۱)

پاسخ معادله حرارت زیر را بدست آورید.

$$u_t = 9u_{xx} \qquad 0 \le x \le 2$$

$$\begin{cases} u(0,t) = 0 \\ u(2,t) = 0 \end{cases}, \qquad u(x,0) = x^2 + x$$

$$\text{BC1, BC2, } L = 2: u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{2}x\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\dot{T}_n(t) + \frac{9n^2\pi^2}{4} T_n(t)\right) \sin\left(\frac{n\pi}{2}x\right) = 0 \Rightarrow \dot{T}_n(t) + \frac{9n^2\pi^2}{4} T_n(t) = 0 \Rightarrow T_n(t) = c_n e^{-\frac{9n^2\pi^2}{4}t}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{9n^2\pi^2}{4}t} \sin\left(\frac{n\pi}{2}x\right) \Rightarrow \text{IC: } u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{2}x\right) = x^2 + x$$

$$\Rightarrow c_n = \int_0^2 (x^2 + x) \sin\left(\frac{n\pi}{2}x\right) dx = \frac{(16 - 12n^2\pi^2)(-1)^n - 16}{n^3\pi^3}$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{(16 - 12n^2\pi^2)(-1)^n - 16}{n^3\pi^3} e^{-\frac{9n^2\pi^2}{4}t} \sin\left(\frac{n\pi}{2}x\right)}$$

پاسخ معادله حرارت زیر را بدست آورید.

$$u_t = 5u_{xx} \qquad 0 \le x \le 3$$

$$\begin{cases} u_x(0,t) = 0 \\ u_x(3,t) = 0 \end{cases}, \qquad u(x,0) = x + 5$$

$$\text{BC1, BC2, } L = 3: u(x,t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t) \cos\left(\frac{n\pi}{3}x\right)$$

$$\Rightarrow \dot{T}_0(t) + \sum_{n=1}^{\infty} \left(\dot{T}_n(t) + \frac{5n^2\pi^2}{9} T_n(t)\right) \cos\left(\frac{n\pi}{3}x\right) = 0 \Rightarrow \begin{cases} \dot{T}_0(t) = 0 \\ \dot{T}_n(t) + \frac{5n^2\pi^2}{9} T_n(t) \end{cases} \Rightarrow \begin{cases} T_0(t) = c_0 \\ T_n(t) = c_n e^{-\frac{5n^2\pi^2}{9}t} \end{cases}$$

$$\Rightarrow u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\frac{5n^2\pi^2}{9}t} \cos\left(\frac{n\pi}{3}x\right) \Rightarrow \text{IC: } u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{3}x\right) = x + 5$$

$$\Rightarrow \begin{cases} c_0 = \frac{1}{3}\int_0^3 (x+5)dx = \frac{13}{2} \\ c_n = \frac{2}{3}\int_0^3 (x+5)\cos\left(\frac{n\pi}{3}x\right) = \frac{6((-1)^n - 1)}{n^2\pi^2} \end{cases}$$

$$\Rightarrow \boxed{u(x,t) = \frac{13}{2} + \sum_{n=1}^{\infty} \frac{6((-1)^n - 1)}{n^2\pi^2} e^{-\frac{5n^2\pi^2}{9}t} \cos\left(\frac{n\pi}{3}x\right)}$$

پاسخ معادله موج زیر را بدست آورید.

$$u_{tt} = 25u_{xx} \qquad 0 \le x \le 4$$

$$\begin{cases} u(0,t) = 0 \\ u(4,t) = 0 \end{cases}, \qquad \begin{cases} u(x,0) = x^2 + 4 \\ u_t(x,0) = x \end{cases}$$

$$\text{BC1, BC2, } L = 4: u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{4}x\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\ddot{T}_n(t) + \frac{25n^2\pi^2}{16} T_n(t)\right) \sin\left(\frac{n\pi}{4}x\right) = 0$$

$$\Rightarrow \ddot{T}_n(t) + \frac{25n^2\pi^2}{16} T_n(t) = 0 \Rightarrow T_n(t) = a_n \cos\left(\frac{5n\pi}{4}t\right) + b_n \sin\left(\frac{5n\pi}{4}t\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{5n\pi}{4}t\right) + b_n \sin\left(\frac{5n\pi}{4}t\right)\right) \sin\left(\frac{n\pi}{4}x\right)$$

$$\text{IC1: } u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{4}x\right) = x^2 + 4$$

$$\Rightarrow a_n = \frac{1}{2}\int_0^4 (x^2 + 4) \sin\left(\frac{n\pi}{4}x\right) = \frac{64((-1)^n - 1)}{n^3\pi^3} - \frac{8(5(-1)^n - 1)}{n\pi}$$

$$\text{IC2: } u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{5n\pi}{4} \sin\left(\frac{n\pi}{4}x\right) = x$$

$$\Rightarrow b_n = \frac{2}{5n\pi}\int_0^4 x \sin\left(\frac{n\pi}{4}\right) dx = -\frac{32(-1)^n}{5n^2\pi^2}$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} \left(\left(\frac{64((-1)^n - 1)}{n^3\pi^3} - \frac{8(5(-1)^n - 1)}{n\pi}\right) \cos\left(\frac{5n\pi}{4}t\right) - \frac{32(-1)^n}{5n^2\pi^2} \sin\left(\frac{5n\pi}{4}t\right)\right) \sin\left(\frac{n\pi}{4}x\right)}$$

