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$$f(x)=\sinh(x)\,,\qquad A=\sum_{n=1}^{\infty}\frac{1}{n^2+\frac{1}{4}},\qquad B=\sum_{n=1}^{\infty}\frac{1}{\left(n^2+\frac{1}{4}\right)^2}$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}\sinh(x)\,dx=\frac{\cosh(\pi)-1}{2}=\frac{2\sinh^2\left(\frac{\pi}{2}\right)}{\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\sinh(x)\cos(2nx)\,dx$$

$$\int\sinh(x)\cos(2nx)\,dx=\frac{1}{2n}\sinh(x)\sin(2nx)-\frac{1}{2n}\int\cosh(x)\sin(2nx)\,dx$$

$$=\frac{1}{2n}\sinh(x)\sin(2nx)-\frac{1}{2n}\left(-\frac{1}{2n}\cosh(x)\cos(2nx)+\frac{1}{2n}\int\sinh(x)\cos(2nx)\,dx\right)$$

$$\Longrightarrow a_n=\frac{2}{\pi}\left[\frac{2n\sinh(x)\sin(2nx)+\cosh(x)\cos(2nx)}{4n^2+1}\right]_0^{\pi}=\frac{2(\cosh(\pi)-1)}{(4n^2+1)\pi}=\frac{4\sinh^2\left(\frac{\pi}{2}\right)}{(4n^2+1)\pi}=\frac{\sinh^2\left(\frac{\pi}{2}\right)}{\left(n^2+\frac{1}{4}\right)\pi}$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}\sinh(x)\cos(2nx)\,dx$$

$$\int\sinh(x)\sin(2nx)\,dx=-\frac{1}{2n}\sinh(x)\cos(2nx)+\frac{1}{2n}\int\cosh(x)\cos(2nx)\,dx$$

$$=-\frac{1}{2n}\sinh(x)\cos(2nx)+\frac{1}{2n}\left(\frac{1}{2n}\cosh(x)\sin(2nx)-\frac{1}{2n}\int\sinh(x)\sin(2nx)\,dx\right)$$

$$\Longrightarrow b_n=\frac{2}{\pi}\left[\frac{\cosh(x)\sin(2nx)-2n\sinh(x)\cos(2nx)}{4n^2+1}\right]_0^{\pi}=-\frac{n\sinh(\pi)}{\left(n^2+\frac{1}{4}\right)\pi}$$

$$\Longrightarrow\boxed{f(x)=\frac{2\sinh^2\left(\frac{\pi}{2}\right)}{\pi}+\sum_{n=1}^{\infty}\left(\frac{\sinh^2\left(\frac{\pi}{2}\right)}{\left(n^2+\frac{1}{4}\right)\pi}\right)\cos(2nx)+\left(-\frac{n\sinh(\pi)}{\left(n^2+\frac{1}{4}\right)\pi}\right)\sin(2nx)}$$

$$f(\pi)=\frac{\sinh(\pi)}{2}=\frac{2\sinh^2\left(\frac{\pi}{2}\right)}{\pi}+\sum_{n=1}^{\infty}\frac{\sinh^2\left(\frac{\pi}{2}\right)}{\left(n^2+\frac{1}{4}\right)\pi}\Longrightarrow\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2+\frac{1}{4}}=\pi\coth\left(\frac{\pi}{2}\right)-2}$$

$$\frac{2}{\pi}\int_0^{\pi}\sinh^2(x)\,dx=\frac{1}{\pi}\int_0^{\pi}(\cosh(2x)-1)dx=\frac{\sinh(2\pi)}{2\pi}-1=\frac{8\sinh^4\left(\frac{\pi}{2}\right)}{\pi^2}+\sum_{n=1}^{\infty}\frac{\sinh^4\left(\frac{\pi}{2}\right)}{\left(n^2+\frac{1}{4}\right)^2\pi^2}+\frac{n^2\sinh^2(\pi)}{\left(n^2+\frac{1}{4}\right)^2\pi^2}$$

$$=\frac{8\sinh^4\left(\frac{\pi}{2}\right)}{\pi^2}+\sum_{n=1}^{\infty}\frac{\sinh^4\left(\frac{\pi}{2}\right)}{\left(n^2+\frac{1}{4}\right)^2\pi^2}+\frac{\sinh^2(\pi)}{\left(n^2+\frac{1}{4}\right)\pi^2}-\frac{\frac{1}{4}\sinh^2(\pi)}{\left(n^2+\frac{1}{4}\right)^2\pi^2}$$

$$\frac{\sinh(2\pi)}{2\pi}-1-\frac{8\sinh^4\left(\frac{\pi}{2}\right)}{\pi^2}-\frac{1}{\pi}4\sinh\left(\frac{\pi}{2}\right)\cosh^3\left(\frac{\pi}{2}\right)+\frac{2}{\pi^2}\sinh^2(\pi)=-\sum_{n}^{\infty}\frac{\sinh^2\left(\frac{\pi}{2}\right)}{\left(n^2+\frac{1}{4}\right)^2\pi^2}$$

