تمرین ۸: معادله لاپلاس در مختصات قطبی


مدرس: دکتر مهدی طالع ماسوله- حل تمرین: گلمهر خسروخاور- نیما هاشمی

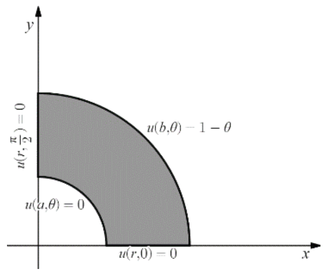
برای سوالات خود در خصوص این تمرین با رایانامه gkhosrokhavar@gmail.com، nimahashemi57@gmail.com مکاتبه نمایید.


۱) معادله لاپلاس را در ناحیه مشخص شده و با شرایط مرزی زیر حل کنید.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0,\qquad a\le r\le b,\qquad 0\le\theta\le\frac{\pi}{2}$$

$$u(r,0)=0,\qquad u\left(r,\frac{\pi}{2}\right)=0$$

$$u(a,\theta)=0,\qquad u(b,\theta)=1-\theta$$




از تغییر متغیر $r=e^x$ استفاده می‌کنیم.

$$u(r,\theta)=\hat{u}(x,\theta)\Longrightarrow \hat{u}_{xx}+\hat{u}_{\theta\theta}=0$$

$$\text{BC: }\begin{cases}u(r,0)=0\\ u\left(r,\frac{\pi}{2}\right)=0\end{cases}\Longrightarrow \hat{u}(x,\theta)=\sum_{n=1}^{\infty}X_n(x)\sin(2n\theta)$$

$$\Longrightarrow\sum_{n=1}^{\infty}\left(X_n''(x)-4n^2X_n(x)\right)\sin(2n\theta)=0$$

$$\Longrightarrow X_n''(x)-4n^2X_n(x)=0$$

$$\Longrightarrow X_n(x)=c_ne^{2nx}+d_ne^{-2nx}$$

$$\Longrightarrow\hat{u}(x,\theta)=\sum_{n=1}^{\infty}\left(c_ne^{2nx}+d_ne^{-2nx}\right)\sin(2n\theta)$$

$$\Longrightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(c_nr^{2n}+d_nr^{-2n}\right)\sin(2n\theta)$$

$$u(a,\theta)=\sum_{n=1}^{\infty}\left(c_na^{2n}+d_na^{-2n}\right)\sin(2n\theta)=0$$

$$\Longrightarrow c_na^{2n}+d_na^{-2n}=\mathcal{FS}_{\sin(2n\theta)}\{u(a,\theta)\}=\mathcal{FS}_{\sin(2n\theta)}\{0\}=0$$

$$u(b,\theta)=\sum_{n=1}^{\infty}\left(c_nb^{2n}+d_nb^{-2n}\right)\sin(2n\theta)=1-\theta$$

$$\Longrightarrow c_nb^{2n}+d_nb^{-2n}=\mathcal{FS}_{\sin(2n\theta)}\{u(b,\theta)\}=\mathcal{FS}_{\sin(2n\theta)}\{1-\theta\}=\frac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(1-\theta)\sin(2n\theta)\,d\theta=\frac{2+(\pi-2)(-1)^n}{n\pi}$$

$$\Longrightarrow\begin{cases}c_na^{2n}+d_na^{-2n}=\mathcal{FS}_{\sin(2n\theta)}\{u(a,\theta)\}\\ c_nb^{2n}+d_nb^{-2n}=\mathcal{FS}_{\sin(2n\theta)}\{u(b,\theta)\}\end{cases}\Longrightarrow\begin{cases}c_n=\dfrac{\mathcal{FS}_{\sin(2n\theta)}\{u(a,\theta)\}a^{2n}-\mathcal{FS}_{\sin(2n\theta)}\{u(b,\theta)\}b^{2n}}{a^{4n}-b^{4n}}\\ d_n=\dfrac{\mathcal{FS}_{\sin(2n\theta)}\{u(a,\theta)\}a^{-2n}-\mathcal{FS}_{\sin(2n\theta)}\{u(b,\theta)\}b^{-2n}}{a^{-4n}-b^{-4n}}\end{cases}$$

$$\Longrightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(\frac{\mathcal{FS}_{\sin(2n\theta)}\{u(a,\theta)\}a^{2n}-\mathcal{FS}_{\sin(2n\theta)}\{u(b,\theta)\}b^{2n}}{a^{4n}-b^{4n}}r^{2n}+\frac{\mathcal{FS}_{\sin(2n\theta)}\{u(a,\theta)\}a^{-2n}-\mathcal{FS}_{\sin(2n\theta)}\{u(b,\theta)\}b^{-2n}}{a^{-4n}-b^{-4n}}r^{-2n}\right)\sin(2n\theta)$$

