Introduction to Quantum Computing Workshop

Lesson 2: Mathematical background for quantum computing

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Mathematical background for quantum computing

Mathematical background for quantum computing:

- Linear Algebra Basics.
  - Linear equations.
  - Vectors and matrices.
  - Vector and matrix operations.
- Complex numbers.
  - The complex plane.
  - Operations on complex numbers.
  - Euler's formula.
- Matrix notation of a qbit.

Every 30 min or so we will have a feed from other QTLC groups.
Feel free to ask questions at any time!
It is an interactive workshop, we all learn from each other!

What is a matrix?

- A matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns.
- The individual items in a matrix are called its elements or entries.
- The horizontal and vertical lines of entries in a matrix are called rows and columns, respectively.
- The size of a matrix is defined by the number of rows and columns that it contains.
- A matrix with m rows and n columns is called an $m \times n$ matrix.
- A matrix can be used to represent a linear map.
- A matrix can be used to represent a property of a mathematical object.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- $a_{11}, a_{12}, \cdots, a_{1n}$ are the elements of the first row of the matrix A.
- $a_{21}, a_{22}, \cdots, a_{2n}$ are the elements of the second row of the matrix A.
- $a_{m1}, a_{m2}, \cdots, a_{mn}$ are the elements of the mth row of the matrix A.

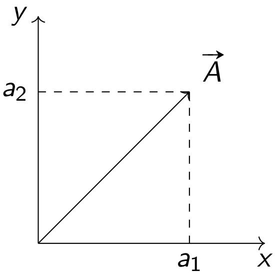
What is a vector?

- A matrix with 1 row and n columns is called an row vector of length n.
- A matrix with n rows and 1 column is called an column vector of length n.

What is a vector?

A vector $\vec{A}$ in a two-dimensional space can be written as a column matrix:

$$\vec{A} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

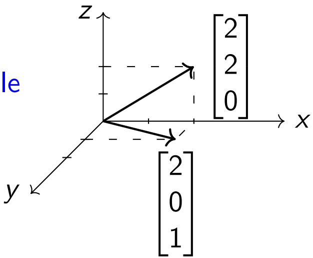


What is a vector?

Similarly, a vector $\vec{B}$ in a three-dimensional space can be written as a column matrix:

$$\vec{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Example



The norm of a vector

The norm of a 2D vector is:

$$||A|| = \sqrt{a_1^2 + a_2^2}$$

Similarly, the norm of a 3D vector is:

$$||B|| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

In general, the norm of a nD vector is:

$$||V|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

The norm of a vector is also called the length of a vector

Why are matrices and vectors convenient?

Given the following set of linear equations:

$$\begin{cases} 4x - 5y = 3 \\ 2x + y = 5 \end{cases}$$

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Given the following set of linear equations:

$$\begin{cases} 4x - 5y = 3 \\ 2x + y = 5 \end{cases}$$

In matrix notation it will become:

$$\begin{bmatrix} 4 & -5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Matrix operations: multiplication

How to perform matrix multiplication?

For example, acting with 2x2 matrix on a 2x1 column matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{bmatrix}$$

For example, acting with 2x2 matrix on a 2x2 column matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Matrix operations: multiplication

How to perform matrix multiplication?

This can be extended to multiplication of 3x3 matrices.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj + bm + cp & ak + bn + cq & al + bo + cr \\ dj + em + fp & dk + en + fq & dl + eo + fr \\ gj + hm + ip & gk + hn + iq & gl + ho + ir \end{bmatrix}$$

Matrix operations: multiplication

Example: multiply the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Matrix operations: multiplication

Example: multiply the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

Matrix operations: multiplication

Example: multiply the following matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: multiply the following matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 0 & 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 0 & 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 1 \\ 4 \cdot 1 + 5 \cdot 0 + 6 \cdot 0 & 4 \cdot 0 + 5 \cdot 1 + 6 \cdot 0 & 4 \cdot 0 + 5 \cdot 0 + 6 \cdot 1 \\ 7 \cdot 1 + 8 \cdot 0 + 9 \cdot 0 & 7 \cdot 0 + 8 \cdot 1 + 9 \cdot 0 & 7 \cdot 0 + 8 \cdot 0 + 9 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Matrix operations: multiplication

Beware: matrices should have the proper dimensions (sizes). Not all multiplication are possible

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

In the above example the dimensions don't fit, since the left matrix has only 2 columns while the right matrix has 3 rows!

Matrix operations: multiplication by a scalar

We can also multiply a matrix by a scalar.

