(١

$$a_0=\frac{1}{2L}\int_0^{2L} f(x)dx\,,\qquad a_n=\frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx\,,\qquad b_n=\frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)+n\cos(nx)}{n^3+1}\Longrightarrow T=2\pi,\qquad L=\pi$$

$$\Longrightarrow a_0=\frac{1}{2\pi}\int_0^{2\pi} f(x)dx=0,\qquad a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx=\frac{n}{n^3+1},\qquad b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx=\frac{1}{n^3+1}$$

$$\cos^4(x)=\left(\frac{1+\cos(2x)}{2}\right)^2=\frac{1+\cos^2(2x)+2\cos(2x)}{4}=\frac{1+\frac{1+\cos(4x)}{2}+2\cos(2x)}{4}=\frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)$$

$$\Longrightarrow\int_0^{2\pi} f(x)\cos^4(x)\,dx=\frac{3}{8}\int_0^{2\pi} f(x)dx+\frac{1}{2}\int_0^{2\pi} f(x)\cos(2x)\,dx+\frac{1}{8}\int_0^{2\pi} f(x)\cos(4x)\,dx$$

$$=\frac{3\pi}{4}a_0+\frac{\pi}{2}a_2+\frac{\pi}{8}a_4=\frac{\pi}{9}+\frac{\pi}{130}=\frac{139\pi}{1170}$$

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)+2n\cos(nx)}{n^3+9}\Longrightarrow T=2\pi,\qquad L=\pi$$

$$\Longrightarrow a_0=\frac{1}{2\pi}\int_0^{2\pi} f(x)dx=0,\qquad a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx=\frac{2n}{n^3+9},\qquad b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx=\frac{1}{n^3+9}$$

$$\sin^4(x)=\left(\frac{1-\cos(2x)}{2}\right)^2=\frac{1+\cos^2(2x)-2\cos(2x)}{4}=\frac{1+\frac{1+\cos(4x)}{2}-2\cos(2x)}{4}=\frac{3}{8}-\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)$$

$$\Longrightarrow\int_0^{2\pi} f(x)\cos^4(x)\,dx=\frac{3}{8}\int_0^{2\pi} f(x)dx-\frac{1}{2}\int_0^{2\pi} f(x)\cos(2x)\,dx+\frac{1}{8}\int_0^{2\pi} f(x)\cos(4x)\,dx$$

$$=\frac{3\pi}{4}a_0-\frac{\pi}{2}a_2+\frac{\pi}{8}a_4=-\frac{2\pi}{17}+\frac{\pi}{73}=-\frac{129\pi}{1241}$$

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)+n\cos(nx)}{n^3+6}\Longrightarrow T=2\pi,\qquad L=\pi$$

$$\Longrightarrow a_0=\frac{1}{2\pi}\int_0^{2\pi} f(x)dx=0,\qquad a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx=\frac{n}{n^3+6},\qquad b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx=\frac{1}{n^3+6}$$

$$\cos^3(x)=\cos(x)\left(\frac{1+\cos(2x)}{2}\right)=\frac{1}{2}\cos(x)+\frac{1}{4}(\cos(3x)+\cos(x))=\frac{3}{4}\cos(x)+\frac{1}{4}\cos(3x)$$

$$\Longrightarrow\int_0^{2\pi} f(x)\cos^4(x)\,dx=\frac{3}{4}\int_0^{2\pi} f(x)\cos(x)\,dx+\frac{1}{4}\int_0^{2\pi} f(x)\cos(3x)\,dx$$

$$=\frac{3\pi}{4}a_1+\frac{\pi}{4}a_3=\frac{3\pi}{28}+\frac{\pi}{44}=\frac{10\pi}{77}$$

۲)

$f(x) = |x| \quad -2 < x < 2 \Longrightarrow T = 4, \qquad L = 2$

$f(x) = f(-x) \Longrightarrow b_n = 0$

$$a_0 = \frac{1}{4}\int_{-2}^{2}|x|dx = \frac{1}{2}\int_{0}^{2}xdx = 1$$

$$a_n = \frac{1}{2}\int_{-2}^{2}|x|\cos\left(\frac{n\pi}{2}x\right)dx = \int_{0}^{2}x\cos(\frac{n\pi}{2}x)dx = \frac{4}{n^2\pi^2}((-1)^n - 1) = \begin{cases} 0, & n = 2k \\ -\dfrac{8}{(2k-1)^2\pi^2}, & n = 2k-1 \end{cases}$$

