مدرس: دکتر مهدی طالع ماسوله- حل تمرین: نگین سفاری و آرمان اکبری برای سوالات خود درخصوص این تمرین بارایانامه های
sneginsafari@gmail.com مکاتبه نمایید.

۱)

$$\int_0^\infty y(\omega)\sin(\omega x)\,d\omega = \text{sign}(x)\begin{cases} x^2 & |x| \le 1 \\ \delta(|x|-1) & |x| = 1 \\ 0 & \text{o.w} \end{cases}$$

تابع فرد است، بنابراین:

$$A(\omega) = 0$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx = \frac{2}{\pi}\int_0^1 x^2\sin(\omega x)\,dx + \frac{2}{\pi}\sin(\omega)$$

$$= \frac{2}{\pi}\left(\left[-\frac{x^2\cos(\omega x)}{\omega}\right]_0^1 + \int_0^1 \frac{2x\cos(\omega x)}{\omega}dx\right) + \frac{2}{\pi}\sin(\omega)$$

$$= \frac{2}{\pi}\left(\left[-\frac{x^2\cos(\omega x)}{\omega} + \frac{2x\sin(\omega x)}{\omega^2}\right]_0^1 - \int_0^1 \frac{2\sin(\omega x)}{\omega^2}dx\right)$$

$$= \frac{2}{\pi}\left[-\frac{x^2\cos(\omega x)}{\omega} + \frac{2x\sin(\omega x)}{\omega^2} + \frac{2\cos(\omega x)}{\omega^3}\right]_0^1 = \frac{2}{\pi}\left(-\frac{\cos(\omega)}{\omega} + \frac{2\sin(\omega)}{\omega^2} + \frac{2(\cos(\omega)-1)}{\omega^3}\right)$$

$$\Longrightarrow \boxed{f(x) = \int_0^\infty \left(-\frac{2\cos(\omega)}{\pi\omega} + \frac{4\sin(\omega)}{\pi\omega^2} + \frac{4(\cos(\omega)-1)}{\omega^3}\right)\sin(\omega x)\,dx} \Longrightarrow y(\omega) = B(\omega)$$

۲)

$$f(x) = \begin{cases} b\left(1-\frac{|x|}{a}\right) & |x| \le a \\ 0 & \text{o.w} \end{cases}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\,e^{-j\omega x}dx = \int_{-a}^{0} b\left(1+\frac{x}{a}\right)e^{-j\omega x}dx + \int_0^a b\left(1-\frac{x}{a}\right)e^{-j\omega x}dx$$

$$= \left[\frac{b}{-j\omega}\left(1+\frac{x}{a}\right)e^{-j\omega x}\right]_{-a}^{0} - \frac{b}{-ja\omega}\int_{-a}^{0} e^{-j\omega x}dx + \left[\frac{b}{-j\omega}\left(1-\frac{x}{a}\right)e^{-j\omega x}\right]_0^{+a} + \frac{b}{-ja\omega}\int_0^{+a} e^{-j\omega x}dx$$

$$= -\frac{b}{j\omega} + \frac{b(1-e^{j\omega a})}{a\omega^2} + \frac{b}{j\omega} - \frac{b(e^{-j\omega a}-1)}{a\omega^2} = \frac{2b}{a}\left(\frac{1-\cos(a\omega)}{\omega^2}\right) = \frac{b}{a}\frac{4\sin^2\left(\frac{a}{2}\omega\right)}{\omega^2}$$

$$\Longrightarrow \boxed{F(\omega) = \frac{b}{a}\frac{4\sin^2\left(\frac{a}{2}\omega\right)}{\omega^2}}$$

$$\text{parseval} \Longrightarrow \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{4b}{a\omega^2}\sin^2\left(\frac{a\omega}{2}\right)\right)^2 d\omega = \int_{-\infty}^{+\infty}|f(x)|^2\,dx = 2\int_0^a\left(-\frac{b}{a}x+b\right)^2 dx = \frac{2ab^2}{3}$$

$$\int_{-\infty}^{+\infty}\left(\frac{4b}{a\omega^2}sin^2\left(\frac{a\omega}{2}\right)\right)^2 d\omega = \frac{4\pi ab^2}{3}$$

$$a=4 \Longrightarrow \boxed{\int_{-\infty}^{+\infty}\left(\frac{1}{x^2}\sin^2(2x)\right)^2 dx = \frac{16\pi b^2}{3}}$$

