1.

a)

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_0^x f_{XY}(x,y)dy = (xy + 2y^2)|_0^x = 3x^2 : \; 0 < x < 1$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{x + 4y}{3x^2} \; : \; 0 < y < x$$

$$f_{Y|X}(y|x) = \frac{x + 4y}{3x^2} \; : \; 0 < y < x$$

$$E[Y|X = x] = \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) \, dy =$$

$$\int_0^x \frac{y(x + 4y)}{3x^2} \, dy = \frac{1}{3x^2}(\frac{1}{2}xy^2 + \frac{4}{3}y^3)|_0^x = \frac{11}{18}x$$

$$E\left[Y\middle|X = \frac{9}{11}\right] = \frac{11}{18} \times \frac{9}{11} = \frac{1}{2}$$

b)

استفاده از متغیر کمکی $W = Y$

حل دستگاه $\begin{cases} z = x/y \\ w = y \end{cases}$ نتیجه می‌دهد:

$$x_1 = wz, \quad y_1 = w$$

$$J(x,y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1/y & -x/y^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{y} = \frac{1}{w}$$

$$\Rightarrow \; f_{ZW}(z,w) = |w| \, f_{XY}\left(wz \, , w\right) = w(wz + 4w)$$

$$z > 1 \, , \; 0 < w < wz < 1 \rightarrow \; 0 < w < 1/z$$

$$\Rightarrow \; f_{\boldsymbol{z}}(z) = \int_{-\infty}^{+\infty} f_{ZW}(z,w)dw = \int_0^{\frac{1}{z}} (z + 4)w^2 \, dw = \frac{1}{3}(z + 4)w^3$$

$$= \frac{1}{3} \times \frac{(z + 4)}{z^3} = \frac{1}{3z^2} + \frac{4}{3z^3} \; : \; z > 1$$

2.

a)

$$\phi_Z(s) = E[e^{sZ}] = E[e^{sZ}|Y=0]P(Y=0) + E[e^{sZ}|Y \neq 0]P(Y \neq 0)$$

$$P(Y=0) = e^{-\lambda} \;\; \rightarrow \;\; P(Y \neq 0) = 1 - e^{-\lambda}$$

$$E[e^{sZ}|Y=0] = E[e^{0}|Y=0] = 1$$

$$E[e^{sZ}|Y \neq 0] = \phi_{X_1+\cdots+X_n}(s) = \left[\phi_{X_i}(s)\right]^n = \left(\frac{1}{1-s}\right)^n$$

$$\phi_Z(s) = e^{-\lambda} + \left(1 - e^{-\lambda}\right)(1-s)^{-n}$$

b)

$$\phi_Z'(s) = n\left(1 - e^{-\lambda}\right)(1-s)^{-n-1}$$

$$E[Z] = \phi_Z'(0) = n(1 - e^{-\lambda})$$

$$\phi_Z''(s) = n(n+1)\left(1 - e^{-\lambda}\right)(1-s)^{-n-2}$$

$$E[Z^2] = \phi_Z''(0) = n(n+1)(1 - e^{-\lambda})$$

$$Var(Z) = E[Z^2] - (E[Z])^2$$

3.

a)

$$P(X=i|X+Y=n) = \frac{P(X=i,\, X+Y=n)}{P(X+Y=n)} = \frac{P(X=i,\, Y=n-i)}{P(X+Y=n)}$$

$$= \frac{P(X=i)\, P(Y=n-i)}{P(X+Y=n)}$$

$$P(X+Y=n) = \sum_{i=1}^{n-1} P(X=i, Y=n-i) = \sum_{i=1}^{n-1} P(X=i)P(Y=n-i)$$

$$= \sum_{i=1}^{n-1} (1-p)^{i-1}p \times (1-p)^{n-i-1}p = \sum_{i=1}^{n-1} p^2(1-p)^{n-1}$$

$$= (n-1)p^2(1-p)^{n-1}$$

$$P(X=i|X+Y=n) = \frac{P(X=i)\, P(Y=n-i)}{P(X+Y=n)} = \frac{p^2(1-p)^{n-1}}{(n-1)p^2(1-p)^{n-1}}$$

$$= \frac{1}{n-1}$$

b)

$$L(\theta_1,\ \theta_2) = \prod_{i=1}^{n} \frac{1}{\theta_1} e^{\frac{\theta_2 - x_i}{\theta_1}} = \frac{1}{\theta_1^n}\ e^{\frac{n\theta_2}{\theta_1}} \times e^{-\frac{x_1+\cdots+x_n}{\theta_1}}$$