پاسخ معادله موج زیر را بدست آورید.

$$u_{tt} = 16u_{xx} \qquad 0 \le x \le 5$$

$$\begin{cases} u_x(0,t) = 0 \\ u_x(5,t) = 0 \end{cases}, \quad \begin{cases} u(x,0) = x \\ u_t(x,0) = 10 \end{cases}$$

$$\text{BC1, BC2, } L = 5: u(x,t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos\left(\frac{n\pi}{5}x\right)$$

$$\Longrightarrow \ddot{T}_0(t) + \sum_{n=1}^{\infty}\left(\ddot{T}_n(t) + \frac{16n^2\pi^2}{25}T_n(t)\right)\cos\left(\frac{n\pi}{5}x\right) = 0$$

$$\Longrightarrow \begin{cases} \ddot{T}_0(t) = 0 \\ \ddot{T}_n(t) + \dfrac{16n^2\pi^2}{25}T_n(t) = 0 \end{cases} \Longrightarrow \begin{cases} T_0(t) = a_0 + b_0 t \\ T_n(t) = a_n\cos\left(\dfrac{4n\pi}{5}t\right) + b_n\sin\left(\dfrac{4n\pi}{5}t\right) \end{cases}$$

$$\Longrightarrow u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{4n\pi}{5}t\right) + b_n\sin\left(\frac{4n\pi}{5}t\right)\right)\cos\left(\frac{n\pi}{5}x\right)$$

$$\text{IC1: } u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{5}x\right) = x$$

$$\Longrightarrow \begin{cases} a_0 = \dfrac{1}{5}\displaystyle\int_0^5 x\,dx = \dfrac{5}{2} \\ a_n = \dfrac{2}{5}\displaystyle\int_0^5 x\cos\left(\dfrac{n\pi}{5}x\right)dx = \dfrac{10((-1)^n - 1)}{n^2\pi^2} \end{cases}$$

$$\text{IC2: } u_t(x,0) = b_0 + \sum_{n=1}^{\infty} b_n\frac{4n\pi}{5}\cos\left(\frac{n\pi}{5}x\right) = 10$$

$$\Longrightarrow \begin{cases} b_0 = \dfrac{1}{5}\displaystyle\int_0^5 10\,dx = 10 \\ b_n = \dfrac{5}{2n\pi}\displaystyle\int_0^5 10\cos\left(\dfrac{n\pi}{5}x\right) = 0 \end{cases}$$

$$\Longrightarrow \boxed{u(x,t) = \frac{5}{2} + \sum_{n=1}^{\infty}\frac{10((-1)^n - 1)}{n^2\pi^2}\cos\left(\frac{4n\pi}{5}t\right)\cos\left(\frac{n\pi}{5}x\right)}$$

۲)

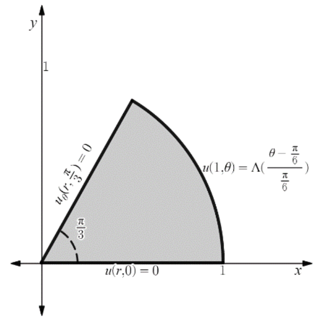
معادله لاپلاس زیر را حل کنید.

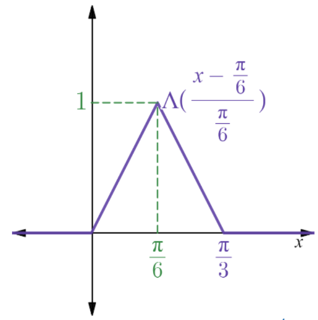
$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

$$0\le r\le 1,\qquad 0\le\theta\le\frac{\pi}{3}$$

$$\begin{cases} u(r,0)=0 \\ u_\theta\left(r,\frac{\pi}{3}\right)=0 \end{cases},\qquad u(1,\theta)=\Lambda\left(\frac{\theta-\frac{\pi}{6}}{\frac{\pi}{6}}\right)$$

$$\Lambda(x)=\begin{cases}1+x, & -1\le x\le 0\\ 1-x, & 0\le x\le 1\\ 0, & \text{otherwise}\end{cases}$$





با تغییر متغیر $r=e^{-t}$ داریم:

$$u_{tt}+u_{\theta\theta}=0\Longrightarrow\frac{\ddot{X}(t)}{X(t)}=-\frac{Y''(\theta)}{Y(\theta)}=\lambda^2\Longrightarrow\begin{cases}Y''(\theta)+\lambda^2Y(\theta)=0 & \text{(I)}\\ \ddot{X}(t)-\lambda^2X(t)=0 & \text{(II)}\end{cases}$$

$$\text{(I)}\Longrightarrow\ Y(\theta)=a\cos(\lambda\theta)+b\sin(\lambda\theta)\Longrightarrow\begin{cases}u(r,0)=0\Longrightarrow a=0\\ u_\theta\left(r,\frac{\pi}{3}\right)=0\Longrightarrow b\lambda\cos\left(\frac{\lambda\pi}{3}\right)=0\Longrightarrow\lambda=\frac{3}{2}(2n-1)\end{cases}$$

