Math 141 Tutorial 8

University of Wollongong in Dubai (UOWD)

Problem 1

Use Gauss elimination method to solve the linear system

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0. \end{aligned}$$

Solution

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 2 & 1 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 2 & 1 & -17 \\ 0 & 3 & 1 & -27 \end{bmatrix}$$

Multiply the second row by $\dfrac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 1 & \frac{1}{2} & \frac{-17}{2} \\ 0 & 3 & 1 & -27 \end{bmatrix}$$

Add -3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 1 & \frac{1}{2} & \frac{-17}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{-3}{2} \end{bmatrix}$$

Multiply the third row by -2 to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 1 & \frac{1}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now doing the back substitution, we get

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 9 \\ x_2 + \tfrac{1}{2}x_3 &= \tfrac{-17}{2} \\ x_3 &= 3. \end{aligned}$$

Thus,

$$x_2 = \frac{-17}{2} - \frac{1}{2}(3) = -10 \text{ and } x_1 = 9 - (-10) + 2(3) = 25$$

$$x_1 = 25, \quad x_2 = -10 \text{ and } \quad x_3 = 3.$$

Problem 2.

Let

$$A = \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix},\ B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -6 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix}.$$

Perform the following calculations

1. $A + B$
2. $C + B^t$
3. $-2AC$

Solution

1.

$$\begin{aligned} A+B &= \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ -2 & -6 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 & -2 \\ 4 & -3 & 5 \end{bmatrix} \end{aligned}$$

2.

$$
\begin{aligned}
& C + B^t \\
= & \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & -6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 6 & -6 \\ -1 & 7 \end{bmatrix}
\end{aligned}
$$

3.

$$
\begin{aligned}
-2AC &= (-2)\begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -48 & 4 \\ -48 & -48 \end{bmatrix}
\end{aligned}
$$

Problem 3

Solve the given vector equation for x, or explain why no solution exists:

$$2\begin{bmatrix}1 & 2 & 3\\ 0 & 4 & 2\end{bmatrix} - 3\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & x\end{bmatrix} = \begin{bmatrix}-1 & 1 & 0\\ 0 & 5 & -2\end{bmatrix}.$$

Solution

$$
\begin{aligned}
& (2)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix} - (3)\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & x \end{bmatrix} \\
= \quad & \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}
\end{aligned}
$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & 4-3x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}$$

This equation is valid only if

$$4 - 3x = -2 \Longrightarrow x = 2.$$

Problem 4
Show that the system

$$
\begin{aligned}
2x_1 + 3x_2 - x_3 - 9x_4 &= -16 \\
x_1 + 2x_2 + x_3 &= 0 \\
-x_1 + 2x_2 + 3x_3 + 4x_4 &= 8
\end{aligned}
$$

has infinitely many solution and find the solution set.

Solution

The augmented matrix of the system of equations is

$$\begin{bmatrix} 2 & 3 & -1 & -9 & -16 \\ 1 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 4 & 8 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 2 & 3 \\ 0 & \boxed{1} & 0 & -3 & -5 \\ 0 & 0 & \boxed{1} & 4 & 7 \end{bmatrix}$$

After doing the back substitution, we see that the solution set is given by

$$S = \left\{ \begin{bmatrix} 3 - 2x_4 \\ 5 + 3x_4 \\ 7 - 4x_4 \\ x_4 \end{bmatrix} : \quad x_4 \in \mathbb{R} \right\}$$

Problem 5

Use the Gauss Jordan method to find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}.$$

Solution

To find A^{-1} we consider the array

$$(A \mid I_3) = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We now perform elementary row operations on this array and try to reduce the left hand half to the matrix I_3.

Note that if we succeed, then the final array is clearly in reduced row echelon form.

We therefore follow the same procedure as reducing an array to reduced row echelon form.

Adding -3 times row 1 to row 2, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Adding 2 times row 1 to row 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 5 & 4 & 2 & 0 & 1 \end{pmatrix}.$$

Multiplying row 3 by 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 15 & 12 & 6 & 0 & 3 \end{pmatrix}.$$

Adding 5 times row 2 to row 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Multiplying row 1 by 3, we obtain

$$\begin{pmatrix} 3 & 3 & 6 & 3 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Adding 2 times row 3 to row 1, we obtain

$$\begin{pmatrix} 3 & 3 & 0 & -15 & 10 & 6 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Adding -1 times row 3 to row 2, we obtain

$$\begin{pmatrix} 3 & 3 & 0 & -15 & 10 & 6 \\ 0 & -3 & 0 & 6 & -4 & -3 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Adding 1 times row 2 to row 1, we obtain

$$\begin{pmatrix} 3 & 0 & 0 & -9 & 6 & 3 \\ 0 & -3 & 0 & 6 & -4 & -3 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Multiplying row 1 by 1/3, row 2 by -1/3 and row 3 by -1/3, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & 1 & 0 & -2 & 4/3 & 1 \\ 0 & 0 & 1 & 3 & -5/3 & -1 \end{pmatrix}.$$