$$LL(\theta_1,\ \theta_2) = -n\ln(\theta_1) + \frac{n\theta_2}{\theta_1} - \frac{x_1 + \cdots + x_n}{\theta_1}$$

$$\frac{\partial LL}{\partial \theta_1} = -\frac{n}{\theta_1} - \frac{n\theta_2}{\theta_1^2} + \frac{x_1 + \cdots + x_n}{\theta_1^2} = 0 \ \ \rightarrow \ \ \theta_1 +\ \theta_2 = \frac{x_1 + \cdots + x_n}{n} = \bar{x}$$

$$\frac{\partial LL}{\partial \theta_2} = \frac{n}{\theta_1}$$

همانطور که مشاهده می‌شود مشتق LL نسبت به θ_2 همواره مثبت است و برابر با صفر نمی‌شود، پس باید θ_2 را تا آنجا که ممکن است افزایش داد.

از آنجا که $\theta_2 \leq x_i$ برای هر مقدار i، داریم:

$$\theta_{2ML} = \min\{x_1,\ x_2,\ \ldots,\ x_n\}$$

$$\theta_{1ML} = \bar{x} - \theta_{2ML}$$

4.

$$Y_n = X_{max} = \max(X_1,\ X_2,\ \ldots,\ X_n)$$

بنابراین:

$$F_{Y_n}(y) = P\{Y_n \leq y\} = P\{X_{max} \leq y\}$$

$$= P\{X_1 \leq y\ ,\ X_2 \leq y\ ,\ \ldots,\ X_n \leq y\}$$

$$= P\{X_1 \leq y\}P\{X_2 \leq y\} \ldots P\{X_n \leq y\}$$

$$= [F_X(y)]^n$$

$$\Rightarrow f_{Y_n}(y) = n[F_X(y)]^{n-1} f_X(y)$$

ابتدا توزیع $M|Y = y$ را به دست می‌آوریم:

$$f_{M|Y}(m|Y = y) = y[F_U(m)]^{y-1} f_U(m) = y.m^{y-1}\ :\ 0 < m < 1$$

$$f_M(m) = \sum_{y=1}^{\infty} f(m|Y = y)P(Y = y)$$

$$f_M(m) = \sum_{y=1}^{\infty} y\, m^{y-1} \times \frac{1}{(e-1)y!} = \frac{1}{e-1}\sum_{y=1}^{\infty} \frac{m^{y-1}}{(y-1)!}$$

$$f_M(m) = \frac{e^m}{e-1} \ : \ 0 < m < 1 \ \rightarrow \ F_M(m) = \frac{e^m - 1}{e-1}$$

حال توزیع $Z|X = k$ را محاسبه می‌کنیم:

$$Z = k - M \ \rightarrow \ F_Z(z) = P(k - M \le z) = P(M \ge k - z) =$$

$$= 1 - P(M \le k - z)$$

$$= \frac{e - e^{k-z}}{e-1} \ : \ 0 < k - z < 1$$

توزیع $Z|X = k$ را محاسبه می‌کنیم:

$$Z = k - M \ \rightarrow \ f_Z(z) = \frac{e^{k-z}}{e-1} \ : \ k - 1 < z < k$$

$$f_Z(z) = \frac{e^{k-z}}{e-1} \ \Big(u\big(z - (k-1)\big) - u(z-k)\Big)$$

$$f_Z(z) = \sum_{k=1}^{\infty} f(Z|X = k)P(X = k)$$

$$= \sum_{k=1}^{\infty} \frac{e^{k-z}}{e-1} \times e^{-k}(e-1)\Big(u\big(z-(k-1)\big) - u(z-k)\Big) = e^{-z} : z > 0$$

5.

a)

$$\alpha = 1 - 0.9 = 0.1 \rightarrow 1 - \frac{\alpha}{2} = 0.95 \rightarrow z_{1-\frac{\alpha}{2}} = 1.65$$

بازه اطمینان برابر است با:

$$\left(7.6 - 1.65 \times \frac{1.4}{\sqrt{49}} \ , \ 7.6 - 1.65 \times \frac{1.4}{\sqrt{49}} \right)$$

$$= \big(7.6 - 0.33 \ , \ 7.6 + 0.33\big) = (7.27, \ 7.93)$$

b)

$$H_0: \ \mu = 8$$

$$H_A: \ \mu < 8$$

p-value $= P(\bar{X} < 7.6|\mu = 8) = P\left(Z < \frac{7.6-8}{\frac{1.4}{\sqrt{49}}}\right) = P(Z < -2)$

$$= 1 - P(Z < 2) = 1 - 0.977 = 0.023 > 0.02$$

بنابراین فرض H_0 را رد نمی‌کنیم.