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\left(\frac{(2k-1)\pi}{2}x\right)$$

Parseval: $\frac{1}{L}\int_{-L}^{L}f^2(x)dx = 2a_0^2 + \sum_{1}^{\infty}a_n^2 + b_n^2 = \frac{1}{2}\int_{-2}^{2}|x|^2dx = \int_{0}^{2}x^2dx = \frac{8}{3} = 2 + \frac{64}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$

$$\Longrightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} = \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

$$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right) = \frac{\pi^4}{96} + \frac{S}{2^4} = S \Longrightarrow S = \frac{\pi^4}{90}$$

۳)

$$f(-x) = -f(x), \qquad f(1) = 1, \qquad \int_{0}^{\infty}f(x)\sin(ax)\,dx + \int_{0}^{\infty}xf(x)\cos(ax)\,dx = 0$$

$$f(x) = \int_{0}^{\infty}B(a)\sin(ax)\,dx\,, \qquad B(a) = \frac{2}{\pi}\int_{0}^{\infty}f(x)\sin(ax)\,dx \Longrightarrow B'(a) = \frac{2}{\pi}\int_{0}^{\infty}xf(x)\cos(ax)\,dx$$

$$\Longrightarrow \frac{2}{\pi}\big(B(a) + B'(a)\big) = 0 \Longrightarrow B(a) = ce^{-a} \Longrightarrow f(x) = \int_{0}^{\infty}ce^{-x}\sin(ax)\,dx = \frac{cx}{1+x^2}$$

$$f(1) = \frac{c}{2} = 1 \Longrightarrow c = 2 \Longrightarrow f(x) = \frac{2x}{1+x^2}$$

۴)

$$\dot{y}+2y=e^{-t}u(t)\Longrightarrow i\omega Y+2Y=\frac{1}{1+i\omega}\Longrightarrow Y=\frac{1}{(1+i\omega)(2+i\omega)}=\frac{1}{1+i\omega}-\frac{1}{2+i\omega}$$

$$\Longrightarrow y(t)=e^{-t}u(t)-e^{-2t}u(t)$$

$$F(\omega)=\frac{e^{-2i\omega}}{(3+i\omega)^2}$$

$$\mathcal{F}\{e^{-3t}u(t)\}=\frac{1}{3+i\omega}\Longrightarrow \mathcal{F}\{te^{-3t}u(t)\}=\frac{1}{(3+i\omega)^2}\Longrightarrow \mathcal{F}\big\{(t-2)e^{-3(t-2)}u(t-2)\big\}=\frac{e^{-2i\omega}}{(3+i\omega)^2}$$

$$f(x)=\frac{x+2}{x+4x+5}=\frac{x+2}{(x+2)^2+1}$$

$$g(x)\triangleq\frac{x}{x^2+1}\Longrightarrow f(x)=g(x+2)\Longrightarrow F(\omega)=G(\omega)e^{2i\omega}$$

$$\frac{x}{x^2+1}=\frac{\frac{i}{2}}{1+ix}-\frac{\frac{i}{2}}{1-ix}$$

$$\begin{cases}\mathcal{F}\{e^{-x}u(x)\}=\dfrac{1}{1+i\omega}\\ \mathcal{F}\{e^{x}u(-x)\}=\dfrac{1}{1-i\omega}\end{cases}\Longrightarrow\begin{cases}\mathcal{F}\left\{\dfrac{1}{1+ix}\right\}=2\pi e^{\omega}u(-\omega)\\ \mathcal{F}\left\{\dfrac{1}{1-ix}\right\}=2\pi e^{-\omega}u(\omega)\end{cases}$$

$$\begin{aligned}\Longrightarrow G(\omega)=\pi i\big(e^{\omega}u(-\omega)-e^{-\omega}u(\omega)\big)\Longrightarrow F(\omega)&=\pi i\big(e^{\omega}u(-\omega)-e^{-\omega}u(\omega)\big)e^{2i\omega}\\ &=-\pi i\,\mathrm{sign}(\omega)e^{2i\omega-|\omega|}\end{aligned}$$

