دانشگاه تهران- دانشکده مهندسی برق و کامپیوتر

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تمرین ۱: سری فوریه

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۱)

$$x(t) \xrightarrow{\text{periodic}} \int_{T}^{2T} x(t)dt = \int_{0}^{T} x(t)dt = 2 \Longrightarrow a_0 = \frac{1}{T}\int_{0}^{T} x(t)dt = \frac{2}{T}$$

$$g(t) = \frac{dx(t)}{dt} \Longrightarrow \sum_{k=-\infty}^{\infty} b_k\, e^{jk\frac{2\pi}{T}t} = \frac{dx(t)}{dt} = \frac{d}{dt}\sum_{k=-\infty}^{\infty} a_k\, e^{jk\frac{2\pi}{T}t}$$

$$\Longrightarrow \sum_{k=-\infty}^{\infty} b_k\, e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} a_k \left(jk\frac{2\pi}{T}\right) e^{jk\frac{2\pi}{T}t}$$

$$\Longrightarrow \boxed{a_k = \begin{cases} \dfrac{2}{T} & k = 0 \\ \dfrac{b_k}{jk\frac{2\pi}{T}} & k \neq 0 \end{cases}}$$

۲)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-jnx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x^2}{4}e^{-jnx}\,dx$$

$$\int x^2 e^{-jnx}dx = -\frac{e^{-jnx}}{jn}x^2 + \frac{2}{jn}\int xe^{-jnx}dx = -\frac{e^{-jnx}}{jn}x^2 + \frac{2}{jn}\left(-\frac{e^{-jnx}}{jn}x + \frac{1}{jn}\int e^{-jnx}dx\right)$$

$$= -\frac{e^{-jnx}}{jn}x^2 + \frac{2e^{-jnx}}{n^2} + \frac{2e^{-jnx}}{jn^3}$$

$$c_n = \frac{1}{8\pi}\left[-\frac{e^{-jnx}}{jn}x^2 + \frac{2e^{-jnx}}{n^2}x + \frac{2e^{-jnx}}{jn^3}\right]_{-\pi}^{\pi} = \frac{(-1)^n}{2n^2} \qquad (n \neq 0)$$

$$c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{4}dx = \left.\frac{x^3}{24}\right|_{-\pi}^{\pi} = \frac{\pi^2}{12}$$

$$\Longrightarrow \boxed{f(x) = \frac{\pi^2}{12} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n^2}e^{jnx}}$$

$$x = \pi \Longrightarrow f(\pi) = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{12} + A$$

$$f(\pi) = \frac{f(\pi^+) + f(\pi^-)}{2} = \frac{\pi^2}{4} \Longrightarrow \boxed{A = \frac{\pi^2}{6}}$$

$$x = 0 \Longrightarrow f(0) = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots = \frac{\pi^2}{12} - B \Longrightarrow \boxed{B = \frac{\pi^2}{12}}$$

۳)

$$c_n = \frac{1}{T}\int_T f(x)e^{-jnx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{ax} + e^{-ax}}{2} e^{-jnx}\, dx = \frac{1}{4\pi}\int_{-\pi}^{\pi} \left(e^{(a-jn)x} + e^{-(a+jn)x}\right)dx$$

$$= \frac{1}{4\pi}\left[\frac{e^{(a-jn)x}}{a-jn} - \frac{e^{-(a+jn)x}}{a+jn}\right]_{-\pi}^{\pi} = \frac{1}{4\pi}\left(\frac{(e^{a\pi} - e^{-a\pi})(-1)^n}{a-jn} + \frac{(e^{a\pi} - e^{-a\pi})(-1)^n}{a+jn}\right)$$

$$= \frac{a\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)}$$

$$\Longrightarrow \cosh(ax) = \sum_{n=-\infty}^{\infty} \frac{a\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)} e^{-jnx}$$

$$\frac{d\cosh(ax)}{dx} = \sinh(ax) = -j\sin(jax) = -j\sum_{n=-\infty}^{\infty} \frac{an\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)} e^{-jnx}$$

$$\Longrightarrow \boxed{\sin(jax) = \sum_{n=-\infty}^{\infty} \frac{an\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)} e^{-jnx}}$$

