۱)

الف)

$$f(x) = \begin{cases} \sin(\pi x), & 0 \le x \le 2 \\ 0, & 2 < x \le 4 \end{cases} \qquad T = 4$$

$$c_n = \frac{1}{4}\int_0^2 \sin \pi x \, e^{-j\frac{n\pi}{2}x}\, dx = \frac{1}{8j}\int_0^2 \left(e^{j\left(\pi-\frac{n\pi}{2}\right)x} - e^{-j\left(\pi+\frac{n\pi}{2}\right)x}\right) dx = \frac{(-1)^n-1}{\left(\pi-\pi\frac{n^2}{4}\right)} \qquad n \neq 2,-2$$

$$c_2 = \frac{1}{4}\int_0^2 \sin \pi x \, e^{-j\pi x}\, dx = \frac{1}{4j}$$

$$c_{-2} = \frac{1}{4}\int_0^2 \sin \pi x \, e^{+j\pi x}\, dx = \frac{-1}{4j}$$

ب)

$$g(x) = x^2 \qquad -1 < x < 1 \qquad T = 2$$

$$c_n = \frac{1}{2}\int_{-1}^{1} x^2 e^{-jn\pi x}\, dx = \frac{1}{2}\left(\left(\frac{-x^2}{jn\pi}e^{-jn\pi}\right)\Big|_{-1}^{1} + \frac{2}{jn\pi}\int_{-1}^{1} x e^{-jn\pi x}\, dx\right)$$

$$= \frac{1}{jn\pi}\int_{-1}^{1} x e^{-jn\pi x}\, dx = \frac{1}{jn\pi}\left(\left(\frac{-x}{jn\pi}e^{-jn\pi}\right)\Big|_{-1}^{1} + \frac{1}{jn\pi}\int_{-1}^{1} e^{-jn\pi x}\, dx\right) = \frac{2(-1)^n}{\pi^2 n^2} \qquad n \neq 0$$

$$c_0 = \frac{1}{2}\int_{-1}^{1} x^2\, dx = \frac{x^3}{6}\Big|_{-1}^{1} = \frac{1}{3}$$

ج)

$$h(x) = e^{-|x|}\cos(20\pi x) \qquad -2 < x < 2 \qquad T = 4$$

$$c_n = \frac{1}{8}\left(\int_{-2}^{0} e^{x}\left(e^{j20\pi x} + e^{-j20\pi x}\right)e^{-j\frac{n\pi}{2}x}dx + \int_{0}^{2} e^{-x}\left(e^{j20\pi x} + e^{-j20\pi x}\right)e^{-j\frac{n\pi}{2}x}dx \right)$$

$$= \frac{1}{8}\left(\frac{e^{\left(1+20j\pi - j\frac{n\pi}{2}\right)x}}{\left(1+20j\pi - j\frac{n\pi}{2}\right)} + \frac{e^{\left(1-20j\pi - j\frac{n\pi}{2}\right)x}}{\left(1-20j\pi - j\frac{n\pi}{2}\right)} \right)\Big|_{-2}^{0}$$

$$+ \frac{1}{8}\left(\frac{e^{\left(-1+20j\pi - j\frac{n\pi}{2}\right)x}}{\left(-1+20j\pi - j\frac{n\pi}{2}\right)} + \frac{e^{\left(-1-20j\pi - j\frac{n\pi}{2}\right)x}}{\left(-1-20j\pi - j\frac{n\pi}{2}\right)} \right)\Big|_{0}^{2}$$

$$= \frac{(e^{-2}+1)}{2(1 + 400\pi^2 + \frac{n^2\pi^2}{4} - 20\pi^2 n)}$$

۲)

الف)

$$T = 4 \Rightarrow \omega_0 = \frac{k\pi}{2}$$

$$c_n = \begin{cases} jk & , \quad |k| < 3 \\ 0 & , \quad \text{در غیر این صورت} \end{cases}$$

$$f(x) = -2je^{-j\pi x} + -je^{-j\frac{\pi}{2}x} + 0 + je^{j\frac{\pi}{2}x} + 2je^{j\pi x} = -2\sin\left(\frac{\pi}{2}x\right) - 4\sin(\pi x)$$

