۱)

۱-۱) سری فوریه تابع $f(x)$ را در بازه $[-1,1]$ بدست آورید و با استفاده از آن حاصل سری A و B را محاسبه کنید.

$$f(x)=\operatorname{sign}(x)\Lambda(x)=\begin{cases}1-x, & 0\le x<1\\ -1-x, & -1\le x<0\\ 0, & \text{otherwise}\end{cases},\qquad A=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots,\qquad B=\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$$

$$f(x)=-f(-x)\Longrightarrow f(x)\text{ is odd}\Longrightarrow a_0=0,\qquad a_n=0$$

$$b_n=\frac{2}{T}\int_T f(x)\sin\left(\frac{2n\pi}{T}x\right)dx=\int_{-1}^{+1}f(x)\sin(n\pi x)\,dx=2\int_0^{+1}f(x)\sin(n\pi x)\,dx=2\int_0^{+1}(1-x)\sin(n\pi x)\,dx$$

$$=2\left[-\frac{1}{n\pi}(1-x)\cos(n\pi x)-\frac{1}{n^2\pi^2}\sin(n\pi x)\right]_0^{+1}=\frac{2}{n\pi}$$

$$\Longrightarrow \boxed{f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x)}$$

$$f\left(\frac{1}{2}\right)=\frac{1}{2}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}$$

$$\sin\left(\frac{n\pi}{2}\right)=\begin{cases}0, & n=2k\\ (-1)^{k+1}, & n=2k-1\end{cases}\Longrightarrow f\left(\frac{1}{2}\right)=\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}=\frac{1}{2}\Longrightarrow \boxed{A=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}=\frac{\pi}{4}}$$

$$\text{Parseval: }\frac{2}{T}\int_T|f(x)|^2dx=2a_0^2+\sum_{n=1}^{\infty}a_n^2+b_n^2\Longrightarrow\int_{-1}^{+1}|f(x)|^2=2\int_0^{+1}(1-x)^2dx=\frac{2}{3}=\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Longrightarrow\boxed{B=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}}$$

۱-۲) سری فوریه تابع $f(x)$ را در بازه $[-1,1]$ بدست آورید و با استفاده از آن حاصل سری A و B را محاسبه کنید.

$$f(x)=\Pi(x)=\begin{cases}1, & |x|\le\frac{1}{2}\\ 0, & \text{otherwise}\end{cases},\qquad A=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots,\qquad B=\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots$$

$$f(x)=f(-x)\Longrightarrow f(x)\text{ is even}\Longrightarrow b_n=0$$

$$a_0=\frac{1}{T}\int_T f(x)dx=\frac{1}{2}\int_{-1}^{+1}f(x)dx=\int_0^{+1}f(x)dx=\frac{1}{2}$$

$$a_n=\frac{2}{T}\int_T f(x)\cos\left(\frac{2n\pi}{T}x\right)dx=\int_{-1}^{+1}f(x)\cos(n\pi x)\,dx=2\int_0^{+1}f(x)\cos(n\pi x)\,dx=2\int_0^{+\frac{1}{2}}\cos(n\pi x)\,dx=\frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$\Longrightarrow\boxed{f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos(n\pi x)}$$

$$f(0)=1=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}$$

$$\sin\left(\frac{n\pi}{2}\right)=\begin{cases}0, & n=2k\\ (-1)^{k+1}, & n=2k-1\end{cases}\Longrightarrow f(0)=\frac{1}{2}+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}=1\Longrightarrow\boxed{A=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}=\frac{\pi}{4}}$$

$$\text{Parseval: }\frac{2}{T}\int_T|f(x)|^2dx=2a_0^2+\sum_{n=1}^{\infty}a_n^2+b_n^2\Longrightarrow\int_{-1}^{+1}|f(x)|^2=2\int_0^{+\frac{1}{2}}dx=1=\frac{1}{2}+\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

$$\Longrightarrow\boxed{B=\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}}$$

۱-۳) سری فوریه تابع $f(x)$ را در بازه $[-1,1]$ بدست آورید و با استفاده از آن حاصل سری A و B را محاسبه کنید.

