دانشگاه تهران- دانشکده مهندسی برق و کامپیوتر

ریاضیات مهندسی- نیم سال اول سال ۱۴۰۰-۱۴۰۱

تمرین ۱: سری فوریه

مدرس: دکتر مهدی طالع ماسوله- حل تمرین: وصال بخت آزاد و هلیا حسینی

برای سوالات خود در خصوص این تمرین با رایانامه های

bakhtazad.v@gmail.com و helia.ho3eini@gmail.com مکاتبه نمایید.

۱)

الف)

$f(x)$ is odd $\Longrightarrow a_n = 0 \;\&\; a_0 = 0$

$$b_n = \frac{2}{T}\int_T f(x)\sin\left(\frac{2n\pi x}{T}\right)dx = \frac{2}{2\pi}\int_{-\pi}^{+\pi} x\cos(x)\sin\left(\frac{2n\pi}{2\pi}x\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{+\pi} x\cos(x)\sin(nx)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{+\pi} x\sin\big((n+1)x\big)\,dx + \int_{-\pi}^{+\pi} x\sin\big((n-1)x\big)\,dx\right]$$

طبق جزء به جزء:

$$\int udv = uv - \int vdu$$

$$b_n = \frac{1}{2\pi}\left[\frac{-x\cos\big((n+1)x\big)}{n+1}\Big|_{-\pi}^{\pi} + \frac{1}{n+1}\int_{-\pi}^{+\pi}\cos\big((n+1)x\big)\,dx + \frac{-x\cos\big((n-1)x\big)}{n-1}\Big|_{-\pi}^{\pi} + \frac{1}{n-1}\int_{-\pi}^{+\pi}\cos\big((n-1)x\big)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{-x\cos\big((n+1)x\big)}{n+1}\Big|_{-\pi}^{\pi} + \frac{-x\cos\big((n-1)x\big)}{n-1}\Big|_{-\pi}^{\pi}\right] \Longrightarrow b_n = \frac{(-1)^n 2n}{n^2-1} \qquad n \neq 1$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{+\pi} x\cos(x)\sin(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{+\pi} x\sin(2x)\,dx = -\frac{1}{2}$$

$$\boxed{f(x) = -\frac{1}{2}\sin(x) + \sum_{n=2}^{\infty}\frac{(-1)^n 2n}{n^2-1}\sin(nx)}$$

ب)

$$a_0 = \frac{1}{T}\int_T g(x)dx = \frac{1}{2}\left[\int_0^1 xdx + \int_1^2 (1-x)dx\right] = 0$$

$$a_n = \frac{2}{T}\int_T g(x)\cos\left(\frac{2n\pi}{T}x\right)dx$$

$$= \int_0^1 x\cos(n\pi x)\,dx + \int_1^2 (1-x)\cos(n\pi x)\,dx$$

$$= \frac{x\sin(n\pi x)}{n\pi}\Big|_0^1 - \frac{\cos(n\pi x)}{n^2\pi^2}\Big|_0^1 + \frac{\sin(n\pi x)}{n\pi}\Big|_1^2 - \frac{x\sin(n\pi x)}{n\pi}\Big|_1^2 + \frac{\cos(n\pi x)}{n^2\pi^2}\Big|_1^2$$

$$= 0 + \frac{1-(-1)^n}{n^2\pi^2} + 0 + 0 + \frac{1-(-1)^n}{n^2\pi^2} = \frac{2(1-(-1)^n)}{n^2\pi^2}$$

$$b_n = \frac{2}{T}\int_T g(x)\sin\left(\frac{2n\pi}{T}x\right)dx$$

$$= \int_0^1 x\sin(n\pi x)\,dx + \int_1^2 (1-x)\sin(n\pi x)\,dx$$

$$= \frac{-x}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx + \frac{-1}{n\pi}\cos(n\pi x)\Big|_1^2 + \frac{x}{n\pi}\cos(n\pi x)\Big|_1^2 - \frac{1}{n\pi}\int_1^2 \cos(n\pi x)\,dx$$

$$= -\frac{(-1)^n}{n\pi} + 0 - \frac{1-(-1)^n}{n\pi} + \frac{2-(-1)^n}{n\pi} + 0 = \frac{1-(-1)^n}{n\pi}$$

$$\Rightarrow \boxed{g(x) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n^2\pi^2}\cos(n\pi x) + \frac{1-(-1)^n}{n\pi}\sin(n\pi x)}$$

