مدرس: دکتر مهدی طالع ماسوله -حل تمرین: نگین سفاری – وصال بختآزاد– آرمان اکبری
برای سوالات خود در خصوص این تمرین با رایانامه bakhtazad.v@gmail.com مکاتبه نمایید.

1)

a)

$$f(t) = \begin{cases} 1 & -T < t < T \\ 0 & T < t < \frac{T_0}{2}, \frac{-T_0}{2} < t < -T \end{cases}$$

$$f(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(t) e^{\frac{-in\pi t}{L}}\, dt = \frac{\sin n\omega_0 T}{n\pi} \qquad\qquad C_0 = \frac{2T}{T_0}$$

$$f(t) = \sum_{n=-\infty}^{\infty} \left(\frac{\sin n\omega_0 T}{n\pi}\right) e^{jn\omega_0 T}$$

DC voltage reference to constant part of series which is: C_0.

Therefore amount of DC voltage equal to $C_0 = \frac{2T}{T_0}$.

b)

As we know:

$$\sum_{-\infty}^{\infty} |C_n|^2 = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2\, dt$$

$$\sum_{-\infty}^{\infty} |C_n|^2 = \frac{2T}{T_0}$$

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2) From example solved in slide 175 of power point we know that if:

$$f(x) = \begin{cases} A, & |x| < \dfrac{d}{2} \\ 0, & \dfrac{d}{2} < |x| < T \end{cases}$$

$$c_k = \frac{A}{T}\int_{-d/2}^{d/2} e^{-jk\omega_0 x}\,dx = \frac{A}{T}\frac{1}{-jk\omega_0}e^{-jk\omega_0 x}\Big|_{-d/2}^{d/2}$$

$$= \frac{A}{T}\left(\frac{1}{-jk\omega_0}e^{-jk\omega_0 d/2} - \frac{1}{-jk\omega_0}e^{jk\omega_0 d/2}\right)$$

$$c_k = \frac{\sin k\,{}^{\pi}/_{8}}{2k\pi} \quad \Rightarrow T = 4\,, d = \frac{1}{2}\,, A = 1/2$$

3) a) $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-jnx}\,dx = \frac{2}{n^2}\cos n\pi$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} = \sum_{n=-\infty}^{\infty} \frac{2}{n^2}\cos n\pi\, e^{inx}$$

b)

$C_n = \frac{2}{n^2}\cos n\pi \qquad C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^3}{3}$

$a_0 = \frac{2\pi^3}{3} \qquad a_n = \frac{4}{n^2}\cos n\pi$

According to page 173 of notes:

$C_0 = \frac{a_0}{2} \qquad$ and $\quad C_n = \frac{1}{2}(a_n - jb_n)$

Relations between coefficients are same as we expect to be.

4)

$$cos^4(x) = \left(\frac{e^{jx}+e^{-jx}}{2}\right)^4 = \frac{e^{4jx}+4e^{2jx}+4e^{-2jx}+6+e^{-4jx}}{16}$$

$$\frac{1}{T}\int_T |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_k|^2$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} cos^8(x)dx$$

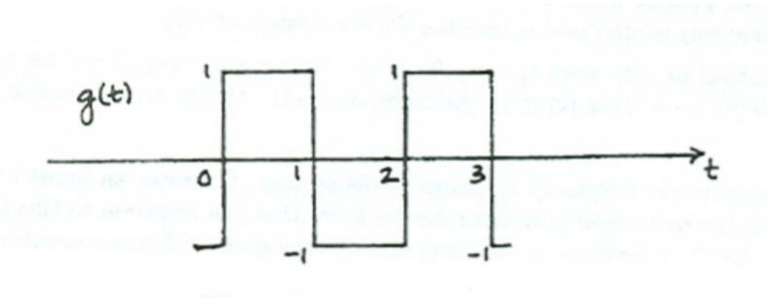
$$= \frac{1}{\pi}\int_0^{\pi} cos^8(x)dx = \left(\frac{1}{16}\right)^2 + \left(\frac{4}{16}\right)^2 + \left(\frac{4}{16}\right)^2 + \left(\frac{6}{16}\right)^2 + \left(\frac{1}{16}\right)^2$$

$$\int_0^{\pi} cos^8(x)dx = \frac{35\pi}{128}$$

5) a)

$$c_0 = \frac{1}{2}\int_0^1 t dt + \frac{1}{2}\int_1^2 (2-t)dt = \frac{1}{2}$$

b) The function $g(t) = \frac{dx(t)}{dt}$ is as shown in the figure below:



The Fourier series coefficients a_k of $g(t)$ may be found as follows:

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt - \frac{1}{2}\int_1^2 e^{-j\pi kt}dt = \frac{1}{j\pi k}\left[1 - e^{-j\pi k}\right] \qquad k \neq 0$$

$$a_0 = \frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt = 0$$

c) Note that:

$$g(t) = \frac{dx(t)}{dt} \leftrightarrow a_k = jk\pi c_k$$

$$c_k = \frac{1}{jk\pi} a_k = -\frac{1}{k^2\pi^2}\{1 - e^{-j\pi k}\} \qquad k \neq 0$$

6)

$$f'(x) = \frac{\sinh(a\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n . in}{(a-in)} e^{inx}$$

$$A = \int_{-\pi}^{\pi} f'(x) e^{2ix} dx + \int_{-\pi}^{\pi} f(x)^2 dx$$

$$= 2\pi \times \frac{\sinh(a\pi)}{\pi} \times \frac{(-1)^{-2} . (-2)i}{(a - i(-2))} + 2\pi \times \sum_{n=-\infty}^{\infty} \left| \frac{\sinh(a\pi)}{\pi(a-in)} \right|^2$$

$$= 2\pi \times \frac{\sinh(a\pi)}{\pi} \times \frac{-2i}{(a+2i)} + 2\pi \times \sum_{n=-\infty}^{\infty} \frac{sinh^2(a\pi)}{\pi^2(a^2+n^2)}$$

$$f(x) = \frac{\sinh(a\pi)}{\pi} \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n a e^{inx}}{a^2+n^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n . in . e^{inx}}{a^2+n^2} \right)$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n . in . e^{inx}}{a^2+n^2} = 0$$

$$x = \pi \Rightarrow f(\pi) = a \frac{\sinh(a\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2} \quad \Rightarrow \quad \sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2} = \frac{\pi f(\pi)}{a \sinh(a\pi)}$$

7)

$$c_{gn} = \begin{cases} \pi/2, & n = 0 \\ \dfrac{(-1)^n - 1}{n^2\pi}, & n \neq 0 \end{cases}$$

$$sin^3(x) = \frac{3\sin(x) - \sin(3x)}{4} = \frac{1}{8i}(3e^{ix} - 3e^{-ix} - e^{3ix} + e^{-3ix})$$

$$c_{fn} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(x)sin^3(x)e^{-inx}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(G(x)\frac{3}{8i}e^{-(n-1)ix} - G(x)\frac{3}{8i}e^{-(n+1)ix} - G(x)\frac{1}{8i}e^{-(n-3)ix} + G(x)\frac{1}{8i}e^{-(n+3)ix}$$

$$= \frac{3}{8i}c_{g(n-1)} - \frac{3}{8i}c_{g(n+1)} - \frac{1}{8i}c_{g(n-3)} + \frac{1}{8i}c_{g(n+3)}$$

$$f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{3}{8i}\left(c_{g(n-1)} - c_{g(n+1)}\right) + \frac{1}{8i}\left(c_{g(n-3)} + c_{g(n+3)}\right)e^{inx}\right.$$

8) The only unknown FS coefficients are $a_1, a_{-1}, a_2, and\ a_{-2}$.Since $x(t)$ is real $a_1 = a^*{}_{-1}$ and $a_2 = a^*{}_{-2}$.Since a_1 is real, $a_1 = a_{-1}$ Now $x(t)$ is of the form:

$a_1 = a_{-1} = a \qquad a_2 = b + cj \qquad a_{-2} = b - cj$

$x(t) = a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{2j\omega_0 t} + a_{-2} e^{-2j\omega_0 t}$

$= 2\,\mathrm{a}\cos(\omega_0 t) + 2b cos(2\omega_0 t) - 2c sin(2\omega_0 t)$

$x(t) = -x(t-3) \qquad \omega_0 = \pi/3$

$2\,\mathrm{a}\cos(\pi/3t - \pi) + 2b cos(2\pi/3t - 2\pi) - 2c sin(2\pi/3t - 2\pi)$

Now, if we need$x(t) = -x(t-3)$ then b and c must be zero therefore,

$a_2 = a_{-2} = 0.$

Now, using Parseval's relation on Clue 5 we get:

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = |a_1|^2 + |a_{-1}|^2 = 1/2$$

Therefore, $|a_1| = \frac{1}{2}$. Since a_1 is positive, we have $a_1 = a_{-1} = 1/2$.Therefore,

$$x(t) = {}^1\!/_2 \cos(\frac{\pi}{3} t)$$

9)

$$f(x) = \frac{1 - \ k\cos x}{1 - 2k\cos x + k^2}$$

As we know:

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$f(x) = \frac{k.\frac{e^{ix}+e^{-ix}}{2}}{1-2k.\frac{e^{ix}+e^{-ix}}{2}+k^2} = \frac{1}{2}.\frac{k\left(e^{ix}+e^{-ix}\right)}{1-k(e^{ix}+e^{-ix})+k^2}$$
$$= \frac{1}{2}.\frac{k\left(e^{ix}+e^{-ix}\right)}{1-ke^{ix}-ke^{-ix}+k^2e^{ix}e^{-ix}} = \frac{1}{2}.\frac{k\left(e^{ix}+e^{-ix}\right)}{(1-ke^{ix})(1-ke^{-ix})}$$
$$= \frac{1}{2}\left(\frac{1}{(1-ke^{ix})}+\frac{1}{(1-ke^{-ix})}\right)$$

$$| ke^{ix} |= |k| \ | e^{ix} |= |k|\sqrt{cosx^2 + sinx^2} = |k| < 1.$$

And in the same way:

$$| ke^{-ix} |= |k| < 1.$$

$$\frac{1}{1-ke^{ix}} = (1-ke^{ix})^{-1} = \sum_{n=0}^{\infty} k^n \, e^{inx}$$

$$\frac{1}{1-ke^{-ix}} = (1-ke^{-ix})^{-1} = \sum_{n=0}^{\infty} k^n \, e^{-inx}$$

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty} k^n (\, e^{inx} + e^{-inx}) = \sum_{n=0}^{\infty} k^n \cos nx = \sum_{n=1}^{\infty} k^n \cos nx$$

10)

a) $\ln(\sin(x)) = \ln\left(\frac{e^{ix}-e^{-ix}}{2i}\right)$

$$(*)\quad \ln\left(\frac{e^{ix}-e^{-ix}}{2i}\right) = \ln(\frac{e^{ix}}{2i}\left(1-e^{-2ix}\right) = ix - \ln(2) - \frac{i\pi}{2} + \ln\left(1-e^{-2ix}\right)$$

$$(**)\quad \ln\left(\frac{e^{ix}-e^{-ix}}{2i}\right) = \ln(\frac{e^{-ix}}{2i}\left(1-e^{2ix}\right) = -ix - \ln(2) + \frac{i\pi}{2} + \ln\left(1-e^{2ix}\right)$$

$$\ln(\sin(x)) = \frac{(*)+(**)}{2} = -\ln(2) + \frac{1}{2}\left(-\sum_{n=1}^{\infty}\frac{e^{-2inx}}{n} - \sum_{n=1}^{\infty}\frac{e^{2inx}}{n}\right)$$

$$= -\ln(2) - \sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}$$

b)The solution is like part a

$$\ln(\cos(x)) = \ln\left(\frac{e^{ix}+e^{-ix}}{2}\right)$$

$$(*)\quad \ln\left(\frac{e^{ix}+e^{-ix}}{2}\right) = ix - \ln(2) + \ln\left(1+e^{-2ix}\right)$$

$$(**)\quad \ln\left(\frac{e^{ix}+e^{-ix}}{2}\right) = ix - \ln(2) + \ln\left(1+e^{2ix}\right)$$

$$\ln(\cos(x)) = \frac{(*)+(**)}{2} = -\ln(2) + \sum_{k=1}^{\infty}\frac{(-1)^{n-1}\cos(2kx)}{k}$$

