Intro to quantum computing

for the brilliant minds and the curious

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Chapter 1

Introduction

1.1 Quantum computing workflow

Quantum computer is a new type of computer that uses quantum mechanics to perform computations. The quantum computer is based on the principles of quantum mechanics, which is the branch of physics that deals with the behavior of particles at the atomic and subatomic levels. Quantum mechanics is a very complex and counterintuitive theory, but it has been experimentally verified to be correct. Quantum computers are able to perform certain types of computations much faster than classical computers. This is because quantum computers can exploit the quantum mechanical properties of particles to perform computations in parallel. In this section, we will discuss the basic workflow of a quantum computer.

Quantum computing workflow consists of the following steps:

1. Quantum state preparation: In this step, the quantum computer prepares the initial state of the qubits. The qubits are the basic building blocks of a quantum computer. The initial state of the qubits is usually a superposition of the 0 and 1 states.

2. Quantum gate operations: In this step, the quantum computer performs quantum gate operations on the qubits. Quantum gates are the basic building blocks of quantum circuits. They are used to manipulate the state of the qubits.

3. Measurement: In this step, the quantum computer measures the final state of the qubits. The measurement collapses the superposition of the qubits into a classical state. This is done to extract the final result of the computation.

4. Classical post-processing: In this step, the classical computer processes the measurement results to obtain the final result of the computation.

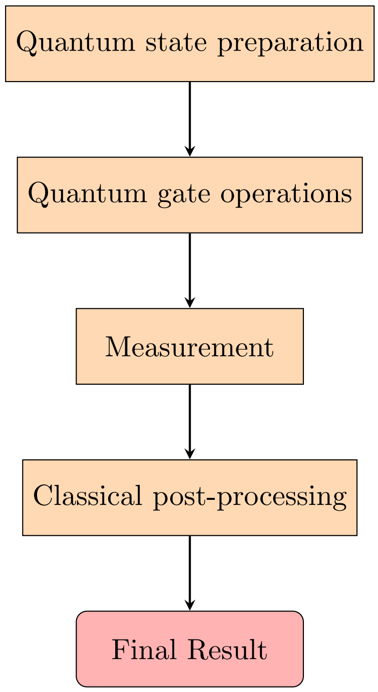


Figure 1.1: Quantum computing workflow

1.2 Quantum bits (qubits)

1.2.1 Classical bits

Quantum bits, or qubits, are the basic building blocks of a quantum computer. A qubit is a two-level quantum system that can exist in a superposition of the 0 and 1 states. This is in contrast to classical bits, which can only exist in one of two states: 0 or 1. The superposition of the qubits allows quantum computers to perform computations in parallel, which can lead to significant speedups over classical computers. In this section, we will discuss the properties of qubits and how they are represented mathematically.

Data storage in classical computers is based on bits, which can be in one of two states: 0 or 1.

Word	**Binary**
Hello	01001000 01100101 01101100 01101100 01101111
World	01010111 01101111 01110010 01101100 01100100

Table 1.1: Words and their binary codes.

Another example of classical bits is the representation of numbers in binary. For example:

Decimal	**Binary**
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Table 1.2: Decimal numbers and their binary representation.

1.2.2 Quantum bits

Quantum bits, or qubits, are the basic building blocks of a quantum computer. A qubit is a two-level quantum system that can exist in a superposition of the 0 and 1 states. A qubit state representation:

$$q = \alpha(\text{state } 0) + \beta(\text{state } 1) \tag{1.1}$$

where α and β are complex numbers, and $|\alpha|^2 + |\beta|^2 = 1$.

As any physical system, the qubit state has to be normalized to unity. The normalization condition is given by:

$$|\alpha|^2 + |\beta|^2 = 1 \tag{1.2}$$

A qubit is a classical bit when it has a definite state, either 0 or 1. However, it has the ability to exist in a superposition of both states. The superposition of qubits allows quantum computers to perform computations in parallel, which can lead to significant speedups over classical computers.

mathematically, a qubit can be represented as a vector in a two-dimensional complex vector space. The state of a qubit can be represented as:

$$|q\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{1.3}$$

where $|0\rangle$ and $|1\rangle$ are the basis states of the qubit, and α and β are complex numbers. The qubit state is normalized to unity, which means that $|\alpha|^2 + |\beta|^2 = 1$.

1.2.3 Qubit representation

Quantum states can be represented as:

- Analytical representation: $\Psi(x) = A\sin(kx) + B\cos(kx)$, where A and B are constants, and k is the wave number.
- The vector representation: $|\Psi\rangle = A\,|\sin(kx)\rangle + B\,|\cos(kx)\rangle = \begin{bmatrix} A \\ B \end{bmatrix}$. In this case, $|\sin(kx)\rangle$ and $|\cos(kx)\rangle$ are the basis states of the quantum system.

1.2.4 Dirac notation of vectors

- Easy to represent quantum states and operations.
- The state of a qubit can be represented as:

$$|q\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{1.4}$$

  where $|0\rangle$ and $|1\rangle$ are the basis states of the qubit, and α and β are complex numbers.
- Dirac notation is inspired by the vector representation and column matrix representation of vectors.