۳)

تابعی که انتخاب می‌کنیم باید در بازه $x \geq 0$ برابر با e^{-kx} باشد و با توجه به رابطه انتگرال فوریه باید تابعی زوج باشد. بنابراین:

$$f(x) = e^{-k|x|}$$

تابع یک تابع زوج است.

$$A(\omega) = \frac{2}{\pi}\int_0^\infty e^{-kx}\cos(\omega x)\,dx = \frac{2}{\pi}\int_0^\infty \frac{e^{-kx}(\,e^{j\omega x}+e^{-j\omega x})}{2}dx = \frac{1}{\pi}\int_0^\infty\left(e^{(j\omega-k)x}+e^{(-j\omega-k)x}\right)\,dx$$
$$= \frac{1}{\pi}\left(\frac{-1}{j\omega-k}-\frac{1}{-j\omega-k}\right) = \frac{2k}{\pi(k^2+\omega^2)}$$

$$e^{-k|x|} = \int_0^\infty \frac{2k\cos ax}{\pi(a^2+k^2)}da$$

از قسمت قبل نسبت به k مشتق میگیریم.

$$\frac{d\left(e^{-k|x|}\right)}{dk} = -|x|e^{-k|x|} = \frac{d}{dk}\left(\int_0^\infty \frac{2k\cos ax}{\pi(a^2+k^2)}da\right) = \int_0^\infty \frac{2\cos ax}{\pi(a^2+k^2)}da - \int_0^\infty \frac{4k^2\cos ax}{\pi(a^2+k^2)^2}da$$
$$= \int_0^\infty \frac{2(a^2-k^2)\cos ax}{\pi(a^2+k^2)^2}da$$

$$e^{-k|x|} = \int_0^\infty \frac{2k\cos ax}{\pi(a^2+k^2)}da$$

$$2|x|e^{-2|x|} - e^{-2|x|} = \int_0^\infty \frac{4(4-a^2)\cos ax}{\pi(a^2+4)^2}da - \int_0^\infty \frac{4\cos ax}{\pi(a^2+4)}da = -\int_0^\infty \frac{8a^2\cos ax}{\pi(a^2+4)^2}da$$

$$x \geq 0 \Longrightarrow \boxed{(2x-1)e^{-2x} = -\int_0^\infty \frac{8a^2\cos ax}{\pi(a^2+4)^2}da}$$

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۴)

$$f(x)=\begin{cases} a^2-x^2 & |x|<a \\ 0 & o.w \end{cases}$$

$$F(\omega)=\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx=\int_{-a}^{a}(a^2-x^2)e^{-j\omega x}dx=\frac{4}{\omega^3}\sin(a\omega)-\frac{4a}{\omega^2}\cos(a\omega)$$

$$F(\omega)=\frac{2a^2}{\omega}\sin(a\omega)-\frac{2a^2}{\omega}\sin(a\omega)+\frac{4}{\omega^3}\sin(a\omega)-\frac{4a}{\omega^2}\cos(a\omega)=\frac{4}{\omega^3}\sin(a\omega)-\frac{4a}{\omega^2}\cos(a\omega)$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega x}dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{4}{\omega^3}\sin(a\omega)-\frac{4a}{\omega^2}\cos(a\omega)\right)e^{j\omega x}d\omega$$

$$\text{even}\Longrightarrow f(x)=\frac{2}{2\pi}\int_{0}^{\infty}\left(\frac{4}{\omega^3}\sin(a\omega)-\frac{4a}{\omega^2}\cos(a\omega)\right)e^{j\omega x}d\omega$$