$$f(x)=\Lambda^2(x)=\begin{cases}(1-x)^2, & 0\le x<1\\ (1+x)^2, & -1\le x<0\\ 0, & \text{otherwise}\end{cases},\qquad A=\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots,\qquad B=\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots$$

$$f(x)=f(-x)\Longrightarrow f(x)\text{ is even}\Longrightarrow b_n=0$$

$$a_0=\frac{1}{T}\int_T f(x)dx=\frac{1}{2}\int_{-1}^{+1}f(x)dx=\int_0^{+1}(1-x)^2dx=\left[x-x^2+\frac{x^3}{3}\right]_0^{+1}=\frac{1}{3}$$

$$a_n=\frac{2}{T}\int_T f(x)\cos\left(\frac{2n\pi}{T}x\right)dx=\int_{-1}^{+1}f(x)\cos(n\pi x)\,dx=2\int_0^{+1}(1-x)^2\cos(n\pi x)\,dx$$

$$=2\left[\frac{1}{n\pi}(1-x)^2\sin(n\pi x)\right]_0^{+1}+\frac{4}{n\pi}\int_0^{+1}(1-x)\sin(n\pi x)\,dx$$

$$=2\left[\frac{1}{n\pi}(1-x)^2\sin(n\pi x)\right]_0^{+1}+\frac{4}{n\pi}\left(\left[-\frac{1}{n\pi}(1-x)\cos(n\pi x)\right]_0^{+1}-\frac{1}{n\pi}\int_0^{+1}\cos(n\pi x)\,dx\right)=\frac{4}{n^2\pi^2}$$

$$\Longrightarrow\boxed{f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\pi x)}$$

$$f(0)=1=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Longrightarrow\boxed{A=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}}$$

$$f(1)=0=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Longrightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$$

$$\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\right)=\boxed{B=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}}$$

۱-۴) سری فوریه تابع $f(x)$ را در بازه $[-1,1]$ بدست آورید و با استفاده از آن حاصل سری A و B را محاسبه کنید.

$$f(x)=x^2,\qquad A=\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots,\qquad B=\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots$$

$$f(x)=f(-x)\Longrightarrow f(x)\text{ is even}\Longrightarrow b_n=0$$

$$a_0=\frac{1}{T}\int_T f(x)dx=\frac{1}{2}\int_{-1}^{+1}f(x)dx=\int_0^{+1}x^2dx=\left[\frac{x^3}{3}\right]_0^{+1}=\frac{1}{3}$$

$$a_n=\frac{2}{T}\int_T f(x)\cos\left(\frac{2n\pi}{T}x\right)dx=\int_{-1}^{+1}f(x)\cos(n\pi x)\,dx=2\int_0^{+1}x^2\cos(n\pi x)\,dx$$

$$=2\left[\frac{1}{n\pi}x^2\sin(n\pi x)\right]_0^{+1}-\frac{4}{n\pi}\int_0^{+1}x\sin(n\pi x)\,dx$$

$$=2\left[\frac{1}{n\pi}x^2\sin(n\pi x)\right]_0^{+1}-\frac{4}{n\pi}\left(\left[-\frac{1}{n\pi}x\cos(n\pi x)\right]_0^{+1}+\frac{1}{n\pi}\int_0^{+1}\cos(n\pi x)\,dx\right)=\frac{4(-1)^n}{n^2\pi^2}$$

$$\Longrightarrow\boxed{f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)}$$

$$f(1)=1=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}\Longrightarrow\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}}$$

$$f(0)=0=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Longrightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$$

$$\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\right)=\boxed{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}}$$

۲)

۲-۱) سری فوریه مختلط تابع $f(x)$ را در بازه $[0,1]$ بدست آورید و با استفاده از آن ضرایب سری فوریه، a_n و b_n را محاسبه کنید.

