Counterfactuals and Mediation

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Counterfactuals Basics

Important Application: Mediation

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Fundamental Problem of Causal Inference

$\mathrm{do}(T = 1)$

$Y_i(1) = 1$

$\mathrm{do}(T = 0)$

$Y_i(0) = 0$

T	: observed treatment
Y	: observed outcome
i	: used in subscript to denote a specific unit/individual
$Y_i(1)$	: potential outcome under treatment
$Y_i(0)$	: potential outcome under no treatment

Causal effect

$$Y_i(1) - Y_i(0) = 1$$

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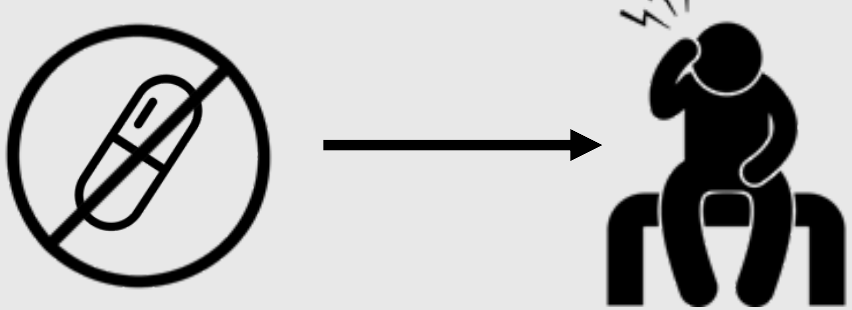
Factual

$\mathrm{do}(T=0)$

$Y_i(0) = 0$



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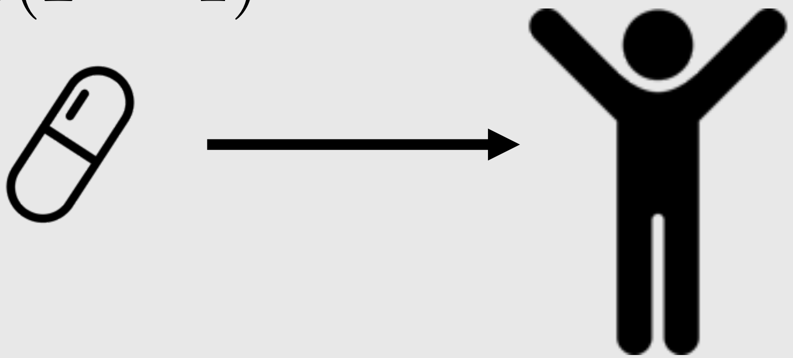

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Counterfactual

$\mathrm{do}(T = 0)$ $\qquad$ $Y_i(0) = 0$

Causal effect

$$Y_i(1) - Y_i(0) = 1$$

We can compute counterfactuals using a parametric SCM.

Counterfactuals

Counterfactual: $P(Y(t) \mid T = t', Y = y')$

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Cannot express counterfactuals using do-notation

Roadmap for Computing Counterfactuals

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Given: Observation of (T, Y) (observation of potential outcome $Y(t)$ where t is the observed value of T)

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Result: access to counterfactuals Y(t') at the unit-level

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Step 1: Solve for U

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$Y = UT + (1-U)(1-T) \quad Y_u(1)?$

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$$T := 1$$

$$Y := T$$

$$Y_u(1) = 1$$

ITE: $Y_u(1) - Y_u(0) = 1 - 0 = 1$

General Steps for Deterministic Counterfactuals

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3. Prediction: Use the value of U from step 1 and the modified SCM from step 2 to compute the value of Y(t)

Question:

Given the observation T = 1 and Y = 0, compute Y(0) for this individual given the following SCM:

$$
\begin{aligned}
T &:= \dots \\
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\end{aligned}
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$$P(U = \text{always happy}) = 0.3$$
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Observation: T = 1 and Y = 0

$$(Y_u(1) = 0)$$

$$P(U = \text{never happy} \mid T = 1, Y = 0) = \frac{0.2}{0.2 + 0.1} = \frac{2}{3}$$
$$P(U = \text{dog-hater} \mid T = 1, Y = 0) = \frac{0.1}{0.2 + 0.1} = \frac{1}{3}$$

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$$Y_u(0) = ?$$

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$$P(Y_u(0) = 1) = \frac{1}{3}$$

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Main ingredient necessary for computing counterfactuals: parametric model for the structural equation for Y

Strong assumption

Without it, we are stuck with the fundamental problem of causal inference.

Question:

Given the observation T = 1 and Y = 1, compute Y(0) for this individual given the following SCM and prior:

$$Y := \begin{cases} 1 & U = \text{always happy} \\ 0 & U = \text{never happy} \\ T & U = \text{dog-needer} \\ 1 - T & U = \text{dog-hater} \end{cases}$$

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