Directed Graphical Models: Bayesian Networks

Probabilistic Graphical Models

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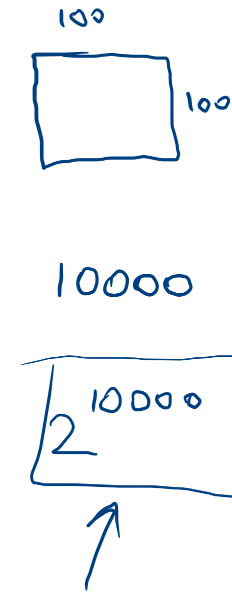
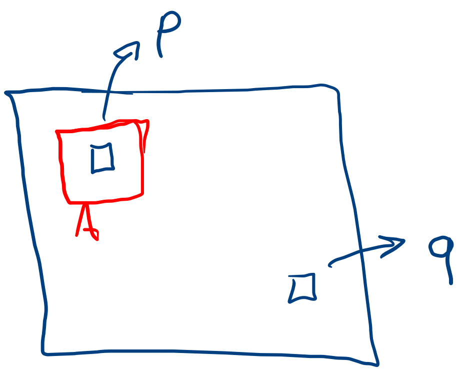
Slides from Dr. Soleymani's PGM course, Sharif University of Technology

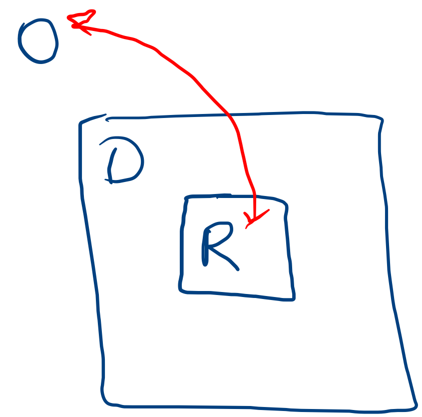
Basics

- Multivariate distributions with large number of variables
- Independency assumptions are useful
  - Independence and conditional independence relationships simplify representation and alleviate inference complexities





$P \perp q \mid A$



$R \perp O \mid D$

Conditional and marginal independence

- X and Y are **conditionally independent** given Z if:

  $X \perp Y | Z$

$$P(X, Y|Z) = P(X|Z)P(Y|Z) \longleftrightarrow \begin{array}{l} P(X|Y,Z) = P(X|Z) \\ P(Y|X,Z) = P(Y|Z) \end{array}$$

- X and Y are **marginal independent** if:

  $X \perp Y | \emptyset$

$$P(X, Y) = P(X)P(Y) \longleftrightarrow \begin{array}{l} P(X|Y) = P(X) \\ P(Y|X) = P(Y) \end{array}$$

$X \perp Y$

✓ $\boxed{X \perp Y}$

$$Z = X + Y$$

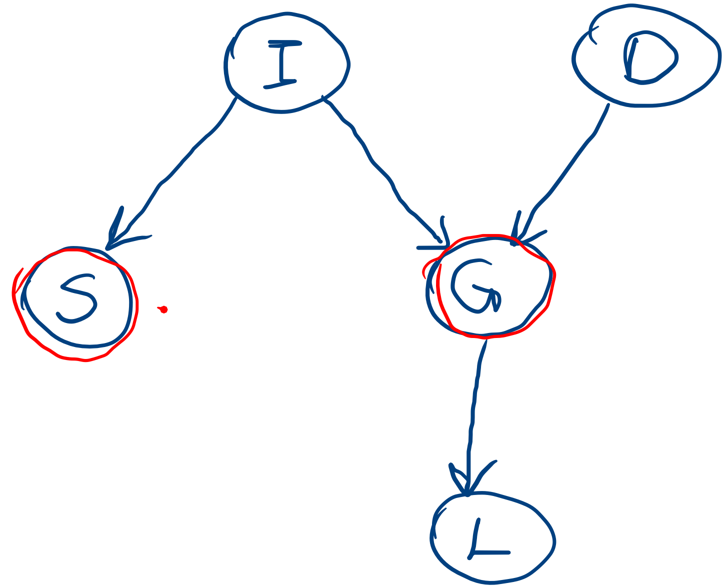
$$X \not\perp Y \mid Z$$

$$X \perp Y \mid Z$$

Example

DAG

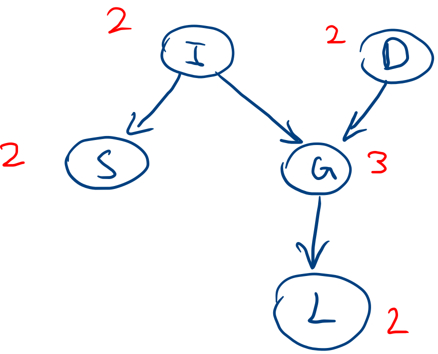
- Random variables:
  - D Course difficulty
  - L Quality of recommendation letter
  - I Intelligence
  - G Grade
  - S SAT score



