به نام خدا

پاسخ تمرین سری اول

درس سیگنال‌ها و سیستم‌ها - دکتر اخوان

دانشگاه تهران – دانشکده مهندسی برق و کامپیوتر

۱)

$$(\boldsymbol{i})\quad z^* = \left(re^{j\theta}\right)^* = (r\cos\theta + jr\sin\theta)^* = (r\cos\theta - jr\sin\theta) = (r\cos(-\theta) + jr\sin(-\theta)) = \left(\boldsymbol{re^{-j\theta}}\right)$$

$$(\boldsymbol{ii})\quad z^2 = re^{j\theta} \times re^{j\theta} = \boldsymbol{r^2 e^{2j\theta}}$$

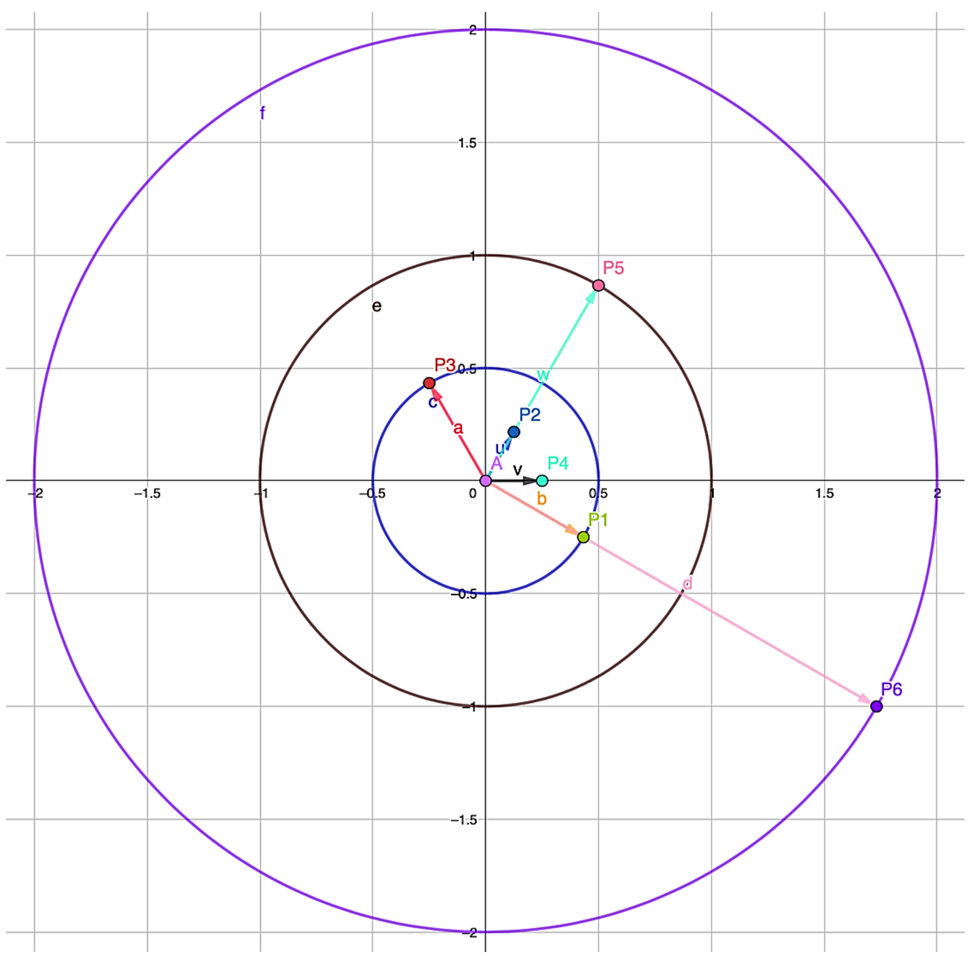
$$(\boldsymbol{iii})\quad jz = e^{j\frac{\pi}{2}} \times re^{j\theta} = \boldsymbol{re^{j(\theta+\frac{\pi}{2})}}$$

$$(\boldsymbol{iv})\quad zz^* = re^{j\theta} \times re^{-j\theta} = \boldsymbol{r^2}$$

$$(\boldsymbol{v})\quad \frac{z}{z^*} = \frac{re^{j\theta}}{re^{-j\theta}} = \boldsymbol{e^{j2\theta}}$$

$$(\boldsymbol{vi})\quad \frac{1}{z} = \frac{1}{re^{j\theta}} = \boldsymbol{\frac{1}{r}e^{-j\theta}}$$

$$(\boldsymbol{vii})\quad \boldsymbol{r = \frac{1}{2}, \theta = \frac{\pi}{6}}$$



۲)

