دانشگاه تهران- دانشکده مهندسی برق و کامپیوتر

ریاضیات مهندسی- نیم سال اول سال ۱۴۰۰-۱۴۰۱

کوییز ۳: معادله لاپلاس در مختصات قطبی و کارتزین

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۱-۱) معادله زیر را حل کنید. (راهنمایی: در راستای x همگن سازی کنید) (۵۰ نمره)

$$u_{xx}+u_{yy}=\sin(y)\sin(\pi x),\qquad 0\le x\le 1,\qquad 0\le y\le 1$$
$$u(0,y)=1,\qquad u(1,y)=2$$
$$u(x,0)=1+x+\sin(2\pi x),\qquad u(x,1)=1+x$$

$$u=v+w\rightarrow w=1+x,v=\sum_{n=1}^{\infty}Y_n(y)\sin(n\pi x),\quad u_{xx}=v_{xx},u_{yy}=v_{yy}\rightarrow v_{xx}+v_{yy}=\sin(y)\sin(\pi x)$$
$$\sum_{n=1}^{\infty}[-n^2\pi^2Y_n(y)+Y_n''(y)]\sin(n\pi x)=\sin(y)\sin(\pi x)\ \rightarrow n=1,X(x)=\sin(\pi x)\rightarrow\ -\pi^2Y_1(y)+Y_1''(y)=\sin(y)$$

$$Y_1(y)=c_1e^{-\pi y}+c_2e^{\pi y}-\frac{\sin(y)}{1+\pi^2}\rightarrow u(x,y)=X(x)Y(y)=\left[c_1e^{-\pi y}+c_2e^{\pi y}-\frac{\sin(y)}{1+\pi^2}\right]\sin(\pi x)$$

$$\xRightarrow{u(1,0)=1,u(1,1)=2}\begin{cases}c_1+c_2=1\\ c_1e^{\pi}+c_2e^{-\pi}=2\end{cases}\rightarrow c_1=\frac{2-e^{-\pi}}{e^{\pi}-e^{-\pi}},\quad c_2=\frac{e^{\pi}-2}{e^{\pi}-e^{-\pi}}$$

۱-۲) معادله زیر را حل کنید.

$$u_{xx}+u_{yy}=\cos(x)\cos(\pi y),\qquad 0\le x\le\frac{1}{2},\qquad 0\le y\le\frac{1}{2}$$
$$u(0,y)=y-\frac{1}{2},\qquad u\left(\frac{1}{2},y\right)=y-\frac{1}{2}+\cos(3\pi y)$$
$$u_y(x,0)=1,\qquad u\left(x,\frac{1}{2}\right)=0$$

$$u=v+w\rightarrow w=y-\frac{1}{2},v=\sum_{n=1}^{\infty}X_n(x)\cos\big((2n-1)\pi y\big),\quad u_{xx}=v_{xx},u_{yy}=v_{yy}\rightarrow v_{xx}+v_{yy}=\cos(x)\cos(\pi y)$$
$$\sum_{n=1}^{\infty}[-4n^2\pi^2X_n(x)+X_n''(x)]\cos\big((2n-1)\pi y\big)=\cos(x)\cos(\pi y)\ \rightarrow n=1,Y(y)=\cos(\pi y)\rightarrow\ -4\pi^2X_1(x)+X_1''(x)=\cos(x)$$

$$X_1(x)=c_1e^{-2\pi x}+c_2e^{2\pi x}-\frac{\cos(x)}{1+4\pi^2}\rightarrow u(x,y)=X(x)Y(y)=\left[c_1e^{-2\pi x}+c_2e^{2\pi x}-\frac{\cos(x)}{1+4\pi^2}\right]\cos(\pi y)$$