$$u_t - u_{xx} = 0 \quad (0 < x < \pi, \qquad t > 0)$$

$$\begin{cases} u(0,t) = 2(1-e^{-t}) + \frac{1}{2}e^{-t} \\ u(\pi,t) = \frac{1}{2}e^{-t} \end{cases}, \qquad u(x,0) = 1 - \frac{2}{\pi^2}\left(x - \frac{\pi}{2}\right)^2$$

$$u(x,t) = v(x,t) + w(x,t)$$

$$\begin{cases} m = 0 \\ n = 0 \end{cases} \Longrightarrow \begin{cases} a = \frac{2}{\pi}(e^{-t} - 1) \\ b = 2(1-e^{-t}) + \frac{1}{2}e^{-t} \end{cases} \Longrightarrow w(x,t) = \frac{2}{\pi}(e^{-t}-1)x + 2(1-e^{-t}) + \frac{1}{2}e^{-t}$$

$$\begin{cases} u_t = v_t + w_t = v_t - \frac{2}{\pi}e^{-t}x + 2e^{-t} - \frac{1}{2}e^{-t} = v_t + \left(\frac{3}{2} - \frac{2}{\pi}x\right)e^{-t} \\ u_{xx} = v_{xx} + w_{xx} = v_{xx} \end{cases} \Longrightarrow$$

$$v_t + \left(\frac{3}{2} - \frac{2}{\pi}x\right)e^{-t} - v_{xx} = 0$$

$$\begin{cases} v(0,t) = 0 \\ v(\pi,t) = 0 \end{cases}, \qquad v(x,0) = u(x,0) - w(x,0) = \frac{1}{2} - \frac{2}{\pi^2}\left(x - \frac{\pi}{2}\right)^2$$

$$\text{BC:} \quad v(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin(nx) \Longrightarrow \begin{cases} v_t = \sum_{n=1}^{\infty} \dot{G}_n(t)\sin(nx) \\ v_{xx} = -\sum_{n=1}^{\infty} n^2 G_n(t)\sin(nx) \end{cases} \Longrightarrow$$

$$\sum_{n=1}^{\infty}\left(\dot{G}_n(t) + n^2 G_n(t)\right)\sin(nx) = \left(\frac{2}{\pi}x - \frac{3}{2}\right)e^{-t}$$

$$\dot{G}_n(t) + n^2 G_n(t) = \frac{2}{\pi}e^{-t}\int_0^{\pi}\left(\frac{2}{\pi}x - \frac{3}{2}\right)dx = -e^{-t}\frac{(-1)^n + 3}{n\pi} \Longrightarrow \text{Solve this ODE:}$$

$$G_n(t)e^{n^2 t} = \int \left(-e^{-t}\frac{(-1)^n + 3}{n\pi}\right)e^{n^2 t}dt + c_n = -\frac{(-1)^n + 3}{n\pi}\int e^{(n^2-1)t}dt + c_n$$

$$\begin{cases} n = 1 \Longrightarrow G_1(t)e^{t} = -\frac{2}{\pi}\int dt + c_1 = -\frac{2}{\pi}t + c_1 \\ n \neq 1 \Longrightarrow G_n(t)e^{n^2 t} = -\frac{(-1)^n + 3}{n\pi}\int e^{(n^2-1)t}dt + c_n = -\frac{(-1)^n + 3}{n(n^2-1)\pi}e^{(n^2-1)t} + c_n \end{cases}$$

$$\begin{cases} n = 1: \quad G_1(t) = -\frac{2}{\pi}te^{-t} + c_1 e^{-t} \\ n \neq 1: \quad G_n(t) = -\frac{(-1)^n + 3}{n(n^2-1)\pi}e^{-t} + c_n e^{-n^2 t} \end{cases}$$