1.2.5 Vector representation in 2D space

- Representation of a vector in 2D space:

$$\vec{A} = A_x \vec{i} + A_y \vec{j}. \tag{1.5}$$

where $\vec{A}$ is the vector, A_x and A_y are the projections of the vector on the x and y axes, and $\vec{i}$ and $\vec{j}$ are the unit vectors along the x and y axes, respectively. The vector $\vec{A}$ is shown in Figure 1.2.

- The vector $\vec{A}$ can be written as:

$$\vec{A} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}. \tag{1.6}$$

- In Dirac notation, the vector $\vec{A}$ can be represented as:

$$|A\rangle = A_x |i\rangle + A_y |j\rangle. \tag{1.7}$$

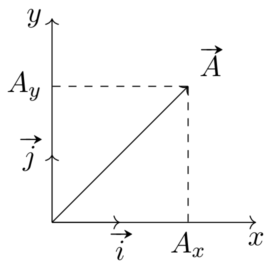


Figure 1.2: Vector $\vec{A}$ in 2D space

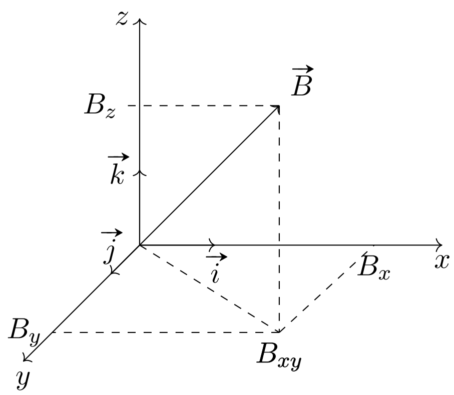


Figure 1.3: Vector $\vec{B}$ in 3D space

1.2.6 Vector representation in 3D space

- Representation of a vector in 3D space:

$$\vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k}. \tag{1.8}$$

where $\vec{B}$ is the vector, B_x, B_y, and B_z are the projections of the vector on the x, y, and z axes, and $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit vectors along the x, y, and z axes, respectively. The vector $\vec{B}$ is shown in Figure 1.3.

- The vector $\vec{B}$ can be written as:

$$\vec{B} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}. \tag{1.9}$$

- In Dirac notation, the vector $\vec{B}$ can be represented as:

$$|B\rangle = B_x |i\rangle + B_y |j\rangle + B_z |k\rangle. \tag{1.10}$$

1.2.7 Qubit representation in 2D space

- A qubit can be represented as a vector in a two-dimensional complex vector space.
- The state of a qubit can be represented as:

$$|q\rangle = \alpha |0\rangle + \beta |1\rangle \tag{1.11}$$

  where $|0\rangle$ and $|1\rangle$ are the basis states of the qubit, and α and β are complex numbers.

- The qubit state is normalized to unity, which means that $|\alpha|^2 + |\beta|^2 = 1$.
- The qubit state can be represented as a vector in a two-dimensional complex vector space:

$$|q\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha |0\rangle + \beta |1\rangle \tag{1.12}$$

$$\begin{array}{cc} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} & \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \end{array}$$

Hence, the states $|0\rangle$ and $|1\rangle$ are the basis states of the qubit are written as:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1.13}$$

The geometic representation of the qubit state $|\Psi\rangle$ in the 2D space.

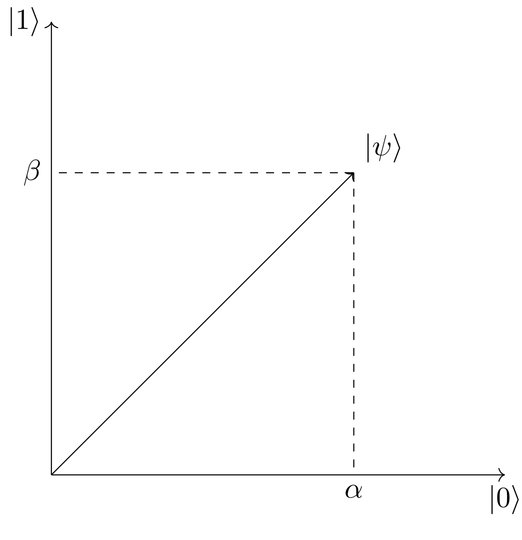


Figure 1.4: Quantum state $|\Psi\rangle$

Hence, the bra-ket notation is a convenient way to represent quantum states and operations.

- The ket is a column vector:
$$|\Psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{1.14}$$
- The bra is a row vector:
$$\langle\Psi| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \tag{1.15}$$
- the inner (dot) product of the bra and ket is:
$$\langle\Psi|\Psi\rangle = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = |\alpha|^2 + |\beta|^2 = 1 \tag{1.16}$$
- The outer product of the bra and ket is:
$$|\Psi\rangle \langle\Psi| = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} \tag{1.17}$$
- The $|\alpha|^2$ and $|\beta|^2$ terms represent the probabilities of measuring the qubit in the 0 and 1 states, respectively.
- $|\alpha|^2$ and $|\beta|^2$ are evaluated as the square of the absolute values of α and β:
$$|\alpha|^2 = \alpha^*\alpha \quad \text{and} \quad |\beta|^2 = \beta^*\beta \tag{1.18}$$
where α^* and β^* are the complex conjugates of α and β.