ج)

$$h(x) = \begin{cases} 2\cos\left(\frac{\pi}{4}x\right), & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

$h(x)$ is even so $b_n = 0$

$$a_0 = \frac{1}{T}\int_T h(x)dx = \frac{1}{8}\int_{-2}^{2} 2\cos\left(\frac{\pi}{4}x\right)dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{T}\int_T h(x)\cos\left(\frac{2n\pi}{T}x\right)dx = \frac{2}{8}\int_{-2}^{2} 2\cos\left(\frac{\pi}{4}x\right)\cos\left(\frac{n\pi}{4}x\right)dx = \frac{1}{4}\int_{-2}^{2}\left(\cos\left(\frac{\pi(n+1)}{4}x\right) + \cos\left(\frac{\pi(n-1)}{4}x\right)\right)dx$$

$$= \frac{1}{\pi}\left[\frac{\sin\left(\frac{\pi(n+1)}{4}x\right)}{n+1} + \frac{\sin\left(\frac{\pi(n-1)}{4}x\right)}{n-1}\right]_{-2}^{+2}, \qquad n \neq 1$$

$$a_1 = \frac{2}{8}\int_{-2}^{2} 2\cos^2\left(\frac{\pi}{4}x\right)dx = \frac{1}{4}\int_{-2}^{2}\left(\cos\left(\frac{\pi}{2}x\right) + 1\right)dx = 1$$

$$a_n = \begin{cases} 0, & n = 2m+1,\ n \neq 1 \\ 1, & n = 1 \\ \frac{4}{\pi}\left[\frac{(-1)^{\frac{n}{2}+1}}{n^2-1}\right], & n = 2m \end{cases}$$

$$\Rightarrow \boxed{h(x) = \frac{2}{\pi} + \cos\left(\frac{\pi}{4}x\right) + \frac{4}{\pi}\sum_{\substack{n=2 \\ n\text{ even}}}^{\infty} \frac{(-1)^{\frac{n}{2}+1}}{n^2-1}\cos\left(\frac{n\pi}{4}x\right) = \frac{2}{\pi} + \cos\left(\frac{\pi}{4}x\right) + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{4m^2-1}\cos\left(\frac{m\pi}{2}x\right)}$$

د)

$$a_0 = \frac{1}{T}\int_T f(x)dx = \frac{1}{3}\int_{-2}^{1} f(x)\,dx = 1$$

$$a_n = \frac{2}{T}\int_T f(x)\cos\left(\frac{2n\pi}{T}x\right)dx = \frac{2}{3}\int_{-2}^{0}(2+x)\cos\left(\frac{2n\pi}{3}x\right)dx + \frac{2}{3}\int_{0}^{1}(-2x+2)\cos\left(\frac{2n\pi}{3}x\right)dx$$

$$= \frac{2}{3}\left(\frac{27}{4n^2\pi^2} - \frac{18\cos\left(\frac{2}{3}n\pi\right)}{4n^2\pi^2} - \frac{9\cos\left(\frac{4}{3}n\pi\right)}{4n^2\pi^2}\right) = \frac{9}{2n^2\pi^2} - \frac{3\cos\left(\frac{2}{3}n\pi\right)}{n^2\pi^2} - \frac{3\cos\left(\frac{4}{3}n\pi\right)}{2n^2\pi^2}$$

$$b_n = \frac{2}{T}\int_{-2}^{1} f(x)\sin\left(\frac{2n\pi}{3}x\right)dx = \frac{2}{3}\int_{-2}^{0}(2+x)\sin\left(\frac{2n\pi}{3}x\right)dx + \frac{2}{3}\int_{0}^{1}(-2x+2)\ \cos\left(\frac{2n\pi}{3}x\right)dx$$

