مدرس: دکتر مهدی طالع ماسوله- حل تمرین: نیکا امامی- سروش مس فروش- حسین عطرسایی
برای سوالات خود در خصوص این تمرین با رایانامه emami.nika@gmail.com، sorush.mes@gmail.com، hatrsaei@gmail.com مکاتبه نمایید.

۱) معادله موج زیر را حل کنید

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad 0 \le x \le l$$

$$u(0,t) = 0, \qquad u(l,t) = 0$$

$$u(x,0) = e^{-x}, \qquad u_t(x,0) = 0$$

$$u(x,t) = X(x)T(t) \Longrightarrow X''(x)T(x) - \frac{1}{c^2}\,X(x)\ddot{T}(t) = 0 \Longrightarrow \frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{c^2T(t)} = -k^2 \Longrightarrow \begin{cases} X''(x) + k^2X(x) = 0 & \text{(I)} \\ \ddot{T}(t) + c^2k^2T(t) = 0 & \text{(II)} \end{cases}$$

$$\text{(I)} \Longrightarrow X(x) = A\cos(kx) + B\sin(kx)$$

$$\text{BC1: } u(0,t) = 0 \Longrightarrow A = 0,\ \text{BC2: } u(l,t) = 0 \Longrightarrow B\sin(kl) = 0 \Longrightarrow k = \frac{n\pi}{l}$$

$$\Longrightarrow X_n(x) = B_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\text{(II)} \Longrightarrow T_n(t) = C_n\cos\left(\frac{cn\pi}{l}t\right) + D_n\sin\left(\frac{cn\pi}{l}t\right)$$

$$\Longrightarrow u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{cn\pi}{l}t\right) + b_n\sin\left(\frac{cn\pi}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right)$$

$$\text{IC1: } u(x,0) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi}{l}x\right) = e^{-x}$$

$$a_n = \frac{2}{l}\int_0^l e^{-x}\sin\left(\frac{n\pi}{l}x\right)dx = \left[-\frac{2e^{-x}\left(n\pi\cos\left(\frac{n\pi}{l}x\right) + l\sin\left(\frac{n\pi}{l}x\right)\right)}{l^2 + n^2\pi^2}\right]_0^l = \frac{2n\pi((-1)^{n+1}e^{-l} + 1)}{l^2 + n^2\pi^2}$$

$$\text{IC2: } u_t(x,0) = \sum_{n=1}^{\infty}\frac{b_n cn\pi}{l}\sin\left(\frac{n\pi}{l}x\right) = 0 \Longrightarrow b_n = 0$$

$$\Longrightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{2n\pi((-1)^{n+1}e^{-l} + 1)}{l^2 + n^2\pi^2}\cos\left(\frac{cn\pi}{l}t\right)\sin\left(\frac{n\pi}{l}x\right)}$$

۲) معادله موج زیر را حل کنید.

$$u_{tt} = 9u_{xx}, \qquad 0 \le x \le \pi$$
$$u_x(0,t) = 0, \qquad u_x(\pi,t) = 0$$
$$u(x,0) = x^2, \qquad u_t(x,0) = e^{-x}$$

$$u(x,t) = X(x)T(t) \Longrightarrow X(x)\ddot{T}(t) = 9X''(x)T(x) \Longrightarrow \frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{9T(t)} = -k^2 \Longrightarrow \begin{cases} X''(x) + k^2X(x) = 0 & \text{(I)} \\ \ddot{T}(t) + 9k^2T(t) = 0 & \text{(II)} \end{cases}$$

$$\text{(I)} \Longrightarrow X(x) = A\cos(kx) + B\sin(kx)$$

$$\text{BC1: } u_x(0,t) = 0 \Longrightarrow B = 0,\ \text{BC2: } u_x(\pi,t) = 0 \Longrightarrow -Ak\sin(k\pi) = 0 \Longrightarrow k = n$$

$$\Longrightarrow X_n(x) = A_n\cos(nx)$$

$$\text{(II)} \Longrightarrow \begin{cases} n \neq 0 \Longrightarrow \ddot{T}_n(t) + 9n^2T_n(t) = 0 \Longrightarrow T_n(t) = C_n\cos(3nt) + D_n\sin(3nt) \\ n = 0 \Longrightarrow \ddot{T}_n(t) = 0 \Longrightarrow T_n(t) = C_0 + D_0t \end{cases}$$

