۱)

۱-۱) انتگرال فوریه تابع $f(x)$ را بدست آورید و با استفاده از آن حاصل انتگرال I_1 و I_2 را محاسبه کنید.

$$f(x)=\Pi\left(\frac{x}{2}\right), \qquad \Pi(x)=\begin{cases}1, & |x|\le \frac{1}{2}\\ 0, & \text{otherwise}\end{cases}, \qquad I_1=\int_{-\infty}^{+\infty}\frac{\sin(\omega)}{\omega}d\omega, \qquad I_2=\int_{-\infty}^{+\infty}\frac{\sin^2(\omega)}{\omega^2}d\omega$$

$$f(x)=f(-x)\Longrightarrow f(x)\text{ is even}\Longrightarrow B(\omega)=0$$

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty}f(x)\cos(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+\infty}f(x)\cos(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+\infty}\Pi\left(\frac{x}{2}\right)\cos(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+1}\cos(\omega x)\,dx=\frac{2\sin(\omega)}{\pi\omega}$$

$$\Longrightarrow \boxed{f(x)=\int_{0}^{\infty}\frac{2\sin(\omega)}{\pi\omega}\cos(\omega x)\,d\omega}$$

$$f(1)=\frac{f(1^+)+f(1^-)}{2}=\frac{1}{2}=\int_{0}^{\infty}\frac{2\sin(\omega)}{\pi\omega}\cos(\omega)\,d\omega=\int_{0}^{\infty}\frac{\sin(2\omega)}{\pi\omega}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin(2\omega)}{\omega}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin(\nu)}{\nu}d\nu$$

$$\Longrightarrow \boxed{I_1=\int_{-\infty}^{+\infty}\frac{\sin(\omega)}{\omega}d\omega=\pi}$$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}|f(x)|^2dx=\int_{0}^{+\infty}\left(A^2(\omega)+B^2(\omega)\right)d\omega$$

$$\text{Parseval: }\frac{1}{\pi}\int_{-\infty}^{+\infty}|f(x)|^2dx=\int_{0}^{+\infty}\left(A^2(\omega)+B^2(\omega)\right)d\omega\Longrightarrow\frac{1}{\pi}\int_{-\infty}^{+\infty}\left|\Pi\left(\frac{x}{2}\right)\right|^2dx=\frac{2}{\pi}\int_{0}^{+1}dx=\frac{2}{\pi}=\int_{0}^{+\infty}\frac{4\sin^2(\omega)}{\pi^2\omega^2}d\omega=\frac{2}{\pi^2}\int_{-\infty}^{+\infty}\frac{\sin^2(\omega)}{\omega^2}d\omega$$

$$\Longrightarrow \boxed{I_2=\int_{-\infty}^{+\infty}\frac{\sin^2(\omega)}{\omega^2}d\omega=\pi}$$

۱-۲) انتگرال فوریه تابع $f(x)$ را بدست آورید و با استفاده از آن حاصل انتگرال I_1 و I_2 را محاسبه کنید.

$$f(x)=\Pi(x), \qquad \Pi(x)=\begin{cases}1, & |x|\le \frac{1}{2}\\ 0, & \text{otherwise}\end{cases}, \qquad I_1=\int_{-\infty}^{+\infty}\frac{\sin^3(\omega)\cos(\omega)}{\omega}d\omega, \qquad I_2=\int_{-\infty}^{+\infty}\frac{\sin^4(\omega)}{\omega^2}d\omega$$

$$f(x)=-f(-x)\Longrightarrow f(x)\text{ is odd}\Longrightarrow A(\omega)=0$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty}f(x)\sin(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+\infty}f(x)\sin(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+\infty}\Pi\left(\frac{x}{2}\right)\sin(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+1}\sin(\omega x)\,dx=\frac{4\sin^2\left(\frac{\omega}{2}\right)}{\pi\omega}$$

$$\Longrightarrow \boxed{f(x)=\int_{0}^{\infty}\frac{4\sin^2\left(\frac{\omega}{2}\right)}{\pi\omega}\sin(\omega x)\,d\omega}$$

