1.

a)

Since f(t) is even, $b_n$=0

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{4}{T}\int_0^3 f(t)\,dt = 1$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\frac{n\pi t}{3})\,dt = \frac{4}{6}\int_0^3 f(t)\cos\left(\frac{n\pi t}{3}\right)dt$$
$$= \frac{4}{6}(\int_0^1 \cos(\frac{n\pi t}{3})\,dt + \int_1^2 (2-t)\cos(\frac{n\pi t}{3})\,dt)$$
$$= \frac{2}{3}(\frac{3}{n\pi}\sin\left(\frac{n\pi}{3}\right) - \frac{3}{n\pi}\sin\left(\frac{n\pi}{3}\right) + \frac{9}{n^2\pi^2}(\cos\left(\frac{n\pi}{3}\right) - \cos\left(\frac{2n\pi}{3}\right)))$$

$$= \frac{12}{n^2\pi^2}\sin(\frac{n\pi}{2})\sin(\frac{n\pi}{6})$$

$$f(t) = 0.5 + \sum_{n=1}^{\infty}\frac{12}{n^2\pi^2}\sin(\frac{n\pi}{2})\sin(\frac{n\pi}{6})\cos(\frac{n\pi t}{3})$$

b)

even function, so: $b_n = 0$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}dx$$

So:

$$a_0 = 2\int_0^{0.5} f(x)dx = \frac{1}{4} \quad a_n = 4\int_0^{0.5} f(x)\cos 2n\pi\,dx = \frac{1}{2}\cos 2n\pi$$

$$a_0 = \frac{1}{4} \qquad a_n = \frac{1}{2}\cos 2n\pi$$

2.

$$a_0 = 0\,, a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\,(\pi - x)\,.\sin nx\,dx = \frac{8\sin^2(\frac{n\pi}{2})}{\pi n^3}$$

$$x(\pi - x) = \sum_{n=1}^{\infty} \frac{\sin(2n-1)\,x}{(2n-1)^3}$$

if we insert $x = \frac{\pi}{3}$, we get

$$\frac{\pi}{3}\left(\pi - \frac{\pi}{3}\right) = \sum_{n=1}^{\infty} \frac{\sin(2n-1)\frac{\pi}{3}}{(2n-1)^3}$$
$$= \frac{4\sqrt{3}}{\pi}\left(\left(\frac{1}{1^3} - \frac{1}{5^3}\right) + \left(\frac{1}{7^3} - \frac{1}{11^3}\right) + \left(\frac{1}{13^3} - \frac{1}{17^3}\right) + \cdots\right)$$

$$A = \frac{\pi^3}{18\sqrt{3}}$$

3.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cosh ax\,dx = \frac{1}{a\pi}\,(\sinh a\pi - \sinh(-a\pi)) = 2\frac{\sinh(a\pi)}{\pi a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cosh ax\,\cos nx\,dx = 2\mathrm{Re}(\int_0^{\pi} \cosh ax\;e^{inx}dx)$$
$$= \mathrm{Re}(\int_0^{\pi}(e^{(a+in)x} + e^{-(a-in)x})\,dx)$$
$$= \mathrm{Re}\,((-1)^n\,(\frac{e^{a\pi}}{a+in} - \frac{e^{-a\pi}}{a-in}))$$
$$= (-1)^n\,\frac{2a\sinh a\pi}{a^2+n^2}$$

Since cosh (ax) is even, $b_n = 0$

f(x)=

$$\frac{\sinh(a\pi)}{\pi a} + \sum_{n=1}^{\infty} (-1)^n \frac{2a \sinh a\pi}{a^2+n^2} \cos nx$$

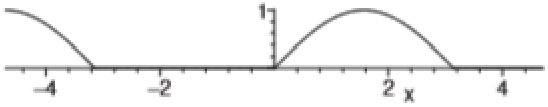
Finally, if we insert x=π, we get:

$$\sum_{n=1}^{\infty} (-1)^n \frac{2a \sinh a\pi}{a^2+n^2} (-1)^n = \cosh a\pi - \frac{\sinh(a\pi)}{\pi a}$$

$$A = \sum_{n=1}^{\infty} \frac{1}{a^2+n^2} = \frac{1}{2a \sinh a\pi} (\cosh a\pi - \frac{\sinh(a\pi)}{\pi a})$$