$$\boxed{S \not\perp G}$$

$X_1, \dots, X_n$

$$P(X_1,\dots,X_n) = \prod_{i=1}^{n} P(X_i \mid Pa(X_i))$$



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$$P(I,D,S,G,L) = \underbrace{P(I)}_{1}\,\underbrace{P(D)}_{1}\,\underbrace{P(S \mid I)}_{2}\,\underbrace{P(G \mid I,D)}_{8}\,\underbrace{P(L \mid G)}_{3}$$

حالت 48

I	D	S	G	L	Prob.
0	0	0	0	0	○
0	0	0	0	1	○
		⋮			⋮
1	1	1	2	1	○

48

47

$P(S \mid I=0)$ → $P(S=0 \mid I=0)$ ✓, $P(S=1 \mid I=0)$

$P(S \mid I=1)$ 1

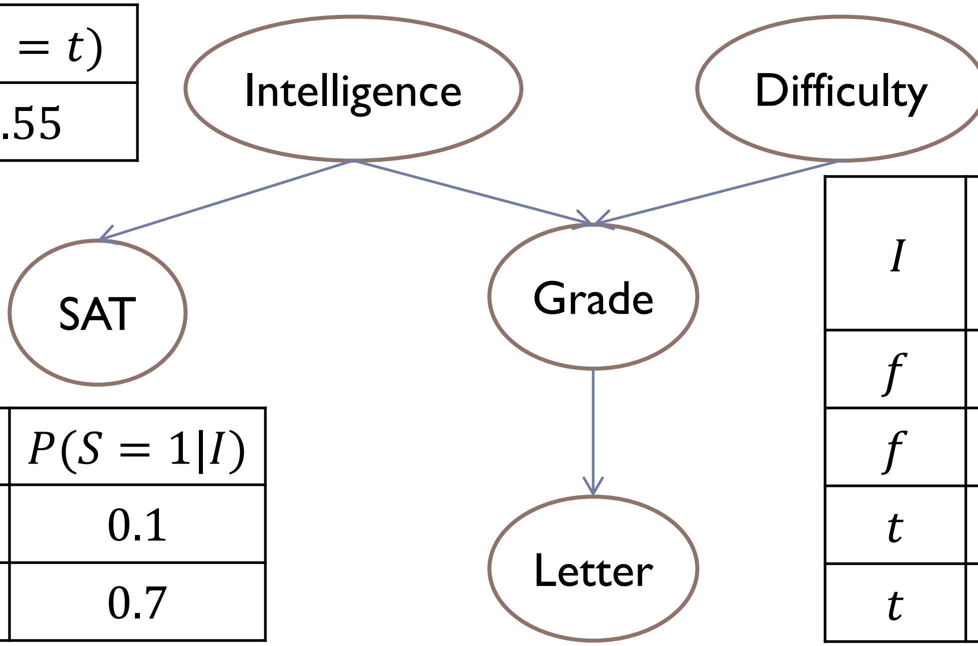
Discovery

Example

$$\boxed{P(A|B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

$P(I=t)$
0.55

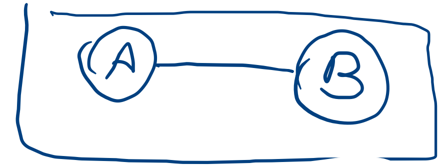
$P(D=t)$
0.65



I	D	$P(G\|I,D)$ $G=1$	$G=2$	$G=3$
f	f	0.3	0.4	0.3
f	t	0.05	0.25	0.7
t	f	0.9	0.08	0.02
t	t	0.5	0.3	0.2

