۱)

۱-۱) معادله موج زیر را حل کنید.

$$u_{tt} = 4u_{xx}, \qquad 0 \le x \le 1, \qquad 0 \le t$$
$$u(0,t) = \sin(t), \qquad u(1,t) = \cos(t)$$
$$u(x,0) = \Pi(x), \qquad u_t(x,0) = \Pi(x) - 1$$

$$u(x,t) = v(x,t) + w(x,t)$$
$$\boxed{w(x,t) = \sin(t) + x(\cos(t) - \sin(t))}$$
$$\Longrightarrow v_{tt} = 4u_{xx} + \sin(t) + x(\cos(t) - \sin(t))$$
$$v(0,t) = 0, \qquad v(1,t) = 0$$
$$v(x,0) = \Pi(x) - x, \qquad v_t(x,0) = \Pi(x) - 2 + x$$
$$\text{BC: } \begin{cases} v(0,t) = 0 \\ v(1,t) = 0 \end{cases} \Longrightarrow v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$$
$$\Longrightarrow \sum_{n=1}^{\infty}\left(\ddot{T}_n(t) + 4n^2\pi^2 T_n(t)\right)\sin(n\pi x) = \sin(t) + x(\cos(t) - \sin(t))$$
$$\Longrightarrow \ddot{T}_n(t) + 4n^2\pi^2 T_n(t) = 2\sin(t)\int_0^1 \sin(n\pi x)\,dx + 2(\cos(t) - \sin(t))\int_0^1 x\sin(n\pi x)\,dx = \frac{2((-1)^{n+1}+1)}{n\pi}\sin(t) + \frac{2(-1)^{n+1}}{n\pi}(\cos(t) - \sin(t))$$
$$= \frac{2(-1)^{n+1}}{n\pi}\cos(t) + \frac{2}{n\pi}\sin(t)$$
$$\Longrightarrow T_n(t) = a_n\cos(2n\pi t) + b_n\sin(2n\pi t) + \frac{2(-1)^{n+1}}{n\pi(4n^2\pi^2 - 1)}\cos(t) + \frac{2}{n\pi(4n^2\pi^2 - 1)}\sin(t)$$
$$\Longrightarrow \boxed{v(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos(2n\pi t) + b_n\sin(2n\pi t) + \frac{2(-1)^{n+1}}{n\pi(4n^2\pi^2 - 1)}\cos(t) + \frac{2}{n\pi(4n^2\pi^2 - 1)}\sin(t)\right)\sin(n\pi x)}$$
$$v(x,0) = \sum_{n=1}^{\infty}\left(a_n + \frac{2(-1)^{n+1}}{n\pi(4n^2\pi^2 - 1)}\right)\sin(n\pi x) = \Pi(x) - x$$
$$\Longrightarrow a_n + \frac{2(-1)^{n+1}}{n\pi(4n^2\pi^2 - 1)} = 2\int_0^{\frac{1}{2}}\sin(n\pi x)\,dx - 2\int_0^1 x\sin(n\pi x)\,dx = \frac{2\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)}{n\pi} + \frac{2(-1)^n}{n\pi}$$
$$\Longrightarrow \boxed{a_n = \frac{2\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)}{n\pi} + \frac{2(-1)^n}{n\pi} - \frac{2(-1)^{n+1}}{n\pi(4n^2\pi^2 - 1)}}$$
$$v_t(x,0) = \sum_{n=1}^{\infty}\left(2b_n n\pi + \frac{2}{n\pi(4n^2\pi^2 - 1)}\right)\sin(n\pi x) = \Pi(x) - 2 + x$$
$$\Longrightarrow 2b_n n\pi + \frac{2}{n\pi(4n^2\pi^2 - 1)} = 2\int_0^{\frac{1}{2}}\sin(n\pi x)\,dx - 4\int_0^1 \sin(n\pi x)\,dx + 2\int_0^1 \sin(n\pi x)\,dx = \frac{2\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)}{n\pi} - \frac{4((-1)^{n+1}+1)}{n\pi} + \frac{2(-1)^{n+1}}{n\pi}$$
$$= \frac{-2}{n\pi} - \frac{2\cos\left(\frac{n\pi}{2}\right)}{n\pi} - \frac{2(-1)^{n+1}}{n\pi} = -\frac{2\left((-1)^{n+1} + 1 + \cos\left(\frac{n\pi}{2}\right)\right)}{n\pi}$$
$$\Longrightarrow \boxed{b_n = -\frac{(-1)^{n+1} + 1 + \cos\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - \frac{1}{n^2\pi^2(4n^2\pi^2 - 1)}}$$
$$\Longrightarrow \boxed{w(x,t) = \sin(t) + x(\cos(t) - \sin(t)) + \sum_{n=1}^{\infty}\left(a_n\cos(2n\pi t) + b_n\sin(2n\pi t) + \frac{2(-1)^{n+1}}{n\pi(4n^2\pi^2 - 1)}\cos(t) + \frac{2}{n\pi(4n^2\pi^2 - 1)}\sin(t)\right)\sin(n\pi x)}$$