$$\Longrightarrow u(x,t) = a_0 + b_0t + \sum_{n=1}^{\infty}(a_n\cos(3nt) + b_n\sin(3nt))\cos(nx)$$

$$\text{IC1: } u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) = x^2$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2dx = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_0^{\pi} = \frac{4(-1)^n}{n^2}$$

$$\text{IC2: } u_t(x,0) = b_0 + \sum_{n=1}^{\infty} 3b_nn\cos(nx) = e^{-x}$$

$$b_0 = \frac{1}{\pi}\int_0^{\pi} e^{-x}dx = \frac{1-e^{-\pi}}{\pi}$$

$$b_n = \frac{2}{3n\pi}\int_0^{\pi} e^{-x}\cos(nx)\,dx = \frac{2}{3n\pi}\left[\frac{e^{-x}(n\sin(nx) - \cos(nx))}{1+n^2}\right]_0^{\pi} = \frac{2(1-e^{-\pi}(-1)^n)}{3n\pi(1+n^2)}$$

$$\Longrightarrow \boxed{u(x,t) = \frac{\pi^2}{3} + \frac{1-e^{-\pi}}{\pi}t + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\cos(3nt) + \frac{2(1-e^{-\pi}(-1)^n)}{3n\pi(1+n^2)}\sin(3nt)\right)\cos(nx)}$$

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۳) معادله گرمای زیر را حل کنید.

$$u_t = 2u_{xx}, \qquad 0 \le x \le \pi$$
$$u(0,t) = 0, \qquad u_x(\pi,t) = 0$$
$$u(x,0) = \Pi\left(\frac{x}{\pi}\right)$$

$$u(x,t) = X(x)T(t) \Longrightarrow X(x)\dot{T}(t) = 2X''(x)T(t) \Longrightarrow \frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{2T(t)} = -k^2 \Longrightarrow \begin{cases} X''(x) + k^2X(x) = 0 & \text{(I)} \\ \dot{T}(t) + 2k^2T(t) = 0 & \text{(II)} \end{cases}$$

$$\text{(I)} \Longrightarrow X(x) = A\cos(kx) + B\sin(kx)$$

$$\text{BC1: } u(0,t) = 0 \Longrightarrow A = 0, \ \text{BC2: } u_x(\pi,t) = 0 \Longrightarrow Bk\cos(k\pi) = 0 \Longrightarrow k = \frac{2n-1}{2}$$

$$\Longrightarrow X(x) = B_n \sin\left(\frac{2n-1}{2}x\right)$$

$$\text{(II)} \Longrightarrow \dot{T}_n(t) + 2\left(\frac{2n-1}{2}\right)^2 T_n(t) = 0 \Longrightarrow T_n(t) = C_n e^{-2\left(\frac{2n-1}{2}\right)^2 t}$$

$$\Longrightarrow u(x,t) = \sum_{n=1}^{\infty} a_n e^{-2\left(\frac{2n-1}{2}\right)^2 t} \sin\left(\frac{2n-1}{2}x\right)$$

$$\text{IC: } u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{2n-1}{2}x\right) = \Pi\left(\frac{x}{\pi}\right)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \Pi\left(\frac{x}{\pi}\right)\sin\left(\frac{2n-1}{2}x\right)dx = \frac{2}{\pi}\left[-\frac{2}{2n-1}\cos\left(\frac{2n-1}{2}x\right)\right]_0^{\frac{\pi}{2}} = \frac{4}{(2n-1)\pi}\left(1 - \cos\left(\frac{(2n-1)\pi}{4}\right)\right)$$

$$\Longrightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\left(1 - \cos\left(\frac{(2n-1)\pi}{4}\right)\right) e^{-2\left(\frac{2n-1}{2}\right)^2 t} \sin\left(\frac{2n-1}{2}x\right)}$$

