۱)

$$U_{xx} + U_{yy} = 0, \qquad 0 < x < a, \qquad 0 < y < b$$

$$\begin{cases} U(0, y) = 0 \\ U_x(a, y) = 0 \end{cases}, \qquad \begin{cases} U(x, 0) = 0 \\ U(x, b) = U_0 \sin\left(\frac{\pi}{2a} x\right) \end{cases}$$

Guess: $U(x, y) = \sum_{n=1}^{\infty} Y(y) \sin\left(\frac{2n-1}{2a} \pi x\right) \Longrightarrow U_{xx} + U_{yy} = \sum_{n}^{\infty} \left(Y''(y) - \frac{\pi^2 (2n-1)^2}{4a^2} Y(y)\right) \sin\left(\frac{2n-1}{2a} \pi x\right) = 0$

ODE: $Y''(y) - \frac{\pi^2 (2n-1)^2}{4a^2} Y(y) = 0$, BC1: $Y(0) = 0 \Longrightarrow Y_n(y) = a_n \sinh\left(\frac{2n-1}{2a} \pi y\right)$

$$\Longrightarrow U(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{2n-1}{2a} \pi y\right) \sin\left(\frac{2n-1}{2a} \pi x\right)$$

BC2: $U(x, b) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{2n-1}{2a} \pi b\right) \sin\left(\frac{2n-1}{2a} \pi x\right) = U_0 \sin\left(\frac{\pi}{2a} x\right) \Longrightarrow b_n = \begin{cases} U_0 \operatorname{csch}\left(\frac{\pi b}{2a}\right), & n = 1 \\ 0, & n \neq 1 \end{cases}$

$$\Longrightarrow \boxed{U(x, y) = U_0 \operatorname{csch}\left(\frac{\pi b}{2a}\right) \sinh\left(\frac{\pi y}{2a}\right) \sin\left(\frac{\pi x}{2a}\right)}$$

$$U_{xx} + U_{yy} = 0, \qquad 0 < x < a, \qquad 0 < y < b$$

$$\begin{cases} U(0, y) = 0 \\ U_x(a, y) = 0 \end{cases}, \qquad \begin{cases} U(x, 0) = U_0 \sin\left(\frac{\pi}{2a} x\right) \\ U(x, b) = 0 \end{cases}$$

Guess: $U(x, y) = \sum_{n=1}^{\infty} Y(y) \sin\left(\frac{2n-1}{2a} \pi x\right) \Longrightarrow U_{xx} + U_{yy} = \sum_{n}^{\infty} \left(Y''(y) - \frac{\pi^2 (2n-1)^2}{4a^2} Y(y)\right) \sin\left(\frac{2n-1}{2a} \pi x\right) = 0$

ODE: $Y''(y) - \frac{\pi^2 (2n-1)^2}{4a^2} Y(y) = 0$, BC2: $Y(b) = 0 \Longrightarrow Y_n(y) = a_n \, sinh\left(\frac{2n-1}{2a} \pi (b - y)\right)$

$$\Longrightarrow U(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{2n-1}{2a} \pi (b - y)\right) \sin\left(\frac{2n-1}{2a} \pi x\right)$$

BC1: $U(x, 0) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{2n-1}{2a} \pi b\right) \sin\left(\frac{2n-1}{2a} \pi x\right) = U_0 \sin\left(\frac{\pi}{2a} x\right) \Longrightarrow a_n = \begin{cases} U_0 \operatorname{csch}\left(\frac{\pi b}{2a}\right), & n = 1 \\ 0, & n \neq 1 \end{cases}$

$$\Longrightarrow \boxed{U(x, y) = U_0 \operatorname{csch}\left(\frac{\pi b}{2a}\right) \sinh\left(\frac{\pi (b - y)}{2a}\right) \sin\left(\frac{\pi x}{2a}\right)}$$

$$U_{xx}+U_{yy}=0,\qquad 0<x<a,\qquad 0<y<b$$

$$\begin{cases} U_x(0,y)=0 \\ U(a,y)=0 \end{cases},\qquad \begin{cases} U(x,0)=U_0\sin\left(\frac{\pi}{2a}x\right) \\ U(x,b)=0 \end{cases}$$

Guess: $U(x,y)=\sum_{n=1}^{\infty} Y(y)\cos\left(\frac{2n-1}{2a}\pi x\right) \Longrightarrow U_{xx}+U_{yy}=\sum_{n}^{\infty}\left(Y''(y)-\frac{\pi^2(2n-1)^2}{4a^2}Y(y)\right)\cos\left(\frac{2n-1}{2a}\pi x\right)=0$

