Boolean Quantum Circuit

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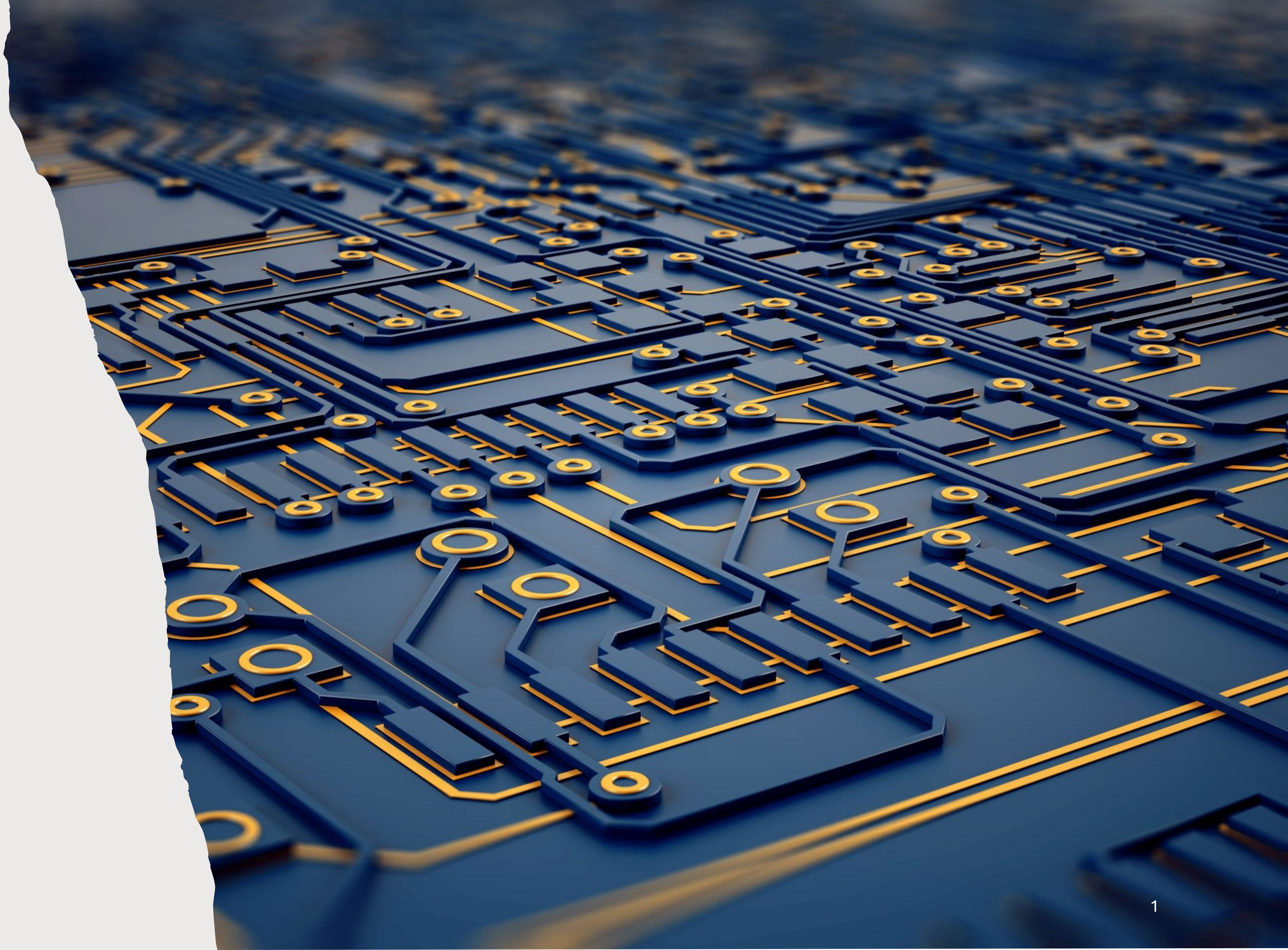


Boolean Function

Boolean Functions:

$$f\colon \{0,1\}^n \rightarrow \{0,1\}$$

e.g.:

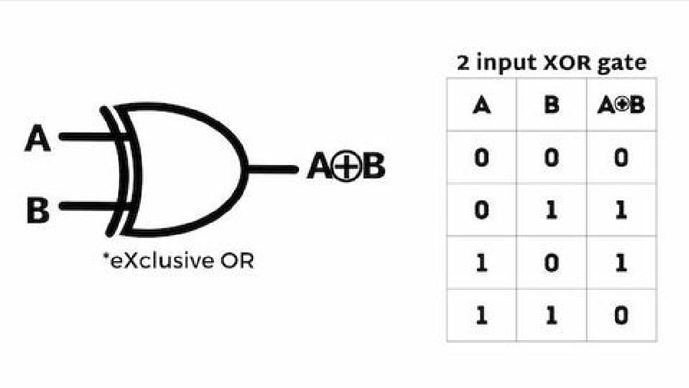
$$f(x_0, x_1, x_2) = \bar{x}_0 x_1 + x_0 x_2$$

Aim: Implement a Quantum Circuit for the given Boolean function.