$$\sum_{n}^{\infty}\frac{\sinh^2\left(\frac{\pi}{2}\right)}{\left(n^2+\frac{1}{4}\right)^2\pi^2}=1+\frac{8\sinh^4\left(\frac{\pi}{2}\right)}{\pi^2}+\frac{1}{\pi}4\sinh\left(\frac{\pi}{2}\right)\cosh^3\left(\frac{\pi}{2}\right)-\frac{2}{\pi^2}\sinh^2(\pi)-\frac{\sinh(2\pi)}{2\pi}$$

$$\Longrightarrow\sum_{n}^{\infty}\frac{1}{\left(n^2+\frac{1}{4}\right)^2}=\pi^2\operatorname{csch}^2\left(\frac{\pi}{2}\right)+8\sinh^2\left(\frac{\pi}{2}\right)+4\pi\coth\left(\frac{\pi}{2}\right)\cosh^2\left(\frac{\pi}{2}\right)-8\cosh^2\left(\frac{\pi}{2}\right)-2\pi\left(2\cosh^2\left(\frac{\pi}{2}\right)-1\right)\coth\left(\frac{\pi}{2}\right)$$

$$=\pi^2\operatorname{csch}^2\left(\frac{\pi}{2}\right)-8+2\pi\coth\left(\frac{\pi}{2}\right)$$

$$\Longrightarrow\boxed{\sum_{n}^{\infty}\frac{1}{\left(n^2+\frac{1}{4}\right)^2}=\pi^2\operatorname{csch}^2\left(\frac{\pi}{2}\right)+2\pi\coth\left(\frac{\pi}{2}\right)-8}$$

۱-۱) سری فوریه تابع $f(x)$ را در بازه $[0,2\pi]$ بدست آورید و با استفاده از آن حاصل سری A و B را بدست آورید.

$$f(x)=\cosh(x)\,,\qquad A=\sum_{n=1}^{\infty}\frac{1}{n^2+1},\qquad B=\sum_{n=1}^{\infty}\frac{1}{(n^2+1)^2}$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}\cosh(x)\,dx=\frac{\sinh(2\pi)}{2\pi}$$

$$a_n=\frac{2}{2\pi}\int_0^{2\pi}\cosh(x)\cos(nx)\,dx$$

$$\int\cosh(x)\cos(nx)\,dx=\frac{1}{n}\cosh(x)\sin(nx)-\frac{1}{n}\int\sinh(x)\sin(nx)\,dx$$

$$=\frac{1}{n}\cosh(x)\sin(nx)-\frac{1}{n}\left(-\frac{1}{n}\sinh(x)\cos(nx)+\frac{1}{n}\int\cosh(x)\cos(nx)\,dx\right)$$

$$\Rightarrow a_n=\frac{1}{\pi}\left[\frac{n\cosh(x)\sin(nx)+\sinh(x)\cos(nx)}{n^2+1}\right]_0^{2\pi}=\frac{\sinh(2\pi)}{\pi(n^2+1)}$$

$$b_n=\frac{2}{2\pi}\int_0^{2\pi}\cosh(x)\sin(nx)\,dx$$

$$\int\cosh(x)\sin(nx)\,dx=-\frac{1}{n}\cosh(x)\cos(nx)+\frac{1}{n}\int\sinh(x)\cos(nx)\,dx$$

$$=-\frac{1}{n}\cosh(x)\cos(nx)+\frac{1}{n}\left(\frac{1}{n}\sinh(x)\sin(nx)-\frac{1}{n}\int\cosh(x)\sin(nx)\,dx\right)$$

$$\Rightarrow\int\cosh(x)\sin(nx)\,dx=\frac{-n\cosh(x)\cos(nx)+\sinh(x)\sin(nx)}{n^2+1}$$

$$\Rightarrow b_n=\frac{1}{\pi}\left[\frac{-n\cosh(x)\cos(nx)+\sinh(x)\sin(nx)}{n^2+1}\right]_0^{2\pi}=\frac{-n(\cosh(2\pi)-1)}{\pi(n^2+1)}=\frac{-2n\sinh^2(\pi)}{\pi(n^2+1)}$$

$$\Rightarrow\boxed{f(x)=\frac{\sinh(2\pi)}{2\pi}+\sum_{n=1}^{\infty}\left(\frac{\sinh(2\pi)}{\pi(n^2+1)}\right)\cos(nx)+\left(-\frac{2n\sinh^2(\pi)}{\pi(n^2+1)}\right)\sin(nx)}$$

$$f(2\pi)=\frac{\cosh(2\pi)+1}{2}=\cosh^2(\pi)=\frac{\sinh(2\pi)}{2\pi}+\sum_{n=1}^{\infty}\frac{\sinh(2\pi)}{\pi(n^2+1)}\Rightarrow\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2+1}=\frac{\pi}{2}\coth(\pi)-\frac{1}{2}}$$