Note now that the array is in reduced row echelon form, and that the left hand half is the identity matrix I_3.
It follows that the right hand half of the array represents the inverse A^{-1}.
Hence

$$A^{-1} = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 4/3 & 1 \\ 3 & -5/3 & -1 \end{pmatrix}.$$

Problem 6

Find the value of h for which the matrix A does not have an inverse.

$$A = \begin{bmatrix} -3 & 2 & 2 & 3 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 3 & 0 & 2 & h \end{bmatrix}$$

Solution

$$\begin{aligned}
\det A &= \det \begin{bmatrix} -3 & 2 & 2 & 3 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & 4 & h+3 \end{bmatrix} \\
&= (-3) \det \begin{bmatrix} 0 & -2 & 0 \\ -2 & -2 & 0 \\ 2 & 4 & h+3 \end{bmatrix} = (-3)\,(h+3) \det \begin{bmatrix} 0 & -2 \\ -2 & -2 \end{bmatrix} \\
&= (-3)\,(h+3)\,(-4)
\end{aligned}$$

$$\det A = 0 \text{ if and only if } h = -3$$

Hence, the value of h for which the matrix A does not have an inverse is $h = -3$.

Problem 7

Use cofactor expansion to find the determinant of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & -2 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ -6 & 0 & 0 & -3 \end{bmatrix}.$$

Solution

$$\begin{aligned}\det A &= \det\begin{bmatrix} 2 & 0 & -2 \\ 1 & -2 & 1 \\ -6 & 0 & -3 \end{bmatrix} = (-2)\det\begin{bmatrix} 2 & -2 \\ -6 & -3 \end{bmatrix} \\ &= (-2)(-6-12) = 36.\end{aligned}$$

Problem 8

Let A be the matrix given by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Suppose that $\det A = 9$ and $\det B = 4$. Find

1. $\det(-3A)$
2. $\det\left(BA^{-1}\right)$
3. $\det \begin{bmatrix} d & e & f \\ g & h & i \\ 2a & 2b & 2c \end{bmatrix}$
4. $\det \begin{bmatrix} -2a & -2b & -2c \\ d & e & f \\ g-d & h-e & i-f \end{bmatrix}$

Solution

1. Since A is a 3×3 matrix

$$\begin{aligned} \det(-2A) &= (-3)^3 \det(A) \\ &= (-27)(9) \\ &= -243\ . \end{aligned}$$

2.

$$\begin{aligned} \det\left(BA^{-1}\right) &= \det B \det\left(A^{-1}\right) \\ &= \frac{\det B}{\det A} \\ &= \frac{4}{9}. \end{aligned}$$

3. The matrix

$$\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$$

is obtained from A by interchanging row 1 and row, then row 2 and row 3 and multiply row 3 by 2.

$$\begin{aligned} \det \begin{bmatrix} d & e & f \\ g & h & i \\ 2a & 2b & 2c \end{bmatrix} &= (2)(-1)(-1)\det A \\ &= \det A = 18 \end{aligned}$$

4. The matrix

$$\begin{bmatrix} -2a & -2b & -2c \\ d & e & f \\ g-d & h-e & i-f \end{bmatrix}$$

is obtained from the matrix A by multiplying the first row of A by (-2) and by adding (-1) times row 2 to row 3.
As a result of these elementary operations, we get

$$\begin{aligned} \det \begin{bmatrix} -2a & -2b & -2c \\ d & e & f \\ g-d & h-e & i-f \end{bmatrix} &= (-2)\det A \\ &= -18. \end{aligned}$$

Problem 9

Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{bmatrix}$$

by <u>using elementary row operations</u> .

Solution

Solution:

$$\text{Interchange } R_2 \text{ and } R_3: \qquad \det\mathbf{A} = -\begin{vmatrix} 2 & 3 & 3 & 1 \\ 2 & -1 & -1 & -3 \\ 0 & 4 & 3 & -3 \\ 0 & -4 & -3 & 2 \end{vmatrix}$$

$$R_2 - R_1 \longrightarrow R_2 \qquad = -\begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ 0 & 4 & 3 & -3 \\ 0 & -4 & -3 & 2 \end{vmatrix}$$

$$R_4 + R_3 \longrightarrow R_4 \qquad = - \begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ 0 & 4 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$R_3 + R_2 \longrightarrow R_3 \qquad = - \begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

Compute the determinant of the upper triangular matrix:
$= -2 \cdot (-4) \cdot (-1) \cdot (-1) = 8$