ODE: $Y''(y)-\frac{\pi^2(2n-1)^2}{4a^2}Y(y)=0$, BC2: $Y(b)=0 \Longrightarrow Y_n(y)=a_n\sinh\left(\frac{2n-1}{2a}\pi(b-y)\right)$

$$\Longrightarrow U(x,y)=\sum_{n=1}^{\infty} a_n\sinh\left(\frac{2n-1}{2a}\pi(b-y)\right)\cos\left(\frac{2n-1}{2a}\pi x\right)$$

BC1: $U(x,0)=\sum_{n=1}^{\infty} a_n\sinh\left(\frac{2n-1}{2a}\pi b\right)\cos\left(\frac{2n-1}{2a}\pi x\right)=U_0\sin\left(\frac{\pi}{2a}x\right) \Longrightarrow a_n=\frac{2}{a}\operatorname{csch}\left(\frac{2n-1}{2a}\pi b\right)\int_0^a U_0\sin\left(\frac{\pi}{2a}x\right)\cos\left(\frac{2n-1}{2a}\pi x\right)dx$

$$a_n=\begin{cases} \left[U_0\operatorname{csch}\left(\frac{2n-1}{2a}\pi b\right)\left(\frac{\cos\left(\frac{n-1}{a}\pi x\right)}{(n-1)\pi}-\frac{\cos\left(\frac{n}{a}\pi x\right)}{n\pi}\right)\right]_0^a=U_0\operatorname{csch}\left(\frac{2n-1}{2a}\pi b\right)\left(\frac{(-1)^{n-1}-1}{(n-1)\pi}+\frac{(-1)^{n-1}+1}{n\pi}\right), & n=\neq 1 \\ \frac{2U_0}{\pi}\operatorname{csch}\left(\frac{2n-1}{2a}\pi b\right), & n=1 \end{cases}$$

$$\Longrightarrow \boxed{U(x,y)=\frac{2U_0}{\pi}\operatorname{csch}\left(\frac{\pi b}{2a}\right)\sinh\left(\frac{\pi(b-y)}{2a}\right)\cos\left(\frac{\pi x}{2a}\right)+\sum_{n=2}^{\infty}U_0\operatorname{csch}\left(\frac{2n-1}{2a}\pi b\right)\left(\frac{(-1)^{n-1}-1}{(n-1)\pi}+\frac{(-1)^{n-1}+1}{n\pi}\right)\sinh\left(\frac{2n-1}{2a}\pi b\right)\cos\left(\frac{2n-1}{2a}\pi x\right)}$$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\qquad 1<r<2,\qquad 0<\theta<2\pi$$

$$\begin{cases} u_\theta(r,0)=0 \\ u_\theta(r,2\pi)=0 \end{cases},\qquad \begin{cases} u(1,\theta)=1-\Pi\left(\dfrac{\theta-\pi}{\pi}\right) \\ u(2,\theta)=\dfrac{1}{\pi^2}(\theta-\pi)^2 \end{cases}$$

$$\Pi(x)=\begin{cases} 1, & -\frac{1}{2}\le x\le\frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

با تغییر متغیر $r=e^{-t}$ داریم:

$$u_{tt}+u_{\theta\theta}=0$$

$$\text{Guess: } u(r,\theta)=T_0(t)+\sum_{n=1}^{\infty}T_n(t)\cos\left(\frac{n}{2}\theta\right)\Longrightarrow u_{tt}+u_{\theta\theta}=\ddot{T}_0(t)+\sum_{n=1}^{\infty}\left(\ddot{T}_n(t)-\frac{n^2}{4}T_n(t)\right)\cos\left(\frac{n}{2}\theta\right)=0$$

$$\text{ODE: }\begin{cases} \ddot{T}_0=0\Longrightarrow T_0(t)=c_0t+d_0=-c_0\ln(r)+d_0 \\ \ddot{T}_n(t)-\frac{n^2}{4}T_n(t)=0\Longrightarrow T_n(t)=c_ne^{-\frac{n}{2}t}+d_ne^{\frac{n}{2}t}=c_nr^{\frac{n}{2}}+d_nr^{-\frac{n}{2}} \end{cases}$$

$$\Longrightarrow u(r,\theta)=-c_0\ln(r)+d_0+\sum_{n=1}^{\infty}\left(c_nr^{\frac{n}{2}}+d_nr^{-\frac{n}{2}}\right)\cos\left(\frac{n}{2}\theta\right)$$

$$u(1,\theta)=d_0+\sum_{n=1}^{\infty}(c_n+d_n)\cos\left(\frac{n}{2}\theta\right)=1-\Pi\left(\frac{\theta-\pi}{\pi}\right)$$

$$\Longrightarrow d_0=\frac{1}{2\pi}\int_0^{2\pi}\left(1-\Pi\left(\frac{\theta-\pi}{\pi}\right)\right)d\theta=\frac{1}{2\pi}\left(\int_0^{\frac{\pi}{2}}d\theta+\int_{\frac{3\pi}{2}}^{2\pi}d\theta\right)=\frac{1}{2}\Longrightarrow d_0=\frac{1}{2}$$