۴) معادله موج زیر را حل کنید

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{16}\frac{\partial^2 u}{\partial t^2}=0, \qquad 0\le x\le 1$$

$$u_x(0,t)=u(1,t)=0$$

$$u(x,0)=0, \qquad u_t(x,0)=\delta\left(x-\frac{1}{2}\right)$$

$$u(x,t)=X(x)T(t)\Longrightarrow X''(x)T(t)-\frac{1}{16}X(x)\ddot{T}(t)=0\Longrightarrow\frac{X''(x)}{X(x)}=\frac{\ddot{T}(t)}{16T(t)}=-k^2\Longrightarrow\begin{cases}X''(x)+k^2X(x)=0 & \text{(I)}\\ \ddot{T}(t)+16k^2T(t)=0 & \text{(II)}\end{cases}$$

$$\text{(I)}\Longrightarrow X(x)=A\cos(kx)+B\sin(kx)$$

$$\text{BC!}: u_x(0,t)=0\Longrightarrow B=0,\ \text{BC2}: u(1,t)=0\Longrightarrow A\cos(k)=0\Longrightarrow k=\frac{(2n-1)\pi}{2}$$

$$\Longrightarrow X_n(x)=A_n\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

$$\text{(II)}\Longrightarrow \ddot{T}_n(t)+16\left(\frac{(2n-1)\pi}{2}\right)^2T_n(t)=0\Longrightarrow T_n(t)=C_n\cos\big((2n-1)2\pi t\big)+D_n\sin\big((2n-1)2\pi t\big)$$

$$\Longrightarrow u(x,t)=\sum_{n=1}^{\infty}\big(a_n\cos\big((2n-1)2\pi t\big)+b_n\sin\big((2n-1)2\pi t\big)\big)\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

$$\text{IC1}: u(x,0)=\sum_{n=1}^{\infty}a_n\cos\left(\frac{(2n-1)\pi}{2}x\right)=0$$

$$a_n=0$$

$$\text{IC2}: u_t(x,0)=\sum_{n=1}^{\infty}b_n(2n-1)2\pi\cos\left(\frac{(2n-1)\pi}{2}x\right)=\delta\left(x-\frac{1}{2}\right)$$

$$b_n=\frac{1}{(2n-1)\pi}\int_0^1\delta\left(x-\frac{1}{2}\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)dx=\frac{\cos\left(\frac{(2n-1)\pi}{4}\right)}{(2n-1)\pi}$$

$$\Longrightarrow\boxed{u(x,t)=\sum_{n=1}^{\infty}\frac{\cos\left(\frac{(2n-1)\pi}{4}\right)}{(2n-1)\pi}\sin\big((2n-1)2\pi t\big)\cos\left(\frac{(2n-1)\pi}{2}x\right)}$$

۵) معادله حرارت زیر را حل کنید.

$u_t = 4u_{xx}, \qquad 0 \le x \le 2\pi$

$u_x(0,t) = u_x(2\pi,t) = 0$

$u(x,0) = \cos(2x)$

$$u(x,t) = X(x)T(t) \Longrightarrow X(x)\dot{T}(t) = 4X''(x)T(t) \Longrightarrow \frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{4T(t)} = -k^2 \Longrightarrow \begin{cases} X''(x) + k^2X(x) = 0 & \text{(I)} \\ \dot{T}(t) + 4k^2T(t) = 0 & \text{(II)} \end{cases}$$

$$\text{(I)} \Longrightarrow X(x) = A\cos(kx) + B\sin(kx)$$

$$\text{BC1: } u_x(0,t) = 0 \Longrightarrow B = 0,\ \text{BC2: } u_x(2\pi,t) = 0 \Longrightarrow -Ak\sin(k2\pi) = 0 \Longrightarrow k = \frac{n}{2}$$

$$\Longrightarrow X_n(x) = A_n\cos\left(\frac{n}{2}x\right)$$

$$\text{(II)} \Longrightarrow \begin{cases} n \neq 0 \Longrightarrow \dot{T}_n(t) + n^2T_n(t) = 0 \Longrightarrow T_n(t) = C_ne^{-n^2t} \\ n = 0 \Longrightarrow \dot{T}_0(t) = 0 \Longrightarrow T(0) = C_0 \end{cases}$$

$$\Longrightarrow u(x,t) = a_0 + \sum_{n=1}^{\infty} a_ne^{-n^2t}\cos\left(\frac{n}{2}x\right)$$

$$\text{IC: } u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n}{2}x\right) = \cos(2x)$$

$$a_0 = 0$$

$$a_n = \begin{cases} 0, & n \neq 4 \\ 1, & n = 4 \end{cases}$$

$$\Longrightarrow \boxed{u(x,t) = e^{-16t}\cos(2x)}$$

۶) معادله گرمای زیر را حل کنید.

