MRW (1992)

- Solow model (s, n, g are exogenous)

$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$, $0<\alpha<1$, $L_t = L_0 \cdot e^{nt}$, $A_t = A_0 e^{gt}$

$k = \frac{K}{AL} \rightarrow \dot{k} = \frac{\dot{K}AL - K\dot{A}L - KA\dot{L}}{A^2L^2} = \frac{\dot{K}}{AL} - (n+g)k$

$\dot{K} = I - \delta K \rightarrow \frac{\dot{K}}{AL} = \frac{I-\delta K}{AL} = \frac{sY - \delta K}{AL} = sy - \delta k$

$\Rightarrow \dot{k} = s.y - (n+g+\delta)k \Rightarrow SS \Rightarrow \dot{k} = 0 \qquad sy^* = (n+g+\delta)k^*$

since $\frac{Y_t}{A_t L_t} = k_t^{\alpha} \Rightarrow sk^{*\alpha} = (n+g+\delta)k^* \Rightarrow k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$

$\Rightarrow \boxed{y^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}} \xrightarrow{\ln \log} \ln y^* = \ln A + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(n+g+\delta)$

(Assume $y = \frac{Y}{L}$ not $\frac{Y}{AL}$)

Assumption $\Rightarrow \alpha \approx 0.3 \Rightarrow$ El. y wrt s & $(n+g+\delta)$ is $\underline{0.5}$.

$\ln y_t = \ln A_0 + gt + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(n+g+\delta)$, $\ln A_0 = a + \epsilon$

a: constant & ϵ: country specific shock → not just tech but resources, institutions, climate etc.

$\Rightarrow \boxed{\ln y_t = a + gt + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(n+g+\delta) + \epsilon}$ (Table 1)

($g+\delta = 0.05$ assumption)

- restrictions → same coefficient for s & $n+g+\delta$
- without restriction

(A) It is valid only if countries are in their SS or if deviations from SS are random.

- Adding human capital accumulation to Solow Model

$Y_t = K_t^{\alpha} H_t^{\beta}(A_t L_t)^{1-\alpha-\beta} \rightarrow \frac{Y}{AL} = \frac{K^{\alpha}}{(AL)^{\alpha}} \cdot \frac{H^{\beta}}{AL^{\beta}} \rightarrow y = k^{\alpha}h^{\beta}$

$\dot{k} = \frac{\dot{K}}{AL} - (g+n)k$ & $\dot{h} = \frac{\dot{H}}{AL} - (g+n)h$,

$\dot{K} = I_K - \delta K$
$\dot{K} = s_k Y - \delta_k K$
$\dot{H} = I_H - \delta H$
$\dot{H} = s_H Y - \delta H$

$SS \rightarrow \begin{array}{l} \dot{k} = s_k y - k(n+g+\delta) \\ \dot{h} = s_H y - h(n+g+\delta) \end{array}$

$\Rightarrow \begin{array}{l} s_k k^{\alpha} h^{\beta} = k(n+g+\delta) \\ s_H k^{\alpha} h^{\beta} = h(n+g+\delta) \end{array}$

$\frac{s_k}{s_H} = \frac{k}{h}$

$\Rightarrow h^{\beta} = \left(\frac{k \cdot s_H}{s_k}\right)^{\beta}$

Since $k^* = \left(\frac{s_k h^{\beta}}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$

$k^* = \left[\frac{s_k\left(\frac{s_H k}{s_k}\right)^{\beta}}{n+g+\delta}\right]^{\frac{1}{1-\alpha}} = \left[\frac{s_k^{1-\beta} s_H^{\beta}}{n+g+\delta}\right]^{\frac{1}{1-\alpha-\beta}}$

$h^* = \left[\frac{s_k^{\alpha} s_H^{1-\alpha}}{n+g+\delta}\right]^{\frac{1}{1-\alpha-\beta}}$

Prod. function $\rightarrow$ $y_t = k^{\alpha} h^{\beta}$ $\Rightarrow$ $\left(\frac{s_k^{1-\beta} s_H^{\beta}}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{s_k^{\alpha} s_H^{1-\alpha}}{\delta+n+g}\right)^{\frac{\beta}{1-\alpha-\beta}}$

After arrangements $\rightarrow$ $y_t = \dfrac{s_k^{\frac{\alpha}{1-\alpha-\beta}} s_H^{\frac{\beta}{1-\alpha-\beta}}}{(\delta+n+g)^{\frac{\alpha+\beta}{1-\alpha-\beta}}}$

