تمرین ۶: معادله حرارت و موج غیر همگن

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۱) معادله موج زیر را حل کنید.

$$9\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=u, \qquad 0\le x\le \pi, \qquad 0\le t$$
$$u(0,t)=u(\pi,t)=0$$
$$u(x,0)=0, \qquad u_t(x,0)=x$$

$$\text{BC: } \begin{cases} u(0,t)=0 \\ u(\pi,t)=0 \end{cases} \Longrightarrow u(x,t)=\sum_{n=1}^{\infty} T_n(t)\sin(nx)$$

$$\Longrightarrow \sum_{n=1}^{\infty}\left(-9n^2T_n(t)-\ddot{T}_n(t)\right)\sin(nx)=\sum_{n=1}^{\infty}T_n(t)\sin(nx) \Longrightarrow \sum_{n=1}^{\infty}\left(\ddot{T}_n(t)+(9n^2+1)T_n(t)\right)\sin(nx)=0$$

$$\Longrightarrow \ddot{T}_n(t)+(9n^2+1)T_n(t)=0 \Longrightarrow T_n(t)=a_n\cos\left(\sqrt{9n^2+1}t\right)+b_n\sin\left(\sqrt{9n^2+1}t\right)$$

$$\Longrightarrow u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\left(\sqrt{9n^2+1}t\right)+b_n\sin\left(\sqrt{9n^2+1}t\right)\right)\sin(nx)$$

$$\text{IC: } u(x,0)=0 \Longrightarrow a_n=0$$

$$\text{IC: } u_t(x,0)=\sum_{n=1}^{\infty} b_n\sqrt{9n^2+1}\sin(nx)=x \Longrightarrow b_n=\frac{2}{\pi\sqrt{9n^2+1}}\int_0^{\pi} x\sin(nx)\,dx=\frac{2}{\pi\sqrt{9n^2+1}}\left[-\frac{x}{n}\cos(nx)+\frac{1}{n^2}\sin(nx)\right]_0^{\pi}=\frac{2(-1)^{n+1}}{n\sqrt{9n^2+1}}$$

$$\Longrightarrow \boxed{u(x,t)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\sqrt{9n^2+1}}\sin\left(\sqrt{9n^2+1}t\right)\sin(nx)}$$

۲) معادله حرارت زیر را حل کنید.

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{2}\frac{\partial u}{\partial t}=0, \qquad 0\le x, \qquad 0\le t$$
$$u_x(0,t)=0$$
$$u(x,0)=e^{-x^2}$$

$$u(x,t)=T(t)X(x) \Longrightarrow \frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{2\dot{T}(t)}=-k^2 \Longrightarrow \begin{cases} X''(x)+k^2X(x)=0 \\ \dot{T}(t)+2k^2T(t)=0 \end{cases} \Longrightarrow \begin{cases} X(x)=A(k)\cos(kx)+B(k)\sin(kx) \\ T(t)=C(k)e^{-2k^2t} \end{cases}$$

$$\Longrightarrow u(x,t)=\int_0^{\infty} e^{-2\omega^2 t}(A(\omega)\cos(\omega x)+B(\omega)\sin(\omega x))d\omega$$

$$u_x(0,t)=\int_0^{\infty} e^{-2\omega^2 t}\omega B(\omega)d\omega=0 \Longrightarrow B(\omega)=0$$

$$u(x,0)=\int_0^{\infty} A(\omega)\cos(\omega x)\,d\omega=e^{-x^2}$$

$$e^{-x^2}\text{ is an even function} \Longrightarrow \mathcal{F}\left[e^{-x^2}\right]=\pi A(\omega) \Longrightarrow A(\omega)=\frac{1}{\pi}\mathcal{F}\left[e^{-x^2}\right]=\frac{1}{\sqrt{\pi}}e^{-\frac{\omega^2}{4}}$$

$$\Longrightarrow u(x,t)=\int_0^{\infty}\frac{1}{\sqrt{\pi}}e^{-\left(2t+\frac{1}{4}\right)\omega^2}\cos(\omega x)\,d\omega$$

$$\Longrightarrow \boxed{u(x,t)=\frac{e^{-\frac{x^2}{8t+1}}}{\sqrt{8t+1}}}$$

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۳) معادله گرمای زیر را حل کنید.