$$\frac{1}{\pi}\int_0^{2\pi}\cosh^2(x)\,dx=1+\frac{\sinh(4\pi)}{4\pi}=\frac{\sinh^2(2\pi)}{2\pi^2}+\sum_{n=1}^{\infty}\frac{\sinh^2(2\pi)}{\pi^2(n^2+1)^2}+\frac{4n^2\sinh^4(\pi)}{\pi^2(n^2+1)^2}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{4\sinh^2(\pi)\cosh^2(\pi)+4(n^2+1)\sinh^4(\pi)-4\sinh^4(\pi)}{\pi^2(n^2+1)^2}=\sum_{n=1}^{\infty}\frac{4\sinh^2(\pi)}{\pi^2(n^2+1)^2}+\sum_{n=1}^{\infty}\frac{4\sinh^4(\pi)}{\pi^2(n^2+1)}=1+\frac{\sinh(4\pi)}{4\pi}-\frac{\sinh^2(2\pi)}{2\pi^2}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{4\sinh^2(\pi)}{\pi^2(n^2+1)^2}=1+\frac{\sinh(4\pi)}{4\pi}-\frac{\sinh^2(2\pi)}{2\pi^2}-\frac{2\sinh^3(\pi)\cosh(\pi)}{\pi}+\frac{2\sinh^4(\pi)}{\pi^2}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{(n^2+1)^2}=\frac{\pi^2}{4}\operatorname{csch}^2(\pi)+\frac{\pi}{4}\cosh(2\pi)\coth(\pi)-\frac{1}{2}\cosh^2(\pi)-\frac{\pi}{4}\sinh(2\pi)+\frac{1}{2}\sinh^2(\pi)$$

$$\Rightarrow\boxed{\sum_{n=1}^{\infty}\frac{1}{(n^2+1)^2}=\frac{\pi^2}{4}\operatorname{csch}^2(\pi)+\frac{\pi}{4}\coth(\pi)-\frac{1}{2}}$$

۲-۱) سری فوریه تابع $f(x)$ را در بازه $[0,\frac{\pi}{2}]$ بدست آورید و با استفاده از آن حاصل سری A و B را بدست آورید.

$$f(x)=\cos(x)\,,\qquad A=\sum_{n=1}^{\infty}\frac{1}{n^2-\frac{1}{16}}\,,\qquad B=\sum_{n=1}^{\infty}\frac{1}{\left(n^2-\frac{1}{16}\right)^2}$$

$$a_0=\frac{1}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos(x)\,dx=\frac{2}{\pi}\sin\left(\frac{\pi}{2}\right)=\frac{2}{\pi}$$

$$a_n=\frac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos(x)\cos(4nx)\,dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\Big(\cos\big((4n+1)x\big)+\cos\big((4n-1)x\big)\Big)dx=-\frac{4}{(16n^2-1)\pi}$$

$$b_n=\frac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos(x)\sin(4nx)\,dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\Big(\sin\big((4n+1)x\big)+\sin\big((4n-1)x\big)\Big)dx=\frac{16n}{(16n^2-1)\pi}$$

$$\Rightarrow\boxed{f(x)=\frac{2}{\pi}+\sum_{n=1}^{\infty}\left(-\frac{1}{4\pi\left(n^2-\frac{1}{16}\right)}\right)\cos(4n\pi x)+\left(\frac{n}{\pi\left(n^2-\frac{1}{16}\right)}\right)\sin(4n\pi x)}$$

$$f(0)=\frac{1}{2}=\frac{2}{\pi}-\sum_{n=1}^{\infty}\frac{1}{4\pi\left(n^2-\frac{1}{16}\right)}\Rightarrow\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2-\frac{1}{16}}=8-2\pi}$$

$$\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\cos^2(x)\,dx=1=\frac{8}{\pi^2}+\sum_{n=1}^{\infty}\frac{1}{16\pi^2\left(n^2-\frac{1}{16}\right)^2}+\frac{n^2}{\pi^2\left(n^2-\frac{1}{16}\right)^2}=\frac{8}{\pi^2}+\sum_{n=1}^{\infty}\frac{1}{16\pi^2\left(n^2-\frac{1}{16}\right)^2}+\frac{1}{\pi^2\left(n^2-\frac{1}{16}\right)}+\frac{1}{16\pi^2\left(n^2-\frac{1}{16}\right)^2}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{8\pi^2(16n^2-1)^2}=1-\frac{8}{\pi^2}-\sum_{n=1}^{\infty}\frac{1}{\pi^2\left(n^2-\frac{1}{16}\right)}=1-\frac{8}{\pi^2}-\frac{8}{\pi^2}+\frac{2}{\pi}=1+\frac{2}{\pi}-\frac{16}{\pi^2}$$

$$\Rightarrow\boxed{\sum_{n=1}^{\infty}\frac{1}{\left(n^2-\frac{1}{16}\right)^2}=8\pi^2+16\pi-128}$$

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۲-۰) انتگرال فوریه تابع $f(x)$ را محاسبه کنید و به کمک آن تابع $g(x)$ را بدست آورید.

$$f(x)=\frac{\pi}{2}e^{-k|x|},\qquad g(x)=\int_0^{\infty}\frac{1}{\omega^2+1024}\cos(\omega x)\,d\omega$$

$$B(\omega)=0,\qquad A(\omega)=\frac{2}{\pi}\int_0^{\infty}\frac{\pi}{2}e^{-k|x|}\cos(\omega x)\,dx=\mathcal{L}[\cos(\omega x)]_{s=k}=\frac{k}{k^2+\omega^2}$$

$$\Rightarrow\boxed{f(x)=\int_0^{\infty}\frac{k}{k^2+\omega^2}\cos(\omega x)\,d\omega}$$

$$\frac{1}{\omega^2+1024}=\frac{1}{i64}\left(\frac{1}{\omega^2-i32}-\frac{1}{\omega^2+i32}\right)=\frac{1}{i64}\left(\frac{1}{\omega^2+(4-i4)^2}-\frac{1}{\omega^2+(4+i4)^2}\right)$$