$16u_t = u_{xx}, \qquad 0 \le x \le 1$

$u_x(0,t) = u(1,t) = 0$

$u(x,0) = x\cos(\pi x)$

$$u(x,t) = X(x)T(t) \Rightarrow 16X(x)\dot{T}(t) = X''(x)T(t) \Rightarrow \frac{X''(x)}{X(x)} = 16\frac{\dot{T}(t)}{T(t)} = -k^2 \Rightarrow \begin{cases} X''(x) + k^2X(x) = 0 & \text{(I)} \\ \dot{T}(t) + \dfrac{k^2}{16}T(t) = 0 & \text{(II)} \end{cases}$$

$$\text{(I)} \Rightarrow X(x) = A\cos(kx) + B\sin(kx)$$

$$\text{BC1: } u_x(0,t) = 0 \Rightarrow B = 0,\ \text{BC2: } u(1,t) = 0 \Rightarrow A\cos(k) = 0 \Rightarrow k = \frac{(2n-1)\pi}{2}$$

$$\Rightarrow X_n(x) = A_n\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

$$\text{(II)} \Rightarrow \dot{T}_n(t) + \left(\frac{(2n-1)\pi}{8}\right)^2 T_n(t) = 0 \Rightarrow T_n(t) = C_n e^{-\left(\frac{(2n-1)\pi}{8}\right)^2 t}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{(2n-1)\pi}{8}\right)^2 t}\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

$$\text{IC: } u(x,0) = \sum_{n=1}^{\infty} a_n\cos\left(\frac{(2n-1)\pi}{2}x\right) = x\cos(\pi x)$$

$$a_n = 2\int_0^1 x\cos(\pi x)\cos\left(\frac{(2n-1)\pi}{2}x\right)dx = \int_0^1 x\cos\left(\frac{(2n+1)\pi}{2}x\right)dx + \int_0^1 x\cos\left(\frac{(2n-3)\pi}{2}x\right)dx$$

$$= \left[\frac{2x}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi}{2}x\right) + \frac{4}{(2n+1)^2\pi^2}\cos\left(\frac{(2n+1)\pi}{2}x\right) + \frac{2x}{(2n-3)\pi}\sin\left(\frac{(2n-3)\pi}{2}x\right)\right.$$

$$\left. + \frac{4}{(2n-3)^2\pi^2}\cos\left(\frac{(2n-3)\pi}{2}x\right)\right]_0^1 = (-1)^n\left(\frac{2}{(2n+1)\pi} + \frac{2}{(2n-3)\pi}\right) - \left(\frac{4}{(2n+1)^2\pi^2} + \frac{4}{(2n-3)^2\pi^2}\right)$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty}\left((-1)^n\left(\frac{2}{(2n+1)\pi} + \frac{2}{(2n-3)\pi}\right) - \left(\frac{4}{(2n+1)^2\pi^2} + \frac{4}{(2n-3)^2\pi^2}\right)\right)e^{-\left(\frac{(2n-1)\pi}{8}\right)^2 t}\cos\left(\frac{(2n-1)\pi}{2}x\right)}$$