۱-۲) معادله موج زیر را حل کنید.

$$u_{tt} = 4u_{xx}, \qquad 0 \le x \le 1, \qquad 0 \le t$$
$$u_x(0,t) = e^{-t}, \qquad u_x(1,t) = -e^{-t}$$
$$u(x,0) = \Pi(x), \qquad u_t(x,0) = \Pi(x) - 1$$

$$u(x,t) = v(x,t) + w(x,t)$$
$$w(x,t) = xe^{-t} - x^2e^{-t}$$
$$\Longrightarrow v_{tt} = 4u_{xx} - 8e^{-t} - xe^{-t} + x^2e^{-t}$$
$$v_x(0,t) = 0, \qquad v_x(1,t) = 0$$
$$v(x,0) = \Pi(x) - x + x^2, \qquad v_t(x,0) = \Pi(x) - 1 + x - x^2$$
$$\text{BC: } \begin{cases} v_x(0,t) = 0 \\ v_x(1,t) = 0 \end{cases} \Longrightarrow v(x,t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x)$$
$$\Longrightarrow \ddot{T}_0(t) + \sum_{n=1}^{\infty}\left(\ddot{T}_n(t) + 4n^2\pi^2T_n(t)\right)\cos(n\pi x) = -8e^{-t} - xe^{-t} + x^2e^{-t}$$
$$\Longrightarrow \ddot{T}_0(t) = -8e^{-t} \Longrightarrow T_0(t) = -8e^{-t} + a_0 + b_0t$$
$$\Longrightarrow \ddot{T}_n(t) + 4n^2\pi^2T_n(t) = -2e^{-t}\int_0^1 x\cos(n\pi x)\,dx + 2e^{-t}\int_0^1 x^2\cos(n\pi x)\,dx = \frac{2((-1)^{n+1}+1)}{n^2\pi^2}e^{-t} + \frac{4(-1)^n}{n^2\pi^2}e^{-t} = \frac{2((-1)^n+1)}{n^2\pi^2}e^{-t}$$
$$\Longrightarrow T_n(t) = a_n\cos(2n\pi t) + b_n\sin(2n\pi t) + \frac{2((-1)^n+1)}{n^2\pi^2(4n^2\pi^2+1)}e^{-t}$$
$$\Longrightarrow \boxed{v(x,t) = -8e^{-t} + a_0 + b_0t + \sum_{n=1}^{\infty}\left(a_n\cos(2n\pi t) + b_n\sin(2n\pi t) + \frac{2((-1)^n+1)}{n^2\pi^2(4n^2\pi^2+1)}e^{-t}\right)\cos(n\pi x)}$$
$$v(x,0) = -8 + a_0 + \sum_{n=1}^{\infty}\left(a_n + \frac{2((-1)^n+1)}{n^2\pi^2(4n^2\pi^2+1)}\right)\cos(n\pi x) = \Pi(x) - x + x^2$$
$$\Longrightarrow -8 + a_0 = \int_0^1 \Pi(x) - x + x^2\,dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} \Longrightarrow \boxed{a_0 = \frac{25}{3}}$$
$$\Longrightarrow a_n + \frac{2((-1)^n+1)}{n^2\pi^2(4n^2\pi^2+1)} = 2\int_0^{\frac{1}{2}}\cos(n\pi x)\,dx - 2\int_0^1 x\cos(n\pi x)\,dx + 2\int_0^1 x^2\cos(n\pi x)\,dx = \frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi} + \frac{2((-1)^n+1)}{n^2\pi^2}$$
$$\Longrightarrow \boxed{a_n = \frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi} + \frac{2((-1)^n+1)}{n^2\pi^2} - \frac{2((-1)^n+1)}{n^2\pi^2(4n^2\pi^2+1)}}$$
$$v_t(x,0) = 8 + b_0 + \sum_{n=1}^{\infty}\left(2b_nn\pi - \frac{2((-1)^n+1)}{n^2\pi^2(4n^2\pi^2+1)}\right)\cos(n\pi x) = \Pi(x) - 1 + x - x^2$$
$$\Longrightarrow 8 + b_0 = \int_0^1 \Pi(x) - 1 + x - x^2\,dx = \frac{1}{2} - 1 + \frac{1}{2} - \frac{1}{3} \Longrightarrow \boxed{b_0 = -\frac{25}{3}}$$
$$\Longrightarrow 2b_nn\pi - \frac{2((-1)^n+1)}{n^2\pi^2(4n^2\pi^2+1)} = 2\int_0^{\frac{1}{2}}\cos(n\pi x)\,dx + 2\int_0^1 x\cos(n\pi x)\,dx - 2\int_0^1 x^2\cos(n\pi x)\,dx = \frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi} - \frac{2((-1)^n+1)}{n^2\pi^2}$$
$$\Longrightarrow \boxed{b_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - \frac{((-1)^n+1)}{n^3\pi^3} + \frac{((-1)^n+1)}{n^3\pi^3(4n^2\pi^2+1)}}$$
$$\Longrightarrow \boxed{w(x,t) = -8e^{-t} + xe^{-t} - x^2e^{-t} + \frac{25}{3} - \frac{25}{3}t + \sum_{n=1}^{\infty}\left(a_n\cos(2n\pi t) + b_n\sin(2n\pi t) + \frac{2((-1)^n+1)}{n^2\pi^2(4n^2\pi^2+1)}e^{-t}\right)\cos(n\pi x)}$$