$$u_t = 4u_{xx} + \Lambda\left(\frac{x-\frac{\pi}{2}}{\frac{\pi}{2}}\right), \qquad 0 \le x \le \pi$$

$$u(0,t) = \sin(t), \qquad u_x(\pi,t) = \cos(t)$$

$$u(x,0) = \Pi\left(\frac{x}{\pi}\right)$$

$$u(x,t) = w(x,t) + v(x,t)$$

$$\text{BC:} \begin{cases} u(0,t) = \sin(t) \\ u_x(\pi,t) = \cos(t) \end{cases} \Rightarrow w(x,t) = \sin(t) + x\cos(t)$$

$$\Rightarrow v_t + \cos(t) - x\sin(t) = 4v_{xx} + \Lambda\left(\frac{x-\frac{\pi}{2}}{\frac{\pi}{2}}\right)$$

$$v(0,t) = 0, \qquad v_x(0,t) = 0$$

$$v(x,0) = \Pi\left(\frac{x}{\pi}\right) - x$$

$$\text{BC:} \begin{cases} v(0,t) = 0 \\ v_x(\pi,t) = 0 \end{cases} \Rightarrow v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\left(\frac{2n-1}{2}x\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \dot{T}_n(t)\sin\left(\frac{2n-1}{2}x\right) + \cos(t) - x\sin(t) = -\sum_{n=1}^{\infty} T_n(t)(2n-1)^2\sin\left(\frac{2n-1}{2}x\right) + \Lambda\left(\frac{x-\frac{\pi}{2}}{\frac{\pi}{2}}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\dot{T}_n(t) + (2n-1)^2T_n(t)\right)\sin\left(\frac{2n-1}{2}x\right) = x\sin(t) - \cos(t) + \Lambda\left(\frac{x-\frac{\pi}{2}}{\frac{\pi}{2}}\right)$$

$$\Rightarrow \dot{T}_n(t) + (2n-1)^2T_n(t) = \frac{2}{\pi}\int_0^{\pi}\left(x\sin(t) - \cos(t) + \Lambda\left(\frac{x-\frac{\pi}{2}}{\frac{\pi}{2}}\right)\right)\sin\left(\frac{2n-1}{2}x\right)$$

$$= \frac{2}{\pi}\sin(t)\int_0^{\pi} x\sin\left(\frac{2n-1}{2}x\right)dx - \frac{2}{\pi}\cos(t)\int_0^{\pi}\sin\left(\frac{2n-1}{2}x\right)dx + \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} x\sin\left(\frac{2n-1}{2}x\right)dx + \frac{4}{\pi^2}\int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin\left(\frac{2n-1}{2}x\right)dx$$

$$= -\frac{8(-1)^n}{\pi(2n-1)^2}\sin(t) - \frac{4}{\pi(2n-1)}\cos(t) + \frac{16\left((-1)^n + 2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^2}$$

$$\Rightarrow \dot{T}_n(t) + (2n-1)^2T_n(t) = -\frac{8(-1)^n}{\pi(2n-1)^2}\sin(t) - \frac{4}{\pi(2n-1)}\cos(t) + \frac{16\left((-1)^n + 2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^2}$$

$$T_{h_n}(t) = c_n e^{-(2n-1)^2 t}$$

$$T_{p_n}(t) = A\cos(t) + B\sin(t) + \frac{16\left((-1)^n + 2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}$$

$$\Rightarrow \left(B + A(2n-1)^2\right)\cos(t) + \left(-A + B(2n-1)^2\right)\sin(t) = -\frac{8(-1)^n}{\pi(2n-1)^2}\sin(t) - \frac{4}{\pi(2n-1)}\cos(t) \Rightarrow \begin{cases} B + A(2n-1)^2 = -\frac{4}{\pi(2n-1)} \\ -A + B(2n-1)^2 = -\frac{8(-1)^n}{\pi(2n-1)^2} \end{cases}$$

$$\Rightarrow \begin{cases} A = \frac{8(-1)^n}{\pi(2n-1)^2(1+(2n-1)^4)} - \frac{4(2n-1)}{\pi(1+(2n-1)^4)} = \frac{8(-1)^n - 4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)} \\ B = -\frac{8(-1)^n}{\pi(1+(2n-1)^4)} - \frac{4}{\pi(2n-1)(1+(2n-1)^4)} = -\frac{8(-1)^n(2n-1)+4}{\pi(2n-1)(1+(2n-1)^4)} \end{cases}$$

$$\Rightarrow T_n(t) = c_n e^{-(2n-1)^2 t} + \frac{8(-1)^n - 4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)}\cos(t) - \frac{8(-1)^n(2n-1)+4}{\pi(2n-1)(1+(2n-1)^4)}\sin(t) + \frac{16\left((-1)^n + 2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}$$