ب)

$$c_k = \begin{cases} 0 & , \quad k = 0 \\ \dfrac{2}{k\pi}\sin\left(\dfrac{9k\pi}{40}\right)\cos\left(\dfrac{k\pi}{40}\right) & , \quad \text{در غیر این صورت} \end{cases}$$

$$c_k = \begin{cases} 0 & , \quad k = 0 \\ \dfrac{\sin\left(\frac{k\pi}{4}\right) + \sin\left(\frac{k\pi}{5}\right)}{k\pi} & , \quad \text{در غیر این صورت} \end{cases}$$

$$c_k = c_1 + c_2$$

$$c_{1k} = \begin{cases} 0 & , \quad k = 0 \\ \dfrac{\sin\left(\frac{k\pi}{4}\right)}{k\pi} & , \quad \text{در غیر این صورت} \end{cases}$$

با توجه به اسلاید ۱۷۵ می‌دانیم :

$$\mathrm{f(x)} = \begin{cases} A, & \mathrm{x} < \left|\dfrac{d}{2}\right| \\ 0, & \left|\dfrac{d}{2}\right| < \mathrm{x} < \left|\dfrac{T}{2}\right| \end{cases} \quad \Leftrightarrow \quad c_k = \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\frac{n\pi d}{T}}$$

پس داریم:

$$\mathrm{f}_1(\mathrm{x}) = \begin{cases} 1, & \mathrm{x} < \left|\dfrac{1}{2}\right| \\ 0, & \left|\dfrac{1}{2}\right| < \mathrm{x} < |2| \end{cases} \quad \Leftrightarrow \quad c_k = \frac{\sin\left(\frac{k\pi}{4}\right)}{k\pi} \qquad c_0 = \frac{1}{4}$$

با جمع کردن این تابع با تابعی که تنها افست متفاوتی با این تابع دارد می توان مقدار c_0 را صفر کرد:

$$g_1(x) = f_1(x) - \frac{1}{2} \qquad \Leftrightarrow \qquad c_k = \frac{\sin\left(\frac{k\pi}{4}\right)}{k\pi} \qquad c_0 = -\frac{1}{4}$$

$$h_1(x) = \frac{f_1(x)+g_1(x)}{2} \quad \Leftrightarrow \quad c_{1k}$$

به همین ترتیب برای c_{2k} داریم:

$$c_{2k} = \begin{cases} 0 & , \quad k = 0 \\ \dfrac{\sin\left(\frac{k\pi}{5}\right)}{k\pi} & , \quad \text{در غیر این صورت} \end{cases}$$

$$\mathrm{f}_2(\mathrm{x}) = \begin{cases} 1, & \mathrm{x} < \left|\frac{4}{5}\right| \\ 0, & \left|\frac{4}{5}\right| < \mathrm{x} < |2| \end{cases} \qquad \Leftrightarrow \qquad c_k = \frac{\sin\left(\frac{k\pi}{5}\right)}{k\pi} \qquad c_0 = \frac{2}{5}$$

با جمع کردن این تابع با تابعی که تنها افست متفاوتی با این تابع دارد می توان مقدار c_0 را صفر کرد:

$$g_2(x) = f_2(x) - \frac{4}{5} \qquad \Leftrightarrow \qquad c_k = \frac{\sin\left(\frac{k\pi}{4}\right)}{k\pi} \qquad c_0 = -\frac{2}{5}$$

$$h_2(x) = \frac{f_2(x)+g_2(x)}{2} \quad \Leftrightarrow \quad c_{2k}$$

۳)

الف)

$$(t) = \sum_{k}^{odd} C_k e^{jk\frac{2\pi t}{T}}$$

$$x\left(t+\frac{T}{2}\right) = \sum_{k}^{odd} C_k e^{jk\frac{2\pi\left(t+\frac{T}{2}\right)}{T}} = = \sum_{k}^{odd} C_k e^{jk\frac{2\pi t}{T}} e^{jk\pi}$$

As we know: $e^{jk\pi} = -1 \rightarrow \quad x(t+T/2) = -x(t)$

ب)

$$c_k = \frac{1}{T}\int_0^{T/2} x(t)e^{-jkw_0 t}\,dt + \frac{1}{T}\int_{\frac{T}{2}}^{T} x(t)e^{-jkw_0 t}\,dt$$

$$= \frac{1}{T}\int_0^{T/2} \left(x(t) + x\left(t+\frac{T}{2}\right)e^{-jk\pi}\right)e^{-jkw_0 t}\,dt$$

$$= \frac{1}{T}\int_0^{T/2} \left(x(t) + x\left(t+\frac{T}{2}\right)(-1)^k\right)e^{-jkw_0 t}\,dt$$ $\rightarrow$ if $k \in even \rightarrow c_k = 0$