$$\Longrightarrow Y_n(\theta)=b_n\sin\left(\frac{3}{2}(2n-1)\theta\right)$$

$$\text{(II)}\Longrightarrow\ \ddot{X}_n(t)-\frac{9}{4}(2n-1)^2X_n(t)=0\Longrightarrow X_n(t)=c_ne^{-\frac{3}{2}(2n-1)t}+d_ne^{\frac{3}{2}(2n-1)t}$$

$$\Longrightarrow X_n=c_nr^{\frac{3}{2}(2n-1)}+d_nr^{-\frac{3}{2}(2n-1)}$$

$$\Longrightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(c_nr^{\frac{3}{2}(2n-1)}+d_nr^{-\frac{3}{2}(2n-1)}\right)b_n\sin\left(\frac{3}{2}(2n-1)\theta\right),\qquad\begin{cases}\tilde{c}_n=c_nb_n\\ \tilde{d}_n=d_nb_n\end{cases}$$

$$\Longrightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(\tilde{c}_nr^{\frac{3}{2}(2n-1)}+\tilde{d}_nr^{-\frac{3}{2}(2n-1)}\right)\sin\left(\frac{3}{2}(2n-1)\theta\right)\Longrightarrow\{r=0\}\in\text{Domain}\Longrightarrow\tilde{d}_n=0$$

$$\Longrightarrow u(1,\theta)=\sum_{n=1}^{\infty}\tilde{c}_n\sin\left(\frac{3}{2}(2n-1)\theta\right)\Longrightarrow\ \tilde{c}_n=\frac{6}{\pi}\int_0^{\frac{\pi}{6}}\Lambda\left(\frac{\theta-\frac{\pi}{6}}{\frac{\pi}{6}}\right)\sin\left(\frac{3}{2}(2n-1)\theta\right)d\theta$$

$$=\frac{6}{\pi}\left(\int_0^{\frac{\pi}{6}}\frac{6}{\pi}\theta\sin\left(\frac{3}{2}(2n-1)\theta\right)d\theta+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(2-\frac{6}{\pi}\theta\right)\sin\left(\frac{3}{2}(2n-1)\theta\right)d\theta\right)$$

$$\int_0^{\frac{\pi}{6}}\frac{6}{\pi}\theta\sin\left(\frac{3}{2}(2n-1)\theta\right)\,d\theta=-\frac{2}{3(2n-1)}\cos\left(\frac{\pi}{4}(2n-1)\right)+\frac{6}{\pi}\frac{4}{9(2n-1)^2}\sin\left(\frac{\pi}{4}(2n-1)\right)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(2-\frac{6}{\pi}\theta\right)\sin\left(\frac{3}{2}(2n-1)\theta\right)d\theta$$

$$=\frac{2}{3(2n-1)}\cos\left(\frac{\pi}{4}(2n-1)\right)-\frac{6}{\pi}\frac{4}{9(2n-1)^2}(-1)^{n+1}+\frac{6}{\pi}\frac{4}{9(2n-1)^2}\sin\left(\frac{\pi}{4}(2n-1)\right)$$

$$\Longrightarrow\tilde{c}_n=\frac{16\left(2\sin\left(\frac{\pi}{4}(2n-1)\right)+(-1)^n\right)}{\pi^2(2n-1)^2}$$

$$\Longrightarrow \boxed{u(r,\theta)=\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{2\sin\left(\frac{\pi}{4}(2n-1)\right)+(-1)^n}{(2n-1)^2}r^{\frac{3}{2}(2n-1)}\sin\left(\frac{3}{2}(2n-1)\theta\right)}$$

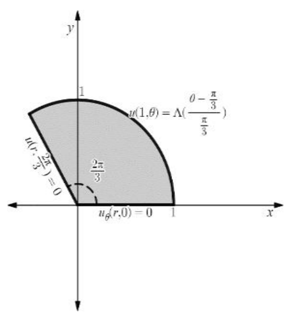
معادله لاپلاس زیر را حل کنید.

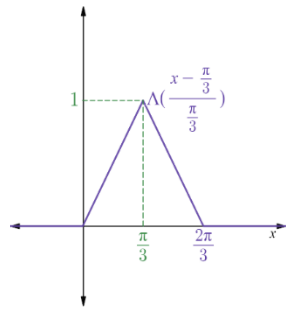
$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

$$0\le r\le 1,\qquad 0\le\theta\le\frac{2\pi}{3}$$

$$\begin{cases}u_\theta(r,0)=0\\ u\left(r,\frac{2\pi}{3}\right)=0\end{cases},\qquad u(1,\theta)=\Lambda\left(\frac{\theta-\frac{\pi}{3}}{\frac{\pi}{3}}\right)$$

$$\Lambda(x)=\begin{cases}1+x, & -1\le x\le 0\\ 1-x, & 0\le x\le 1\\ 0, & \text{otherwise}\end{cases}$$