$$f(1)=\frac{f(1^+)+f(1^-)}{2}=\frac{1}{2}=\int_{0}^{\infty}\frac{4\sin^2\left(\frac{\omega}{2}\right)}{\pi\omega}\sin(\omega)\,d\omega=\int_{0}^{\infty}\frac{8\sin^3\left(\frac{\omega}{2}\right)\cos\left(\frac{\omega}{2}\right)}{\pi\omega}d\omega=\frac{4}{\pi}\int_{-\infty}^{+\infty}\frac{\sin^3\left(\frac{\omega}{2}\right)\cos\left(\frac{\omega}{2}\right)}{\omega}d\omega=\frac{4}{\pi}\int_{-\infty}^{+\infty}\frac{\sin^3(\nu)\cos(\nu)}{\nu}d\nu$$

$$\Longrightarrow \boxed{I_1=\int_{-\infty}^{+\infty}\frac{\sin^3(\omega)\cos(\omega)}{\omega}d\omega=\frac{\pi}{8}}$$

$$\text{Parseval: }\frac{1}{\pi}\int_{-\infty}^{+\infty}|f(x)|^2dx=\int_{0}^{+\infty}\left(A^2(\omega)+B^2(\omega)\right)d\omega\Longrightarrow\frac{1}{\pi}\int_{-\infty}^{+\infty}\left|\operatorname{sign}(x)\Pi\left(\frac{x}{2}\right)\right|^2dx=\frac{2}{\pi}\int_{0}^{+1}dx=\frac{2}{\pi}=\int_{0}^{+\infty}\frac{16\sin^4\left(\frac{\omega}{2}\right)}{\pi^2\omega^2}d\omega$$

$$=\frac{8}{\pi^2}\int_{-\infty}^{+\infty}\frac{\sin^4\left(\frac{\omega}{2}\right)}{\omega^2}d\omega=\frac{4}{\pi^2}\int_{-\infty}^{+\infty}\frac{\sin^4(\nu)}{\nu^2}d\nu$$

$$\Longrightarrow \boxed{I_2=\int_{-\infty}^{+\infty}\frac{\sin^4(\omega)}{\omega^2}d\omega=\frac{\pi}{2}}$$

۱-۳) انتگرال فوریه تابع $f(x)$ را بدست آورید و با استفاده از آن حاصل انتگرال I_1 و I_2 را محاسبه کنید.

$$f(x)=\cos(\pi x)\,\Pi\left(\frac{x}{2}\right),\qquad \Pi(x)=\begin{cases}1, & |x|\le \frac{1}{2}\\ 0, & \text{otherwise}\end{cases},\qquad I_1=\int_{-\infty}^{+\infty}\frac{\omega\sin(2\omega)}{\omega^2-\pi^2}d\omega,\qquad I_2=\int_{-\infty}^{+\infty}\frac{\omega^2\sin^2(\omega)}{(\omega^2-\pi^2)^2}d\omega$$

$$f(x)=f(-x)\Longrightarrow f(x)\text{ is even}\Longrightarrow B(\omega)=0$$

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty}f(x)\cos(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+\infty}f(x)\cos(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+\infty}\cos(\pi x)\,\Pi\left(\frac{x}{2}\right)\cos(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{1}\cos(\pi x)\cos(\omega x)\,dx$$

$$=\frac{1}{\pi}\left[\frac{\sin\big((\omega+\pi)x\big)}{\omega+\pi}+\frac{\sin\big((\omega-\pi)x\big)}{\omega-\pi}\right]_0^1=\frac{2\omega\sin(\omega)}{\pi^3-\pi\omega^2}$$

$$\Longrightarrow\boxed{f(x)=\int_0^{\infty}\frac{2\omega\sin(\omega)}{\pi^3-\pi\omega^2}\cos(\omega x)\,d\omega}$$

$$f(1)=\frac{f(1^+)+f(1^-)}{2}=-\frac{1}{2}=\int_0^{\infty}\frac{2\omega\sin(\omega)}{\pi^3-\pi\omega^2}\cos(\omega)\,d\omega=\frac{1}{\pi}\int_0^{\infty}\frac{\omega\sin(2\omega)}{\pi^2-\omega^2}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega\sin(2\omega)}{\pi^2-\omega^2}d\omega$$