$$\text{IC:} \quad v(x,0) = \sum_{n=1}^{\infty} G_n(0)\sin(nx) = \frac{1}{2} - \frac{2}{\pi^2}\left(x - \frac{\pi}{2}\right)^2 \Longrightarrow G_n(0) = \frac{2}{\pi}\int_0^{\pi}\left(\frac{1}{2} - \frac{2}{\pi^2}\left(x - \frac{\pi}{2}\right)^2\right)\sin(nx)\,dx = \frac{8(1-(-1)^n)}{n^3\pi^3}$$

$$\begin{cases} G_1(0) = c_1 = \frac{16}{\pi^3} \\ G_n(0) = -\frac{(-1)^n + 3}{n(n^2-1)\pi} + c_n = \frac{8(1-(-1)^n)}{n^3\pi^3} \Longrightarrow c_n = \frac{8(1-(-1)^n)}{n^3\pi^3} + \frac{(-1)^n + 3}{n(n^2-1)\pi} \end{cases}$$

$$u(x,t) = \frac{2}{\pi}(e^{-t}-1)x + 2(1-e^{-t}) + \frac{1}{2}e^{-t} + \left(-\frac{2}{\pi}t + \frac{16}{\pi^3}\right)e^{-t}\sin(x) + \sum_{n=2}^{\infty}\left(-\frac{(-1)^n + 3}{n(n^2-1)\pi}e^{-t} + \left(\frac{8(1-(-1)^n)}{n^3\pi^3} + \frac{(-1)^n + 3}{n(n^2-1)\pi}\right)e^{-n^2 t}\right)\sin(nx)$$

$$u_t - u_{xx} = 1 + x\cos(t) \qquad (0 < x < 1, \qquad t > 0)$$

$$\begin{cases} u_x(0,t) = \sin(t) \\ u_x(1,t) = \sin(t) \end{cases}, \qquad u(x,0) = 1 + \cos(2\pi x)$$

$$u(x,t) = v(x,t) + w(x,t)$$

$$\begin{cases} m = 1 \\ n = 1 \end{cases} \Longrightarrow \begin{cases} a = 0 \\ b = \sin(t) \end{cases} \Longrightarrow w(x,t) = x\sin(t)$$

$$\begin{cases} u_t = v_t + w_t = v_t + x\cos(t) \\ u_{xx} = v_{xx} + w_{xx} = v_{xx} \end{cases} \Longrightarrow$$

$$v_t - v_{xx} = 1$$

$$\begin{cases} v_x(0,t) = 0 \\ v_x(1,t) = 0 \end{cases}, \qquad v(x,0) = 1 + \cos(2\pi x)$$

BC: $v(x,t) = G_0(t) + \sum_{n=1}^{\infty} G_n(t)\cos(n\pi x) \Longrightarrow \begin{cases} v_t = \dot{G}_0(t) + \sum_{n=1}^{\infty} \dot{G}_n(t)\cos(n\pi x) \\ v_{xx} = -\sum_{n=1}^{\infty} n^2\pi^2 G_n(t)\cos(n\pi x) \end{cases} \Longrightarrow$

$\dot{G}_0(t) + \sum_{n=1}^{\infty} \left(\dot{G}_n(t) + n^2\pi^2 G_n(t)\right)\cos(n\pi x) = 1 \Longrightarrow \begin{cases} \dot{G}_0(t) = 1 \\ \dot{G}_n(t) + n^2\pi^2 G_n(t) = 0 \end{cases} \Longrightarrow$ Solve these ODEs:

$$\begin{cases} G_0(t) = t + c_0 \\ G_n(t) = c_n e^{-n^2\pi^2 t} \end{cases}$$

IC: $v(x,0) = c_0 + \sum_{n=1}^{\infty} c_n\cos(n\pi x) = 1 + \cos(2\pi x) \Longrightarrow c_n = \begin{cases} 1, & n = 0,2 \\ 0, & n \neq 0,2 \end{cases}$

$$u(x,t) = x\sin(t) + t + 1 + e^{-4\pi^2 t}\cos(2\pi x)$$

$u_{tt} - u_{xx} = x + t \quad (0 < x < \pi, \qquad t > 0)$

$\begin{cases} u(0,t) = 2t \\ u(\pi,t) = t \end{cases}, \quad \begin{cases} u(x,0) = 2 \\ u_t(x,0) = x \end{cases}$