$$=\sum_{n=1}^{\infty}\left(\left(\frac{a^{2n}r^{2n}}{a^{4n}-b^{4n}}+\frac{a^{-2n}r^{-2n}}{a^{-4n}-b^{-4n}}\right)\mathcal{FS}_{\sin(2n\theta)}\{u(a,\theta)\}+\left(\frac{b^{2n}r^{2n}}{b^{4n}-a^{4n}}+\frac{b^{-2n}r^{-2n}}{b^{-4n}-a^{-4n}}\right)\mathcal{FS}_{\sin(2n\theta)}\{u(b,\theta)\}\right)\sin(2n\theta)$$

$$=\sum_{n=1}^{\infty}\left(\left(\frac{\left(\frac{r}{b}\right)^{2n}}{\left(\frac{a}{b}\right)^{2n}-\left(\frac{a}{b}\right)^{-2n}}+\frac{\left(\frac{r}{b}\right)^{-2n}}{\left(\frac{a}{b}\right)^{-2n}-\left(\frac{a}{b}\right)^{2n}}\right)\mathcal{FS}_{\sin(2n\theta)}\{u(a,\theta)\}+\left(\frac{\left(\frac{r}{a}\right)^{2n}}{\left(\frac{b}{a}\right)^{2n}-\left(\frac{b}{a}\right)^{-2n}}+\frac{\left(\frac{r}{a}\right)^{-2n}}{\left(\frac{b}{a}\right)^{-2n}-\left(\frac{b}{a}\right)^{2n}}\right)\mathcal{FS}_{\sin(2n\theta)}\{u(b,\theta)\}\right)\sin(2n\theta)$$

$$=\sum_{n=1}^{\infty}\left(\frac{\left(\frac{r}{b}\right)^{2n}-\left(\frac{r}{b}\right)^{-2n}}{\left(\frac{a}{b}\right)^{2n}-\left(\frac{a}{b}\right)^{-2n}}\mathcal{FS}_{\sin(2n\theta)}\{u(a,\theta)\}+\frac{\left(\frac{r}{a}\right)^{2n}-\left(\frac{r}{a}\right)^{-2n}}{\left(\frac{b}{a}\right)^{2n}-\left(\frac{b}{a}\right)^{-2n}}\mathcal{FS}_{\sin(2n\theta)}\{u(b,\theta)\}\right)\sin(2n\theta)$$

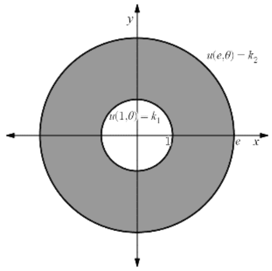
بنابراین می‌توان فرم کلی معادله را به صورت زیر نوشت

$$u(r,\theta)=\sum_{n=1}^{\infty}\left(\frac{\left(\frac{r_2}{r}\right)^{\frac{n\pi}{L}}-\left(\frac{r_2}{r}\right)^{-\frac{n\pi}{L}}}{\left(\frac{r_2}{r_1}\right)^{\frac{n\pi}{L}}-\left(\frac{r_2}{r_1}\right)^{-\frac{n\pi}{L}}}\mathcal{FS}_{\sin\left(\frac{n\pi}{L}\theta\right)}\{u(r_1,\theta)\}+\frac{\left(\frac{r}{r_1}\right)^{\frac{n\pi}{L}}-\left(\frac{r}{r_1}\right)^{-\frac{n\pi}{L}}}{\left(\frac{r_2}{r_1}\right)^{\frac{n\pi}{L}}-\left(\frac{r_2}{r_1}\right)^{-\frac{n\pi}{L}}}\mathcal{FS}_{\sin\left(\frac{n\pi}{L}\theta\right)}\{u(r_2,\theta)\}\right)\sin\left(\frac{n\pi}{L}\theta\right)$$

$$\Longrightarrow \boxed{u(r,\theta)=\sum_{n=1}^{\infty}\left(\frac{\left(\frac{r}{a}\right)^{2n}-\left(\frac{r}{a}\right)^{-2n}}{\left(\frac{b}{a}\right)^{2n}-\left(\frac{b}{a}\right)^{-2n}}\frac{2+(\pi-2)(-1)^n}{n\pi}\right)\sin(2n\theta)}$$

۲) معادله زیر را حل کنید.