با تغییر متغیر $r=e^{-t}$ داریم:

$$u_{tt}+u_{\theta\theta}=0\Longrightarrow\frac{\ddot{X}(t)}{X(t)}=-\frac{Y''(\theta)}{Y(\theta)}=\lambda^2\Longrightarrow\begin{cases}Y''(\theta)+\lambda^2Y(\theta)=0\quad(\mathrm{I})\\ \ddot{X}(t)-\lambda^2X(t)=0\quad(\mathrm{II})\end{cases}$$

$$(\mathrm{I})\Longrightarrow Y(\theta)=a\cos(\lambda\theta)+b\sin(\lambda\theta)\Longrightarrow\begin{cases}u(r,0)=0\Longrightarrow b=0\\ u\left(r,\frac{2\pi}{3}\right)=0\Longrightarrow a\cos\left(\frac{2\lambda\pi}{3}\right)=0\Longrightarrow\lambda=\frac{3}{4}(2n-1)\end{cases}$$

$$\Longrightarrow Y_n(\theta)=a_n\cos\left(\frac{3}{4}(2n-1)\theta\right)$$

$$(\mathrm{II})\Longrightarrow\ddot{X}_n(t)-\frac{9}{16}(2n-1)^2X_n(t)=0\Longrightarrow X_n(t)=c_ne^{-\frac{3}{4}(2n-1)t}+d_ne^{\frac{3}{4}(2n-1)t}$$

$$\Longrightarrow X_n=c_nr^{\frac{3}{4}(2n-1)}+d_nr^{-\frac{3}{4}(2n-1)}$$

$$\Longrightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(c_nr^{\frac{3}{4}(2n-1)}+d_nr^{-\frac{3}{4}(2n-1)}\right)a_n\cos\left(\frac{3}{4}(2n-1)\theta\right),\qquad\begin{cases}\tilde{c}_n=c_na_n\\ \tilde{d}_n=d_na_n\end{cases}$$

$$\Longrightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(\tilde{c}_nr^{\frac{3}{4}(2n-1)}+\tilde{d}_nr^{-\frac{3}{4}(2n-1)}\right)\cos\left(\frac{3}{4}(2n-1)\theta\right)\Longrightarrow\{r=0\}\in\text{Domain}\Longrightarrow\tilde{d}_n=0$$

$$\Longrightarrow u(1,\theta)=\sum_{n=1}^{\infty}\tilde{c}_n\cos\left(\frac{3}{4}(2n-1)\theta\right)\Longrightarrow\tilde{c}_n=\frac{3}{\pi}\int_0^{\frac{2\pi}{3}}\Lambda\left(\frac{\theta-\frac{\pi}{3}}{\frac{\pi}{3}}\right)\cos\left(\frac{3}{4}(2n-1)\theta\right)d\theta$$

$$=\frac{3}{\pi}\left(\int_0^{\frac{\pi}{3}}\frac{3}{\pi}\theta\cos\left(\frac{3}{4}(2n-1)\theta\right)d\theta+\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\left(2-\frac{3}{\pi}\theta\right)\cos\left(\frac{3}{4}(2n-1)\theta\right)d\theta\right)$$

$$\int_0^{\frac{\pi}{3}}\frac{3}{\pi}\theta\cos\left(\frac{3}{4}(2n-1)\theta\right)d\theta=\frac{4}{3(2n-1)}\sin\left(\frac{\pi}{4}(2n-1)\right)+\frac{3}{\pi}\frac{16}{9(2n-1)^2}\left(1-\cos\left(\frac{\pi}{4}(2n-1)\right)\right)$$

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\left(2-\frac{3}{\pi}\theta\right)\cos\left(\frac{3}{4}(2n-1)\theta\right)d\theta=\frac{-4}{3(2n-1)}\cos\left(\frac{\pi}{4}(2n-1)\right)+\frac{3}{\pi}\frac{16}{9(2n-1)^2}\cos\left(\frac{\pi}{4}(2n-1)\right)$$

$$\Longrightarrow\tilde{c}_n=\frac{16\left(2\cos\left(\frac{\pi}{4}(2n-1)\right)-1\right)}{\pi^2(2n-1)^2}$$

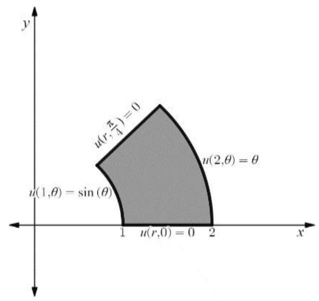
$$\Rightarrow \boxed{u(r,\theta)=\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{2\cos\left(\frac{\pi}{4}(2n-1)\right)-1}{(2n-1)^2}r^{\frac{3}{4}(2n-1)}\cos\left(\frac{3}{4}(2n-1)\theta\right)}$$

معادله لاپلاس زیر را حل کنید.