۳-۱) معادله موج زیر را حل کنید.

$$u_{tt} = u_{xx}, \quad 0 \le x \le 1, \quad 0 \le t$$
$$u_x(0,t) = e^{-t}, \quad u(1,t) = \sin(t)$$
$$u(x,0) = \Pi(x), \quad u_t(x,0) = \Pi(x) - 1$$

$$u(x,t) = v(x,t) + w(x,t)$$
$$w(x,t) = (x-1)e^{-t} + \sin(t)$$
$$\Longrightarrow v_{tt} = 4u_{xx} + (1-x)e^{-t} + \sin(t)$$
$$v_x(0,t) = 0, \quad v(1,t) = 0$$
$$v(x,0) = \Pi(x) + 1 - x, \quad v_t(x,0) = \Pi(x) - 3 + x$$

$$\text{BC: } \begin{cases} v_x(0,t) = 0 \\ v(1,t) = 0 \end{cases} \Longrightarrow v(x,t) = \sum_{n=1}^{\infty} T_n(t)\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

$$\Longrightarrow \sum_{n=1}^{\infty}\left(\ddot{T}_n(t) + \frac{(2n-1)^2\pi^2}{4}T_n(t)\right)\cos\left(\frac{(2n-1)\pi}{2}x\right) = (1-x)e^{-t} + \sin(t)$$

$$\Longrightarrow \ddot{T}_n(t) + \frac{(2n-1)^2\pi^2}{4}T_n(t) = 2e^{-t}\int_0^1 \cos\left(\frac{(2n-1)\pi}{2}x\right)dx - 2e^{-t}\int_0^1 x\cos\left(\frac{(2n-1)\pi}{2}x\right)dx + 2\sin(t)\int_0^1 \cos\left(\frac{(2n-1)\pi}{2}x\right)dx$$
$$= \frac{4(-1)^{n+1}}{(2n-1)\pi}\sin(t) + \frac{8}{(2n-1)^2\pi^2}e^{-t}$$

$$\Longrightarrow T_n(t) = a_n\cos\left(\frac{(2n-1)\pi}{2}t\right) + b_n\sin\left(\frac{(2n-1)\pi}{2}t\right) + \frac{4(-1)^{n+1}}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4}-1\right)}\sin(t) + \frac{8}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4}+1\right)}e^{-t}$$

$$\Longrightarrow \boxed{v(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{(2n-1)\pi}{2}t\right) + b_n\sin\left(\frac{(2n-1)\pi}{2}t\right) + \frac{4(-1)^{n+1}}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4}-1\right)}\sin(t) + \frac{8}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4}+1\right)}e^{-t}\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)}$$

$$v(x,0) = \sum_{n=1}^{\infty}\left(a_n + \frac{8}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4}+1\right)}\right)\cos\left(\frac{(2n-1)\pi}{2}x\right) = \Pi(x) + 1 - x$$