$$\xRightarrow{u(0,0)=-\frac{1}{2},u\left(\frac{1}{2},\ 0\right)=\frac{1}{2}}\begin{cases}c_1+c_2-\dfrac{1}{1+4\pi^2}=-\dfrac{1}{2}\\ c_1e^{-\pi}+c_2e^{\pi}-\dfrac{\cos\left(\frac{1}{2}\right)}{1+4\pi^2}=\dfrac{1}{2}\end{cases}\rightarrow c_1=\frac{\mathrm{b}-\mathrm{a}e^{\pi}}{e^{-\pi}-e^{\pi}},\quad c_2=\frac{-b+ae^{-\pi}}{e^{-\pi}-e^{\pi}},\qquad a=-\frac{1}{2}+\frac{1}{1+4\pi^2},\qquad b=\frac{1}{2}+\frac{\cos\left(\frac{1}{2}\right)}{1+4\pi^2}$$

۳-۱) معادله زیر را حل کنید. (راهنمایی: در راستای y همگن سازی کنید) (۵۰ نمره)

$$u_{xx}+u_{yy}=\sinh(x)\sin(3\pi y),\qquad 0\le x\le \frac{1}{2},\qquad 0\le y\le \frac{1}{2}$$
$$u(0,y)=1+y+\sin(\pi y),\qquad u(1,y)=1+y$$
$$u(x,0)=1,\qquad u_y\left(x,\frac{1}{2}\right)=1$$

$$u=v+w\to w=1+y,\ v=\sum_{n=1}^{\infty}X_n(x)\sin\big((2n-1)\pi y\big),\quad u_{xx}=v_{xx},u_{yy}=v_{yy}\to v_{xx}+v_{yy}=\sinh(x)\sin(3\pi y)$$

$$\sum_{n=1}^{\infty}[-4n^2\pi^2X_n(x)+X_n''(x)]\sin\big((2n-1)\pi y\big)=\sinh(x)\sin(3\pi y)\ \to n=2,Y(y)=\sin(3\pi y)\to\ -16\pi^2X_2(x)+X_2''(x)=\sinh(x)$$

$$X_2(x)=c_1e^{-4\pi x}+c_2e^{4\pi x}+\frac{\sinh(x)}{1-16\pi^2}\to u(x,y)=X(x)Y(y)=\left[c_1e^{-4\pi x}+c_2e^{4\pi x}+\frac{\sinh(x)}{1-16\pi^2}\right]\sin(3\pi y)$$

$$\xRightarrow{u\left(0,\frac{1}{2}\right)=\frac{5}{2},u\left(1,\frac{1}{2}\right)=\frac{3}{2}}\begin{cases}c_1+c_2-\dfrac{1}{1+4\pi^2}=\dfrac{5}{2}\\ c_1e^{-4\pi}+c_2e^{4\pi}+\dfrac{\sinh(1)}{1-16\pi^2}=\dfrac{3}{2}\end{cases}\to c_1=\frac{\mathrm{b}-\mathrm{a}e^{4\pi}}{e^{-4\pi}-e^{4\pi}},\quad c_2=\frac{-b+ae^{-4\pi}}{e^{-4\pi}-e^{4\pi}},a=\frac{1}{1+4\pi^2}+\frac{5}{2},\qquad b=-\frac{\sinh(1)}{1-16\pi^2}+\frac{3}{2}$$

۴-۱) معادله زیر را حل کنید. (راهنمایی: در راستای x همگن سازی کنید) (۵۰ نمره)

$$u_{xx}+u_{yy}=1+\cosh(y)\cos(\pi x),\qquad 0\le x\le 1,\qquad 0\le y\le 1$$
$$u_x(0,y)=1,\qquad u_x(1,y)=2$$
$$u(x,0)=x+\frac{x^2}{2},\qquad u(x,1)=x+\frac{x^2}{2}+\cos(2\pi x)$$

$$u=v+w\to w=x+\frac{x^2}{2},v=\sum_{n=1}^{\infty}Y_n(y)\cos(n\pi x),\quad u_{xx}=v_{xx}+1,u_{yy}=v_{yy}\to v_{xx}+v_{yy}=\cosh(y)\cos(\pi x)$$

