1.

$$a)\ f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases} \qquad (fourier\ \boldsymbol{cosine}\ integral)$$

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos wv\, dv = \frac{2}{\pi}\int_0^{1} \cos wv\, dv = \frac{2\sin w}{\pi w}$$

$$f(x) = \int_0^{\infty} \frac{2\sin w}{\pi w}\cos wx\, dw$$

$$b)\ f(x) = \begin{cases} x^2 & 0 < x < 1 \\ 0 & x > 1 \end{cases} \qquad (fourier\ \boldsymbol{cosine}\ integral)$$

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos wv\, dv = \frac{2}{\pi}\int_0^{1} \mathrm{x}^2 \cos wv\, dv$$
$$= \frac{2((w^2-2)\sin w + 2w\cos w)}{\pi w^3}$$

$$f(x) = \int_0^{\infty} \frac{2((w^2-2)\sin w + 2w\cos w)}{\pi w^3}\cos wx\, dw$$

پشت رو چاپ شود. برگه اضافه تصحیح نخواهد شد.

$$c)\ f(x)=\frac{\pi}{2}e^{-x}\cos x\ ,\ x>0\quad (fourier\ \boldsymbol{sine}\ integral)$$

$$B(w)=\frac{2}{\pi}\int_0^{\infty} f(v)\sin wv\,dv=\int_0^{\infty}\mathrm{e}^{-\mathrm{v}}\cos \mathrm{v}\sin wv\,dv$$

$$=\int_0^{\infty}\mathrm{e}^{-\mathrm{v}}\ 0.5\,(\sin(w-1)v+\sin(w+1)\,v)dv$$

$$=0.5\,Laplace\{\sin(w-1)v+\sin(w+1)\}_{s=1}$$

$$=0.5(\frac{w-1}{1+(w-1)^2}+\frac{w+1}{1+(w+1)^2}=\frac{w^3}{w^4+4}$$

$$f(x)=\int_0^{\infty}\frac{w^3}{w^4+4}\sin wx\,dw$$

2.

$$I)\ \ f(ax)=\frac{1}{a}\int_0^{\infty}A\left(\frac{w}{a}\right)\cos wx\,dw\quad (a>0)$$

$$wa=p\rightarrow\quad f(ax)=\int_0^{\infty}A(w)\cos wax\,dw=\int_0^{\infty}A\left(\frac{p}{a}\right)\cos px\ \frac{dp}{a}$$

$$replacing\ p\ with\ w\rightarrow\ f(ax)=\frac{1}{a}\int_0^{\infty}A\left(\frac{w}{a}\right)\cos wx\,dw$$

$$II)\ \ xf(x) = \int_0^\infty -\frac{dA}{dw}\sin wx\, dw$$

$$replacing\ f(v)\ with\ vf(v) \rightarrow\ B^*(w) = \frac{2}{\pi}\int_0^\infty vf(v)\sin wv\, dv = -\frac{dA}{dw}\ \rightarrow$$

$$xf(x) = \int_0^\infty -\frac{dA}{dw}\sin wx\, dw$$

$$III)\ \ x^2 f(x) = \int_0^\infty -\frac{d^2A}{dw^2}\cos wx\, dw$$

$$differentiating\ \ A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv\ \ twice\ with\ respect\ to\ w:$$

$$\frac{d^2A(w)}{dw^2} = -\frac{2}{\pi}\int_0^\infty v^2 f(v)\cos wv\, dv\ \rightarrow\ x^2 f(x) = \int_0^\infty -\frac{d^2A}{dw^2}\cos wx\, dw$$

$$1a\ \rightarrow\ \ A(w) = \frac{2\sin w}{\pi w} \rightarrow\quad A'' = \frac{2}{\pi}\left(\frac{2}{w^3}\sin w - \frac{2}{w^2}\cos w - \frac{1}{w}\sin w\right)$$

$$III \rightarrow\ \ x^2.f(x)$$
$$= \int_0^\infty -\frac{2}{\pi}\left(\frac{2}{w^3}\sin w - \frac{2}{w^2}\cos w - \frac{1}{w}\sin w\right)\cos wx\, dw$$
$$= \int_0^\infty \frac{2((w^2-2)\sin w + 2w\cos w)}{\pi w^3}\cos wx\, dw$$

3.

