۱. الف)

$$H_0:\ \mu = 16.5$$
$$H_A:\ \mu \neq 16.5$$
$$\text{p-value} = P(\overline{X} < 16 \ \text{ or } \overline{X} > 17\ |\mu = 16.5) = 2P\left(Z > \frac{17-16.5}{\frac{1.5}{\sqrt{36}}}\right)$$
$$= 2P(Z > 2) = 2(1 - 0.97725) = 0.0455 < 0.05$$

بنابراین فرض H_0 را رد می‌کنیم.

ب)

$$\alpha = 1 - 0.96 = 0.04 \rightarrow 1 - \frac{\alpha}{2} = 0.98 \rightarrow z_{1-\frac{\alpha}{2}} = 2.05$$

بازه اطمینان برابر است با:

$$\left(16 - 2.05 \times \frac{1.5}{\sqrt{36}}\ ,\ 16 - 2.05 \times \frac{1.5}{\sqrt{36}}\right) = (15.4875,\ 16.5125)$$

۲.

الف)

$$L(\alpha) = \prod_{i=1}^{n} f\,(x_i|\alpha) = \prod_{i=1}^{n} \alpha(1+x_i)^{-\alpha-1} = \alpha^n \prod_{i=1}^{n}(1+x_i)^{-\alpha-1}$$

$$LL(\alpha) = n.\ln(\alpha) - (\alpha+1)\sum_{i=1}^{n} ln(1+x_i)$$

$$\frac{\partial LL}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} ln(1+x_i) = 0 \ \Rightarrow\ \hat{\alpha}_{ML} = \frac{n}{\sum_{i=1}^{n} ln(1+x_i)}$$

ب)

$$X_i \sim Pareto\left(\frac{9}{4}\right) \rightarrow \ \ E[X_i] = \frac{1}{\frac{9}{4}-1} = \frac{4}{5}\ , \qquad Var(X_i) = \frac{9/4}{\left(\frac{5}{4}\right)^2\left(\frac{1}{4}\right)} = 144/25$$

طبق قضیه حد مرکزی:

$$Y = X_1 + X_2 + \cdots + X_{100} \sim N\left(100 \times \frac{4}{5}, \qquad 100 \times \frac{144}{25}\right) = N(80\,, 24^2)$$

$$P(Y > 104) = P\left(Z > \frac{104-80}{24}\right) = P(Z > 1) = 1 - 0.841 = 0.159$$

.۳

الف)

$$Var(X|Y) = E[X^2|Y] - (E[X|Y])^2$$

$$\rightarrow \ E[Var(X|Y)] = E\big[E[X^2|Y]\big] - E[(E[X|Y])^2] = E[X^2] - E[(E[X|Y])^2]$$

$$Var(E[X|Y]) = E[(E[X|Y])^2] - \big(E\big[E[X|Y]\big]\big)^2 = E[(E[X|Y])^2] - (E[X])^2$$

$$\rightarrow E[Var(X|Y)] + Var(E[X|Y]) = E[X^2] - (E[X])^2 = Var(X)$$

ب)

با استفاده از فرمول واریانس مجموع‌های تصادفی داریم:

$$Var(S) = E[N]Var(X_i) + Var(N)(E[X])^2 = 2 \times 9 + 0 \times 1 = 18$$

پ)

$$E[S|N=n] = E[X_1 + X_2 + \cdots + X_n] = n.E[X_i] = 0$$

$$\rightarrow E[S|N] = 0 \rightarrow \ \ E[S] = E\big[E[S|N]\big] = 0$$

$$E[SN] = E\big[E[SN|N]\big] = E[N.E[S|N] = E[N \times 0] = 0$$

$$\rightarrow Cov(S,N) = 0 - 0 = 0 \rightarrow \ \ \rho_{SN} = 0$$

.۴

الف)

$$f_X(x) = \int_x^1 3(y-x)dy + \int_0^x 3(x-y)\,dy = 3\left(x^2 - x + \frac{1}{2}\right): \ \ 0 < x < 1$$

$$E[X] = \int_0^1 x \times 3\left(x^2 - x + \frac{1}{2}\right)dx \ = \ \frac{1}{2} \ = \ E[Y] \text{ (because of symmetry)}$$

$$E[X^2] = \int_0^1 x^2 \times 3\left(x^2 - x + \frac{1}{2}\right)dx \ = \frac{7}{20} \ = \ E[Y^2] \text{ (because of symmetry)}$$

$$Var(X) = \frac{7}{20} - \left(\frac{1}{2}\right)^2 = \frac{7}{20} - \frac{5}{20} = \frac{1}{10} = Var(Y)$$

$$E[XY] = \int_0^1 \left(\int_x^1 3xy(y-x)dy + \int_0^x 3xy(x-y)\,dy\right)dx = \frac{1}{5}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{5} - \left(\frac{1}{2}\right)^2 = -\frac{1}{20}$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-1/20}{1/10} = -\frac{1}{2}$$

ب)

$$E[\min\{X,Y\}] = \int_0^1 \left(\int_x^1 3\min\{x,y\}\,(y-x)dy + \int_0^x 3\min\{x,y\}\,(x-y)\,dy \right) dx$$

$$= \int_0^1 \left(\int_x^1 3x(y-x)dy + \int_0^x 3y(x-y)\,dy \right) dx = \frac{1}{4}$$

.۵

الف) ابتدا تابع چگالی احتمال مشترک X و Z را به دست می‌آوریم:

$Z = X + Y$

$W = X$

از حل دستگاه:

$x = w \;, \qquad y = z - w$

$$J(x,y) = \begin{vmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\Rightarrow f_{ZW}(z,w) = \frac{f_{XY}(w\,,z-w)}{|-1|} = \frac{z}{2}e^{-z} \;\; ,z > 0\,, z-w > 0 \to w < z$$

از آنجایی که W همان X است:

$$f_{ZX}(z,x) = \frac{1}{2}z\,e^{-z} \;:\;\; 0 < z\,, \qquad 0 < x < z$$

$$f_Z(z) = \int f_{XZ}(x,z)dx = \int_0^z \frac{1}{2}z\,e^{-z}dx = \frac{1}{2}z^2e^{-z}$$

$$f_X(x|z) = \frac{f_{XZ}(x,z)}{f_Z(z)} = \frac{1}{z} :\;\; 0 < x < z$$

$$E[X^2|Z=z] = \int_0^z \frac{1}{z}x^2dx = \frac{1}{3}z^2 \to \;\; E[X^2|Z] = \frac{1}{3}Z^2$$

ب) با توجه به تقارن:

$$E[X^2|Z] = E[Y^2|Z]$$

$$E[Z^2|Z] = E[(X+Y)^2|Z] = E[X^2|Z] + 2E[XY|Z] + E[Y^2|Z] = 2E[X^2|Z] + 2E[XY|Z]$$

$$E[Z^2|Z] = Z^2 \;\; \to E[XY|Z] = \frac{1}{2}Z^2 - E[X^2|Z] = \frac{1}{2}Z^2 - \frac{1}{3}Z^2 = \frac{1}{6}Z^2$$