$$\Longrightarrow c_n+d_n=\frac{1}{\pi}\int_0^{2\pi}\left(1-\Pi\left(\frac{\theta-\pi}{\pi}\right)\right)\cos\left(\frac{n}{2}\theta\right)d\theta=\frac{1}{\pi}\left(\int_0^{\frac{\pi}{2}}\cos\left(\frac{n}{2}\theta\right)d\theta+\int_{\frac{3\pi}{2}}^{2\pi}\cos\left(\frac{n}{2}\theta\right)d\theta\right)=\frac{2}{n\pi}\left(\sin\left(\frac{n\pi}{4}\right)-\sin\left(\frac{3n\pi}{4}\right)\right)=g_n$$

$$u(2,\theta)=-c_0\ln(2)+d_0+\sum_{n=1}^{\infty}\left(c_n2^{\frac{n}{2}}+d_n2^{-\frac{n}{2}}\right)\cos\left(\frac{n}{2}\theta\right)=\frac{1}{\pi^2}(\theta-\pi)^2$$

$$\Longrightarrow -c_0\ln(2)+d_0=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\pi^2}(\theta-\pi)^2d\theta=\frac{1}{3}\Longrightarrow c_0=\frac{1}{\ln(2)}\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{6\ln(2)}$$

$$\Longrightarrow c_n2^{\frac{n}{2}}+d_n2^{-\frac{n}{2}}=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{\pi^2}(\theta-\pi)^2\cos\left(\frac{n}{2}\theta\right)d\theta=\frac{8}{n^2\pi^2}(1+(-1)^n)=h_n$$

$$\Longrightarrow\begin{cases} c_n+d_n=g_n \\ c_n2^{\frac{n}{2}}+d_n2^{-\frac{n}{2}}=h_n \end{cases}$$

$$\Longrightarrow\boxed{u(r,\theta)=\frac{1}{2}-\frac{1}{6}\frac{\ln(r)}{\ln(2)}+\sum_{n=1}^{\infty}\left(\frac{r^{\frac{n}{2}}-r^{-\frac{n}{2}}}{2^{\frac{n}{2}}-2^{-\frac{n}{2}}}\left(\frac{8}{n^2\pi^2}(1+(-1)^n)\right)+\frac{\left(\frac{2}{r}\right)^{\frac{n}{2}}-\left(\frac{2}{r}\right)^{-\frac{n}{2}}}{2^{\frac{n}{2}}-2^{-\frac{n}{2}}}\left(\frac{2}{n\pi}\left(\sin\left(\frac{n\pi}{4}\right)-\sin\left(\frac{3n\pi}{4}\right)\right)\right)\right)\cos\left(\frac{n}{2}\theta\right)}$$

$$\frac{d^2u(r,\theta)}{dr^2}+\frac{1}{r}\frac{du(r,\theta)}{dr}+\frac{1}{r^2}\frac{d^2u(r,\theta)}{d\theta^2}=0,\qquad 1<r<2,\qquad 0<\theta<2\pi$$

$$\begin{cases} u(r,0)=0\\ u(r,2\pi)=0\end{cases},\qquad \begin{cases} u(1,\theta)=\Pi\left(\frac{\theta-\pi}{\pi}\right)\\ u(2,\theta)=1-\frac{1}{\pi^2}(\theta-\pi)^2\end{cases}$$

$$\Pi(x)=\begin{cases}1, & -\frac{1}{2}\le x\le\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

با تغییر متغیر $r=e^{-t}$ داریم:

$$u_{tt}+u_{\theta\theta}=0$$

$$\text{Guess: } u(r,\theta)=\sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{n}{2}\theta\right)\Longrightarrow u_{tt}+u_{\theta\theta}=\sum_{n=1}^{\infty}\left(\ddot{T}_n(t)-\frac{n^2}{4}T_n(t)\right)\sin\left(\frac{n}{2}\theta\right)=0$$

$$\text{ODE: } \ddot{T}_n(t)-\frac{n^2}{4}T_n(t)=0\Longrightarrow T_n(t)=c_ne^{-\frac{n}{2}t}+d_ne^{\frac{n}{2}t}=c_nr^{\frac{n}{2}}+d_nr^{-\frac{n}{2}}$$

$$\Longrightarrow u(r,\theta)=\sum_{n=1}^{\infty}\left(c_nr^{\frac{n}{2}}+d_nr^{-\frac{n}{2}}\right)\sin\left(\frac{n}{2}\theta\right)$$

$$u(1,\theta)=\sum_{n=1}^{\infty}(c_n+d_n)\sin\left(\frac{n}{2}\theta\right)=\Pi\left(\frac{\theta-\pi}{\pi}\right)$$