$$\Longrightarrow\boxed{I_1=\int_{-\infty}^{+\infty}\frac{\omega\sin(2\omega)}{\omega^2-\pi^2}d\omega=\pi}$$

$$\text{Parseval: }\frac{1}{\pi}\int_{-\infty}^{+\infty}|f(x)|^2dx=\int_0^{+\infty}\big(A^2(\omega)+B^2(\omega)\big)d\omega\Longrightarrow\frac{1}{\pi}\int_{-\infty}^{+\infty}\left|\cos(\pi x)\,\Pi\left(\frac{x}{2}\right)\right|^2dx=\frac{2}{\pi}\int_0^{+1}|\cos(\pi x)|^2dx=\frac{1}{\pi}=\int_0^{+\infty}\frac{4\omega^2\sin^2(\omega)}{(\pi^3-\pi\omega^2)^2}d\omega$$

$$=\frac{2}{\pi^2}\int_{-\infty}^{+\infty}\frac{\omega^2\sin^2(\omega)}{(\pi^2-\omega^2)^2}d\omega$$

$$\Longrightarrow\boxed{I_2=\int_{-\infty}^{+\infty}\frac{\omega^2\sin^2(\omega)}{(\omega^2-\pi^2)^2}d\omega=\frac{\pi}{2}}$$

۱-۴) انتگرال فوریه تابع $f(x)$ را بدست آورید و با استفاده از آن حاصل انتگرال I_1 و I_2 را محاسبه کنید.

$$f(x)=\sin(\pi x)\,\Pi(x),\qquad \Pi(x)=\begin{cases}1, & |x|\le \frac{1}{2}\\ 0, & \text{otherwise}\end{cases},\qquad I_1=\int_{-\infty}^{+\infty}\frac{\omega\sin(\omega)}{\pi^2-\omega^2}d\omega,\qquad I_2=\int_{-\infty}^{+\infty}\frac{\omega^2\cos^2\left(\frac{\omega}{2}\right)}{(\pi^2-\omega^2)^2}d\omega$$

$$f(x)=-f(-x)\Longrightarrow f(x)\text{ is odd}\Longrightarrow A(\omega)=0$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty}f(x)\sin(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+\infty}f(x)\sin(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+\infty}\sin(\pi x)\,\Pi(x)\sin(\omega x)\,dx=\frac{2}{\pi}\int_{0}^{+\frac{1}{2}}\sin(\pi x)\sin(\omega x)\,dx$$

$$=\frac{1}{\pi}\left[\frac{\sin\big((\omega-\pi)x\big)}{\omega-\pi}-\frac{\sin\big((\omega+\pi)x\big)}{\omega+\pi}\right]_0^{+\frac{1}{2}}=\frac{2\omega\cos\left(\frac{\omega}{2}\right)}{\pi^3-\pi\omega^2}$$

$$\Longrightarrow\boxed{f(x)=\int_0^{\infty}\frac{2\omega\cos\left(\frac{\omega}{2}\right)}{\pi^3-\pi\omega^2}\sin(\omega x)\,d\omega}$$

$$f\left(\frac{1}{2}\right)=\frac{f\left(\frac{1^+}{2}\right)+f\left(\frac{1^-}{2}\right)}{2}=\frac{1}{2}=\int_0^{\infty}\frac{2\omega\cos\left(\frac{\omega}{2}\right)}{\pi^3-\pi\omega^2}\sin\left(\frac{\omega}{2}\right)d\omega=\int_0^{\infty}\frac{\omega\sin(\omega)}{\pi^3-\pi\omega^2}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega\sin(\omega)}{\pi^2-\omega^2}d\omega$$

$$\Longrightarrow\boxed{I_1=\int_{-\infty}^{+\infty}\frac{\omega\sin(\omega)}{\pi^2-\omega^2}d\omega=\pi}$$