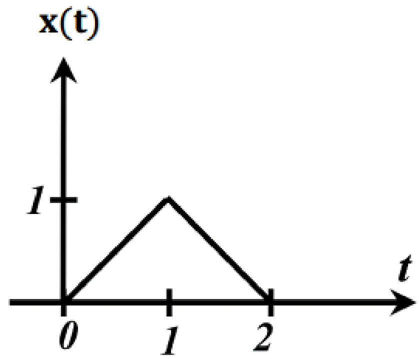
$$-2j\sin\left(\frac{\theta}{2}\right)e^{j\frac{\theta}{2}} = -2j\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right)+j\sin\left(\frac{\theta}{2}\right)\right) = -\,2j\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)+2\sin\left(\frac{\theta}{2}\right)^2 = -j\sin\theta+2\sin\left(\frac{\theta}{2}\right)^2$$

$$\rightarrow\ \cos\theta = (\cos\frac{\theta}{2})^2-(\sin\frac{\theta}{2})^2 = 1-2\,(\sin\frac{\theta}{2})^2 \Longrightarrow\ 2\,(\sin\frac{\theta}{2})^2 = 1-\cos\theta\quad (*)$$

جایگذاری رابطه‌ی $(*)$ در رابطه‌ی بدست آمده در ابتدا:

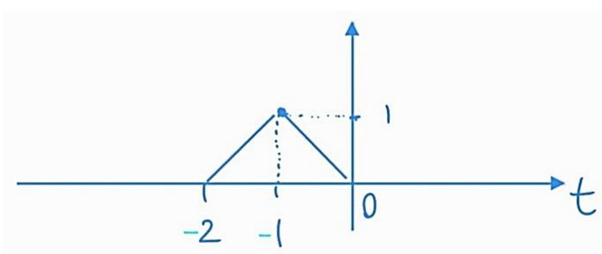
$$-j\sin\theta+2\sin\left(\frac{\theta}{2}\right)^2 = 1-j\sin\theta-\cos\theta = \mathbf{1-e^{j\theta}}$$

۳)



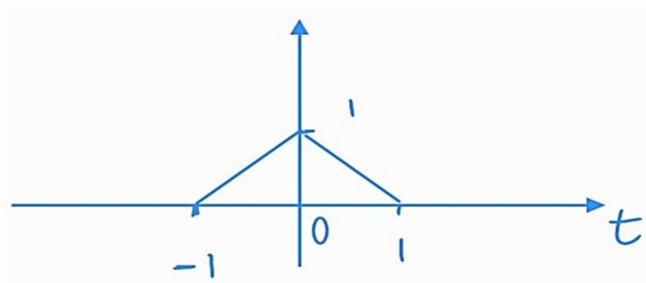
$(a)\ \ x(-t)$

قرینه نسبت به محور عمودی (y)



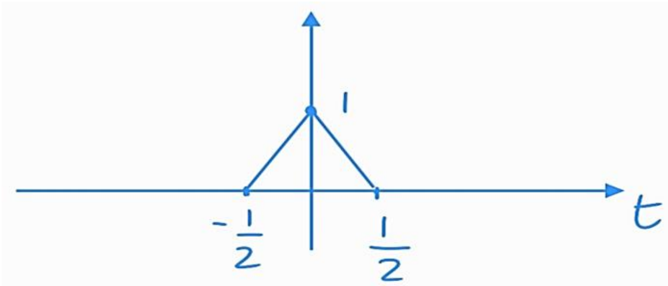
$(b)\ \ x(t+1)$

یک واحد شیفت به چپ

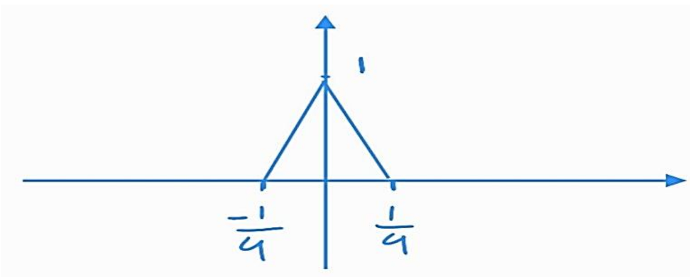


$(c)\ \ \boldsymbol{x(2t+1)}$

یک واحد شیفت به چپ سپس دامنه در $\frac{1}{2}$ ضرب میشود.