$u(x,t) = v(x,t) + w(x,t)$

$\begin{cases} m = 0 \\ n = 0 \end{cases} \Longrightarrow \begin{cases} a = -\dfrac{t}{\pi} \\ b = 2t \end{cases} \Longrightarrow w(x,t) = -\dfrac{t}{\pi}x + 2t$

$\begin{cases} u_{tt} = v_{tt} + w_{tt} = v_{tt} \\ u_{xx} = v_{xx} + w_{xx} = v_{xx} \end{cases} \Longrightarrow$

$v_{tt} - v_{xx} = x + t$

$\begin{cases} v(0,t) = 0 \\ v(\pi,t) = 0 \end{cases}, \quad \begin{cases} v(x,0) = u(x,0) - w(x,0) = 2 \\ v_t(x,0) = u_t(x,0) - w_t(x,0) = \left(1 + \dfrac{1}{\pi}\right)x - 2 \end{cases}$

BC: $v(x,t) = \displaystyle\sum_{n=1}^{\infty} G_n(t)\sin(nx) \Longrightarrow \begin{cases} v_{tt} = \displaystyle\sum_{n=1}^{\infty} \ddot{G}_n(t)\sin(nx) \\ v_{xx} = -\displaystyle\sum_{n=1}^{\infty} n^2 G_n(t)\sin(nx) \end{cases} \Longrightarrow$

$\displaystyle\sum_{n=1}^{\infty} \left(\ddot{G}_n(t) + n^2 G_n(t)\right)\sin(nx) = x + t \Longrightarrow \ddot{G}_n(t) + n^2 G_n(t) = \frac{2}{\pi}\int_0^{\pi} (x+t)\sin(nx)\,dx = \frac{2t(1-(-1)^n) - 2\pi(-1)^n}{\pi n}$ $\Longrightarrow$ Solve this ODE:

$G_{n,p}(t) = Ct + D \Longrightarrow G_{n,p}(t) = \dfrac{2t(1-(-1)^n) - 2\pi(-1)^n}{\pi n^3}$

$G_{n,h}(t) = A_n\cos(nt) + B_n\sin(nt)$

IC: $v(x,0) = \displaystyle\sum_{n=1}^{\infty} G_n(0)\sin(nx) = \sum_{n=1}^{\infty}\left(A_n - \frac{(-1)^n}{2n^3}\right)\sin(nx) = 2 \Longrightarrow A_n - \frac{2(-1)^n}{n^3} = \frac{2}{\pi}\int_0^{\pi} 2\sin(nx)\,dx = \frac{4(1-(-1)^n)}{\pi n}$

$\Longrightarrow A_n = \dfrac{4(1-(-1)^n)}{\pi n} + \dfrac{2(-1)^n}{n^3}$

$v_t(x,0) = \displaystyle\sum_{n=1}^{\infty} \dot{G}_n(0)\sin(nx) = \sum_{n=1}^{\infty}\left(nB_n + \frac{2(1-(-1)^n)}{\pi n^3}\right)\sin(nx) = \left(1 + \frac{1}{\pi}\right)x - 2$

$\Longrightarrow nB_n + \dfrac{2(1-(-1)^n)}{\pi n^3} = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}\left(\left(1+\frac{1}{\pi}\right)x - 2\right)\sin(nx)\,dx = \frac{2(1-\pi)(-1)^n - 4}{\pi n}$

$\Longrightarrow B_n = \dfrac{2((-1)^n - 1)}{\pi n^4} + \dfrac{2(1-\pi)(-1)^n - 4}{\pi n^2}$

$u(x,t) = -\dfrac{t}{\pi}x + 2t + \displaystyle\sum_{n=1}^{\infty}\left(\frac{2t(1-(-1)^n) - 2\pi(-1)^n}{\pi n^3} + \left(\frac{4(1-(-1)^n)}{\pi n} + \frac{2(-1)^n}{n^3}\right)\cos(nt) + \left(\frac{2((-1)^n - 1)}{\pi n^4} + \frac{2(1-\pi)(-1)^n - 4}{\pi n^2}\right)\sin(nt)\right)\sin(nx)$