$$\sum_{n=1}^{\infty}[-n^2\pi^2Y_n(y)+Y_n''(y)]\cos(n\pi x)=\cosh(y)\cos(\pi x)\ \to n=1,X(x)=\cos(\pi x)\to\ -\pi^2Y_1(y)+Y_1''(y)=\cosh(y)$$

$$Y_1(y)=c_1e^{-\pi y}+c_2e^{\pi y}-\frac{\cosh(y)}{-1+\pi^2}\to u(x,y)=X(x)Y(y)=\left[c_1e^{-\pi y}+c_2e^{\pi y}-\frac{\cosh(y)}{-1+\pi^2}\right]\cos(\pi x)$$

$$\xRightarrow{u(0,0)=0,u(0,1)=1}\begin{cases}c_1+c_2-\dfrac{\cosh(0)}{-1+\pi^2}=0\\ c_1e^{-\pi}+c_2e^{\pi}-\dfrac{\cosh(1)}{-1+\pi^2}=1\end{cases}\to c_1=\frac{\mathrm{b}-\mathrm{a}e^{\pi}}{e^{-\pi}-e^{\pi}},\quad c_2=\frac{-b+ae^{-\pi}}{e^{-\pi}-e^{\pi}},a=\frac{\cosh(0)}{-1+\pi^2},\qquad b=\frac{\cosh(1)}{-1+\pi^2}+1$$

۱-۲) معادله زیر را حل کنید.

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \qquad r\le r_0\ ,\ 0\le\theta\le\frac{\pi}{4}$$
$$u(r,0)=u\left(r,\frac{\pi}{4}\right)=0$$
$$u(r_0,\theta)=1$$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \quad , \quad r=e^{-t} \implies u_{tt}+u_{\theta\theta}=0 \implies u(t,\theta)=X(t)Y(\theta) \implies X''(t)Y(\theta)+X(t)Y''(\theta)=0 \implies \frac{X''}{X}=-\frac{Y''}{Y}=\lambda$$

$$From\ \ Y''(\theta)+\lambda Y(\theta)=0 \implies Y(\theta)=a_n\cos(\sqrt{\lambda}\theta)+b_n\sin(\sqrt{\lambda}\theta) \implies boundry\ conditions\ give:$$

$$u(r,0)=0 \implies X(r)Y(0)=0 \implies Y(0)=0 \qquad , \qquad u\left(r,\frac{\pi}{4}\right)=0 \implies X(r)Y\left(\frac{\pi}{4}\right)=0 \implies Y\left(\frac{\pi}{4}\right)=0$$

$$\implies Y(0)=Y\left(\frac{\pi}{4}\right)=0$$

$$0=Y(0)=a_n \ \ , \ \ Y\left(\frac{\pi}{4}\right)=b_n\sin\left(\sqrt{\lambda}\frac{\pi}{4}\right)=0 \implies \sqrt{\lambda}=4n \implies \lambda_n=16n^2 \qquad Thus,\ \ Y(\theta)=b_n\sin(4n\theta) \quad n=1,2,\ldots$$

$$with\ this\ values\ of\ \lambda_n\ we\ solve\ \ X''(t)-16n^2X(t)=0\ :$$

$$If\ \ n=0:\ X(t)=c_0t+d_0 \implies X(r)=-c_0Ln(r)+d_0 \ \ ,\ \ Y(\theta)=0 \implies X(r)Y(\theta)=0$$

$$If\ \ n>0:\ X(t)=c_ne^{4nt}+d_ne^{-4nt} \implies X(r)=c_nr^{-4n}+d_nr^{4n} \ \ ,\ \ Y(\theta)=b_n\sin(4n\theta) \implies X(r)Y(\theta)=(c_nr^{-4n}+d_nr^{4n})b_n\sin(4n\theta)$$