$$f(x) = x\Pi\left(\frac{x-\frac{1}{4}}{\frac{1}{2}}\right) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$c_n = \frac{1}{T}\int_T f(x)e^{-i\frac{2n\pi}{T}x}dx = \int_0^1 f(x)e^{-i2n\pi x}dx = \int_0^{\frac{1}{2}} xe^{-i2n\pi x}dx = \left[\frac{xe^{-i2n\pi x}}{-i2n\pi} + \frac{e^{-i2n\pi x}}{4n^2\pi^2}\right]_0^{\frac{1}{2}} = \frac{\frac{1}{2}e^{-in\pi}}{-i2n\pi} + \frac{e^{-in\pi}-1}{4n^2\pi^2}$$

$$\Rightarrow c_n = \frac{(-1)^n - 1}{4n^2\pi^2} + i\frac{(-1)^n}{4n\pi}, \qquad n \neq 0$$

$$\Rightarrow c_0 = \frac{1}{T}\int_T f(x)dx = \int_0^1 f(x)dx = \int_0^{\frac{1}{2}} xdx = \frac{1}{8}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{8} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\frac{(-1)^n - 1}{4n^2\pi^2} + i\frac{(-1)^n}{4n\pi}\right)e^{i2n\pi x}}$$

$$\begin{cases} a_n = c_n + c_{-n} = \dfrac{(-1)^n - 1}{4n^2\pi^2} + i\dfrac{(-1)^n}{4n\pi} + \dfrac{(-1)^{-n} - 1}{4(-n)^2\pi^2} - i\dfrac{(-1)^{-n}}{4n\pi} = \boxed{\dfrac{(-1)^n - 1}{2n^2\pi^2}} \\ b_n = i(c_n - c_{-n}) = i\left(\dfrac{(-1)^n - 1}{4n^2\pi^2} + i\dfrac{(-1)^n}{4n\pi} - \dfrac{(-1)^{-n} - 1}{4(-n)^2\pi^2} + i\dfrac{(-1)^{-n}}{4n\pi}\right) = \boxed{-\dfrac{(-1)^n}{2n\pi}} \end{cases}$$

۲-۲) سری فوریه مختلط تابع $f(x)$ را در بازه $[0,1]$ بدست آورید و با استفاده از آن ضرایب سری فوریه، a_n و b_n را محاسبه کنید.

$$f(x) = \left(\frac{1}{2} - x\right)\Pi\left(\frac{x-\frac{1}{4}}{\frac{1}{2}}\right) = \begin{cases} \frac{1}{2} - x, & 0 \le x \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$c_n = \frac{1}{T}\int_T f(x)e^{-i\frac{2n\pi}{T}x}dx = \int_0^1 f(x)e^{-i2n\pi x}dx = \int_0^{\frac{1}{2}}\left(\frac{1}{2} - x\right)e^{-i2n\pi x}dx = \left[\frac{e^{-i2n\pi x}}{-i4n\pi} - \frac{xe^{-i2n\pi x}}{-i2n\pi} - \frac{e^{-i2n\pi x}}{4n^2\pi^2}\right]_0^{\frac{1}{2}}$$

$$= \frac{e^{-in\pi} - 1}{-i4n\pi} - \frac{\frac{1}{2}e^{-in\pi}}{-i2n\pi} - \frac{e^{-in\pi} - 1}{4n^2\pi^2}$$

$$\Rightarrow c_n = \frac{1 + (-1)^{n+1}}{4n^2\pi^2} - i\frac{1}{4n\pi}, \qquad n \neq 0$$

$$\Rightarrow c_0 = \frac{1}{T}\int_T f(x)dx = \int_0^1 f(x)dx = \int_0^{\frac{1}{2}} xdx = \frac{1}{8}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{8} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\frac{1 + (-1)^{n+1}}{4n^2\pi^2} - i\frac{1}{4n\pi}\right)e^{i2n\pi x}}$$

$$\begin{cases} a_n = c_n + c_{-n} = \dfrac{1 + (-1)^{n+1}}{4n^2\pi^2} - i\dfrac{1}{4n\pi} + \dfrac{1 + (-1)^{-n+1}}{4(-n)^2\pi^2} + i\dfrac{1}{4n\pi} = \boxed{\dfrac{1 + (-1)^{n+1}}{2n^2\pi^2}} \\ b_n = i(c_n - c_{-n}) = i\left(\dfrac{1 + (-1)^{n+1}}{4n^2\pi^2} - i\dfrac{1}{4n\pi} - \dfrac{1 + (-1)^{-n+1}}{4(-n)^2\pi^2} - i\dfrac{1}{4n\pi}\right) = \boxed{\dfrac{1}{2n\pi}} \end{cases}$$