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\qquad 1\le r\le e,\qquad 0\le\theta\le 2\pi$$
$$u(1,\theta)=k_1,\qquad u(e,\theta)=k_2$$




از تغییر متغیر $r=e^x$ استفاده می‌کنیم.

$$\Longrightarrow \hat{u}_{xx}+\hat{u}_{\theta\theta}=0$$

$$\Longrightarrow \frac{X''(x)}{X(x)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}=k^2\Longrightarrow\begin{cases}X_k(x)=\begin{cases}c_k e^{-kx}+d_k e^{kx}, & k\neq 0\\ c_0x+d_0, & k=0\end{cases}\\ \Theta_k(\theta)=\begin{cases}a_k\cos(k\theta)+b_k\sin(k\theta), & k\neq 0\\ a_0+b_0\theta, & k=0\end{cases}\end{cases}$$

$$\Longrightarrow u(r,\theta)=c_0\ln(r)+d_0+\sum_{n=1}^{\infty}(c_nr^{-n}+d_nr^n)(a_n\cos(n\theta)+b_n\sin(n\theta))$$

$$u(1,\theta)=d_0+\sum_{k=1}^{\infty}(c_n+d_n)(a_n\cos(n\theta)+b_n\sin(n\theta))=k_1$$

$$\Longrightarrow d_0=\frac{1}{2\pi}\int_0^{2\pi}k_1d\theta=k_1\Longrightarrow d_0=k_1$$

$$\Longrightarrow (c_n+d_n)a_n=\frac{2}{2\pi}\int_0^{2\pi}k_1\cos(n\theta)\,d\theta=0$$

$$\Longrightarrow (c_n+d_n)b_n=\frac{2}{2\pi}\int_0^{2\pi}k_2\sin(n\theta)\,d\theta=0$$

$$u(e,\theta)=c_0+k_1=\frac{1}{2\pi}\int_0^{2\pi}k_2d\theta=k_2\Longrightarrow c_0=k_2-k_1$$

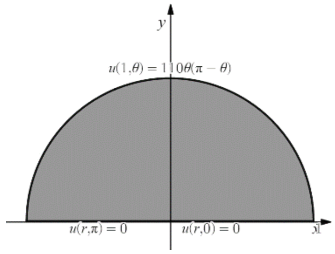
$$\Longrightarrow (c_ne^{-n}+d_ne^n)a_n=\frac{2}{2\pi}\int_0^{2\pi}k_2\cos(n\theta)\,d\theta=0$$

$$\Longrightarrow (c_ne^{-n}+d_ne^n)b_n=\frac{2}{2\pi}\int_0^{2\pi}k_2\sin(n\theta)\,d\theta=0$$

$$\Longrightarrow \boxed{u(r,\theta)=(k_2-k_1)\ln(r)+k_1}$$

۳) می‌دانیم که پتانسیل الکترواستاتیک، معادله لاپلاس $\nabla^2 u = 0$ را در هر منطقه‌ای که بار الکتریکی در آن صفر باشد، ارضا می‌کند. پتانسیل الکتریکی را در شکل زیر بدست آورید.

$\nabla^2 u = 0, \quad r \le 1, \quad 0 \le \theta \le \pi$
$u(r,0) = 0, \quad u(r,\pi) = 0$
$u(1,\theta) = 110\theta(\pi - \theta)$




از تغییر متغیر $r = e^x$ استفاده می‌کنیم.

$$\Rightarrow \hat{u}_{xx} + \hat{u}_{\theta\theta} = 0$$

$$\text{BC: } \begin{cases} u(r,0) = 0 \\ u(r,\pi) = 0 \end{cases} \Rightarrow \hat{u}(x,\theta) = \sum_{n=1}^{\infty} X_n(x) \sin(nx)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(X_n''(x) - n^2 X_n(x)\right) \sin(nx) = 0$$

$$\Rightarrow X_n(x) = c_n e^{-nx} + d_n e^{nx}$$

$$\Rightarrow u(r,\theta) = \sum_{n=1}^{\infty} (c_n r^{-n} + d_n r^n) \sin(nx)$$

با توجه به مقدار داشتن $u$ در $r = 0$، ضریب $r^{-n}$ باید صفر باشد.

$$\Rightarrow u(r,\theta) = \sum_{n=1}^{\infty} d_n r^n \sin(nx)$$

$$u(1,\theta) = \sum_{n=1}^{\infty} d_n \sin(nx) = 110\theta(\pi - \theta)$$

$$\Rightarrow d_n = \mathcal{FS}_{\sin(nx)}\{110\theta(\pi - \theta)\} = \frac{440(1-(-1)^n)}{n^3\pi}$$

$$\Rightarrow \boxed{u(r,\theta) = \sum_{n=1}^{\infty} \frac{440(1-(-1)^n)}{n^3\pi} r^n \sin(nx)}$$