$$\Longrightarrow a_n + \frac{8}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4}+1\right)} = 2\int_0^{\frac{1}{2}}\cos\left(\frac{(2n-1)\pi}{2}x\right)dx + 2\int_0^1 \cos\left(\frac{(2n-1)\pi}{2}x\right)dx - 2\int_0^1 x\cos\left(\frac{(2n-1)\pi}{2}x\right)dx$$

$$= \frac{8}{(2n-1)^2\pi^2} + \frac{4\sin\left(\frac{(2n-1)\pi}{4}\right)}{(2n-1)\pi}$$

$$\Longrightarrow \boxed{a_n = \frac{8}{(2n-1)^2\pi^2} + \frac{4\sin\left(\frac{(2n-1)\pi}{4}\right)}{(2n-1)\pi} - \frac{8}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4}+1\right)}}$$

$$v_t(x,0) = \sum_{n=1}^{\infty}\left(b_n\frac{(2n-1)\pi}{2} + \frac{4(-1)^{n+1}}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4}-1\right)} - \frac{8}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4}+1\right)}\right)\cos\left(\frac{(2n-1)\pi}{2}x\right) = \Pi(x) - 2 + x$$

$$\Longrightarrow b_n\frac{(2n-1)\pi}{2} + \frac{4(-1)^{n+1}}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4}-1\right)} - \frac{8}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4}+1\right)} = 2\int_0^{\frac{1}{2}}\cos\left(\frac{(2n-1)\pi}{2}x\right)dx - 2\int_0^1 (3-x)\cos\left(\frac{(2n-1)\pi}{2}x\right)dx$$

$$= \frac{4\sin\left(\frac{(2n-1)\pi}{4}\right)}{(2n-1)\pi} + \frac{8(-1)^n}{(2n-1)\pi} - \frac{8}{(2n-1)^2\pi^2}$$

$$\Longrightarrow \boxed{b_n = \frac{8\sin\left(\frac{(2n-1)\pi}{4}\right)}{(2n-1)^2\pi^2} + \frac{16(-1)^n}{(2n-1)^2\pi^2} - \frac{16}{(2n-1)^3\pi^3} - \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4}-1\right)} + \frac{16}{(2n-1)^3\pi^3\left(\frac{(2n-1)^2\pi^2}{4}+1\right)}}$$

$$\Longrightarrow \boxed{w(x,t) = (x-1)e^{-t} + \sin(t) + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{(2n-1)\pi}{2}t\right) + b_n\sin\left(\frac{(2n-1)\pi}{2}t\right) + \frac{4(-1)^{n+1}}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4}-1\right)}\sin(t) + \frac{8}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4}+1\right)}e^{-t}\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)}$$

۴-۱) معادله موج زیر را حل کنید.

$$u_{tt} = u_{xx}, \qquad 0 \le x \le 1, \qquad 0 \le t$$
$$u(0,t) = \cos(t)\,, \qquad u_x(1,t) = -e^{-t}$$
$$u(x,0) = \Pi(x), \qquad u_t(x,0) = \Pi(x) - 1$$

$$u(x,t) = v(x,t) + w(x,t)$$
$$w(x,t) = \cos(t) - xe^{-t}$$
$$\Longrightarrow v_{tt} = 4u_{xx} + \cos(t) + xe^{-t}$$
$$v(0,t) = 0, \qquad v_x(1,t) = 0$$
$$v(x,0) = \Pi(x) + x - 1, \qquad v_t(x,0) = \Pi(x) - 1 - x$$

$$\text{BC: } \begin{cases} v(0,t) = 0 \\ v_x(1,t) = 0 \end{cases} \Longrightarrow v(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{(2n-1)\pi}{2}x\right)$$

$$\Longrightarrow \sum_{n=1}^{\infty} \left(\ddot{T}_n(t) + \frac{(2n-1)^2\pi^2}{4} T_n(t)\right) \sin\left(\frac{(2n-1)\pi}{2}x\right) = \cos(t) + xe^{-t}$$

$$\Longrightarrow \ddot{T}_n(t) + \frac{(2n-1)^2\pi^2}{4} T_n(t) = 2\cos(t) \int_0^1 \sin\left(\frac{(2n-1)\pi}{2}x\right) dx + 2e^{-t} \int_0^1 x \sin\left(\frac{(2n-1)\pi}{2}x\right) dx$$

$$= \frac{4}{(2n-1)\pi} \cos(t) + \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2} e^{-t}$$