1.3 Quantum gates

1.3.1 Definition

- Quantum gates are the basic building blocks of quantum circuits.
- They are used to manipulate the state of the qubits.
- Quantum gates are unitary operators that act on the state of the qubits.
- The action of a quantum gate on a qubit can be represented as a matrix multiplication:

$$|q'\rangle = U\,|q\rangle \tag{1.19}$$

  where U is the unitary matrix representing the quantum gate, $|q\rangle$ is the input state of the qubit, and $|q'\rangle$ is the output state of the qubit.

A sketch of the quantum gate acting on a qubit is shown in Figure 1.5.

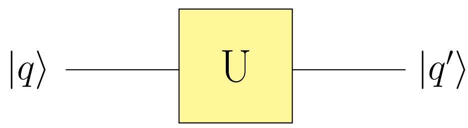


Figure 1.5: Quantum gate acting on a qubit.

The gate-matrix is a linear transformation between the old state and the new state of the qubit.

$$U = \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix} \tag{1.20}$$

where U_{00}, U_{01}, U_{10}, and U_{11} are the elements of the matrix representing the quantum gate.

$$U = \begin{array}{c} \\ \langle 0|_{q'} \\ \langle 1|_{q'} \end{array} \begin{array}{c} \begin{array}{cc} |0\rangle_q & |1\rangle_q \end{array} \\ \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix} \end{array}$$

Figure 1.6: Matrix representation of quantum gate U with element labels.

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{array}{c} \\ \langle 0|_{q'} \\ \langle 1|_{q'} \end{array} \begin{array}{c} \begin{array}{cc} |0\rangle_q & |1\rangle_q \end{array} \\ \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix} \end{array}$$

Figure 1.7: Matrix representation of quantum gate U with element labels.

The matrix representation of the quantum gate is a unitary matrix, which means that the inverse of the matrix is equal to its conjugate transpose:

$$U^{\dagger}U = I \tag{1.21}$$

Example 1: Pauli-X gate

- The Pauli-X gate is a quantum gate that flips the state of a qubit from 0 to 1 and vice versa:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.22}$$

- The action of the Pauli-X gate on a qubit can be represented as:

$$|q'\rangle = X\,|q\rangle \tag{1.23}$$

- Hence, the action of the Pauli-X gate on a qubit in the state $|0\rangle$ is:

$$X\,|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \tag{1.24}$$

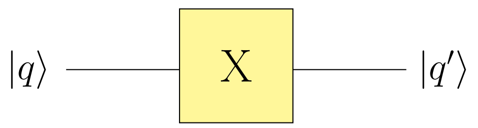
- matrix multiplication in detail is done as follows:

$$X\,|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \times 1 + 1 \times 0 \\ 1 \times 1 + 0 \times 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \tag{1.25}$$

- Pauli-X gate is also known as the NOT gate and/or the bit-flip gate.
- The action of the Pauli-X gate on a qubit in the state $|1\rangle$ is:

$$X\,|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \tag{1.26}$$

- The circuit representation of the Pauli-X gate is shown in Figure 1.8a and table 1.8b.



(a) Circuit representation of the Pauli-X gate.

Input	**Output**
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$

(b) Truth table of the Pauli-X gate.

Example 2: Hadamard gate

- The Hadamard gate is a quantum gate that creates a superposition of the 0 and 1 states:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1.27}$$

- The action of the Hadamard gate on a qubit can be represented as:

$$|q'\rangle = H\,|q\rangle \tag{1.28}$$

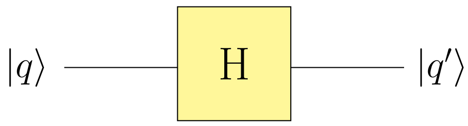
- Hence, the action of the Hadamard gate on a qubit in the state $|0\rangle$ is:

$$H\,|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{1.29}$$

- The action of the Hadamard gate on a qubit in the state $|1\rangle$ is:

$$H\,|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{1.30}$$

- The circuit representation of the Hadamard gate is shown in Figure 1.9a and table 1.9b.



(a) Circuit representation of the Hadamard gate.

Input	**Output**
$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$\|1\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$

(b) Truth table of the Hadamard gate.

1.3.2 Tensor product

To perform operations on multiple qubits, we need to use the tensor product. The tensor product is a way to combine two or more quantum states into a single quantum state, represented by Hilbert space.

The tensor product of two quantum states $|\psi\rangle$ and $|\phi\rangle$ is denoted as $|\psi\rangle \otimes |\phi\rangle$.