۴) معادله لاپلاس زیر را حل کنید.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0, \qquad 1\le r\le 2, \qquad 0\le\theta\le 2\pi$$
$$u_\theta(r,0)=0, \qquad u_\theta(r,2\pi)=0$$
$$u(1,\theta)=1-\Pi\left(\frac{\theta-\pi}{\pi}\right), \qquad u(2,\theta)=\theta$$

از تغییر متغیر $r=e^x$ استفاده می‌کنیم

$$\Longrightarrow \hat{u}_{xx}+\hat{u}_{\theta\theta}=0$$

$$\text{BC: }\begin{cases}u_\theta(r,0)=0\\ u_\theta(r,2\pi)=0\end{cases}\Longrightarrow \hat{u}(x,\theta)=X_0(x)+\sum_{n=1}^{\infty}X_n(x)\cos\left(\frac{n}{2}\theta\right)$$

$$\Longrightarrow X_0''(x)+\sum_{n=1}^{\infty}\left(X_n''(x)-\frac{n^2}{4}X_n(x)\right)\cos\left(\frac{n}{2}\theta\right)=0$$

$$\Longrightarrow\begin{cases}X_0(x)=c_0x+d_0\\ X_n(x)=c_ne^{-\frac{n}{2}x}+d_ne^{\frac{n}{2}x}\end{cases}$$

$$\Longrightarrow u(r,\theta)=c_0\ln(r)+d_0+\sum_{n=1}^{\infty}\left(c_nr^{-\frac{n}{2}}+d_nr^{\frac{n}{2}}\right)\cos\left(\frac{n}{2}\theta\right)$$

$$u(1,\theta)=d_0+\sum_{n=1}^{\infty}(c_n+d_n)\cos\left(\frac{n}{2}\theta\right)=1-\Pi\left(\frac{\theta-\pi}{\pi}\right)$$

$$\Longrightarrow d_0=\frac{1}{2\pi}\int_0^{2\pi}\left(1-\Pi\left(\frac{\theta-\pi}{\pi}\right)\right)d\theta=\frac{1}{2}$$

$$c_n+d_n=\frac{2}{2\pi}\int_0^{2\pi}\left(1-\Pi\left(\frac{\theta-\pi}{\pi}\right)\right)\cos\left(\frac{n}{2}\theta\right)d\theta=\frac{2\left(\sin\left(\frac{n\pi}{4}\right)-\sin\left(\frac{3n\pi}{4}\right)\right)}{n\pi}$$

$$u(2,\theta)=c_0\ln(2)+d_0+\sum_{n=1}^{\infty}\left(c_n2^{-\frac{n}{2}}+d_n2^{\frac{n}{2}}\right)\cos\left(\frac{n}{2}\theta\right)=\theta$$

$$\Longrightarrow c_0\ln(2)+d_0=\frac{1}{2\pi}\int_0^{2\pi}\theta d\theta=\pi$$

$$\Longrightarrow c_n2^{-\frac{n}{2}}+d_n2^{\frac{n}{2}}=\frac{2}{2\pi}\int_0^{2\pi}\theta\cos\left(\frac{n}{2}\theta\right)d\theta=\frac{4((-1)^n-1)}{n^2\pi}$$

$$\Longrightarrow \boxed{u(r,\theta)=\left(\pi-\frac{1}{2}\right)\frac{\ln(r)}{\ln(2)}+\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\left(\frac{2}{r}\right)^{\frac{n}{2}}-\left(\frac{2}{r}\right)^{-\frac{n}{2}}}{2^{\frac{n}{2}}-2^{-\frac{n}{2}}}\frac{2\left(\sin\left(\frac{n\pi}{4}\right)-\sin\left(\frac{3n\pi}{4}\right)\right)}{n\pi}+\frac{r^{\frac{n}{2}}-r^{-\frac{n}{2}}}{2^{\frac{n}{2}}-2^{-\frac{n}{2}}}\frac{4((-1)^n-1)}{n^2\pi}\right)\cos\left(\frac{n}{2}\theta\right)}$$

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۵) معادله لاپلاس زیر را حل کنید. (امتیازی)

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\qquad 1\le r,\qquad 0\le\theta\le 2\pi$$
$$u(r,0)=u(r,2\pi),\qquad u_\theta(r,0)=u_\theta(r,2\pi)$$
$$u(1,\theta)=|\cos(2\theta)|+\cos(2\theta)$$