$(c)\ \ \boldsymbol{x(1-4t)}$



۴)

الف) $x(t)=\cos(\omega_x(t+\tau_x)+\theta_x) \rightarrow T=\frac{2\pi}{\omega_x}$

$(\boldsymbol{i})\ \ \omega_x=\frac{\pi}{2}, \tau_x=0, \theta_x=2\pi \rightarrow x(t)=\cos(\frac{\pi}{2}(t+0)+2\pi)=\cos\left(\frac{\pi}{2}t+2\pi\right)=\cos\left(\frac{\pi}{2}t\right)$

$$\boldsymbol{T}=\frac{2\pi}{\frac{\pi}{2}}=4\text{ sec},\qquad \boldsymbol{f}=\frac{1}{4}\ Hz$$

$(\boldsymbol{ii})\ \ \omega_x=\frac{3\pi}{2}, \tau_x=\frac{1}{2}, \theta_x=\frac{\pi}{7} \rightarrow x(t)=\cos(\frac{3\pi}{2}\left(t+\frac{1}{2}\right)+\frac{\pi}{7})=\cos\left(\frac{3\pi}{2}t+\frac{25\pi}{28}\right)$

$$\boldsymbol{T}=\frac{2\pi}{\frac{3\pi}{2}}=\frac{4}{3}\text{ sec},\qquad \boldsymbol{f}=\frac{3}{4}\ Hz$$

$(\boldsymbol{iii})\ \ \omega_x=\frac{3}{4}, \tau_x=\frac{1}{2}, \theta_x=\frac{1}{7} \rightarrow x(t)=\cos(\frac{3}{4}\left(t+\frac{1}{2}\right)+\frac{1}{7})=\cos\left(\frac{3}{4}t+\frac{29}{56}\right)$

$$\boldsymbol{T}=\frac{2\pi}{\frac{3}{4}}=\frac{8\pi}{3}\text{ sec},\quad \boldsymbol{f}=\frac{3}{8\pi}\ Hz$$

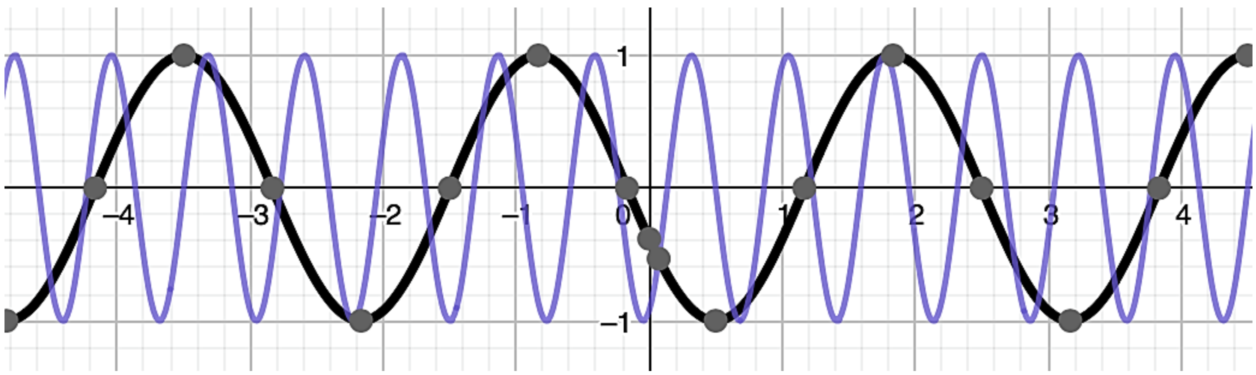
ب)

$$x(t) = \cos(\omega_x(t+\tau_x)+\theta_x)\,; \qquad y(t) = \cos\left(\omega_y\left(t+\tau_y\right)+\theta_y\right)$$

$(\boldsymbol{i})$ $\quad x(t) = \cos\left(\frac{\pi}{3}t+2\pi\right), \quad y(t) = \cos\left(\frac{\pi}{3}(t+1)-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}t\right)$ ✓

$(\boldsymbol{ii})$ $\quad x(t) = \cos\left(\frac{3\pi}{4}\left(t+\frac{1}{2}\right)+\frac{\pi}{4}\right), \quad y(t) = \cos\left(\frac{11\pi}{4}(t+1)+\frac{3\pi}{8}\right)$

از آنجا که فرکانس‌ها متفاوت است، آنها یکسان نیستند.