Example: multiply the following matrix by a scalar:

$$2 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Matrix operations: multiplication by a scalar

We can also multiply a matrix by a scalar.

Example: multiply the following matrix by a scalar:

$$2 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution

$$= \begin{bmatrix} 2 \cdot 1 & 2 \cdot 2 \\ 2 \cdot 3 & 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}.$$

This is equivalent to multiplying each element of the matrix by the scalar.
This is called scaling a matrix.

Matrix operations: addition

We can also add two matrices.

Example: add the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Matrix operations: addition

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Example: add the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution

$$= \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}.$$

Beware: matrices should have the same sizes, otherwise an addition is not possible.

Matrix operations: subtraction

We can also subtract two matrices.

Example: subtract the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Matrix operations: subtraction

We can also subtract two matrices.

Example: subtract the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution

$$= \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix} = -4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Beware: matrices should have the same sizes, otherwise an subtraction is not possible.

Matrix operations: transpose

We can also find the transpose of a matrix.

General rule is to swap the rows and columns of the matrix.
Let A be an $m \times n$ matrix.

The transpose of A, denoted by A^T, is an $n \times m$ matrix.
Example:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Matrix operations: transpose

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Matrix operations: transpose

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$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T$$

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Solution

$$= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Matrix operations: transpose

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Solution

$$= \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Complex numbers: Carthesian notation

A complex number c is a number $a + b \cdot i$,
with a, and b real numbers,
and i the imaginary number, which is defined as $i^2 = -1$.

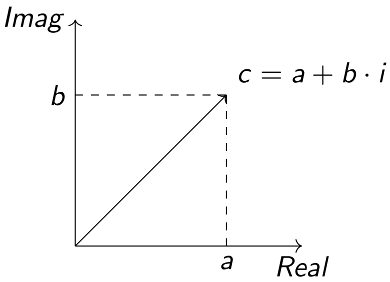
a is called the real part of c, and
b is called the imaginary part of c.

This notation is called the Carthesian notation. Later we will see the polar notation.

Complex numbers: the complex plane (the Argand plane)

A complex number c can be represented as a 2D vector:

$$c = \begin{bmatrix} a \\ b \end{bmatrix}$$



Complex numbers: the modulus or amplitude

The modulus of complex number $= a + b \cdot i$ is:

$$|c| = \sqrt{a^2 + b^2}$$

which is actually the length of the vector

Complex numbers: the argument

The argument of complex number $c = a + b \cdot i$ is defined as the angle of the vector with the positive real axis.

$$arg(c) = \begin{cases} arctan(\frac{b}{a}), & \text{if } a > 0 \\ arctan(\frac{b}{a}) \pm \pi, & \text{if } a < 0 \\ \frac{\pi}{2}, & \text{if } a = 0 \text{ and } b > 0 \\ -\frac{\pi}{2}, & \text{if } a = 0 \text{ and } b < 0 \\ undefined, & \text{if } a = 0 \text{ and } b = 0 \end{cases}$$

Complex numbers: adding and subtracting

Let $c_1 = a_1 + b_1 \cdot i$ and $c_2 = a_2 + b_2 \cdot i$.

Then $c_1 + c_2 = (a_1 + a_2) + (b_1 + b_2) \cdot i$.
Then $c_1 - c_2 = (a_1 - a_2) + (b_1 - b_2) \cdot i$.

Complex numbers: multiplying

Let $c_1 = a_1 + b_1 \cdot i$ and $c_2 = a_2 + b_2 \cdot i$.

Then $c_1 \cdot c_2 =$

$$(a_1 + b_1 \cdot i) \cdot (a_2 + b_2 \cdot i) =$$
$$a_1 \cdot a_2 - b_1 \cdot b_2 + (a_1 \cdot b_2 + a_2 \cdot b1) \cdot i.$$

Complex numbers: the conjugate

Let $c = a + b \cdot i$.

Then the conjugate c^* is defined as $c^* = a - b \cdot i$.
(the conjugate of a vector, is the vector mirrored in the real axis)

NB: $c \cdot c^* = (a + b \cdot i) \cdot (a - b \cdot i) = a^2 + b^2 = |c|^2$.
Note that $|c|^2$ is a real number, since $|c|$ is a real number too.