از تغییر متغیر $r=e^x$ استفاده می‌کنیم

$$\Longrightarrow \hat{u}_{xx}+\hat{u}_{\theta\theta}=0$$
$$\Longrightarrow \frac{X''(x)}{X(x)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}=k^2\Longrightarrow\begin{cases}X''(x)-k^2X(x)=0\\ \Theta''(\theta)+k^2\Theta(\theta)=0\end{cases}$$
$$\Longrightarrow\begin{cases}X(x)=\begin{cases}c_0x+d_0, & k=0\\ c_ke^{-kx}+d_ke^{kx}, & k\neq 0\end{cases}\\ \Theta(\theta)=\begin{cases}a_0+b_0\theta, & k=0\\ a_k\cos(k\theta)+b_k\sin(k\theta), & k\neq 0\end{cases}\end{cases}$$
$$u(r,0)=u(r,2\pi)\Longrightarrow\Theta(0)=\Theta(2\pi)=\begin{cases}a_0, & k=0\\ a_k, & k\neq 0\end{cases}=\begin{cases}a_0+2\pi b_0, & k=0\\ a_k\cos(2\pi k)+b_k\sin(2\pi k)\end{cases}\Longrightarrow b_0=0$$
$$u_\theta(r,0)=u_\theta(r,2\pi)\Longrightarrow\Theta'(0)=\Theta'(2\pi)=b_kk=-a_kk\sin(2\pi k)+b_kk\cos(2\pi k)$$
$$\Longrightarrow k=n$$
$$\Longrightarrow u(r,\theta)=c_0\ln(r)+d_0+\sum_{n=1}^{\infty}(c_nr^{-n}+d_nr^n)(a_n\cos(n\theta)+b_n\sin(n\theta))$$

با توجه به اینکه $1\le r$ پس $d_n=0$ و $c_0=0$

$$\Longrightarrow u(r,\theta)=d_0+\sum_{n=1}^{\infty}r^{-n}(a_n\cos(n\theta)+b_n\sin(n\theta))$$
$$u(1,\theta)=d_0+\sum_{n=1}^{\infty}a_n\cos(n\theta)+b_n\sin(n\theta)=|\cos(2\theta)|+\cos(2\theta)$$
$$d_0=\frac{1}{2\pi}\int_0^{2\pi}(|\cos(2\theta)|+\cos(2\theta))d\theta=\frac{2}{\pi}$$
$$a_n=\frac{2}{2\pi}\int_0^{2\pi}(|\cos(2\theta)|+\cos(2\theta))\cos(n\theta)\,d\theta=\begin{cases}-\dfrac{4\left(\cos\left(\frac{n\pi}{4}\right)+\cos\left(\frac{3n\pi}{4}\right)+\cos\left(\frac{5n\pi}{4}\right)+\cos\left(\frac{7n\pi}{4}\right)\right)}{\pi(n^2-4)}, & n>2\\ 1, & n=2\\ 0, & n=1\end{cases}$$
$$b_n=\frac{2}{2\pi}\int_0^{2\pi}(|\cos(2\theta)|+\cos(2\theta))\sin(n\theta)\,d\theta=\begin{cases}-\dfrac{4\left(\sin\left(\frac{n\pi}{4}\right)+\sin\left(\frac{3n\pi}{4}\right)+\sin\left(\frac{5n\pi}{4}\right)+\sin\left(\frac{7n\pi}{4}\right)\right)}{\pi(n^2-4)}, & n>2\\ 0, & n=1,2\end{cases}$$
$$\Longrightarrow u(r,\theta)=\frac{2}{\pi}+r^{-2}\cos(2\theta)-\sum_{n=3}^{\infty}4r^{-n}\left(\frac{\cos\left(\frac{n\pi}{4}\right)+\cos\left(\frac{3n\pi}{4}\right)+\cos\left(\frac{5n\pi}{4}\right)+\cos\left(\frac{7n\pi}{4}\right)}{\pi(n^2-4)}\cos(n\theta)+\frac{\sin\left(\frac{n\pi}{4}\right)+\sin\left(\frac{3n\pi}{4}\right)+\sin\left(\frac{5n\pi}{4}\right)+\sin\left(\frac{7n\pi}{4}\right)}{\pi(n^2-4)}\sin(n\theta)\right)$$

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۶) معادله لاپلاس زیر را حل کنید.