$(\boldsymbol{iii})$ $\quad x(t) = \cos\left(\frac{3}{4}\left(t+\frac{1}{2}\right)+\frac{1}{4}\right) = \cos(\frac{3}{4}t+\frac{5}{8}), \quad y(t) = \cos\left(\frac{3}{4}(t+1)+\frac{3}{8}\right) = \cos(\frac{3}{4}t+\frac{9}{8})$

$Set\ t=0 \;\Rightarrow\; x(0) = \cos\left(\frac{5}{8}\right) \approx 0.81\,, \quad y(0) = \cos\left(\frac{9}{8}\right) \approx 0.43 \to x(0) \neq y(0)$

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۵)

الف) $x[n] = cos(\omega_x(n+m_x)+\theta_x) \to T = \frac{2\pi}{\omega_x}\times m$

که در عبارت بالا m عددی صحیح است به گونه‌ای که: $\omega_0 = \frac{2\pi}{N}m$

$(\boldsymbol{i})$ $\quad \omega_x = \frac{\pi}{3}\,, m_x = 0\,, \theta_x = 2\pi \to x[n] = \cos(\frac{\pi}{3}(n+0)+2\pi) = \cos\left(\frac{\pi}{3}n+2\pi\right) = \cos\left(\frac{\pi}{3}n\right)$

$$\boldsymbol{N} = \frac{2\pi}{\frac{\pi}{3}} = 6\,, \qquad \boldsymbol{f} = \frac{1}{6}\,Hz$$

$(\boldsymbol{ii})$ $\quad \omega_x = \frac{3\pi}{4}\,, m_x = 2\,, \theta_x = \frac{\pi}{4} \to x[n] = \cos(\frac{3\pi}{4}(n+2)+\frac{\pi}{4}) = \cos\left(\frac{3\pi}{4}n+\frac{7\pi}{4}\right)$

$$\frac{2\pi}{\frac{3\pi}{4}} = \frac{8}{3} \Rightarrow m = 3 \Rightarrow \boldsymbol{N} = \frac{8}{3}\times 3 = 8\ \mathrm{sec}\,, \qquad \boldsymbol{f} = \frac{1}{8}\,Hz$$

$(\boldsymbol{iii})\quad \omega_x = \frac{3}{4}\,, m_x = 1\,, \theta_x = \frac{1}{4} \rightarrow x[n] = \cos(\frac{3}{4}(n+1) + \frac{1}{4}) = \cos\left(\frac{3}{4}n + 1\right)$

$$\frac{2\pi}{\frac{3}{4}} = \frac{8\pi}{3}$$

← در این بخش به دلیل آنکه $\frac{8\pi}{3}$ گویا نمی‌باشد، سیگنال مورد نظر متناوب نخواهد بود.

ب)

$$x[n] = \cos(\omega_x(n + m_x) + \theta_x)\,, \quad y[n] = \cos\left(\omega_y\left(n + m_y\right) + \theta_y\right)$$

$(\boldsymbol{i})\quad x[n] = \cos\left(\frac{\pi}{3}n + 2\pi\right), \quad y[n] = \cos\left(\frac{8\pi}{3}n\right) = \cos\left(\frac{2\pi}{3}n + 2\pi n\right)$

$$\Rightarrow\ therefore\ x[n] \neq y[n]$$

$(\boldsymbol{ii})\quad x[n] = \cos\left(\frac{3\pi}{4}(n+2) + \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}n + \frac{7\pi}{4}\right)$

$y[n] = \cos\left(\frac{3\pi}{4}(n+1) - \pi\right) = \cos\left(\frac{3\pi}{4}n - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}n - \frac{\pi}{4} + 2\pi\right) = \cos\left(\frac{3\pi}{4}n + \frac{7\pi}{4}\right)$

$$\Rightarrow x[n] = y[n]$$

$(\boldsymbol{iii})\quad x[n] = \cos\left(\frac{3}{4}(n+1) + \frac{1}{4}\right) = \cos\left(\frac{3}{4}n + 1\right), \quad y[n] = \cos\left(\frac{3}{4}n + 1\right)$

$$\Rightarrow x[n] = y[n]$$

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۶)

$$x(t) = \sqrt{2}(1+j)e^{j\frac{\pi}{4}}e^{(-1+j2\pi)t}$$

$\rightarrow e^{j\frac{\pi}{4}} = \cos\frac{\pi}{4} + j\ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+j)$