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty}\left(c_n e^{-(2n-1)^2 t} + \frac{8(-1)^n - 4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)}\cos(t) - \frac{8(-1)^n(2n-1)+4}{\pi(2n-1)(1+(2n-1)^4)}\sin(t) + \frac{16\left((-1)^n + 2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}\right)\sin\left(\frac{2n-1}{2}x\right)$$

$$\text{IC: } v(x,0)=\sum_{n=1}^{\infty}\left(c_n+\frac{8(-1)^n-4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)}+\frac{16\left((-1)^n+2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}\right)\sin\left(\frac{2n-1}{2}x\right)=\Pi\left(\frac{x}{\pi}\right)-x$$

$$\Rightarrow c_n+\frac{8(-1)^n-4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)}+\frac{16\left((-1)^n+2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}=\frac{2}{\pi}\int_0^{\pi}\left(\Pi\left(\frac{x}{\pi}\right)-x\right)\sin\left(\frac{2n-1}{2}x\right)dx$$

$$=\frac{8(-1)^n}{\pi(2n-1)^2}+\frac{4\left(1-\cos\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi(2n-1)}$$

$$\Rightarrow c_n=\frac{8(-1)^n}{\pi(2n-1)^2}+\frac{4\left(1-\cos\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi(2n-1)}-\frac{8(-1)^n-4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)}-\frac{16\left((-1)^n+2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}$$

$$\Rightarrow v(x,t)=\sum_{n=1}^{\infty}\left(\left(\frac{8(-1)^n}{\pi(2n-1)^2}+\frac{4\left(1-\cos\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi(2n-1)}-\frac{8(-1)^n-4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)}-\frac{16\left((-1)^n+2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}\right)e^{-(2n-1)^2t}\right.$$
$$\left.+\frac{8(-1)^n-4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)}\cos(t)-\frac{8(-1)^n(2n-1)+4}{\pi(2n-1)(1+(2n-1)^4)}\sin(t)+\frac{16\left((-1)^n+2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}\right)\sin\left(\frac{2n-1}{2}x\right)$$

$$\Rightarrow u(x,t)=\sin(t)+x\cos(t)$$
$$+\sum_{n=1}^{\infty}\left(\left(\frac{8(-1)^n}{\pi(2n-1)^2}+\frac{4\left(1-\cos\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi(2n-1)}-\frac{8(-1)^n-4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)}-\frac{16\left((-1)^n+2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}\right)e^{-(2n-1)^2t}\right.$$
$$\left.+\frac{8(-1)^n-4(2n-1)^3}{\pi(2n-1)^2(1+(2n-1)^4)}\cos(t)-\frac{8(-1)^n(2n-1)+4}{\pi(2n-1)(1+(2n-1)^4)}\sin(t)+\frac{16\left((-1)^n+2\sin\left(\frac{(2n-1)\pi}{4}\right)\right)}{\pi^2(2n-1)^4}\right)\sin\left(\frac{2n-1}{2}x\right)$$

۴) معادله با مشتقات جزئی زیر را حل کنید

$u_t-t^2u_{xx}-u=0,\qquad 0\le x\le 1,\qquad 0\le t$
$u(0,t)=u(1,t)=0$
$u(x,0)=1$

$$\text{BC: }\begin{cases}u(0,t)=0\\u(1,t)=0\end{cases}\Rightarrow u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin(n\pi x)$$

$$\Rightarrow\sum_{n=1}^{\infty}\left(\dot{T}_n(t)+(t^2n^2\pi^2-1)T_n(t)\right)\sin(n\pi x)=0$$

