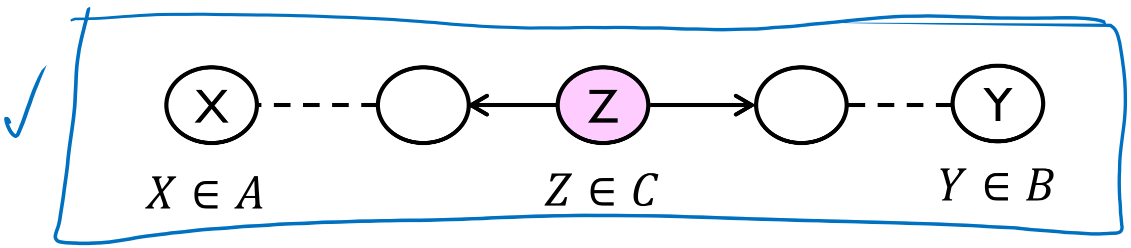
1

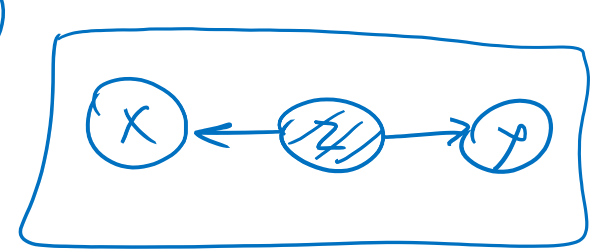




- Tail-to-tail at a node $Z \in C$

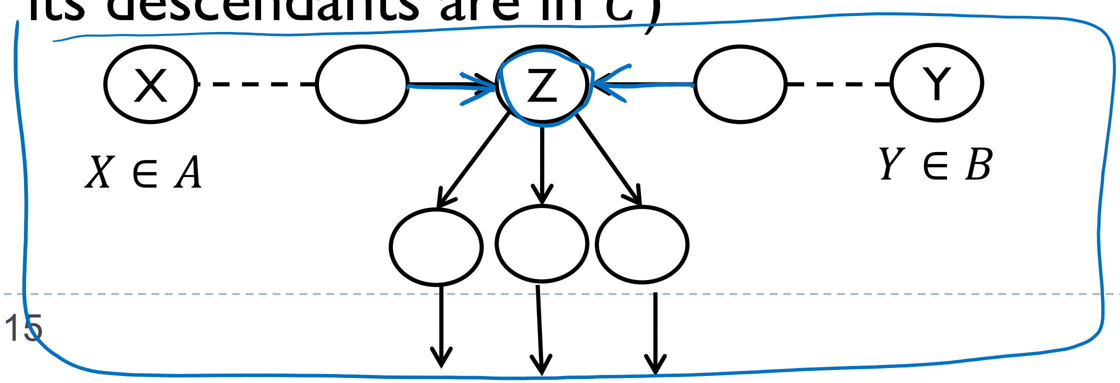
2





- Head-to-head (i.e., v-structure) at a node Z ($Z \notin C$ & none of its descendants are in C)

