MATH 141 EXAM 1 - UOWD AUTUMN 2018

THE SOLUTION

$(12pts)$ **Problem 1**
Evaluate the following limits

$$(a)\quad \lim_{x\to -1}\frac{x^2-4}{3x-2} \qquad\qquad (b)\quad \lim_{x\to -\infty}\frac{3\left|x^3\right|+x+1}{1+2x^3}$$

$$(c)\quad \lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x} \qquad\qquad (d)\quad \lim_{x\to 0^+}\frac{\ln x^2}{x^2}$$

Solution

(a)

$$\lim_{x\to -1}\frac{x^2-4}{3x-2} = \frac{1-4}{-3-2} = \frac{3}{5} = 0.6 \qquad (3pts)$$

(b)

$$\lim_{x\to -\infty}\frac{3\left|x^3\right|+x+1}{1+2x^3} = \lim_{x\to -\infty}\frac{-3x^3+x+1}{1+2x^3} = \frac{-3}{2} = -1.5 \qquad (3pts)$$

(c)

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x} &= \lim_{x\to 0}\frac{\left(\sqrt{2+x}-\sqrt{2}\right)\left(\sqrt{2+x}+\sqrt{2}\right)}{x\left(\sqrt{2+x}+\sqrt{2}\right)} \\
&= \lim_{x\to 0}\frac{\cancel{2}+x-\cancel{2}}{x\left(\sqrt{2+x}+\sqrt{2}\right)} = \lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}\left(\sqrt{2+x}+\sqrt{2}\right)} \\
&= \lim_{x\to 0}\frac{1}{\left(\sqrt{2+x}+\sqrt{2}\right)} = \frac{1}{2\sqrt{2}} = 0.353\,55 \qquad (3pts)
\end{aligned}$$

(d)

$$\lim_{x\to 0^+}\frac{\ln x^2}{x^2} = \lim_{x\to 0^+}\frac{1}{x^2}\cdot\ln x^2 = (+\infty)(-\infty) = -\infty \qquad (3pts)$$

$(9pts)$ **Problem 2**
Consider the function

$$g(x) = \frac{x^2 - 1}{|x| - 1}.$$

Evaluate the following limits
(a) $\lim_{x \to 1^-} g(x)$ (b) $\lim_{x \to 1^+} g(x)$
(c) $\lim_{x \to 1} g(x)$

Solution
(a)

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \frac{x^2 - 1}{|x| - 1} = \lim_{x \to 1^-} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1^-} \frac{x+1}{1} = 2 \qquad (3pts)$$

(b)

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \frac{x^2 - 1}{|x| - 1} = \lim_{x \to 1^+} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1^+} \frac{x+1}{1} = 2 \qquad (3pts)$$

(c)

$$\lim_{x \to 1} g(x) = 2 \qquad (3pts)$$

($9pts$) **Problem 3**

For what value (s) of the constant a is the function f continuous at $x = 3$.

$$f(x) = \begin{cases} 5ax^2 + x, & \text{if } x < 3 \\ x - 1, & \text{if } x \geq 3 \end{cases}$$

Solution

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \left(5ax^2 + x\right) = 45a + 3 \qquad (2pts)$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x - 1) = 2 \qquad (2pts)$$

For the function to be continuous at $x = 3$, we must have

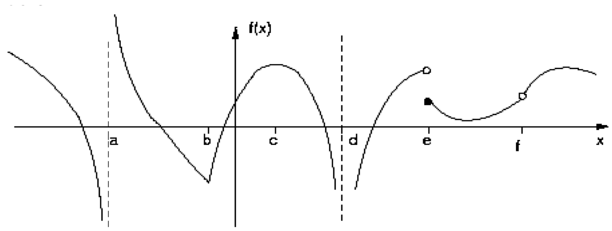
$$\begin{aligned} \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^+} f(x). \text{ i.e.} \\ 45a + 3 &= 2 \qquad (3pts) \end{aligned}$$

Solving for a, we get

$$a = \frac{-1}{45} \qquad (2pts)$$

(10*pts*)**Problem 4**

Consider the function $f(x)$ graphed below.



(a) Find the points where the function is discontinuous and classify the discontinuities as removable, jump or infinite.

(b) Find the point(s) where the function is NOT differentiable.