$$\dot{T}_n(t)+(t^2n^2\pi^2-1)T_n(t)=0\Rightarrow\frac{\dot{T}_n(t)}{T_n(t)}=1-t^2n^2\pi^2\Rightarrow\ln\left(T_n(t)\right)=t-\frac{1}{3}t^3n^2\pi^2+C\Rightarrow T_n(t)=c_ne^{t-\frac{1}{3}t^3n^2\pi^2}$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}c_ne^{t-\frac{1}{3}t^3n^2\pi^2}\sin(n\pi x)$$

$$u(x,0)=\sum_{n=1}^{\infty}c_n\sin(n\pi x)=1\Rightarrow c_n=2\int_0^1\sin(n\pi x)\,dx=\frac{2(1-(-1)^n)}{n\pi}$$

$$\Rightarrow\boxed{u(x,t)=\sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}e^{t-\frac{1}{3}t^3n^2\pi^2}\sin(n\pi x)}$$

۵) سیمی به طول l در دو سر خود بدون تغییر مکان بوده و تا قبل از زمان صفر در حال سکون می‌باشد. در زمان صفر در وسط خود تحت شدت نیرویی برابر با $\frac{P}{2\epsilon}$ در طول 2ϵ قرار می‌گیرد. تغییر مکان سیم، $u(x,t)$ را در هر لحظه بیابید. نتیجه را برای زمانی که $\epsilon \to 0$ ساده کنید. (امتیازی)

$c^2u_{xx} - u_{tt} = g(x,t), \qquad 0 \le x \le l, \qquad 0 \le t$
$u(0,t) = u(l,t) = 0$
$u(x,0) = u_t(x,0) = 0$

$$g(x,t) = \begin{cases} 0 & 0 \le x < \frac{l}{2} - \epsilon \\ \frac{P}{2\epsilon} & \frac{l}{2} - \epsilon \le x \le \frac{l}{2} + \epsilon \\ 0 & \frac{l}{2} + \epsilon < x \le l \end{cases}$$

$$\text{BC: } \begin{cases} u(0,t) = 0 \\ u(l,t) = 0 \end{cases} \Rightarrow u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(-\frac{c^2n^2\pi^2}{l^2}T_n(t) - \ddot{T}_n(t)\right)\sin\left(\frac{n\pi}{l}x\right) = g(x,t)$$

$$\Rightarrow -\frac{c^2n^2\pi^2}{l^2}T_n(t) - \ddot{T}_n(t) = \frac{2}{l}\int_{\frac{l}{2}-\epsilon}^{\frac{l}{2}+\epsilon} \frac{P}{2\epsilon}\sin\left(\frac{n\pi}{l}x\right)dx = -\frac{P}{n\pi\epsilon}\left(\cos\left(\frac{n\pi}{2} + \frac{n\pi\epsilon}{l}\right) - \cos\left(\frac{n\pi}{2} - \frac{n\pi\epsilon}{l}\right)\right) = \frac{2P}{n\pi\epsilon}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\epsilon}{l}\right)$$

$$\Rightarrow \ddot{T}_n(t) + \frac{c^2n^2\pi^2}{l^2}T_n(t) = -\frac{2P}{n\pi\epsilon}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\epsilon}{l}\right)$$

$$\Rightarrow T_n(t) = a_n\cos\left(\frac{cn\pi}{l}t\right) + b_n\sin\left(\frac{cn\pi}{l}t\right) - \frac{2Pl^2}{c^2n^3\pi^3\epsilon}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\epsilon}{l}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{cn\pi}{l}t\right) + b_n\sin\left(\frac{cn\pi}{l}t\right) - \frac{2Pl^2}{c^2n^3\pi^3\epsilon}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\epsilon}{l}\right)\right)\sin\left(\frac{n\pi}{l}x\right)$$

$$u(x,0) = 0 \Rightarrow a_n = \frac{2Pl^2}{c^2n^3\pi^3\epsilon}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\epsilon}{l}\right)$$

$$u_t(x,0) = 0 \Rightarrow b_n = 0$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2Pl^2}{c^2n^3\pi^3\epsilon}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\epsilon}{l}\right)\cos\left(\frac{cn\pi}{l}t\right) - \frac{2Pl^2}{c^2n^3\pi^3\epsilon}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\epsilon}{l}\right)\right)\sin\left(\frac{n\pi}{l}x\right)}$$

$$\lim_{\epsilon\to 0} u(x,t) = \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\left(\frac{2Pl}{c^2n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\operatorname{sinc}\left(\frac{n\epsilon}{l}\right)\cos\left(\frac{cn\pi}{l}t\right) - \frac{2Pl}{c^2n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\operatorname{sinc}\left(\frac{n\epsilon}{l}\right)\right)\sin\left(\frac{n\pi}{l}x\right)$$

$$= \boxed{\sum_{n=1}^{\infty}\left(\frac{2Pl}{c^2n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{cn\pi}{l}t\right) - \frac{2Pl}{c^2n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi}{l}x\right)}$$

۶) معادله گرمای زیر را حل کنید.