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$$g(x)=\int_0^{\infty}\frac{1}{i64}\left(\frac{1}{\omega^2+(4-i4)^2}-\frac{1}{\omega^2+(4+i4)^2}\right)\cos(\omega x)\,dx=\frac{1}{i64}\left(\frac{\pi}{8(1-i)}e^{-4(1-i)|x|}-\frac{\pi}{8(1+i)}e^{-4(1+i)|x|}\right)$$

$$=\frac{\pi}{i1024}\left((1+i)e^{-4|x|}e^{i4|x|}-(1-i)e^{-4|x|}e^{-i4|x|}\right)=\frac{\pi e^{-4|x|}}{i1024}\big(i2\sin(4|x|)+i2\cos(4|x|)\big)$$

$$=\boxed{\frac{\pi e^{-4|x|}}{512}\big(\sin(4|x|)+\cos(4|x|)\big)}$$

۱-۲) انتگرال فوریه تابع $f(x)$ را محاسبه کنید.

$$f(x)=e^{-2|x|}\cos(x)$$

$$B(\omega)=0,\qquad A(\omega)=\frac{2}{\pi}\int_0^{\infty}e^{-2|x|}\cos(x)\cos(\omega x)\,dx=\frac{1}{\pi}\int_0^{\infty}e^{-2x}\Big(\cos\big((\omega+1)x\big)+\cos\big((\omega-1)x\big)\Big)dx$$

$$=\frac{1}{\pi}\Big(\mathcal{L}\big[\cos\big((\omega+1)x\big)\big]_{s=2}+\mathcal{L}\big[\cos\big((\omega-1)x\big)\big]_{s=2}\Big)=\frac{1}{\pi}\left(\frac{2}{4+(\omega+1)^2}+\frac{2}{4+(\omega-1)^2}\right)=\frac{4(5+\omega^2)}{\pi(25+6\omega^2+\omega^4)}$$

$$\Rightarrow f(x)=\int_0^{\infty}\frac{4(5+\omega^2)}{\pi(25+6\omega^2+\omega^4)}\cos(\omega x)\,d\omega$$

۲-۲) انتگرال فوریه تابع $f(x)$ را محاسبه کنید.

$$f(x) = e^{-2|x|}\sin(x)$$

$$A(\omega) = 0, \qquad B(\omega) = \frac{2}{\pi}\int_0^\infty e^{-2|x|}\sin(x)\sin(\omega x)\,dx = \frac{1}{\pi}\int_0^\infty e^{-2x}\big(\cos((\omega-1)x) - \cos((\omega+1)x)\big)dx$$

$$= \frac{1}{\pi}\Big(\mathcal{L}\big[\cos((\omega-1)x)\big]_{s=2} - \mathcal{L}\big[\cos((\omega+1)x)\big]_{s=2}\Big) = \frac{1}{\pi}\left(\frac{2}{4+(\omega-1)^2} - \frac{2}{4+(\omega+1)^2}\right) = \frac{8\omega}{\pi(25+6\omega^2+\omega^4)}$$

$$\Longrightarrow f(x) = \int_0^\infty \frac{8\omega}{\pi(25+6\omega^2+\omega^4)}\sin(\omega x)\,d\omega$$

۳-۰) معادله دیفرانسیل زیر را به کمک تبدیل فوریه حل کنید. ۳

$$y''' + 2y'' - 1y' - 2y = -e^{2x}u(-x)$$

$$-i\omega^3 Y(\omega) - 2\omega^2 Y(\omega) - i\omega Y(\omega) - 2Y(\omega) = -\frac{1}{2-i\omega}$$

$$\Longrightarrow Y(\omega) = \frac{1}{(2+i\omega+2\omega^2+i\omega^3)(2-i\omega)} = \frac{1}{(1+\omega^2)(2+i\omega)(2-i\omega)} = \frac{1}{(1+\omega^2)(4+\omega^2)} = \frac{\frac{1}{3}}{1+\omega^2} - \frac{\frac{1}{3}}{4+\omega^2}$$

$$= \frac{1}{6}\frac{2}{1+\omega^2} - \frac{1}{12}\frac{4}{4+\omega^2}$$

$$\Longrightarrow \boxed{y(x) = \frac{1}{6}e^{-|x|} - \frac{1}{12}e^{-2|x|}}$$

۳-۱) معادله دیفرانسیل زیر را به کمک تبدیل فوریه حل کنید.

$$y''' + y'' - 9y' - 9y = -e^{x}u(-x)$$

$$-i\omega^3 Y(\omega) - \omega^2 Y(\omega) - i9\omega Y(\omega) - 9Y(\omega) = -\frac{1}{1-i\omega}$$

$$\Longrightarrow Y(\omega) = \frac{1}{(9+i9\omega+\omega^2+i\omega^3)(2-i\omega)} = \frac{1}{(9+\omega^2)(1+i\omega)(1-i\omega)} = \frac{1}{(9+\omega^2)(1+\omega^2)} = \frac{\frac{1}{8}}{1+\omega^2} - \frac{\frac{1}{8}}{9+\omega^2}$$

$$= \frac{1}{16}\frac{2}{1+\omega^2} - \frac{1}{48}\frac{6}{9+\omega^2}$$

$$\Longrightarrow \boxed{y(x) = \frac{1}{16}e^{-|x|} - \frac{1}{48}e^{-3|x|}}$$

۳-۲) معادله دیفرانسیل زیر را به کمک تبدیل فوریه حل کنید.