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\qquad \frac{1}{2}\le r\le 2,\qquad 0\le\theta\le\pi$$
$$u_\theta(r,0)=0,\qquad u(r,\pi)=0$$
$$u\left(\frac{1}{2},\theta\right)=\Pi\left(\frac{\theta}{\pi}\right),\qquad u(2,\theta)=1-\Pi\left(\frac{\theta}{\pi}\right)$$

از تغییر متغیر $r=e^x$ استفاده می‌کنیم

$$\Rightarrow \hat{u}_{xx}+\hat{u}_{\theta\theta}=0$$

$$\text{BC: }\begin{cases}u_\theta(r,0)=0\\ u_\theta(r,2\pi)=0\end{cases}\Rightarrow \hat{u}(x,\theta)=\sum_{n=1}^{\infty}X_n(x)\cos\left(\frac{2n-1}{2}\theta\right)$$

$$\Rightarrow X_0''(x)+\sum_{n=1}^{\infty}\left(X_n''(x)-\frac{(2n-1)^2}{4}X_n(x)\right)\cos\left(\frac{2n-1}{2}\theta\right)=0$$

$$\Rightarrow X_n(x)=c_ne^{-\frac{2n-1}{2}x}+d_ne^{\frac{2n-1}{2}x}$$

$$\Rightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(c_nr^{-\frac{2n-1}{2}}+d_nr^{\frac{2n-1}{2}}\right)\cos\left(\frac{2n-1}{2}\theta\right)$$

$$u\left(\frac{1}{2},\theta\right)=\sum_{n=1}^{\infty}\left(c_n\left(\frac{1}{2}\right)^{-\frac{2n-1}{2}}+d_n\left(\frac{1}{2}\right)^{\frac{2n-1}{2}}\right)\cos\left(\frac{2n-1}{2}\theta\right)=\Pi\left(\frac{\theta}{\pi}\right)$$

$$\Rightarrow c_n\left(\frac{1}{2}\right)^{-\frac{2n-1}{2}}+d_n\left(\frac{1}{2}\right)^{\frac{2n-1}{2}}=\frac{2}{\pi}\int_0^{\pi}\Pi\left(\frac{\theta}{\pi}\right)\cos\left(\frac{2n-1}{2}\theta\right)d\theta=-\frac{4\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}{\pi(2n-1)}$$

$$u(2,\theta)=\sum_{n=1}^{\infty}\left(c_n(2)^{-\frac{2n-1}{2}}+d_n(2)^{\frac{2n-1}{2}}\right)\cos\left(\frac{2n-1}{2}\theta\right)=1-\Pi\left(\frac{\theta}{\pi}\right)$$

$$\Rightarrow c_n(2)^{-\frac{2n-1}{2}}+d_n(2)^{\frac{2n-1}{2}}=\frac{2}{\pi}\int_0^{\pi}\left(1-\Pi\left(\frac{\theta}{\pi}\right)\right)\cos\left(\frac{2n-1}{2}\theta\right)d\theta=-\frac{4\left((-1)^n-\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right)}{\pi(2n-1)}$$

$$\Rightarrow \boxed{u(r,\theta)=\sum_{n=1}^{\infty}\left(\frac{\left(\frac{2}{r}\right)^{\frac{2n-1}{2}}-\left(\frac{2}{r}\right)^{-\frac{2n-1}{2}}}{4^{\frac{2n-1}{2}}-4^{-\frac{2n-1}{2}}}\left(-\frac{4\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}{\pi(2n-1)}\right)+\frac{(2r)^{\frac{2n-1}{2}}-(2r)^{-\frac{2n-1}{2}}}{4^{\frac{2n-1}{2}}-4^{-\frac{2n-1}{2}}}\left(-\frac{4\left((-1)^n-\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right)}{\pi(2n-1)}\right)\right)\cos\left(\frac{2n-1}{2}\theta\right)}$$

۷) معادله لاپلاس زیر را حل کنید.

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\qquad 1\le r\le 2,\qquad 0\le\theta\le\frac{3\pi}{2}$$

$$u(r,0)=0,\qquad u_\theta\left(r,\frac{3\pi}{2}\right)=0$$

$$u(1,\theta)=\Lambda\left(\frac{\theta-\frac{3\pi}{4}}{\frac{3\pi}{2}}\right),\qquad u(2,\theta)=\Pi\left(\frac{\theta}{\frac{3\pi}{2}}\right)$$

