ریاضی مهندسی

نیم‌سال دوم ۱۴۰۰-۱۴۰۱	**پاسخ تکلیف شماره ۷**	

حل معادلات مشتقات جزئی با تبدیلات

پاسخ سوال ۱ : (۲۵ نمره)

ب

$$\frac{d^2U}{dx^2} - s^2 U = -s e^x \Rightarrow \begin{cases} U_h = Ae^{xs} + Be^{-xs} \\ U_p = \dfrac{s}{s^2-1} e^x \end{cases} \Rightarrow U_{(x,s)} = U_h + U_p$$

$$\begin{cases} u(0,t) = 1 \xrightarrow{L} U(0,s) = 1/s \\ \lim\limits_{x\to-\infty} u(x,t) = 0 \xrightarrow{L} \lim\limits_{x\to-\infty} U(x,s) = 0 \Rightarrow B = 0 \end{cases} \Rightarrow A = 1/s - \frac{s}{s^2-1}$$

$$\Rightarrow U(x,s) = \frac{1}{s} e^{xs} - \frac{1}{2}\frac{e^{xs}}{s-1} - \frac{1}{2}\frac{e^{xs}}{s+1} + \frac{1}{2}\frac{e^{x}}{s-1} + \frac{1}{2}\frac{e^{x}}{s+1}$$

$$u(x,t) = L^{-1}\{U(x,s)\} = \left(1 - \frac{1}{2}e^{t+x} - \frac{1}{2}e^{-(t+x)}\right)\overbrace{u(t+x)}^{1} + \frac{1}{2}\left(e^{t} + e^{-t}\right)e^{x}\, u(t)$$

$$= \left(1 - \cosh(t+x)\right)u(t+x) + e^{x}\cosh t\; u(t)$$

پاسخ سوال ۲: (۲۵ نمره)

ج ه

$$\frac{d^2U}{dx^2} - s^2 U = -\sin\pi x\left(\frac{1}{s^2+1}\right) \Rightarrow \begin{cases} U_h = Ae^{sx} + Be^{-sx} \\ U_p = \left(\dfrac{1}{s^2+1}\right)\left(\dfrac{1}{s^2+\pi^2}\right)\sin\pi x \end{cases} \Rightarrow U(x,s) = U_p + U_h$$

$$\begin{cases} U(0,s) = 0 \\ U(1,s) = 0 \end{cases} \Rightarrow A = B = 0 \Rightarrow U(x,s) = \frac{\sin\pi x}{(s^2+1)(s^2+\pi^2)}$$

$$U(x,s) = \sin\pi x\left(\frac{\frac{1}{\pi^2-1}}{s^2+1} - \frac{\frac{1}{\pi^2-1}}{s^2+\pi^2}\right) = \frac{\sin\pi x}{\pi^2-1}\left(\frac{1}{s^2+1} - \frac{1}{s^2+\pi^2}\right)$$

$$\Rightarrow u(x,t) = \frac{\sin\pi x}{\pi^2-1}\left(\sin t - \frac{1}{\pi}\sin\pi t\right) \qquad t>0$$

پاسخ سوال ۳: (۲۵ نمره)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \xrightarrow{\mathcal{L}} \frac{d^2U}{dx^2}(x,s) = sU(x,s) - u(x,0) = sU(x,s) - 3\sin\left(\frac{2\pi x}{l}\right)$$

$$\Rightarrow U'' - sU = -3\sin\left(\frac{2\pi x}{l}\right) \qquad U(x,s) = \begin{cases} U_h(x,s) \\ U_p(x,s) \end{cases} = U_h + U_p$$

$$U_h(x,s) = c_1 e^{x\sqrt{s}} + c_2 e^{-x\sqrt{s}}$$

$$U_p = A\cos\frac{2\pi x}{l} + B\sin\frac{2\pi x}{l} \longrightarrow U_p' = -\frac{2\pi}{l}A\sin\left(\frac{2\pi x}{l}\right) + \frac{2\pi B}{l}\cos 2\pi x \longrightarrow$$

$$U_p'' = -\left(\frac{4\pi^2}{l^2}\right)A\cos 2\pi x - \left(\frac{4\pi^2}{l^2}\right)B\sin 2\pi x$$

$$U_p'' - sU_p = -\left(\frac{4\pi^2}{l^2} + s\right)\left[A\cos 2\pi x + B\sin 2\pi x\right] = -3\sin 2\pi x \longrightarrow \begin{cases} \left(s + \left(\frac{2\pi}{l}\right)^2\right)A = 0 \\ B\left(s + \frac{4\pi^2}{l^2}\right) = 3 \end{cases}$$

$$\rightarrow \begin{cases} A = 0 \\ B = \dfrac{3}{s + \frac{4\pi^2}{l^2}} \end{cases} \Longrightarrow \boxed{U(x,s) = c_1 e^{x\sqrt{s}} + c_2 e^{-x\sqrt{s}} + \frac{3l^2}{l^2 s + 4\pi^2}\sin\frac{2\pi x}{l}}$$

$$\left.\begin{matrix} u(0,t) = 0 \Rightarrow U(0,s) = 0 \\ u(l,t) = 0 \Rightarrow U(l,s) = 0 \end{matrix}\right\} \rightarrow \left.\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{l\sqrt{s}} + c_2 e^{-l\sqrt{s}} = 0 \end{cases}\right\} \Rightarrow c_1 = c_2 = 0$$

$$\rightarrow U(x,s) = \frac{3l^2}{sl^2 + 4\pi^2}\sin\frac{2\pi x}{l} \xrightarrow{\mathcal{L}^{-1}} u(x,t) = \mathcal{L}^{-1}\left\{\frac{3\sin\frac{2\pi x}{l}}{s + \frac{4\pi^2}{l^2}}\right\} = \boxed{3e^{-\frac{4\pi^2}{l^2}t}\sin\frac{2\pi x}{l}}$$

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پاسخ سوال ۴: (۲۵ نمره)

ت[illegible]یه نسبت به:

$$F(u_t) = c^2 F(u_{xx}) \longrightarrow \hat{u}_t + c^2 \omega^2 \hat{u} = 0$$

$$\longrightarrow \hat{u}(\omega, t) = A(\omega)\, e^{-c^2\omega^2 t}$$

$$u(x,0) = f(x) \longrightarrow \hat{u}(\omega, 0) = \hat{f}(\omega) \longrightarrow A(\omega)\, e^{-c^2\omega^2(0)} = \hat{f}(\omega)$$

$$\longrightarrow A(\omega) = \hat{f}(\omega) \qquad \longrightarrow \hat{u}(\omega, t) = \hat{f}(\omega)\, e^{-c^2\omega^2 t}$$

$$\xrightarrow{F^{-1}} u(x,t) = f(x) * \frac{e^{-x^2/4c^2t}}{2c\sqrt{\pi t}}$$

موفق باشید – خان چرلی