۴)

$$\sin^4(x) = \left(\frac{e^{jx} - e^{-jx}}{2j}\right)^4 = \frac{e^{4jx} - 4e^{2jx} + 6 - 4e^{-2jx} + e^{-4jx}}{16}$$

$$\frac{1}{T}\int_T |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} sin^8(x)dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$= \frac{1}{\pi}\int_0^{\pi} sin^8(x)dx = \left|\frac{1}{16}\right|^2 + \left|\frac{-4}{16}\right|^2 + \left|\frac{6}{16}\right|^2 + \left|\frac{-4}{16}\right|^2 + \left|\frac{1}{16}\right|^2 = \frac{35}{128}$$

$$\int_0^{\pi} sin^8(x)dx = \frac{35\pi}{128}$$

۵)

$$f(x) = e^{-|x|}\cos(20\pi x) \qquad -2 < x < 2 \qquad T = 4$$

$$c_{n=} \frac{1}{8}\left(\int_{-2}^{0} e^{x}\left(e^{j20\pi x} + e^{-j20\pi x}\right)e^{-j\frac{n\pi}{2}x}dx + \int_{0}^{2} e^{-x}\left(e^{j20\pi x} + e^{-j20\pi x}\right)e^{-j\frac{n\pi}{2}x}dx\right)$$

$$= \frac{1}{8}\left[\frac{e^{\left(1+20j\pi-j\frac{n\pi}{2}\right)x}}{1+20j\pi-j\frac{n\pi}{2}} + \frac{e^{\left(1-20j\pi-j\frac{n\pi}{2}\right)x}}{1-20j\pi-j\frac{n\pi}{2}}\right]_{-2}^{0} + \frac{1}{8}\left[\frac{e^{\left(-1+20j\pi-j\frac{n\pi}{2}\right)x}}{-1+20j\pi-j\frac{n\pi}{2}} + \frac{e^{\left(-1-20j\pi-j\frac{n\pi}{2}\right)x}}{-1-20j\pi-j\frac{n\pi}{2}}\right]_{0}^{2}$$

$$= \frac{1}{8}\left(\frac{1-e^{-2}(-1)^n}{1+20j\pi-j\frac{n\pi}{2}} + \frac{1-e^{-2}(-1)^n}{1-20j\pi-j\frac{n\pi}{2}}\right) + \frac{1}{8}\left(\frac{e^{-2}(-1)^n-1}{-1+20j\pi-j\frac{n\pi}{2}} + \frac{e^{-2}(-1)^n-1}{-1-20j\pi-j\frac{n\pi}{2}}\right)$$

$$= \frac{1}{8}\left(\frac{(1-e^{-2}(-1)^n)(2-jn\pi)}{1-jn\pi-\frac{n^2\pi^2}{4}+400\pi^2}\right) + \frac{1}{8}\left(\frac{(1-e^{-2}(-1)^n)(2+jn\pi)}{1+jn\pi-\frac{n^2\pi^2}{4}+400\pi^2}\right)$$

$$= \frac{(1-e^{-2}(-1)^n)\left((2-jn\pi)\left(1+jn\pi-\frac{n^2\pi^2}{4}+400\pi^2\right)+(2+jn\pi)\left(1-jn\pi-\frac{n^2\pi^2}{4}+400\pi^2\right)\right)}{8\left(1-jn\pi-\frac{n^2\pi^2}{4}+400\pi^2\right)\left(1+jn\pi-\frac{n^2\pi^2}{4}+400\pi^2\right)}$$

$$= \frac{(1-e^{-2}(-1)^n)(4+\pi^2(n^2+1600))}{8\left(1+\pi^2\left(800+\frac{n^2}{2}\right)+\pi^4\left(160000-200n^2+\frac{n^4}{16}\right)\right)}$$

$$\Longrightarrow \boxed{f(x)=\frac{1}{8}\sum_{n=-\infty}^{\infty}\frac{(1-e^{-2}(-1)^n)(4+\pi^2(n^2+1600))}{\left(1+\pi^2\left(800+\frac{n^2}{2}\right)+\pi^4\left(160000-200n^2+\frac{n^4}{16}\right)\right)}e^{-j\frac{n\pi}{2}x}}$$