Example 1: Assume a two-qubit system with the two qubits in state $|0\rangle$ and $|0\rangle$. The combined state of the two qubits is:

$$|q_0q_1\rangle = |q_0\rangle \otimes |q_1\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}1\\0\end{bmatrix}\\0\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00\rangle\,. \tag{1.31}$$

Example 2: Assume a two-qubit system with the two qubits in state $|0\rangle$ and $|1\rangle$. The combined state of the two qubits is:

$$|q_0q_1\rangle = |q_0\rangle \otimes |q_1\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}0\\1\end{bmatrix}\\0\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = |01\rangle\,. \tag{1.32}$$

Example 3: Assume a two-qubit system with the two qubits in state $|1\rangle$ and $|0\rangle$. The combined state of the two qubits is:

$$|q_0q_1\rangle = |q_0\rangle \otimes |q_1\rangle = |1\rangle \otimes |0\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\begin{bmatrix}1\\0\end{bmatrix}\\1\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = |10\rangle\,. \tag{1.33}$$

Example 4: Assume a two-qubit system with the two qubits in state $|1\rangle$ and $|1\rangle$. The combined state of the two qubits is:

$$|q_0q_1\rangle = |q_0\rangle \otimes |q_1\rangle = |1\rangle \otimes |1\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\begin{bmatrix}0\\1\end{bmatrix}\\1\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = |11\rangle\,. \tag{1.34}$$

1.3.3 Multi-qubit gates

Gates like X and H are single-qubit gates that act on a single qubit. However, quantum computers can have multiple qubits, and gates can act on multiple qubits at the same time. These gates are called multi-qubit gates.

- CNOT gate: The CNOT gate is a two-qubit gate that flips the second qubit if the first qubit is in the state $|1\rangle$.
- The action of the CNOT gate denoted as CX on two qubits can be represented as:

$$|q_0'q_1'\rangle = CX\,|q_1\rangle \otimes |q_2\rangle \tag{1.35}$$

- The CNOT gate is represented by the matrix:

$$CX = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.36}$$

- The action of the CNOT gate on two qubits in the state $|00\rangle$ is:

$$CX\,|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle \tag{1.37}$$

- The action of the CNOT gate on two qubits in the state $|01\rangle$ is:

$$CX\,|01\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle \tag{1.38}$$

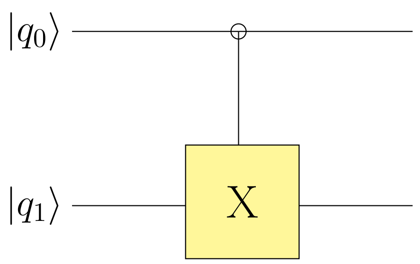
- The action of the CNOT gate on two qubits in the state $|10\rangle$ is:

$$CX\,|10\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle \tag{1.39}$$

- The action of the CNOT gate on two qubits in the state $|11\rangle$ is:

$$CX\,|11\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle \tag{1.40}$$

- The circuit representation of the CNOT gate is shown in Figure 1.10a and table 1.10b.



(a) Circuit representation of the CNOT gate.

Input	**Output**
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

(b) Truth table of the CNOT gate.

1.4 Quantum circuits

1.4.1 Definition

A quantum circuit is typically represented as a sequence of quantum gates that act on qubits. The quantum gates manipulate the state of the qubits, and the final state of the qubits is measured to obtain the result of the computation. Quantum circuits are used to perform various quantum algorithms and simulations. In this section, we will discuss the basic components of a quantum circuit.

A typical quantum circuit is composed of the following components:

- Quantum register: A quantum register is a collection of qubits that are used to store the quantum information. The qubits in the quantum register can be in a superposition of the 0 and 1 states.
- Quantum gates: Quantum gates are the basic building blocks of quantum circuits. They are used to manipulate the state of the qubits. The quantum gates are represented as unitary operators that act on the state of the qubits.
- Measurement gates: The measurement operation collapses the superposition of the qubits into a classical state. The measurement results are stored in the classical register.
- Classical register: A classical register is a collection of classical bits that are used to store the measurement results of the qubits. The classical bits are used to extract the final result of the computation.

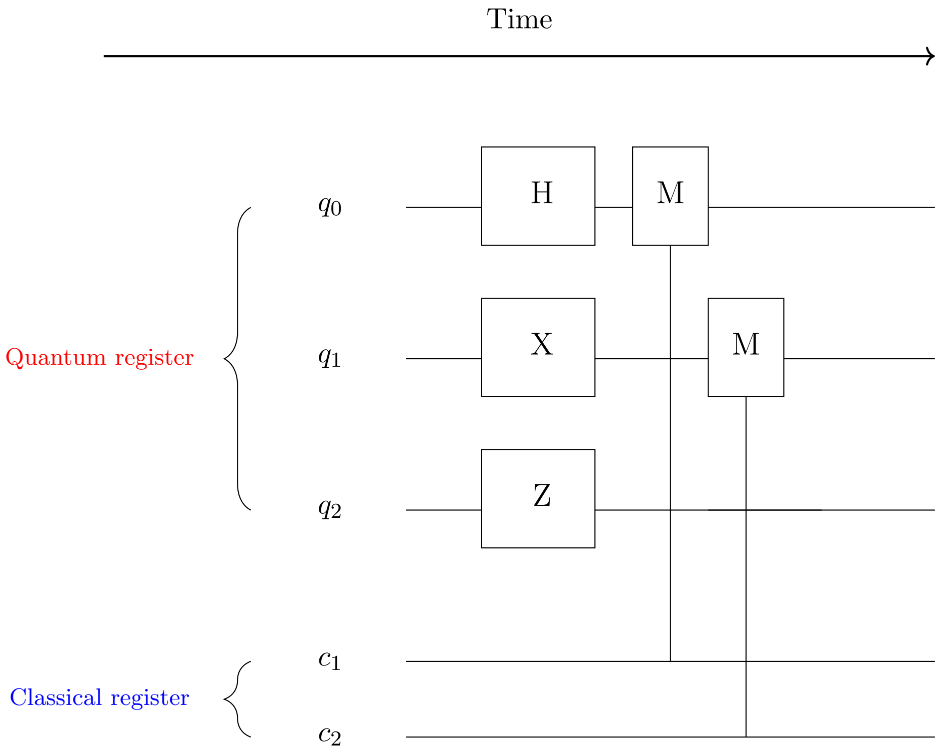
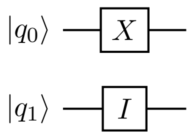