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$$y''' + 3y'' - 4y' - 12y = -e^{3x}u(-x)$$

$$-i\omega^3 Y(\omega) - 3\omega^2 Y(\omega) - i4\omega Y(\omega) - 12Y(\omega) = -\frac{1}{3-i\omega}$$

$$\Longrightarrow Y(\omega) = \frac{1}{(12+i4\omega+3\omega^2+i\omega^3)(2-i\omega)} = \frac{1}{(4+\omega^2)(3+i\omega)(3-i\omega)} = \frac{1}{(4+\omega^2)(9+\omega^2)} = \frac{\frac{1}{5}}{4+\omega^2} - \frac{\frac{1}{5}}{9+\omega^2}$$

$$= \frac{1}{20}\frac{4}{4+\omega^2} - \frac{1}{30}\frac{6}{9+\omega^2}$$

$$\Longrightarrow \boxed{y(x) = \frac{1}{20}e^{-2|x|} - \frac{1}{30}e^{-3|x|}}$$

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۴-۰) معادله موج زیر را حل کنید.

$$u_{tt}=4u_{xx}-x\cos(t)\,,\qquad 0\le x\le 1,\qquad 0\le t$$
$$u(0,t)=\sin(t)\,,\qquad u(1,t)=\cos(t)$$
$$u(x,0)=x^2+x,\qquad u_t(x,0)=1-3x$$

$$u(x,t)=w(x,t)+v(x,t)$$
$$w(x,t)=(1-x)\sin(t)+x\cos(t)$$
$$\Longrightarrow v_{tt}=4u_{xx}+(1-x)\sin(t)$$
$$v(0,t)=0,\qquad v(1,t)=0$$
$$v(x,0)=x^2,\qquad u_t(x,0)=-2x$$

$$\Longrightarrow v(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin(n\pi x)\Longrightarrow\sum_{n=1}^{\infty}\left(\ddot{T}_n(t)+4n^2\pi^2T_n(t)\right)\sin(n\pi x)=(1-x)\sin(t)$$

$$\Longrightarrow \ddot{T}_n(t)+4n^2\pi^2T_n(t)=2\sin(t)\int_0^1(1-x)\sin(n\pi x)\,dx=\frac{2}{n\pi}\sin(t)$$

$$\Longrightarrow T_n(t)=a_n\cos(2n\pi t)+b_n\sin(2n\pi t)+\frac{\frac{2}{n\pi}}{4n^2\pi^2-1}\sin(t)$$

$$v(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos(2n\pi t)+b_n\sin(2n\pi t)+\frac{\frac{2}{n\pi}}{4n^2\pi^2-1}\sin(t)\right)\sin(n\pi x)$$

$$v(x,0)=\sum_{n=1}^{\infty}a_n\sin(n\pi x)=x^2$$

$$\Longrightarrow a_n=2\int_0^1x^2\sin(n\pi x)\,dx=\frac{2(-1)^{n+1}}{n\pi}-\frac{4((-1)^{n+1}+1)}{n^3\pi^3}$$

$$v_t(x,0)=\sum_{n=1}^{\infty}\left(2b_nn\pi+\frac{\frac{2}{n\pi}}{4n^2\pi^2-1}\right)\sin(n\pi x)=-2x$$

$$\Longrightarrow 2b_nn\pi+\frac{\frac{2}{n\pi}}{4n^2\pi^2-1}=-4\int_0^1x\sin(n\pi x)\,dx=\frac{4(-1)^n}{n\pi}$$

$$\Longrightarrow b_n=\frac{2(-1)^n}{n^2\pi^2}-\frac{\frac{1}{n^2\pi^2}}{4n^2\pi^2-1}$$

$$\Longrightarrow\boxed{u(x,t)=(1-x)\sin(t)+x\cos(t)+\sum_{n=1}^{\infty}\left(\left(\frac{2(-1)^{n+1}}{n\pi}-\frac{4((-1)^{n+1}+1)}{n^3\pi^3}\right)\cos(2n\pi t)+\left(\frac{2(-1)^n}{n^2\pi^2}-\frac{\frac{1}{n^2\pi^2}}{4n^2\pi^2-1}\right)\sin(2n\pi t)+\frac{\frac{2}{n\pi}}{4n^2\pi^2-1}\sin(t)\right)\sin(n\pi x)}$$

۴-۱) معادله موج زیر را حل کنید.