Solution

(a)

f is discontinuous at a, d, e and f.	(4pts).
Infinite discontinuity at a and d.	(1pt)
Jump discontinuity at e	(1pt)
Removable discontinuity at f	(1pt)

(b) The function is NOT differentiable at a, b, d, e and f. (3pts)

$(16pts)$ **Problem 5**

Find $\dfrac{dy}{dx}$ for

$$(a) \quad y = (1+2x)(1+\sin x) \qquad (b) \quad y = \ln\sqrt[3]{3x-1}$$

$$(c)\ y = e^{x^2+3x-\frac{1}{x}} \qquad (d)\ y = \frac{4x+3}{5x-1}$$

(a) $y = (1+2x)(1+\sin x)$

$$\begin{aligned} \frac{dy}{dx} &= (2)(1+\sin x) + (1+2x)\cos x \\ &= \cos x + 2\sin x + 2x\cos x + 2 \qquad (4pts) \end{aligned}$$

(b)

$$y = \ln\sqrt[3]{3x-1} = \frac{1}{3}\ln(3x-1)$$

$$\frac{dy}{dx} = \frac{1}{3}\frac{3}{3x-1} = \frac{1}{3x-1} \qquad (4pts)$$

(c) $y = e^{x^2+3x-\frac{1}{x}}$

$$\frac{dy}{dx} = \left(2x+3+\frac{1}{x^2}\right)e^{x^2+3x-\frac{1}{x}} \qquad (4pts)$$

(d) $y = \dfrac{4x+3}{5x-1}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{4(5x-1)-5(4x+3)}{(5x-1)^2} \\ &= \frac{19}{(5x-1)^2} \qquad (4pts) \end{aligned}$$

$(12pts)$ **Problem 6**
Find the critical numbers and the local extrema of

$$f(x) = 6x^5 + 33x^4 - 30x^3 + 100.$$

Solution

$$\begin{aligned} f\,'(x) &= 30x^4 + 132x^3 - 90x^2 \\ &= 6x^2\,(x+5)\,(5x-3) \qquad\qquad (2pts) \end{aligned}$$

The critical numbers are

$$0, \quad -5, \quad \frac{3}{5} \qquad\qquad (3pts)$$

$$f\,''(x) = 120x^3 + 396x^2 - 180x \quad \text{or Table Variation} \qquad (3pts)$$

$$f\,''(0) = 0$$

$$\begin{aligned} f\,''(-5) &= 120(-5)^3 + 396(-5)^2 - 180(-5) \\ &= -4200 < 0 \end{aligned}$$

$$\begin{aligned} f(-5) &= 6\,(-5)^5 + 33\,(-5)^4 - 30\,(-5)^3 + 100 \\ &= 5725 \end{aligned}$$

is a **local maximum.** $(2pts)$

$$\begin{aligned} f\,''(\frac{3}{5}) &= 120(\frac{3}{5})^3 + 396(\frac{3}{5})^2 - 180(\frac{3}{5}) \\ &= \frac{1512}{25} > 0. \end{aligned}$$

$$\begin{aligned} f(\frac{3}{5}) &= 6\left(\frac{3}{5}\right)^5 + 33\left(\frac{3}{5}\right)^4 - 30\left(\frac{3}{5}\right)^3 + 100 \\ &= \frac{307\,073}{3125} = 98.263 \end{aligned}$$

is a **local minimum.** $(2pts)$

$(6pts)$**Problem 7**

If $G(x) = f(g(x))$ where $f(-1)= 4,\ f\,'(-1) = 3,\ f\,'(-2) = 3,\ \ g(2) = -2,\ g'(2) = \dfrac{1}{2}$. $G'(2)$ is equal to

(a) $\quad 9$

(b) $\quad -6$

(c) $\quad \dfrac{3}{2}$

(d) $\quad -4$

(e) $\quad \dfrac{-5}{2}$

Solution

$$G'(x) = g'(x) \cdot f\,'(g(x))$$

$$\begin{aligned} G'(2) &= g'(2) \cdot f\,'(g(2)) \\ &= \frac{1}{2} f\,'(-2) \\ &= \frac{3}{2}. \end{aligned}$$

Answer is (c)