I	$P(S=1\|I)$
f	0.1
t	0.7

G	$P(L=t\|G)$
1	0.9
2	0.5
3	0.05

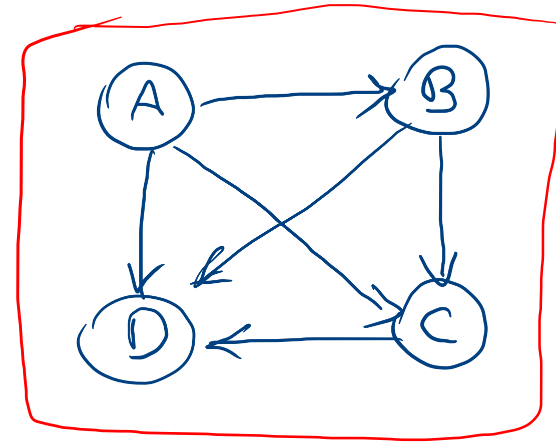


Missing edges

- Chain Rule

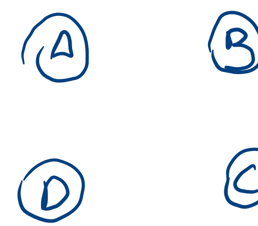
$$P(A,B,C,D) = \underline{P(A)}\,\underline{P(B|A)}\,\underline{P(C|B,A)}\,\underline{P(D|A,B,C)}$$

$$= P(B)\,P(C|B)\,P(A|B,C)\,P(D|A,B,C)$$



- Missing edges imply conditional independencies.

- The more the sparse DAG, the more conditional independencies.



Compact representation

- A BN for a Boolean variables with *k* <u>Boolean parents</u>

$n: X_1, \dots, X_n$

$\boxed{2^n - 1}$

$$P(X_1, \dots, X_n) = \prod_{i=1}^{n} P(X_i \mid \underbrace{Pa(X_i)}_{\leq k})$$

$\boxed{n 2^k}$

$\boxed{k \ll n}$

Factorization & independence

- Let G be a graph over $X_1, \dots, X_n$, distribution P **factorizes** over G if:

$$\boxed{P(X_1, \dots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid Pa(X_i)\right)}$$

$I(P) = \{ A \perp B \mid C,\ A, D \mid H, \dots \}$

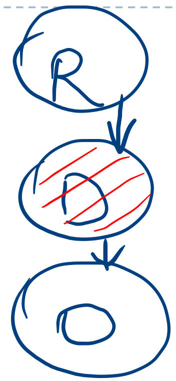
$I(G)$

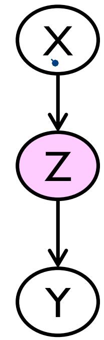
$\boxed{I(G) \subseteq I(P)}$

Basic structures

D_Separation

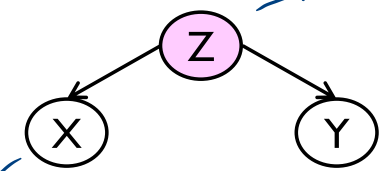
- $\boxed{X \perp Y | Z}$





- $X \perp Y | Z$

باهوش

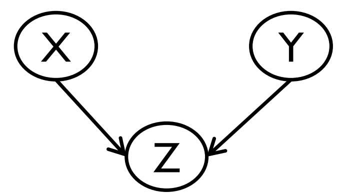


نمره فیزیک

نمره ریاضی

- $X \perp Y$

V-structure

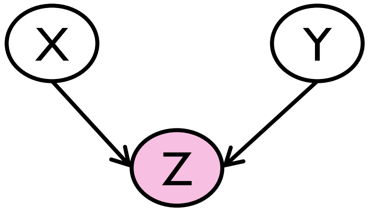


Explaining away

$Z = X + Y$

Explaining away

- When we condition on Z are X and Y are independent?



$$P(X,Y,Z) = P(X)P(Y)P(Z|X,Y)$$

- X and Y are marginally independent but given Z they are conditionally dependent

- This is called explaining away

- Two coins example

D-separation

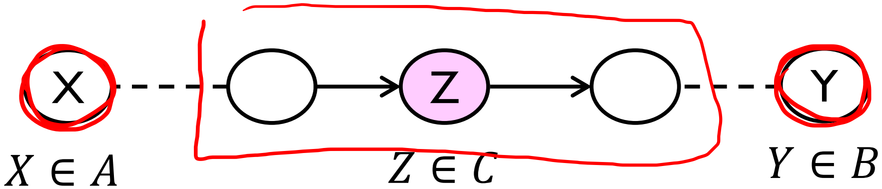
- Let A, B, C denote three disjoint sets of nodes, A is **d-separated** from B by C then $\boldsymbol{A} \perp \boldsymbol{B}|\boldsymbol{C}$