$\rightarrow e^{(-1+j2\pi)t} = e^{-t} \times e^{(j2\pi)t} = e^{-t} \times (\cos(2\pi t) + j\sin(2\pi t))$

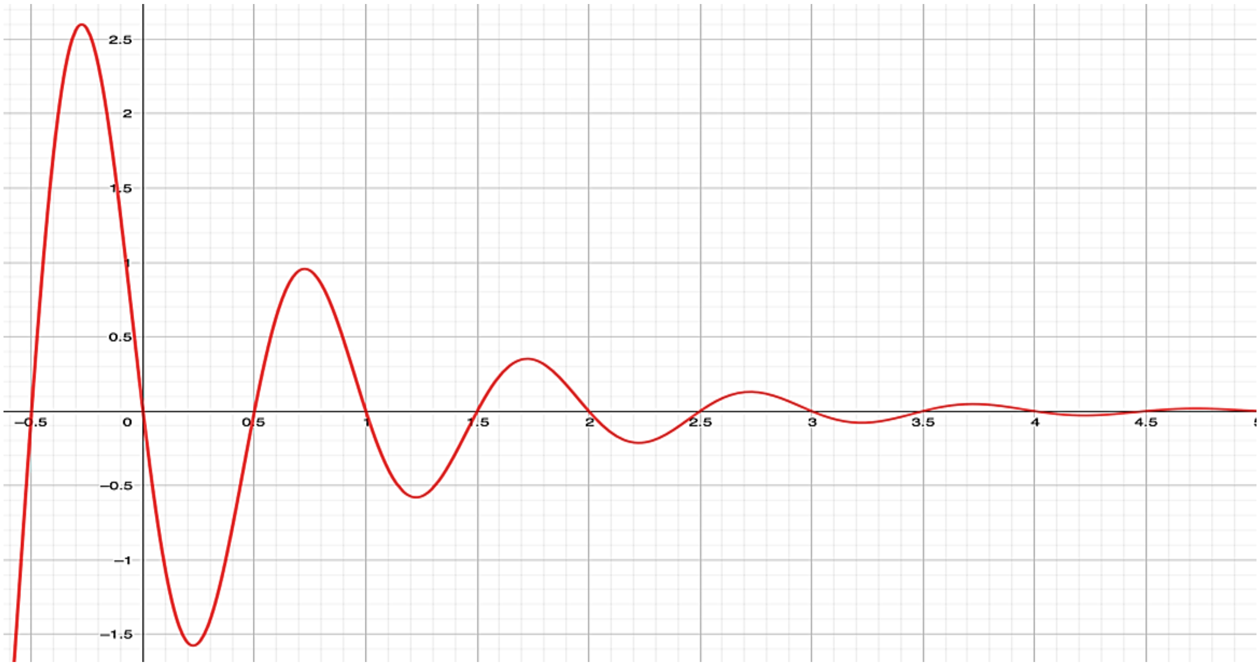
جایگذاری در $x(t)$:

$\Rightarrow x(t) = \sqrt{2}(1+j) \times \frac{1}{\sqrt{2}}(1+j) \times e^{-t} \times (\cos(2\pi t) + j\sin(2\pi t))$