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

$$1\le r\le 2,\qquad 0\le\theta\le\frac{\pi}{4}$$

$$\begin{cases}u(r,0)=0\\ u\left(r,\frac{\pi}{4}\right)=0\end{cases},\qquad \begin{cases}u(1,\theta)=sin(\theta)\\ u(2,\theta)=\theta\end{cases}$$



با تغییر متغیر $r=e^{-t}$ داریم:

$$u_{tt}+u_{\theta\theta}=0\Rightarrow\frac{\ddot{X}(t)}{X(t)}=-\frac{Y''(\theta)}{Y(\theta)}=\lambda^2\Rightarrow\begin{cases}Y''(\theta)+\lambda^2Y(\theta)=0 & (\mathrm{I})\\ \ddot{X}(t)-\lambda^2X(t)=0 & (\mathrm{II})\end{cases}$$

$$(\mathrm{I})\Rightarrow Y(\theta)=a\cos(\lambda\theta)+b\sin(\lambda\theta)\Rightarrow\begin{cases}u(r,0)=0\to a=0\\ u_\theta\left(r,\frac{\pi}{4}\right)=0\Rightarrow b\lambda\cos\left(\frac{\lambda\pi}{4}\right)=0\Rightarrow\lambda=4n\end{cases}$$

$$\Rightarrow Y_n(\theta)=b_n\sin(4n\theta)$$

$$(\mathrm{II})\Rightarrow\ddot{X}_n(t)-16n^2X_n(t)=0\Rightarrow\begin{cases}n\neq0\Rightarrow X_n(t)=c_ne^{-4nt}+d_ne^{4nt}\Rightarrow X_n=c_nr^{4n}+d_nr^{-4n}\\ n=0\Rightarrow X_0(t)=c_0t+d_0\Rightarrow -c_0\,ln(r)+d_0\end{cases}$$

$$\Rightarrow u(r,\theta)=(-c_0\,ln(r)+d_0)\sin(0\times\theta)+\sum_{n=1}^{\infty}(c_nr^{4n}+d_nr^{-4n})b_n\sin(4n\theta)\,,\qquad\begin{cases}\tilde{c}_n=c_nb_n\\ \tilde{d}_n=d_nb_n\end{cases}$$

$$\Rightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(\tilde{c}_nr^{4n}+\tilde{d}_nr^{-4n}\right)\sin(4n\theta)$$

$$u(2,\theta)=\sum_{n=1}^{\infty}\left(\tilde{c}_n2^{4n}+\tilde{d}_n2^{-4n}\right)\sin(4n\theta)\ \Rightarrow\tilde{c}_n2^{4n}+\tilde{d}_n2^{-4n}=\frac{8}{\pi}\int_0^{\frac{\pi}{4}}\theta\sin(4n\theta)\,d\theta=\frac{(-1)^{n+1}}{2n}=g_n$$

$$u(1,\theta)=\sum_{n=1}^{\infty}\left(\tilde{c}_n2^{4n}+\tilde{d}_n2^{-4n}\right)\sin(4n\theta)\Rightarrow\frac{8}{\pi}\int_0^{\frac{\pi}{4}}\sin(\theta)\sin(4n\theta)\,d\theta=$$

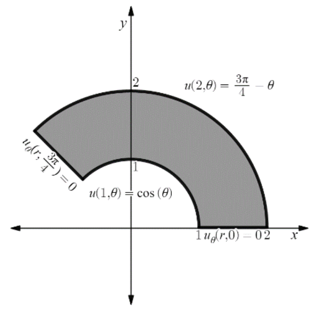
$$\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\cos\big((4n-1)\theta\big)-\cos\big((4n+1)\theta\big)\,d\theta=\frac{4}{\pi}\left(\frac{\sin\left((4n-1)\frac{\pi}{4}\right)}{4n-1}-\frac{\sin\left((4n+1)\frac{\pi}{4}\right)}{4n+1}\right)=h_n$$

$$\begin{cases}\tilde{c}_n2^{4n}+\tilde{d}_n2^{-4n}=g_n\\ \tilde{c}_n+\tilde{d}_n=h_n\end{cases}\Rightarrow\begin{cases}\tilde{c}_n=\dfrac{g_n-h_n2^{-4n}}{2^{4n}-2^{-4n}}\\ \tilde{d}_n=\dfrac{g_n-h_n2^{4n}}{2^{-4n}-2^{4n}}\end{cases}$$

$$\Rightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(g_n\frac{r^{4n}-r^{-4n}}{2^{4n}-2^{-4n}}+h_n\frac{\left(\frac{2}{r}\right)^{4n}-\left(\frac{2}{r}\right)^{-4n}}{2^{4n}-2^{-4n}}\right)\sin(4n\theta)$$

$$\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\left(\frac{r^{4n}-r^{-4n}}{2^{4n}-2^{-4n}}\frac{(-1)^{n+1}}{2n}+\frac{\left(\frac{2}{r}\right)^{4n}-\left(\frac{2}{r}\right)^{-4n}}{2^{4n}-2^{-4n}}\frac{4}{\pi}\left(\frac{\sin\left((4n-1)\frac{\pi}{4}\right)}{4n-1}-\frac{\sin\left((4n+1)\frac{\pi}{4}\right)}{4n+1}\right)\right)\sin(4n\theta)}$$

معادله لاپلاس زیر را حل کنید.