از تغییر متغیر $r=e^x$ استفاده می‌کنیم

$$\Rightarrow \hat{u}_{xx}+\hat{u}_{\theta\theta}=0$$

$$\text{BC: }\begin{cases}u_\theta(r,0)=0\\ u_\theta(r,2\pi)=0\end{cases}\Rightarrow \hat{u}(x,\theta)=\sum_{n=1}^{\infty}X_n(x)\sin\left(\frac{2n-1}{3}\theta\right)$$

$$\Rightarrow X_0''(x)+\sum_{n=1}^{\infty}\left(X_n''(x)-\frac{(2n-1)^2}{9}X_n(x)\right)\sin\left(\frac{2n-1}{3}\theta\right)=0$$

$$\Rightarrow X_n(x)=c_ne^{-\frac{2n-1}{3}x}+d_ne^{\frac{2n-1}{3}x}$$

$$\Rightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(c_nr^{-\frac{2n-1}{3}}+d_nr^{\frac{2n-1}{3}}\right)\sin\left(\frac{2n-1}{3}\theta\right)$$

$$u(1,\theta)=\sum_{n=1}^{\infty}(c_n+d_n)\sin\left(\frac{2n-1}{3}\theta\right)=\Lambda\left(\frac{\theta-\frac{3\pi}{4}}{\frac{3\pi}{2}}\right)$$

$$\Rightarrow c_n+d_n=\frac{2}{\frac{3\pi}{2}}\int_0^{\frac{3\pi}{2}}\Lambda\left(\frac{\theta-\frac{3\pi}{4}}{\frac{3\pi}{2}}\right)\sin\left(\frac{2n-1}{3}\theta\right)d\theta=\frac{2\left(4(-1)^n+(2n-1)\pi-8\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right)}{(2n-1)^2\pi^2}$$

$$u(2,\theta)=\sum_{n=1}^{\infty}\left(c_n2^{-\frac{2n-1}{3}}+d_n2^{\frac{2n-1}{3}}\right)\sin\left(\frac{2n-1}{3}\theta\right)=\Pi\left(\frac{\theta}{\frac{3\pi}{2}}\right)$$

$$\Rightarrow c_n2^{-\frac{2n-1}{3}}+d_n2^{\frac{2n-1}{3}}=\frac{2}{\frac{3\pi}{2}}\int_0^{\frac{3\pi}{2}}\Pi\left(\frac{\theta}{\frac{3\pi}{2}}\right)\sin\left(\frac{2n-1}{3}\theta\right)d\theta=\frac{8\cos^2\left(\frac{n\pi}{4}+\frac{3\pi}{8}\right)}{(2n-1)\pi}$$

$$\Rightarrow \boxed{u(r,\theta)=\sum_{n=1}^{\infty}\left(\frac{\left(\frac{2}{r}\right)^{\frac{2n-1}{3}}-\left(\frac{2}{r}\right)^{-\frac{2n-1}{3}}}{2^{\frac{2n-1}{3}}-2^{-\frac{2n-1}{3}}}\frac{2\left(4(-1)^n+(2n-1)\pi-8\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right)}{(2n-1)^2\pi^2}+\frac{(r)^{\frac{2n-1}{3}}-(r)^{-\frac{2n-1}{3}}}{2^{\frac{2n-1}{3}}-2^{-\frac{2n-1}{3}}}\frac{8\cos^2\left(\frac{n\pi}{4}+\frac{3\pi}{8}\right)}{(2n-1)\pi}\right)\sin\left(\frac{2n-1}{3}\theta\right)}$$

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۸) معادله لاپلاس زیر را حل کنید.

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\qquad r\le 2,\qquad 0\le\theta\le 2\pi$$
$$u(2,\theta)=\Lambda\left(\frac{\theta-\pi}{2\pi}\right)$$

از تغییر متغیر $r=e^x$ استفاده می‌کنیم.

$$\Rightarrow \hat{u}_{xx}+\hat{u}_{\theta\theta}=0$$

$$\Rightarrow \frac{X''(x)}{X(x)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}=k^2\Rightarrow\begin{cases}X_k(x)=\begin{cases}c_k e^{-kx}+d_k e^{kx}, & k\neq 0\\ c_0x+d_0, & k=0\end{cases}\\ \Theta_k(\theta)=\begin{cases}a_k\cos(k\theta)+b_k\sin(k\theta), & k\neq 0\\ a_0+b_0\theta, & k=0\end{cases}\end{cases}$$

$$\Rightarrow u(r,\theta)=c_0\ln(r)+d_0+\sum_{n=1}^{\infty}(c_nr^{-n}+d_nr^n)(a_n\cos(n\theta)+b_n\sin(n\theta))$$

با توجه به این که $r\le 2$، نقطه $r=0$ در دامنه است و با توجه به آن $c_0=0$ و $c_n=0$.