- A is **d-separated** from B by C if <u>all undirected paths between A and B are **blocked** by C</u>

Undirected path blocking

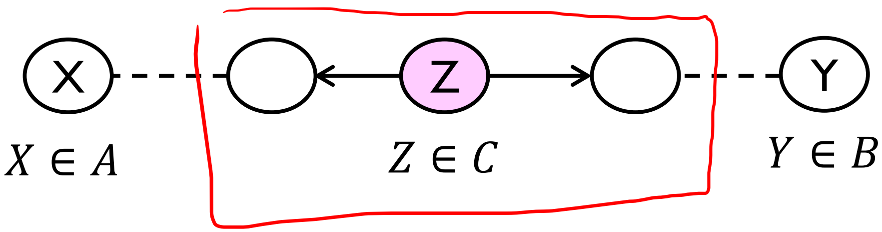
- Head-to-tail at a node $Z \in C$

1



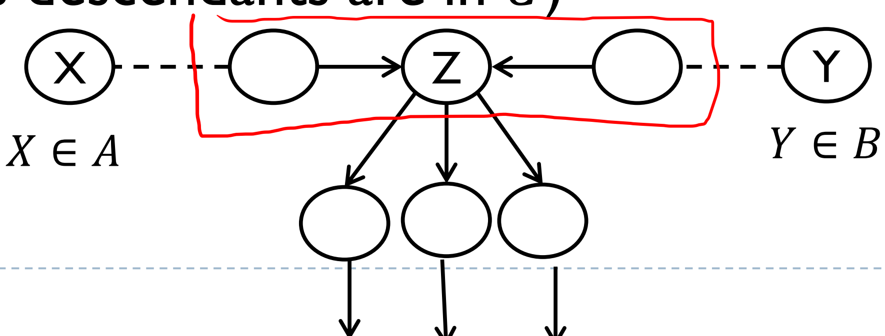
- Tail-to-tail at a node $Z \in C$

2



- Head-to-head (i.e., v-structure) at a node Z ($Z \notin C$ & none of its descendants are in C)