$$\Longrightarrow T_n(t) = a_n \cos\left(\frac{(2n-1)\pi}{2}t\right) + b_n \sin\left(\frac{(2n-1)\pi}{2}t\right) + \frac{4}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4} - 1\right)} \cos(t) + \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4} + 1\right)} e^{-t}$$

$$\Longrightarrow \boxed{v(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{(2n-1)\pi}{2}t\right) + b_n \sin\left(\frac{(2n-1)\pi}{2}t\right) + \frac{4}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4} - 1\right)} \cos(t) + \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4} + 1\right)} e^{-t}\right) \sin\left(\frac{(2n-1)\pi}{2}x\right)}$$

$$v(x,0) = \sum_{n=1}^{\infty} \left(a_n + \frac{4}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4} - 1\right)} + \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4} + 1\right)}\right) \sin\left(\frac{(2n-1)\pi}{2}x\right) = \Pi(x) + x - 1$$

$$\Longrightarrow a_n + \frac{4}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4} - 1\right)} + \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4} + 1\right)} = 2\int_0^{\frac{1}{2}} \sin\left(\frac{(2n-1)\pi}{2}x\right) dx - 2\int_0^1 \sin\left(\frac{(2n-1)\pi}{2}x\right) dx + 2\int_0^1 x \sin\left(\frac{(2n-1)\pi}{2}x\right) dx$$

$$= \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2} - \frac{4}{(2n-1)\pi} + \frac{4\left(1 - \cos\left(\frac{(2n-1)\pi}{4}\right)\right)}{(2n-1)\pi}$$

$$\Longrightarrow \boxed{a_n = \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2} - \frac{4\cos\left(\frac{(2n-1)\pi}{4}\right)}{(2n-1)\pi} - \frac{4}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4} - 1\right)} - \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4} + 1\right)}}$$

$$v_t(x,0) = \sum_{n=1}^{\infty} \left(b_n \frac{(2n-1)\pi}{2} - \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4} + 1\right)}\right) \sin\left(\frac{(2n-1)\pi}{2}x\right) = \Pi(x) - 1 - x$$

$$\Longrightarrow b_n \frac{(2n-1)\pi}{2} - \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4} + 1\right)} = 2\int_0^{\frac{1}{2}} \sin\left(\frac{(2n-1)\pi}{2}x\right) dx - 2\int_0^1 \sin\left(\frac{(2n-1)\pi}{2}x\right) dx - 2\int_0^1 x \sin\left(\frac{(2n-1)\pi}{2}x\right) dx$$

$$= \frac{8(-1)^{n}}{(2n-1)^2\pi^2} - \frac{4}{(2n-1)\pi} + \frac{4\left(1 - \cos\left(\frac{(2n-1)\pi}{4}\right)\right)}{(2n-1)\pi}$$

$$\Longrightarrow \boxed{b_n = \frac{16(-1)^{n}}{(2n-1)^3\pi^3} - \frac{8\cos\left(\frac{(2n-1)\pi}{4}\right)}{(2n-1)^2\pi^2} + \frac{16(-1)^{n+1}}{(2n-1)^3\pi^3\left(\frac{(2n-1)^2\pi^2}{4} + 1\right)}}$$

$$\Longrightarrow \boxed{w(x,t) = \cos(t) - xe^{-t} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{(2n-1)\pi}{2}t\right) + b_n \sin\left(\frac{(2n-1)\pi}{2}t\right) + \frac{4}{(2n-1)\pi\left(\frac{(2n-1)^2\pi^2}{4} - 1\right)} \cos(t) + \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2\left(\frac{(2n-1)^2\pi^2}{4} + 1\right)} e^{-t}\right) \sin\left(\frac{(2n-1)\pi}{2}x\right)}$$

۲)

۲-۱) معادله حرارت زیر را حل کنید. (۵۰ نمره)

$u_t = u_{xx}, \qquad 0 \le x, \qquad 0 \le t$
$u(0,t) = e^{-4t}$
$u(x,0) = \Pi(x)$

$$u(x,t) = X(x)T(t) \Longrightarrow \frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\omega^2 \Longrightarrow \begin{cases} \dot{T}(t) + \omega^2 T(t) = 0 \\ X''(x) + \omega^2 X(x) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} T(t) = c(\omega)e^{-\omega^2 t} \\ X(x) = a(\omega)\cos(\omega x) + b(\omega)\sin(\omega x) \end{cases}$$

$$\Longrightarrow u(x,t) = \int_0^\infty (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))e^{-\omega^2 t} d\omega$$

$$u(0,t) = \int_0^\infty A(\omega)e^{-\omega^2 t} d\omega = e^{-4t} \Longrightarrow A(\omega) = \delta(\omega - 2)$$

$$\Longrightarrow u(x,t) = \cos(2x)\, e^{-4t} + \int_0^\infty B(\omega)\sin(\omega x)\, e^{-\omega^2 t} d\omega$$

$$u(x,0) = \cos(2x) + \int_0^\infty B(\omega)\sin(\omega x)\, d\omega = \Pi(x)$$

$$\Longrightarrow \int_0^\infty B(\omega)\sin(\omega x)\, d\omega = \Pi(x) - \cos(2x)$$