Complex numbers: dividing

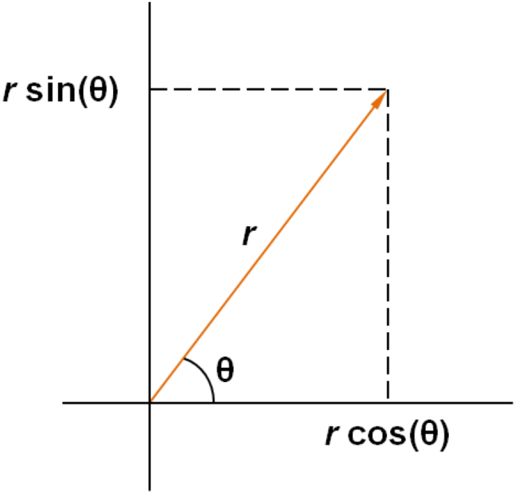
Let $c_1 = a_1 + b_1 \cdot i$ and $c_2 = a_2 + b_2 \cdot i$.

Then $\frac{c_1}{c_2} = \frac{c_1}{c_2} \cdot \frac{{c_2}^*}{{c_2}^*} = \frac{c_1 \cdot c_2^*}{|c_2|^2}$.

So, a division of complex numbers can be rewritten as a multiplication divided by a real number, since $|c_2|^2$ is a real number.

Complex numbers: a and b expressed in r and θ

$c = a + b \cdot i$, with $a = r \cdot cos(\theta)$ and $b = r \cdot sin(\theta)$



Complex numbers: Euler's formula

$$e^{i \cdot \theta} = cos(\theta) + i \cdot sin(\theta)$$

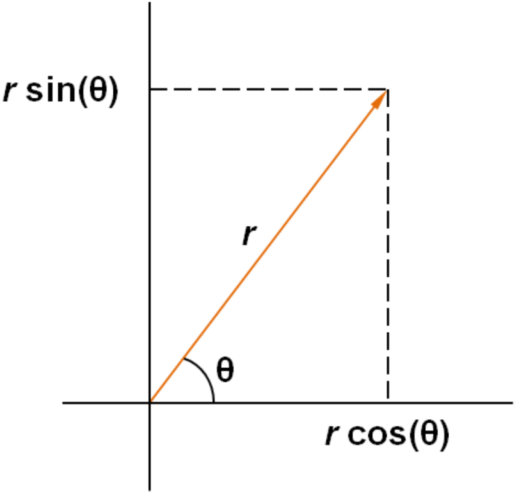
Complex numbers: Polar notation

$a = r \cdot cos(\theta)$
$b = r \cdot sin(\theta)$

So, $c = a + b \cdot i = r \cdot cos(\theta) + r \cdot sin(\theta) \cdot i$
So, $c = r \cdot (cos(\theta) + i \cdot sin(\theta))$
So, $c = r \cdot e^{i \cdot \theta}$

Complex numbers: r and θ expressed in a and b

$c = a + b \cdot i = r \cdot e^{i \cdot \theta}$, with $r = \sqrt{a^2 + b^2}$ and $\theta = arg(c)$



Complex numbers: Polar notation: multiplication

$$c_1 = r_1 \cdot e^{i \cdot \theta_1}$$
$$c_2 = r_2 \cdot e^{i \cdot \theta_2}$$

$$c_1 \cdot c_2 = r_1 \cdot r_2 \cdot e^{i \cdot (\theta_1 + \theta_2)}$$

Complex numbers: Polar notation: division

$$c_1 = r_1 \cdot e^{i \cdot \theta_1}$$
$$c_2 = r_2 \cdot e^{i \cdot \theta_2}$$

$$\frac{c_1}{c_2} = \frac{r_1}{r_2} \cdot e^{i \cdot (\theta_1 - \theta_2)}$$

Matrix operations: multiplication

A matrix also can have complex numbers as elements.

Example: multiply the following matrices:

$$\begin{bmatrix} 1 & 2i & 3 \\ 4 & 5 & 6i \\ 7 & 8i & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix operations: multiplication

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$$\begin{bmatrix} 1 & 2i & 3 \\ 4 & 5 & 6i \\ 7 & 8i & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$= \begin{bmatrix} 1 \cdot 1 + 2i \cdot 0 + 3 \cdot 0 & 1 \cdot 0 + 2i \cdot 1 + 3 \cdot 0 & 1 \cdot 0 + 2i \cdot 0 + 3 \cdot 1 \\ 4 \cdot 1 + 5 \cdot 0 + 6i \cdot 0 & 4 \cdot 0 + 5 \cdot 1 + 6i \cdot 0 & 4 \cdot 0 + 5 \cdot 0 + 6i \cdot 1 \\ 7 \cdot 1 + 8i \cdot 0 + 9 \cdot 0 & 7 \cdot 0 + 8i \cdot 1 + 9 \cdot 0 & 7 \cdot 0 + 8i \cdot 0 + 9 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 2i & 3 \\ 4 & 5 & 6i \\ 7 & 8i & 9 \end{bmatrix}.$$

Matrix operations: conjugate

Also, we can find the conjugate of a matrix.