$$u_{tt}=4u_{xx}-\cos(t)\,,\qquad 0\le x\le 1,\qquad 0\le t$$
$$u_x(0,t)=\sin(t)\,,\qquad u(1,t)=\cos(t)$$
$$u(x,0)=x^2+1,\qquad u_t(x,0)=-1$$

$$u(x,t)=w(x,t)+v(x,t)$$
$$w(x,t)=(x-1)\sin(t)+\cos(t)$$
$$\Longrightarrow v_{tt}=4u_{xx}+(x-1)\sin(t)$$
$$v_x(0,t)=0,\qquad v(1,t)=0$$
$$v(x,0)=x^2,\qquad u_t(x,0)=-x$$

$$\Longrightarrow v(x,t)=\sum_{n=1}^{\infty}T_n(t)\cos\left(\frac{(2n-1)\pi}{2}x\right)\Longrightarrow\sum_{n=1}^{\infty}\left(\ddot{T}_n(t)+(2n-1)^2\pi^2T_n(t)\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)=(x-1)\sin(t)$$

$$\Longrightarrow \ddot{T}_n(t)+(2n-1)^2\pi^2T_n(t)=2\sin(t)\int_0^1(x-1)\cos\left(\frac{(2n-1)\pi}{2}x\right)dx=-\frac{8}{(2n-1)^2\pi^2}\sin(t)$$

$$\Longrightarrow T_n(t)=a_n\cos\big((2n-1)\pi t\big)+b_n\sin\big((2n-1)\pi t\big)-\frac{\frac{8}{(2n-1)^2\pi^2}}{(2n-1)^2\pi^2-1}\sin(t)$$

$$v(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\big((2n-1)\pi t\big)+b_n\sin\big((2n-1)\pi t\big)-\frac{\frac{8}{(2n-1)^2\pi^2}}{(2n-1)^2\pi^2-1}\sin(t)\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

$$v(x,0)=\sum_{n=1}^{\infty}a_n\cos\left(\frac{(2n-1)\pi}{2}x\right)=x^2$$

$$\Longrightarrow a_n=2\int_0^1x^2\cos\left(\frac{(2n-1)\pi}{2}x\right)dx=\frac{32(-1)^n}{(2n-1)^3\pi^3}+\frac{4(-1)^{n+1}}{(2n-1)\pi}$$

$$v_t(x,0)=\sum_{n=1}^{\infty}\left(b_n(2n-1)\pi-\frac{\frac{8}{(2n-1)^2\pi^2}}{(2n-1)^2\pi^2-1}\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)=-x$$

$$\Longrightarrow b_n(2n-1)\pi - \frac{\frac{8}{(2n-1)^2\pi^2}}{(2n-1)^2\pi^2-1} = -2\int_0^1 x\cos\left(\frac{(2n-1)\pi}{2}x\right)dx = \frac{8}{(2n-1)^2\pi^2} + \frac{4(-1)^n}{(2n-1)\pi}$$

$$\Longrightarrow b_n = \frac{8}{(2n-1)^5\pi^5-(2n-1)^3\pi^3} + \frac{8}{(2n-1)^3\pi^3} + \frac{4(-1)^n}{(2n-1)^2\pi^2}$$

$$\Longrightarrow \boxed{u(x,t) = (x-1)\sin(t)+\cos(t)+\sum_{n=1}^{\infty}\left(\left(\frac{32(-1)^n}{(2n-1)^3\pi^3}+\frac{4(-1)^{n+1}}{(2n-1)\pi}\right)\cos\big((2n-1)\pi t\big)+\left(\frac{8}{(2n-1)^5\pi^5-(2n-1)^3\pi^3}+\frac{8}{(2n-1)^3\pi^3}+\frac{4(-1)^n}{(2n-1)^2\pi^2}\right)\sin\big((2n-1)\pi t\big)-\frac{\frac{8}{(2n-1)^2\pi^2}}{(2n-1)^2\pi^2-1}\sin(t)\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)}$$

۴-۲) معادله موج زیر را حل کنید.

$u_{tt} = 4u_{xx} - x(\sin(t)+\cos(t)), \qquad 0 \le x \le 1, \qquad 0 \le t$
$u(0,t) = \sin(t), \qquad u_x(1,t) = \cos(t)$
$u(x,0) = x^2 + x, \qquad u_t(x,0) = 1 - x$

$$w(x,t) = \sin(t) + x\cos(t)$$
$$\Longrightarrow v_{tt} = 4u_{xx} + (1-x)\sin(t)$$
$$v(0,t) = 0, \qquad v_x(1,t) = 0$$
$$v(x,0) = x^2, \qquad u_t(x,0) = -x$$

$$\Longrightarrow v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\left(\frac{(2n-1)\pi}{2}x\right) \Longrightarrow \sum_{n=1}^{\infty}\left(\ddot{T}_n(t)+(2n-1)^2\pi^2 T_n(t)\right)\sin\left(\frac{(2n-1)\pi}{2}x\right) = (1-x)\sin(t)$$

$$\Longrightarrow \ddot{T}_n(t) + (2n-1)^2\pi^2 T_n(t) = 2\sin(t)\int_0^1 (1-x)\sin\left(\frac{(2n-1)\pi}{2}x\right)dx = \left(\frac{8(-1)^n}{(2n-1)^2\pi^2}+\frac{4}{(2n-1)\pi}\right)\sin(t)$$

$$\Longrightarrow T_n(t) = a_n\cos\big((2n-1)\pi t\big) + b_n\sin\big((2n-1)\pi t\big) + \left(\frac{8(-1)^n}{(2n-1)^4\pi^4-(2n-1)^2\pi^2}+\frac{4}{(2n-1)^3\pi^3-(2n-1)\pi}\right)\sin(t)$$