$$\frac{\pi}{4}f(x)=\int_{0}^{\infty}\left(\frac{1}{\omega^3}\sin(a\omega)-\frac{a}{\omega^2}\cos(a\omega)\right)e^{j\omega x}d\omega$$

$$a=3,\qquad x=0\Longrightarrow\frac{\pi}{4}f(0)=\int_{0}^{\infty}\left(\frac{1}{\omega^3}\sin(3\omega)-\frac{3}{\omega^2}\cos(3\omega)\right)d\omega$$

$$\Longrightarrow\boxed{\int_{0}^{\infty}\left(\frac{1}{\omega^3}\sin(3\omega)-\frac{3}{\omega^2}\cos(3\omega)\right)d\omega=\frac{9\pi}{4}}$$

۵)

$$\int_{0}^{\infty}\frac{\omega\sin(\omega x)}{\omega^2+k^2}=\frac{\pi}{2}e^{-kx}$$

میخواهیم ثابت کنیم:

$$\int_{0}^{\infty}\frac{\omega\sin(\omega x)}{\omega^4+64}=\frac{\pi}{16}e^{-2x}\sin(2x)$$

$$\sin(2x)\,e^{-2x}=\left(\frac{1}{2j}\left(e^{j2x}-e^{-j2x}\right)\right)e^{-2x}=\frac{1}{2j}e^{-x(2-2j)}-\frac{1}{2j}e^{-x(2+2j)}$$

اکنون در رابطه اول یکبار $k=2-2j$ و یکبار $k=2+2j$ را جایگذاری میکنیم. بنابراین داریم:

$$\int_{0}^{\infty}\frac{\omega\sin(\omega x)}{\omega^2+k^2}=\frac{\pi}{2}e^{-kx}\Longrightarrow\int_{0}^{\infty}\frac{\omega\sin(\omega x)}{\omega^2+(2-2j)^2}=\frac{\pi}{2}e^{-(2-2j)x}\Longrightarrow e^{-(2-2j)x}=\frac{2}{\pi}\int_{0}^{\infty}\frac{\omega\sin(\omega x)}{\omega^2-8j}$$

$$\int_{0}^{\infty}\frac{\omega\sin(\omega x)}{\omega^2+k^2}=\frac{\pi}{2}e^{-kx}\Longrightarrow\int_{0}^{\infty}\frac{\omega\sin(\omega x)}{\omega^2+(2+2j)^2}=\frac{\pi}{2}e^{-(2+2j)x}\Longrightarrow e^{-(2+2j)x}=\frac{2}{\pi}\int_{0}^{\infty}\frac{\omega\sin(\omega x)}{\omega^2+8j}$$

بنابراین:

$$\sin(2x)\,e^{-2x} = \frac{1}{2j}e^{-x(2-2j)} - \frac{1}{2j}e^{-x(2+2j)} = \frac{1}{j\pi}\left(\int_0^\infty \frac{\omega\sin(\omega x)}{\omega^2-8j} - \int_0^\infty \frac{\omega\sin(\omega x)}{\omega^2+8j}\right) = \frac{16}{\pi}\int_0^\infty \frac{\omega\sin(\omega x)}{\omega^4+64}$$

$$\Longrightarrow \boxed{\int_0^\infty \frac{\omega\sin(\omega x)}{\omega^4+64} = \frac{\pi}{16}e^{-2x}\sin(2x)}$$

۶)

$$\int_0^\infty f(\omega)\cos(\omega x)d\omega = \begin{cases} e^{-ax} & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

$$f(\omega) = \frac{2}{\pi}\int_0^1 e^{-ax}\cos(\omega x)\,dx = \frac{2}{\pi}\left[\frac{e^{-ax}(\omega\sin(\omega x) - \mathrm{a}\cos(\omega x))}{a^2+\omega^2}\right]_0^1$$

$$= \boxed{\frac{2}{\pi}\left[\frac{e^{-a}(\omega\sin(\omega) - \mathrm{a}\cos(\omega))}{a^2+\omega^2} + \frac{a}{a^2+\omega^2}\right]}$$