۴)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-jnx}\,dx$$

$$c_n = \frac{1}{2\pi}\left(\int_{0}^{\pi} e^{(a-jn)x}\,dx + \int_{-\pi}^{0} e^{-(a+jn)x}\,dx\right)$$

$$c_n = \frac{1}{2\pi}\left[\frac{1}{a-jn}\left(e^{(a-jn)\pi}-1\right) + \frac{1}{a+jn}\left(e^{(a+jn)\pi}-1\right)\right]$$

$$c_n = \frac{1}{2\pi(a^2+n^2)}\left[(a+jn)e^{(a-jn)\pi} - (a+jn) + (a-jn)e^{(a+jn)\pi} - (a-jn)\right]$$

As you know $e^{jn\pi} = (-1)^n$ so we have:

$$c_n = \frac{1}{2\pi\,(a^2+n^2)}\left(2a(-1)^n e^{a\pi} - 2a\right)$$

$$\rightarrow c_n = \frac{a}{\pi(a^2+n^2)}\left((-1)^n e^{a\pi} - 1\right)$$

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^n e^{5\pi}-1}{n^2+25}\right]^2 = Y \qquad (a=5)$$

از رابطه پارسوال می دانیم

$$\frac{1}{T}\int_{T} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = c_0^2 + 2\sum_{n=1}^{\infty} |c_n|^2$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2a|x|}\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} e^{-2ax}\,dx + \int_{0}^{\pi} e^{2ax}\,dx\right) = c_0^2 + 2\sum_{n=1}^{\infty}(|c_n|)^2$$

$$= \frac{1}{10\pi}(e^{10\pi} - 1) = \frac{\left(e^{5\pi}-1\right)^2}{25\pi^2} + \frac{50}{\pi^2}Y$$

$$\Rightarrow Y = \frac{e^{10\pi}(2.5\pi-1)+2e^{5\pi}-(2\cdot 5\pi+1)}{1250}$$

۵)

$$\sin^3(x) = \left(\frac{e^{jx} - e^{-jx}}{2j}\right)^3 = \frac{e^{3jx} - 3e^{jx} + 3e^{-jx} - e^{-3jx}}{-8j}$$

$$\frac{1}{T}\int_T |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} sin^6(x)dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$= \frac{1}{\pi}\int_0^{\pi} sin^6(x)dx = \left(\left|\frac{1}{-8j}\right|\right)^2 + \left(\left|\frac{3}{8j}\right|\right)^2 + \left(\left|\frac{-3}{8j}\right|\right)^2 + \left(\left|\frac{1}{8j}\right|\right)^2$$

$$\Rightarrow \int_0^{\pi} sin^6(x)dx = \frac{5\pi}{16}$$

۶)

$$f'(x) = \frac{\sinh(a\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n . in}{(a-in)} e^{inx}$$

$$A = \int_{-\pi}^{\pi} f'(x)e^{2ix}dx + \int_{-\pi}^{\pi} |f(x)|^2 dx$$

$$= 2\pi \times \frac{\sinh(a\pi)}{\pi} \times \frac{(-1)^{-2}.(-2)i}{(a-i(-2))} + 2\pi \times \sum_{n=-\infty}^{\infty} \left|\frac{\sinh(a\pi)}{\pi(a-in)}\right|^2$$

$$= 2\pi \times \frac{\sinh(a\pi)}{\pi} \times \frac{-2i}{(a+2i)} + 2\pi \times \sum_{n=-\infty}^{\infty} \frac{sinh^2(a\pi)}{\pi^2(a^2+n^2)}$$

۷)

$$f(x,y) = x^2y - xy^2 \qquad\qquad 0 < x < \pi \quad , \quad 0 < y < \pi$$

$$f(x,y) = A_0(y) + \sum_{n=1}^{\infty} [A_n(y)\cos(2nx) + B_n(y)\sin(2nx)]$$

$$A_0(y) = \frac{1}{\pi}\int_0^{\pi} x^2y - xy^2\,dx = \frac{\pi^2}{3}y - \frac{\pi}{2}y^2$$

$$A_n(y) = \frac{2}{\pi}\int_0^{\pi} (x^2y - xy^2)\cos(2nx)\,dx$$

$$= \frac{2}{\pi}\left(\int_0^\pi x^2 y\cos(2nx)\,dx + \int_0^\pi xy^2\cos(2nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(y\int_0^\pi x^2\cos(2nx)\,dx + y^2\int_0^\pi x\cos(2nx)\,dx\right)$$