$$\Longrightarrow c_n+d_n=\frac{1}{\pi}\int_0^{2\pi}\Pi\left(\frac{\theta-\pi}{\pi}\right)\sin\left(\frac{n}{2}\theta\right)d\theta=\frac{1}{\pi}\left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin\left(\frac{n}{2}\theta\right)d\theta\right)=\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{4}\right)-\cos\left(\frac{3n\pi}{4}\right)\right)=g_n$$

$$u(2,\theta)=1-\frac{1}{\pi^2}(\theta-\pi)^2=\sum_{n=1}^{\infty}\left(c_n2^{\frac{n}{2}}+d_n2^{-\frac{n}{2}}\right)\sin\left(\frac{n}{2}\theta\right)$$

$$\Longrightarrow c_n2^{\frac{n}{2}}+d_n2^{-\frac{n}{2}}=\frac{1}{\pi}\int_0^{2\pi}\left(1-\frac{1}{\pi^2}(\theta-\pi)^2\right)\sin\left(\frac{n}{2}\theta\right)d\theta=\frac{16}{n^3\pi^3}(1-(-1)^n)=h_n$$

$$\Longrightarrow\begin{cases} c_n+d_n=g_n\\ c_n2^{\frac{n}{2}}+d_n2^{-\frac{n}{2}}=h_n\end{cases}$$

$$\Longrightarrow\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\left(\frac{r^{\frac{n}{2}}-r^{-\frac{n}{2}}}{2^{\frac{n}{2}}-2^{-\frac{n}{2}}}\left(\frac{16}{n^3\pi^3}(1-(-1)^n)\right)+\frac{\left(\frac{2}{r}\right)^{\frac{n}{2}}-\left(\frac{2}{r}\right)^{-\frac{n}{2}}}{2^{\frac{n}{2}}-2^{-\frac{n}{2}}}\left(\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{4}\right)-\cos\left(\frac{3n\pi}{4}\right)\right)\right)\right)\sin\left(\frac{n}{2}\theta\right)}$$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

$$u(a,\theta)=\begin{cases}u_0, & 0\le\theta<\pi\\ 0, & \pi\le\theta<2\pi\end{cases}$$

$$u_{tt}+u_{\theta\theta}=0\Longrightarrow\frac{\ddot{T}(t)}{T(t)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}=n^2\Longrightarrow\begin{cases}\ddot{T}(t)-n^2T(t)=0\\ \Theta''(\theta)+n^2\Theta(\theta)=0\end{cases}$$

$$\Longrightarrow T_n(t)=\begin{cases}c_0-d_0t=c_0+d_0\ln(r)\\ c_ne^{-nt}+d_ne^{nt}=c_nr^n+d_nr^{-n}\end{cases}$$

$$\Longrightarrow\Theta_n(\theta)=\begin{cases}a_0\theta+b_0\\ a_n\cos(n\theta)+b_n\sin(n\theta)\end{cases}$$

$r=0$ is in the domain $\Longrightarrow d_0=0,\qquad d_n=0,\qquad u(r,\theta)$ has to be periodic $\Longrightarrow a_0=0$

$$\Longrightarrow u(r,\theta)=\tilde{c}_0+\sum_{n=1}^{\infty}r^n\big(\tilde{a}_n\cos(n\theta)+\tilde{b}_n\sin(n\theta)\big)$$

$$u(a,\theta)=\begin{cases}u_0, & 0\le\theta<\pi\\ 0, & \pi\le\theta<2\pi\end{cases}\Longrightarrow\tilde{c}_0=\frac{1}{2\pi}\int_0^{\pi}u_0d\theta=\frac{u_0}{2}$$

بررسی تعامد:

$$\int_0^{2\pi}u(a,\theta)\cos(m\theta)\,d\theta=\frac{u_0}{2}\int_0^{2\pi}\cos(m\theta)\,d\theta+\int_0^{2\pi}a^n\big(\tilde{a}_n\cos(n\theta)+\tilde{b}_n\sin(n\theta)\big)\cos(m\theta)\,d\theta=u_0\int_0^{\pi}\cos(m\theta)\,d\theta=0+a_ma^m\pi\Longrightarrow a_m=0$$

$$\int_0^{2\pi}u(a,\theta)\sin(m\theta)\,d\theta=\frac{u_0}{2}\int_0^{2\pi}\sin(m\theta)\,d\theta+\int_0^{2\pi}a^n\big(\tilde{a}_n\cos(n\theta)+\tilde{b}_n\sin(n\theta)\big)\sin(m\theta)\,d\theta=u_0\int_0^{\pi}\sin(m\theta)\,d\theta=\frac{u_0}{2}\left[\frac{\cos(m\theta)}{m}\right]_0^{2\pi}+b_ma^m\pi$$

$$\Longrightarrow\frac{2u_0}{m}=b_ma^m\pi\text{ for odd }m=2n-1$$

$$\Longrightarrow\boxed{u(r,\theta)=\frac{u_0}{2}+\sum_{n=1}^{\infty}\frac{2u_0}{(2n-1)\pi}\left(\frac{r}{a}\right)^{2n-1}\sin\big((2n-1)\theta\big)}$$