4.

f(t) :




$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} \sin x \; dx = \frac{2}{\pi},$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx \;\; dx = \frac{1}{2\pi}\int_0^{\pi} (\sin(n+1)x - \sin(n-1)x)\, dx$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \;\; dx = \frac{1}{2\pi}[\sin^2 x]_0^{\pi} \neq = 0,$$

$for\ n \neq 1$:

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx \;\; dx = \frac{1}{2\pi}\int_0^{\pi} (\sin(n+1)x - \sin(n-1)x)\, dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{n-1}\cos(n-1)\, x - \frac{1}{n+1}\cos(n+1)\, x\right]_0^{\pi} = \frac{-1}{\pi}\,\frac{1+(-1)^n}{n^2-1} \quad for\ n > 1$$

hence $a_{2n+1}$=0 for $n \geq 1$,

$a_{2n}=$

$\frac{-2}{\pi}\frac{1}{4n^2-1}\ ,\quad n\in N$

$b_n=\frac{1}{\pi}\int_0^\pi \sin x\sin nx\ dx=\frac{1}{2\pi}\int_0^\pi(\cos(n-1)x-\cos(n+1)x)\,dx$

$for\ \ n\neq 1\,,\qquad b_n=0$

$for\ \ n=1\,,\qquad b_1=\frac{1}{\pi}\int_0^\pi \sin x\sin x\ dx=\frac{1}{2\pi}\int_0^\pi(\cos^2 x+\sin^2 x)dx=0.5$

$f(x)=\frac{1}{\pi}+0.5\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx$

$$A=\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

For x= 0.5 π: $\qquad f(0.5\pi)=1=\frac{1}{\pi}+0.5+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}$

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}=\frac{\pi}{4}-0.5$

5.

$a_n=\frac{(-1)^n}{n}\ ,\ b_n=\frac{1}{n^2}$

$$I=\int_{-\pi}^{\pi}f(x)(\sin(3x)+\cos(3x))^2\cos(6x)\,dx$$

$=\ \int_{-\pi}^{\pi}f(x)\ (\cos 6x+0.5\sin 12x)dx=\ \pi a_6+0.5\,\pi\, b_{12}=\frac{\pi}{6}+\frac{\pi}{288}=\frac{49\pi}{288}$

6.

As mentioned in problem:

$$a_0 = \frac{sin\pi a}{\pi a} \quad a_n = \frac{2a\, sin\pi a}{\pi}\frac{(-1)^n}{a^2-n^2} \qquad b_n = 0$$

According to Parseval's theorem in Fourier series:

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(x)dx = a_0^2 + \sum_{n=1}^{\infty}\left(\frac{a_n^2+b_n^2}{2}\right)$$

So:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{2} \qquad \text{and} \qquad a_0^2 = (\frac{sin\pi a}{\pi a})^2$$

$${a_n}^2 = \frac{(2a\, sin\pi a)^2}{\pi^2(a^2-n^2)^2}$$

$$\frac{1}{2} = (\frac{sin\pi a}{\pi a})^2 + \sum_{n=1}^{\infty}\left(\frac{\frac{(2a\, sin\pi a)^2}{\pi^2(a^2-n^2)^2}}{2}\right) \qquad \text{so}$$

$$\sum_{n=1}^{\infty}\frac{1}{(n^2-a^2)^2} = (\frac{1}{2} - \left(\frac{sin\pi a}{\pi a}\right)^2)(\frac{\pi^2}{(2a\, sin\pi a)^2})$$

a = ½ ,then: $\sum_{n=1}^{\infty}\frac{1}{(n^2-1/4)^2} = \left(\frac{1}{2}-\left(\frac{4}{\pi^2}\right)\right)(\pi^2) = \frac{\pi^2}{2} - 4 = S$

$$S = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

7.

a)