$$v(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\big((2n-1)\pi t\big) + b_n\sin\big((2n-1)\pi t\big)\right.$$
$$\left.+\left(\frac{8(-1)^n}{(2n-1)^4\pi^4-(2n-1)^2\pi^2}+\frac{4}{(2n-1)^3\pi^3-(2n-1)\pi}\right)\sin(t)\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

$$v(x,0) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{(2n-1)\pi}{2}x\right) = x^2$$

$$\Longrightarrow a_n = 2\int_0^1 x^2\sin\left(\frac{(2n-1)\pi}{2}x\right)dx = -\frac{32}{(2n-1)^3\pi^3} - \frac{16(-1)^n}{(2n-1)^2\pi^2}$$

$$v_t(x,0) = \sum_{n=1}^{\infty}\left(b_n(2n-1)\pi + \frac{8(-1)^n}{(2n-1)^4\pi^4-(2n-1)^2\pi^2}+\frac{4}{(2n-1)^3\pi^3-(2n-1)\pi}\right)\cos\left(\frac{(2n-1)\pi}{2}x\right) = -x$$

$$\Longrightarrow b_n(2n-1)\pi + \frac{8(-1)^n}{(2n-1)^4\pi^4-(2n-1)^2\pi^2}+\frac{4}{(2n-1)^3\pi^3-(2n-1)\pi} = -2\int_0^1 x\sin\left(\frac{(2n-1)\pi}{2}x\right)dx = \frac{8(-1)^n}{(2n-1)^2\pi^2}$$

$$\Longrightarrow b_n = \frac{8(-1)^n}{(2n-1)^3\pi^3} - \frac{8(-1)^n}{(2n-1)^5\pi^5-(2n-1)^3\pi^3} - \frac{4}{(2n-1)^4\pi^4-(2n-1)^2\pi^2}$$

$$\Longrightarrow \boxed{u(x,t) = \sin(t)+x\cos(t)+\sum_{n=1}^{\infty}\left(\left(-\frac{32}{(2n-1)^3\pi^3}-\frac{16(-1)^n}{(2n-1)^2\pi^2}\right)\cos\big((2n-1)\pi t\big)+\left(\frac{8(-1)^n}{(2n-1)^3\pi^3}-\frac{8(-1)^n}{(2n-1)^5\pi^5-(2n-1)^3\pi^3}-\frac{4}{(2n-1)^4\pi^4-(2n-1)^2\pi^2}\right)\sin\big((2n-1)\pi t\big)+\left(\frac{8(-1)^n}{(2n-1)^4\pi^4-(2n-1)^2\pi^2}+\frac{4}{(2n-1)^3\pi^3-(2n-1)\pi}\right)\sin(t)\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)}$$

۵-۰) معادله حرارت زیر را حل کنید. ۵

$u_t = 4u_{xx} + x - \frac{\pi}{2} + t\cos(x), \qquad 0 \le x \le \frac{\pi}{2}, \qquad 0 \le t$
$u_x(0,t) = t, \qquad u\left(\frac{\pi}{2},t\right) = 1$
$u(x,0) = \cos(3x) + 1$

$$u(x,t) = w(x,t) + v(w,t)$$
$$w(x,t) = \left(x-\frac{\pi}{2}\right)t + 1$$
$$\Longrightarrow v_t = v_{xx} + t\cos(x)$$
$$v(0,t) = 0, \qquad v\left(\frac{\pi}{2},t\right) = 0$$
$$v(x,0) = \cos(3x)$$

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$$\Longrightarrow v(x,t) = \sum_{n=1}^{\infty} T_n(t)\cos\big((2n-1)x\big) \Longrightarrow \sum_{n=1}^{\infty}\left(\dot{T}_n(t)+4(2n-1)^2 T_n(t)\right)\cos\big((2n-1)x\big) = t\cos(x)$$

$$\Longrightarrow \dot{T}_n(t) + 4(2n-1)^2 T_n(t) = t\delta[n-1] = \begin{cases} t & n=1 \\ 0 & n\ne 1\end{cases} \Longrightarrow T_n(t) = c_n e^{-4(2n-1)^2 t} + \left(\frac{1}{4(2n-1)^2}t - \frac{1}{16(2n-1)^4}\right)\delta[n-1]$$

$$= c_n e^{-4(2n-1)^2 t} + \begin{cases}\frac{1}{4}t - \frac{1}{16} & n=1 \\ 0 & n \ne 1\end{cases}$$

$$\Longrightarrow v(x,t) = \sum_{n=1}^{\infty} c_n e^{-4(2n-1)^2 t}\cos\big((2n-1)x\big) + \left(\frac{1}{4}t - \frac{1}{16}\right)\cos(x)$$

$$v(x,0)=\sum_{n=1}^{\infty} c_n\cos\big((2n-1)x\big)-\frac{1}{16}\cos(x)=\cos(3x)\Rightarrow c_n=\delta[n-2]+\frac{1}{16}\delta[n-1]=\begin{cases}\frac{1}{16} & n=1\\ 1 & n=2\\ 0 & n\notin\{1,2\}\end{cases}$$

$$\Rightarrow v(x,t)=e^{-36t}\cos(3x)+\frac{1}{16}e^{-4t}\cos(x)+\left(\frac{1}{4}t-\frac{1}{16}\right)\cos(x)$$

$$\Rightarrow \boxed{u(x,t)=\left(x-\frac{\pi}{2}\right)t+1+e^{-36t}\cos(3x)+\frac{1}{16}e^{-4t}\cos(x)+\left(\frac{1}{4}t-\frac{1}{16}\right)\cos(x)}$$