$$f(x)=\int_{0}^{\infty} e^{-w}\cos wx\,dw=\frac{1}{1+x^2}\quad ,\ x=>w$$

$$\rightarrow\quad f(w)=\int_{0}^{\infty} e^{-x}\cos wx\,dx$$

$$g(x)=\int_{0}^{\infty}\arctan w\sin wx\,dw=\int_{0}^{\infty}\mathrm{B}(w)\sin wx\,dw$$

$$g(x)\ is\ odd\rightarrow\quad g(x)=\begin{cases}k(x) & x>0\\ -k(-x) & x<0\end{cases}$$

$$B(w)=\frac{1}{\pi}\int_{-\infty}^{\infty}g(x)\sin wx\,dx=\frac{2}{\pi}\int_{0}^{\infty}k(x)\sin wx\,dx=\arctan w$$

$$differentiating\ \ \frac{2}{\pi}\int_{0}^{\infty}k(x)\sin wx\,dx=\arctan w\ \ with\ respect\ to\ w$$

$$\frac{2}{\pi}\int_{0}^{\infty}x\,k(x)\cos wx\,dx=\frac{1}{1+w^2}=\int_{0}^{\infty}e^{-x}\cos wx\,dx$$

$$x\,k(x)=\frac{\pi}{2}\frac{e^{-x}}{x}\quad\rightarrow\qquad g(x)=\begin{cases}\frac{\pi}{2}\frac{e^{-x}}{x} & x>0\\ \frac{\pi}{2}\frac{e^{x}}{x} & x<0\end{cases}$$

4.

$$\int_0^\infty x\,f(x)\cos ax\,dx + 2\int_0^\infty f(x)\sin\,ax\,dx = \int_0^\infty \frac{1}{x} f(x)\cos ax\,dx$$

$$y = \int_0^\infty f(x)\sin\,ax\,dx \quad \to \quad y' + 2y = -\int y\,da \quad =>$$

$$y_1 = c.e^{-a} \quad ,y_2 = d.ae^{-a}$$

$y_1 = c.e^{-a}$:

$$ce^{-a} = \int_0^\infty f(x)\sin\,ax\,dx \;\to\; ce^{-a} = \int_0^\infty f(w)\sin\,aw\,dw = g(a)$$

$$f1(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(a)\sin\,aw\,da \;= \frac{2}{\pi}\int_0^\infty ce^{-a}\sin\,aw\,da = \frac{2c}{\pi}\frac{w}{1+w^2}$$

$$f1(w) = \frac{2c}{\pi}\frac{w}{1+w^2} \quad \to \quad f1(x) = \frac{2c}{\pi}\frac{x}{1+x^2}$$

$y_2 = d.ae^{-a}$:

$$dae^{-a} = \int_0^\infty f(x)\sin\,ax\,dx \;\to\; dae^{-a} = \int_0^\infty f(w)\sin\,aw\,dw = g(a)$$

$$f2(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(a)\sin\,aw\,da \;= \frac{2}{\pi}\int_0^\infty ce^{-a}\sin\,aw\,da = \frac{4d}{\pi}\frac{w}{(1+w^2)^2}$$

$$f2(w) = \frac{4d}{\pi}\frac{w}{(1+w^2)^2} \quad \to \quad f2(x) = \frac{4d}{\pi}\frac{x}{(1+x^2)^2}$$

$$f(x) = f1(x) + f2(x) = \frac{2c}{\pi}\frac{x}{1+x^2} + \frac{4d}{\pi}\frac{x}{(1+x^2)^2}\,,\quad f(2) = 0.8 \quad ,$$

$,f(1) = 1$

=> d=0, c=π

$$f(x) = \frac{2x}{1+x^2}$$

پشت رو چاپ شود. برگه اضافه تصحیح نخواهد شد.

5.