$$y=\sum_{n=-\infty}^{\infty} C_n\, e^{\frac{jn\pi}{2}x}$$

$$y'=\sum_{n=-\infty}^{\infty}\frac{jn\pi}{2}C_n\, e^{\frac{jn\pi}{2}x}$$

$$y'+2y=e^{-|x|}\cos(20\pi x)$$

$$\Longrightarrow \sum_{n=-\infty}^{\infty}\frac{jn\pi}{2}C_n\, e^{\frac{jn\pi}{2}x}+2\sum_{n=-\infty}^{\infty}C_n\, e^{\frac{jn\pi}{2}x}=\sum_{n=-\infty}^{\infty}\left(\frac{jn\pi}{2}C_n+2C_n\right)e^{\frac{jn\pi}{2}x}$$

$$=\frac{1}{8}\sum_{n=-\infty}^{\infty}\frac{(1-e^{-2}(-1)^n)(4+\pi^2(n^2+1600))}{\left(1+\pi^2\left(800+\frac{n^2}{2}\right)+\pi^4\left(160000-200n^2+\frac{n^4}{16}\right)\right)}e^{-j\frac{n\pi}{2}x}$$

$$\Longrightarrow C_n=\frac{(1-e^{-2}(-1)^n)(4+\pi^2(n^2+1600))}{4\left(1+\pi^2\left(800+\frac{n^2}{2}\right)+\pi^4\left(160000-200n^2+\frac{n^4}{16}\right)\right)(jn\pi+4)}$$

$$\Longrightarrow \boxed{y=\frac{1}{4}\sum_{n=-\infty}^{\infty}\frac{(1-e^{-2}(-1)^n)(4+\pi^2(n^2+1600))}{\left(1+\pi^2\left(800+\frac{n^2}{2}\right)+\pi^4\left(160000-200n^2+\frac{n^4}{16}\right)\right)(jn\pi+4)}e^{-j\frac{n\pi}{2}x}}$$

۶)

$$f(x) = \sinh(ax) \qquad -\pi < x < \pi \qquad a > 0$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-jnx}dx \Longrightarrow c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sinh(ax)\, e^{-jnx}dx$$

$$\sinh(ax) = \frac{e^{ax} - e^{-ax}}{2}$$

$$\Longrightarrow c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{ax} - e^{-ax}}{2}\right)e^{-jnx}dx = \frac{1}{4\pi}\left[\frac{e^{(a-jn)x}}{a-jn} + \frac{e^{-(a+jn)x}}{a+jn}\right]_{-\pi}^{\pi} = \frac{jn\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)}$$

اکنون برای به دست آوردن ضرایب سری فوریه حقیقی از روابط تعریف شده در درس استفاده می‌کنیم:

$$C_{-n} = \frac{-jn\sinh(a\pi)}{(-1)^n\pi(a^2+n^2)}$$

$$\Longrightarrow \boxed{\begin{cases} a_n = C_n + C_{-n} = \dfrac{jn\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)} + \dfrac{-jn\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)} = 0 \\ b_n = \dfrac{C_n - C_{-n}}{j} = \dfrac{n\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)} + \dfrac{n\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)} = \dfrac{2n\sinh(a\pi)\,(-1)^n}{\pi(a^2+n^2)} \end{cases}}$$

۷)

$$f(x) = \sin^4(\pi x)\cos(2\pi x) = \left(\frac{e^{j\pi x} - e^{-j\pi x}}{2j}\right)^4 \left(\frac{e^{j2\pi x} + e^{-j2\pi x}}{2}\right) =$$

$$= \left(\frac{e^{4jx} - 4e^{2jx} + 6 - 4e^{-2jx} + e^{-4jx}}{16}\right)\left(\frac{e^{j2\pi x} + e^{-j2\pi x}}{2}\right)$$

$$= \frac{1}{32}\left(e^{j6\pi x} - 4e^{j4\pi x} + 7e^{j2\pi x} + e^{-j6\pi x} - 4e^{-j4\pi x} + 7e^{-j2\pi x} - 8\right)$$

با توجه به فرم کلی سری فوریه مختلط توابع:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{jnx} \Longrightarrow C_0 = -\frac{8}{32}, \qquad C_2 = C_{-2} = \frac{7}{32}, \qquad C_4 = C_{-4} = -\frac{4}{32},$$

$$C_6 = C_{-6} = \frac{1}{32}, \qquad C_n = 0 \text{ for } n \notin \{-6,-4,-2,0,2,4,6\}$$

