به نام خدا

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درس سیگنال‌ها و سیستم‌ها - دکتر اخوان

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الف)

$$y(t) = Re\{x(t)\} + Ev\{x(t)\} \overset{F.s}{\Rightarrow} b_k$$

$$Re\big(x(t)\big) = \frac{x(t) + x^*(t)}{2} \qquad Ev\big(x(t)\big) = \frac{x(t) + x(-t)}{2}$$

$$\begin{cases} x(t) \overset{F.s}{\Rightarrow} a_k \\ x(-t) \overset{F.s}{\Rightarrow} a_{-k} \\ x^*(t) \overset{F.s}{\Rightarrow} a^*_{-k} \end{cases} \Rightarrow \frac{x(t)+x^*(t)}{2} + \frac{x(t)+x(-t)}{2} \overset{F.s}{\Rightarrow} \frac{a_k+a^*_{-k}}{2} + \frac{a_k+a_{-k}}{2}$$

$$b_k = a_k + \frac{a^*_{-k}}{2} + \frac{a_{-k}}{2}$$

$$b_k = a_k + Re\{a_{-k}\}$$

ب)

$$y(t) = \sum_{m=1}^{M}\left(x(t+mt_0)+x(t-mt_0)\right) \overset{F.s}{\Rightarrow} b_k$$

$$x(t) \to a_k \quad , x(t) \to T_0 - periodic$$

$$\begin{cases} x(t-mT_0) \to e^{-j\frac{2\pi}{T_0}kmt_0}a_k \\ x(t+mT_0) \to e^{+j\frac{2\pi}{T_0}kmt_0}a_k \end{cases} \Longrightarrow F.s\left\{\sum_{m=1}^{M}\left(x(t+mt_0)+x(t-mt_0)\right)\right\}$$

$$= \sum_{m=1}^{M} F.s\left\{\left(x(t+mt_0)+x(t-mt_0)\right)\right\}$$

$$\Rightarrow b_k = \sum_{m=1}^{M} e^{-j\frac{2\pi}{T_0}kmt_0}a_k + e^{+j\frac{2\pi}{T_0}kmt_0}a_k$$

$$b_k = 2a_k\sum_{m=1}^{M}\frac{e^{-j\frac{2\pi}{T_0}kmt_0}a_k + e^{+j\frac{2\pi}{T_0}kmt_0}a_k}{2}$$

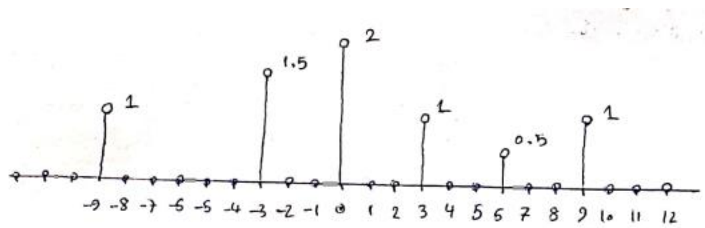
$$b_k = 2a_k\sum_{m=1}^{M}\cos\left(\frac{2\pi}{T_0}kmt_0\right)$$

ج)

$$x(t) = x(t+T_0) \xrightarrow{F.s} a_k$$

$$x(t) = x(t+MT_0) \xrightarrow{F.s} b_k = \begin{cases} a_n & k = Mn \\ 0 & otherwise \end{cases}$$

$$M = 3 \Rightarrow b_k = \begin{cases} a_n & k = 3n \\ 0 & otherwise \end{cases}$$