Properties of XOR:

- $x \oplus 0 = x$
- $x \oplus 1 = \bar{x}$
- $x \oplus x = 0$

XOR Truth Table



A	B	A⊕B
0	0	0
0	1	1
1	0	1
1	1	0

x_0	x_1	x_2	$f(x0, x1, x2)$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

CNOT & Toffoli gate (CCNOT);

Properties of XOR:

- $x \oplus 0 = x$
- $x \oplus 1 = \bar{x}$
- $x \oplus x = 0$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \& \ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

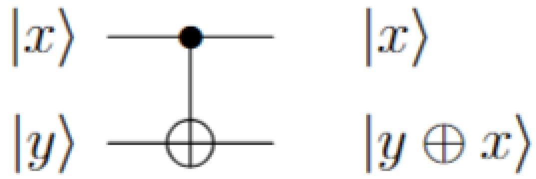
- $U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$;

$|11\rangle \xrightarrow{maps} |10\rangle$
$|10\rangle \xrightarrow{maps} |11\rangle$
Otherwise do nothing.

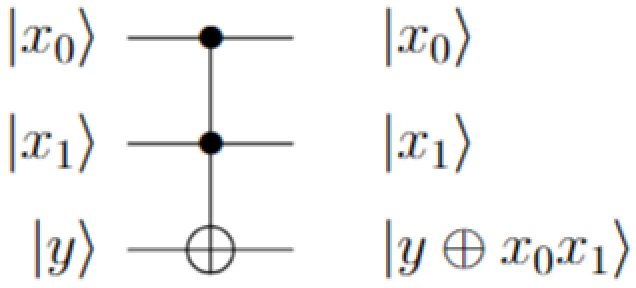
- $U_{CCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

$|111\rangle \xrightarrow{maps} |110\rangle$
$|110\rangle \xrightarrow{maps} |111\rangle$
Otherwise do nothing.

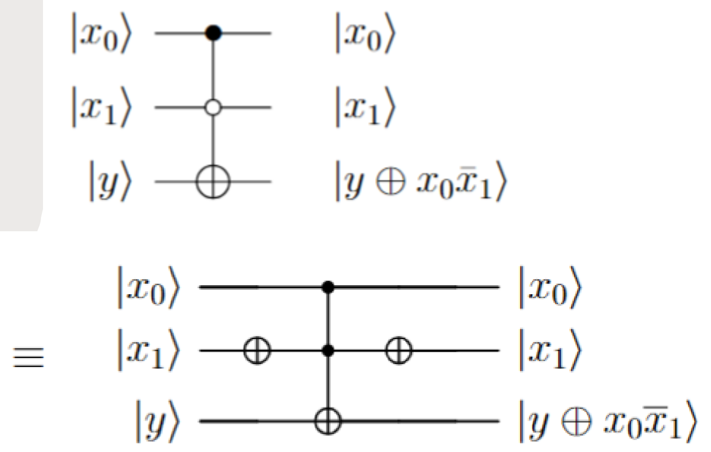
CNOT Gate



Toffoli Gate



Generalized Toffoli Gate



Classical Logic gates:

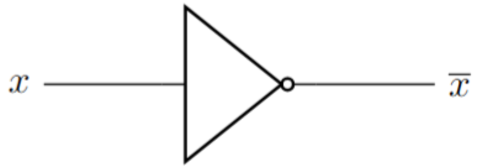
- Any Boolean function $f: \{0,1\}^n \rightarrow \{0,1\}$ can be implemented classically in a logic gate circuit using the universal gate set (AND, OR, NOT).

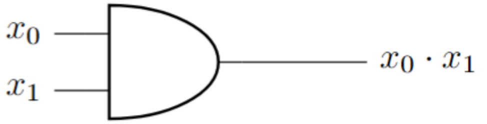
- NAND gate is a universal classical gate, currently used in processors.

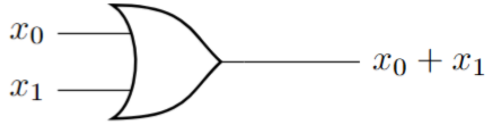
x	$\bar{x}$
0	1
1	0

x_0	x_1	$x_0 . x_1$
0	0	0
0	1	0
1	0	0
1	1	1

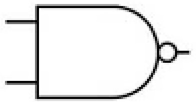
x_0	x_1	$x_0 + x_1$
0	0	0
0	1	1
1	0	1
1	1	1