$u_t = u_{xx} + e^{-t} + \cos(3\pi x), \qquad 0 \le x \le 1, \qquad 0 \le t$
$u_x(0,t) = u_x(1,t) = 1$
$u(x,0) = x + \sin^2(\pi x)$

$$u(x,t) = w(x,t) + v(x,t)$$

$$\text{BC: } \begin{cases} u_x(0,t) = 1 \\ u_x(1,t) = 1 \end{cases} \Rightarrow w(x,t) = x$$

$$\Rightarrow v_{tt} = v_{xx} + e^{-t} + \cos(3\pi x)$$

$$v_x(0,t) = v_x(1,t) = 0$$

$$v(x,0) = \sin^2(\pi x)$$

$$\text{BC: } \begin{cases} v_x(0,t) = 0 \\ v_x(1,t) = 0 \end{cases} \Rightarrow v(x,t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x)$$

$$\Rightarrow \dot{T}_0(t) + \sum_{n=1}^{\infty}\left(\dot{T}_n(t) + n^2\pi^2T_n(t)\right)\cos(n\pi x) = e^{-t} + \cos(3\pi x)$$

$$\dot{T}_0(t) = \int_0^1 (e^{-t} + \cos(3\pi x))dx = e^{-t}$$

$$\Rightarrow T_0(t) = -e^{-t} + c_0$$

$$\dot{T}_n(t) + n^2\pi^2 T_n(t) = 2\int_0^1 \left(e^{-t} + \cos(3\pi x)\right)\cos(n\pi x)\,dx = \begin{cases} 1 & n = 3 \\ 0 & n \neq 3 \end{cases} = \delta[n-3]$$

$$\Rightarrow T_n(t) = c_n e^{-n^2\pi^2 t} + \frac{1}{n^2\pi^2}\delta[n-3]$$

$$\Rightarrow v(x,t) = -e^{-t} + c_0 + \sum_{n=1}^{\infty}\left(c_n e^{-n^2\pi^2 t} + \frac{1}{n^2\pi^2}\delta[n-3]\right)\cos(n\pi x)$$

$$\text{IC: } v(x,0) = -1 + c_0 + \sum_{n=1}^{\infty}\left(c_n + \frac{1}{n^2\pi^2}\delta[n-3]\right)\cos(n\pi x) = \sin^2(x)$$

$$\Rightarrow -1 + c_0 = \int_0^1 \sin^2(x)\,dx = \int_0^1\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx = \frac{1}{2} - \frac{\sin(2)}{4}$$

$$\Rightarrow c_0 = \frac{3}{2} - \frac{\sin(2)}{4}$$

$$c_n + \frac{1}{n^2\pi^2}\delta[n-3] = 2\int_0^1 \sin^2(x)\cos(n\pi x)\,dx = 2\int_0^1\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\cos(n\pi x)\,dx = \frac{2(-1)^n\sin(2)}{n^2\pi^2 - 4}$$

$$\Rightarrow c_n = \frac{2(-1)^n\sin(2)}{n^2\pi^2 - 4} - \frac{1}{n^2\pi^2}\delta[n-3]$$

$$\Rightarrow v(x,t) = -e^{-t} + \frac{3}{2} - \frac{\sin(2)}{4} + \sum_{n=1}^{\infty}\left(\left(\frac{2(-1)^n\sin(2)}{n^2\pi^2 - 4} - \frac{1}{n^2\pi^2}\delta[n-3]\right)e^{-n^2\pi^2 t} + \frac{1}{n^2\pi^2}\delta[n-3]\right)\cos(n\pi x)$$

$$= -e^{-t} + \frac{3}{2} - \frac{\sin(2)}{4} + \frac{1 - e^{-9\pi^2 t}}{9\pi^2}\cos(3\pi x) + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n\sin(2)}{n^2\pi^2 - 4}e^{-n^2\pi^2 t}\right)\cos(n\pi x)$$

$$\Rightarrow \boxed{u(x,t) = x - e^{-t} + \frac{3}{2} - \frac{\sin(2)}{4} + \frac{1 - e^{-9\pi^2 t}}{9\pi^2}\cos(3\pi x) + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n\sin(2)}{n^2\pi^2 - 4}e^{-n^2\pi^2 t}\right)\cos(n\pi x)}$$

۷) معادله حرارت غیر همگن زیر را حل کنید.