Figure 1.11: Quantum circuit with quantum and classical registers.

1.4.2 Quantum circuit representation

The matrix representation of the quantum circuit is the tensor product of the matrices representing the quantum gates applied to the qubits. The matrix representation of the quantum circuit is obtained by multiplying the matrices of the quantum gates applied to the qubits.

Example: Assume a two-qubit quantum circuit with X gate applied to qubit 0. Both qubits are initialized in the state $|0\rangle$. The matrix representation of the quantum circuit is obtained by multiplying the matrix of the X gate, acting on qubit 0, with the identity matrix acting on qubit 1:

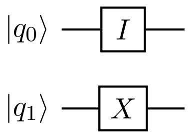


$$U|q_0q_1\rangle = U|00\rangle = X \otimes I|00\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle \tag{1.41}$$

$$U = X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.42}$$

Notice the order of the tensor product is important. The X gate is applied to qubit 0, and the identity matrix is applied to qubit 1. The matrix representation of the quantum circuit is obtained by multiplying the matrix of the X gate with the identity matrix.

Example: Assume a two-qubit quantum circuit with X gate applied to qubit 1. Both qubits are initialized in the state $|0\rangle$. The matrix representation of the quantum circuit is obtained by multiplying the matrix of the X gate, acting on qubit 1, with the identity matrix acting on qubit 0:



$$U|q_0q_1\rangle = U|00\rangle = I \otimes X|00\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle \tag{1.43}$$

1.4.3 Hilbert space

Assuming a two-qubit system, the Hilbert space of the quantum system is the tensor product of the Hilbert spaces of the individual qubits. The Hilbert space of a single qubit is a two-dimensional complex vector space, and the Hilbert space of a two-qubit system is a four-dimensional complex vector space.

Back to the example:

$$|0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle \tag{1.44}$$

The column vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ represents the state of the two-qubit system in the computational basis. The Hilbert space of the two-qubit system is a four-dimensional complex vector space. The states are labeled as in table 1.3.

State	Label	Coefficient
$\|00\rangle$	00	1
$\|01\rangle$	01	0
$\|10\rangle$	10	0
$\|11\rangle$	11	0

Table 1.3: States of a two-qubit system.

Example: Apply a Hadamard gate to qubit 0 and nothing to qubit 1. The matrix representation of the quantum circuit is obtained by multiplying the matrix of the Hadamard gate H, acting on qubit 0, with the identity matrix acting on qubit 1:

$$|q_0'q_1'\rangle = H \otimes I\,|q_0q_1\rangle = H \otimes I\,|00\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \tag{1.45}$$

The corresponding quantum circuit is shown in Figure 1.12.

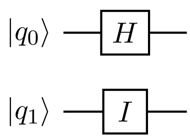


Figure 1.12: Quantum circuit with Hadamard gate applied to qubit 0.

The corresponding table of the Hilbert vector space is shown in table 1.4.

State	**Label**	**Coefficient**
$\|00\rangle$	00	$\frac{1}{\sqrt{2}}$
$\|01\rangle$	01	0
$\|10\rangle$	10	$\frac{1}{\sqrt{2}}$
$\|11\rangle$	11	0

Table 1.4: States of a two-qubit system after applying the Hadamard gate to qubit 0.

Normalization: The coefficients of the states in the Hilbert space are normalized to ensure that the sum of the probabilities of all possible states is equal to 1. The normalization condition is given by:

$$\sum_i |c_i|^2 = 1 \tag{1.46}$$

1.4.4 Quantum Measurements

Quantum measurements are used to extract the classical information from the quantum system. The measurement operation collapses the superposition of the qubits into a classical state. The measurement results are stored in the classical register.

Example: Consider a two-qubit quantum circuit with the following quantum gates applied to the qubits:

- Hadamard gate H applied to qubit 0.
- Pauli-X gate X applied to qubit 1.

The circuit representation of the quantum circuit is shown in Figure 1.13.

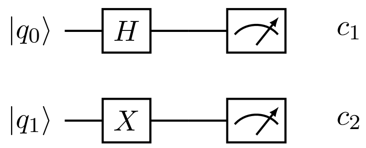


Figure 1.13: Quantum circuit with quantum measurements.