۸)

$$a_0 = \frac{1}{T}\int_T f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(x)\,dx = \frac{1}{2\pi}[\sin(x) - x\cos(x)]_{-\pi}^{\pi} = 1$$

$$\begin{aligned}
a_n &= \frac{2}{T}\int_T f(x)\cos\left(\frac{2n\pi}{T}x\right)dx = \frac{2}{\pi}\int_0^{\pi} x\sin(x)\cos(nx)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} x\big(\sin((n+1)x) - \sin((n-1)x)\big)\,dx \\
&= \frac{1}{\pi}\left[x\left(-\frac{\cos(nx+x)}{n+1} + \frac{\cos(nx-x)}{n-1}\right) + \left(\frac{\sin(nx+x)}{(n+1)^2} - \frac{\sin(nx-x)}{(n-1)^2}\right)\right]_0^{\pi}
\end{aligned}$$

$$\text{when } n \neq 1 \Longrightarrow a_n = -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = \frac{2(-1)^{n-1}}{n^2-1}$$

$$\begin{aligned}
\text{when } n = 1 \Longrightarrow a_1 &= \frac{2}{\pi}\int_0^{\pi} x\sin(x)\cos(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin(2x)\,dx \\
&= \frac{1}{\pi}\left[x\left(-\frac{\cos(2x)}{2}\right) + \left(\frac{\sin(2x)}{4}\right)\right]_0^{\pi} = -\frac{1}{2}
\end{aligned}$$

$$b_n = 0$$

$$\Longrightarrow \boxed{f(x) = 1 - \frac{1}{2}\cos(x) + \sum_{n=2}^{\infty}\frac{2(-1)^{n-1}}{n^2-1}\cos(nx)}$$

$$\begin{aligned}
\text{Parseval} \to\ 2(1)^2 + \frac{1}{4} + 4\left(\frac{1}{9} + \frac{1}{64} + \frac{1}{225} + \cdots\right) &= \frac{9}{4} + 4\sum_{k=2}^{\infty}\frac{1}{(k^2-1)^2} \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(x\sin(x))^2\,dx
\end{aligned}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(x\sin(x))^2\,dx = \frac{2\pi^2-3}{6}$$

$$\Longrightarrow A = \sum_{n=2}^{\infty}\frac{1}{(k^2-1)^2} = \frac{1}{4}\left(\frac{2\pi^2-3}{6} - \frac{9}{4}\right) = \frac{1}{4}\left(\frac{8\pi^2-66}{24}\right) = \boxed{\frac{4\pi^2-33}{48}}$$

۹)

نکته: اگر ضرایب سری فوریه تابعی برابر با C_n باشد، ضریب سری فوریه شیفت یافته به اندازه a همان تابع از رابطه $e^{-\frac{j2n\pi a}{T_0}}C_n$ به دست می‌آید.

اثبات:

$$f(x)=\sum_{n=-\infty}^{\infty}C_n\,e^{\frac{j2\pi nx}{T_0}} \Longrightarrow f(x-a)=\sum_{n=-\infty}^{\infty}C_n\,e^{\frac{j2\pi n(x-a)}{T_0}}=\sum_{n=-\infty}^{\infty}e^{-\frac{j2n\pi a}{T_0}}C_n\,e^{\frac{j2\pi nx}{T_0}}$$

$$\Longrightarrow C_n'=e^{-\frac{j2n\pi a}{T_0}}C_n$$

هم چنین، اگر ضرایب سری فوریه تابعی برابر با C_n باشد، ضریب سری فوریه قرینه تابع از رابطه زیر به دست می‌آید:

$$f(x)=\sum_{n=-\infty}^{\infty}C_n\,e^{\frac{j2\pi nx}{T_0}} \Longrightarrow f(-x)=\sum_{n=-\infty}^{\infty}C_n\,e^{-\frac{j2\pi nx}{T_0}} \to C'_n=C_{-n}$$

در اینجا داریم:

$$g(x)=f(x-1)+f(1-x)$$

$$f(x-1): C_n'=e^{\frac{-j2n\pi}{T_0}}C_n$$

$$f(1-x): C_n''=e^{\frac{-j2n\pi}{T_0}}C_{-n}$$

بنابراین با توجه به خطی بودن سری فوریه، ضریب سری فوریه تابع $g(x)$ از حاصل جمع ضرایب سری فوریه توابع تشکیل دهنده آن به دست می‌آید، بنابراین اگر ضرایب سری فوریه $g(x)$ را با D_n نمایش دهیم داریم:

$$D_n=C_n'+C_n''=e^{\frac{-j2n\pi}{T_0}}(C_n+C_{-n})$$