د)

$$y(t) = x(t) + x\left(\frac{3}{2}t\right)$$

$$\begin{cases} x(t) \rightarrow T_0 - periodic \\ x\left(\frac{3}{2}t\right) \rightarrow \frac{2}{3}T_0 - periodic \end{cases} \Rightarrow \ y(t) \ \rightarrow \ \ 2T_0 - periodic$$

$$x(t) = x(t+T_0) \xrightarrow{F.s} a_k$$

$$x(t) = x(t+2T_0) \xrightarrow{F.s} b_k = \begin{cases} a_n & k = 2n \\ 0 & otherwise \end{cases}$$

$$x\left(\frac{3}{2}t\right) = x\left(\frac{3}{2}t + \frac{2}{3}T_0\right) \xrightarrow{F.s} a_k$$

$$x\left(\frac{3}{2}t\right) = x\left(\frac{3}{2}t + 2T_0\right) \xrightarrow{F.s} d_k = \begin{cases} a_n & k = 3n \\ 0 & otherwise \end{cases}$$

$$linearity \Rightarrow \ x(t) + x\left(\frac{3}{2}t\right) \xrightarrow{T=2T_0, F.s} c_k = b_k + d_k$$

$$c_2 = b_2 + d_2 = a_1$$

$$c_6 = b_6 + d_6 = a_3 + a_2$$

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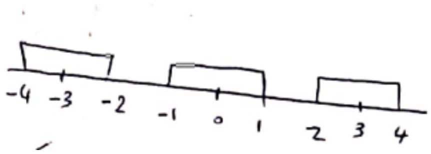
$$x_1(t) = \cos(20\pi t) \sum_{k=-\infty}^{+\infty} \Pi\left(\frac{1}{2}(t-3k)\right)$$

$$\cos(20\pi t) \;\rightarrow\; T = \frac{1}{10}$$

$$\sum_{k=-\infty}^{+\infty} \Pi\left(\frac{1}{2}(t-3k)\right) \;\rightarrow$$



$$\rightarrow\; T = 3$$

برای استفاده از رابطه کانوولوشن ضرایب سری فوریه دوره تناوب باید یکسان باشد.

$$T = 3$$

$$b_k = \frac{1}{3}\int_{-\frac{3}{2}}^{\frac{3}{2}} \Pi\left(\frac{1}{2}t\right) e^{-j\frac{2\pi}{3}kt} dt \;=\; \frac{1}{3}\int_{-1}^{1} e^{-j\frac{2\pi}{3}kt}\, dt = \frac{2}{3} \times \frac{1}{\frac{2\pi}{3}k} \times \frac{e^{j\frac{2\pi}{3}k} - e^{-j\frac{2\pi}{3}k}}{j2} = \frac{\sin\left(\frac{2\pi}{3}k\right)}{\pi k}$$

$$a_k \;\Rightarrow \cos(20\pi t) = \frac{1}{2}e^{j20\pi t} + \frac{1}{2}e^{-j20\pi t} \;\Rightarrow\; \pm 20\pi = \frac{2\pi}{T}k = \frac{2\pi}{3}k \;\Rightarrow\; k = \pm 30$$

$$a_{30} = a_{-30} = \frac{1}{2}$$

$$x_1(t) \overset{F.s}{\rightarrow} c_k \qquad \Rightarrow c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n} \xrightarrow{n=30,-30} a_{30}b_{k-30} + a_{-30}b_{k+30}$$

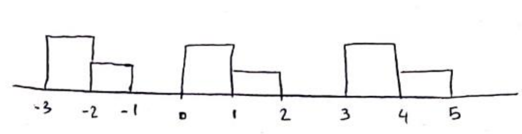
$$c_k = \frac{1}{2}\left(\frac{\sin\left(\frac{2\pi}{3}(k-30)\right)}{\pi(k-30)} + \frac{\sin\left(\frac{2\pi}{3}(k+30)\right)}{\pi(k+30)}\right)$$

*

$$x_2(t) = \cos(20\pi t)\sum_{k=-\infty}^{+\infty} 2\Pi\left(t-\frac{1}{2}-3k\right)+\Pi\left(t-\frac{3}{2}-3k\right)$$