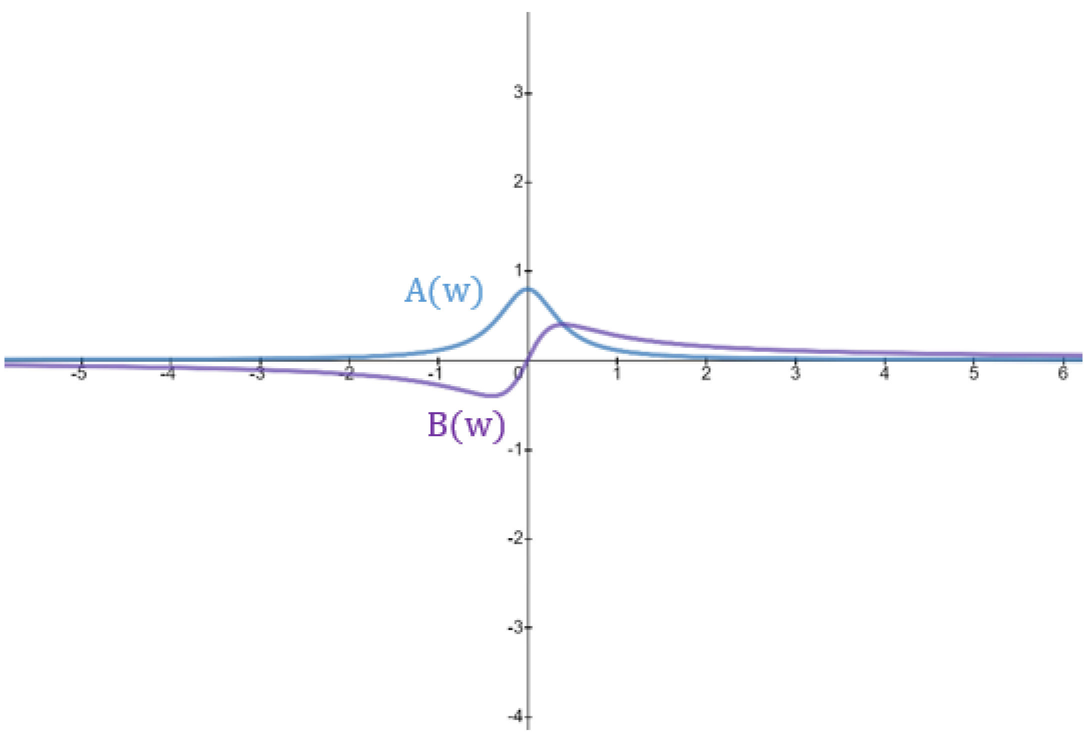
I)

$$f(x)=\begin{cases}0 & x<0\\ e^{-ax} & x>0\end{cases}$$

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx=\frac{1}{\pi}\int_{0}^{\infty} e^{-ax}\cos(\omega x)\,dx=\frac{1}{\pi}\frac{a}{\omega^2+a^2}$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx=\frac{1}{\pi}\int_{0}^{\infty} e^{-ax}\sin(\omega x)\,dx=\frac{1}{\pi}\frac{\omega}{\omega^2+a^2}$$

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{\omega^2+a^2}\cos(\omega x)+\frac{\omega}{\omega^2+a^2}\sin(\omega x)\,d\omega$$



مدرس: دکتر مهدی طالع ماسوله- حل تمرین: وصال بخت آزاد- آرمان اکبری- نگین سفاری
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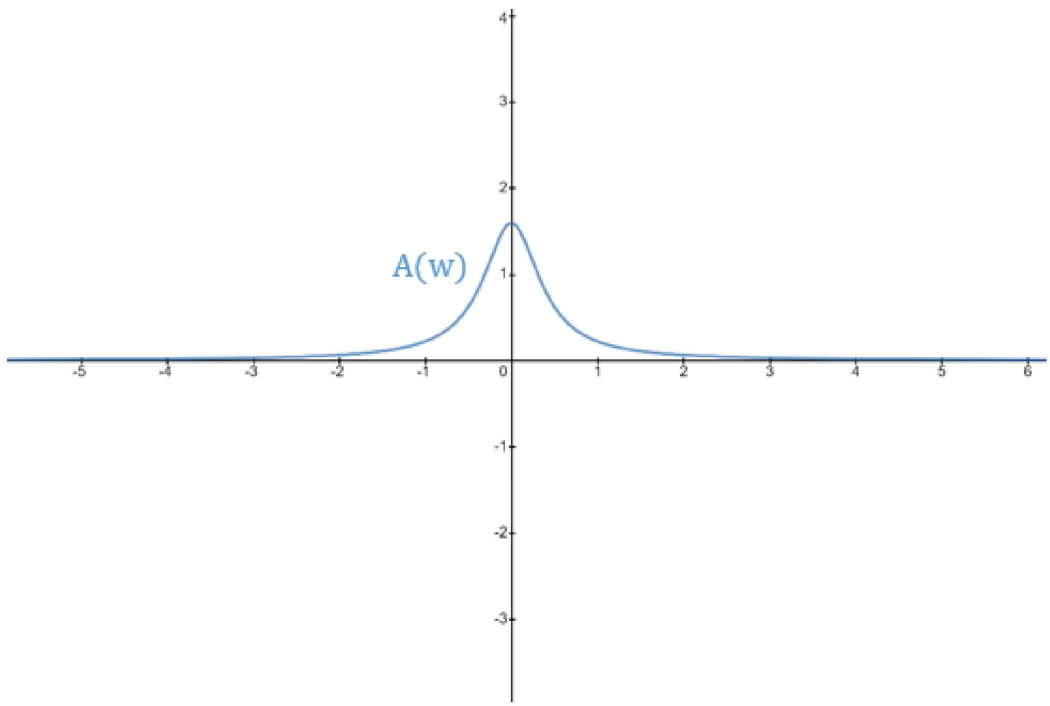
II)

$$g(x) = \begin{cases} e^{ax} & x < 0 \\ e^{-ax} & x > 0 \end{cases}$$

Function is even → $B(\omega) = 0$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx = \frac{2}{\pi}\int_{0}^{\infty} e^{-ax}\cos(\omega x)\,dx = \frac{2}{\pi}\frac{a}{\omega^2 + a^2}$$

$$g(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{a}{\omega^2 + a^2}\cos(\omega x)\,d\omega$$