Again, take log & rewrite y_t as $\frac{Y}{L}$ rather than $\frac{Y}{AL}$

$$\Rightarrow \ln y_t = \ln A_0 + gt + \frac{\alpha}{1-\alpha-\beta} \ln s_k + \frac{\beta}{1-\alpha-\beta} \cdot \ln s_H - \frac{\alpha+\beta}{1-\alpha-\beta} \ln(n+g+\delta)$$

OR if s_H is difficult to calculate, we can rewrite the model using $y_t = k^{\alpha} h^{\beta} \Rightarrow \ln y_t = \alpha \ln k_t + \beta \ln h_t$

$$\Rightarrow \boxed{\ln y_t = \ln A_0 + gt + \frac{\alpha}{1-\alpha} \ln s_k + \beta \ln h^* - \frac{\alpha}{1-\alpha} \ln(n+g+\delta)}$$ (Table 2)

Endogenous Growth & Convergence

non decreasing returns to the set of reproducible factors of production

($\alpha+\beta=1$ model with K&H)

OR — endogenous growth model.

($\alpha=1$ model with K only)

$\Rightarrow$ With endogenous growth model $\rightarrow$ countries that save more grow faster indefinitely

$\rightarrow$ countries need not converge in income per capita even if they have same preferences & technology.

Whereas in neoclassical growth model, (with diminishing returns) a country's per capita income growth tends to be inversely related to its initial level of income per capita. Hence, in absence of shocks, poor & rich converge in per capita income.

Let y^* be SS income per effective labor

Approximating around the SS $\Rightarrow$ $\dfrac{d \ln y_t}{dt} = \lambda [\ln y^* - \ln y_t]$

gives the speed of convergence $\lambda = (n+g+\delta)(1-\alpha-\beta)$

How?

(Assumption without H) $\dot{k} = sy - (n+g+\delta)k$, $y = k^{\alpha} \rightarrow \hat{y} = \alpha\hat{k}$

(since $\frac{\dot{x}}{x} = \hat{x}$) $\Rightarrow \hat{k} = \frac{\dot{k}}{k} = \frac{\hat{y}}{\alpha}$

divide by $k \Rightarrow \frac{\dot{k}}{k} = \frac{sy}{k} - (n+g+\delta)$

$$\frac{\hat{y}}{\alpha} = s\frac{y}{y^{1/\alpha}} - (n+g+\delta)$$

$$\Rightarrow \hat{y} = \alpha\left[s \cdot y^{-\frac{(1-\alpha)}{\alpha}} - (n+g+\delta)\right]$$

Since $y = e^{\ln y}$

$$\Rightarrow \hat{y} = \alpha\left[s\, e^{-\frac{(1-\alpha)}{\alpha}\ln y} - (n+g+\delta)\right] \xrightarrow[\text{series}]{\text{1st order Taylor}} \hat{y} \simeq \underbrace{\phi(\ln y^*)}_{1^{st}} + \underbrace{\phi'(\ln y^*)(\ln y - \ln y^*)}_{2^{nd}}$$

$1^{st} \rightarrow \phi(\ln y^*) = \alpha\left[s e^{-\frac{(1-\alpha)}{\alpha}\ln y^*} - (n+g+\delta)\right]$

in SS $\phi \ln y^* = 0 \Rightarrow s\, e^{-\frac{(1-\alpha)}{\alpha}\ln y^*} = n+g+\delta \Rightarrow \boxed{s y^{*-\frac{(1-\alpha)}{\alpha}} = n+g+\delta}$ (1)

(since $\hat{y} = 0$ in SS)

$2^{nd} \rightarrow \phi'(\ln y^*) = \alpha s \frac{-(1-\alpha)}{\alpha} \cdot e^{\frac{-(1-\alpha)}{\alpha}\ln y^*} = \alpha s \cdot \frac{-(1-\alpha)}{\alpha} \cdot y^{*\frac{-(1-\alpha)}{\alpha}}$

1 & 2 together $\Rightarrow \hat{y} = 0 + \alpha s \frac{-(1-\alpha)}{\alpha} y^{*\frac{-(1-\alpha)}{\alpha}} \Rightarrow \hat{y} = -s(1-\alpha)\, y^{*\frac{-(1-\alpha)}{\alpha}}(\ln y - \ln y^*)$

Since (1) $s y^{*\frac{-(1-\alpha)}{\alpha}} = n+g+\delta \Rightarrow \boxed{\hat{y} = -(1-\alpha)(n+g+\delta)(\ln y - \ln y^*)}$

$$\hat{y} = \frac{\dot{y}}{y} = \frac{d\ln y_t}{dt} = \lambda\left[\ln y^* - \ln y_t\right] \longrightarrow \lambda = (1-\alpha)(n+g+\delta)$$