با توجه به انتگرال، تابع باید تابعی فرد باشد و از طرفی در بازه $0 \le x$ برابر با $\Pi(x) - \cos(2x)$ باشد. تابع $\mathrm{sgn}(x)(\Pi(x) - \cos(2x))$ در این شرایط صدق می‌کند.

$$\mathrm{sgn}(x)(\Pi(x) - \cos(2x)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}[\mathrm{sgn}(x)(\Pi(x) - \cos(2x))]e^{i\omega x} d\omega = \frac{i}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}[\mathrm{sgn}(x)(\Pi(x) - \cos(2x))]\sin(\omega x)\, d\omega$$

$$= \frac{i}{\pi}\int_0^\infty \mathcal{F}[\mathrm{sgn}(x)(\Pi(x) - \cos(2x))]\sin(\omega x)\, d\omega$$

$$\Longrightarrow B(\omega) = \frac{i}{\pi}\mathcal{F}[\mathrm{sgn}(x)(\Pi(x) - \cos(2x))] = \frac{i}{\pi}\int_{-\infty}^{+\infty} \mathrm{sgn}(x)\Pi(x)e^{-i\omega x}dx - \frac{i}{\pi}\mathcal{F}[\mathrm{sgn}(x)\cos(2x)] = \frac{2}{\pi}\int_0^{\frac{1}{2}} \sin(\omega x)\, dx - \frac{i}{\pi}\left(\frac{2}{i\omega} * \pi(\delta(\omega - 2) + \delta(\omega + 2))\right)$$

$$= \frac{4\sin^2\left(\frac{\omega}{4}\right)}{\omega\pi} + \frac{2\omega}{\pi(\omega^2 - 4)}$$

$$\Longrightarrow \boxed{u(x,t) = \cos(2x)\, e^{-4t} + \int_0^\infty \left(\frac{4\sin^2\left(\frac{\omega}{4}\right)}{\omega\pi} + \frac{2\omega}{\pi(\omega^2 - 4)}\right)\sin(\omega x)\, e^{-\omega^2 t} d\omega}$$

۲-۲) معادله حرارت زیر را حل کنید.

$$u_t = 4u_{xx}, \qquad 0 \le x, \qquad 0 \le t$$
$$u_x(0,t) = e^{-4t}$$
$$u(x,0) = \Pi(x)$$

$$u(x,t) = X(x)T(t) \Longrightarrow \frac{\dot{T}(t)}{4T(t)} = \frac{X''(x)}{X(x)} = -\omega^2 \Longrightarrow \begin{cases} \dot{T}(t) + 4\omega^2 T(t) = 0 \\ X''(x) + \omega^2 X(x) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} T(t) = c(\omega)e^{-4\omega^2 t} \\ X(x) = a(\omega)\cos(\omega x) + b(\omega)\sin(\omega x) \end{cases}$$

$$\Longrightarrow u(x,t) = \int_0^\infty (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))e^{-4\omega^2 t} d\omega$$

$$u_x(0,t) = \int_0^\infty B(\omega)\omega e^{-4\omega^2 t} d\omega = e^{-4t} \Longrightarrow B(\omega) = \delta(\omega - 1)$$

$$\Longrightarrow u(x,t) = \int_0^\infty A(\omega)\cos(\omega x)\, e^{-4\omega^2 t} d\omega + \sin(x)\, e^{-4t}$$

$$u(x,0) = \int_0^\infty A(\omega)\cos(\omega x)\, d\omega + \sin(x) = \Pi(x)$$

$$\Longrightarrow \int_0^\infty A(\omega)\cos(\omega x)\, d\omega = \Pi(x) - \sin(x)$$

با توجه به انتگرال، تابع باید تابعی زوج باشد و از طرفی در بازه $0 \le x$ برابر با $\Pi(x) - \sin(x)$ باشد. تابع $\Pi(x) - \operatorname{sgn}(x)\sin(x)$ در این شرایط صدق می‌کند.