3



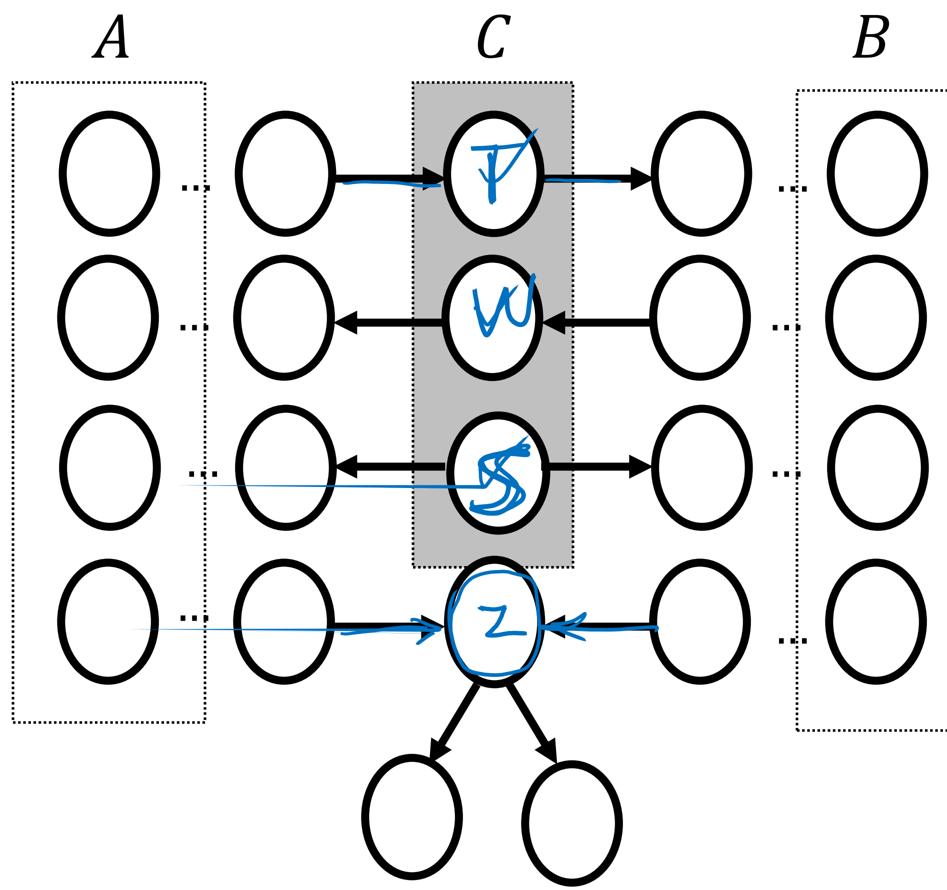
$A \perp B \mid T, S, W$

Undirected path blocking

In all trails (undirected paths) between A and B:

- A node in the path is in C and the path at the node do not meet head-to-head.
- Or a head-to-head node in the path, and neither the node, nor any of its descendants, is in C