۷) معادله موج زیر را حل کنید

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{4}\frac{\partial^2 u}{\partial t^2}=0, \qquad 0\le x\le 1$$

$$u(0,t)=u_x(1,t)=0$$

$$u(x,0)=0, \qquad u_t(x,0)=4x(1-x)$$

$$u(x,t)=X(x)T(t) \Rightarrow X''(x)T(t)-\frac{1}{4}X(x)\ddot{T}(t)=0 \Rightarrow \frac{X''(x)}{X(x)}=\frac{\ddot{T}(t)}{4T(t)}=-k^2 \Rightarrow \begin{cases} X''(x)+k^2X(x)=0 & \text{(I)} \\ \ddot{T}(t)+4k^2T(t)=0 & \text{(II)} \end{cases}$$

$$\text{(I)} \Rightarrow X(x)=A\cos(kx)+B\sin(kx)$$

$$\text{BC!: } u(0,t)=0 \Rightarrow A=0,\ \text{BC2: } u_x(1,t)=0 \Rightarrow Bk\cos(k)=0 \Rightarrow k=\frac{(2n-1)\pi}{2}$$

$$\Rightarrow X_n(x)=B_n\sin\left(\frac{(2n-1)\pi}{2}x\right)$$

$$\text{(II)} \Rightarrow \ddot{T}_n(t)+4\left(\frac{(2n-1)\pi}{2}\right)^2 T_n(t)=0 \Rightarrow T_n(t)=C_n\cos\big((2n-1)\pi t\big)+D_n\sin\big((2n-1)\pi t\big)$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}\Big(a_n\cos\big((2n-1)\pi t\big)+b_n\sin\big((2n-1)\pi t\big)\Big)\sin\left(\frac{(2n-1)\pi}{2}x\right)$$

$$\text{IC1: } u(x,0)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{(2n-1)\pi}{2}x\right)=0$$

$$a_n=0$$

$$\text{IC2: } u_t(x,0)=\sum_{n=1}^{\infty}b_n(2n-1)\pi\sin\left(\frac{(2n-1)\pi}{2}x\right)=4x(1-x)$$

$$b_n=\frac{2}{(2n-1)\pi}\int_0^1 4x(1-x)\sin\left(\frac{(2n-1)\pi}{2}x\right)dx$$

$$=\frac{2}{(2n-1)\pi}\left[\frac{8(x-1)x}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{2}x\right)+\frac{16(1-2x)}{(2n-1)^2\pi^2}\sin\left(\frac{(2n-1)\pi}{2}x\right)\right.$$

$$\left.-\frac{64}{(2n-1)^3\pi^3}\cos\left(\frac{(2n-1)\pi}{2}x\right)\right]_0^1=\frac{2}{(2n-1)\pi}\left(\frac{16(-1)^n}{(2n-1)^2\pi^2}+\frac{64}{(2n-1)^3\pi^3}\right)=\frac{32(-1)^n}{(2n-1)^3\pi^3}+\frac{128}{(2n-1)^4\pi^4}$$

$$\Rightarrow \boxed{u(x,t)=\sum_{n=1}^{\infty}\left(\frac{32(-1)^n}{(2n-1)^3\pi^3}+\frac{128}{(2n-1)^4\pi^4}\right)\sin\big((2n-1)\pi t\big)\sin\left(\frac{(2n-1)\pi}{2}x\right)}$$

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۸) ابتدا و انتهای یک میله به طول l در مخلوط آب و یخ قرار دارد. اگر توزیع دما در لحظه اولیه روی میله به صورت $U(x,0)=\Lambda\left(\frac{x-\frac{l}{2}}{\frac{l}{2}}\right)$ باشد، دما در میانه میله را حساب کنید. $(c^2=1)$

$$u_t=u_{xx},\qquad 0\le x\le l$$
$$u(0,t)=u(l,t)=0$$
$$u(x,0)=\Lambda\left(\frac{x-\frac{l}{2}}{\frac{l}{2}}\right)$$

$$u(x,t)=X(x)T(t)\Longrightarrow X(x)\dot{T}(t)=X''(x)T(t)\Longrightarrow\frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{T(t)}=-k^2\Longrightarrow\begin{cases}X''(x)+k^2X(x)=0 & \text{(I)}\\ \dot{T}(t)+k^2T(t)=0 & \text{(II)}\end{cases}$$

$$\text{(I)}\Longrightarrow X(x)=A\cos(kx)+B\sin(kx)$$

$$\text{BC1}: u(0,t)=0\Longrightarrow A=0,\ \text{BC2}: u(l,t)=0\Longrightarrow B\sin(kl)=0\Longrightarrow k=\frac{n\pi}{l}$$