$$By\ superposition:\ u(r,\theta)=\sum_{n=1}^{\infty}(\tilde{c}_nr^{-4n}+\tilde{d}_nr^{4n})\sin(4n\theta)$$

$$\lim_{r\to0}u(r,\theta)<\infty \implies \tilde{c}_n=0 \implies u(r,\theta)=\sum_{n=1}^{\infty}(\tilde{d}_nr^{4n})\sin(4n\theta)$$

$$Using\ other\ \ Boundry\ condition:$$

$$1=u(r_0,\theta)=\sum_{n=1}^{\infty}(\tilde{d}_n{r_0}^{4n})\sin(4n\theta) \implies \tilde{d}_n{r_0}^{4n}=\frac{2}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\sin(4n\theta)\,d\theta=\frac{8}{\pi}\frac{-1}{4n}[(-1)^n-1]=\frac{2}{n\pi}[1-(-1)^n]$$

$$\implies \tilde{d}_n{r_0}^{4n}=\frac{2}{n\pi}[1-(-1)^n] \implies \tilde{d}_n=\frac{2{r_0}^{-4n}}{n\pi}[1-(-1)^n]$$

$$\implies \boxed{u(r,\theta)=\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}[1-(-1)^n]\right)\left(\frac{r}{r_0}\right)^{4n}\sin(4n\theta)}$$

۲-۲) معادله زیر را حل کنید.

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \qquad r\le r_0\ ,\ 0\le\theta\le\frac{\pi}{2}$$
$$u(r,0)=u_\theta\left(r,\frac{\pi}{2}\right)=0$$
$$u(r_0,\theta)=\Pi\left(\frac{\theta}{\frac{\pi}{2}}\right)$$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0\quad,\quad r=e^{-t}\quad\Longrightarrow\quad u_{tt}+u_{\theta\theta}=0\ \Longrightarrow u(t,\theta)=X(t)Y(\theta)\ \Longrightarrow\ X''(t)Y(\theta)+X(t)Y''(\theta)=0\ \Longrightarrow\ \frac{X''}{X}=-\frac{Y''}{Y}=\lambda$$

$$\text{From } Y''(\theta)+\lambda Y(\theta)=0\ \Longrightarrow Y(\theta)=a_n\cos(\sqrt{\lambda}\theta)+b_n\sin(\sqrt{\lambda}\theta)\ \Longrightarrow \text{boundry conditions give:}$$

$$u(r,0)=0\ \Longrightarrow\ X(r)Y(0)=0\ \Longrightarrow\ Y(0)=0 \qquad , \qquad u_\theta\left(r,\frac{\pi}{2}\right)=0\ \Longrightarrow\ X(r)Y'\left(\frac{\pi}{2}\right)=0\ \Longrightarrow\ Y'\left(\frac{\pi}{2}\right)=0$$

$$\Longrightarrow\ Y(0)=Y'\left(\frac{\pi}{2}\right)=0$$

$$0=Y(0)=a_n\quad,\quad Y'\left(\frac{\pi}{2}\right)=b_n\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right)=0\quad\Longrightarrow\ \sqrt{\lambda}=(2n-1)\ \Longrightarrow\ \lambda_n=(2n-1)^2$$

$$\text{Thus,}\quad Y(\theta)=b_n\sin\big((2n-1)\,\theta\big)\qquad n=1,2,\ldots$$

$$\text{with this values of } \lambda_n \text{ we solve }\ X''(t)-(2n-1)^2X(t)=0\ :$$

$$X(t)=c_ne^{(2n-1)t}+d_ne^{-(2n-1)t}\Longrightarrow X(r)=c_nr^{-(2n-1)}+d_nr^{(2n-1)}\quad,\ Y(\theta)=b_n\sin\big((2n-1)\,\theta\big)$$
$$\Longrightarrow\ X(r)Y(\theta)=(c_nr^{-(2n-1)}+d_nr^{(2n-1)})b_n\sin((2n-1)\,\theta)$$