$A \perp B \mid C$



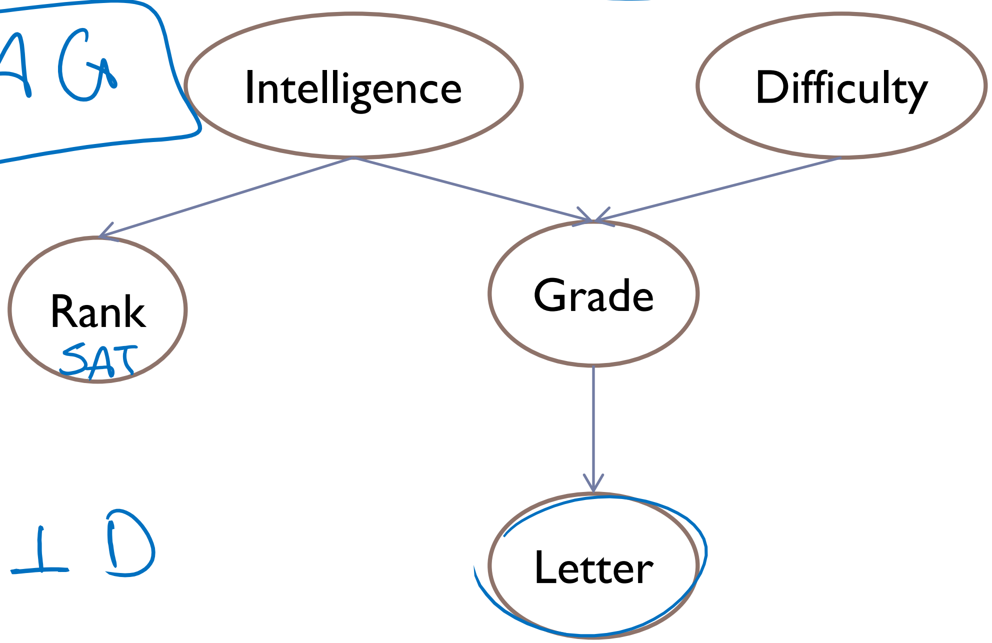
$A \perp B | C$

x y

D-separation: example

$I(G) \subseteq I(P)$

DAG



$R \perp G \mid I$ ✓

$R \perp D \mid I$ ✓

$R \perp D \mid G$ ✗

$R \perp D \mid L$ ✗ ✓

$R \perp L \mid G$

$D \perp L \mid G$

$R \perp D$ ✓

$R \perp D$

$R \perp D \mid I$

$R \perp D \mid G$ ✗

$R \perp D \mid I, G$

Markov Blanket in Bayesian Network

- A variable is <u>conditionally independent of all other variables given its Markov blanket</u>

- Markov blanket of a node:
  - All parents
  - Children
  - Co-parents of children

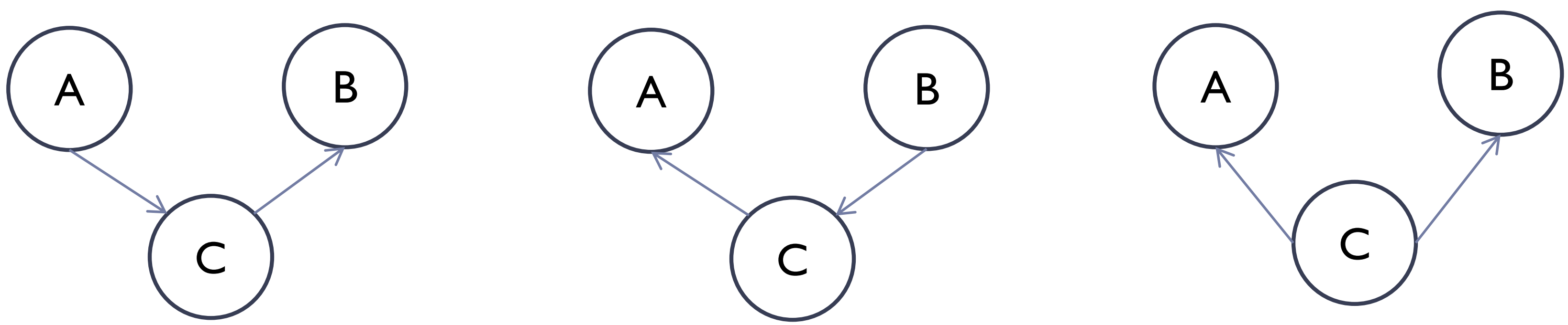
D-Separation: soundness & completeness

- **Soundness**: Any conditional independence properties that we can derive from G should hold for the probability distribution that factorize over G
  - **Theorem**: If P factorizes over G, and d-sep$_G(\boldsymbol{X}, \boldsymbol{Y}|\boldsymbol{Z})$ then P satisfies $\boldsymbol{X} \perp \boldsymbol{Y}|\boldsymbol{Z}$

- Weak **completeness**:
  - For almost all distributions P that factorize over G, if $\boldsymbol{X} \perp \boldsymbol{Y}|\boldsymbol{Z}$ in P then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are d-separated given $\boldsymbol{Z}$ in the graph G
    - There can be independencies in P that are not found by conditional independence properties of G

I-equivalence

- Definition: Two graphs G_1 and G_2 over a set of variables are I-equivalent if $I(G_1) = I(G_2)$



- Most graphs have many I-equivalent variants

I-map

- $I(G) = \{(\boldsymbol{X} \perp \boldsymbol{Y} | \boldsymbol{Z}) : \text{d-sepG}(\boldsymbol{X}, \boldsymbol{Y} | \boldsymbol{Z})\}$