$$\Pi(x) - \operatorname{sgn}(x)\sin(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}[\Pi(x) - \operatorname{sgn}(x)\sin(x)]e^{i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}[\Pi(x) - \operatorname{sgn}(x)\sin(x)]\cos(\omega x)\, d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \mathcal{F}[\Pi(x) - \operatorname{sgn}(x)\sin(x)]\cos(\omega x)\, d\omega$$

$$\Longrightarrow A(\omega) = \frac{1}{\pi}\mathcal{F}[\Pi(x) - \operatorname{sgn}(x)\sin(x)] = \frac{1}{\pi}\operatorname{sinc}\left(\frac{\omega}{2\pi}\right) - \frac{1}{\pi}\left(\frac{2}{i\omega} * \left(-i\pi\left(\delta(\omega - 1) - \delta(\omega + 1)\right)\right)\right)$$

$$= \frac{1}{\pi}\operatorname{sinc}\left(\frac{\omega}{2\pi}\right) + \frac{2}{\pi(\omega^2 - 1)}$$

$$\Longrightarrow \boxed{u(x,t) = \int_0^\infty \left(\frac{1}{\pi}\operatorname{sinc}\left(\frac{\omega}{2\pi}\right) + \frac{2}{\pi(\omega^2 - 1)}\right)\cos(\omega x)\, e^{-4\omega^2 t} d\omega + \sin(x)\, e^{-4t}}$$

۲-۳) معادله حرارت زیر را حل کنید.

$u_t = 4u_{xx}, \qquad 0 \le x, \qquad 0 \le t$
$u(0,t) = e^{-8t}$
$u(x,0) = \Lambda(x)$

$$u(x,t) = X(x)T(t) \Longrightarrow \frac{\dot{T}(t)}{4T(t)} = \frac{X''(x)}{X(x)} = -\omega^2 \Longrightarrow \begin{cases} \dot{T}(t) + 4\omega^2 T(t) = 0 \\ X''(x) + \omega^2 X(x) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} T(t) = c(\omega)e^{-4\omega^2 t} \\ X(x) = a(\omega)\cos(\omega x) + b(\omega)\sin(\omega x) \end{cases}$$

$$\Longrightarrow u(x,t) = \int_0^\infty (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))e^{-4\omega^2 t} d\omega$$

$$u(0,t) = \int_0^\infty A(\omega)e^{-4\omega^2 t} d\omega = e^{-8t} \Longrightarrow A(\omega) = \delta(\omega - \sqrt{2})$$

$$\Longrightarrow u(x,t) = \cos(\sqrt{2}x)\, e^{-8t} + \int_0^\infty B(\omega)\sin(\omega x)\, e^{-4\omega^2 t} d\omega$$

$$u(x,0) = \cos(\sqrt{2}x) + \int_0^\infty B(\omega)\sin(\omega x)\, d\omega = \Lambda(x)$$

$$\Longrightarrow \int_0^\infty B(\omega)\sin(\omega x)\, d\omega = \Lambda(x) - \cos(\sqrt{2}x)$$

با توجه به انتگرال، تابع باید تابعی فرد باشد و از طرفی در بازه $0 \le x$ برابر با $\Lambda(x) - \cos(\sqrt{2}x)$ باشد. تابع $\mathrm{sgn}(x)(\Lambda(x) - \cos(\sqrt{2}x))$ در این شرایط صدق می‌کند.

$$\mathrm{sgn}(x)(\Lambda(x) - \cos(\sqrt{2}x)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}[\mathrm{sgn}(x)(\Lambda(x) - \cos(\sqrt{2}x))]e^{i\omega x} d\omega = \frac{i}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}[\mathrm{sgn}(x)(\Lambda(x) - \cos(\sqrt{2}x))]\sin(\omega x)\, d\omega$$

$$= \frac{i}{\pi}\int_0^\infty \mathcal{F}[\mathrm{sgn}(x)(\Lambda(x) - \cos(\sqrt{2}x))]\sin(\omega x)\, d\omega$$

$$\Longrightarrow B(\omega) = \frac{i}{\pi}\mathcal{F}\left[\mathrm{sgn}(x)\left(\Lambda(x) - \cos\left(\sqrt{2}x\right)\right)\right] = \frac{i}{\pi}\int_{-\infty}^{+\infty} \mathrm{sgn}(x)\Lambda(x)e^{-i\omega x} dx - \frac{i}{\pi}\mathcal{F}\left[\mathrm{sgn}(x)\cos\left(\sqrt{2}x\right)\right]$$

