تمرین ۷: معادله لاپلاس در مختصات کارتزین

مدرس: دکتر مهدی طالع ماسوله- حل تمرین: نیما هاشمی- حسین عطرسائی- نیکا امامی

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۱) معادله زیر را حل کنید.

$u_{xx} + u_{yy} = x + 2y, \qquad 0 \le x \le \pi, \qquad 0 \le y \le \pi$

$u(x,0) = x, \qquad u(x,\pi) = 2$

$u(0,y) = y, \qquad u(\pi,y) = \cos(y)$

$$u(x,y) = w(x,y) + v(x,y)$$

$$\text{BC: } \begin{cases} u(x,0) = x \\ u(x,\pi) = 2 \end{cases} \Rightarrow w(x,y) = x + \frac{y}{\pi}(2 - x)$$

$$\Rightarrow v_{xx} + v_{yy} = x + 2y$$

$$v(x,0) = 0, \qquad v(x,\pi) = 0$$

$$v(0,y) = y\left(1 - \frac{2}{\pi}\right), \qquad v(\pi,y) = \cos(y) - \pi + \frac{y}{\pi}(\pi - 2)$$

$$\text{BC: } \begin{cases} v(x,0) = 0 \\ v(x,\pi) = 0 \end{cases} \Rightarrow v(x,y) = \sum_{n=1}^{\infty} X(x)\sin(ny)$$

$$\sum_{n=1}^{\infty} \left(\ddot{X}(x) - n^2 X(x)\right)\sin(ny) = x + 2y$$

$$\Rightarrow \ddot{X}(x) - n^2 X(x) = \frac{2}{\pi}\int_0^{\pi} (x + 2y)\sin(ny)\,dy = \frac{4(-1)^{n+1}}{n} + x\frac{2(1 - (-1)^n)}{n\pi}$$

$$\Rightarrow X(x) = a_n\cosh(nx) + b_n\sinh(nx) - \frac{4(-1)^{n+1}}{n^3} - x\frac{2(1 - (-1)^n)}{n^3\pi}$$

$$\Rightarrow v(x,y) = \sum_{n=1}^{\infty}\left(a_n\cosh(nx) + b_n\sinh(nx) - \frac{4(-1)^{n+1}}{n^3} - x\frac{2(1 - (-1)^n)}{n^3\pi}\right)\sin(ny)$$

$$v(0,y) = \sum_{n=1}^{\infty}\left(a_n - \frac{4(-1)^{n+1}}{n^3}\right)\sin(ny) = y\left(1 - \frac{2}{\pi}\right)$$

$$\Rightarrow a_n - \frac{4(-1)^{n+1}}{n^3} = \frac{2}{\pi}\int_0^{\pi} y\left(1 - \frac{2}{\pi}\right)\sin(ny)\,dy = \frac{2(-1)^{n+1}(\pi - 2)}{n\pi} \Rightarrow a_n = \frac{4(-1)^{n+1}}{n^3} + \frac{2(-1)^{n+1}(\pi - 2)}{n\pi}$$

$$v(\pi,y) = \sum_{n=1}^{\infty}\left(a_n\cosh(n\pi) + b_n\sinh(n\pi) - \frac{4(-1)^{n+1}}{n^3} - \frac{2(1 - (-1)^n)}{n^3}\right)\sin(ny) = \cos(y) - \pi + \frac{y}{\pi}(\pi - 2)$$

$$\Rightarrow a_n\cosh(n\pi) + b_n\sinh(n\pi) - \frac{4(-1)^{n+1}}{n^3} - \frac{2(1 - (-1)^n)}{n^3} = \frac{2}{\pi}\int_0^{\pi}\left(\cos(y) - \pi + \frac{y}{\pi}(\pi - 2)\right)\sin(ny)\,dy$$

$$= \frac{2(\pi - 2(-1)^n)}{n(n^2 - 1)\pi} + \frac{2n(1 - \pi + 3(-1)^n)}{(n^2 - 1)\pi}$$

$$\Rightarrow b_n = \left(\frac{2(\pi - 2(-1)^n)}{n(n^2 - 1)\sinh(n\pi)\,\pi} + \frac{2n(1 - \pi + 3(-1)^n)}{(n^2 - 1)\sinh(n\pi)\,\pi}\right)(1 - \delta[n - 1]) - a_n\coth(n\pi)$$

$$\Rightarrow \boxed{u(x,y) = x + 2y + \sum_{n=1}^{\infty}\left(\left(\frac{4(-1)^{n+1}}{n^3} + \frac{2(-1)^{n+1}(\pi - 2)}{n\pi}\right)\cosh(nx) + \left(\left(\frac{2(\pi - 2(-1)^n)}{n(n^2 - 1)\sinh(n\pi)\,\pi} + \frac{2n(1 - \pi + 3(-1)^n)}{(n^2 - 1)\sinh(n\pi)\,\pi}\right)(1 - \delta[n - 1]) - \left(\frac{4(-1)^{n+1}}{n^3} + \frac{2(-1)^{n+1}(\pi - 2)}{n\pi}\right)\coth(nx)\right)\right)\sin(ny)}$$