NAND

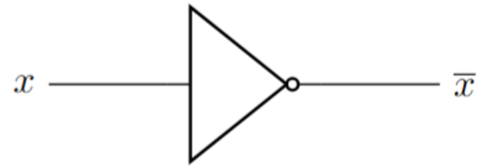


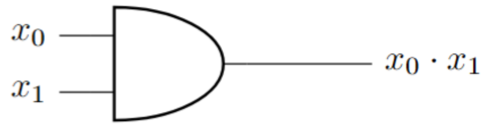
A	B	Output
0	0	1
1	0	1
0	1	1
1	1	0

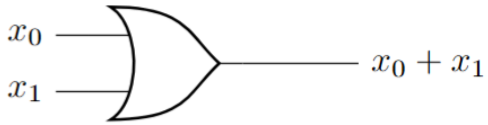
Reversible Computing

- By looking at the output column of the tables of the $\boldsymbol{AND}$ and $\boldsymbol{OR}$ gates, we can not guess what the input is. We can say that the information or the entropy is lost by applying those gates and those operations are called **irreversible.**
  - Irreversible computation dissipates heat to the environment.

- On the other hand, this is not the case for the $\boldsymbol{NOT}$ gate as the input can be constructed by looking at the output. Such gates are called reversible and a computation which consists of only reversible operations is called a **reversible computation**.

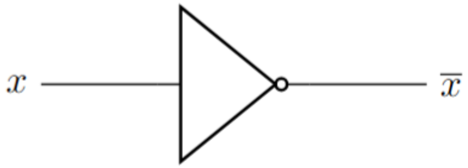


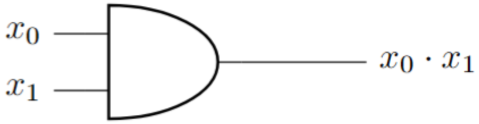


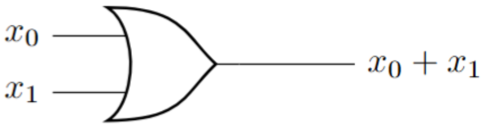


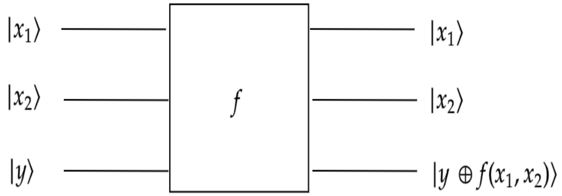
Reversible Computing

- A set of gates is called **universal** if it is possible to implement any other gate using the gates in the set. Theoretically, it is possible to build a universal computer that only uses reversible gates. For instance, $\boldsymbol{AND}$ and $\boldsymbol{NOT}$ gates or the Toffoli gate ($\boldsymbol{CCNOT}$) itself are universal sets of gates for classical computing.
  - ➢ Note that since $\boldsymbol{CCNOT}$ is also a quantum gate, we conclude that a quantum computer can simulate any classical operation.
- Since quantum computing is reversible according to the laws of physics $\boldsymbol{AND}$ and $\boldsymbol{NOT}$ gates should be implemented in a reversible way as well. The idea is to create a 3-qubit circuit, which does not modify the input bits and writes the output to the third bit. When the output bit is set to 0, then you exactly get the same output.



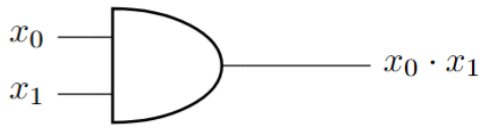




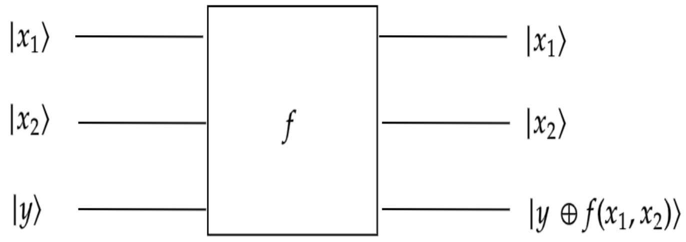


Example:

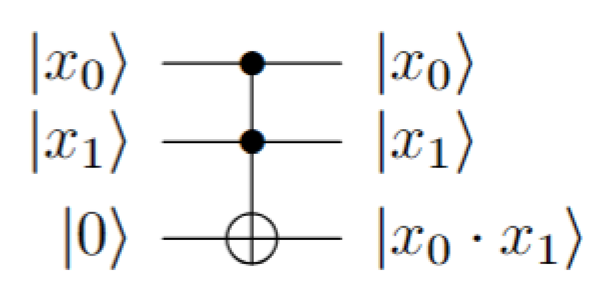
Complete the following table that corresponds to reversible AND gate, where $|x_1\rangle$ and $|x_2\rangle$ are the inputs of the AND gate and the $|y\rangle$ = $|0\rangle$ is the output.



Which three-qubit quantum gate can we use to implement the AND operator in a reversible manner?



IN			OUT		
x_0	x_1	y	x_0	x_1	$x_0 . x_1$
0	0	0	0	0	0
0	1	0	0	1	0
1	0	0	1	0	0
1	1	0	1	1	1



We can use CCNOT (Toffoli) gate.