$$\text{By superposition}:\ u(r,\theta)=\sum_{n=1}^{\infty}(\tilde{c}_nr^{-(2n-1)}+\tilde{d}_nr^{(2n-1)})\sin\big((2n-1)\,\theta\big)$$

$$\lim_{r\to0}u(r,\theta)<\infty\quad\Longrightarrow\ \tilde{c}_n=0\quad\Longrightarrow\ u(r,\theta)=\sum_{n=1}^{\infty}\tilde{d}_nr^{(2n-1)}\ \sin\big((2n-1)\,\theta\big)$$

$$\text{Using other Boundry condition:}$$

$$\Pi\left(\frac{\theta}{\frac{\pi}{2}}\right)=u(r_0,\theta)=\sum_{n=1}^{\infty}\tilde{d}_n{r_0}^{(2n-1)}\ \sin\big((2n-1)\,\theta\big)$$

$$\Longrightarrow\ \tilde{d}_n{r_0}^{(2n-1)}=\frac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\Pi\left(\frac{\theta}{\frac{\pi}{2}}\right)\sin\big((2n-1)\theta\big)\ d\theta=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}1\,.\sin\big((2n-1)\theta\big)\ d\theta+\frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}0\,.\sin\big((2n-1)\theta\big)\ d\theta$$

$$\tilde{d}_n{r_0}^{(2n-1)}=\frac{4}{\pi}\frac{1}{(2n-1)}\left[1-\cos\left((2n-1)\frac{\pi}{4}\right)\right]\ \Longrightarrow\ \tilde{d}_n=\frac{4}{\pi}\frac{{r_0}^{-(2n-1)}}{(2n-1)}\left[1-\cos\left((2n-1)\frac{\pi}{4}\right)\right]$$

$$\Longrightarrow\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\left[1-\cos\left((2n-1)\frac{\pi}{4}\right)\right]\left(\frac{r}{r_0}\right)^{2n-1}\sin\big((2n-1)\theta\big)}$$

۳-۲) معادله زیر را حل کنید.

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \qquad r_0\le r\ ,\ 0\le\theta\le\frac{\pi}{4}$$
$$u_\theta(r,0)=u_\theta\left(r,\frac{\pi}{4}\right)=0$$
$$u(r_0,\theta)=1$$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0\ \ ,\ \ r=e^{-t}\ \ \Longrightarrow\ \ u_{tt}+u_{\theta\theta}=0\ \Longrightarrow u(t,\theta)=X(t)Y(\theta)\ \Longrightarrow\ X''(t)Y(\theta)+X(t)Y''(\theta)=0\ \Longrightarrow\ \frac{X''}{X}=-\frac{Y''}{Y}=\lambda$$
$$From\ \ Y''(\theta)+\lambda Y(\theta)=0\ \ \Longrightarrow Y(\theta)=\ a_n\cos(\sqrt{\lambda}\theta)+b_n\sin(\sqrt{\lambda}\theta)\ \ \Rightarrow boundry\ conditions\ give:$$

