Part 1 MCQ (50%) Master Version

$(5pts)$**Problem 1**

Let
$$z = \frac{-9+3i}{1-2i}.$$
The modulus of z is equal to

$$(a) \quad |z| = \sqrt{18}$$

$$(b) \quad |z| = 3\sqrt{8}$$

$$(c) \quad |z| = 9$$

$$(d) \quad |z| = 2\sqrt{5}$$

$$(e) \quad |z| = \sqrt{10}$$

Solution

$$\begin{aligned} |z| &= \left|\frac{-9+3i}{1-2i}\right| \\ &= \frac{|-9+3i|}{|1-2i|} \\ &= \frac{\sqrt{81+9}}{\sqrt{1+4}} \\ &= \frac{\sqrt{90}}{\sqrt{5}} \\ &= \frac{\sqrt{5}\sqrt{18}}{\sqrt{5}} = \sqrt{18}. \end{aligned}$$

(5*pts*)**Problem 2**
Solve the quadratic equation

$$2z^2 - 2iz - 5 = 0.$$

If z_1 and z_2 are the solutions, then $z_1^2 + z_2^2$ is equal to

$$(a) \quad 4$$

$$(b) \quad 2$$

$$(c) \quad 6/4$$

$$(d) \quad 3/2$$

$$(e) \quad 3$$

Solution

$$2z^2 - 2iz - 5 = 0.$$

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2i) \pm \sqrt{(-2i)^2 - 4(2)(-5)}}{4} \\ &= \frac{2i \pm \sqrt{36}}{4} \\ &= \frac{2i \pm 6}{4} \end{aligned}$$

$$z_1 = \frac{3}{2} + \frac{1}{2}i, \qquad z_2 = -\frac{3}{2} + \frac{1}{2}i$$

$$\begin{aligned} z_1^2 + z_2^2 &= \left(\frac{3}{2} + \frac{1}{2}i\right)^2 + \left(-\frac{3}{2} + \frac{1}{2}i\right)^2 \\ &= 2 + \frac{3}{2}i + 2 - \frac{3}{2}i \\ &= 4. \end{aligned}$$

$(5pts)$**Problem 3**

If $x\,(1+i)^2 + y\,(2-i)^2 = 3+10i$, then $x+y$ is equal to

(a) 8

(b) 7

(c) 6

(d) 5

(e) 4

Solution

$$x\,(1+i)^2 + y\,(2-i)^2 = 3+10i$$

$$2ix + 3y - 4iy = 3+10i$$

$$3y + (2x-4y)\,i = 3+10i$$

$$\left\{\begin{array}{c} 3y = 3 \\ 2x-4y = 10 \end{array}\right. \Rightarrow y = 1 \quad \text{and } x = 7$$

$$x+y = 1+7 = 8$$

$(5pts)$**Problem 4**
Let

$$z = \frac{-9+3i}{1-2i}.$$

The argument of z is equal to

(a) $\frac{-3\pi}{4}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$

(d) $\frac{-2\pi}{3}$

(e) π

Solution

$$\begin{aligned} z &= \frac{(-9+3i)(1+2i)}{(1-2i)(1+2i)} \\ &= \frac{-15-15i}{5} \\ &= -3-3i \end{aligned}$$

$$\tan\theta = 1 \Rightarrow \quad \theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{-3\pi}{4}$$

Because $z = -3-3i$ is in the third quadrant, $\theta = \dfrac{-3\pi}{4}$.

(5*pts*)**Problem 5**
Find k so that $\mathbf{u} = \langle 3,\ -2\rangle$ and $\mathbf{v} = \langle 1, k\rangle$ are perpendicular

(a) $\frac{3}{2}$

(b) $\frac{1}{4}$

(c) $\frac{5}{6}$

(d) $\frac{4}{3}$

(e) $\frac{1}{2}$

Solution
$\mathbf{u} = \langle 3,\ -2\rangle$ and $\mathbf{v} = \langle 1, k\rangle$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

$$\langle 3,\ -2\rangle \cdot \langle 1, k\rangle = 0 \Leftrightarrow 3 - 2k = 0$$

$$k = \frac{3}{2}.$$

$(5pts)$**Problem 6** Find the vector projection of $\mathbf{u} = \langle 2, 7\rangle$ on $\mathbf{v} = \langle -3, 1\rangle$.