$$= \frac{2}{\pi}\int_0^1 (1-x)\sin(\omega x)\, dx - \frac{i}{\pi}\left(\frac{2}{i\omega} * \pi\left(\delta(\omega - \sqrt{2}) + \delta(\omega + \sqrt{2})\right)\right)$$

$$= \frac{2(\omega - \sin(\omega))}{\pi\omega^2} + \frac{2\omega}{\pi(\omega^2 - 2)}$$

$$\Longrightarrow \boxed{u(x,t) = \cos\left(\sqrt{2}x\right)e^{-8t} + \int_0^\infty \left(\frac{2(\omega - \sin(\omega))}{\pi\omega^2} + \frac{2\omega}{\pi(\omega^2 - 2)}\right)\sin(\omega x)\, e^{-4\omega^2 t} d\omega}$$

۴-۲) معادله حرارت زیر را حل کنید.

$$u_t = u_{xx}, \qquad 0 \le x, \qquad 0 \le t$$
$$u_x(0,t) = e^{-4t}$$
$$u(x,0) = \Lambda(x)$$

$$u(x,t) = X(x)T(t) \Rightarrow \frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\omega^2 \Rightarrow \begin{cases} \dot{T}(t) + \omega^2 T(t) = 0 \\ X''(x) + \omega^2 X(x) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} T(t) = c(\omega) e^{-\omega^2 t} \\ X(x) = a(\omega)\cos(\omega x) + b(\omega)\sin(\omega x) \end{cases}$$

$$\Rightarrow u(x,t) = \int_0^\infty (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)) e^{-\omega^2 t} d\omega$$

$$u_x(0,t) = \int_0^\infty B(\omega)\omega e^{-\omega^2 t} d\omega = e^{-4t} \Rightarrow B(\omega) = \frac{1}{2}\delta(\omega - 2)$$

$$\Rightarrow u(x,t) = \int_0^\infty A(\omega)\cos(\omega x)\, e^{-\omega^2 t} d\omega + \frac{1}{2}\sin(2x)\, e^{-4t}$$

$$u(x,0) = \int_0^\infty A(\omega)\cos(\omega x)\, d\omega + \frac{1}{2}\sin(2x) = \Lambda(x)$$

$$\Rightarrow \int_0^\infty A(\omega)\cos(\omega x)\, d\omega = \Lambda(x) - \frac{1}{2}\sin(2x)$$

با توجه به انتگرال، تابع باید تابعی زوج باشد و از طرفی در بازه $x \ge 0$ برابر با $\Lambda(x) - \frac{1}{2}\sin(2x)$ باشد. تابع $\Lambda(x) - \frac{1}{2}\operatorname{sgn}(x)\sin(2x)$ در این شرایط صدق می‌کند.

$$\Lambda(x) - \frac{1}{2}\operatorname{sgn}(x)\sin(2x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}\left[\Lambda(x) - \frac{1}{2}\operatorname{sgn}(x)\sin(2x)\right] e^{i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}\left[\Lambda(x) - \frac{1}{2}\operatorname{sgn}(x)\sin(2x)\right]\cos(\omega x)\, d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \mathcal{F}\left[\Lambda(x) - \frac{1}{2}\operatorname{sgn}(x)\sin(2x)\right]\cos(\omega x)\, d\omega$$

$$\Rightarrow A(\omega) = \frac{1}{\pi}\mathcal{F}\left[\Lambda(x) - \frac{1}{2}\operatorname{sgn}(x)\sin(2x)\right] = \frac{1}{\pi}\operatorname{sinc}^2\left(\frac{\omega}{2\pi}\right) - \frac{1}{2\pi}\left(\frac{2}{i\omega} * \left(-i\pi(\delta(\omega-2) - \delta(\omega+2))\right)\right)$$

$$= \frac{1}{\pi}\operatorname{sinc}^2\left(\frac{\omega}{2\pi}\right) + \frac{2}{\pi(\omega^2 - 4)}$$

$$\Rightarrow \boxed{u(x,t) = \int_0^\infty \left(\frac{1}{\pi}\operatorname{sinc}^2\left(\frac{\omega}{2\pi}\right) + \frac{2}{\pi(\omega^2-4)}\right)\cos(\omega x)\, e^{-\omega^2 t} d\omega + \frac{1}{2}\sin(2x)\, e^{-4t}}$$