به نام خدا

**تمرین سری هفتم**

درس سیگنال‌ها و سیستم‌ها

دکتر اخوان

**پاسخ نامه**

1)

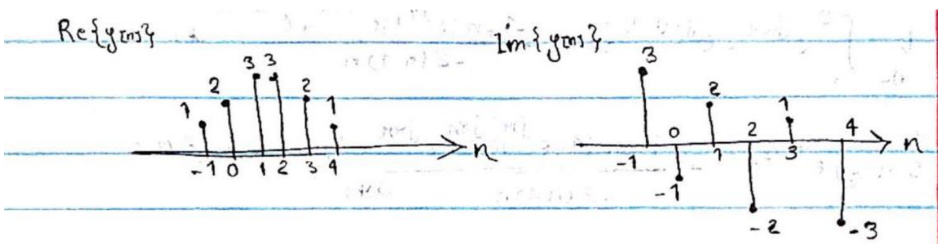
$$X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n} \qquad\qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

الف) $X(e^{j\omega})_{\omega=0} = \sum_n x[n]e^{-j.0.n} = \sum_n x[n] = \sum_n Re\{x[n]\} + \sum_n Im\{x[n]\} = 12$

ب) $X(e^{j\omega})_{\omega=\pi} = \sum_n x[n](-1)^n = -j12$

ج) $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi\, x[0] = 2\pi(2-j) = 4\pi - j2\pi$

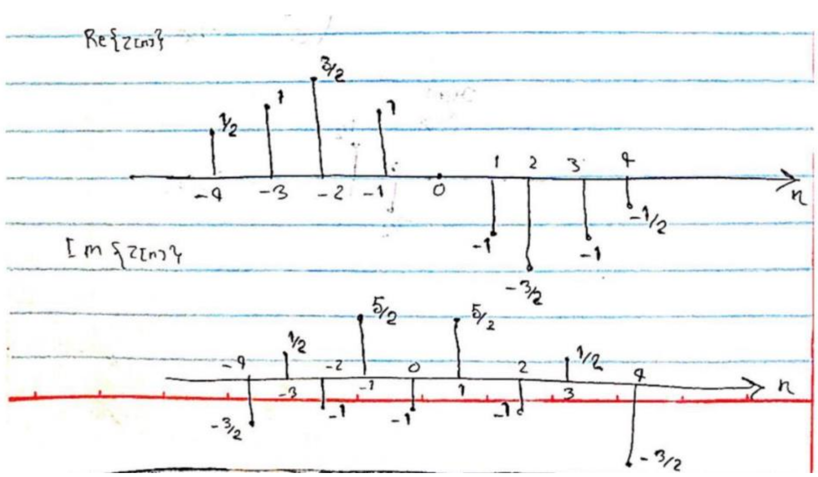
د) $Y(e^{j\omega}) = X(e^{-j\omega}) \rightarrow y[n] = x[-n]$




ه) $Z(e^{j\omega}) = jIm\{X(e^{j\omega})\} = \frac{X(e^{j\omega}) - X^*(e^{j\omega})}{2}$

$$z[n] = \frac{x[n] - x^*[-n]}{2}$$

$$Re\{z[n]\} = \frac{Re\{x[n]\} - Re\{x[-n]\}}{2} \qquad Im\{z[n]\} = \frac{Im\{x[n]\} + Im\{x[-n]\}}{2}$$



2)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega \ = \frac{1}{2\pi}\int_{-\pi}^{0} \sin\left(\frac{\omega}{2}\right)e^{\frac{j\pi}{2}}\, e^{-\frac{j\omega}{2}}e^{j\omega n}\, d\omega \ + \ \frac{1}{2\pi}\int_{0}^{\pi} \sin\left(\frac{\omega}{2}\right)e^{-\frac{j\pi}{2}}\, e^{-\frac{j\omega}{2}}e^{j\omega n}\, d\omega$$

