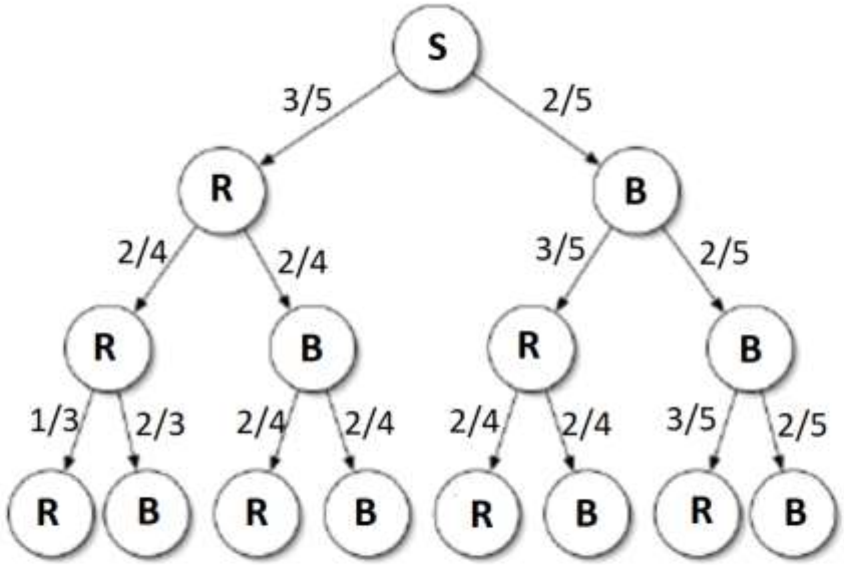
1.

a)



$$P(R_3) = \frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} + \frac{2}{4} \times \frac{2}{4} \times \frac{3}{5} + \frac{2}{4} \times \frac{3}{5} \times \frac{2}{5} + \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{233}{500}$$

b)

$$P(R_1|R_2) = ?$$

$$P(R_2) = \frac{2}{4} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{5} = \frac{27}{50}$$

$$P(R_2 \cap R_1) = \frac{2}{4} \times \frac{3}{5} = \frac{3}{10}$$

$$\rightarrow P(R_1|R_2) = \frac{3/10}{27/50} = \frac{5}{9}$$

2.

a)

x_i = number of books for student i

$$x_1 + x_2 + x_3 + x_4 = 10$$

Number of non-negative solutions:

$$\binom{10+4-1}{4-1} = \binom{13}{3} = 286$$

Conditions: $x_1 = 2,\ x_4 = 0, 1, 2, 3$

$x_4 = 0 \rightarrow\ x_2 + x_3 = 8 \rightarrow\ \binom{8+2-1}{2-1} = 9$

$$x_4 = 1 \rightarrow \quad x_2 + x_3 = 7 \rightarrow \quad \binom{7+2-1}{2-1} = 8$$

$$x_4 = 2 \rightarrow \quad x_2 + x_3 = 6 \rightarrow \quad \binom{6+2-1}{2-1} = 7$$

$$x_4 = 3 \rightarrow \quad x_2 + x_3 = 5 \rightarrow \quad \binom{5+2-1}{2-1} = 6$$

$$p = \frac{9+8+7+6}{286} = \frac{15}{143}$$

b)

number of ways to assign 10 books to 4 students: 4^{10}

choose two of the books for the first student: $\binom{10}{2}$

choose k of the remaining books for the fourth student: $\binom{8}{k}$

number of ways to assign $8-k$ books to 2 students: 2^{8-k}

$$p = \frac{\sum_{k=0}^{3} \binom{10}{2}\binom{8}{k} 2^{8-k}}{4^{10}} = \frac{855}{4096}$$

3.

$$X = \min\{X_1, X_2, X_3\}$$

a)

$$P(X > k) = P(X_1 > k \ \cap \ X_2 > k \ \cap X_3 > k) = [P(X_1 > k)]^3 = \left(\frac{6-k}{6}\right)^3 \quad , \qquad 0 \le k < 6$$

Since X is a positive random variable:

$$E[X] = \sum\big(1 - F_X(k)\big) = \sum P(X > k) = \frac{6^3 + 5^3 + \cdots + 1^3}{6^3} = \frac{\left(\frac{6(6+1)}{2}\right)^2}{6^3} = \frac{49}{24}$$

b)

$$T = X_1 + X_2 + X_3 \rightarrow \quad E[T] = E[X_1] + E[X_2] + E[X_3] = 3 \times \frac{7}{2} = \frac{21}{2}$$

$$S = T - X \rightarrow \quad E[S] = E[T] - E[X] = \frac{21}{2} - \frac{49}{24} = \frac{203}{24}$$

4.

