Quantum AI Lab

A collobration between iQafé and Bibliotheca Alexandrina
Quiz 1, Hackathon Warm-up

Available Time: 30 minutes

Permitted Materials: Simple calculator, graphic and advanced calculators are not permitted

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Question 1 Find the complex conjugate transpose of the following matrices:

$$\begin{pmatrix} 1 & 2 & -3 \\ 4i & -5i & 6 \\ 7 & 8 & 9i \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 3+1i & 2-2i & 1+3i \\ 4-2i & 5+1i & 6-3i \\ 7+1i & 8-2i & 9+3i \end{pmatrix} \tag{2}$$

Answer

First, for the given matrix

$$\begin{pmatrix} 1 & 2 & -3 \\ 4i & -5i & 6 \\ 7 & 8 & 9i \end{pmatrix}$$

the transpose of the matrix is:

$$\begin{pmatrix} 1 & 4i & 7 \\ 2 & -5i & 8 \\ -3 & 6 & 9i \end{pmatrix}$$

and the complex conjugate of the transposed matrix is:

$$\begin{pmatrix} 1 & -4i & 7 \\ 2 & 5i & 8 \\ -3 & 6 & -9i \end{pmatrix}$$

Hence the final answer is

$$\begin{pmatrix} 1 & -4i & 7 \\ 2 & 5i & 8 \\ -3 & 6 & -9i \end{pmatrix}.$$

Second, for the given matrix

$$\begin{pmatrix} 3+1i & 2-2i & 1+3i \\ 4-2i & 5+1i & 6-3i \\ 7+1i & 8-2i & 9+3i \end{pmatrix}$$

the transpose of the matrix is:

$$\begin{pmatrix} 3+1i & 4-2i & 7+1i \\ 2-2i & 5+1i & 8-2i \\ 1+3i & 6-3i & 9+3i \end{pmatrix}$$

and the complex conjugate of the transposed matrix is:

$$\begin{pmatrix} 3-1i & 4+2i & 7-1i \\ 2+2i & 5-1i & 8+2i \\ 1-3i & 6+3i & 9-3i \end{pmatrix}$$

Question 2 Which of the following expressions represent the state $|q_0q_1\rangle$ of the two qubits $|q_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|q_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$?

1. $|q_0q_1\rangle = \begin{pmatrix}1\\0\\1\\1\end{pmatrix}$.

2. $|q_0q_1\rangle = \begin{pmatrix}1\\0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$.

3. $|q_0q_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$.

4. $|q_0q_1\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\\0\end{pmatrix}$.

Answer

$$|q_0q_1\rangle = |q_0\rangle \otimes |q_1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = |q_0q_1\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1.\begin{pmatrix}1\\1\end{pmatrix}\\0.\begin{pmatrix}1\\1\end{pmatrix}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\\0\end{pmatrix}.$$

Question 3 Which of the following expressions represent quantum state(s), you can choose more than one if applicable:

1. $|\psi\rangle = |0\rangle$.
2. $|\psi\rangle = |1\rangle$.
3. $|\psi\rangle = |0\rangle + |1\rangle$.
4. $|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$.
5. $|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i\,|1\rangle]$.

Answer

For $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$, they are quantum states. The reason is, quantum states should be normalized.

The expression $|\psi\rangle = |0\rangle + |1\rangle$ does not represent a quantum state since it is not normalized.

The expression $|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$ is a quantum state since it is normalized.

Similarly, the expression $|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i\,|1\rangle]$ is a quantum state since it is normalized.

Question 4 Which of the following matrices are unitary: Note: multiple answers are possible.

1. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
2. $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.
3. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
4. $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.
5. $\begin{pmatrix} 1 & 0 \\ 0 & \frac{i}{\sqrt{2}} \end{pmatrix}$.

Answer

General rule: A matrix is unitary if its conjugate transpose multiplied by the matrix itself gives an identity matrix:

$$U^{\dagger}U = I. \tag{3}$$

By applying the rule to the given matrices, we can determine which of them are unitary.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a unitary matrix.

The matrix $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ is a unitary matrix.

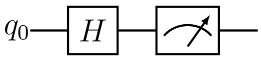
The matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a unitary matrix.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & i/\sqrt{2} \end{pmatrix}$ is not a unitary matrix.

Question 5 Let's find the state of the qubit after applying the Hadamard gate as shown in the figure.

The Hadamard gate is defined as:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{4}$$



First, apply the Hadamard gate to the qubit state $|\psi\rangle = |0\rangle$ and determine the new state of the qubit.

Answer

The Hadamard gate is applied to the qubit state $|\psi\rangle = |0\rangle$ as follows:

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \tag{5}$$

Hence, the new state of the qubit is $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Second, apply the Hadamard gate to the qubit state $|\psi\rangle = |1\rangle$ and determine the new state of the qubit.

Answer

The Hadamard gate is applied to the qubit state $|\psi\rangle = |1\rangle$ as follows:

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{6}$$

Hence, the new state of the qubit is $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Third, what is the probability of measuring the qubit in the state $|0\rangle$ after applying the Hadamard gate to the qubit state $|\psi\rangle = |1\rangle$?