$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

$$1\le r\le 2,\qquad 0\le\theta\le\frac{3\pi}{4}$$

$$\begin{cases} u_\theta(r,0)=0 \\ u_\theta\left(r,\frac{3\pi}{4}\right)=0 \end{cases}, \begin{cases} u(1,\theta)=cos(\theta) \\ u(2,\theta)=\frac{3\pi}{4}-\theta \end{cases}$$

با تغییر متغیر $r=e^{-t}$ داریم:

$$u_{tt}+u_{\theta\theta}=0 \Longrightarrow \frac{\ddot{X}(t)}{X(t)}=-\frac{Y''^{(\theta)}}{Y(\theta)}=\lambda^2 \Longrightarrow \begin{cases} Y''(\theta)+\lambda^2 Y(\theta)=0 \quad (\mathrm{I}) \\ \ddot{X}(t)-\lambda^2 X(t)=0 \quad (\mathrm{II}) \end{cases}$$

$$(\mathrm{I}) \Longrightarrow\ Y(\theta)=a\cos(\lambda\theta)+b\sin(\lambda\theta) \Longrightarrow \begin{cases} u_\theta(r,0)=0 \Longrightarrow b=0 \\ u_\theta\left(r,\frac{\pi}{4}\right)=0 \Longrightarrow -a\lambda\sin\left(\lambda\frac{3\pi}{4}\right)=0 \Longrightarrow \lambda=\frac{4n}{3} \end{cases}$$

$$\Longrightarrow Y_n(\theta)=a_n\cos\left(\frac{4n\theta}{3}\right)$$

$$(\mathrm{II}) \Longrightarrow \ddot{X}_n(t)-\frac{16}{9}n^2X_n(t)=0 \Longrightarrow \begin{cases} n\neq 0 \Longrightarrow X_n(t)=c_ne^{-\frac{4n}{3}t}+d_ne^{\frac{4n}{3}t} \Longrightarrow X_n=c_nr^{\frac{4n}{3}}+d_nr^{-\frac{4n}{3}} \\ n=0 \Longrightarrow X_0(t)=c_0t+d_0 \Longrightarrow -c_0\ln(r)+d_0 \end{cases}$$

$$\Longrightarrow u(r,\theta)=(-c_0\ln(r)+d_0)a_0\cos(0\times\theta)+\sum_{n=1}^{\infty}\left(c_nr^{\frac{4n}{3}}+d_nr^{-\frac{4n}{3}}\right)a_n\cos\left(\frac{4n}{3}\theta\right), \qquad \begin{cases} \tilde{c}_n=c_na_n \\ \tilde{d}_n=d_na_n \end{cases}$$

$$\Longrightarrow u(r,\theta)=-\tilde{c}_0\ln(r)+\tilde{d}_0+\sum_{n=1}^{\infty}\left(\tilde{c}_nr^{\frac{4}{3}n}+\tilde{d}_nr^{-\frac{4}{3}n}\right)\cos\left(\frac{4}{3}n\theta\right)$$

$$u(2,\theta)=-\tilde{c}_0\ln(2)+\tilde{d}_0+\sum_{n=1}^{\infty}\left(\tilde{c}_n2^{\frac{4}{3}n}+\tilde{d}_n2^{-\frac{4}{3}n}\right)\cos\left(\frac{4}{3}n\theta\right)$$

$$\Longrightarrow -\tilde{c}_0\ln(2)+\tilde{d}_0=\frac{4}{3\pi}\int_0^{\frac{3\pi}{4}}\left(\frac{3\pi}{4}-\theta\right)d\theta=\frac{3\pi}{8},$$

$$\Longrightarrow \tilde{c}_n2^{\frac{4}{3}n}+\tilde{d}_n2^{-\frac{4}{3}n}=\frac{4}{3\pi}\int_0^{\frac{3\pi}{4}}\left(\frac{3\pi}{4}-\theta\right)\cos\left(\frac{4}{3}n\theta\right)d\theta=\frac{\frac{3}{2\pi}(1-(-1)^n)}{n^2}=g_n$$

$$u(1,\theta)=\tilde{d}_0+\sum_{n=1}^{\infty}(\tilde{c}_n+\tilde{d}_n)\cos\left(\frac{4}{3}n\theta\right)$$

$$\Longrightarrow \tilde{d}_0=\frac{4}{3\pi}\int_0^{\frac{3\pi}{4}}\cos(\theta)\,d\theta=\frac{2\sqrt{2}}{3\pi}$$

$$\Longrightarrow \tilde{c}_n+\tilde{d}_n=\frac{8}{3\pi}\int_0^{\frac{3\pi}{4}}\cos(\theta)\cos\left(\frac{4}{3}n\theta\right)d\theta=\frac{4}{3\pi}\int_0^{\frac{3\pi}{4}}\cos\left(\left(\frac{4}{3}n-1\right)\theta\right)+\cos\left(\left(\frac{4}{3}n+1\right)\theta\right)d\theta$$

$$=\frac{4}{3\pi}\left(\frac{\sin\left(\left(\frac{4}{3}n-1\right)\frac{3\pi}{4}\right)}{\frac{4}{3}n-1}+\frac{\sin\left(\left(\frac{4}{3}n+1\right)\frac{3\pi}{4}\right)}{\frac{4}{3}n+1}\right)=h_n$$

$$\begin{cases} \tilde{c}_n 2^{\frac{4}{3}n} + \tilde{d}_n 2^{-\frac{4}{3}n} = g_n \\ \tilde{c}_n + \tilde{d}_n = h_n \end{cases} \Longrightarrow \begin{cases} \tilde{c}_n = \dfrac{g_n - h_n 2^{-\frac{4}{3}n}}{2^{\frac{4}{3}n} - 2^{-\frac{4}{3}n}} \\ \tilde{d}_n = \dfrac{g_n - h_n 2^{\frac{4}{3}n}}{2^{-\frac{4}{3}n} - 2^{\frac{4}{3}n}} \end{cases}$$