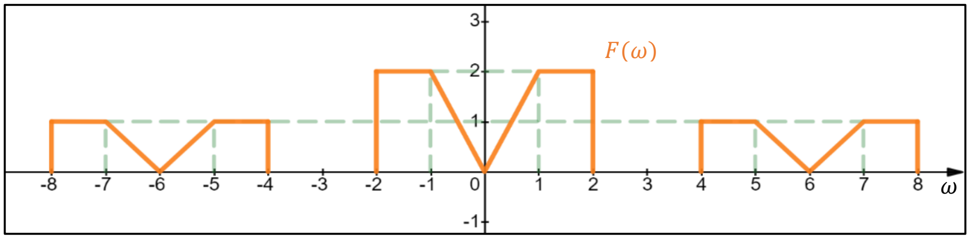
$$\text{Parseval: }\frac{1}{\pi}\int_{-\infty}^{+\infty}|f(x)|^2dx=\int_0^{+\infty}\big(A^2(\omega)+B^2(\omega)\big)d\omega\Longrightarrow\frac{1}{\pi}\int_{-\infty}^{+\infty}|\sin(\pi x)\,\Pi(x)|^2dx=\frac{2}{\pi}\int_0^{+\frac{1}{2}}|\sin(\pi x)|^2dx=\frac{1}{2\pi}=\int_0^{+\infty}\frac{4\omega^2\cos^2\left(\frac{\omega}{2}\right)}{(\pi^3-\pi\omega^2)^2}d\omega$$

$$=\frac{2}{\pi^2}\int_{-\infty}^{+\infty}\frac{\omega^2\cos^2\left(\frac{\omega}{2}\right)}{(\pi^2-\omega^2)^2}d\omega$$

$$\Longrightarrow\boxed{I_2=\int_{-\infty}^{+\infty}\frac{\omega^2\cos^2\left(\frac{\omega}{2}\right)}{(\pi^2-\omega^2)^2}d\omega=\frac{\pi}{4}}$$

۲)

۲-۱) تبدیل فوریه معکوس تابع $F(\omega)$ را با استفاده از خواص تبدیل فوریه و تبدیل فوریه توابع مهم بدست آورید.



$$\mathcal{F}^{-1}\{F(\omega)\} = f(x), \qquad F(\omega) = G(\omega-6) + G(\omega+6) + 2G(\omega) = G(\omega-3-3) + G(\omega-3+3) + G(\omega+3-3) + G(\omega+3+3)$$

$$\Rightarrow f(x) = 4\cos^2(3x)\, g(x)$$

$$G(\omega) = \begin{cases} |\omega| & |\omega| \le 1 \\ 1 & 1 \le |\omega| \le 2 \\ 0 & \text{otherwise} \end{cases} = \Pi\left(\frac{\omega}{4}\right) - \Lambda(\omega)$$

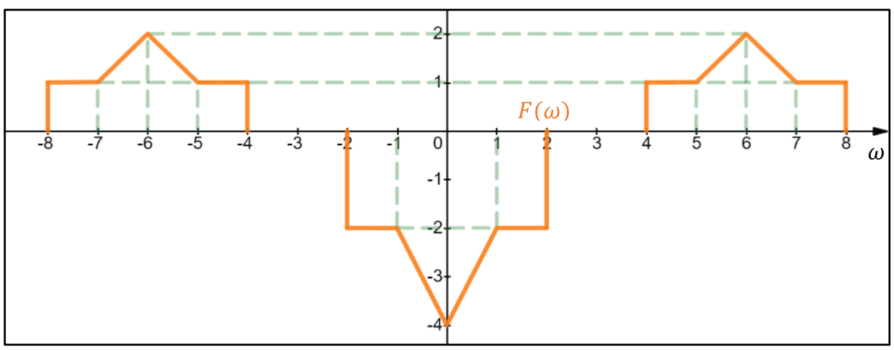
$$\mathcal{F}\{\operatorname{sinc}(ax)\} = \frac{1}{|a|}\Pi\left(\frac{\omega}{2\pi a}\right) \Rightarrow \mathcal{F}^{-1}\left\{\Pi\left(\frac{\omega}{4}\right)\right\} = \frac{2}{\pi}\operatorname{sinc}\left(\frac{2x}{\pi}\right)$$

$$\mathcal{F}\{\operatorname{sinc}^2(ax)\} = \frac{1}{|a|}\Lambda\left(\frac{\omega}{2\pi a}\right) \Rightarrow \mathcal{F}^{-1}\{\Lambda(\omega)\} = \frac{1}{2\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)$$

$$\Rightarrow g(x) = \frac{2}{\pi}\operatorname{sinc}\left(\frac{2x}{\pi}\right) - \frac{1}{2\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)$$

$$\Rightarrow \boxed{f(x) = \left(\frac{8}{\pi}\operatorname{sinc}\left(\frac{2x}{\pi}\right) - \frac{2}{\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)\right)\cos^2(3x)}$$