۲) معادله زیر را حل کنید.

$$u_{xx}+u_{yy}=0,\qquad 0\le x,\qquad 0\le y$$
$$u_x(0,y)=0$$
$$u(x,0)=\Pi\left(\frac{x}{2}\right)$$

$$u(x,y)=X(x)Y(y)\Rightarrow \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-k^2\Rightarrow \begin{cases}X''(x)+k^2X(x)=0\\ Y''(y)-k^2Y(y)=0\end{cases}\Rightarrow \begin{cases}X(x)=a\cos(kx)+b\sin(kx)\\ Y(y)=ce^{-ky}+\cancel{be^{ky}}\end{cases}$$

$$\Rightarrow u(x,y)=\int_0^\infty (A(\omega)\cos(\omega x)+B(\omega)\sin(\omega x))e^{-\omega y}d\omega$$

$$u_x(0,y)=\int_0^\infty B(\omega)\omega e^{-\omega y}d\omega=0\ (\mathrm{I})\Rightarrow B(\omega)=0$$

$$u(x,0)=\int_0^\infty A(\omega)\cos(\omega x)\,d\omega=\Pi\left(\frac{x}{2}\right)\Rightarrow A(\omega)=\frac{1}{\pi}\mathcal{F}\left[\Pi\left(\frac{x}{2}\right)\right]=\frac{2}{\pi}\operatorname{sinc}\left(\frac{\omega}{\pi}\right)$$

$$\Rightarrow u(x,y)=\int_0^\infty \frac{2}{\pi}\operatorname{sinc}\left(\frac{\omega}{\pi}\right)\cos(\omega x)\,e^{-\omega y}d\omega$$

نمره مثبت:

معادله (I) جواب‌های دیگری دارد که باید در نظر گرفته شود.

$$u_x(0,y)=\int_0^\infty B(\omega)\omega e^{-\omega y}d\omega=0\Rightarrow \int_0^\infty B(\omega)(-\omega)e^{-\omega y}d\omega=0\Rightarrow B(\omega)=L_n^{(-1)}(\omega)=\sum_{k=0}^{n}(-1)^k\binom{n-1}{n-k}\frac{\omega^k}{k!},\qquad n\in\{0,2,3,4,\dots\}$$

$$\Rightarrow u(x,y)=\int_0^\infty \frac{2}{\pi}\operatorname{sinc}\left(\frac{\omega}{\pi}\right)\cos(\omega x)\,e^{-\omega y}d\omega+\int_0^\infty\left(\frac{2}{\pi}\operatorname{sinc}\left(\frac{\omega}{\pi}\right)\cos(\omega x)+\sum_{\substack{n=0\\ n\neq 1}}^{\infty}L_n^{(-1)}(\omega)\sin(\omega x)\right)e^{-\omega y}d\omega$$

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۳) معادله زیر را حل کنید.

$u_{xx}+u_{yy}=0, \quad 0\le x\le a, \quad 0\le x\le b$
$u(x,0)-u_y(x,0)=0, \quad u(x,b)=x$
$u(0,y)=0, \quad u(a,y)=0$

$$\text{BC: } \begin{cases} u(0,y)=0 \\ u(a,y)=0 \end{cases} \Rightarrow u(x,y)=\sum_{n=1}^{\infty} Y_n(y)\sin\left(\frac{n\pi}{a}x\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(Y_n''(y)-\frac{n^2\pi^2}{a^2}Y_n(y)\right)\sin\left(\frac{n\pi}{a}x\right)=0 \Rightarrow Y_n''(y)-\frac{n^2\pi^2}{a^2}Y_n(y)=0 \Rightarrow Y_n(y)=a_n\cosh\left(\frac{n\pi}{a}y\right)+b_n\sinh\left(\frac{n\pi}{a}y\right)$$

$$\Rightarrow u(x,y)=\sum_{n=1}^{\infty}\left(a_n\cosh\left(\frac{n\pi}{a}y\right)+b_n\sinh\left(\frac{n\pi}{a}y\right)\right)\sin\left(\frac{n\pi}{a}x\right)$$

$$u(x,0)=u_y(x,0)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi}{a}x\right)=\sum_{n=1}^{\infty}\frac{n\pi}{a}b_n\sin\left(\frac{n\pi}{a}x\right)\Rightarrow a_n=\frac{n\pi}{a}b_n$$

$$u(x,b)=\sum_{n=1}^{\infty}\left(a_n\cosh\left(\frac{n\pi b}{a}\right)+b_n\sinh\left(\frac{n\pi b}{a}\right)\right)\sin\left(\frac{n\pi}{a}x\right)=x\Rightarrow a_n\cosh\left(\frac{n\pi b}{a}\right)+b_n\sinh\left(\frac{n\pi b}{a}\right)=\frac{2}{a}\int_0^a x\sin\left(\frac{n\pi}{a}x\right)dx=\frac{2a(-1)^{n+1}}{n\pi}$$