Since $\frac{d\ln y^*}{dt} = 0$ in SS we can rewrite above eq. as follows:

$$\frac{d(\ln y_t - \ln y^*)}{dt} = \lambda\left[\ln y^* - \ln y_t\right] = -\lambda\left[\ln y_t - \ln y^*\right]$$

assume $\ln y_t - \ln y^* = x_t$

here $\Rightarrow \dot{x} = -\lambda x_t$

$\Rightarrow \frac{\dot{x}}{x} = -\lambda$

$x_t = x_0 e^{-\lambda t}$

$$\Rightarrow \underbrace{\ln y_t - \ln y^*}_{x_t} = e^{-\lambda t}\left[\underbrace{\ln y_0 - \ln y^*}_{x_0}\right]$$

(Table 3)

$$\Rightarrow \ln y_t - \ln y_0 = \ln y^* + e^{-\lambda t}\ln y_0 - \ln y_0 - e^{-\lambda t}\ln y^* = \boxed{\ln y^*(1-e^{-\lambda t}) - \ln y_0(1-e^{-\lambda t})}$$

$$\Rightarrow \boxed{\ln y_t - \ln y_0 = (1-e^{-\lambda t})\frac{\alpha}{1-\alpha}\ln s - (1-e^{-\lambda t})\frac{\alpha}{1-\alpha}\ln(\delta+n+g) - \ln y_0(1-e^{-\lambda t})}$$

[using (1)] (Table 4)

(3) (with H) → $\boxed{\ln y_t - \ln y_0 = (1-e^{-\lambda t})\frac{\alpha}{1-\alpha-\beta}\ln s_k + (1-e^{-\lambda t})\frac{\beta}{1-\alpha-\beta}\ln s_H - (1-e^{-\lambda t})\frac{\alpha+\beta}{1-\alpha-\beta}\ln(n+g+\delta) - \ln y_0 (1-e^{-\lambda t})}$

(Table 5)

Table 3 → unconditional convergence
Solow model of income growth, which is a function of SS and the initial level of income.

Table 4&5 → endogenous growth models, where there is no SS level of income; differences among countries in per capita income can persist indefinitely

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Table 6 imposes the restriction that the coefficients on $\ln s_k$, $\ln s_H$ and $\ln(n+g+\delta)$ sum to zero.

Lastly, how to calculate λs from the estimated parameter?

Ex: $-(1-e^{-\lambda t}) = -0.228$ from Table 4, intermediate countries initial y's coefficient

$t = 25$ because $\underbrace{1985-1960}_{\text{data range}}$,

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Growth Accounting

How much of a country's growth can be explained by
- labor force growth
- capital accumulation
- technical progress

$Y = F(K,L,A) \rightarrow \dot{Y} = F_K\dot{K} + F_L\dot{L} + F_A\dot{A}$

$\Rightarrow \frac{\dot{Y}}{Y} = F_K\cdot\frac{\dot{K}}{Y} + F_L\frac{\dot{L}}{Y} + F_A\frac{\dot{A}}{Y}$

$\Rightarrow g_y = \alpha_K\frac{\dot{K}}{K} + \alpha_L\frac{\dot{L}}{L} + \alpha_A\frac{\dot{A}}{A}$

Factor shares

$\frac{\partial F}{\partial K} = \alpha_K\frac{Y}{K}$, $\frac{\partial F}{\partial L} = \alpha_L\frac{Y}{L}$

$\frac{\partial F}{\partial A} = \alpha_A\frac{Y}{A}$.

Since $\mathcal{L}: F(K,L,A) - F_K K - F_L L - F_A A$

$\frac{\partial \mathcal{L}}{\partial K}: \alpha_K\frac{Y}{K} - F_K = 0$

...

$\alpha_K = \frac{\partial F}{\partial K}\frac{K}{Y}$

$\alpha_L = \frac{\partial F}{\partial L}\cdot\frac{L}{Y}$

$\alpha_A = \frac{\partial F}{\partial A}\frac{A}{Y}$

If we have observation on the growth rate of output, labor force & capital stock, we can estimate growth rate of TFP

$\alpha_A\frac{\dot{A}}{A} = g_y - \alpha_K g_K - g_L g_L = \underbrace{\alpha_A g_A}$, Solow residual: part of growth which is not explained by capital accumulation & labor force expansion

If perfect competition & CRTS

And if we have α_K, then, having g_K, g_y, g_L is sufficient to estimate Solow residual since $\alpha_L = 1-\alpha_K$.

Can we measure α_K?, We have a good measure for α_L, which is simply total payments to labor (total wages & compensations) over GDP $\Rightarrow \alpha_L = \frac{w.L}{Y}$

$\alpha_A g_A = g_y - (1-\alpha_L)g_K - \alpha_L g_L$ (without human capital etc.)