با استفاده از روش جزء به جزء داریم:

$$A_n(y) = \frac{y}{n^2}$$

$$B_n(y) = \frac{2}{\pi}\int_0^\pi (x^2y - xy^2)\sin(2nx)\,dx$$

$$= \frac{2}{\pi}\left(\int_0^\pi x^2 y\sin(2nx)\,dx + \int_0^\pi xy^2\sin(2nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(y\int_0^\pi x^2\sin(2nx)\,dx + y^2\int_0^\pi x\sin(2nx)\,dx\right)$$

با استفاده از روش جزء به جزء داریم:

$$B_n(y) = \frac{-\pi y}{n} + \frac{y^2}{n}$$

$$A_0(y) = a_{00} + \sum_{m=1}^{\infty}[a_{m0}\cos(2my) + b_{m0}sin(2my)]$$

$$a_{00} = \frac{1}{\pi}\int_0^{\pi} \frac{\pi^2}{3}y - \frac{\pi}{2}y^2\, dy = 0$$

$$a_{m0} = \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi^2}{3}y - \frac{\pi}{2}y^2\right)\cos(2my)\, dy$$

با استفاده از روش جزء به جزء داریم:

$$a_{m0} = \frac{-\pi}{2m^2}$$

$$b_{m0} = \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi^2}{3}y - \frac{\pi}{2}y^2\right)\sin(2my)\, dy$$

با استفاده از روش جزء به جزء داریم:

$$b_{m0} = \frac{\pi^2}{6m}$$

$$A_n(y) = a_{0n} + \sum_{m=1}^{\infty}[a_{mn}\cos(2my) + b_{mn}\sin(2my)]$$

$$a_{0n} = \frac{1}{\pi}\int_0^{\pi} \frac{y}{n^2}\, dy = \frac{\pi}{2n^2}$$

$$a_{mn} = \frac{2}{\pi}\int_0^{\pi} \frac{y}{n^2}\cos(2my)\, dy$$

با استفاده از روش جزء به جزء داریم:

$$a_{mn} = 0$$

$$b_{mn} = \frac{2}{\pi}\int_0^{\pi} \frac{y}{n^2}\sin(2my)\, dy$$

با استفاده از روش جزء به جزء داریم:

$$b_{mn} = \frac{-1}{mn^2}$$

$$B_n(y) = c_{0n} + \sum_{m=1}^{\infty}\left[c_{mn}\cos(2my) + d_{mn}\sin(2my)\right]$$

$$c_{0n} = \frac{1}{\pi}\int_0^{\pi}\left(\frac{-\pi y}{n} + \frac{y^2}{n}\right)dy = \frac{-\pi^2}{6n}$$

$$c_{mn} = \frac{2}{\pi}\int_0^{\pi}\left(\frac{-\pi y}{n} + \frac{y^2}{n}\right)\cos(2my)\, dy$$

با استفاده از روش جزء به جزء داریم:

$$c_{mn} = \frac{1}{m^2 n}$$

$$d_{mn} = \frac{2}{\pi}\int_0^{\pi}\left(\frac{-\pi y}{n} + \frac{y^2}{n}\right)\sin(2my)\,dy$$

با استفاده از روش جزء به جزء داریم:

$$d_{mn} = 0$$

و در آخر ضرایب بدست آمده را در رابطه ی زیر جایگذاری می کنیم:

$$\begin{aligned} f(x,y) = a_{00} &+ \sum_{n=1}^{\infty}\left(a_{0n}\cos(2nx) + c_{0n}\sin(2nx)\right) \\ &+ \sum_{m=1}^{\infty}\left(a_{m0}\cos(2my) + b_{m0}\sin(2my)\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\big(a_{mn}\cos(2nx)\cos(2my) + b_{mn}\cos(2nx)\sin(2my) \\ &+ c_{mn}\sin(2nx)\cos(2my) + d_{mn}\sin(2nx)\sin(2my)\big) \end{aligned}$$

$$\begin{aligned} f(x,y) = \sum_{n=1}^{\infty}&\left(\frac{\pi}{2n^2}\cos(2nx) - \frac{\pi^2}{6n}\sin(2nx)\right) \\ &+ \sum_{m=1}^{\infty}\left(-\frac{\pi}{2m^2}\cos(2my) + \frac{\pi^2}{6m}\sin(2my)\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(-\frac{1}{mn^2}\cos(2nx)\sin(2my) + \frac{1}{m^2n}\sin(2nx)\cos(2my)\right) \end{aligned}$$