$$\Longrightarrow \boxed{u(r,\theta) = \left(\frac{3\pi}{8} - \frac{2\sqrt{2}}{3\pi}\right)\frac{\ln(r)}{\ln(2)} + \frac{2\sqrt{2}}{3\pi} + \sum_{n=1}^{\infty}\left(g_n \frac{r^{\frac{4}{3}n} - r^{-\frac{4}{3}n}}{2^{\frac{4}{3}n} - 2^{-\frac{4}{3}n}} + h_n \frac{\left(\frac{2}{r}\right)^{\frac{4}{3}n} - \left(\frac{2}{r}\right)^{-\frac{4}{3}n}}{2^{\frac{4}{3}n} - 2^{-\frac{4}{3}n}}\right)\cos\left(\frac{4}{3}n\theta\right)}$$

۳)

معادله زیر را با تبدیل لاپلاس حل کنید.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x^2 + 2, \qquad x > 0, \qquad t > 0$$

$$u(0,t) = 0, \qquad u(x,0) = 0$$

$$U(x,s) = \mathcal{L}\{u(x,t)\} \Longrightarrow \mathcal{L}\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right\} = \mathcal{L}\{x^2 + 2\} = \frac{dU(x,s)}{dx} + sU(x,s) - u(x,0) = \frac{x^2 + 2}{s}$$

جواب معادله دیفرانسیلی فوق به صورت جمع جواب های همگن و ناهمگن است: $U = U_h + U_p$

$$\xrightarrow{u(x,0)=0} \frac{dU(x,s)}{dx} + sU(x,s) = \frac{x^2 + 2}{s} \ \text{(I)}$$

$$U_h = c_0 e^{-\int s dx} = c_0 e^{-sx} \ \text{(II)}$$

$$U_p = ax^2 + bx + c \xrightarrow{\text{(I)}} 2ax + b + s(ax^2 + bx + c) = \frac{x^2 + 2}{s}$$

$$\begin{cases} as = \dfrac{1}{s} \\ 2a + bs = 0 \\ b + cs = \dfrac{2}{s} \end{cases} \Longrightarrow \begin{cases} a = \dfrac{1}{s^2} \\ b = -\dfrac{2}{s^3} \\ c = \dfrac{2}{s^2} + \dfrac{2}{s^4} \end{cases};$$

$$U(x,s) = \frac{x^2}{s^2} - \frac{2x}{s^3} + \left(\frac{2}{s^2} + \frac{2}{s^4}\right) + c_0 e^{-sx} \xrightarrow{U(0,s)=0} c_0 = -\left(\frac{2}{s^2} + \frac{2}{s^4}\right)$$

$$u(x,t) = \mathcal{L}^{-1}\{U(x,s)\} = \boxed{x^2 tH(t) - xt^2 H(t) + 2tH(t) + \frac{t^3}{6}H(t) - 2(t-x)H(t-x) - \frac{(t-x)^3}{3}H(t-x)}$$

$$\xrightarrow{\Delta} H(x) = \begin{cases} 0, x < 0 \\ 1, x > 0 \end{cases}$$

معادله زیر را با تبدیل لاپلاس حل کنید.

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=2x^2+x, \qquad x>0, \qquad t>0$$

$$u(0,t)=0, \qquad u(x,0)=0$$

$$U(x,s)=\mathcal{L}\{u(x,t)\} \Longrightarrow \mathcal{L}\left\{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}\right\}=\mathcal{L}\{2x^2+x\}=\frac{dU(x,s)}{dx}+sU(x,s)-u(x,0)=\frac{2x^2+x}{s}$$

جواب معادله دیفرانسیلی فوق به صورت جمع جواب های همگن و ناهمگن است: $U=U_h+U_p$

$$\xrightarrow{u(x,0)=0} \frac{dU(x,s)}{dx}+sU(x,s)=\frac{2x^2+x}{s} \text{ (I)}$$

$$U_h=c_0e^{-\int s dx}=c_0e^{-sx} \text{ (II)}$$

$$U_p=ax^2+bx+c \overset{\text{(I)}}{\rightarrow} 2ax+b+s(ax^2+bx+c)=\frac{2x^2+x}{s}$$

$$\begin{cases} as=\frac{2}{s} \\ 2a+bs=\frac{1}{s} \\ b+cs=0 \end{cases} \Longrightarrow \begin{cases} a=\frac{2}{s^2} \\ b=-\frac{4}{s^3}+\frac{1}{s^2} \\ c=\frac{4}{s^4}-\frac{1}{s^3} \end{cases}$$

$$U(x,s)=\frac{2x^2}{s^2}-x\left(-\frac{4}{s^3}+\frac{1}{s^2}\right)+\left(\frac{4}{s^4}-\frac{1}{s^3}\right)+c_0e^{-sx} \xrightarrow{U(0,s)=0} c_0=-\left(\frac{4}{s^4}-\frac{1}{s^3}\right)$$

$$u(x,t)=\mathcal{L}^{-1}\{U(x,s)\}$$

$$=\boxed{2x^2tH(t)+2xt^2H(t)-xtH(t)+\frac{2t^3}{3}H(t)-\frac{t^2}{2}H(t)-2\frac{(t-x)^3}{3}H(t-x)+\frac{(t-x)^2}{2}H(t-x)}$$

$$\overset{\Delta}{\rightarrow} H(x)=\begin{cases}0, x<0\\1, x>0\end{cases}$$

معادله زیر را با تبدیل لاپلاس حل کنید.