- $I(G) \subseteq I(P)$

Minimal I-map

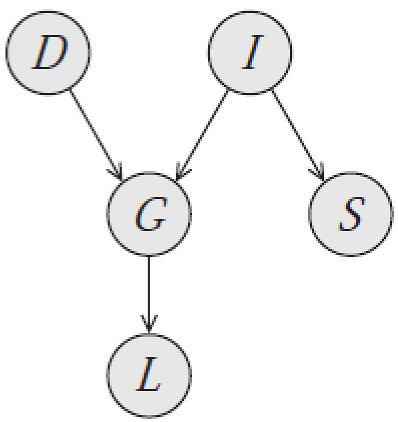
- When more independence relations exist in the graph
  - $\Rightarrow$ sparser representation (fewer parameters)
  - $\Rightarrow$ more informative or intuitive representation

- We want a graph that captures as much of the structure (conditional independence relations) in P as possible

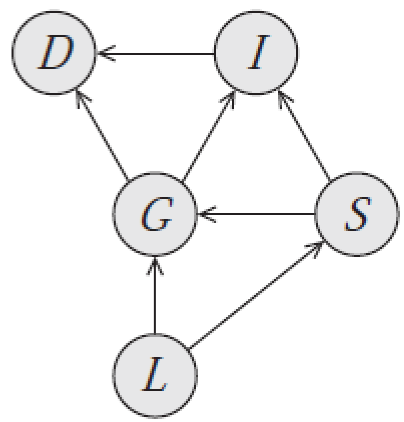
- G is a **minimal I-map** for P if it is an I-map for P, and also the removal of each edge from G renders it not an I-map.

Minimal I-map

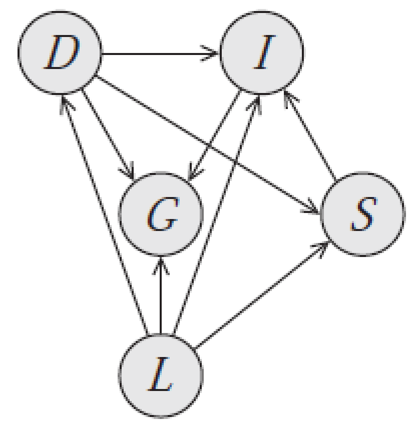
- The fact that G is a minimal I-map for P is far from a guarantee that G captures the independence structure in P



Perfect map of a distribution P



Minimal I-map of P



Minimal I-map of P

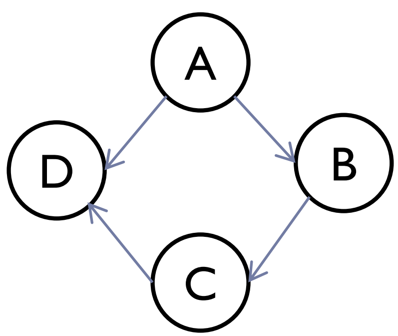
[Koller and Friedman Book]

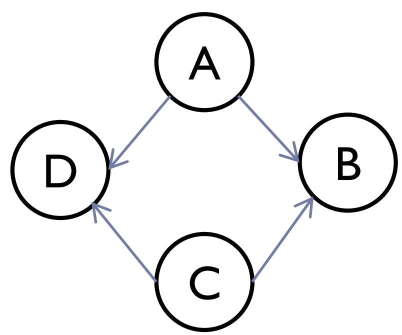
Perfect map

- Theorem: not every distribution has a perfect map as a DAG.
  - A distribution P with the independencies

    $I(P) = \{A \perp C | \{B, D\},\ B \perp D | \{A, C\}\}$

    cannot be represented by any Bayesian network.





Bayesian networks: summary

- *Bayesian network* is a pair (G, CPDs) where G is a DAG and CPDs can be used to find a joint distribution P that factorizes over G
  - Each CPD is the conditional distribution $P(X_i|Pa(X_i))$ associated to the graph node X_i.

- We can show "causality", "generative schemes", "asymmetric influences", etc., between variables via a Bayesian network

- We can find conditional independencies from the graph structure via d-separation criteria.

Reference

- D. Koller and N. Friedman, “Probabilistic Graphical Models: Principles and Techniques”, MIT Press, 2009 [Chapter 3].