$u_t = u_{xx} + x^2 + t(1+\cos(\pi x)), \qquad 0 \le x \le 1, \qquad 0 \le t$

$u_x(0,t) = 2, \qquad u_x(1,t) = 2+2t$

$u(x,0) = 2x$

$$u(x,t) = w(x,t) + v(x,t)$$

$$\text{BC: } \begin{cases} u_x(0,t) = 2 \\ u_x(1,t) = 2+2t \end{cases} \Rightarrow w(x,t) = 2x + x^2 t$$

$$\Rightarrow v_t + x^2 = v_{xx} + 2t + x^2 + t + t\cos(\pi x)$$

$$\Rightarrow v_t = v_{xx} + 3t + t\cos(\pi x)$$

$$v_x(0,t) = 0, \qquad v_x(1,t) = 0$$

$$v(x,0) = 0$$

$$\text{BC: } \begin{cases} v_x(0,t) = 0 \\ v_x(1,t) = 0 \end{cases} \Rightarrow v(x,t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x)$$

$$\Rightarrow \dot{T}_0(t) + \sum_{n=1}^{\infty}\left(\dot{T}_n(t) + n^2\pi^2 T_n(t)\right)\cos(n\pi x) = 3t + t\cos(\pi x)$$

$$\dot{T}_0(t) = 3t \Rightarrow T_0(t) = \frac{3}{2}t^2 + c_0$$

$$\dot{T}_n(t) + n^2\pi^2 T_n(t) = t\delta[n-1] \Rightarrow T_n(t) = c_n e^{-n^2\pi^2 t} + \left(\frac{1}{n^2\pi^2}t - \frac{1}{n^4\pi^4}\right)\delta[n-1]$$

$$\Rightarrow v(x,t) = \frac{3}{2}t^2 + c_0 + \sum_{n=1}^{\infty}\left(c_n e^{-n^2\pi^2 t} + \left(\frac{1}{n^2\pi^2}t - \frac{1}{n^4\pi^4}\right)\delta[n-1]\right)\cos(n\pi x)$$

$$v(x,0) = c_0 + \sum_{n=1}^{\infty}\left(c_n - \frac{1}{n^4\pi^4}\delta[n-1]\right)\cos(n\pi x) = 0$$

$$\Rightarrow c_0 = 0, \qquad c_n = \frac{1}{n^4\pi^4}\delta[n-1]$$

$$\Rightarrow v(x,t) = \frac{3}{2}t^2 + \sum_{n=1}^{\infty}\left(\frac{1}{n^4\pi^4}\delta[n-1]e^{-n^2\pi^2 t} + \left(\frac{1}{n^2\pi^2}t - \frac{1}{n^4\pi^4}\right)\delta[n-1]\right)\cos(n\pi x)$$

$$v(x,t) = \frac{3}{2}t^2 + \left(\frac{1}{\pi^4}e^{-\pi^2 t} - \frac{1}{\pi^4} + \frac{1}{\pi^2}t\right)\cos(\pi x)$$

$$\Rightarrow \boxed{u(x,t) = 2x + x^2 t + \frac{3}{2}t^2 + \left(\frac{1}{\pi^4}e^{-\pi^2 t} - \frac{1}{\pi^4} + \frac{1}{\pi^2}t\right)\cos(\pi x)}$$

مدرس: دکتر مهدی طالع ماسوله- حل تمرین: گلمهر خسروخاور- سروش مس فروش- حسین عطرسایی
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۸) یک میله نیمه محدود را در نظر بگیرید. حرارت، $u(x,t)$، را در طول میله با شرایط زیر بدست آورید.