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < 2, \qquad t > 0$$
$$u(0,t) = 0, \qquad u(2,t) = 0$$
$$u(x,0) = 3\sin(2\pi x)$$

بعد از اعمال تبدیل لاپلاس به معادله داده شده، داریم:

$$\frac{d^2U(x,s)}{dx^2} - sU(x,s) = -u(x,0) = -3\sin(2\pi x)$$

جواب معادله دیفرانسیلی فوق به صورت جمع جواب های همگن و ناهمگن است: $U = U_h + U_p$

$$U_h = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x}, \qquad U_p = A\cos(2\pi x) + B\sin(2\pi x)$$
$$\frac{d^2U_p}{dx^2}(x,s) = -4\pi^2\big(A\cos(2\pi x) + B\sin(2\pi x)\big)$$
$$\frac{d^2U_p}{dx^2}(x,s) - sU_p(x,s) = (-4\pi^2 - s)(A\cos(2\pi x) + B\sin(2\pi x)) = -3\sin(2\pi x) \Longrightarrow \begin{cases} A = 0 \\ B = \dfrac{3}{4\pi^2 + s} \end{cases}$$
$$U(x,s) = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x} + \frac{3}{4\pi^2 + s}\sin(2\pi x)$$
$$\begin{cases} u(0,t) = 0 \Longrightarrow U(0,s) = 0 \\ u(2,t) = 0 \Longrightarrow U(2,s) = 0 \end{cases} \Longrightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 e^{-\sqrt{s}2} + c_2 e^{\sqrt{s}2} = 0 \end{cases} \Longrightarrow c_1 = c_2 = 0 \Longrightarrow U(x,s) = \frac{3}{4\pi^2 + s}\sin(2\pi x)$$
$$\mathcal{L}^{-1}\{U(x,s)\} = u(x,t) = \mathcal{L}^{-1}\left\{\frac{3}{4\pi^2 + s}\sin(2\pi x)\right\} = \boxed{3e^{-4\pi^2 t}\sin(2\pi x)\,H(t)}$$
$$\xrightarrow{\Delta} H(x) = \begin{cases} 0, x < 0 \\ 1, x > 0 \end{cases}$$

معادله زیر را با تبدیل لاپلاس حل کنید.

$$\frac{\partial u}{\partial t}(x,t)=\frac{\partial^2 u}{\partial x^2}(x,t),\qquad 0<x<7,\qquad t>0$$

$$u(0,t)=0,\qquad u(7,t)=0$$

$$u(x,0)=4\sin(7\pi x)$$

بعد از اعمال تبدیل لاپلاس به معادله داده شده، داریم:

$$\frac{d^2U(x,s)}{dx^2}-sU(x,s)=-u(x,0)=-4\sin(7\pi x)$$

جواب معادله دیفرانسیلی فوق به صورت جمع جواب های همگن و ناهمگن است: $U=U_h+U_p$

$$U_h=c_1e^{-\sqrt{s}x}+c_2e^{\sqrt{s}x},\qquad U_p=A\cos(7\pi x)+B\sin(7\pi x)$$

$$\frac{d^2U_p(x,s)}{dx^2}=-49\pi^2(A\cos(7\pi x)+B\sin(7\pi x))$$

$$\frac{d^2U_p}{dx^2}(x,s)-sU_p(x,s)=(-49\pi^2-s)(A\cos(7\pi x)+B\sin(7\pi x))=-4\sin(7\pi x)\Longrightarrow\begin{cases}A=0\\ B=\dfrac{4}{49\pi^2+s}\end{cases}$$

$$U(x,s)=c_1e^{-\sqrt{s}x}+c_2e^{\sqrt{s}x}+\frac{4}{49\pi^2+s}\sin(7\pi x)$$

$$\begin{cases}u(0,t)=0\Longrightarrow U(0,s)=0\\ u(7,t)=0\Longrightarrow U(7,s)=0\end{cases}\Longrightarrow\begin{cases}c_1+c_2=0\\ c_1e^{-\sqrt{s}2}+c_2e^{\sqrt{s}2}=0\end{cases}\Longrightarrow c_1=c_2=0\Longrightarrow U(x,s)=\frac{4}{49\pi^2+s}\sin(7\pi x)$$

$$\mathcal{L}^{-1}\{U(x,s)\}=u(x,t)=L^{-1}\left\{\frac{4}{49\pi^2+s}\sin(7\pi x)\right\}=\boxed{4e^{-49\pi^2t}\sin(7\pi x)\,H(t)}$$

$$\xrightarrow{\Delta}H(x)=\begin{cases}0,x<0\\ 1,x>0\end{cases}$$