$$\Rightarrow\begin{cases} a_n=\frac{n\pi}{a}b_n \\ a_n\cosh\left(\frac{n\pi b}{a}\right)+b_n\sinh\left(\frac{n\pi b}{a}\right)=\frac{2a(-1)^{n+1}}{n\pi}\end{cases}\Rightarrow\begin{cases} a_n=\dfrac{2(-1)^{n+1}}{\frac{n\pi}{a}\cosh\left(\frac{n\pi b}{a}\right)+\sinh\left(\frac{n\pi b}{a}\right)} \\ b_n=\dfrac{2\frac{a}{n\pi}(-1)^{n+1}}{\frac{n\pi}{a}\cosh\left(\frac{n\pi b}{a}\right)+\sinh\left(\frac{n\pi b}{a}\right)}\end{cases}$$

$$\Rightarrow u(x,y)=\sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{\frac{n\pi}{a}\cosh\left(\frac{n\pi b}{a}\right)+\sinh\left(\frac{n\pi b}{a}\right)}\cosh\left(\frac{n\pi}{a}y\right)+\frac{2\frac{a}{n\pi}(-1)^{n+1}}{\frac{n\pi}{a}\cosh\left(\frac{n\pi b}{a}\right)+\sinh\left(\frac{n\pi b}{a}\right)}\sinh\left(\frac{n\pi}{a}y\right)\right)\sin\left(\frac{n\pi}{a}x\right)$$

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۴) معادله زیر را حل کنید

$$u_{xx}+u_{yy}=2-4x,\qquad 0\le x\le \pi,\qquad 0\le y\le 1$$
$$u(x,0)=x,\qquad u(x,1)=2x+1$$
$$u_x(0,y)=-y,\qquad u_x(\pi,y)=\pi+y$$

$$u(x,y)=w(x,y)+v(x,y)$$
$$w(x,y)=-xy+\frac{x^2}{2\pi}(\pi+2y)$$
$$v_{xx}+v_{yy}=1-4x-\frac{2}{\pi}y$$
$$v(x,0)=x-\frac{x^2}{2},\qquad v(x,1)=3x+1-\frac{(\pi+2)}{2\pi}x^2$$
$$v_x(0,y)=0,\qquad v_x(\pi,y)=0$$
$$\text{BC: }\begin{cases}v_x(0,y)=0\\ v_x(\pi,y)=0\end{cases}\Rightarrow v(x,t)=Y_0(y)+\sum_{n=1}^{\infty}Y_n(y)\cos(nx)$$
$$Y_0''(y)+\sum_{n=1}^{\infty}\left(Y_n''(y)-n^2Y_n(y)\right)\cos(nx)=1-4x-\frac{2}{\pi}y$$
$$\Rightarrow Y_0''(y)=\frac{1}{\pi}\int_0^{\pi}\left(1-4x-\frac{2}{\pi}y\right)dx=1-2\pi-\frac{2}{\pi}y\Rightarrow Y_0(y)=-\frac{1}{3\pi}y^3+\left(\frac{1}{2}-\pi\right)y^2+b_0y+a_0$$
$$\Rightarrow Y_n''(y)-n^2Y_n(y)=\frac{2}{\pi}\int_0^{\pi}\left(1-4x-\frac{2}{\pi}y\right)\cos(nx)\,dx=\frac{2}{\pi}\left(1-\frac{2}{\pi}y\right)\int_0^{\pi}\cos(nx)\,dx-\frac{8}{\pi}\int_0^{\pi}x\cos(nx)\,dx=\frac{8(1-(-1)^n)}{\pi n^2}$$
$$\Rightarrow Y_n(y)=a_n\cosh(ny)+b_n\sinh(ny)+\frac{8((-1)^n-1)}{\pi n^4}$$
$$\Rightarrow v(x,y)=-\frac{1}{3\pi}y^3+\left(\frac{1}{2}-\pi\right)y^2+b_0y+a_0+\sum_{n=1}^{\infty}\left(a_n\cosh(ny)+b_n\sinh(ny)+\frac{8((-1)^n-1)}{\pi n^4}\right)\cos(nx)$$
$$v(x,0)=a_0+\sum_{n=1}^{\infty}\left(a_n+\frac{8((-1)^n-1)}{\pi n^4}\right)\cos(nx)=x-\frac{x^2}{2}$$
$$a_0=\frac{1}{\pi}\int_0^{\pi}\left(x-\frac{x^2}{2}\right)dx=\frac{\pi}{2}-\frac{\pi^2}{6}$$
$$a_n+\frac{8((-1)^n-1)}{\pi n^4}=\frac{2}{\pi}\int_0^{\pi}\left(x-\frac{x^2}{2}\right)\cos(nx)\,dx=\frac{2((-1)^{n+1}(\pi-1)-1)}{n^2\pi}$$
$$\Rightarrow a_n=\frac{2((-1)^{n+1}(\pi-1)-1)}{n^2\pi}-\frac{8((-1)^n-1)}{\pi n^4}$$
$$v(x,1)=-\frac{1}{3\pi}+\frac{1}{2}-\pi+b_0+a_0+\sum_{n=1}^{\infty}\left(a_n\cosh(n)+b_n\sinh(n)+\frac{8((-1)^n-1)}{\pi n^4}\right)\cos(nx)=3x+1-\frac{(\pi+2)}{2\pi}x^2$$
$$-\frac{1}{3\pi}+\frac{1}{2}-\pi+b_0+a_0=\frac{1}{\pi}\int_0^{\pi}\left(3x+1-\frac{(\pi+2)}{2\pi}x^2\right)dx=1+\frac{3\pi}{2}-\frac{1}{6}\pi(2+\pi)\Rightarrow b_0=\frac{1}{3\pi}+\frac{1}{2}+\frac{11}{6}\pi-\frac{1}{6}\pi^2-a_0=\frac{1}{3\pi}+\frac{1}{2}+\frac{8}{6}\pi$$
$$a_n\cosh(n)+b_n\sinh(n)+\frac{8((-1)^n-1)}{\pi n^4}=\frac{2}{\pi}\int_0^{\pi}\left(3x+1-\frac{(\pi+2)}{2\pi}x^2\right)\cos(nx)\,dx=\frac{2((-1)^{n+1}(\pi-1)-3)}{n^2\pi}$$
$$\Rightarrow b_n=\frac{\frac{2((-1)^{n+1}(\pi-1)-3)}{n^2\pi}-a_n\cosh(n)-\frac{8((-1)^n-1)}{\pi n^4}}{\sinh(n)}$$