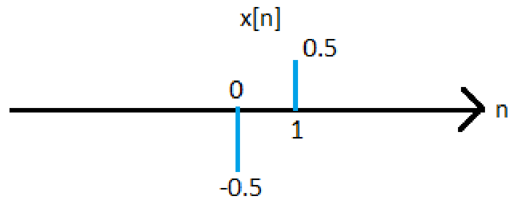
$$I_{*} = \int_{-\pi}^{0} \sin\left(\frac{\omega}{2}\right)\, e^{-\frac{j\omega}{2}}e^{j\omega n}\, d\omega \ =$$

$$= \frac{1}{j2}\int_{-\pi}^{0}\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)e^{\frac{j\pi}{2}}\, e^{-\frac{j\omega}{2}}e^{j\omega n}\, d\omega$$

$$= \ \frac{1}{j2}\int_{-\pi}^{0} e^{j\omega n} \ - e^{j\omega(n-1)}d\omega = \frac{-1 + n\left(e^{j\pi} - 1\right)e^{-j\pi n} + e^{-j\pi n}}{-2(n-1)n}$$

$$I_{**} = \int_{0}^{\pi} \sin\left(\frac{\omega}{2}\right)\, e^{-\frac{j\omega}{2}}e^{j\omega n}\, d\omega \ = \ \frac{1 + n\left(1 - e^{j\pi}\right)e^{-j\pi n} - e^{-j\pi n}}{-2(n-1)n}$$

$$x[n] = \frac{1}{2\pi}e^{\frac{j\pi}{2}}I_{*} \ + \ \frac{1}{2\pi}e^{-\frac{j\pi}{2}}I_{**} \ = \ -\frac{1}{2\pi}\left(\frac{\sin(n\pi)}{n} - \frac{\sin\left((n-1)\pi\right)}{n-1}\right) = \ -\frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$$



3)

الف)

$$X(e^{j\omega})Y^*(e^{j\omega}) \rightarrow IDTFT \rightarrow \; x[n] * y^*[-n]$$

$$Y^*(e^{j\omega}) \rightarrow IDTFT \rightarrow \; y^*[-n]$$

ب)

$$x[n] = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \quad , \qquad x[n] * y^*[-n] \rightarrow DTFT \rightarrow X(e^{j\omega})Y^*(e^{j\omega})$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})\, d\omega = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})\, e^{j\omega n}\, d\omega\right)_{n=0} = (x[n] * y^*[-n])_{n=0}$$

$$= \left(\sum\nolimits_m x[m]y^*[-(n-m)]\right)_{n=0} = \left(\sum\nolimits_m x[m]y^*[m-n]\right)_{n=0} = \sum\nolimits_m x[m]y^*[m]$$

ج)

$$F\left\{\frac{\sin(\omega_0 n)}{\pi n}\right\} = 1\,;|\omega| < \omega_0 < \pi$$

$$F\left\{\frac{\sin\left(\frac{\pi}{4}n\right)}{2\pi n}\right\} = \frac{1}{2}\,F\left\{\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\right\} = \frac{1}{2}\,;|\omega| < \frac{\pi}{4}$$

$$F\left\{\frac{\sin\left(\frac{\pi}{6}n\right)}{5\pi n}\right\} = \frac{1}{5}\,F\left\{\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n}\right\} = \frac{1}{5};|\omega| < \frac{\pi}{6}$$

$$S = \sum\nolimits_n \frac{\sin\left(\frac{\pi}{4}n\right)}{2\pi n}.\frac{\sin\left(\frac{\pi}{6}n\right)}{5\pi n} = \frac{1}{10}\sum\nolimits_n \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}.\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n} = \frac{1}{10}\sum\nolimits_n x[n]y^*[n] \; =$$

$$= \frac{1}{20\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})\, d\omega = \frac{1}{20\pi}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1\times 1\, d\omega \; = \frac{1}{60}$$