$$F_Y(y) = P\{X^2 u(X) \le y\}$$

$$X^2u(X) \geq 0 \rightarrow \ \ if\ y < 0\ : F_Y(y) = 0$$

$$y = 0 \rightarrow \ \ F_Y(0) = P\{X^2u(X) \leq 0\} = P\{u(X) = 0\} = P\{X \leq 0\} = \int_{-\infty}^{0} \frac{1}{2b} e^{-\frac{|x|}{b}} dx$$

$$F_Y(0) = \frac{1}{2b}\int_{-\infty}^{0} e^{\frac{x}{b}} dx = \frac{b}{2b} e^{\frac{x}{b}} \ \ |_{-\infty}^{0} = \frac{1}{2}$$

$$y > 0 \rightarrow \ \ F_Y(y) = P\{X^2u(X) < y\} = \int_{-\infty}^{\sqrt{y}} \frac{1}{2b} e^{-\frac{|x|}{b}} dx = \frac{1}{2} + \int_{0}^{\sqrt{y}} \frac{1}{2b} e^{-\frac{x}{b}} dx = 1 - \frac{1}{2} e^{-\frac{\sqrt{y}}{b}}$$

$$f_Y(y) = \frac{dF}{dy} = \frac{1}{2}\delta(y) + \frac{1}{4b\sqrt{y}} e^{-\frac{\sqrt{y}}{b}} u(y)$$

b)

$$P(Y \leq b|X|) = P(X^2u(X) \leq b|X|)$$

$$X < \ 0 \rightarrow \ \ X^2u(X) = 0 \leq b|X|$$

$$X > 0 \rightarrow \ \ u(X) = 1\,, |X| = X \rightarrow \ \ X^2 < bX \rightarrow \ \ 0 < X < b$$

$$\rightarrow P(Y \leq b|X|) = \int_{-\infty}^{b} \frac{1}{2b} e^{-\frac{|x|}{b}} dx = \int_{-\infty}^{0} \frac{1}{2b} e^{-\frac{|x|}{b}} dx + \int_{0}^{b} \frac{1}{2b} e^{-\frac{|x|}{b}} dx = 1 - \frac{1}{2e}$$

5.

a)

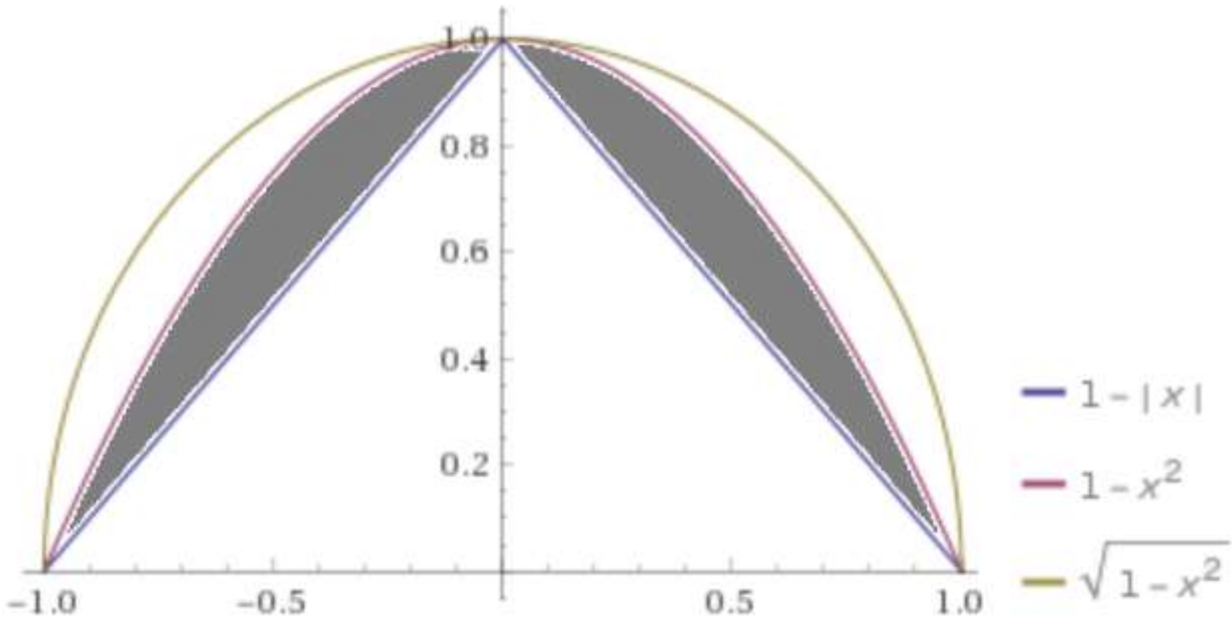
$$f_{XY}(x, y) = C \ \ , -1 \leq x \leq 1 \ \ , 0 \leq y \leq 1 - x^2$$

$$\int_{-1}^{1}\int_{0}^{1-x^2} C \ \ dy \ \ dx = 1 \rightarrow \ \ \int_{-1}^{1} C(1 - x^2) dx = C\left(x - \frac{1}{3}x^3\right) |_{-1}^{1} = \frac{4}{3}C = 1 \rightarrow \ \ C = \frac{3}{4}$$

$$f_X(x) = \int_{0}^{1-x^2} \frac{3}{4} \ dy = \frac{3}{4}(1 - x^2) : \ -1 < x < 1$$

$$f_Y(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{4} \ dx = \frac{3}{2}\sqrt{1 - y} : \ 0 < y < 1$$

b)



We have to find the area of the gray region:

$$\int_{-1}^{1}(1-x^2)dx=\frac{4}{3}$$

The area of the triangle:

$$1\times\frac{1-(-1)}{2}=1$$

$$S=\frac{4}{3}-1=\frac{1}{3}$$

$$p=S\times C=\frac{1}{3}\times\frac{3}{4}=\frac{1}{4}$$