۵) معادله لاپلاس زیر را حل کنید. (امتیازی)

$$u_{xx}+u_{yy}=0,\qquad 0\le x,\qquad 0\le y$$
$$u(x,0)=e^{-x^2}$$
$$u(0,y)=\frac{a}{y^2+a^2}$$

$$u(x,y)=X(x)Y(y)\Rightarrow \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-k^2\Rightarrow\begin{cases}X''(x)+k^2X(x)=0\\ Y''(y)-k^2Y(y)=0\end{cases}\Rightarrow\begin{cases}X(x)=a\cos(kx)+b\sin(kx)\\ Y(y)=ce^{-ky}+be^{ky}\end{cases}$$

$$\Rightarrow u(x,y)=\int_0^\infty (A(\omega)\cos(\omega x)+B(\omega)\sin(\omega x))e^{-\omega y}d\omega$$

$$u(0,y)=\int_0^\infty A(\omega)e^{-\omega y}d\omega=\frac{a}{y^2+a^2}\Rightarrow A(\omega)=\sin(\omega a)$$

$$u(x,0)=\int_0^\infty (\sin(\omega a)\cos(\omega x)+B(\omega)\sin(\omega x))d\omega=e^{-x^2}\Rightarrow\int_0^\infty B(\omega)\sin(\omega x)\,d\omega=e^{-x^2}-\int_0^\infty \sin(\omega a)\cos(\omega x)\,d\omega=e^{-x^2}$$

$$\int_0^\infty \sin(\omega a)\cos(\omega x)\,d\omega=\frac{1}{2}\int_{-\infty}^{+\infty}\operatorname{sgn}(\omega)\sin(\omega a)\cos(\omega x)\,d\omega=\pi\mathcal{F}^{-1}[\operatorname{sgn}(\omega)\sin(\omega a)]=\pi\mathcal{F}^{-1}[\operatorname{sgn}(\omega)]*\mathcal{F}^{-1}[\sin(\omega a)]$$

$$=-\frac{1}{ix}*\left(i\left(\frac{\delta(x-a)-\delta(x+a)}{2}\right)\right)=\frac{1}{2}\left(\frac{1}{x+a}-\frac{1}{x-a}\right)=-\frac{a}{x^2-a^2}$$

$$\Rightarrow\int_0^\infty B(\omega)\sin(\omega x)\,d\omega=e^{-x^2}+\frac{a}{x^2-a^2}$$

$$\mathcal{F}\left[\left(e^{-x^2}+\frac{a}{x^2-a^2}\right)\operatorname{sgn}(x)\right]=\mathcal{F}\left[e^{-x^2}+\frac{a}{x^2-a^2}\right]*\mathcal{F}[\operatorname{sgn}(x)]=\left(\sqrt{\pi}e^{-\frac{\omega^2}{4}}-\pi\operatorname{sgn}(\omega)\sin(\omega a)\right)*\frac{2}{i\omega}$$

$$\Rightarrow B(\omega)=-\frac{i}{\pi}\left(\sqrt{\pi}e^{-\frac{\omega^2}{4}}-\pi\operatorname{sgn}(\omega)\sin(\omega a)\right)*\frac{2}{i\omega}=\left(\operatorname{sgn}(\omega)\sin(\omega a)-\frac{1}{\sqrt{\pi}}e^{-\frac{\omega^2}{4}}\right)*\frac{2}{\omega}$$

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۶) معادله لاپلاس زیر را حل کنید.