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$$f_{xx}-\frac{1}{\pi^2}f_{tt}=\left(\frac{1}{\pi^2}x^2-\frac{1}{\pi}x+2\right)\sin(t)\,u(t)-2tu(t),\qquad 0<x<\pi,\qquad 0<t$$

$$\begin{cases}f(x.0)=0\\ f_t(x,0)=0\end{cases},\qquad \begin{cases}f(0,t)=e^{-t}u(t)\\ f(\pi,t)=e^{-(t-1)}u(t-1)\end{cases}$$

$$u(t)=\begin{cases}1, & 0<t\\ 0, & t<0\end{cases}\Longrightarrow \text{تابع پله}$$

$$F_{xx}(x,s)-\frac{s^2}{\pi^2}F(x,s)=\left(\frac{1}{\pi^2}x^2-\frac{1}{\pi}x+2\right)\frac{1}{s^2+1}-\frac{2}{s^2}\Longrightarrow\begin{cases}F_h(x,s)=k_1e^{-\frac{s}{\pi}x}+k_2e^{\frac{s}{\pi}x}\\ F_p(x,s)=Ax^2+Bx+C\end{cases}$$

$$\begin{cases}-\dfrac{s^2}{\pi^2}A=\dfrac{1}{\pi^2}\dfrac{1}{s^2+1}\\ -\dfrac{s^2}{\pi^2}B=-\dfrac{1}{\pi}\dfrac{1}{s^2+1}\\ 2A-\dfrac{s^2}{\pi^2}C=\dfrac{2}{s^2+1}-\dfrac{2}{s^2}\end{cases}\Longrightarrow\begin{cases}A=-\dfrac{1}{s^2(s^2+1)}\\ B=\dfrac{\pi}{s^2(s^2+1)}\\ C=0\end{cases}\Longrightarrow F_p(x,s)=\frac{\pi x-x^2}{s^2(s^2+1)}$$

$$f(0,t)=e^{-t}u(t)\Longrightarrow F(0,s)=\frac{1}{s+1},\qquad f(\pi,t)=e^{-(t-1)}u(t-1)\Longrightarrow F(\pi,s)=\frac{e^{-s}}{s+1}$$

$$\begin{cases}F(0,s)=k_1+k_2=\dfrac{1}{s+1}\\ F(\pi,s)=k_1e^{-s}+k_2e^{s}=\dfrac{e^{-s}}{s+1}\end{cases}\Longrightarrow\begin{cases}k_1=\dfrac{1}{s+1}\\ k_2=0\end{cases}$$

$$\Longrightarrow F(x,s)=\frac{1}{s+1}e^{-\frac{x}{\pi}s}+\frac{\pi x-x^2}{s^2}-\frac{\pi x-x^2}{s^2+1}$$

$$\Longrightarrow \boxed{f(x,t)=e^{-\left(t-\frac{x}{\pi}\right)}u\left(t-\frac{x}{\pi}\right)+(\pi x-x^2)tu(t)-(\pi x-x^2)\sin(t)\,u(t)}$$

$$f_{xx}-\frac{1}{\pi^2}f_{tt}=\left(\frac{1}{\pi^2}x^2-\frac{1}{\pi}x+\frac{1}{2}\right)\sin(2t)\,u(t)-tu(t),\qquad 0<x<\pi,\qquad 0<t$$

$$\begin{cases} f(x.0)=0 \\ f_t(x,0)=0 \end{cases},\qquad \begin{cases} f(0,t)=e^{-2t}u(t) \\ f(\pi,t)=e^{-2(t-1)}u(t-1) \end{cases}$$

$$u(t)=\begin{cases} 1, & 0<t \\ 0, & t<0 \end{cases} \Longrightarrow \text{تابع پله}$$

$$F_{xx}(x,s)-\frac{s^2}{\pi^2}F(x,s)=\left(\frac{1}{\pi^2}x^2-\frac{1}{\pi}x+\frac{1}{2}\right)\frac{2}{s^2+4}-\frac{1}{s^2}\Longrightarrow \begin{cases} F_h(x,s)=k_1e^{-\frac{s}{\pi}x}+k_2e^{\frac{s}{\pi}x} \\ F_p(x,s)=Ax^2+Bx+C \end{cases}$$