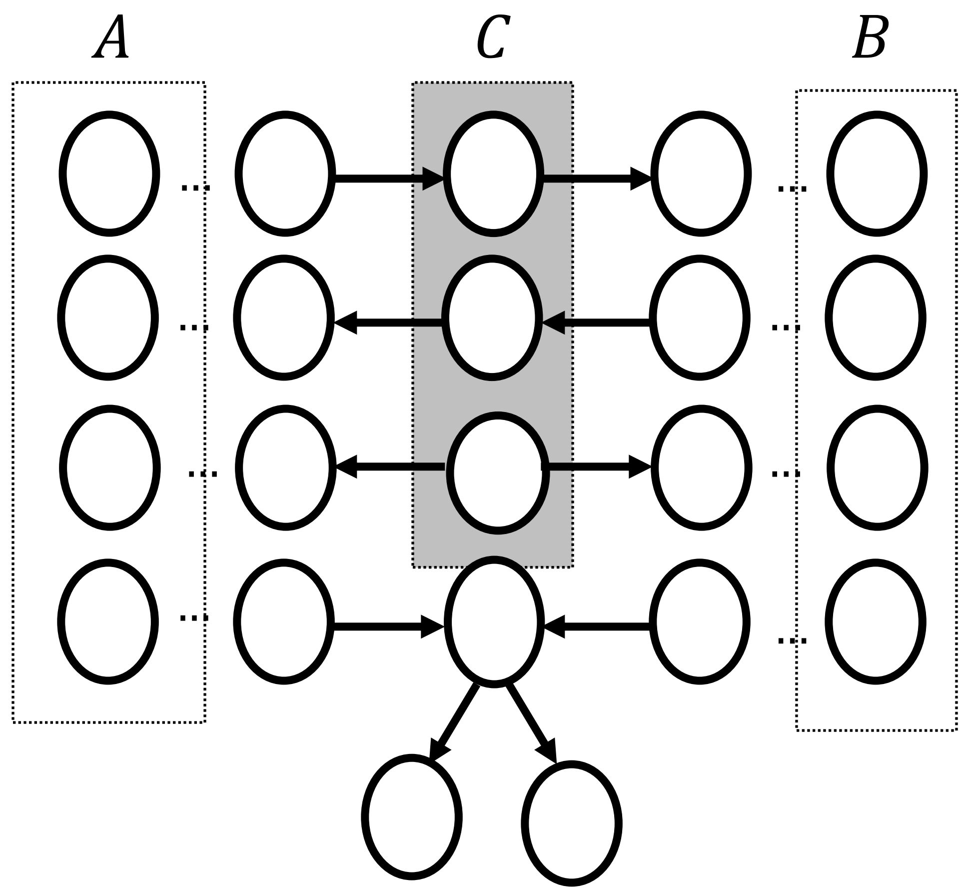
3



Undirected path blocking

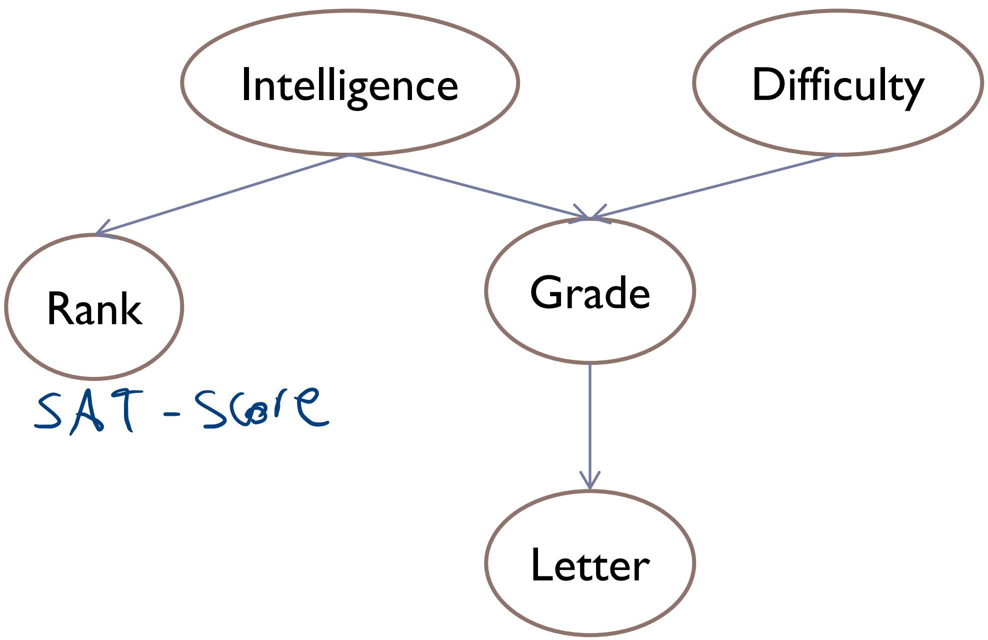
In all trails (undirected paths) between A and B:

- A node in the path is in C and the path at the node do not meet head-to-head.
- Or a head-to-head node in the path, and neither the node, nor any of its descendants, is in C



$A \perp B | C$

D-separation: example



$R \perp G | I$ ✓

$R \perp D | I$ ✓

$R \perp D | G$

$R \perp D | L$ ¿

$R \perp L | G$

$D \perp L | G$

$\boxed{R \perp D}$ ✓

$R \perp D | I, G$ ✓

$R \perp D | L$

Markov Blanket in Bayesian Network

- A variable is <u>conditionally independent of all other variables given its Markov blanket</u>

- Markov blanket of a node:
  - All parents
  - Children
  - Co-parents of children

D-Separation: soundness & completeness

- **Soundness**: Any conditional independence properties that we can derive from G should hold for the probability distribution that factorize over G
  - **Theorem**: If P factorizes over G, and d-sep$_G(\boldsymbol{X}, \boldsymbol{Y}|\boldsymbol{Z})$ then P satisfies $\boldsymbol{X} \perp \boldsymbol{Y}|\boldsymbol{Z}$

$I(G) \subseteq I(P)$

- **Weak completeness**:
  - For almost all distributions P that factorize over G, if $\boldsymbol{X} \perp \boldsymbol{Y}|\boldsymbol{Z}$ in P then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are d-separated given $\boldsymbol{Z}$ in the graph G
    - There can be independencies in P that are not found by conditional independence properties of G