۲-۲) تبدیل فوریه معکوس تابع $F(\omega)$ را با استفاده از خواص تبدیل فوریه و تبدیل فوریه توابع مهم بدست آورید.



$$\mathcal{F}^{-1}\{F(\omega)\} = f(x), \qquad F(\omega) = G(\omega-6) + G(\omega+6) - 2G(\omega) = \big(G(\omega-3-3) - G(\omega-3+3)\big) - \big(G(\omega+3-3) - G(\omega+3+3)\big)$$

$$\Rightarrow f(x) = 4i^2\sin^2(3x)\, g(x) = -4\sin^2(3x)\, g(x)$$

$$G(\omega) = \begin{cases} 2-|\omega| & |\omega| \le 1 \\ 1 & 1 \le |\omega| \le 2 \\ 0 & \text{otherwise} \end{cases} = \Pi\left(\frac{\omega}{4}\right) + \Lambda(\omega)$$

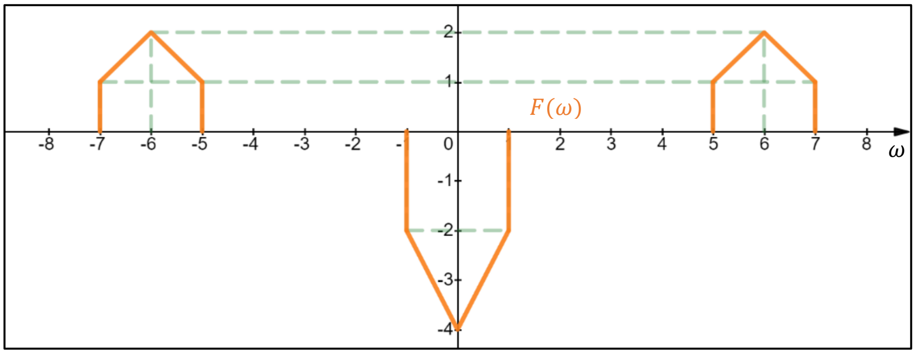
$$\mathcal{F}\{\operatorname{sinc}(ax)\} = \frac{1}{|a|}\Pi\left(\frac{\omega}{2\pi a}\right) \Rightarrow \mathcal{F}^{-1}\left\{\Pi\left(\frac{\omega}{4}\right)\right\} = \frac{2}{\pi}\operatorname{sinc}\left(\frac{2x}{\pi}\right)$$

$$\mathcal{F}\{\operatorname{sinc}^2(ax)\} = \frac{1}{|a|}\Lambda\left(\frac{\omega}{2\pi a}\right) \Rightarrow \mathcal{F}^{-1}\{\Lambda(\omega)\} = \frac{1}{2\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)$$

$$\Rightarrow g(x) = \frac{2}{\pi}\operatorname{sinc}\left(\frac{2x}{\pi}\right) + \frac{1}{2\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)$$

$$\Rightarrow \boxed{f(x) = \left(-\frac{2}{\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right) - \frac{8}{\pi}\operatorname{sinc}\left(\frac{2x}{\pi}\right)\right)\sin^2(3x)}$$

۲-۳) تبدیل فوریه معکوس تابع $F(\omega)$ را با استفاده از خواص تبدیل فوریه و تبدیل فوریه توابع مهم بدست آورید.