$$\begin{cases} -\frac{s^2}{\pi^2}A=\frac{1}{\pi^2}\frac{2}{s^2+4} \\ -\frac{s^2}{\pi^2}B=-\frac{1}{\pi}\frac{2}{s^2+4} \\ 2A-\frac{s^2}{\pi^2}C=\frac{1}{s^2+4}-\frac{1}{s^2} \end{cases}\Longrightarrow \begin{cases} A=-\frac{2}{s^2(s^2+4)} \\ B=\frac{2\pi}{s^2(s^2+4)} \\ C=0 \end{cases}\Longrightarrow F_p(x,s)=\frac{2(\pi x-x^2)}{s^2(s^2+4)}$$

$$f(0,t)=e^{-2t}u(t)\Longrightarrow F(0,s)=\frac{1}{s+2},\qquad f(\pi,t)=e^{-2(t-1)}u(t-1)\Longrightarrow F(\pi,s)=\frac{e^{-s}}{s+2}$$

$$\begin{cases} F(0,s)=k_1+k_2=\frac{1}{s+2} \\ F(\pi,s)=k_1e^{-s}+k_2e^{s}=\frac{e^{-s}}{s+2} \end{cases}\Longrightarrow \begin{cases} k_1=\frac{1}{s+2} \\ k_2=0 \end{cases}$$

$$\Longrightarrow F(x,s)=\frac{1}{s+2}e^{-\frac{x}{\pi}s}+\frac{\pi x-x^2}{2s^2}-\frac{\pi x-x^2}{2(s^2+4)}$$

$$\Longrightarrow \boxed{f(x,t)=e^{-2\left(t-\frac{x}{\pi}\right)}u\left(t-\frac{x}{\pi}\right)+\frac{\pi x-x^2}{2}tu(t)-\frac{\pi x-x^2}{4}\sin(2t)\,u(t)}$$

$$f_{xx} - \frac{1}{\pi^2} f_{tt} = \sin(x)\left(tu(t) - \left(1 + \frac{1}{\pi^2}\right)\sinh(t)\, u(t)\right), \qquad 0 < x < \pi, \qquad 0 < t$$

$$\begin{cases} f(x.0) = 0 \\ f_t(x,0) = 0 \end{cases}, \qquad \begin{cases} f(0,t) = e^{-t}u(t) \\ f(\pi,t) = e^{-(t-1)}u(t-1) \end{cases}$$

$$u(t) = \begin{cases} 1, & 0 < t \\ 0, & t < 0 \end{cases} \Longrightarrow \text{تابع پله}$$

$$F_{xx} - \frac{s^2}{\pi^2}F = \sin(x)\left(\frac{1}{s^2} - \left(1 + \frac{1}{\pi^2}\right)\frac{1}{s^2 - 1}\right) \Longrightarrow \begin{cases} F_h(x,s) = k_1 e^{-\frac{s}{\pi}x} + k_2 e^{\frac{s}{\pi}x} \\ F_p(x,s) = A\cos(x) + B\sin(x) \end{cases}$$

$$\begin{cases} -A\left(1 + \frac{s^2}{\pi^2}\right) = 0 \\ -B\left(1 + \frac{s^2}{\pi^2}\right) = \frac{1}{s^2} - \left(1 + \frac{1}{\pi^2}\right)\frac{1}{s^2 - 1} = -\frac{1 + \frac{s^2}{\pi^2}}{s^2(s^2 - 1)} \end{cases} \Longrightarrow \begin{cases} A = 0 \\ B = \frac{1}{s^2(s^2 - 1)} \end{cases} \Longrightarrow F_p(x,s) = \frac{\sin(x)}{s^2(s^2 - 1)}$$

$$f(0,t) = e^{-t}u(t) \Longrightarrow F(0,s) = \frac{1}{s+1}, \qquad f(\pi,t) = e^{-(t-1)}u(t-1) \Longrightarrow F(\pi,s) = \frac{e^{-s}}{s+1}$$

$$\begin{cases} F(0,s) = k_1 + k_2 = \frac{1}{s+1} \\ F(\pi,s) = k_1 e^{-s} + k_2 e^{s} = \frac{e^{-s}}{s+1} \end{cases} \Longrightarrow \begin{cases} k_1 = \frac{1}{s+1} \\ k_2 = 0 \end{cases}$$

$$\Longrightarrow F(x,s) = \frac{e^{-\frac{x}{\pi}s}}{s+1} + \sin(x)\left(\frac{1}{s^2 - 1} - \frac{1}{s^2}\right)$$

$$\Longrightarrow \boxed{f(x,t) = e^{-\left(t - \frac{x}{\pi}\right)}u\left(t - \frac{x}{\pi}\right) + \sin(x)\left(\sinh(t)\, u(t) - tu(t)\right)}$$