4)

$$jIm\{X(e^{j\omega})\} \rightarrow IDTFT \rightarrow \frac{x[n] - x^*[-n]}{2}$$

$$j\frac{d}{d\omega}\; Im\{X(e^{j\omega})\} \rightarrow IDTFT \rightarrow \; -jn\frac{x[n] - x^*[-n]}{2}$$

$$\int_{-\pi}^{\pi}\left(\left|\frac{d}{d\omega}\; Im\{X(e^{j\omega})\}\right|\right)^2 d\omega = 2\pi\sum\nolimits_n \left(\left|n\;\frac{x[n] - x^*[-n]}{2}\right|\right)^2$$

$$n(x[n] - x^*[-n]) = 4\delta[n+2] - 2\delta[n+1] - 2\delta[n-1] + 4\delta[n-2]$$

$$2\pi\sum\nolimits_n \left(\left|n\;\frac{x[n] - x^*[-n]}{2}\right|\right)^2 = \frac{\pi}{2}40 = 20\pi$$

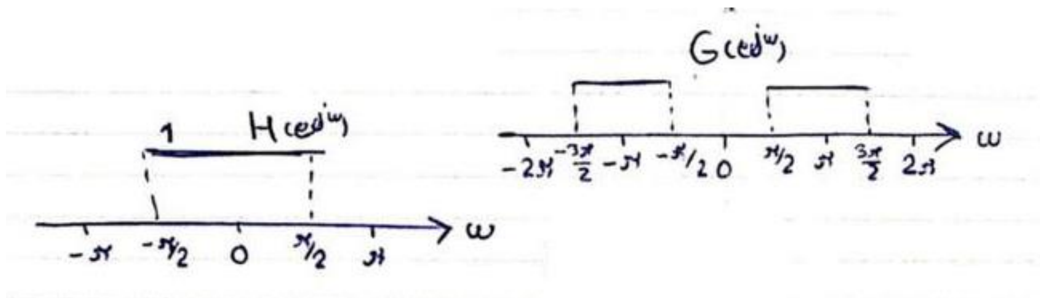
$$\int_{-\pi}^{\pi}\left(\left|\frac{d}{d\omega}\; Im\{X(e^{j\omega})\}\right|\right)^2 d\omega = 20\,\pi$$

5)

الف)

$$g[n] = x[n](-1)^n = x[n]e^{j\pi n} \to G\left(e^{j\omega}\right) = X\left(e^{j(\omega-\pi)}\right)$$

$$h[n] = \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} \ \to \ H\left(e^{j\omega}\right) = \sum_{m=-\infty}^{\infty} \Pi\left(\frac{\omega}{\pi} - 2m\pi\right)$$




$$Y\left(e^{j\omega}\right) = \ G\left(e^{j\omega}\right)H\left(e^{j\omega}\right) = 0 \to y[n] = 0$$

ب)

$$Y\left(e^{j\omega}\right) = \ G\left(e^{j\omega}\right)H\left(e^{j\omega}\right), G\left(e^{j\omega}\right) = 1 - H\left(e^{j(\omega-\pi)}\right) = H\left(e^{j\omega}\right) = \sum_{m=-\infty}^{\infty} \Pi\left(\frac{\omega}{\pi} - 2m\pi\right)$$

$$\to y[n] = h[n] = \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}$$

ج)

$$g[n] = x[n]w[n] = \left(\cos\left(\frac{\pi}{2}n\right)\right)^2 \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} = \frac{1}{2}\delta[n] \to \ G\left(e^{j\omega}\right) = \frac{1}{2}$$

$$y[n] = \frac{1}{2}\ \delta[n] * h[n] = \frac{1}{2}h[n] = \frac{\sin\left(\frac{\pi}{2}n\right)}{2\pi n}$$

د)

$$g[n] = x[n]w[n] = \cos\left(\frac{\pi}{2}n\right)\left(1 + \sin\left(\frac{n\pi}{8}\right) + 2\cos\left(\frac{3\pi}{4}n\right)\right)$$

$$g[n] = \ \cos\left(\frac{\pi}{2}n\right) + \frac{1}{2}\ \sin\left(\frac{5\pi n}{8}\right) - \frac{1}{2}\sin\left(\frac{3\pi n}{8}\right) + \cos\left(\frac{3\pi}{4}n\right) + \cos\left(\frac{\pi}{4}n\right)$$