$$= \frac{2}{3}\left(-\frac{9\sin\left(\frac{2}{3}n\pi\right)}{2n^2\pi^2} + \frac{9\sin\left(\frac{4}{3}n\pi\right)}{4n^2\pi^2}\right) = \frac{3\sin\left(\frac{4}{3}n\pi\right)}{2n^2\pi^2} - \frac{3\sin\left(\frac{2}{3}n\pi\right)}{n^2\pi^2}$$

$$\Longrightarrow \boxed{f(x) = 1 + \sum_{n}^{\infty}\left(\frac{9}{2n^2\pi^2} - \frac{3\cos\left(\frac{2}{3}n\pi\right)}{n^2\pi^2} - \frac{3\cos\left(\frac{4}{3}n\pi\right)}{2n^2\pi^2}\right)\cos\left(\frac{2n\pi}{3}x\right) + \left(\frac{3\sin\left(\frac{4}{3}n\pi\right)}{2n^2\pi^2} - \frac{3\sin\left(\frac{2}{3}n\pi\right)}{n^2\pi^2}\right)\sin\left(\frac{2n\pi}{3}x\right)}$$

۲)

$$\text{Taylor series: } \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n \Longrightarrow \frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n \xrightarrow{z=re^{i\theta}} \frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n r^n e^{in\theta}$$

$$\xrightarrow{e^{in\theta}=\cos(n\theta)+i\sin(n\theta)} \frac{1}{1+re^{i\theta}} = \sum_{n=0}^{\infty}(-1)^n r^n(\cos(n\theta) + i\sin(n\theta))$$

$$\Longrightarrow \operatorname{Im}\left\{\frac{1}{1+re^{i\theta}}\right\} = \frac{-r\sin(\theta)}{1+r^2+2r\cos(\theta)} = \sum_{n=0}^{\infty}(-1)^n r^n\sin(n\theta) = \sum_{n=1}^{\infty}(-1)^n r^n\sin(n\theta)$$

$$\xrightarrow{\int d\theta} \frac{1}{2}\ln(1+r^2+2r\cos(\theta)) + c = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}r^n}{n}\cos(n\theta)$$

$$\text{assume } \theta = 0 \Longrightarrow \frac{1}{2}\ln(1+r^2+2r) + c = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}r^n}{n} = \ln(1+r)$$

$$\Longrightarrow \ln(1+r^2+2r) + 2c = 2\ln(1+r) = \ln(1+r^2+2r) \Longrightarrow c = 0$$

$$\Longrightarrow \boxed{\ln(1+r^2 2r\cos(x)) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}r^n}{n}\cos(n\theta)}$$

۳)

با توجه به تابع پله که چنین تعریف می‌شود:

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

عبارت $u(1-|x|)$ به صورت زیر خواهد بود:

$$u(1-|x|) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

و در نتیجه $f(x)$ چنین است:

$$f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

سری فوریه این تابع در بازه‌ی $-2 < x < 2$ به صورت زیر محاسبه می‌شود.

$$b_n = 0$$

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-1}^{1} 1-x^2 dx = \frac{1}{3}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}\int_{-1}^{1}(1-x^2)\cos\left(\frac{n\pi}{2}x\right)dx =$$

$$-\frac{1}{2}\int_{-1}^{1}(x^2-1)\cos\left(\frac{n\pi}{2}x\right)dx = -\frac{1}{2}\left(\frac{2(x^2-1)\sin\left(\frac{n\pi}{2}x\right)}{n\pi}\Bigg|_{-1}^{1} - \int_{-1}^{1}\frac{4x\sin\left(\frac{n\pi}{2}x\right)}{n\pi}dx\right)$$

$$\int_{-1}^{1}\frac{4x\sin\left(\frac{n\pi}{2}x\right)}{n\pi}dx = \frac{4}{n\pi}\int_{-1}^{1}x\sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{4}{n\pi}\left(\frac{-2x\cos\left(\frac{n\pi}{2}x\right)}{n\pi}\Bigg|_{-1}^{1} + \int_{-1}^{1}\frac{2\cos\left(\frac{n\pi}{2}x\right)}{n\pi}dx\right) = \frac{4}{n\pi}\left(-\frac{4\cos\left(\frac{n\pi}{2}\right)}{n\pi} + \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\right)$$

$$\Rightarrow a_n = \frac{16\sin\left(\frac{n\pi}{2}\right)}{n^3\pi^3} - \frac{8\cos\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{16\sin\left(\frac{n\pi}{2}\right)}{n^3\pi^3} - \frac{8\cos\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\right)\cos\left(\frac{n\pi}{2}x\right)}$$