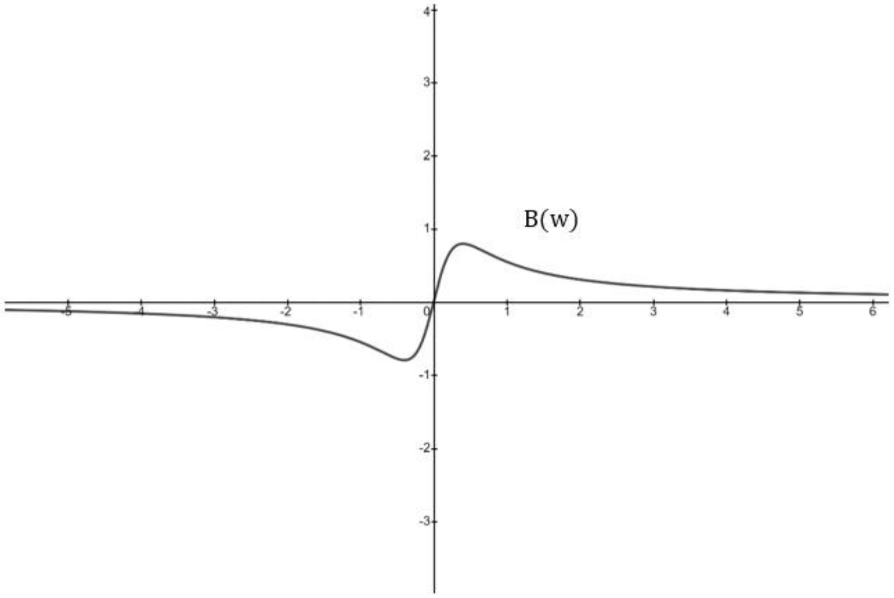
III)

$$h(x) = \begin{cases} -e^{ax} & x < 0 \\ e^{-ax} & x > 0 \end{cases}$$

Function is even → $A(\omega) = 0$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx = \frac{2}{\pi}\int_{0}^{\infty} e^{-ax}\sin(\omega x)\,dx = \frac{2}{\pi}\frac{\omega}{\omega^2 + a^2}$$

$$h(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\omega}{\omega^2 + a^2}\sin(\omega x)\,d\omega$$



با توجه به شکل نمودار ها، میبینم که A(w) ها در کل زودتر میرا می‌شوند و به صفر میل می‌کنند، بنابراین از یک جمله مشخص به بعد می توانیم صفر لحاظش کنیم.
بین سه تابعی که داریم، تابع دوم، $g(x)$ ، که فقط A(w) دارد، ساده تر محاسبه می‌شود.

پشت رو چاپ شود. برگه اضافه تصحیح نخواهد شد.

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6.

$$A(w) = \int_{-\infty}^{\infty} f(t) \cos wt \, dt = \int_{-1}^{1} (1-t) \cos wt \, dt = \frac{2}{w} \sin w$$

$$B(w) = \int_{-\infty}^{\infty} f(t) \sin wt \, dt = \int_{-1}^{1} (1-t) \sin wt \, dt = \frac{2}{w} \cos w - \frac{2}{w^2} \sin w$$

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} [\frac{2}{w} \sin w \ \cos wx + (\frac{2}{w} \cos w - \frac{2}{w^2} \sin w) \sin wx] \, dw$$

at x=0, we have

$$\frac{f(0^+)+f(0^-)}{2} = 1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin w}{w} \, dw \ \rightarrow \ \frac{\pi}{2} = \int_0^{\infty} \frac{\sin w}{w} \, dw$$

7.

$$A(w) = \int_{0}^{\infty} f(x) \cos wx \, dx$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{\pi}{2} \cos x \cos wx \, dx$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{\pi}{4} (\cos(w+1)x + \cos(-1+w)x \, dx = \frac{\pi}{2(1-w^2)} (\cos \frac{w\pi}{2})$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\pi}{2(1-w^2)} \cos(\frac{w\pi}{2}) \cos wx \, dw \ , \ \ w => a$$

Therefore:

$$\int_{0}^{\infty} \frac{\cos(\frac{a\pi}{2})\cos(ax)}{1-a^2} da = \begin{cases} \frac{\pi}{2} \cos(x), & |x| \leq \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$$