$$G\left(e^{j\omega}\right) = \ \pi\left[\delta\left(\omega - \frac{\pi}{2}\right) + \delta\left(\omega + \frac{\pi}{2}\right) + \frac{j}{2}\delta\left(\omega + \frac{5\pi}{8}\right) - \frac{j}{2}\delta\left(\omega - \frac{5\pi}{8}\right) - \frac{j}{2}\delta\left(\omega + \frac{3\pi}{8}\right) + \frac{j}{2}\delta\left(\omega - \frac{3\pi}{8}\right)\right.$$
$$\left. + \delta\left(\omega - \frac{3\pi}{4}\right) + \delta\left(\omega + \frac{3\pi}{4}\right) + \delta\left(\omega - \frac{\pi}{4}\right) + \delta\left(\omega + \frac{\pi}{4}\right)\right]$$

$$H\left(e^{j\omega}\right) = 1\ ;\ \ |\omega| \leq \frac{\pi}{2}$$

$$y[n] = \ \cos\left(\frac{\pi}{2}n\right) - \frac{1}{2}\sin\left(\frac{3\pi n}{8}\right) + \cos\left(\frac{\pi}{4}n\right)$$

6)

$$x[n] = f[n]a^n u[n] = f[n]g[n]$$
$$F\{g[n]\} = \frac{1}{1 - ae^{-j\omega}} \;\rightarrow\; F\{f[n]\} = \sum_k 2\pi\, a_k \delta\left(\omega - \frac{2\pi}{N}k\right)$$

$$X(e^{j\omega}) = \frac{1}{2\pi}\left(G(e^{j\omega}) \circledast F(e^{j\omega})\right) = \sum_k \frac{a_k}{1 - ae^{-j\left(\omega - \frac{2\pi}{N}k\right)}} = \sum_{k=0}^{3} \frac{\left(\frac{1}{2}\right)^k}{1 - \frac{1}{4}e^{-j\left(\omega - \frac{2\pi}{4}k\right)}} \;\rightarrow\; N = 4, a = \frac{1}{4}$$

$g[n]$ *is real.*
$a_k \neq a^*_{-k} \rightarrow f[n]$ *is not real*
$\rightarrow x[n]$ *is not real.*

7)

$$x[n] = 1 + \cos\left(2\pi f_0 n + \frac{\pi}{3}\right) = 1 + \frac{1}{2}e^{\frac{j\pi}{3}}e^{j2\pi f_0 n} + \frac{1}{2}e^{-\frac{j\pi}{3}}e^{-j2\pi f_0 n}$$
$$X(e^{j\omega}) = 2\pi\delta(\omega) + \pi\left(e^{\frac{j\pi}{3}}\delta(\omega - 2\pi f_0) + e^{-\frac{j\pi}{3}}\delta(\omega + 2\pi f_0)\right)$$
$$y[n] = j - e^{j2\pi f_0 n} \rightarrow\; Y(e^{j\omega}) = j2\pi\delta(\omega) - 2\pi\delta(\omega - 2\pi f_0)$$
$$H(e^{j\omega})_{\omega=2\pi f_0} = 2e^{-\frac{j\pi}{3}},\;\; H(e^{j\omega})_{\omega=0} = j, \qquad H(e^{j\omega})_{\omega=-2\pi f_0} = 0$$

توجه داشته باشید در سایر $\omega$ ها مقدار $H(e^{j\omega})$ مشخص نیست!

$$S = \sum_n Re\{h[n]\}\sin(2\pi f_0 n) = \sum_n \frac{h[n] + h^*[n]}{2} \cdot \frac{e^{j2\pi f_0 n} - e^{j2\pi f_0 n}}{j2} =$$
$$= \frac{1}{j4}\left[\sum_n h[n]e^{j2\pi f_0 n} - \sum_n h[n]e^{-j2\pi f_0 n} + \sum_n h^*[n]e^{j2\pi f_0 n} - \sum_n h^*[n]e^{-j2\pi f_0 n}\right] =$$
$$= \frac{1}{j4}\left[H(e^{j\omega})_{\omega=2\pi f_0} - H(e^{j\omega})_{\omega=-2\pi f_0} + H^*(e^{j\omega})_{\omega=2\pi f_0} - H^*(e^{j\omega})_{\omega=-2\pi f_0}\right] =$$
$$= \frac{1}{j4}\left(-2e^{-\frac{j\pi}{3}} - 0 + 0 + 2e^{\frac{j\pi}{3}}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \rightarrow \quad S = \frac{\sqrt{3}}{2}$$