$$u_\theta(r,0)=0\ \Longrightarrow\ X(r)Y'(0)=0\ \ \Longrightarrow\ Y'(0)=0 \qquad , \qquad u_\theta\left(r,\frac{\pi}{4}\right)=0\ \Longrightarrow\ X(r)Y'\left(\frac{\pi}{4}\right)=0\ \ \Longrightarrow\ Y'\left(\frac{\pi}{4}\right)=0$$
$$\Longrightarrow\ Y'(0)=Y'\left(\frac{\pi}{4}\right)=0 \qquad , \qquad Y'(\theta)=\ -a_n\sqrt{\lambda}\sin(\sqrt{\lambda}\theta)+b_n\sqrt{\lambda}\cos(\sqrt{\lambda}\theta)$$
$$0=\ Y'(0)=b_n\sqrt{\lambda}\ \ \Longrightarrow\ b_n=0\quad ,\ \ Y'\left(\frac{\pi}{4}\right)=-a_n\sqrt{\lambda}\sin\left(\sqrt{\lambda}\frac{\pi}{4}\right)=0\quad\Longrightarrow\ \sqrt{\lambda}=4n\ \ \Longrightarrow\ \ \lambda_n=16n^2$$
$$Thus,\ \ Y(\theta)=a_n\cos(4n\theta)\qquad n=0,1,2,\ldots$$
$$with\ this\ values\ of\ \lambda_n\ we\ solve\ \ X''(t)-\ 16n^2X(t)=0\ \ :$$
$$If\ \ \ n=0:\ X(t)=c_0t+\ d_0\ \ \Rightarrow X(r)=-c_0Ln(r)+\ d_0\ \ \ ,\ Y(\theta)=a_0\ \ \ \Rightarrow X(r)Y(\theta)=(-c_0Ln(r)+\ d_0)a_0$$
$$If\ \ \ n>0:\ X(t)=c_ne^{4nt}+d_ne^{-4nt}\Rightarrow X(r)=c_nr^{-4n}+d_nr^{4n}\ \ ,\ Y(\theta)=a_n\cos(4n\theta)\ \ \ \Rightarrow X(r)Y(\theta)=(\ c_nr^{-4n}+d_nr^{4n}\ )a_n\cos(4n\theta)$$

$$By\ superposition:\ \ u(r,\theta)=\tilde{c}_0Ln(r)+\ \tilde{d}_0+\sum_{n=1}^{\infty}(\ \tilde{c}_nr^{-4n}+\tilde{d}_nr^{4n}\ )\cos(4n\theta)$$

$$\lim_{r\to\infty}u(r,\theta)<\infty\quad\Longrightarrow\ \tilde{c}_0=0\quad ,\ \ \tilde{d}_n=0\ \Longrightarrow\ \ u(r,\theta)=\tilde{d}_0+\sum_{n=1}^{\infty}(\ \tilde{c}_nr^{-4n})\cos(4n\theta)$$

$$Using\ other\ \ Boundry\ condition:$$

$$1=\ u(r_0,\theta)=\tilde{d}_0+\sum_{n=1}^{\infty}(\tilde{c}_n{r_0}^{-4n}\ \ )\cos(4n\theta)$$

$$\Longrightarrow\ \tilde{d}_0=\frac{1}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}1\ d\theta=1\quad\Longrightarrow\quad\tilde{d}_0=1$$

$$\Longrightarrow\ \tilde{c}_n{r_0}^{-4n}\ =\frac{2}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}1\,.\cos(4n\theta)\ d\theta=0\quad\Longrightarrow\quad\tilde{c}_n=0\quad\Longrightarrow\boxed{u(r,\theta)=1}$$

۴-۲) معادله زیر را حل کنید.

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \qquad r_0\le r\ ,\ 0\le\theta\le\frac{\pi}{2}$$
$$u_\theta(r,0)=u\left(r,\frac{\pi}{2}\right)=0$$
$$u(r_0,\theta)=\ \Pi\left(\frac{\theta}{\frac{\pi}{2}}\right)$$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0\ \ ,\ \ r=e^{-t}\ \Rightarrow\ u_{tt}+u_{\theta\theta}=0\ \Rightarrow u(t,\theta)=X(t)Y(\theta)\ \Rightarrow\ X''(t)Y(\theta)+X(t)Y''(\theta)=0\ \Rightarrow\ \frac{X''}{X}=-\frac{Y''}{Y}=\lambda$$

$$\textit{From } Y''(\theta)+\lambda Y(\theta)=0\ \Rightarrow Y(\theta)=\ a_n\cos(\sqrt{\lambda}\theta)+b_n\sin(\sqrt{\lambda}\theta)\ \Rightarrow \textit{boundry conditions give}:$$