$$u_{xx}+u_{yy}=0,\qquad 0\le x\le 1,\qquad 0\le y\le 1$$
$$u(x,0)=\Pi(x),\qquad u(x,1)=1-\Pi(x)$$
$$u(0,y)=1-\Pi(y),\qquad u(1,y)=\Pi(y)$$

$$u(x,y)=w(x,y)+v(x,y)$$
$$\text{BC: }\begin{cases}u(x,0)=\Pi(x)\\ u(x,1)=1-\Pi(x)\end{cases}\Rightarrow w(x,y)=\Pi(x)+y\big(1-2\Pi(x)\big)$$
$$\Rightarrow v_{xx}+v_{yy}=(1-2y)\delta'\left(x-\frac{1}{2}\right)$$
$$v(x,0)=0,\qquad v(x,1)=0$$
$$v(0,y)=y-\Pi(y),\qquad v(1,y)=\Pi(y)-y$$
$$\text{BC: }\begin{cases}v(x,0)=0\\ v(x,1)=0\end{cases}\Rightarrow v(x,y)=\sum_{n=1}^{\infty}X_n(x)\sin(n\pi y)$$
$$\Rightarrow \sum_{n=1}^{\infty}\big(X_n''(x)-n^2\pi^2X_n(x)\big)\sin(n\pi y)=(1-2y)\delta'\left(x-\frac{1}{2}\right)$$
$$\Rightarrow X_n''(x)-n^2\pi^2X_n(x)=2\delta'\left(x-\frac{1}{2}\right)\int_0^1(1-2y)\sin(n\pi y)\,dy$$
$$(-\omega^2-n^2\pi^2)\mathcal{F}[X_n(x)]=i2\omega e^{-i\frac{\omega}{2}}\int_0^1(1-2y)\sin(n\pi y)\,dy$$
$$\mathcal{F}[X_n(x)]=-\frac{i2\omega e^{-i\frac{\omega}{2}}}{\omega^2+n^2\pi^2}\int_0^1(1-2y)\sin(n\pi y)\,dy=\left(\frac{1}{n\pi+i\omega}-\frac{1}{n\pi-i\omega}\right)e^{-i\frac{\omega}{2}}\int_0^1(1-2y)\sin(n\pi y)\,dy$$
$$X_n(x)=a_n\cosh(n\pi x)+b_n\sinh(n\pi x)+2e^{-n\pi\left|x-\frac{1}{2}\right|}\operatorname{sgn}\left(x-\frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}$$
$$\Rightarrow v(x,y)=\sum_{n=1}^{\infty}\left(a_n\cosh(n\pi x)+b_n\sinh(n\pi x)+2e^{-n\pi\left|x-\frac{1}{2}\right|}\operatorname{sgn}\left(x-\frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi y)$$
$$v(0,y)=\sum_{n=1}^{\infty}\left(a_n-2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi y)=y-\Pi(y)$$
$$\Rightarrow a_n-2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}=2\int_0^1\big(y-\Pi(y)\big)\sin(n\pi y)=\frac{2\left(\cos\left(\frac{n\pi}{2}\right)-1-(-1)^n\right)}{n\pi}$$
$$\Rightarrow a_n=\frac{2\left(\cos\left(\frac{n\pi}{2}\right)-1-(-1)^n\right)}{n\pi}+2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}$$
$$v(1,y)=\sum_{n=1}^{\infty}\left(a_n\cosh(n\pi)+b_n\sinh(n\pi)+2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi y)=\Pi(y)-y$$
$$\Rightarrow a_n\cosh(n\pi)+b_n\sinh(n\pi)+2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}=-\frac{2\left(\cos\left(\frac{n\pi}{2}\right)-1-(-1)^n\right)}{n\pi}$$
$$\Rightarrow b_n=-a_n\coth(n\pi)-a_n\operatorname{csch}(n\pi)=-a_n\coth\left(\frac{n\pi}{2}\right)$$

همچنین می‌توانستیم همگن سازی را در راستای دیگر انجام دهیم

$$u(x,y)=w(x,y)+v(x,y)$$
$$\text{BC: }\begin{cases}u(0,y)=1-\Pi(y)\\ u(1,y)=\Pi(y)\end{cases}\Rightarrow w(x,y)=1-\Pi(y)+x\big(2\Pi(y)-1\big)$$
$$\Rightarrow v_{xx}+v_{yy}=(1-2x)\delta'\left(y-\frac{1}{2}\right)$$
$$v(x,0)=\Pi(x)-x,\qquad v(x,1)=x-\Pi(x)$$
$$v(0,y)=0,\qquad v(1,y)=0$$
$$\text{BC: }\begin{cases}v(0,y)=0\\ v(1,y)=0\end{cases}\Rightarrow v(x,y)=\sum_{n=1}^{\infty}Y_n(y)\sin(n\pi x)$$
$$\Rightarrow \sum_{n=1}^{\infty}\big(Y_n''(y)-n^2\pi^2Y_n(y)\big)\sin(n\pi x)=(1-2x)\delta'\left(y-\frac{1}{2}\right)$$

$$\Rightarrow Y_n''(y) - n^2\pi^2 Y_n(y) = 2\delta'\left(y - \frac{1}{2}\right)\int_0^1 (1-2x)\sin(n\pi x)\,dx$$