(a) $\left\langle \frac{-3}{10}, \frac{1}{10} \right\rangle$

(b) $\left\langle \frac{-3}{7}, \frac{1}{7} \right\rangle$

(c) $\left\langle \frac{1}{10}, \frac{3}{10} \right\rangle$

(d) $\langle -2, -7\rangle$

(e) $\langle 3, -1\rangle$

Solution

$$\Pr oj_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

$$\mathbf{u}\cdot\mathbf{v} = -6 + 7 = 1, \qquad \|\mathbf{v}\|^2 = 10$$

$$\begin{aligned} \Pr oj_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \\ &= \frac{1}{10}\langle -3, 1\rangle \\ &= \left\langle \frac{-3}{10}, \frac{1}{10} \right\rangle. \end{aligned}$$

(5pts)**Problem 7**
Find k so that $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 1, k \rangle$ are parallel.

(a) $\frac{-2}{3}$

(b) -2

(c) $\frac{5}{3}$

(d) $\frac{7}{3}$

(e) $\frac{9}{2}$

Solution
Two vectors are parallel if one is a scalar multiple of the other

$$\begin{aligned} \mathbf{u} &= t\mathbf{v} \Leftrightarrow \langle 3, -2 \rangle = t \langle 1, k \rangle \\ &\Rightarrow \left\{ \begin{array}{c} 3 = t \\ -2 = tk \end{array} \right. \Rightarrow k = \frac{-2}{3}. \end{aligned}$$

$(5pts)$**Problem 8**

The equation

$$x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$$

represents

(a) a sphere with center $(1, 2, -4)$ and radius 6

(b) a sphere with center $\left(0, -2, \frac{1}{2}\right)$ and radius 7

(c) a sphere with center $(0, 0, 0)$ and radius $\frac{1}{5}$

(d) a point

(e) no graph in R^3

Solution

$$x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$$

$\Leftrightarrow$

$$x^2 - 2x + y^2 - 4y + z^2 + 8z = 15$$

$\Leftrightarrow$

$$(x-1)^2 - 1 + (y-2)^2 - 4 + (z+4)^2 - 16 = 15$$

$\Leftrightarrow$

$$(x-1)^2 + (y-2)^2 + (z+4)^2 = 15 + 1 + 4 + 16$$

$\Leftrightarrow$

$$(x-1)^2 + (y-2)^2 + (z+4)^2 = 36.$$

This is the equation of a sphere centered at $(1,\ 2,\ -4)$ with radius 6.

(5pts)**Problem 9**

The area of the triangle that is determined by
$P_1 = (0,0,0)$ $P_2 = (1,2,3)$ $P_3 = (3,2,1)$
is

(a) $2\sqrt{6}$
(b) $2\sqrt{2}$
(c) $4\sqrt{6}$
(d) $8\sqrt{3}$
(e) $4\sqrt{2}$

Solution

$$\overrightarrow{P_1P_2} = \langle 1,\ 2,\ 3\rangle\,, \qquad \overrightarrow{P_1P_3} = \langle 3,\ 2,\ 1\rangle$$

$$\text{Area of Triangle } = \frac{1}{2}\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\|$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = \langle -4,\ 8,\ -4\rangle$$

$$\begin{aligned} \text{Area of Triangle } &= \frac{1}{2}\left\|\langle -4,\ 8,\ -4\rangle\right\| \\ &= \frac{1}{2}\sqrt{16+64+16} \\ &= \left(\frac{1}{2}\right)\left(4\sqrt{6}\right) = 2\sqrt{6}. \end{aligned}$$

($5pts$)**Problem 10**

At which point does the line with parametric equations

$$x = -1 + 3t \quad y = 2 - 2t \quad z = 3 + t$$

intersect the plane $3x + y - 4z = -4$?

(a) $(8, -4, 6)$

(b) $(0, 0, 1)$

(c) $(1, 1, 2)$

(d) $\left(2, 4, \frac{7}{2}\right)$

(e) they do not intersect

Solution

$$x = -1 + 3t, \qquad y = 2 - 2t, \qquad z = 3 + t.$$

Substituting these in the equation of the plane, we obtain

$$3(-1 + 3t) + (2 - 2t) - 4(3 + t) = -4.$$

$\Leftrightarrow$

$$3t - 13 = -4$$

Now solving for t, we obtain

$$t = 3.$$

The point of intersection is then given by

$$\begin{aligned} x &= -1 + 9 = 8, \\ y &= 2 - 6 = -4, \\ z &= 3 + 3 = 6. \end{aligned}$$

Part 2 Written Questions (50%)

$(10pts)$**Problem 1**
Find the parametric equations of the line that passes through the point $(-1, -2, 3)$ and perpendicular to the plane $x - 2y - 5z = 9$.