۴)

$$b_n = 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right) dx = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right)\cos(nx)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\cos\left(\left(n+\frac{1}{2}\right)x\right) + \cos\left(\left(n-\frac{1}{2}\right)x\right)\right) dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{n+\frac{1}{2}}\left(\sin\left(\left(n+\frac{1}{2}\right)x\right)\right) + \frac{1}{n-\frac{1}{2}}\left(\sin\left(\left(n-\frac{1}{2}\right)x\right)\right)\right]_{-\pi}^{\pi} = \frac{(-1)^{n+1}}{\pi(n^2-\frac{1}{4})}$$

$$\Longrightarrow \boxed{f(x) = \frac{2}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-\frac{1}{4}}\cos(nx)}$$

$$\text{if } x = \pi \Longrightarrow f(\pi) = 0 \Longrightarrow 0 = \frac{2}{\pi} - \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{\left(n^2-\frac{1}{4}\right)}\right)\cos(n\pi)$$

$$\Longrightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2-\frac{1}{4}} = 2}$$

۵)

سری فوریه تابع $f(x) = x^2$ در بازه‌ی $0 < x < \pi$ با گسترش زوج به صورت زیر محاسبه می‌شود:

$$b_n = 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\, dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)\, dx = \frac{2}{\pi}\left(\frac{x^2\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{2x\sin(nx)}{n}dx\right)$$

$$\int_0^{\pi}\frac{2x\sin(nx)}{n}dx = \frac{2}{n}\left(\frac{-x\cos(nx)}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}dx\right) = \frac{2}{n}\left(\frac{-x\cos(nx)}{n}\Big|_0^{\pi} + \frac{\sin(nx)}{n^2}\Big|_0^{\pi}\right) = -\frac{2\pi(-1)^n}{n^2}$$

$$\Longrightarrow a_n = \frac{2}{\pi}\left(0 - \left(-\frac{2\pi(-1)^n}{n^2}\right)\right) = \frac{4(-1)^n}{n^2}$$

$$f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)$$

الف) از دو طرف سری فوریه انتگرال می‌گیریم.

$$\int x^2dx=\int\frac{\pi^2}{3}dx+\int\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)+c=\frac{x^3}{3}=\frac{\pi^2}{3}x+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^3}\sin(nx)+c$$

assume $x=0\Longrightarrow 0=0+c\Longrightarrow c=0$

$$\Longrightarrow\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(nx)=\frac{1}{12}x(\pi^2-x^2)}$$

ب)

$$x=\frac{\pi}{2}\Longrightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi}{2}\right)=\sum_{m=1}^{\infty}\frac{-(-1)^m}{(2m-1)^3}=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}=\frac{\pi^3}{32}$$

۶)

سری فوریه تابع $f(x)=\cos(ax)$ را در بازه‌ی $-\pi\le x\le\pi$، به صورت زیر محاسبه می‌شود.

$$b_n=0$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(ax)\,dx=\frac{1}{\pi}\int_0^{\pi}\cos(ax)\,dx=\frac{\sin(\pi a)}{\pi a}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(ax)\cos(nx)\,dx=\frac{2}{\pi}\int_0^{\pi}\cos(ax)\cos(nx)\,dx=\frac{1}{\pi}\int_0^{\pi}\cos\big((n+a)x\big)\,dx+\frac{1}{\pi}\int_0^{\pi}\cos\big((n-a)x\big)\,dx$$

$$=\frac{1}{\pi}\left(\frac{\sin\big((n+a)x\big)}{n+a}\bigg|_0^{\pi}+\frac{\sin\big((n-a)x\big)}{n-a}\bigg|_0^{\pi}\right)=\frac{1}{\pi}\left(\frac{(-1)^n\sin(a\pi)}{n+a}-\frac{(-1)^n\sin(a\pi)}{n-a}\right)=-\frac{2a\sin(\pi a)}{\pi}\frac{(-1)^n}{n^2-a^2}$$

