1)

a)

$$E[XY] = \int_0^1 \int_x^1 10x^2y^3\,dy\ \,dx = \int_0^1 \frac{5}{2}x^2(1-x^4)dx = \frac{5}{2}\left(\frac{1}{3}x^3 - \frac{1}{7}x^7\right)|_0^1 = \frac{10}{21}$$

$$E[X] = \int_0^1 \int_x^1 10x^2y^2\,dy\ \,dx = \int_0^1 \frac{10}{3}x^2(1-x^3)dx = \frac{10}{3}\left(\frac{1}{3}x^3 - \frac{1}{6}x^6\right)|_0^1 = \frac{5}{9}$$

$$E[Y] = \int_0^1 \int_x^1 10xy^3\,dy\ \,dx = \int_0^1 \frac{5}{2}x(1-x^4)dx = \frac{5}{2}\left(\frac{1}{2}x^2 - \frac{1}{6}x^6\right)|_0^1 = \frac{5}{6}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{10}{21} - \frac{5}{9} \times \frac{5}{6} = \frac{5}{378}$$

b)

$$f_X(x) = \int_x^1 10xy^2\,dy = \frac{10}{3}x(1-x^3)$$

$$f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \frac{3y^2}{1-x^3}\ :\ \ x < y < 1$$

c)

حل دستگاه $\begin{cases} z = y/x \\ w = x \end{cases}$ نتیجه می‌دهد:

$$y_1 = wz, \quad x_1 = w$$

$$0 \le x < y \le 1 \to \ \ w \ge 0\ , \qquad z > 1$$

$$J(x,y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} -y/x^2 & 1/x \\ 1 & 0 \end{vmatrix} = -\frac{1}{x} = -\frac{1}{w}$$

$$\Rightarrow f_{ZW}(z,w) = |w|\ f_{XY}(w,wz) = w(10w(wz)^2):\ \ 0 \le w < wz \le 1$$

$$z > 1\ ,\ 0 \le w < wz \le 1 \to \ \ 0 \le w \le 1/z$$

$$\Rightarrow f_z(z) = \int_{-\infty}^{+\infty} f_{ZW}(z,w)dw = \int_0^{\frac{1}{z}} 10w^4z^2\,dw = 2z^2w^5\ |_0^{1/z} = \frac{2}{z^3}\ :\ z > 1$$

2)

a)

$$f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{+\infty} f_X(z-y) f_Y(y) dy$$

$$f_X(z-y) = c\ e^{-c(z-y)} :\ \ y < z$$

$$f_Z(z) = \int_{-\infty}^{z} c\ e^{-c(z-y)} f_Y(y)\ dy$$

با مشتق‌گیری از طرفین و استفاده از قضیه انتگرال لایبنیتز داریم:

$$\frac{d}{dt}\int_{f_1(t)}^{f_2(t)} g(x,t)dx = \frac{df_2}{dt} g(f_2(t),t) - \frac{df_1}{dt} g(f_1(t),t) + \int_{f_1(t)}^{f_2(t)} \frac{\partial g(x,t)}{\partial t} dx$$

$$f_Z'(z) = 1 \times c\ e^{-c(z-z)} f_Y(z) - 0 + \int_{-\infty}^{z} -c^2 e^{-c(z-y)} f_Y(y) dy = c f_Y(z) - c f_Z(z)$$

$$\rightarrow c^2(e^{-cz} - cz\ e^{-cz}) = c\ f_Y(z) - c^3 z\ e^{-cz}\ :\ \ z > 0$$

$$\rightarrow f_Y(z) = c\ e^{-cz} :\ \ z > 0$$

$$\rightarrow f_Y(y) = c\ e^{-cy}\ u(y)$$

b)

$$f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{+\infty} f_X(z-y) f_Y(y) dy = \int_{0}^{1} f_X(z-y) dy = -F_X(z-y)|_0^1$$

$$\rightarrow f_Z(z) = -F_X(z-1) + F_X(z)$$

3)