$$u_{xx}=\frac{1}{c^2}u_t,\qquad 0\le x,\qquad 0\le t$$

$$u_x(0,t)=0$$

$$u(x,0)=\Pi\left(\frac{x-\frac{a}{2}}{a}\right)$$

$$\frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{c^2T(t)}=-k^2\Rightarrow\begin{cases}X''(x)+k^2X(x)=0\\ \dot{T}(t)+c^2k^2T(t)=0\end{cases}\Rightarrow\begin{cases}X(x)=a(k)\cos(kx)+b(k)\sin(kx)\\ T(t)=c(k)e^{-c^2k^2t}\end{cases}$$

$$\Rightarrow u(x,t)=\int_0^\infty e^{-c^2\omega^2t}(A(\omega)\cos(\omega x)+B(\omega)\sin(\omega x))d\omega$$

$$\text{BC: } u_x(0,t)=\int_0^\infty e^{-c^2\omega^2t}\omega B(\omega)d\omega=0$$

$$\Rightarrow B(\omega)=0$$

$$\Rightarrow u(x,t)=\int_0^\infty A(\omega)e^{-c^2\omega^2t}\cos(\omega x)\,d\omega$$

$$u(x,0)=\int_0^\infty A(\omega)\cos(\omega x)\,d\omega=\Pi\left(\frac{x-\frac{a}{2}}{a}\right)\Rightarrow A(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\Pi\left(\frac{x}{2a}\right)\cos(\omega x)\,dx=\frac{1}{\pi}\int_{-a}^{a}\cos(\omega x)\,dx=\frac{2}{\pi}\int_0^a\cos(\omega x)\,dx=\frac{2\sin(\omega a)}{\pi\omega}$$

$$\Rightarrow u(x,t)=\int_0^\infty\frac{2\sin(\omega a)}{\pi\omega}e^{-c^2\omega^2t}\cos(\omega x)\,d\omega$$

$$\mathcal{F}\left[\frac{1}{\sqrt{4\pi c^2t}}e^{-\frac{x^2}{4c^2t}}\right]=e^{-c^2\omega^2t}$$

$$\mathcal{F}\left[\frac{i}{2\sqrt{4\pi c^2t}}\left(e^{-\frac{(x-a)^2}{4c^2t}}-e^{-\frac{(x+a)^2}{4c^2t}}\right)\right]=\sin(\omega a)\,e^{-c^2\omega^2t}$$

$$\mathcal{F}\left[\frac{1}{\pi\sqrt{4\pi c^2t}}\int_{-\infty}^{x}\left(e^{-\frac{(\xi+a)^2}{4c^2t}}-e^{-\frac{(\xi-a)^2}{4c^2t}}\right)d\xi\right]=\frac{2\sin(\omega a)}{\pi\omega}e^{-c^2\omega^2t}\Rightarrow\mathcal{F}\left[\frac{1}{2\pi}\left(\operatorname{erf}\left(\frac{x+a}{2c\sqrt{t}}\right)-\operatorname{erf}\left(\frac{x-a}{2c\sqrt{t}}\right)\right)\right]=\frac{2\sin(\omega a)}{\pi\omega}e^{-c^2\omega^2t}$$

$$2\pi\mathcal{F}^{-1}\left[\frac{2\sin(\omega a)}{\pi\omega}e^{-c^2\omega^2t}\right]=2\int_0^\infty\frac{2\sin(\omega a)}{\pi\omega}e^{-c^2\omega^2t}\cos(\omega x)\,d\omega=\operatorname{erf}\left(\frac{x+a}{2c\sqrt{t}}\right)-\operatorname{erf}\left(\frac{x-a}{2c\sqrt{t}}\right)$$

$$\Rightarrow\boxed{u(x,t)=\frac{1}{2}\left(\operatorname{erf}\left(\frac{x+a}{2c\sqrt{t}}\right)-\operatorname{erf}\left(\frac{x-a}{2c\sqrt{t}}\right)\right)}$$