$$\Longrightarrow\boxed{f(x)=\frac{\sin(\pi a)}{\pi a}-\frac{2a\sin(\pi a)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2}\cos(nx)}$$

$$x=0\,,a=\frac{3}{2}\Longrightarrow\boxed{\left(\frac{-1}{\frac{9}{4}-1^2}\right)+\left(\frac{1}{\frac{9}{4}-2^2}\right)+\left(\frac{-1}{\frac{9}{4}-3^2}\right)+\cdots=\frac{\pi}{2a\sin(\pi a)}-\frac{1}{2a^2}}$$

۷)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x dx = \frac{e^\pi - e^{-\pi}}{2\pi} = \frac{\sinh(\pi)}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)\, dx$$

$$I_1 = \int e^x \cos(nx)\, dx = \frac{e^x \sin(nx)}{n} - \frac{1}{n}\int e^x \sin(nx)\, dx = \frac{e^x \sin(nx)}{n} + \frac{e^x \cos(nx)}{n^2} - \frac{1}{n^2}\int e^x \cos(nx)\, dx$$

$$= \frac{e^x \sin(nx)}{n} + \frac{e^x \cos(nx)}{n^2} - \frac{1}{n^2} I_1 \Longrightarrow I_1 = \frac{ne^x \sin(nx) + e^x \cos(nx)}{n^2+1}$$

$$a_n = \frac{1}{\pi}\left[\frac{ne^x \sin(nx) + e^x \cos(nx)}{n^2+1}\right]_{-\pi}^{\pi} = \frac{(e^\pi - e^{-\pi})(-1)^n}{\pi(n^2+1)} = \frac{2\sinh(\pi)\,(-1)^n}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\, dx$$

$$I_2 = \int e^x \sin(nx)\, dx = -\frac{e^x \cos(nx)}{n} + \frac{1}{n}\int e^x \cos(nx)\, dx = -\frac{e^x \cos(nx)}{n} + \frac{e^x \sin(nx)}{n^2} - \frac{1}{n^2}\int e^x \sin(nx)\, dx$$

$$= -\frac{e^x \cos(nx)}{n} + \frac{e^x \sin(nx)}{n^2} - \frac{1}{n^2} I_2 \Longrightarrow I_2 = \frac{-ne^x \cos(nx) + e^x \sin(nx)}{n^2+1}$$

$$b_n = \frac{1}{\pi}\left[\frac{-ne^x \cos(nx) + e^x \sin(nx)}{n^2+1}\right]_{-\pi}^{\pi} = -\frac{n(e^\pi - e^{-\pi})(-1)^n}{\pi(n^2+1)} = -\frac{2n\sinh(\pi)\,(-1)^n}{\pi(n^2+1)}$$

$$\Longrightarrow \boxed{f(x) = \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos(nx) - \frac{n(-1)^n}{n^2+1}\sin(nx)}$$

$$x = \pi \xRightarrow{\text{دیریکله}} \frac{e^\pi + e^{-\pi}}{2} = \cosh(\pi) = \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(-1)^n$$

$$\Longrightarrow \boxed{\frac{\pi}{2}\coth(\pi) - \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{(n^2+1)}}$$

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۸)

$$(\sin(7x)+\cos(7x))^2\cos(14x) = (1+2\sin(7x)\cos(7x))\cos(14x) = \cos(14x)+\frac{1}{2}\sin(28x)$$

$$\Rightarrow I = \int_{-\pi}^{\pi} f(x)\left(\cos(14x)+\frac{1}{2}\sin(28x)\right)dx = \pi a_{14}+\frac{\pi}{2}b_{28} = \frac{\pi}{14}+\frac{\pi}{1568} = \boxed{\frac{113\pi}{1568}}$$

۹)

الف)

$$\sin^7(x) = \left(\frac{1-\cos(2x)}{2}\right)^3\sin(x) = \frac{35}{64}\sin(x)-\frac{21}{64}\sin(3x)+\frac{7}{64}\sin(5x)-\frac{1}{64}\sin(7x)$$

ب)