Hilbert space of the two-qubit system is a four-dimensional complex vector space. The states are labeled as in table 1.7.

State	**Label**	**Coefficient**
$\|00\rangle$	00	$\frac{1}{\sqrt{2}}$
$\|01\rangle$	01	0
$\|10\rangle$	10	$\frac{1}{\sqrt{2}}$
$\|11\rangle$	11	0

Table 1.5: States of a two-qubit system with quantum measurements.

The measurement operation collapses the superposition of the qubits into a classical state. The measurement results are stored in the classical register. An instance of the measurement results of the qubits are stored in the classical register as shown in table 1.6.

Qubit	**Measurement Result**
q_0	0
q_1	1

Table 1.6: Quantum measurement results.

Theoretically, or in the ideal case, the measurement results are deterministic. However, in practice, the measurement results are probabilistic due to the noise and errors in the quantum system.

The probability of obtaining a measurement result is given by the Born rule:

$$P(\text{measurement result}) = |c_i|^2 \tag{1.47}$$

The probability to find the qubits q_0q_1 in the state $|00\rangle$ is given by:

$$P(|00\rangle) = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \tag{1.48}$$

The probability to find the qubits q_0q_1 in the state $|01\rangle$ is given by:

$$P(|01\rangle) = |0|^2 = 0 \tag{1.49}$$

The probability to find the qubits q_0q_1 in the state $|10\rangle$ is given by:

$$P(|10\rangle) = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \tag{1.50}$$

The probability to find the qubits q_0q_1 in the state $|11\rangle$ is given by:

$$P(|11\rangle) = |0|^2 = 0 \tag{1.51}$$

Partial Measurements

Sometimes, we are interested in measuring only one of the qubits in a multi-qubit system. This is called a partial measurement. The partial measurement collapses the superposition of the qubits into a classical state, and the measurement results are stored in the classical register.

Measuring qubit q_0:

- The measurement operation collapses the superposition of qubit q_0 into a classical state.
- However, the measurement results of qubit q_1 are not affected.

The measurement results of qubit q_0 are stored in the classical register.

Example: Assume we apply Hadamard gate to both qubits. The circuit representation is:

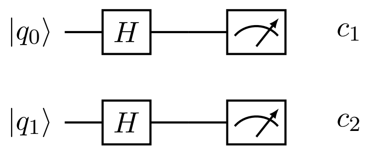


Figure 1.14: Quantum circuit with partial measurements.

The Hilbert space of the two-qubit system is a four-dimensional complex vector space:

$$
\begin{aligned}
|q_1'q_2'\rangle = H \otimes H\,|q_1q_2\rangle = H \otimes H\,|00\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)
\end{aligned}
$$

The states are labeled as in table 1.7.

State	**Label**	**Coefficient**
$\|00\rangle$	00	$\frac{1}{2}$
$\|01\rangle$	01	$\frac{1}{2}$
$\|10\rangle$	10	$\frac{1}{2}$
$\|11\rangle$	11	$\frac{1}{2}$

Table 1.7: States of a two-qubit system with partial measurements.

Measuring qubit q_0: Collapse of the superposition of qubit q_0 into a classical state:

$$\hat{P}_{q_0}(0) \otimes I \, |q_0 q_1\rangle = \hat{P}_{q_0}(0) \otimes I \left(\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle \right) \tag{1.52}$$

- **Assume the measurement result of qubit q_0 is 0**: Hence, only the states $|00\rangle$ and $|01\rangle$ are possible. The new quantum state of the system is:

$$|(q_0' = 0)\ q_1'\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |01\rangle \tag{1.53}$$

-

- **Assume the measurement result of qubit q_0 is 1**: Hence, only the states $|10\rangle$ and $|11\rangle$ are possible. The new quantum state of the system is:

$$|(q_0' = 1)\ q_1'\rangle = \frac{1}{\sqrt{2}} |10\rangle + \frac{1}{\sqrt{2}} |11\rangle \tag{1.54}$$

<u>Be mindful of the renormalization of the state vector after the measurement.</u>

Measuring qubit q_1: Collapse of the superposition of qubit q_1 into a classical state:

$$I \otimes \hat{P}_{q_1}(1) \, |q_0 q_1\rangle = I \otimes \hat{P}_{q_1}(1) \left(\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle \right) \tag{1.55}$$

- **Assume the measurement result of qubit q_1 is 0**: Hence, only the states $|00\rangle$ and $|10\rangle$ are possible. The new quantum state of the system is:

$$|q_0'\ (q_1' = 0)\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |10\rangle \tag{1.56}$$

-

- **Assume the measurement result of qubit q_1 is 1**: Hence, only the states $|01\rangle$ and $|11\rangle$ are possible. The new quantum state of the system is:

$$|q_0'\ (q_1' = 1)\rangle = \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |11\rangle \tag{1.57}$$

<u>Be mindful of the renormalization of the state vector after the measurement.</u>