۲-۳) سری فوریه مختلط تابع $f(x)$ را در بازه $[0,1]$ بدست آورید و با استفاده از آن ضرایب سری فوریه، a_n و b_n را محاسبه کنید.

$$f(x)=\left(x-\frac{1}{2}\right)\Pi\left(\frac{x-\frac{3}{4}}{\frac{1}{2}}\right)=\begin{cases}x-\frac{1}{2}, & \frac{1}{2}\le x\le 1\\ 0, & \text{otherwise}\end{cases}$$

$$c_n=\frac{1}{T}\int_T f(x)e^{-i\frac{2n\pi}{T}x}dx=\int_0^1 f(x)e^{-i2n\pi x}dx=\int_{\frac{1}{2}}^1\left(x-\frac{1}{2}\right)e^{-i2n\pi x}dx=\left[\frac{xe^{-i2n\pi x}}{-i2n\pi}+\frac{e^{-i2n\pi x}}{4n^2\pi^2}-\frac{e^{-i2n\pi x}}{-i4n\pi}\right]_{\frac{1}{2}}^1$$

$$=\frac{e^{-i2n\pi}-\frac{1}{2}e^{-in\pi}}{-i2n\pi}+\frac{e^{-i2n\pi}-e^{-in\pi}}{4n^2\pi^2}-\frac{e^{-i2n\pi}-e^{-in\pi}}{-i4n\pi}=\frac{1-(-1)^n}{4n^2\pi^2}+i\frac{1}{4n\pi}$$

$$\Longrightarrow c_n=\frac{1+(-1)^{n+1}}{4n^2\pi^2}+i\frac{1}{4n\pi},\qquad n\neq 0$$

$$\Longrightarrow c_0=\frac{1}{T}\int_T f(x)dx=\int_0^1 f(x)dx=\int_0^{\frac{1}{2}}xdx=\frac{1}{8}$$

$$\Longrightarrow \boxed{f(x)=\frac{1}{8}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{1+(-1)^{n+1}}{4n^2\pi^2}+i\frac{1}{4n\pi}\right)e^{i2n\pi x}}$$

$$\begin{cases}a_n=c_n+c_{-n}=\frac{1+(-1)^{n+1}}{4n^2\pi^2}+i\frac{1}{4n\pi}+\frac{1+(-1)^{-n+1}}{4(-n)^2\pi^2}-i\frac{1}{4n\pi}=\boxed{\frac{1+(-1)^{n+1}}{2n^2\pi^2}}\\ b_n=i(c_n-c_{-n})=i\left(\frac{1+(-1)^{n+1}}{4n^2\pi^2}+i\frac{1}{4n\pi}-\frac{1+(-1)^{-n+1}}{4(-n)^2\pi^2}+i\frac{1}{4n\pi}\right)=\boxed{-\frac{1}{2n\pi}}\end{cases}$$

۲-۴) سری فوریه مختلط تابع $f(x)$ را در بازه $[0,1]$ بدست آورید و با استفاده از آن ضرایب سری فوریه، a_n و b_n را محاسبه کنید.

$$f(x)=(1-x)\Pi\left(\frac{x-\frac{3}{4}}{\frac{1}{2}}\right)=\begin{cases}1-x, & \frac{1}{2}\le x\le 1\\ 0, & \text{otherwise}\end{cases}$$

$$c_n=\frac{1}{T}\int_T f(x)e^{-i\frac{2n\pi}{T}x}dx=\int_0^1 f(x)e^{-i2n\pi x}dx=\int_{\frac{1}{2}}^1(1-x)e^{-i2n\pi x}dx=\left[\frac{e^{-i2n\pi x}}{-i2n\pi}-\frac{xe^{-i2n\pi x}}{-i2n\pi}-\frac{e^{-i2n\pi x}}{4n^2\pi^2}\right]_{\frac{1}{2}}^1$$

$$=\frac{e^{-i2n\pi}-e^{-in\pi}}{-i2n\pi}-\frac{e^{-i2n\pi}-\frac{1}{2}e^{-in\pi}}{-i2n\pi}-\frac{e^{-i2n\pi}-e^{-in\pi}}{4n^2\pi^2}=\frac{(-1)^n-1}{4n^2\pi^2}-i\frac{(-1)^n}{4n\pi}$$