$$(-\omega^2 - n^2\pi^2)\mathcal{F}[Y_n(y)] = i2\omega e^{-i\frac{\omega}{2}}\int_0^1 (1-2x)\sin(n\pi x)\,dx$$

$$\mathcal{F}[Y_n(y)] = -\frac{i2\omega e^{-i\frac{\omega}{2}}}{\omega^2 + n^2\pi^2}\int_0^1 (1-2x)\sin(n\pi x)\,dx = \left(\frac{1}{n\pi + i\omega} - \frac{1}{n\pi - i\omega}\right)e^{-i\frac{\omega}{2}}\int_0^1 (1-2x)\sin(n\pi x)\,dx$$

$$Y_n(y) = a_n\cosh(n\pi y) + b_n\sinh(n\pi y) + 2e^{-n\pi\left|y-\frac{1}{2}\right|}\mathrm{sgn}\left(y - \frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}$$

$$\Rightarrow v(x,y) = \sum_{n=1}^{\infty}\left(a_n\cosh(n\pi y) + b_n\sinh(n\pi y) + 2e^{-n\pi\left|y-\frac{1}{2}\right|}\mathrm{sgn}\left(y - \frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi x)$$

$$v(x,0) = \sum_{n=1}^{\infty}\left(a_n - 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi x) = \Pi(x) - \mathrm{x}$$

$$\Rightarrow a_n - 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2} = 2\int_0^1 (\Pi(x) - \mathrm{x})\sin(n\pi x) = -\frac{2\left(\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right)}{n\pi}$$

$$\Rightarrow a_n = -\frac{2\left(\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right)}{n\pi} + 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}$$

$$v(x,1) = \sum_{n=1}^{\infty}\left(a_n\cosh(n\pi) + b_n\sinh(n\pi) + 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi x) = x - \Pi(x)$$

$$\Rightarrow a_n\cosh(n\pi) + b_n\sinh(n\pi) + 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2} = \frac{2\left(\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right)}{n\pi}$$

$$\Rightarrow b_n = -a_n\coth(n\pi) - a_n\,\mathrm{csch}(n\pi) = -a_n\coth\left(\frac{n\pi}{2}\right)$$

مدرس: دکتر مهدی طالع ماسوله- حل تمرین: نیما هاشمی- حسین عطرسایی- نیکا امامی
برای سوالات خود در خصوص این تمرین با ایمیل های nimahashemi57@gmail.com، hatrsaei@gmail.com، emami.nika@gmail.com مکاتبه نمایید.

۷) معادله لاپلاس زیر را حل کنید.

$$u_{xx}+u_{yy}=0,\qquad 0\le x\le 1,\qquad 0\le y\le 1$$
$$u_y(x,0)=\Pi(x),\qquad u_y(x,1)=1-\Pi(x)$$
$$u(0,y)=1-\Pi(y),\qquad u(1,y)=\Pi(y)$$

برای حل این سوال مانند روش دوم سوال ۶ عمل می‌کنیم تا همگن سازی یکسانی داشته باشند.

$$u(x,y)=w(x,y)+v(x,y)$$
$$\text{BC: }\begin{cases}u(0,y)=1-\Pi(y)\\ u(1,y)=\Pi(y)\end{cases}\Rightarrow w(x,y)=1-\Pi(y)+x(2\Pi(y)-1)$$
$$\Rightarrow v_{xx}+v_{yy}=(1-2x)\delta'\left(y-\frac{1}{2}\right)$$
$$v_y(x,0)=\Pi(x),\qquad v_y(x,1)=1-\Pi(x)$$
$$v(0,y)=0,\qquad v(1,y)=0$$
$$\text{BC: }\begin{cases}v(0,y)=0\\ v(1,y)=0\end{cases}\Rightarrow v(x,y)=\sum_{n=1}^{\infty}Y_n(y)\sin(n\pi x)$$
$$\Rightarrow\sum_{n=1}^{\infty}\left(Y_n''(y)-n^2\pi^2Y_n(y)\right)\sin(n\pi x)=(1-2x)\delta'\left(y-\frac{1}{2}\right)$$
$$\Rightarrow Y_n''(y)-n^2\pi^2Y_n(y)=2\delta'\left(y-\frac{1}{2}\right)\int_0^1(1-2x)\sin(n\pi x)\,dx$$
$$(-\omega^2-n^2\pi^2)\mathcal{F}[Y_n(y)]=i2\omega e^{-i\frac{\omega}{2}}\int_0^1(1-2x)\sin(n\pi x)\,dx$$
$$\mathcal{F}[Y_n(y)]=-\frac{i2\omega e^{-i\frac{\omega}{2}}}{\omega^2+n^2\pi^2}\int_0^1(1-2x)\sin(n\pi x)\,dx=\left(\frac{1}{n\pi+i\omega}-\frac{1}{n\pi-i\omega}\right)e^{-i\frac{\omega}{2}}\int_0^1(1-2x)\sin(n\pi x)\,dx$$
$$Y_n(y)=a_n\cosh(n\pi y)+b_n\sinh(n\pi y)+2e^{-n\pi\left|y-\frac{1}{2}\right|}\operatorname{sgn}\left(y-\frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}$$
$$\Rightarrow v(x,y)=\sum_{n=1}^{\infty}\left(a_n\cosh(n\pi y)+b_n\sinh(n\pi y)+2e^{-n\pi\left|y-\frac{1}{2}\right|}\operatorname{sgn}\left(y-\frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi x)$$
$$v_y(x,0)=\sum_{n=1}^{\infty}\left(n\pi b_n-2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n\pi}\right)\sin(n\pi x)=\Pi(x)$$
$$\Rightarrow n\pi b_n-2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n\pi}=2\int_0^1(\Pi(x))\sin(n\pi x)=\frac{2\left(1-\cos\left(\frac{n\pi}{2}\right)\right)}{n\pi}$$
$$\Rightarrow b_n=\frac{2\left(1-\cos\left(\frac{n\pi}{2}\right)\right)}{n^2\pi^2}+2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}$$
$$v_y(x,1)=\sum_{n=1}^{\infty}\left(n\pi a_n\sinh(n\pi)+n\pi b_n\cosh(n\pi)-2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n\pi}\right)\sin(n\pi x)=1-\Pi(x)$$
$$\Rightarrow n\pi a_n\sinh(n\pi)+n\pi b_n\cosh(n\pi)-2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n\pi}=\frac{2\left(\cos\left(\frac{n\pi}{2}\right)-(-1)^n\right)}{n\pi}$$
$$\Rightarrow a_n=-b_n\coth(n\pi)+\left(\frac{2\left(\cos\left(\frac{n\pi}{2}\right)-(-1)^n\right)}{n^2\pi^2}+2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}\right)\operatorname{csch}(n\pi)$$