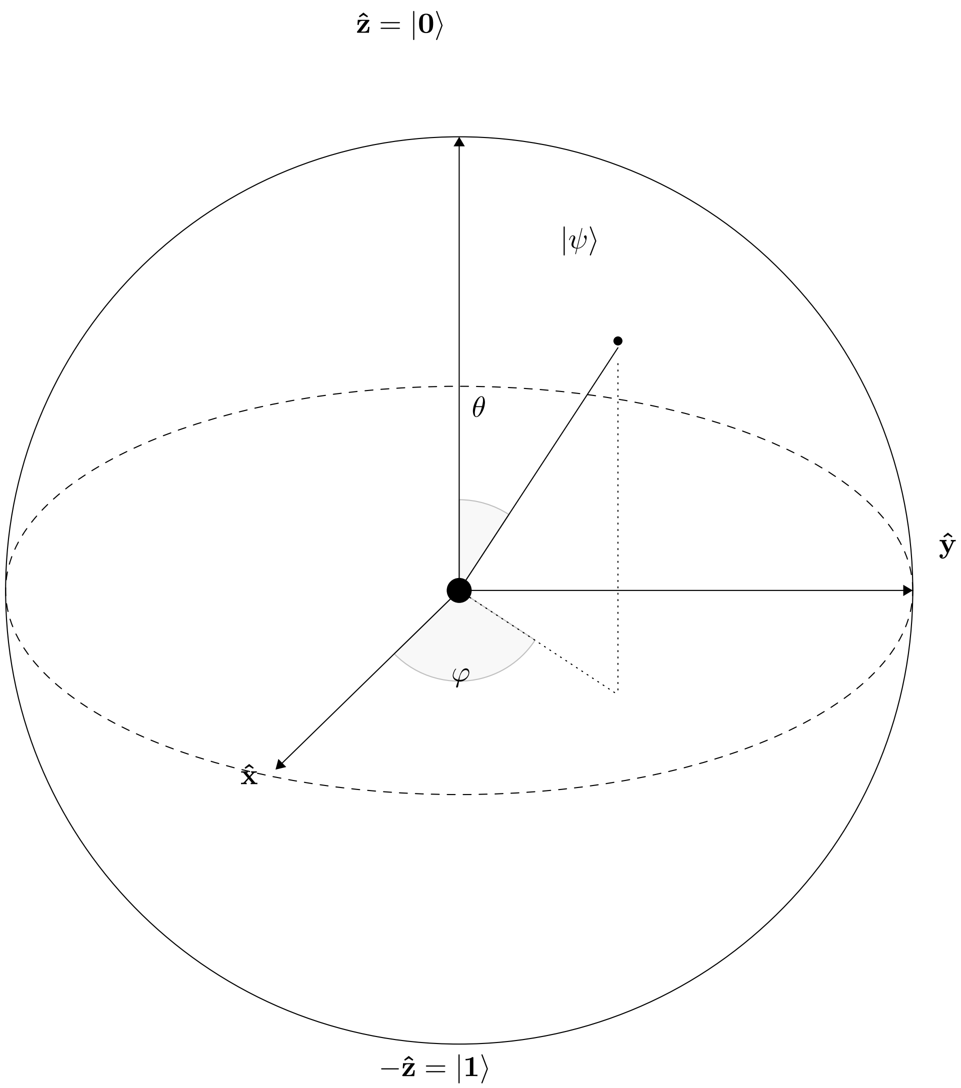
1.4.5 Block Sphere: Representing the quantum state of a qubit

The state of a qubit can be represented by a point on the Blochsphere. The Bloch sphere representation of a qubit state $|q\rangle$ is given by the Bloch vector $\vec{r} = (x, y, z)$, where θ is the polar angle and ϕ is the azimuthal angle.

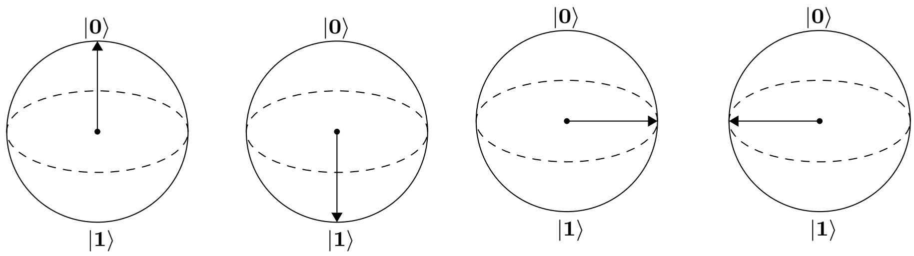
It is convenient to represent the state of a qubit in terms of the polar and azimuthal angles θ and ϕ:

$$|q\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{1.58}$$

The Bloch sphere representation of a qubit state $|q\rangle$ is given by the Bloch vector $\vec{r} = (x, y, z)$, where θ is the polar angle and ϕ is the azimuthal angle.



The state of a qubit can be represented by a point on the Blochsphere. This point is defined by the spherical coordinates (θ, ϕ).
It also gives us an idea about the superposition of the qubit.