۹) معادله موج زیر را حل کنید. (امتیازی)

$$u_{xx}+u_{yy}=\frac{1}{c^2}u_{tt},\qquad 0\le x\le l_x,\qquad 0\le y\le l_y,\qquad 0\le t$$

$$u(0,y,t)=0,\qquad u(l_x,y,t)=0$$

$$u(x,0,t)=0,\qquad u_y(x,l_y,t)=0$$

$$u(x,y,0)=\Lambda\left(\frac{x-\frac{l_x}{2}}{\frac{l_x}{2}}\right)\Lambda\left(\frac{y-\frac{l_y}{2}}{\frac{l_y}{2}}\right),\qquad u_t(x,y)=0$$

$$\mathrm{BC}_x:\begin{cases}u(0,y,t)=0\\ u(l_x,y,t)=0\end{cases},\qquad \mathrm{BC}_y:\begin{cases}u(x,0,t)=0\\ u_y(x,l_y,t)=0\end{cases}\Longrightarrow u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}T_{n,m}(t)\sin\left(\frac{n\pi}{l_x}x\right)\sin\left(\frac{(2m-1)\pi}{2l_y}y\right)$$

$$\Longrightarrow\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{c^2}\ddot{T}_{n,m}(t)+\left(\frac{n^2\pi^2}{l_x^2}+\frac{(2m-1)^2\pi^2}{4l_y^2}\right)T_{n,m}(t)\right)\sin\left(\frac{n\pi}{l_x}x\right)\sin\left(\frac{(2m-1)\pi}{2l_y}y\right)=0$$

$$\Longrightarrow\ddot{T}_{n,m}(t)+c^2\left(\frac{n^2\pi^2}{l_x^2}+\frac{(2m-1)^2\pi^2}{4l_y^2}\right)T_{n,m}(t)=0\Longrightarrow T_{n,m}(t)=a_{n,m}\cos(c\lambda_{n,m}t)+b_{n,m}\sin(c\lambda_{n,m}t),\qquad \lambda_{n,m}=\sqrt{\frac{n^2\pi^2}{l_x^2}+\frac{(2m-1)^2\pi^2}{4l_y^2}}$$

$$\Longrightarrow u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(a_{n,m}\cos(c\lambda_{n,m}t)+b_{n,m}\sin(c\lambda_{n,m}t)\right)\sin\left(\frac{n\pi}{l_x}x\right)\sin\left(\frac{(2m-1)\pi}{2l_y}y\right)$$

$$u_t(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c\lambda_{n,m}b_{n,m}\sin\left(\frac{n\pi}{l_x}x\right)\sin\left(\frac{(2m-1)\pi}{2l_y}y\right)=0\Longrightarrow b_{n,m}=0$$

$$u(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{n,m}\sin\left(\frac{n\pi}{l_x}x\right)\sin\left(\frac{(2m-1)\pi}{2l_y}y\right)=\Lambda\left(\frac{x-\frac{l_x}{2}}{\frac{l_x}{2}}\right)\Lambda\left(\frac{y-\frac{l_y}{2}}{\frac{l_y}{2}}\right)$$

$$a_{n,m}=\frac{4}{l_xl_y}\int_0^{l_x}\int_0^{l_y}\Lambda\left(\frac{x-\frac{l_x}{2}}{\frac{l_x}{2}}\right)\Lambda\left(\frac{y-\frac{l_y}{2}}{\frac{l_y}{2}}\right)\sin\left(\frac{n\pi}{l_x}x\right)\sin\left(\frac{(2m-1)\pi}{2l_y}y\right)dy\,dx$$

$$=\frac{4}{l_xl_y}\left(\int_0^{l_x}\Lambda\left(\frac{x-\frac{l_x}{2}}{\frac{l_x}{2}}\right)\sin\left(\frac{n\pi}{l_x}x\right)dx\right)\left(\int_0^{l_y}\Lambda\left(\frac{y-\frac{l_y}{2}}{\frac{l_y}{2}}\right)\sin\left(\frac{(2m-1)\pi}{2l_y}y\right)dy\right)=128\frac{\sin\left(\frac{n\pi}{2}\right)\left((-1)^m+2\sin\left(\frac{(2m-1)\pi}{4}\right)\right)}{n^2(2m-1)^2\pi^4}$$

$$\Longrightarrow\boxed{u(x,y,t)=\frac{128}{\pi^4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)\left((-1)^m+2\sin\left(\frac{(2m-1)\pi}{4}\right)\right)}{n^2(2m-1)^2}\cos(c\lambda_{n,m}t)\sin\left(\frac{n\pi}{l_x}x\right)\sin\left(\frac{(2m-1)\pi}{2l_y}y\right)}$$