۸) معادله لاپلاس زیر را حل کنید.

$$u_{xx}+u_{yy}=0,\qquad 0\le x\le 1,\qquad 0\le y\le 1$$
$$u_y(x,0)=\Pi(x),\qquad u(x,1)=1-\Pi(x)$$
$$u(0,y)=1-\Pi(y),\qquad u(1,y)=\Pi(y)$$

برای حل این سوال مانند روش دوم سوال ۶ عمل می‌کنیم تا همگن سازی یکسانی داشته باشند.

$$u(x,y)=w(x,y)+v(x,y)$$
$$\text{BC: }\begin{cases}u(0,y)=1-\Pi(y)\\ u(1,y)=\Pi(y)\end{cases}\Rightarrow w(x,y)=1-\Pi(y)+x(2\Pi(y)-1)$$
$$\Rightarrow v_{xx}+v_{yy}=(1-2x)\delta'\left(y-\frac{1}{2}\right)$$
$$v_y(x,0)=\Pi(x),\qquad v(x,1)=x-\Pi(x)$$
$$v(0,y)=0,\qquad v(1,y)=0$$
$$\text{BC: }\begin{cases}v(0,y)=0\\ v(1,y)=0\end{cases}\Rightarrow v(x,y)=\sum_{n=1}^{\infty}Y_n(y)\sin(n\pi x)$$
$$\Rightarrow\sum_{n=1}^{\infty}\left(Y_n''(y)-n^2\pi^2Y_n(y)\right)\sin(n\pi x)=(1-2x)\delta'\left(y-\frac{1}{2}\right)$$
$$\Rightarrow Y_n''(y)-n^2\pi^2Y_n(y)=2\delta'\left(y-\frac{1}{2}\right)\int_0^1(1-2x)\sin(n\pi x)\,dx$$
$$(-\omega^2-n^2\pi^2)\mathcal{F}[Y_n(y)]=i2\omega e^{-i\frac{\omega}{2}}\int_0^1(1-2x)\sin(n\pi x)\,dx$$
$$\mathcal{F}[Y_n(y)]=-\frac{i2\omega e^{-i\frac{\omega}{2}}}{\omega^2+n^2\pi^2}\int_0^1(1-2x)\sin(n\pi x)\,dx=\left(\frac{1}{n\pi+i\omega}-\frac{1}{n\pi-i\omega}\right)e^{-i\frac{\omega}{2}}\int_0^1(1-2x)\sin(n\pi x)\,dx$$
$$Y_n(y)=a_n\cosh(n\pi y)+b_n\sinh(n\pi y)+2e^{-n\pi\left|y-\frac{1}{2}\right|}\operatorname{sgn}\left(y-\frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}$$
$$\Rightarrow v(x,y)=\sum_{n=1}^{\infty}\left(a_n\cosh(n\pi y)+b_n\sinh(n\pi y)+2e^{-n\pi\left|y-\frac{1}{2}\right|}\operatorname{sgn}\left(y-\frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi x)$$
$$v_y(x,0)=\sum_{n=1}^{\infty}\left(n\pi b_n-2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n\pi}\right)\sin(n\pi x)=\Pi(x)$$
$$\Rightarrow n\pi b_n-2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n\pi}=2\int_0^1(\Pi(x))\sin(n\pi x)=\frac{2\left(1-\cos\left(\frac{n\pi}{2}\right)\right)}{n\pi}$$
$$\Rightarrow b_n=\frac{2\left(1-\cos\left(\frac{n\pi}{2}\right)\right)}{n^2\pi^2}+2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}$$
$$v(x,1)=\sum_{n=1}^{\infty}\left(a_n\cosh(n\pi)+b_n\sinh(n\pi)+2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi x)=x-\Pi(x)$$
$$\Rightarrow a_n\cosh(n\pi)+b_n\sinh(n\pi)+2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}=\frac{2\left(\cos\left(\frac{n\pi}{2}\right)-1-(-1)^n\right)}{n\pi}$$
$$\Rightarrow a_n=-b_n\coth(n\pi)+\left(\frac{2\left(\cos\left(\frac{n\pi}{2}\right)-1-(-1)^n\right)}{n\pi}-2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}\right)\operatorname{csch}(n\pi)$$