$$\Longrightarrow c_n=\frac{(-1)^n-1}{4n^2\pi^2}+i\frac{(-1)^{n+1}}{4n\pi},\qquad n\neq 0$$

$$\Longrightarrow c_0=\frac{1}{T}\int_T f(x)dx=\int_0^1 f(x)dx=\int_0^{\frac{1}{2}}xdx=\frac{1}{8}$$

$$\Longrightarrow \boxed{f(x)=\frac{1}{8}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{(-1)^n-1}{4n^2\pi^2}+i\frac{(-1)^{n+1}}{4n\pi}\right)e^{i2n\pi x}}$$

$$\begin{cases}a_n=c_n+c_{-n}=\frac{(-1)^n-1}{4n^2\pi^2}+i\frac{(-1)^{n+1}}{4n\pi}+\frac{(-1)^{-n}-1}{4(-n)^2\pi^2}-i\frac{(-1)^{-n+1}}{4n\pi}=\boxed{\frac{(-1)^n-1}{2n^2\pi^2}}\\ b_n=i(c_n-c_{-n})=i\left(\frac{(-1)^n-1}{4n^2\pi^2}+i\frac{(-1)^{n+1}}{4n\pi}-\frac{(-1)^{-n}-1}{4(-n)^2\pi^2}+i\frac{(-1)^{-n+1}}{4n\pi}\right)=\boxed{\frac{(-1)^n}{2n\pi}}\end{cases}$$

۳)

۱-۳) فرض کنید تابع $f(x)$ تابعی متناوب با دوره تناوب T_f باشد. سری فوریه این تابع را در بازه $[0,T_f]$ محاسبه می‌کنیم و ضرایب سری فوریه آن a_0، a_n و b_n می‌باشد. حال فرض کنید سری فوریه تابع $f(2x)$ را در بازه $[0,T_f]$ محاسبه می‌کنیم و ضرایب سری فوریه آن a'_0، a'_n و b'_n است. نشان دهید روابط زیر بین ضرایب سری فوریه این دو تابع برقرار است.

$$a'_0 = a_0, \qquad a'_n = \frac{1}{2} a_{\frac{n}{2}} (1 + (-1)^n), \qquad b'_n = \frac{1}{2} b_{\frac{n}{2}} (1 + (-1)^n)$$

$$a_0 = \frac{1}{T} \int_T f(x)dx = \frac{1}{T_f} \int_0^{T_f} f(x)dx$$

$$a'_0 = \frac{1}{T_f} \int_0^{T_f} f(2x)dx \xRightarrow{2x=u} \frac{1}{2T_f} \int_0^{2T_f} f(u)du = \frac{1}{2T_f} \left(\int_0^{T_f} f(u)du + \int_{T_f}^{2T_f} f(u)du \right)$$

$$= \frac{1}{2T_f} \left(T_f a_0 + \int_{T_f}^{2T_f} f(u)du \right) \xRightarrow{u=v+T_f} \frac{1}{2T_f} \left(T_f a_0 + \int_0^{T_f} f(v+T_f)dv \right) = \frac{1}{2T_f} \left(T_f a_0 + \int_0^{T_f} f(v)dv \right)$$

$$= \frac{1}{2T_f} (2T_f a_0) = a_0$$

$$a_n = \frac{2}{T} \int_T f(x) \cos\left(\frac{2n\pi}{T} x\right) dx = \frac{2}{T_f} \int_0^{T_f} f(x) \cos\left(\frac{2n\pi}{T_f} x\right) dx$$

$$a'_n = \frac{2}{T_f} \int_0^{T_f} f(2x) \cos\left(\frac{2n\pi}{T_f} x\right) dx \xRightarrow{2x=u} \frac{1}{T_f} \int_0^{2T_f} f(u) \cos\left(\frac{n\pi}{T_f} u\right) du$$

$$= \frac{1}{T_f} \left(\int_0^{T_f} f(u) \cos\left(\frac{n\pi}{T_f} u\right) du + \int_{T_f}^{2T_f} f(u) \cos\left(\frac{n\pi}{T_f} u\right) du \right)$$