T = 0.1

T = 3



$\Rightarrow T = 3$

همانند بخش (الف) : $a_{30} = a_{-30} = \frac{1}{2}$

$$b_k = \frac{1}{3}\int_0^3 (2\Pi\left(t-\frac{1}{2}-3k\right)+\Pi\left(t-\frac{3}{2}-3k\right))e^{-\frac{j2\pi}{3}kt}$$

$$= \frac{1}{3}\int_0^1 2e^{-\frac{j2\pi}{3}kt}\,dt + \int_1^2 e^{-\frac{j2\pi}{3}kt}\,dt$$

$$\frac{j}{\pi k}\left(-1+e^{-\frac{j2\pi}{3}k}\right)+\frac{j}{2\pi k}\left(-e^{-\frac{j2\pi}{3}k}+\ e^{-\frac{j4\pi}{3}k}\right) = \frac{j}{2\pi k}\left(-2+e^{-\frac{j2\pi}{3}k}+e^{-\frac{j4\pi}{3}k}\right)$$

$$c_k = a_k * b_k = \sum a_n b_{k-n} = a_{30}b_{k-30} + a_{-30}b_{k+30}$$

$$c_k = \frac{j}{4\pi(k-30)}\left(-2+e^{-\frac{j2\pi}{3}(k-30)}+e^{-\frac{j4\pi}{3}(k-30)}\right) + \frac{j}{4\pi(k+30)}\left(-2+e^{-\frac{j2\pi}{3}(k+30)}+e^{-\frac{j4\pi}{3}(k+30)}\right)$$

ب) تعمیم رابطه پارسوال:

$$x(t)y^*(t) \xrightarrow{F.s} c_k$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)y^*(t)\, e^{-\frac{j2\pi}{T_0}kt} dt \overset{k=0}{\Longrightarrow} c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)y^*(t)\, dt \qquad (1)$$

$$x(t)y^*(t) \xrightarrow{F.s} a_k * b^*_{-k} = \sum_{n=-\infty}^{\infty} a_n b^*_{-(k-n)} = c_k$$

$$\overset{k=0}{\Longrightarrow} c_0 = \sum_{n=-\infty}^{\infty} a_n b_n^* \overset{n\to k}{\Longrightarrow} c_0 = \sum_{k=-\infty}^{\infty} a_k b_k^* \qquad (2)$$

$$(1),(2) \Rightarrow \frac{1}{T_0}\int_0^{T_0} x(t)y^*(t)\, dt = \sum_{k=-\infty}^{\infty} a_k b_k^*$$

3-

$$x(t) = 1 - |t| \qquad -2 < t < 2$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-\frac{j\pi}{2}kt}$$

- $$\sum_{k=-\infty}^{\infty} j^k a_k = \sum_{k=-\infty}^{\infty} e^{j\frac{\pi}{2}k} a_k = \sum_{k=-\infty}^{\infty} e^{j\frac{\pi}{2}k\times 1} a_k = x(1) = 1 - |1| = 0$$
- $$\sum_{k=0}^{\infty} |a_k|^2 \xrightarrow{x(t)\to even} \frac{1}{2}\sum_{k=-\infty}^{\infty} |a_k|^2 + \frac{a_0^2}{2}$$

$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} a_k^* a_k + \frac{1}{4}\underbrace{\int_{-2}^{2} x(t)dt}_{0} = \frac{1}{2}\sum_{k=-\infty}^{\infty} a_k^* a_k \xrightarrow{\text{بخش ب سوال 2}} \frac{1}{2}\times\frac{1}{4}\int_{-2}^{2} |x(t)|^2 dt$$

$$= \frac{1}{8}\int_{-2}^{2} (1-|t|)^2 dt = \frac{2}{8}\int_0^2 (1-t)^2 dt = \frac{1}{4}\times\frac{2}{3} = \frac{1}{6}$$

- $\sum_{k=-\infty}^{\infty}(-1)^k a_k = \sum_{k=-\infty}^{\infty}\left(e^{-j\pi}\right)^k a_k = \sum_{k=-\infty}^{\infty} e^{-j\pi k} a_k = \sum_{k=-\infty}^{\infty} e^{-j\frac{\pi}{2}k\times 2} a_k = x(2)$