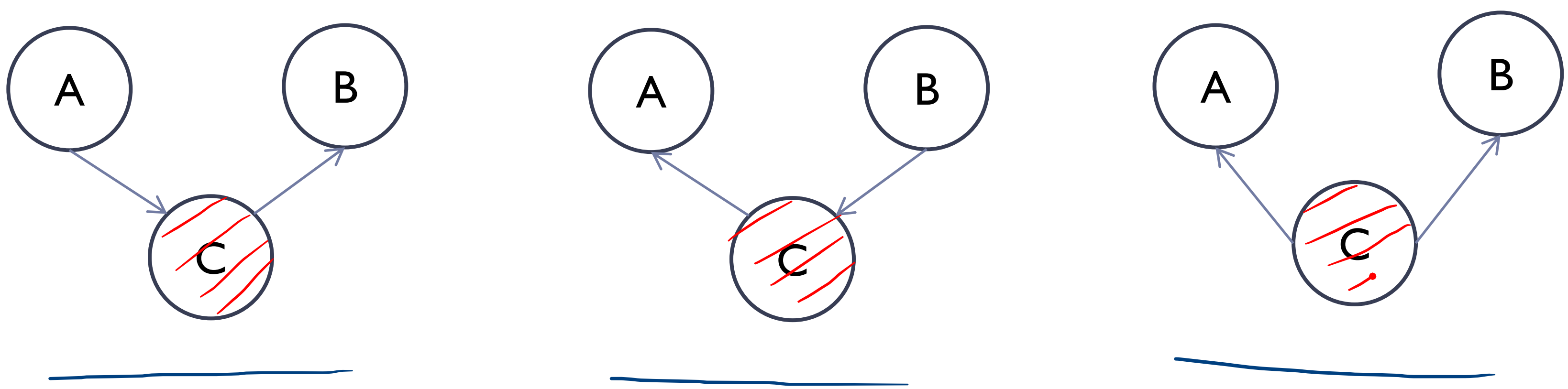
$I(G) = I(P)$

$I(P) \overset{?}{\subseteq} I(G)$

$P: A \perp B \mid C$

I-equivalence

- Definition: Two graphs G_1 and G_2 over a set of variables are I-equivalent if $I(G_1) = I(G_2)$



- Most graphs have many I-equivalent variants

I-map

- $I(G) = \{(\boldsymbol{X} \perp \boldsymbol{Y} | \boldsymbol{Z}) : \text{d-sepG}(\boldsymbol{X}, \boldsymbol{Y} | \boldsymbol{Z})\}$

- $\boxed{I(G) \subseteq I(P)}$

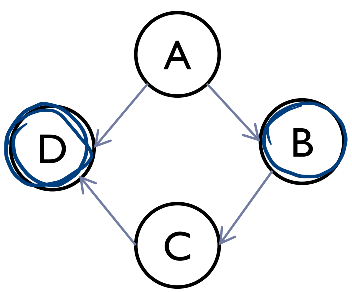
Perfect map

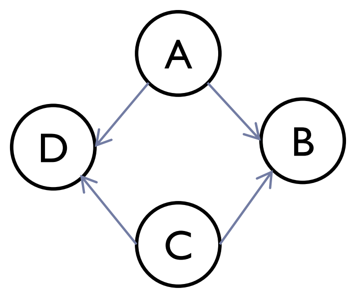
- Theorem: not every distribution has a perfect map as a DAG.
  - A distribution P with the independencies

$$I(P) = \{A \perp C | \{B, D\}, B \perp D | \{A, C\}\}$$

cannot be represented by any Bayesian network.

$$\boxed{I(G) = I(P)}$$





$$\boxed{B \perp D | A, C}$$

$B \perp D | A, C$

Bayesian networks: summary

- *Bayesian network* is a pair (G, CPDs) where G is a DAG and CPDs can be used to find a joint distribution P that factorizes over G
  - Each CPD is the conditional distribution $P(X_i|Pa(X_i))$ associated to the graph node X_i.

- We can show "causality", "generative schemes", "asymmetric influences", etc., between variables via a Bayesian network

- We can find conditional independencies from the graph structure via d-separation criteria.

Reference

- D. Koller and N. Friedman, "Probabilistic Graphical Models: Principles and Techniques", MIT Press, 2009 [Chapter 3].