۱-۵) معادله حرارت زیر را حل کنید.

$$u_t=u_{xx}+2xt+t\sin(x)\,,\qquad 0\le x\le\frac{\pi}{2},\qquad 0\le t$$
$$u(0,t)=1,\qquad u_x\left(\frac{\pi}{2},t\right)=t^2$$
$$u(x,0)=\sin(3x)+1$$

$$u(x,t)=w(x,t)+v(x,t)$$
$$w(x,t)=1+xt^2$$
$$\Rightarrow v_t=v_{xx}+t\sin(x)$$
$$v(0,t)=0,\qquad v\left(\frac{\pi}{2},t\right)=0$$
$$v(x,0)=\sin(3x)$$

$$\Rightarrow v(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\big((2n-1)x\big)\Rightarrow\sum_{n=1}^{\infty}\left(\dot{T}_n(t)+(2n-1)^2T_n(t)\right)\sin\big((2n-1)x\big)=t\sin(x)$$

$$\Rightarrow \dot{T}_n(t)+(2n-1)^2T_n(t)=t\delta[n-1]=\begin{cases}t & n=1\\0 & n\ne1\end{cases}\Rightarrow T_n(t)=c_ne^{-(2n-1)^2t}+\left(\frac{1}{(2n-1)^2}t-\frac{1}{(2n-1)^4}\right)\delta[n-1]$$

$$=c_ne^{-(2n-1)^2t}+\begin{cases}t-1 & n=1\\ 0 & n\ne1\end{cases}$$

$$\Rightarrow v(x,t)=\sum_{n=1}^{\infty}c_ne^{-(2n-1)^2t}\sin\big((2n-1)x\big)+(t-1)\sin(x)$$

$$v(x,0)=\sum_{n=1}^{\infty}c_n\sin\big((2n-1)x\big)-\sin(x)=\sin(3x)\Rightarrow c_n=\delta[n-2]+\delta[n-1]=\begin{cases}1 & n=1\\1 & n=2\\0 & n\notin\{1,2\}\end{cases}$$

$$\Rightarrow v(x,t)=e^{-9t}\sin(3x)+e^{-t}\sin(x)+(t-1)\sin(x)$$

$$\Rightarrow\boxed{u(x,t)=1+xt^2+e^{-9t}\sin(3x)+e^{-t}\sin(x)+(t-1)\sin(x)}$$

۲-۵) معادله حرارت زیر را حل کنید.

$$u_t=u_{xx}+t\sin(2x)\,,\qquad 0\le x\le\pi,\qquad 0\le t$$
$$u(0,t)=1,\qquad u(\pi,t)=2$$
$$u(x,0)=1+\frac{x}{\pi}+\sin(x)$$

$$u(x,t)=w(x,t)+v(x,t)$$
$$w(x,t)=1+\frac{x}{\pi}$$
$$\Rightarrow v_t=v_{xx}+t\sin(2x)$$
$$v(0,t)=0,\qquad v(\pi,t)=0$$
$$v(x,0)=\sin(x)$$

$$\Rightarrow v(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin(nx)\Rightarrow\sum_{n=1}^{\infty}\left(\dot{T}_n(t)+n^2T_n(t)\right)\sin(nx)=t\sin(2x)$$

$$\Rightarrow \dot{T}_n(t)+n^2T_n(t)=t\delta[n-2]=\begin{cases}t & n=2\\0 & n\ne2\end{cases}\Rightarrow T_n(t)=c_ne^{-n^2t}+\left(\frac{1}{n^2}t-\frac{1}{n^4}\right)\delta[n-2]=c_ne^{-n^2t}+\begin{cases}\frac{1}{4}t-\frac{1}{16} & n=2\\ 0 & n\ne2\end{cases}$$

$$\Rightarrow v(x,t)=\sum_{n=1}^{\infty}c_ne^{-n^2t}\sin(nx)+\left(\frac{1}{4}t-\frac{1}{16}\right)\sin(2x)$$

$$v(x,0)=\sum_{n=1}^{\infty}\left(c_n-\frac{1}{16}\delta[n-2]\right)\sin(nx)=\sin(x)\Rightarrow c_n=\delta[n-1]+\frac{1}{16}\delta[n-2]=\begin{cases}1 & n=1\\ \frac{1}{16} & n=2\\ 0 & n\notin\{1,2\}\end{cases}$$

$$\Rightarrow v(x,t)=e^{-t}\sin(x)+\frac{1}{16}e^{-4t}\sin(2x)+\left(\frac{1}{4}t-\frac{1}{16}\right)\sin(2x)$$

$$\Rightarrow\boxed{u(x,t)=1+\frac{x}{2}+e^{-t}\sin(x)+\frac{1}{16}e^{-4t}\sin(2x)+\left(\frac{1}{4}t-\frac{1}{16}\right)\sin(2x)}$$