$= 1 - |2| = -1$

- $\sum_{k=-\infty}^{\infty} a_{2k+1}$

$$\begin{cases} \sum a_k = \cdots + a_{-1} + a_0 + a_1 + a_2 + \cdots \\ \sum (-1)^k a_k = \cdots - a_{-1} + a_0 - a_1 + a_2 + \cdots \end{cases}$$

$$\sum a_k - \sum (-1)^k a_k = 2(\ldots + a_{-1} + a_1 + a_3 + \cdots) = 2\sum_{k=-\infty}^{\infty} a_{2k+1}$$

$$\sum_{k=-\infty}^{\infty} a_{2k+1} = \frac{1}{2}\left(\sum a_k - \sum (-1)^k a_k\right) = \frac{1}{2}\left(x(0) - x(2)\right) = \frac{1}{2}\times 2 = 1$$

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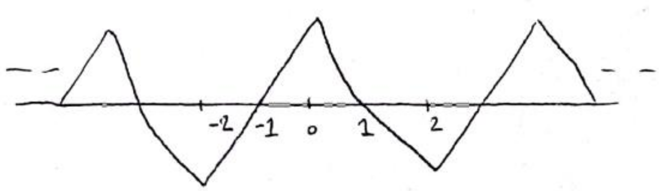
1.

$$\sum_{n=-\infty}^{\infty} (-1)^n \Lambda(t-2n)$$



$$a_k = \frac{1}{4}\int_{-2}^{2} \underbrace{x(t)}_{\text{even}} \underbrace{e^{-j\frac{\pi}{2}kt}}_{\text{even+odd}} dt$$

با توجه به بازه انتگرال فقط بخش زوج زیر انتگرال جواب غیر صفر دارد.

$$a_k = \frac{1}{4}\int_{-2}^{2} x(t)\cos\left(\frac{\pi}{2}kt\right)dt = \frac{2}{4}\int_{0}^{2} x(t)\cos\left(\frac{\pi}{2}kt\right)dt = \frac{2}{4}\int_{0}^{2} (1-t)\cos\left(\frac{\pi}{2}kt\right)dt$$

$$= \frac{1}{2}\int_{0}^{2} \cos\left(\frac{\pi}{2}kt\right)dt - \frac{1}{2}\int_{0}^{2} t\cos\left(\frac{\pi}{2}kt\right)dt$$

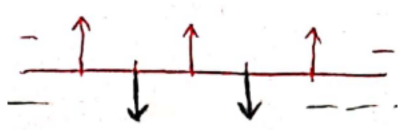
$$= \frac{1}{2}\times\frac{1}{\frac{\pi}{2}k}\times\sin\left(\frac{\pi}{2}kt\right) - \frac{1}{2}\left(\frac{t\sin\left(\frac{\pi}{2}kt\right)}{\frac{\pi}{2}k} + \frac{\cos\left(\frac{\pi}{2}kt\right)}{\frac{\pi^2}{4}k^2}\right)\Big\{\begin{matrix}2\\0\end{matrix}$$

$$= \frac{\sin(\pi k)}{\pi k} - 2\frac{\sin(\pi k)}{\pi k} - \frac{\frac{1}{2}(\cos(\pi k)-1)}{\frac{\pi^2}{4}k^2} = \boxed{\frac{2}{\pi^2 k^2}(1-\cos(\pi k))}$$

2.

$$x_2(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta'(t - nT)$$

$$\begin{cases} x(t) \to a_k \\ \dfrac{d}{dt} x(t) \to \dfrac{j2\pi}{T} k a_k \end{cases} \Rightarrow \sum_{n=-\infty}^{\infty} (-1)^n \frac{d}{dt}\big(\delta(t - nT)\big) = \frac{d}{dt} \sum_{n=-\infty}^{\infty} (-1)^n \big(\delta(t - nT)\big)$$