$|\psi\rangle = |0\rangle$ $\quad$ $|\psi\rangle = |1\rangle$ $\quad$ $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$ $\quad$ $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$

1.5 Quantum Entanglement

- **Quantum entanglement** is a phenomenon where two or more qubits are correlated in such a way that the state of one qubit is dependent on the state of another qubit.
- **Example:** The Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state.
- **Entanglement** is a key resource in quantum computing and quantum communication.
- **Entanglement** is a fundamental property of quantum mechanics that has no classical counterpart.

Assume a quantum computer of two qubits A and B, their state will be:

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \tag{1.59}$$

Now, let's make a trace table for the states of qubits A and B in $|\psi\rangle$ as shown in table 1.8. We have $2^2 = 4$ possible states for the two qubits A and B.

Qubit A	Qubit B	State $\|\psi\rangle$
$\|0\rangle$	$\|0\rangle$	$\frac{1}{2}\|00\rangle$
$\|0\rangle$	$\|1\rangle$	$\frac{1}{2}\|01\rangle$
$\|1\rangle$	$\|0\rangle$	$\frac{1}{2}\|10\rangle$
$\|1\rangle$	$\|1\rangle$	$\frac{1}{2}\|11\rangle$

Table 1.8: States of qubits A and B in the state $|\psi\rangle$.

Let's have a look on the states of qubit B when A is in the state $|0\rangle$ as shown in table 1.9. When qubit A is in the state $|1\rangle$, qubit B is in the states $|0\rangle$ and $|1\rangle$.

Qubit A	Qubit B	State $\|\psi\rangle$
$\|0\rangle$	$\|0\rangle$	$\frac{1}{2}\|00\rangle$
$\|0\rangle$	$\|1\rangle$	$\frac{1}{2}\|01\rangle$
$\|1\rangle$	$\|0\rangle$	$\frac{1}{2}\|10\rangle$
$\|1\rangle$	$\|1\rangle$	$\frac{1}{2}\|11\rangle$

Table 1.9: States of qubit B when qubit A is in the state $|0\rangle$.

Let's have a look on the states of qubit B when A is in the state $|1\rangle$.
When qubit A is in the state $|1\rangle$, qubit B is in the states $|0\rangle$ and $|1\rangle$ as shown in table 1.11.

Qubit A	Qubit B	State $\|\psi\rangle$
$\|0\rangle$	$\|0\rangle$	$\frac{1}{2}\|00\rangle$
$\|0\rangle$	$\|1\rangle$	$\frac{1}{2}\|01\rangle$
$\|1\rangle$	$\|0\rangle$	$\frac{1}{2}\|10\rangle$
$\|1\rangle$	$\|1\rangle$	$\frac{1}{2}\|11\rangle$

Table 1.10: States of qubit B when qubit A is in the state $|1\rangle$.

Let's have a look on the states of qubit B when A is in the state $|1\rangle$.
When qubit A is in the state $|1\rangle$, qubit B is in the states $|0\rangle$ and $|1\rangle$ as shown in table 1.11.

Qubit A	Qubit B	State $\|\psi\rangle$
$\|0\rangle$	$\|0\rangle$	$\frac{1}{2}\|00\rangle$
$\|0\rangle$	$\|1\rangle$	$\frac{1}{2}\|01\rangle$
$\|1\rangle$	$\|0\rangle$	$\frac{1}{2}\|10\rangle$
$\|1\rangle$	$\|1\rangle$	$\frac{1}{2}\|11\rangle$

Table 1.11: States of qubit B when qubit A is in the state $|1\rangle$.

Quantum Entanglement:

Let's trace the action of H gate on qubit A and CNOT gate on qubit A and B, where A is the control qubit and B is the target qubit. The circuit representation of the quantum circuit is shown in Figure 1.15.

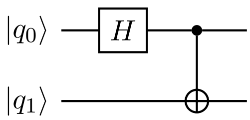


Figure 1.15: Quantum circuit with quantum entanglement.

mathematically,

- First, we act with H gate on qubit A, leaving qubit B unchanged:

$$|\psi_1\rangle = H \otimes I\,|00\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) \tag{1.60}$$

- Second, we act with $CNOT$ gate on qubits A and B:

$$|\psi_2\rangle = CNOT\,|\psi_1\rangle = CNOT\left(\frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \tag{1.61}$$

Depending on the input states of qubits A and B, the output states of qubits A and B are given in table 1.12.

Qubit A	**Qubit B**	**State** $\|\psi\rangle$
$\|0\rangle$	$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
$\|0\rangle$	$\|1\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle)$
$\|1\rangle$	$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$
$\|1\rangle$	$\|1\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$

Table 1.12: States of qubits A and B in the entangled state $|\psi\rangle$.

The states:

- $|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

- $|B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

- $|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

- $|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

In matrix representation:

- The action of the Hadamard gate H on qubit A is given by:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1.62}$$

$$\begin{aligned} |\psi_1\rangle &= H \otimes I \, |00\rangle \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) \end{aligned} \tag{1.63}$$

- Second, the action of the CNOT gate on qubits A and B is given by:

$$|\psi_2\rangle = CNOT \, |\psi_1\rangle = CNOT \left(\frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) \right) = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \tag{1.64}$$

  In matrix representation:

$$|\psi_2\rangle = CNOT \, |\psi_1\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \tag{1.65}$$