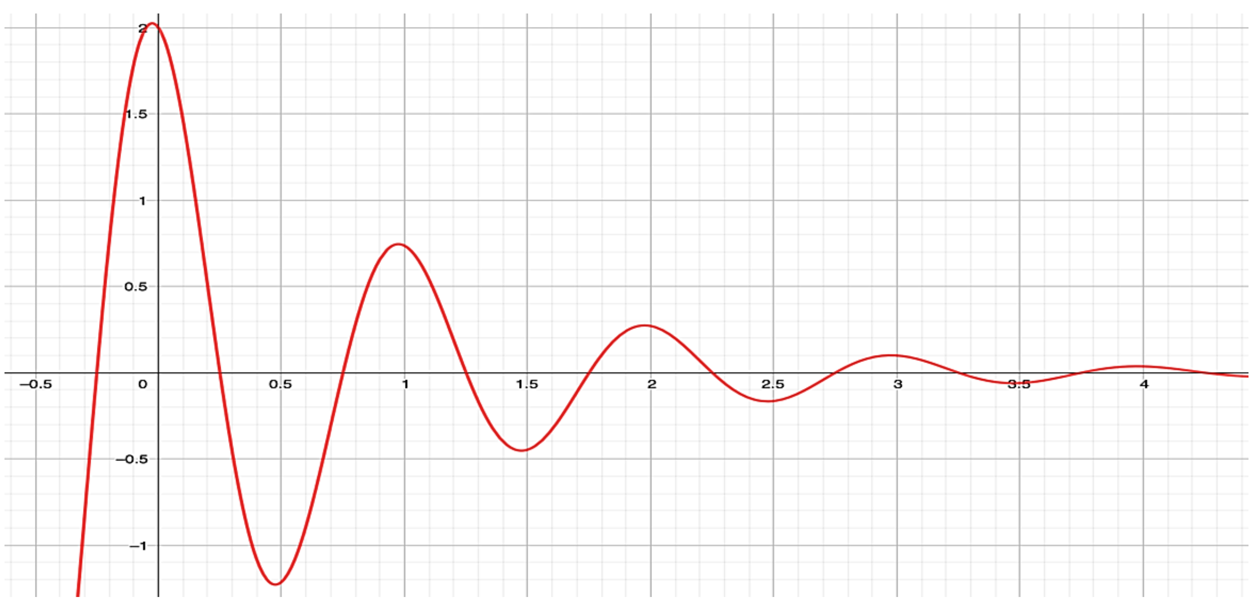
$= (1+j)^2 e^{-t} \times (\cos(2\pi t) + j\sin(2\pi t)) = 2je^{-t} \times (\cos(2\pi t) + j\sin(2\pi t))$

$= \boldsymbol{-2e^{-t}\sin(2\pi t) + 2je^{-t}\cos(2\pi t)}$

$(\boldsymbol{i})\quad \Re\{x(t)\} = -2e^{-t}\sin 2\pi t$

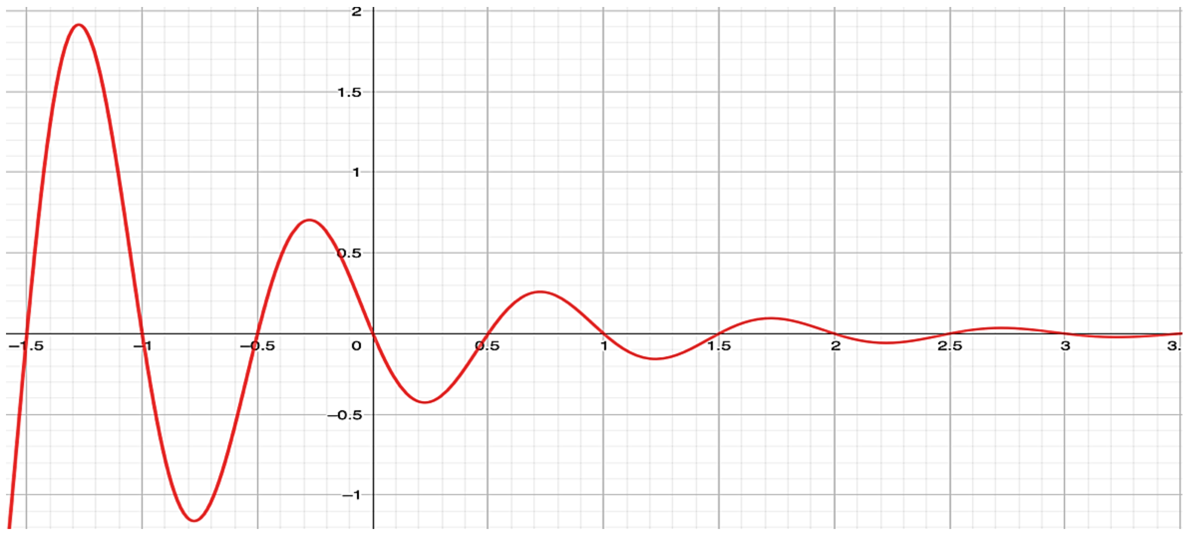


$(\boldsymbol{ii})\quad \Im\{x(t)\} = 2e^{-t}\cos 2\pi t$



$(\boldsymbol{iii})\quad x(t+2)+x^*(t+2)$

$$-2e^{-(t+2)}\sin 2\pi(t+2)+2je^{-(t+2)}\cos 2\pi(t+2)-2e^{-(t+2)}\sin 2\pi(t+2)-2je^{-(t+2)}\cos 2\pi(t+2)$$
$$=\boldsymbol{-4e^{-(t+2)}\sin 2\pi(t+2)}$$