11)

Here the length of interval is 6

(i.e) 2l=6 → l=3

Fourier series is: $f(x) = \sum_{k=0}^{n} a_n \cos\left(\frac{n\pi x}{3}\right) + b_n sin\left(\frac{n\pi x}{3}\right)$

$$f(x) = \frac{a_0}{2} + a_1\cos(\frac{\pi x}{3}) + a_2\cos(\frac{2\pi x}{3}) + a_3\cos(\frac{3\pi x}{3}) + b_1\sin(\frac{\pi x}{3}) + b_2\sin(\frac{2\pi x}{3}) + b_3\sin(\frac{3\pi x}{3})$$

$$f(x) = \frac{a_0}{2} + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta$$

where $\theta = \frac{\pi x}{3}$

x	y	θ=πx/3	ycosθ	ycos2θ	ycos3θ	ysinθ	ysin2θ	ysin3θ
0	9	0	9	9	9	0	0	0
1	18	π/3	9	–9	–18	15.588	15.588	0
2	24	2π/3	–12	–12	24	20.784	–20.784	0
3	28	π	–28	28	–28	0	0	0
4	26	4π/3	–13	–13	26	–22.516	22.516	0
5	20	5π/3	10	–10	–20	–17.32	–17.32	0
Total	**125**		**–25**	**–7**	**–7**	**–3.464**	**0**	**0**

Here n=6

$$a_0 = 2[mean\ value\ of\ y] = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{125}{6}\right] = 41.667$$

$$a_1 = 2[mean\ value\ of\ ycos\theta] = 2\left[\frac{\sum ycos\theta}{n}\right] = 2\left[\frac{-25}{6}\right] = -8.333$$

$$a_2 = 2[mean\ value\ of\ ycos2\theta] = 2\left[\frac{\sum ycos2\theta}{n}\right] = 2\left[\frac{-7}{6}\right] = -2.333$$

$$a_3 = 2[mean\ value\ of\ ycos3\theta] = 2\left[\frac{\sum ycos3\theta}{n}\right] = 2\left[\frac{-7}{6}\right] = -2.333$$

$$b_1 = 2[mean\ value\ of\ ysin\theta] = 2\left[\frac{\sum ysin\theta}{n}\right] = 2\left[\frac{-3.464}{6}\right] = -1.155$$

$$b_2 = 2[mean\ value\ of\ ysin2\theta] = 2\left[\frac{\sum ysin2\theta}{n}\right] = 2\left[\frac{0}{6}\right] = -2.333$$

$$b_3 = 2[mean\ value\ of\ ysin3\theta] = 2\left[\frac{\sum ysin3\theta}{n}\right] = 2\left[\frac{0}{6}\right] = -2.333$$

$$f(x) = 20.833 - 8.333\cos\theta - 2.333\cos 2\theta - 2.333\cos 3\theta - 1.155\sin\theta$$

where $\theta = \frac{\pi x}{3}$

HARMONIC ANALYSIS

The process of finding the Fourier series for a function given by numerical value is known as harmonic analysis. In harmonic analysis the Fourier coefficients a_0, a_n and b_n of the function $y = f(x)$ in $(0, 2\pi)$ are given by

$a_0 = 2$ [mean value of y in $(0, 2\pi)$]
$a_n = 2$ [mean value of y cosnx in $(0, 2\pi)$]
$b_n = 2$ [mean value of y sinnx in $(0, 2\pi)$]