$$= \frac{1}{T_f} \left(\frac{T_f}{2} a_{\frac{n}{2}} + \int_{T_f}^{2T_f} f(u) \cos\left(\frac{n\pi}{T_f} u\right) du \right) \xRightarrow{u=v+T_f} \frac{1}{T_f} \left(\frac{T_f}{2} a_{\frac{n}{2}} + \int_0^{T_f} f(v+T_f) \cos\left(\frac{n\pi}{T_f} v + n\pi\right) du \right)$$

$$= \frac{1}{T_f} \left(\frac{T_f}{2} a_{\frac{n}{2}} + \int_0^{T_f} f(v) \cos\left(\frac{n\pi}{T_f} v\right) (-1)^n du \right) = \frac{1}{T_f} \left(\frac{T_f}{2} a_{\frac{n}{2}} + \frac{T_f}{2} a_{\frac{n}{2}} (-1)^n \right) = \frac{1}{2} a_{\frac{n}{2}} (1 + (-1)^n)$$

$$b_n = \frac{2}{T} \int_T f(x) \sin\left(\frac{2n\pi}{T} x\right) dx = \frac{2}{T_f} \int_0^{T_f} f(x) \sin\left(\frac{2n\pi}{T_f} x\right) dx$$

$$b'_n = \frac{2}{T_f} \int_0^{T_f} f(2x) \sin\left(\frac{2n\pi}{T_f} x\right) dx \xRightarrow{2x=u} \frac{1}{T_f} \int_0^{2T_f} f(u) \sin\left(\frac{n\pi}{T_f} u\right) du$$

$$= \frac{1}{T_f} \left(\int_0^{T_f} f(u) \sin\left(\frac{n\pi}{T_f} u\right) du + \int_{T_f}^{2T_f} f(u) \sin\left(\frac{n\pi}{T_f} u\right) du \right)$$

$$= \frac{1}{T_f} \left(\frac{T_f}{2} b_{\frac{n}{2}} + \int_{T_f}^{2T_f} f(u) \sin\left(\frac{n\pi}{T_f} u\right) du \right) \xRightarrow{u=v+T_f} \frac{1}{T_f} \left(\frac{T_f}{2} b_{\frac{n}{2}} + \int_0^{T_f} f(v+T_f) \sin\left(\frac{n\pi}{T_f} v + n\pi\right) du \right)$$

$$= \frac{1}{T_f} \left(\frac{T_f}{2} b_{\frac{n}{2}} + \int_0^{T_f} f(v) \sin\left(\frac{n\pi}{T_f} v\right) (-1)^n du \right) = \frac{1}{T_f} \left(\frac{T_f}{2} b_{\frac{n}{2}} + \frac{T_f}{2} b_{\frac{n}{2}} (-1)^n \right) = \frac{1}{2} b_{\frac{n}{2}} (1 + (-1)^n)$$

۲-۳) فرض کنید تابع $f(x)$ تابعی متناوب با دوره تناوب T_f باشد. سری فوریه این تابع را در بازه $[0,T_f]$ محاسبه می‌کنیم و ضرایب سری فوریه آن a_0، a_n و b_n می‌باشد. حال فرض کنید سری فوریه تابع $f(x-\frac{T_f}{2})$ را در بازه $[0,T_f]$ محاسبه می‌کنیم و ضرایب سری فوریه آن a_0'، a_n' و b_n' است. نشان دهید روابط زیر بین ضرایب سری فوریه این دو تابع برقرار است.

$$a_0' = a_0, \qquad a_n' = (-1)^n a_n, \qquad b_n' = (-1)^n b_n$$

$$a_0 = \frac{1}{T}\int_T f(x)dx = \frac{1}{T_f}\int_0^{T_f} f(x)dx$$

$$a_0' = \frac{1}{T_f}\int_0^{T_f} f\left(x-\frac{T_f}{2}\right)dx \xRightarrow{x-\frac{T_f}{2}=u} \frac{1}{T_f}\int_{-\frac{T_f}{2}}^{\frac{T_f}{2}} f(u)du = \frac{1}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)du + \int_{-\frac{T_f}{2}}^{0} f(u)du\right) = \frac{1}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)du + \int_{\frac{T_f}{2}}^{T_f} f(u-T_f)du\right)$$

$$= \frac{1}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)du + \int_{\frac{T_f}{2}}^{T_f} f(u-T_f)du\right) = \frac{1}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)du + \int_{\frac{T_f}{2}}^{T_f} f(u)du\right) = \frac{1}{T_f}\int_0^{T_f} f(u)du = a_0$$