Solution
The direction vector of the line is

$$\overrightarrow{u} = \langle 1,\ -2,\ -5 \rangle . \qquad (\mathbf{4pts})$$

The parametric equations are

$$\begin{cases} x = -1 + t \\ y = -2 - 2t \\ z = 3 - 5t \end{cases} \qquad (\mathbf{6pts})$$

$(10pts)$**Problem 2**
Which of the points $A(0,0,0)$ and $B(1,1,1)$ is closer to the plane $3x+2y+z=4$? (Justify your answer and show your work)

Solution
A normal vector of the plane is

$$\overrightarrow{n} = \langle 3,\ 2,\ 1\rangle \quad \text{and } P\,(0,\ 0,\ 4) \text{ is a point on the plane.} \qquad \textbf{(2pts)}$$

$$\overrightarrow{PA} = \langle 0,\ 0,\ -4\rangle\,, \qquad \overrightarrow{PB} = \langle 1,\ 1,\ -3\rangle \qquad \textbf{(2pts)}$$

3+2-3

The distance from A to the plane is

$$D_1 = \frac{\left|\overrightarrow{PA}\cdot\overrightarrow{n}\right|}{\|\overrightarrow{n}\|} = \frac{4}{\sqrt{14}} = 1.069 \qquad \textbf{(2pts)}$$

The distance from B to the plane is

$$D_2 = \frac{\left|\overrightarrow{PB}\cdot\overrightarrow{n}\right|}{\|\overrightarrow{n}\|} = \frac{2}{\sqrt{14}} = 0.534\,52 \qquad \textbf{(2pts)}$$

$$D_2 < D_1 \Rightarrow \text{ B is closer to the plane.} \qquad \textbf{(2pts)}$$

(10pts)**Problem 3**

Let

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -2 & 3 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Find, if possible

a) AB **b)** AC

Solution

a)

A is a 4×2 matrix and B is a 4×2 $\Rightarrow$ AB is not possible (**5pts**)

b)

A is a 4×2 matrix and C is a 2×3 $\Rightarrow$ AC is 4×3

$$AC = \begin{pmatrix} 1 & 3 \\ 0 & -1 \\ -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 11 & 3 & 1 \\ -3 & -1 & 0 \\ 4 & 2 & -1 \\ 6 & 0 & 3 \end{pmatrix} \quad (\mathbf{5pts})$$

$(10pts)$**Problem 4**

Use the **Gauss Jordan method** to find the inverse of the matrix

$$A = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}.$$

Rk: You must use the Gauss Jordan method and show your work.

Solution

$$\begin{pmatrix} 4 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}$$ (−**5pts**) if Gauss Jordan Method is not correct

(−**8pts**) if Gauss Jordan Method is not used.

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -2 \end{pmatrix}$$

(10pts)**Problem 5**
Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be two orthogonal unit vectors. Find

$$\|2\overrightarrow{u} - 3\overrightarrow{v}\|.$$

Solution

$$\begin{aligned}
\|2\overrightarrow{u} - 3\overrightarrow{v}\|^2 &= (2\overrightarrow{u} - 3\overrightarrow{v}) \cdot (2\overrightarrow{u} - 3\overrightarrow{v}) && (\textbf{3pts}) \\
&= 4\overrightarrow{u} \cdot \overrightarrow{u} - 6\overrightarrow{u} \cdot \overrightarrow{v} - 6\overrightarrow{u} \cdot \overrightarrow{v} + 9\overrightarrow{v} \cdot \overrightarrow{v} \\
&= 4\|\overrightarrow{u}\|^2 + 0 + 0 + 9\|\overrightarrow{v}\|^2 && (\textbf{4pts}) \\
&= 4 + 9 = 13
\end{aligned}$$

$$\|2\overrightarrow{u} - 3\overrightarrow{v}\| = \sqrt{13}. \qquad (\textbf{3pts})$$