۷)

طبق صورت سوال انتگرال فوریه سینوسی تابع $f(x) = \frac{x}{x^2+4}$ برابر $e^{-2\omega}$ بنابراین:

$$B(\omega) = e^{-2\omega} = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx\,, \qquad \omega \ge 0$$

$$F(\omega) = \int_{-\infty}^\infty f(x)e^{-j\omega x}dx = \int_{-\infty}^\infty f(x)\big(\cos(\omega x) + j\sin(\omega x)\big)dx = 2j\int_0^\infty f(x)\sin(\omega x)\,dx = 2j\left(\frac{\pi}{2}\right)B(\omega)$$
$$= \pi j B(\omega), \qquad \omega \ge 0$$

تابع $f(x)$ فرد است بنابراین $F(\omega)$ نیز فرد است

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{i\omega x}d\omega$$
$$f(-x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{-i\omega x}d\omega = \frac{1}{2\pi}\int_\infty^{-\infty} F(-v)e^{ivx}d(-v) = \frac{1}{2\pi}\int_{-\infty}^\infty F(-\omega)e^{i\omega x}d\omega = -f(x) = -\frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{i\omega x}d\omega$$
$$\Longrightarrow F(\omega) = -F(-\omega)$$

با توجه به فرد بودن $F(\omega)$:

$$F(\omega) = \begin{cases} \pi j B(\omega), & \omega \ge 0 \\ -\pi j B(-\omega), & \omega < 0 \end{cases} = \pi j\,\mathrm{sign}(\omega)B(|\omega|)$$

حال با استفاده از رابطه پارسوال داریم:

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2d\omega \Longrightarrow 2\int_0^{\infty}f^2(x)dx=\frac{2}{2\pi}\int_0^{\infty}|F(\omega)|^2d\omega \Longrightarrow$$

$$\int_0^{\infty}\frac{x^2}{(x^2+4)^2}dx=\frac{1}{2\pi}\left(\int_0^{\infty}|\pi jB(\omega)|^2d\omega\right) \Longrightarrow \int_0^{\infty}\frac{x^2}{(x^2+4)^2}dx=\frac{\pi}{2}\int_0^{\infty}e^{-4\omega}d\omega=\frac{1}{4}\left(\frac{\pi}{2}\right)=\frac{\pi}{8}$$

$$\Longrightarrow \boxed{\int_0^{\infty}\frac{x^2}{(x^2+4)^2}dx=\frac{\pi}{8}}$$

۸)

$$A(\omega)=\frac{1}{\pi}\int_0^{\infty}\pi e^{-x}\cos(\omega x)\,dx=\left[\frac{e^{-x}}{1+\omega^2}(-\cos(\omega x)+\omega\sin(\omega x))\right]_0^{\infty}=\frac{1}{1+\omega^2}\rightarrow A(\omega)=\frac{1}{1+\omega^2}$$

$$B(\omega)=\frac{1}{\pi}\int_0^{\infty}\pi e^{-x}\sin(\omega x)\,dx=\left[\frac{e^{-x}}{1+\omega^2}(-\sin(\omega x)-\omega\cos(\omega x))\right]_0^{\infty}=\frac{\omega}{1+\omega^2}\rightarrow B(\omega)=\frac{\omega}{1+\omega^2}$$

حال طبق قضیه دیریکله میدانیم:

$$F(x_0)=\frac{f(x^+)+f(x^-)}{2}$$

بنابراین:

$$\boxed{\int_0^{\infty}\frac{\cos(\omega x)+\omega\sin(\omega x)}{1+\omega^2}d\omega=\begin{cases}0 & x<0\\ \frac{\pi}{2} & x=0\\ \pi e^{-x} & x>0\end{cases}}$$

موفق باشید.