۷)

الف) $x[n]=x(nT)=e^{j\omega_0 nT},\quad \frac{2\pi}{\omega_0}=T_0$

$$x[n]=\ x[n+N](\ N\in\ \mathbb{Z})\rightarrow\ e^{j\omega_0 nT}=e^{j\omega_0(n+N)T}\Rightarrow e^{j\omega_0 NT}=1$$

$$also\ we\ know\ that\ for\ m\in\ \mathbb{Z}\ \ e^{j2\pi m}=1\ \Rightarrow\ j\omega_0 NT=j2\pi m\ \rightarrow\ \omega_0 NT=2\pi m\rightarrow\frac{2\pi}{T_0}NT=2\pi m$$

$$\Rightarrow\frac{T}{T_0}=\frac{m}{N}\in\ \mathbb{Q}$$

ب)

$$\frac{T}{T_0}=\frac{p}{q}=\frac{m}{N}\rightarrow N=m\frac{q}{p}=m\frac{\frac{q}{\gcd(p,q)}}{\frac{p}{\gcd(p,q)}}=m\ \frac{a}{b},\qquad cause\ \gcd(a,b)=1\ \rightarrow\ m=b=\frac{p}{\gcd(p,q)}$$

$$\rightarrow N=\frac{p}{\gcd(p,q)}\frac{q}{p}=\frac{q}{\gcd(p,q)}=a$$

$$\Rightarrow fundamental\ period=\frac{q}{\gcd(p,q)}$$

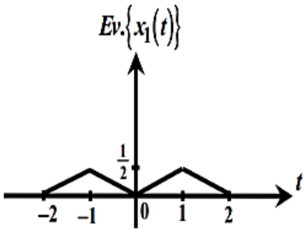
$$\Rightarrow fundamental\ frequency=w=\frac{2\pi}{\frac{q}{\gcd(p,q)}}=\gcd(p,q)\,.\frac{2\pi}{q}$$

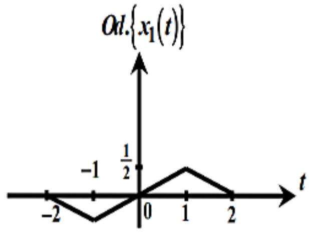
$$we\ alse\ know\ that\ \ \frac{T_0}{T}=\frac{2\pi}{w_0T}=\frac{q}{p}\ \Rightarrow w=\gcd(p,q)\,.\frac{2\pi}{\frac{2\pi}{w_0T}p}=\ \gcd(p,q)\,.\frac{w_0T}{p}$$

ج)

$$\Rightarrow \frac{N\,T}{T_0}=\frac{q}{\gcd(p,q)}\times\frac{p}{q}=\frac{p}{\gcd(p,q)}$$

۸)





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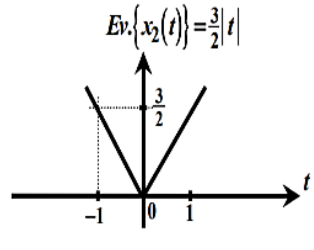

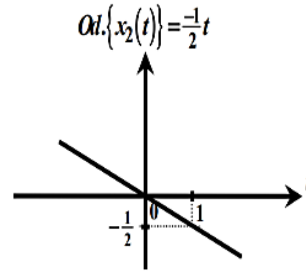

۹)

$$\int_{-\infty}^{\infty} x^2(t)\ dt=\int_{-\infty}^{\infty}\left[x_e(t)+x_o(t)\right]^2\ dt=\int_{-\infty}^{\infty} x_e{}^2(t)\ dt+\int_{-\infty}^{\infty} x_o{}^2(t)\ dt+\underbrace{2\int_{-\infty}^{\infty} x_e(t)\,x_o(t)\ dt}_{\text{Since } x_e(t)\,x_o(t) \text{ is odd then } =0}=\int_{-\infty}^{\infty} x_e{}^2(t)\ dt+\int_{-\infty}^{\infty} x_o{}^2(t)\ dt$$

موفق باشید.