$$u\left(r,\frac{\pi}{2}\right)=0\ \Rightarrow\ X(r)Y\left(\frac{\pi}{2}\right)=0\ \Rightarrow\ Y\left(\frac{\pi}{2}\right)=0 \qquad ,\qquad u_\theta(r,0)=0\ \Rightarrow\ X(r)Y'(0)=0\ \Rightarrow\ Y'(0)=0$$

$$\Rightarrow\ Y\left(\frac{\pi}{2}\right)=Y'(0)=0\quad ,\quad Y'(\theta)=\ -a_n\sqrt{\lambda}\sin(\sqrt{\lambda}\theta)+b_n\sqrt{\lambda}\cos(\sqrt{\lambda}\theta)$$

$$\Rightarrow\ Y'(0)=b_n\sqrt{\lambda}=0\quad\Rightarrow\quad b_n=0\quad ,\quad\Rightarrow\ Y\left(\frac{\pi}{2}\right)=a_n\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right)=0\ \Rightarrow\ \sqrt{\lambda}=(2n-1)\ \Rightarrow\ \lambda_n=(2n-1)^2$$

$$\textit{Thus},\ \ Y(\theta)=a_n\cos((2n-1)\,\theta)\qquad n=1,2,\dots$$
$$\textit{with this values of } \lambda_n \textit{ we solve }\ X''(t)-\ (2n-1)^2X(t)=0\ :$$

$$n>0:\ X(t)=c_ne^{(2n-1)t}+d_ne^{-(2n-1)t}\Rightarrow X(r)=c_nr^{-(2n-1)}+d_nr^{(2n-1)}\quad ,\ \ Y(\theta)=a_n\cos((2n-1)\,\theta)$$
$$\Rightarrow\ X(r)Y(\theta)=(c_nr^{-(2n-1)}+d_nr^{(2n-1)}\ )a_n\cos((2n-1)\,\theta)$$

$$\textit{By superposition}:\ u(r,\theta)=\sum_{n=1}^{\infty}(\ \tilde{c}_nr^{-(2n-1)}+\tilde{d}_nr^{(2n-1)}\ )\cos((2n-1)\,\theta)$$

$$\lim_{r\to\infty}u(r,\theta)<\infty\quad\Rightarrow\ \tilde{d}_n=0\quad\Rightarrow\quad u(r,\theta)=\sum_{n=1}^{\infty}\tilde{c}_nr^{-(2n-1)}\ \cos((2n-1)\,\theta)$$

$$\textit{Using other Boundry condition}:$$

$$\Pi\left(\frac{\theta}{\frac{\pi}{2}}\right)=\ u(r_0,\theta)=\sum_{n=1}^{\infty}\ \tilde{c}_n{r_0}^{-(2n-1)}\ \cos((2n-1)\,\theta)$$

$$\Rightarrow\ \tilde{c}_n{r_0}^{-(2n-1)}\ =\frac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\Pi\left(\frac{\theta}{\frac{\pi}{2}}\right)\cos((2n-1)\theta)\ d\theta=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}1\,.\cos((2n-1)\theta)\ d\theta+\frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}0\,.\cos((2n-1)\theta)\ d\theta$$

$$\tilde{c}_n{r_0}^{-(2n-1)}=\frac{4}{\pi}\,\frac{1}{(2n-1)}\left[\sin\left((2n-1)\frac{\pi}{4}\right)\right]$$

$$\Rightarrow\qquad \tilde{c}_n=\ \frac{4{r_0}^{(2n-1)}}{(2n-1)\pi}\sin\left((2n-1)\frac{\pi}{4}\right)$$

$$\Rightarrow\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin\left((2n-1)\frac{\pi}{4}\right)\ \left(\frac{r_0}{r}\right)^{2n-1}\ \cos((2n-1)\theta)}$$