$$\mathcal{F}^{-1}\{F(\omega)\} = f(x), \qquad F(\omega) = G(\omega-6) + G(\omega+6) - 2G(\omega) = \big(G(\omega-3-3) - G(\omega-3+3)\big) - \big(G(\omega+3-3) - G(\omega+3+3)\big)$$

$$\Longrightarrow f(x) = 4i^2 \sin^2(3x)\, g(x) = -4\sin^2(3x)\, g(x)$$

$$G(\omega) = \begin{cases} 2-|\omega| & |\omega| \le 1 \\ 0 & \text{otherwise} \end{cases} = \Pi\left(\frac{\omega}{2}\right) + \Lambda(\omega)$$

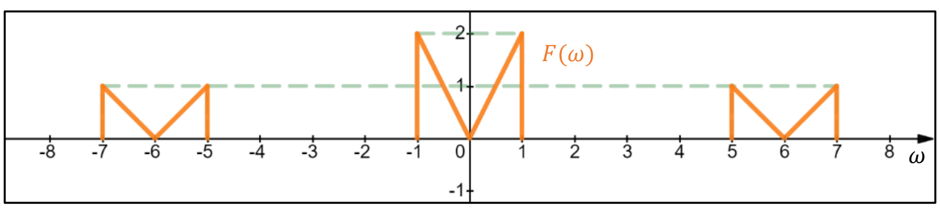
$$\mathcal{F}\{\operatorname{sinc}(ax)\} = \frac{1}{|a|}\Pi\left(\frac{\omega}{2\pi a}\right) \Longrightarrow \mathcal{F}^{-1}\left\{\Pi\left(\frac{\omega}{2}\right)\right\} = \frac{1}{\pi}\operatorname{sinc}\left(\frac{x}{\pi}\right)$$

$$\mathcal{F}\{\operatorname{sinc}^2(ax)\} = \frac{1}{|a|}\Lambda\left(\frac{\omega}{2\pi a}\right) \Longrightarrow \mathcal{F}^{-1}\{\Lambda(\omega)\} = \frac{1}{2\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)$$

$$\Longrightarrow g(x) = \frac{1}{\pi}\operatorname{sinc}\left(\frac{x}{\pi}\right) + \frac{1}{2\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)$$

$$\Longrightarrow \boxed{f(x) = \left(-\frac{2}{\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right) - \frac{4}{\pi}\operatorname{sinc}\left(\frac{x}{\pi}\right)\right)\sin^2(3x)}$$

۲-۴) تبدیل فوریه معکوس تابع $F(\omega)$ را با استفاده از خواص تبدیل فوریه و تبدیل فوریه توابع مهم بدست آورید.



$$\mathcal{F}^{-1}\{F(\omega)\} = f(x), \qquad F(\omega) = G(\omega-6) + G(\omega+6) + 2G(\omega) = \big(G(\omega-3-3) + G(\omega-3+3)\big) + \big(G(\omega+3-3) + G(\omega+3+3)\big)$$

$$\Longrightarrow f(x) = 4\cos^2(3x)\, g(x) = 4\cos^2(3x)\, g(x)$$

$$G(\omega) = \begin{cases} |\omega| & |\omega| \le 1 \\ 0 & \text{otherwise} \end{cases} = \Pi\left(\frac{\omega}{2}\right) - \Lambda(\omega)$$

$$\mathcal{F}\{\operatorname{sinc}(ax)\} = \frac{1}{|a|}\Pi\left(\frac{\omega}{2\pi a}\right) \Longrightarrow \mathcal{F}^{-1}\left\{\Pi\left(\frac{\omega}{2}\right)\right\} = \frac{1}{\pi}\operatorname{sinc}\left(\frac{x}{\pi}\right)$$

$$\mathcal{F}\{\operatorname{sinc}^2(ax)\} = \frac{1}{|a|}\Lambda\left(\frac{\omega}{2\pi a}\right) \Longrightarrow \mathcal{F}^{-1}\{\Lambda(\omega)\} = \frac{1}{2\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)$$

$$\Longrightarrow g(x) = \frac{1}{\pi}\operatorname{sinc}\left(\frac{x}{\pi}\right) - \frac{1}{2\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)$$

$$\Longrightarrow \boxed{f(x) = \left(\frac{4}{\pi}\operatorname{sinc}\left(\frac{x}{\pi}\right) - \frac{2}{\pi}\operatorname{sinc}^2\left(\frac{x}{2\pi}\right)\right)\cos^2(3x)}$$