۴)

$$u - v = (x-y)(x^2+4xy+y^2) \Longrightarrow \begin{cases} u_x - v_x = (x^2+4xy+y^2)+(x-y)(2x+4y) \\ u_y - v_y = -(x^2+4xy+y^2)+(x-y)(4x+2y) \end{cases}$$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Longrightarrow \begin{cases} u_x - v_x + u_y - v_y = 6x^2 - 6y^2 = 2u_y \Longrightarrow u(x,y) = 3x^2y - y^3 + g(x) \\ u_x - v_x - u_y + v_y = 12xy = 2u_x \Longrightarrow u(x,y) = 3x^2y + h(y) \end{cases} \Longrightarrow h(y) = -y^3 + g(x) \Longrightarrow \begin{cases} g(x) = c \\ h(y) = -y^3 + c \end{cases}$$

$$\Longrightarrow \begin{cases} u(x,y) = 3x^2y - y^3 + c \\ v(x,y) = 3xy^2 - x^3 + c \end{cases} \Longrightarrow \boxed{f(z) = -iz^3 + c(1+i)}$$

$$u - v = e^x(\cos y - \sin y) \Longrightarrow \begin{cases} u_x - v_x = e^x(\cos y - \sin y) \\ u_y - v_y = -e^x(\cos y + \sin y) \end{cases}$$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Longrightarrow \begin{cases} u_x - v_x + u_y - v_y = -2e^x \sin y = 2u_y \Longrightarrow u(x,y) = e^x \cos y + g(x) \\ u_x - v_x - u_y + v_y = 2e^x \cos y = 2u_x \Longrightarrow u(x,y) = e^x \cos y + h(y) \end{cases} \Longrightarrow g(x) = h(y) = c \Longrightarrow \begin{cases} u(x,y) = e^x \cos y + c \\ v(x,y) = e^x \sin y + c \end{cases}$$

$$f(z) = e^x(\cos y + i \sin y) + c + ic = \boxed{e^z + c(1+i)}$$

$$f'(z) = u_x + iv_x = v_y - iu_y \Longrightarrow \mathrm{Re}\big(f'(z)\big) = u_x = v_y = 3x^2 - 4y - 3y^2 \Longrightarrow \begin{cases} u(x,y) = x^3 - 4xy - 3xy^2 + g(y) \\ v(x,y) = 3x^2y - 2y^2 - y^3 + h(x) \end{cases}$$

$$u_y = -v_x = -4x - 6xy + g'(y) = -6xy - h'(x) \Longrightarrow \begin{cases} h'(x) = 4x + c \\ g'(y) = -c \end{cases} \Longrightarrow \begin{cases} h(x) = 2x^2 + cx + d \\ g(y) = -cy + e \end{cases}$$

$$\Longrightarrow \begin{cases} u(x,y) = x^3 - 4xy - 3xy^2 - cy + e \\ v(x,y) = 3x^2y - 2y^2 - y^3 + 2x^2 + cx + d \end{cases}$$

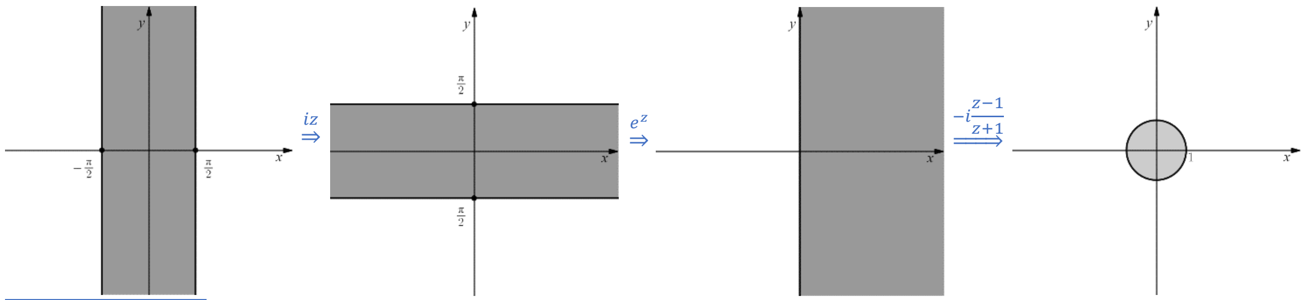
$$f'(0) = 0 \Longrightarrow c = 0$$

$$f(1+i) = 0 \Longrightarrow \begin{cases} 1 - 4 - 3 + e = 0 \\ 3 - 2 - 1 + 2 + d = 0 \end{cases} \Longrightarrow \begin{cases} e = 6 \\ d = -2 \end{cases} \Longrightarrow f(z) = (x^3 - 4xy - 3xy^2 + 6) + i(3x^2y - 2y^2 - y^3 + 2x^2 - 2)$$