8)

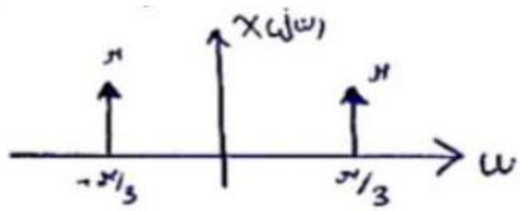
$$y[n] = x[n] * h[n] \rightarrow y[n] \text{ is real and even.}$$

$$y[n] = h[n-1] + h[n+M+1]$$
$$Y(e^{j\omega}) = e^{-j\omega}H(e^{j\omega}) + e^{j\omega(M+1)}H(e^{j\omega}) = H(e^{j\omega})\left[e^{-j\omega} + e^{j\omega(M+1)}\right] =$$
$$= H(e^{j\omega})\left[e^{\frac{j\omega M}{2}}\left(e^{-j\omega\left(\frac{M}{2}+1\right)} + e^{j\omega\left(\frac{M}{2}+1\right)}\right)\right] = 2H(e^{j\omega})e^{\frac{j\omega M}{2}}\cos\left(\omega\left(\frac{M}{2}+1\right)\right)$$
$$\measuredangle Y(e^{j\omega}) = 0 \rightarrow \measuredangle H(e^{j\omega}) = -\frac{M}{2}\omega \rightarrow \text{phase is linear.}$$

9)

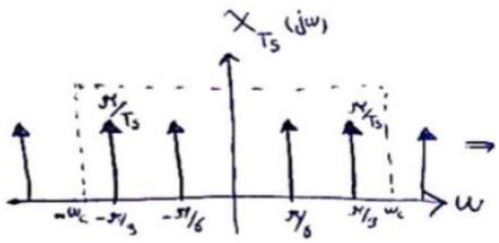
$$x_{Ts}(t) = x(t)\sum_k \delta(t - KT_s) \rightarrow X_{Ts}(j\omega) = \frac{1}{T_s}\sum X\left(j\omega - \frac{k2\pi}{T_s}\right)$$

$$x(t) = \cos\left(\frac{\pi}{3}t\right) \rightarrow X(j\omega) = \pi\left(\delta\left(\omega - \frac{\pi}{3}\right) + \delta\left(\omega + \frac{\pi}{3}\right)\right)$$




الف)

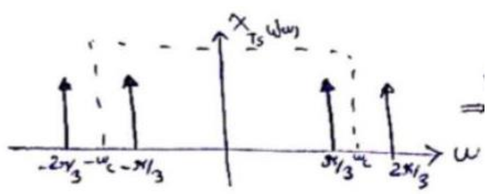
$\omega_s = \frac{\pi}{2}$




$$y_1(t) = \frac{1}{T_s}\left(\cos\left(\frac{\pi}{3}t\right) + \cos\left(\frac{\pi}{6}t\right)\right)$$

ب)

$\omega_s = \pi$




$$y_2(t) = \frac{1}{T_s}\cos\left(\frac{\pi}{3}t\right)$$

ج)

$$y_2(t) = \frac{1}{T_s}x(t) \rightarrow y_2(t)\ is\ similar\ to\ x(t).$$

$$nyquist\ rate: \omega_s \geq 2\omega_{max}$$

$$\omega_{max} = \frac{\pi}{3} \quad \rightarrow \quad \omega_s \geq \frac{2\pi}{3}$$

$$\omega_{s_1} = \frac{\pi}{2} < \omega_s$$

$$\omega_{s_2} = \pi > \omega_s$$

- در حالت دوم نرخ نایکوییست رعایت شده و سیگنال مشابه سیگنال ورودی می باشد در نتیجه سیگنال ورودی قابل بازیابی است.