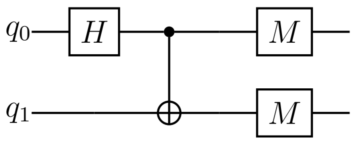
Answer

The probability of measuring the qubit in the state $|0\rangle$ after applying the Hadamard gate to the qubit state $|\psi\rangle = |1\rangle$ is given by:

$$P(|0\rangle) = |\langle 0|H|1\rangle|^2 = \left|\frac{1}{\sqrt{2}}\langle 0|0\rangle - \frac{1}{\sqrt{2}}\langle 0|1\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}. \tag{7}$$

Hence, the probability of measuring the qubit in the state $|0\rangle$ is $\frac{1}{2}$.

Question 6 Let's explore the matrix representation of the following quantum circuit.



The two qubits are initialized in the state $|00\rangle$ and the Hadamard gate is applied to the first qubit, while the CNOT gate is applied to the two qubits where the first qubit is the control qubit and the second qubit is the target qubit. The Hadamard gate is defined as:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{8}$$

The CNOT gate is defined as:

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{9}$$

First, write the statevector of the two qubits after applying the Hadamard gate to the first qubit and show the mathematical steps.

Answer

First the state, $|q_0 q_1\rangle = |00\rangle$, is represented as a column vector which can be calcalated using the tensor product of the two qubits:

$$|q_0 q_1\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

The statevector of the two qubits after applying the Hadamard gate to the first qubit is given by:

$$\begin{aligned} |q_0' q_1'\rangle &= H \otimes I \, |00\rangle \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|00\rangle + |10\rangle \right). \end{aligned}$$

We can label the coefficients of the statevector as follows:

Coefficient	State
c_0	$\|00\rangle$
c_1	$\|01\rangle$
c_2	$\|10\rangle$
c_3	$\|11\rangle$

or

Coefficient	State
$\frac{1}{\sqrt{2}}$	$\|00\rangle$
0	$\|01\rangle$
$\frac{1}{\sqrt{2}}$	$\|10\rangle$
0	$\|11\rangle$

Second, apply the CNOT gate to the two qubits and show the mathematical steps.

Answer

The CNOT gate is applied to the two qubits as follows:

$$CNOT\,|q_0'q_1'\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{11}$$

Hence, the statevector of the two qubits after applying the CNOT gate is

$$|q_0''q_1''\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{12}$$

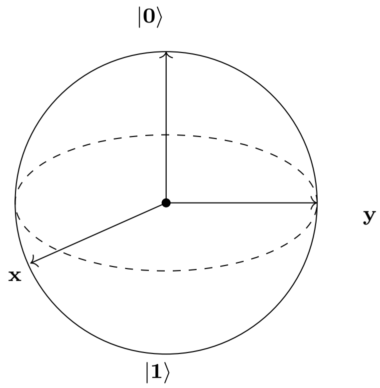
The coefficients of the statevector are:

Coefficient	State
c_0	$\|00\rangle$
c_1	$\|01\rangle$
c_2	$\|10\rangle$
c_3	$\|11\rangle$

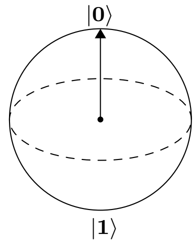
or

Coefficient	State
$\frac{1}{\sqrt{2}}$	$\|00\rangle$
0	$\|01\rangle$
0	$\|10\rangle$
$\frac{1}{\sqrt{2}}$	$\|11\rangle$

Question 7 Let's find the corresponding quantum state to a given Bloch sphere representation. The bloch sphere representation of a qubit is a geometrical representation of the state of a qubit. The state of a qubit can be represented as a point on the bloch sphere as shown in the figure below.



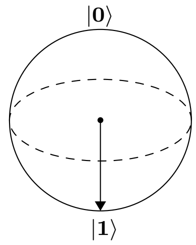
Write below each Bloch sphere representation the corresponding quantum state in the computational basis. Assume the y-axis is horizontal, the z-axis is vertical, and the x-axis is perpendicular to the paper.



$|\psi\rangle =?$

Answer

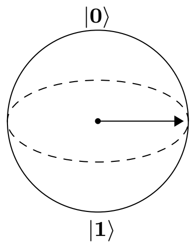
The given Bloch sphere representation corresponds to the quantum state $|\psi\rangle = |0\rangle$.



$|\psi\rangle =?$

Answer

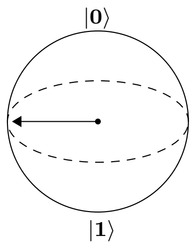
The given Bloch sphere representation corresponds to the quantum state $|\psi\rangle = |1\rangle$.



$|\psi\rangle = ?$

Answer

The given Bloch sphere representation corresponds to the quantum state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$.



$|\psi\rangle = ?$

Answer

The given Bloch sphere representation corresponds to the quantum state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$.

——— *End of Examination* ———