$$a_n = \frac{2}{T}\int_T f(x)\cos\left(\frac{2n\pi}{T}x\right)dx = \frac{2}{T_f}\int_0^{T_f} f(x)\cos\left(\frac{2n\pi}{T_f}x\right)dx$$

$$a_n' = \frac{2}{T_f}\int_0^{T_f} f\left(x-\frac{T_f}{2}\right)\cos\left(\frac{2n\pi}{T_f}x\right)dx \xRightarrow{x-\frac{T_f}{2}=u} \frac{2}{T_f}\int_{-\frac{T_f}{2}}^{\frac{T_f}{2}} f(u)\cos\left(\frac{2n\pi}{T_f}u+n\pi\right)dx$$

$$= \frac{2}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)\cos\left(\frac{2n\pi}{T_f}u+n\pi\right)dx + \int_{-\frac{T_f}{2}}^{0} f(u)\cos\left(\frac{2n\pi}{T_f}u+n\pi\right)dx\right)$$

$$= \frac{2}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)\cos\left(\frac{2n\pi}{T_f}u+n\pi\right)dx + \int_{\frac{T_f}{2}}^{T_f} f(u-T_f)\cos\left(\frac{2n\pi}{T_f}u-n\pi\right)dx\right)$$

$$= \frac{2}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)\cos\left(\frac{2n\pi}{T_f}u+n\pi\right)dx + \int_{\frac{T_f}{2}}^{T_f} f(u)\cos\left(\frac{2n\pi}{T_f}u-n\pi\right)dx\right)$$

$$= \frac{2}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)\cos\left(\frac{2n\pi}{T_f}u\right)(-1)^n dx + \int_{\frac{T_f}{2}}^{T_f} f(u)\cos\left(\frac{2n\pi}{T_f}u\right)(-1)^n dx\right)$$

$$= \frac{2}{T_f}\int_0^{T_f} f(u)\cos\left(\frac{2n\pi}{T_f}u\right)(-1)^n dx = (-1)^n a_n$$

$$b_n = \frac{2}{T}\int_T f(x)\sin\left(\frac{2n\pi}{T}x\right)dx = \frac{2}{T_f}\int_0^{T_f} f(x)\sin\left(\frac{2n\pi}{T_f}x\right)dx$$

$$
\begin{aligned}
b_n' = \frac{2}{T_f}\int_0^{T_f} f\left(x-\frac{T_f}{2}\right)\sin\left(\frac{2n\pi}{T_f}x\right)dx &\xRightarrow{x-\frac{T_f}{2}=u} \frac{2}{T_f}\int_{-\frac{T_f}{2}}^{\frac{T_f}{2}} f(u)\sin\left(\frac{2n\pi}{T_f}u+n\pi\right)dx \\
&= \frac{2}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)\sin\left(\frac{2n\pi}{T_f}u+n\pi\right)dx + \int_{-\frac{T_f}{2}}^{0} f(u)\sin\left(\frac{2n\pi}{T_f}u+n\pi\right)dx\right) \\
&= \frac{2}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)\sin\left(\frac{2n\pi}{T_f}u+n\pi\right)dx + \int_{\frac{T_f}{2}}^{T_f} f(u-T_f)\sin\left(\frac{2n\pi}{T_f}u-n\pi\right)dx\right) \\
&= \frac{2}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)\sin\left(\frac{2n\pi}{T_f}u+n\pi\right)dx + \int_{\frac{T_f}{2}}^{T_f} f(u)\sin\left(\frac{2n\pi}{T_f}u-n\pi\right)dx\right) \\
&= \frac{2}{T_f}\left(\int_0^{\frac{T_f}{2}} f(u)\sin\left(\frac{2n\pi}{T_f}u\right)(-1)^n dx + \int_{\frac{T_f}{2}}^{T_f} f(u)\sin\left(\frac{2n\pi}{T_f}u\right)(-1)^n dx\right) \\
&= \frac{2}{T_f}\int_0^{T_f} f(u)\sin\left(\frac{2n\pi}{T_f}u\right)(-1)^n dx = (-1)^n b_n
\end{aligned}
$$