با توجه به تقارن مساله نسبت به X_1 و X_2:

$$E[X_1|\bar{X}] = E[X_2|\bar{X}]$$

$$E[X_1 + X_2|\bar{X}] = E[2\bar{X}|\bar{X}] = 2\bar{X}$$

$$\rightarrow E[X_1|\bar{X}] + E[X_2|\bar{X}] = 2\bar{X}$$

$$\rightarrow 2E[X_1|\bar{X}] = 2\bar{X} \rightarrow\ \ E[X_1|\bar{X}] = \bar{X} = E[X_2|\bar{X}]$$

$$\rightarrow E[w_1X_1 + w_2X_2|\bar{X}] = w_1E[X_1|\bar{X}] + w_2E[X_2|\bar{X}] = w_1\bar{X} + w_2\bar{X} = (w_1 + w_2)\bar{X} = \bar{X}$$

4)

$$P(Y_1 = 0) = P(X_1 = 0\,, X_2 = 0) = P(X_1 = 0)P(X_2 = 0) = (1-p)^2$$

$$P(Y_1 = 1) = 1 - P(Y_1 = 0) = 2p - p^2$$

$$\rightarrow Y_i \sim Ber(2p - p^2)$$

b)

$$E[Y] = E[Y_1] + E[Y_2] + E[Y_3] = 3E[Y_i] = 3 \times (2p - p^2) = 6p - 3p^2$$

$X =$ number of X_is that are equal to 1

$$X = Bin(3, p)$$

$$E[Y^2] = E[Y^2|X=0]P(X=0) + E[Y^2|X=1]P(X=1) + E[Y^2|X=2]P(X=2) + E[Y^2|X=3]P(X=3)$$

$$= (0+0+0)^2P(X=0) + (1+1+0)^2P(X=1) + (1+1+1)^2P(X=2) + (1+1+1)^2P(X=3)$$

$$= 0 \times (1-p)^3 + 4 \times 3p(1-p)^2 + 9 \times 3p^2(1-p) + 9 \times p^3 = 12p + 3p^2 - 6p^3$$

$$Var(Y) = 12p + 3p^2 - 6p^3 - (6p - 3p^2)^2 = 12p - 33p^2 + 30p^3 - 9p^4$$

5)

a)

$$f(x_i|\alpha) = \frac{1}{\alpha^3}\sqrt{\frac{2}{\pi}}\, x_i^2 e^{-\frac{(x_i)^2}{2\alpha^2}}$$

$$LL(\alpha) = \sum_{i=1}^{n} \ln\left(\frac{1}{\alpha^3}\sqrt{\frac{2}{\pi}}\, x_i^2 e^{-\frac{(x_i)^2}{2\alpha^2}}\right) = \sum_{i=1}^{n}(-3\ln(\alpha) + \ln\left(\sqrt{\frac{2}{\pi}}\, x_i^2\right) - \frac{(x_i)^2}{2\alpha^2})$$

$$\frac{\partial LL}{\partial \alpha} = -\frac{3n}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{n} x_i^2 = 0 \Rightarrow \hat{\alpha}_{ML} = \sqrt{\frac{1}{3n}\sum_{i=1}^{n} x_i^2}$$

b)

$$\bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} < \mu < \bar{X} + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}$$

ابتدا باید $z_{1-\frac{\alpha}{2}}$ را محاسبه کنیم:

$$1 - \alpha = 0.96 \quad \rightarrow \quad \alpha = 0.04$$

$$\rightarrow 1-\frac{\alpha}{2}=0.98 \rightarrow \quad z_{1-\frac{\alpha}{2}}=2.05$$

بازه اطمینان برابر است با:

$$\left(175-2.05\times\frac{16}{\sqrt{64}}\ ,175+2.05\times\frac{16}{\sqrt{64}}\right)$$

$$(171.9\ ,179.1)$$