The complex conjugate of a matrix is obtained by taking the conjugate of each element of the matrix.
Let A be an $m \times n$ matrix.

The complex conjugate of A, denoted by A^*, is an $m \times n$ matrix.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad A^* = \begin{bmatrix} a^* & b^* & c^* \\ d^* & e^* & f^* \end{bmatrix}$$

Matrix operations: conjugate

Example: find the conjugate of the following matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^*$$

Matrix operations: conjugate

Example: find the conjugate of the following matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^*$$

Solution

$$= \begin{bmatrix} 1^* & 2^* \\ 3^* & 4^* \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

It is a real matrix, so the conjugate of a real matrix is the matrix itself.

Matrix operations: conjugate

Example: find the conjugate of the following matrix:

$$\begin{bmatrix} 1 & 2i & 3 \\ 4 & 5 & 6i \\ 7 & 8i & 9 \end{bmatrix}^*$$

Matrix operations: conjugate

Example: find the conjugate of the following matrix:

$$\begin{bmatrix} 1 & 2i & 3 \\ 4 & 5 & 6i \\ 7 & 8i & 9 \end{bmatrix}^*$$

Solution

$$= \begin{bmatrix} 1^* & 2i^* & 3^* \\ 4^* & 5^* & 6i^* \\ 7^* & 8i^* & 9^* \end{bmatrix} = \begin{bmatrix} 1 & -2i & 3 \\ 4 & 5 & -6i \\ 7 & -8i & 9 \end{bmatrix}.$$

Matrix operations: conjugate

We can also find the conjugate transpose of a matrix.

The conjugate transpose of a matrix is obtained by taking the conjugate of each element of the matrix and then taking the transpose of the matrix.
Let A be an $m \times n$ matrix.

The conjugate transpose of A, denoted by $A^\dagger$, is an $n \times m$ matrix.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad A^\dagger = \begin{bmatrix} a^* & d^* \\ b^* & e^* \\ c^* & f^* \end{bmatrix}$$

Matrix operations: conjugate

Example: find the conjugate transpose of the following matrix:

$$\begin{bmatrix} 1i & 2 \\ 3 & 4i \\ 5i & 6 \end{bmatrix}^\dagger$$

Matrix operations: conjugate

Example: find the conjugate transpose of the following matrix:

$$\begin{bmatrix} 1i & 2 \\ 3 & 4i \\ 5i & 6 \end{bmatrix}^\dagger$$

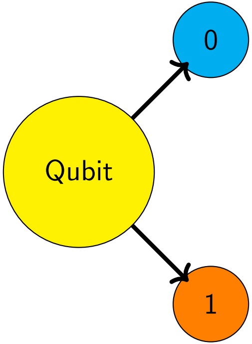
Solution

$$= \begin{bmatrix} 1i^* & 3^* & 5i^* \\ 2^* & 4i^* & 6^* \end{bmatrix} = \begin{bmatrix} -i & 3 & i \\ 2 & -4i & 6 \end{bmatrix}.$$

Qubit vs. classical bit

Qubit is a quantum bit living in a superposition of two states.

Classical bit: can be in one of two states: 0 or 1.
Qubit: can be in a superposition of 0 and 1.



Matrix representation of qubit

We use matrix algebra to represent qubits.
State of a single qubit:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

In a vector form/representation:

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

The states $|0\rangle$ and $|1\rangle$ are represented by the following column matrices:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The coefficients α and β are complex numbers, and they are generally called the amplitudes of the states.

The amplitudes

Amplitudes give the probability of finding the system in a given state when performing a measurement.

The probability of finding the system in state $|0\rangle$ is $|\alpha|^2$, and the probability of finding the system in state $|1\rangle$ is $|\beta|^2$.

The sum of the probabilities of finding the system in the two states must be equal to 1. Hence,

$$|\alpha|^2 + |\beta|^2 = 1$$

The particle exists by itself in a superposition of states.

Thank you

Thank you for your attention!

Start working with Jupyter notebook for Lesson 2.

Contact: `r.j.w.t.tangelder@saxion.nl`