$$\Rightarrow u(r,\theta)=d_0+\sum_{n=1}^{\infty}r^n(a_n\cos(n\theta)+b_n\sin(n\theta))$$

$$u(2,\theta)=d_0+\sum_{k=1}^{\infty}2^n(a_n\cos(n\theta)+b_n\sin(n\theta))=\Lambda\left(\frac{\theta-\pi}{2\pi}\right)$$

$$\Rightarrow d_0=\frac{1}{2\pi}\int_0^{2\pi}\Lambda\left(\frac{\theta-\pi}{2\pi}\right)d\theta=\frac{3}{4}$$

$$\Rightarrow 2^na_n=\frac{2}{2\pi}\int_0^{2\pi}\Lambda\left(\frac{\theta-\pi}{2\pi}\right)\cos(n\theta)\,d\theta=\frac{(-1)^n-1}{n^2\pi^2}\Rightarrow a_n=\frac{(-1)^n-1}{n^2\pi^2}2^{-n}$$

$$\Rightarrow 2^nb_n=\frac{2}{2\pi}\int_0^{2\pi}\Lambda\left(\frac{\theta-\pi}{2\pi}\right)\sin(n\theta)\,d\theta=0\Rightarrow b_n=0$$

$$\Rightarrow \boxed{u(r,\theta)=\frac{3}{4}+\sum_{n=1}^{\infty}\left(\frac{r}{2}\right)^n\frac{(-1)^n-1}{n^2\pi^2}\cos(n\theta)}$$

۹) معادله لاپلاس زیر را حل کنید. (امتیازی)

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\qquad r_0\le r\le r_1,\qquad 0\le\theta\le\frac{\pi}{4}$$
$$u(r,0)=0,\qquad u\left(r,\frac{\pi}{4}\right)=0$$
$$u(r_0,\theta)=\theta\left(\frac{\pi}{4}-\theta\right),\qquad u(r_1,\theta)=\theta^2$$

از تغییر متغیر $r=e^x$ استفاده می‌کنیم.

$$u(r,\theta)=\hat{u}(x,\theta)\Longrightarrow \hat{u}_{xx}+\hat{u}_{\theta\theta}=0$$
$$\text{BC: }\begin{cases}u(r,0)=0\\ u\left(r,\frac{\pi}{4}\right)=0\end{cases}\Longrightarrow \hat{u}(x,\theta)=\sum_{n=1}^{\infty}X_n(x)\sin(4n\theta)$$
$$\Longrightarrow \sum_{n=1}^{\infty}\left(X_n''(x)-16n^2X_n(x)\right)\sin(4n\theta)=0$$
$$\Longrightarrow X_n''(x)-16n^2X_n(x)=0$$
$$\Longrightarrow X_n(x)=c_ne^{4nx}+d_ne^{-4nx}$$
$$\Longrightarrow \hat{u}(x,\theta)=\sum_{n=1}^{\infty}(c_ne^{4nx}+d_ne^{-4nx})\sin(4n\theta)$$
$$\Longrightarrow u(r,\theta)=\sum_{n=1}^{\infty}(c_nr^{4n}+d_nr^{-4n})\sin(4n\theta)$$
$$u(r_0,\theta)=\sum_{n=1}^{\infty}(c_nr_0^{4n}+d_nr_0^{-4n})\sin(4n\theta)=\theta\left(\frac{\pi}{4}-\theta\right)$$
$$\Longrightarrow c_nr_0^{4n}+d_nr_0^{-4n}=\frac{2}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\theta\left(\frac{\pi}{4}-\theta\right)\sin(4n\theta)\,d\theta=\frac{1-(-1)^n}{4n^3\pi}$$
$$u(r_1,\theta)=\sum_{n=1}^{\infty}(c_nr_1^{2n}+d_nr_1^{-2n})\sin(4n\theta)=\theta^2$$
$$\Longrightarrow c_nb^{2n}+d_nb^{-2n}=\frac{2}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\theta^2\sin(4n\theta)\,d\theta=\frac{(-1)^{n+1}(n^2\pi^2-2)-2}{8n^3\pi}$$
$$\Longrightarrow \boxed{u(r,\theta)=\sum_{n=1}^{\infty}\left(\frac{\left(\frac{r_1}{r}\right)^{4n}-\left(\frac{r_1}{r}\right)^{-4n}}{\left(\frac{r_1}{r_0}\right)^{4n}-\left(\frac{r_1}{r_0}\right)^{-4n}}\frac{1-(-1)^n}{4n^3\pi}+\frac{\left(\frac{r}{r_0}\right)^{4n}-\left(\frac{r}{r_0}\right)^{-4n}}{\left(\frac{r_1}{r_0}\right)^{4n}-\left(\frac{r_1}{r_0}\right)^{-4n}}\frac{(-1)^{n+1}(n^2\pi^2-2)-2}{8n^3\pi}\right)\sin(4n\theta)}$$