$(6pts)$**Problem 8**
An equation of the tangent line to the graph of

$$y = \tan x + 2\sin x + 2$$

at $x = 0$ is equal to

$$(a) \qquad y = \frac{x}{3} + 2$$

$$(b) \qquad y = -x + 2$$

$$(c) \qquad y = 3x + 2$$

$$(d) \qquad y = 2x + 3$$

$$(e) \qquad y = 6x + 2$$

Solution

$$\frac{dy}{dx} = \sec^2 x + 2\cos x$$

The slope of the tangent line is

$$m = \sec^2 0 + 2\cos 0 = 3$$

The equation of the tangent line is

$$\begin{aligned} y &= 3\,(x - 0) + 2 \\ &= 3x + 2 \end{aligned}$$

<u>Answer is (c)</u>

$(5pts)$**Problem 9**

The slope of the tangent line to the graph of

$$x^2 + y^2 = 9$$

at the point $\left(2, \sqrt{5}\right)$ is equal to

$$(a) \qquad 9\sqrt{5}$$

$$(b) \qquad \frac{-2\sqrt{5}}{9}$$

$$(c) \qquad \frac{\sqrt{5}}{2}$$

$$(d) \qquad 2\sqrt{5}$$

$$(e) \qquad \frac{-2\sqrt{5}}{5}$$

Solution

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

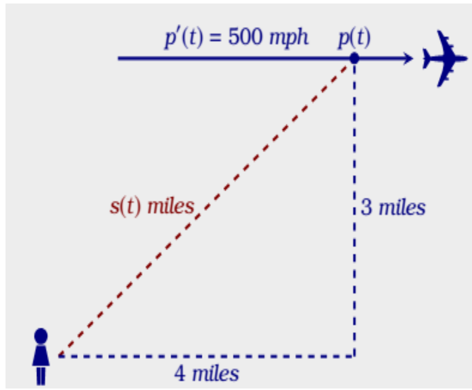
The slope at $\left(2, \sqrt{5}\right)$ is

$$m = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}.$$

Answer is (e).

($5pts$)**Problem 10**

A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?



(a) $\quad s'(t) = 500mph$

(b) $\quad s'(t) = 400mph$

(c) $\quad s'(t) = 300mph$

(d) $\quad s'(t) = 100mph$

(e) $\quad s'(t) = 200mph$

Solution

The related equation is

$$p^2 + 3^2 = s^2$$

The related rate equation is

$$2p'(t)p(t) = 2s'(t)s(t)$$

Now we'll evaluate the equation at the desired values. We are interested in the time at which $p(t) = 4$ and $p'(t) = 500$. Additionally, at this time we know that

$$4^2 + 9 = s^2,$$

so $s(t) = 5$.

Putting together all the information we get

$$2(4)(500) = 2(5)s'(t),$$

thus

$$s'(t) = 400mph.$$

Answer is (b)

($5pts$) **Problem 11**
Suppose that the amount of money in a bank account after t years is given by

$$A(t) = 2000 - 10te^{5-\frac{t^2}{8}}$$

The minimum amount of money in the account during the first 10 years.(i.e. on the interval $[0, 10]$) is equal to:

(a) 1999.94

(b) 199.66

(c) 200

(d) 190.6

(e) 1990

Solution
Here we are really asking for the absolute extrema of $A(t)$ on the interval [0,10].

$$A'(t) = \left(\frac{5}{2}t^2 - 10\right) e^{5-\frac{1}{8}t^2}$$

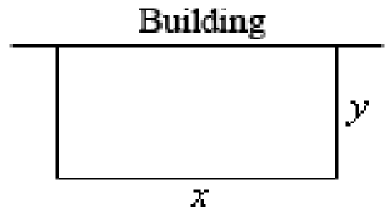
$$A'(t) = 0 \Leftrightarrow \left(\frac{5}{2}t^2 - 10\right) = 0 \Rightarrow t^2 = 4.$$

The critical numbers are 2 and -2. Because -2 is not in $[0, 10]$, the only critical number to be considered is $t = 2$.

$$A(0) = 2000, \qquad A(2) = 199.66 \quad \text{and} \quad A(10) = 1999.94.$$

So, the maximum amount in the account will be \$2000 which occurs at t = 0 and the minimum amount in the account will be \$199.66 which occurs at the 2 year mark.
Answer is (b)

(5*pts*) **Problem 12**
We need to enclose a field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing.



The largest possible area is equal to

$(a)\qquad A = 31250$

$(b)\qquad A = 5503$

$(c)\qquad A = 42730$

$(d)\qquad A = 2225$

$(e)\qquad A = 25000$

Solution
In this problem we want to maximize the area of a field and we know that will use 500 ft of fencing material. So, the area will be the function we are trying to optimize and the amount of fencing is the constraint. The two equations for these are,

$$\text{Maximize: } A = xy$$

$$\text{Subject to: } 500 = x + 2y$$

So, let's solve the constraint for x. Note that we could have just as easily solved for y but that would have led to fractions and so, in this case, solving for x will probably be best.

$$x = 500 - 2y$$

Substituting this into the area function gives a function of y.

$$\begin{aligned} A &= A(y) = y\,(500 - 2y) \\ &= 500y - 2y^2. \end{aligned}$$

Since the area is positive, the feasible domain is $[0, 250]$.
So let's get the derivative and find the critical points.

$$A'(y) = 500 - 4y$$

Setting this equal to zero and solving gives a lone critical point of $y = 125$.

$$A(0) = 0, \qquad A(125) = 31250, \qquad A(250) = 0.$$

The largest possible area is $A = 31250$. It occurs at $y = 125$, and $x = 500 - 2\,(125) = 250$. The dimensions of the field that will give the largest area, subject to the fact that we used exactly 500 ft of fencing material, are 250 x 125.

Answer is (a)