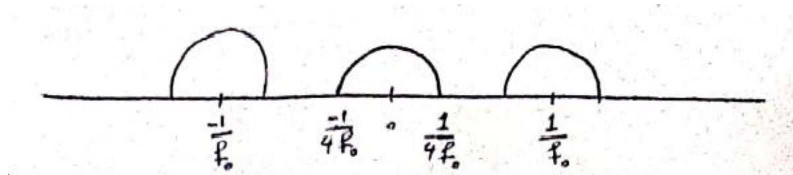


$$\frac{d}{dt}\big(\Psi_{2\mathrm{T}}(t) - \Psi_{2T}(t - T)\big)$$

$$\overset{F.S}{\Rightarrow} j\frac{2\pi}{2T} k \times \left(\frac{1}{2T} - \frac{1}{2T} e^{-j\frac{2\pi}{2T}kT}\right) = j\frac{\pi k}{2T^2}\left(1 - e^{-j\pi k}\right)$$

3.

$$x_3(t) = |\cos(2\pi f_0 t)| + \cos(2\pi f_0 t)$$



$$a_k = \frac{1}{\frac{1}{f_0}} \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 2\cos(2\pi f_0 t)\, e^{-j2\pi f_0 k t} dt = 2f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \frac{1}{2}\left(e^{j2\pi f_0 t(1-k)} + e^{-j2\pi f_0 t(1+k)}\right) dt$$

$$= f_0 \left[\frac{e^{j\frac{\pi}{2}(1-k)} - e^{-j\frac{\pi}{2}(1-k)}}{j2\pi f_0 (1-k)} + \frac{e^{j\frac{\pi}{2}(1+k)} - e^{-j\frac{\pi}{2}(1+k)}}{j2\pi f_0 (1+k)}\right]$$

$$= \frac{\sin\left(\frac{\pi}{2}(1-k)\right)}{\pi(1-k)} + \frac{\sin\left(\frac{\pi}{2}(1+k)\right)}{\pi(1+k)}$$

4.

$$x_4(t) = f^*(t)e^{j\frac{2\pi}{T_0}t}$$

$$\begin{cases} f(t) \to a_k \\ f^*(t) \to a^*_{-k} \\ e^{j\frac{2\pi}{T_0}t} \to \delta(k-1) \end{cases} \Rightarrow c_k = a^*_{-k} * \delta(k-1) = a^*_{-k+1}$$

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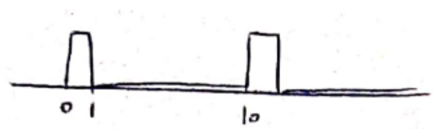
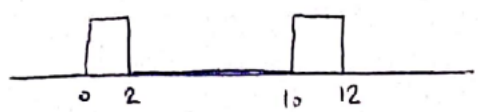
$x_1(t)$



$$a_k = \frac{1}{10}\int_0^{10} e^{-j\frac{2}{10}\pi k t}dt = \frac{1}{10} \times \frac{-e^{-j\frac{\pi}{5}k} + 1}{\frac{j\pi}{5}k} = \frac{1 - e^{-j\frac{\pi}{5}k}}{j2\pi k}$$

$x_2(t)$



$$x_2(t) = x_1(t) + x_1(t-1) \quad \Rightarrow \quad b_k = a_k\left(1 + e^{-j\frac{2\pi}{10}k}\right)$$

$x_3(t)$

$$x_3(t) = x_1(t) - x_1(t-2) \Rightarrow c_k = a_k\left(1 - e^{-j\frac{4\pi}{10}k}\right)$$

$x_4(t)$

$$x_4(t) = \int_{-\infty}^{t} x_3(t)dt$$

$$d_k = \frac{1}{j\frac{2\pi}{10}k} c_k = \frac{1}{j\frac{2\pi}{10}k}\left(1 - e^{-j\frac{4\pi}{10}k}\right) a_k \qquad k \neq 0$$

$$d_0 = \frac{1}{10}\int_0^{10} x_4(t)dt = 0.2$$

$$d_k = \begin{cases} \dfrac{1 - e^{-j\frac{4\pi}{10}k}}{\frac{j2\pi}{10}k} a_k & k \neq 0 \\ 0.2 & k = 0 \end{cases}$$