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\left(\frac{35}{32}-\frac{7}{32}+\frac{7}{160}-\frac{1}{224}\right) = \frac{32}{35\pi}$$

$$b_n = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin^7(x)\cos(2nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{35}{64}\sin(x)\cos(2nx)-\frac{21}{64}\sin(3x)\cos(2nx)+\frac{7}{64}\sin(5x)\cos(2nx)-\frac{1}{64}\sin(7x)\cos(2nx)\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{35}{128}\big(\sin((1+2n)x)+\sin((1-2n)x)\big)-\frac{21}{128}\big(\sin((3+2n)x)+\sin((3-2n)x)\big)\right.$$
$$\left.+\frac{7}{128}\big(\sin((5+2n)x)+\sin((5-2n)x)\big)-\frac{1}{128}\big(\sin((7+2n)x)+\sin((7-2n)x)\big)\right)$$

$$= \frac{2}{\pi}\left(-\frac{35}{128}\left(\frac{-1-1}{1+2n}+\frac{-1-1}{1-2n}\right)+\frac{21}{128}\left(\frac{-1-1}{3+2n}+\frac{-1-1}{3-2n}\right)-\frac{7}{128}\left(\frac{-1-1}{5+2n}+\frac{-1-1}{5-2n}\right)+\frac{1}{128}\left(\frac{-1-1}{7+2n}+\frac{-1-1}{7-2n}\right)\right)$$

$$= \frac{2}{\pi}\left(\frac{35}{32}\left(\frac{1}{1-4n^2}\right)-\frac{21}{32}\left(\frac{3}{9-4n^2}\right)+\frac{7}{32}\left(\frac{5}{25-4n^2}\right)-\frac{1}{32}\left(\frac{7}{49-4n^2}\right)\right)$$

$$\Rightarrow \boxed{f(x) = \frac{32}{35\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{35}{32}\left(\frac{1}{1-4n^2}\right)-\frac{21}{32}\left(\frac{3}{9-4n^2}\right)+\frac{7}{32}\left(\frac{5}{25-4n^2}\right)-\frac{1}{32}\left(\frac{7}{49-4n^2}\right)\right)\cos(2nx)}$$

۱۰)

الف)

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} \pi dx + \int_0^{\pi} (\pi - x) dx\right] = \frac{1}{2\pi}\left(\pi^2 + \pi^2 - \frac{\pi^2}{2}\right) = \frac{3\pi}{4}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi\cos(nx)\,dx + \int_0^{\pi} \pi\cos(nx)\,dx - \int_0^{\pi} x\cos(nx)\,dx\right] = \frac{(-1)^{n+1}+1}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi\sin(nx)\,dx + \int_0^{\pi} \pi\sin(nx)\,dx - \int_0^{\pi} x\sin(nx)\,dx\right] = \frac{(-1)^n}{n}$$

$$\Rightarrow \boxed{f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{\pi n^2}\cos(nx) + \frac{(-1)^n}{n}\sin(nx)}$$

ب)

$$f(0) = \pi = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{\pi n^2}$$

$$f(0) = \pi = \frac{3\pi}{4} + \sum_{n \text{ is odd}} \frac{2}{\pi n^2}$$

$$\Rightarrow \boxed{\sum_{n \text{ is odd}} \frac{1}{n^2} = \frac{\pi^2}{8}}$$

۱۱)

$$b_n = \frac{1}{n^3}$$

$$I = \int_0^{\pi} f(x)\big(\sin(x)\,(2\sin^2(x))\big)dx = 2\int_0^{\pi} f(x)\sin^3 x\,dx$$

$f(x)$ is odd and $\sin^3(x)$ is odd $\Rightarrow$

$$I = \frac{1}{2}\times 2\int_{-\pi}^{\pi} f(x)\sin^3 x\,dx = \int_{-\pi}^{\pi} f(x)\sin^3 x\,dx$$

$$= \int_{-\pi}^{\pi} f(x)\left(\frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)\right)dx = \frac{3}{4}\pi b_1 - \frac{1}{4}\pi b_3 = \frac{3\pi}{4} - \frac{\pi}{4\times 27}$$

$$= \frac{80\pi}{4\times 27} = \boxed{\frac{20\pi}{27}}$$