۹) معادله لاپلاس زیر را حل کنید. (امتیازی)

$$u_{xx} + u_{yy} = 0, \qquad 0 \le x \le 1, \qquad 0 \le y \le 1$$
$$u(x,0) = \Pi(x), \qquad u_y(x,1) = 1 - \Pi(x)$$
$$u(0,y) = 1 - \Pi(y), \qquad u(1,y) = \Pi(y)$$

برای حل این سوال مانند روش دوم سوال ۶ عمل می‌کنیم تا همگن سازی یکسانی داشته باشند.

$$u(x,y) = w(x,y) + v(x,y)$$

$$\text{BC: } \begin{cases} u(0,y) = 1 - \Pi(y) \\ u(1,y) = \Pi(y) \end{cases} \Rightarrow w(x,y) = 1 - \Pi(y) + x(2\Pi(y) - 1)$$

$$\Rightarrow v_{xx} + v_{yy} = (1-2x)\delta'\left(y - \frac{1}{2}\right)$$

$$v(x,0) = \Pi(x) - x, \qquad v_y(x,1) = 1 - \Pi(x)$$
$$v(0,y) = 0, \qquad v(1,y) = 0$$

$$\text{BC: } \begin{cases} v(0,y) = 0 \\ v(1,y) = 0 \end{cases} \Rightarrow v(x,y) = \sum_{n=1}^{\infty} Y_n(y)\sin(n\pi x)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(Y_n''(y) - n^2\pi^2 Y_n(y)\right)\sin(n\pi x) = (1-2x)\delta'\left(y - \frac{1}{2}\right)$$

$$\Rightarrow Y_n''(y) - n^2\pi^2 Y_n(y) = 2\delta'\left(y - \frac{1}{2}\right)\int_0^1 (1-2x)\sin(n\pi x)\,dx$$

$$(-\omega^2 - n^2\pi^2)\mathcal{F}[Y_n(y)] = i2\omega e^{-i\frac{\omega}{2}}\int_0^1 (1-2x)\sin(n\pi x)\,dx$$

$$\mathcal{F}[Y_n(y)] = -\frac{i2\omega e^{-i\frac{\omega}{2}}}{\omega^2 + n^2\pi^2}\int_0^1 (1-2x)\sin(n\pi x)\,dx = \left(\frac{1}{n\pi + i\omega} - \frac{1}{n\pi - i\omega}\right)e^{-i\frac{\omega}{2}}\int_0^1 (1-2x)\sin(n\pi x)\,dx$$

$$Y_n(y) = a_n\cosh(n\pi y) + b_n\sinh(n\pi y) + 2e^{-n\pi\left|y-\frac{1}{2}\right|}\operatorname{sgn}\left(y - \frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}$$

$$\Rightarrow v(x,y) = \sum_{n=1}^{\infty}\left(a_n\cosh(n\pi y) + b_n\sinh(n\pi y) + 2e^{-n\pi\left|y-\frac{1}{2}\right|}\operatorname{sgn}\left(y - \frac{1}{2}\right)\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi x)$$

$$v(x,0) = \sum_{n=1}^{\infty}\left(a_n - 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}\right)\sin(n\pi x) = \Pi(x) - x$$

$$\Rightarrow a_n - 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2} = 2\int_0^1 (\Pi(x) - x)\sin(n\pi x) = -\frac{2\left(\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right)}{n\pi}$$

$$\Rightarrow b_n == -\frac{2\left(\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right)}{n\pi} + 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}$$

$$v_y(x,1) = \sum_{n=1}^{\infty}\left(n\pi a_n\sinh(n\pi) + n\pi b_n\cosh(n\pi) - 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n\pi}\right)\sin(n\pi x) = 1 - \Pi(x)$$

$$\Rightarrow n\pi a_n\sinh(n\pi) + n\pi b_n\cosh(n\pi) - 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n\pi} = \frac{2\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)}{n\pi}$$

$$\Rightarrow a_n = -b_n\coth(n\pi) + \left(\frac{2\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)}{n^2\pi^2} + 2e^{-\frac{n\pi}{2}}\frac{1-(-1)^n}{n^2\pi^2}\right)\operatorname{csch}(n\pi)$$