$x$ Fourier series between $-\pi < x < \pi$:

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nx)}{n} \quad \text{so} \quad b_n = 2\frac{(-1)^{n+1}}{n} \quad a_0 = 0 \quad a_n = 0$$

$x^2$ Fourier series between $-\pi < x < \pi$:

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n^2} \quad \text{so} \quad a_n = \frac{4(-1)^n}{n^2} \quad a_0 = \frac{\pi^2}{3} \quad b_n = 0$$

b)

According to Parseval's theorem in Fourier series:

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(x)dx = a_0^2 + \sum_{n=1}^{\infty}\left(\frac{a_n^2 + b_n^2}{2}\right)$$

So:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x) - g^2(x)dx = \sum_{n=1}^{\infty}\left(\frac{2}{n^2} - \frac{8}{n^4}\right)$$

8.

Fourier series of r(t):

$a_0 = 0\,, \qquad b_n = 0\,, \qquad a_n = \dfrac{4}{n^2\pi}$

r(t)= $\sum_{n=1}^{\infty} \frac{4}{n^2\pi} \cos nt$

general form of the answer: $y(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos nt + B_n \sin nt$

$\sum_{n=1}^{\infty} (-A_n n^2) \cos nt + (-B_n n^2) \sin nt + \sum_{n=1}^{\infty} 24A_n \cos nt + 24B_n \sin nt$
$+24A_0 = \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \cos nt$

$24B_n - B_n n^2 = 0 \quad \rightarrow \quad B_n = 0 \;, A_0 = 0$

$24A_n - A_n n^2 = \dfrac{4}{n^2\pi} \quad \rightarrow \quad A_n = \dfrac{4}{n^2\pi(24-n^2)}$

$$y(t) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi(24-n^2)} \cos nt$$

9.

Since f(t) is odd, $a_n = 0$

$b_n = \frac{2}{\pi}\int_0^{\pi} t.\sin t\,.\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi} t.(\cos(n-1)\,t - \cos(n+1)\,t)dt$

for n=1:

$b_1 = \frac{1}{\pi}\int_0^{\pi} t(1-\cos 2t)\,dt = \left[\frac{1}{\pi}\frac{t^2}{2}\right]_0^{\pi} - \left[\frac{1}{2\pi}\,t.\sin 2t\right]_0^{\pi} + \frac{1}{2\pi}\int_0^{\pi}\sin 2t\,dt = \frac{\pi}{2}$

for n > 1:

$b_n = \frac{1}{\pi}\left[t\left(\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1}\right)\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}\left(\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1}\right)\,dt =$

$0 + \frac{1}{\pi}\left[t\left(\frac{\cos(n-1)t}{(n-1)^2} - \frac{\cos(n+1)t}{(n+1)^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right).((-1)^{n-1} - 1)$

$b_{2n+1}=0$ for $n \geq 1$

therefore:

$f(t)=\frac{\pi}{2}\sin t - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2\,(2n+1)^2}\sin 2nt$

$f'(t)=\begin{cases} \sin t + t\cos t, & for\ t\,\epsilon\,[0,\pi] \\ -\sin t - t\cos t, & for\ t\,\epsilon\,[-\pi,0] \end{cases}$

where $\lim_{n\to 0^+} f'(t) = \lim_{n\to 0^-} f'(t) = 0 \quad and \quad \lim_{n\to \pi^+} f'(t) = \lim_{n\to \pi^-} f'(t) = -\pi$

the continuation of f'(t) is continuous, hence we conclude that

$f'(t)=\frac{\pi}{2}\cos t - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)^2\,(2n+1)^2}\cos 2nt\,,$

and thus by a rearrangement,

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2\,(2n+1)^2} \cos 2nt = \frac{\pi^2}{64} \cos t - \frac{\pi}{32} f'(t) = \frac{\pi^2}{64} \cos t - \frac{\pi}{32} \sin t - \frac{\pi}{32} t \cos t$$

$for\ t \in [0, \pi]$

Finally, if we insert t=0, we get

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2\,(2n+1)^2} = \frac{\pi^2}{64}$$