$$\Longrightarrow X_n(x)=B_n\sin\left(\frac{n\pi}{l}x\right)$$

$$\text{(II)}\Longrightarrow \dot{T}_n(t)+\left(\frac{n\pi}{l}\right)^2T_n(t)=0\Longrightarrow T_n(t)=C_ne^{-\left(\frac{n\pi}{l}\right)^2t}$$

$$\Longrightarrow u(x,t)=\sum_{n=1}^{\infty}a_ne^{-\left(\frac{n\pi}{l}\right)^2t}\sin\left(\frac{n\pi}{l}x\right)$$

$$\text{IC}: u(x,0)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi}{l}x\right)=\Lambda\left(\frac{x-\frac{l}{2}}{\frac{l}{2}}\right)$$

$$a_n=\frac{2}{l}\int_0^l\Lambda\left(\frac{x-\frac{l}{2}}{\frac{l}{2}}\right)\sin\left(\frac{n\pi}{l}x\right)dx=\frac{2}{l}\int_0^{\frac{l}{2}}\frac{2}{l}x\sin\left(\frac{n\pi}{l}x\right)dx+\frac{2}{l}\int_{\frac{l}{2}}^{l}\frac{2}{l}(l-x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$=\frac{4}{l^2}\left[\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}x\right)-\frac{lx}{n\pi}\cos\left(\frac{n\pi}{l}x\right)\right]_0^{\frac{l}{2}}+\frac{4}{l^2}\left[-\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}x\right)-\frac{l(l-x)}{n\pi}\cos\left(\frac{n\pi}{l}x\right)\right]_{\frac{l}{2}}^{l}=\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\Longrightarrow\boxed{u(x,t)=\sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)e^{-\left(\frac{n\pi}{l}\right)^2t}\sin\left(\frac{n\pi}{l}x\right)}$$

۹) معادله حرارت زیر را حل کنید.

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial u}{\partial t} = 0, \qquad -1 \le x \le 1$$

$$u(-1,t) = u(1,t), \qquad u_x(-1\ t) = u_x(1\ t)$$

$$u(x,0) = |x|$$

$$u(x,t) = X(x)T(t) \Longrightarrow X''(x)T(t) - \frac{1}{2}X(x)\dot{T}(t) = 0 \Longrightarrow \frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{2T(t)} = -k^2 \Longrightarrow \begin{cases} X''(x) + k^2X(x) = 0 & \text{(I)} \\ \dot{T}(t) + 2k^2T(t) = 0 & \text{(II)} \end{cases}$$

$$\text{(I)} \Longrightarrow X(x) = A\cos(kx) + B\sin(kx)$$

$$\text{BC1: } u(-1,t) = u(1,t) \Longrightarrow k = n\pi \text{ or } B = 0$$

$$\text{BC2: } u_x(-1,t) = u_x(1,t) \Longrightarrow A = 0 \text{ or } k = n\pi$$

$$\Longrightarrow X_n(x) = A_n\cos(n\pi x) + B_n\sin(n\pi x)$$

$$\text{(II)} \Longrightarrow \begin{cases} n \ne 0 \Longrightarrow \dot{T}_n(t) + 2(n\pi)^2T_n(t) = 0 \Longrightarrow T_n(t) = C_n e^{-2n^2\pi^2 t} \\ n = 0 \Longrightarrow \dot{T}_0(t) = 0 \Longrightarrow T_0(t) = C_0 \end{cases}$$

$$\Longrightarrow u(x,t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\pi x) + b_n\sin(n\pi x))e^{-2n^2\pi^2 t}$$

$$\text{IC: } u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi x) + b_n\sin(n\pi x) = |x|$$

$$a_0 = \frac{1}{2}\int_{-1}^{1}|x|dx = \int_0^1 x dx = \frac{1}{2}$$

$$a_n = \int_{-1}^{1}|x|\cos(n\pi x)\,dx = 2\int_0^1 x\cos(n\pi x)\,dx = 2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x)\right]_0^1 = \frac{2((-1)^n - 1)}{n^2\pi^2}$$

$$b_n = 0$$

$$\Longrightarrow \boxed{u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2((-1)^n - 1)}{n^2\pi^2}\cos(n\pi x)\, e^{-2n^2\pi^2 t}}$$