$$f(z) = (x+iy)^3 + 2i(x+iy)^2 + 6 - 2i = \boxed{z^3 + 2iz^2 + 6 - 2i}$$

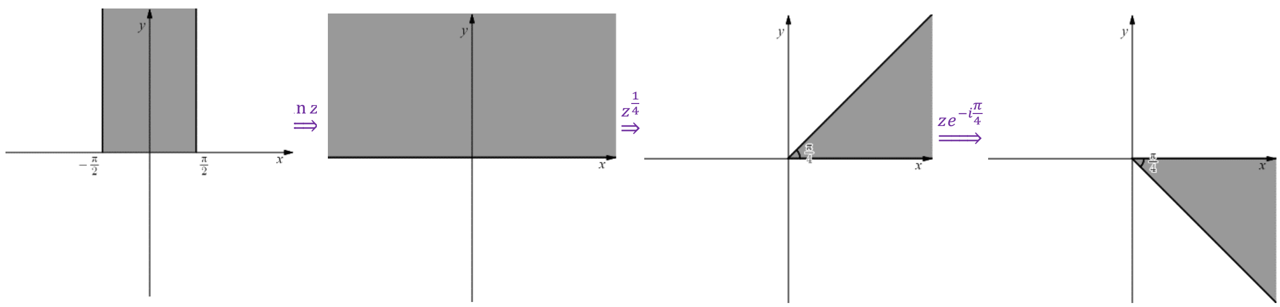
۵)

$$\tan\left(\frac{z}{2}\right)=\frac{\sin\left(\frac{z}{2}\right)}{\cos\left(\frac{z}{2}\right)}=-i\frac{e^{i\frac{z}{2}}-e^{-i\frac{z}{2}}}{e^{i\frac{z}{2}}+e^{-i\frac{z}{2}}}=-i\frac{e^{iz}-1}{e^{iz}+1}$$



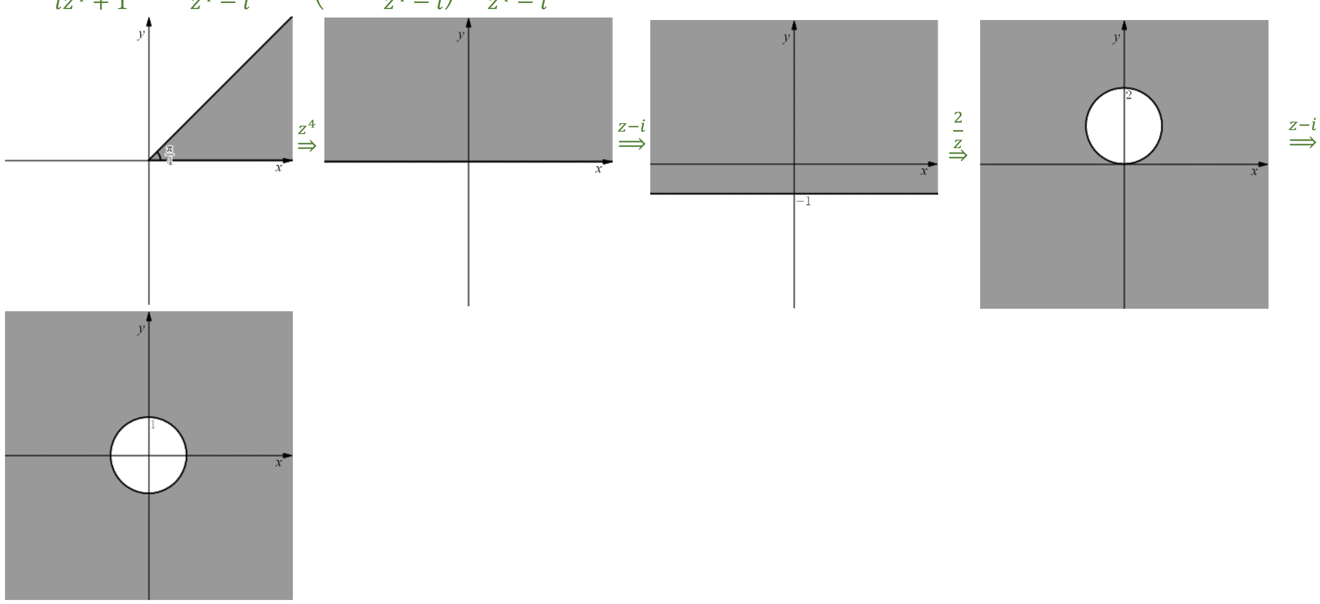
$\boxed{D_w=\{w|\ |w|\le 1\}}$

$$w=e^{-i\frac{\pi}{4}}(\sin z)^{\frac{1}{4}}$$



$\boxed{D_w=\left\{w|-\frac{\pi}{4}\le\theta\le 0\right\}}$

$$w=\frac{z^4+i}{iz^4+1}=-i\frac{z^4+i}{z^4-i}=-i\left(1+2\frac{i}{z^4-i}\right)=\frac{2}{z^4-i}-i$$



$\boxed{D_w=\{w|\ |w|\le 1\}}$