Answer Key Solution

Part 1 MCQ 30% (circle your choice)

(6pts)Problem 1
Find the values of x and y in the following equation

$$(x+iy)(2+i)=3-i.$$

(a) $(2,1)$ (b) $(1,-1)$ (c) $(2,-2)$ (d) $(-1,2)$ (e) $(3,-1)$

Solution of Problem 1

$$(x+iy)(2+i)=3-i.$$

$\Leftrightarrow$

$$2x+ix+2iy-y=3-i$$

$\Leftrightarrow$

$$2x-y+i(x+2y)=3-i$$

$$\begin{cases} 2x-y=3 \\ x+2y=-1 \end{cases}$$

Solution is:

$$[x=1, y=-1] \qquad \text{Answer is (b)}$$

(**6pts**)**Problem 2**

The complex conjugate of $z = x + iy$ is denoted by $\bar{z} = x - iy$. Solve the equation

$$2z - 3\bar{z} = \frac{-27 + 23i}{1 + i}.$$

The modulus of the solution z is equal to

(a) $|z| = \sqrt{29}$ (b) $|z| = \sqrt{27}$ (c) $|z| = \sqrt{23}$ (d) $|z| = 2\sqrt{23}$ (e) $|z| = 3\sqrt{2}$

Solution of Problem 2

$$2z - 3\bar{z} = \frac{-27 + 23i}{1 + i}.$$

$\Leftrightarrow$

$$2(x + iy) - 3(x - iy) = \frac{-27 + 23i}{1 + i}$$
$$2x + 2iy - 3x + 3iy = -2 + 25i$$

$\Leftrightarrow$

$$-x + 5iy = -2 + 25i$$
$$x = 2 \text{ and } y = 5$$
$$z = 2 + 5i$$
$$|z| = \sqrt{4 + 25} = \sqrt{29}. \quad \text{Answer is } (a)$$

(6pts)Problem 3

The vector $\langle a,\ b,\ 0\rangle$ with a and b not equal to zero is perpendicular to the vector $\langle 2,\ -1,\ 3\rangle$.
Then $\dfrac{a^2+b^2}{a^2}$ is equal to

$$(a)\quad 1 \qquad (b)\quad 2 \qquad (c)\quad 3 \qquad (d)\quad 4 \qquad (e)\quad 5$$

Solution of Problem 3

$$\langle a,\ b,\ 0\rangle \cdot \langle 2,\ -1,\ 3\rangle = 0.$$

$\Leftrightarrow$

$$2a - b = 0 \Rightarrow b = 2a$$

$$\frac{a^2+b^2}{a^2} = \frac{a^2+4a^2}{a^2} = 5 \quad \text{Answer is } (e).$$

(6pts)Problem 4

Consider the sphere given by the equation

$$2x^2 + 2y^2 + 2z^2 + x + y + z = 9.$$

The radius of the sphere is equal to

$$(a)\ \frac{1}{4} \qquad (b)\ \frac{3}{4} \qquad (c)\ \frac{5\sqrt{3}}{4} \qquad (d)\ \frac{\sqrt{3}}{5} \qquad (e)\ \frac{\sqrt{3}}{16}$$

Solution of Problem 4

$$2x^2 + 2y^2 + 2z^2 + x + y + z = 9$$

$\Leftrightarrow$

$$x^2 + \frac{x}{2} + y^2 + \frac{y}{2} + z^2 + \frac{z}{2} = \frac{9}{2}$$

Now using

$$t^2 + kt = \left(t + \frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2,$$

we get

$$\left(x + \frac{1}{4}\right)^2 - \frac{1}{16} + \left(y + \frac{1}{4}\right)^2 - \frac{1}{16} + \left(z + \frac{1}{4}\right)^2 - \frac{1}{16} = \frac{9}{2}.$$

$\Leftrightarrow$

$$\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \frac{9}{2} + \frac{3}{16} = \frac{75}{16}.$$

$$R = \sqrt{\frac{75}{16}} = \frac{5\sqrt{3}}{4}. \qquad \text{Answer is } (c).$$

(6pts)Problem 5
Find the value of x for which the matrix A does not have an inverse.

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ x & 0 & 1 \end{pmatrix}$$

$(a)\ \frac{-1}{2}$ $\quad (b)\ 2$ $\quad (c)\ -1$ $\quad (d)\ 0$ $\quad (e)\ \frac{3}{2}$

Solution of Problem 5

$$\det A = 0.$$

$\Leftrightarrow$

$$2\det\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} + x\det\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} = 0$$

$\Leftrightarrow$

$$2 + 4x = 0 \Rightarrow x = \frac{-1}{2} \qquad \text{Answer is } (a).$$

Part 2 Written 70%

(14pts)**Problem 1**
Consider the points $A(1,\ 0,\ -2)$, $B(0,\ -1,\ 2)$, $C(1,\ 1,\ 1)$ and $D(1,1,\ -3)$.
(a) Find the equation of the plane $(\mathcal{P})$ containing the points $A,\ B,$ and C.
(b) Find the distance from D to $(\mathcal{P})$.

Solution
(a)

$$\overrightarrow{AB} = \langle -1,\ -1,\ 4\rangle, \qquad \overrightarrow{AC} = \langle 0,\ 1,\ 3\rangle.$$

The normal vector of the plane is given by

$$\begin{aligned}\overrightarrow{n} &= \overrightarrow{AB} \times \overrightarrow{AC} \\ &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -1 & -1 & 4 \\ 0 & 1 & 3 \end{vmatrix} = \langle -7,\ 3,\ -1\rangle. \qquad (\mathbf{4pts})\end{aligned}$$

Using the point A, the equation of the plane is given by

$$-7(x-1) + 3(y-0) - (z+2) = 0$$

$\Leftrightarrow$

$$-7x + 3y - z = -5 \qquad (\mathbf{3pts})$$

(b)

$$D = \frac{\left|\overrightarrow{AD} \cdot \overrightarrow{n}\right|}{\|\overrightarrow{n}\|}$$

$$\overrightarrow{AD} = \langle 0,\ 1,\ -1\rangle$$

$$\begin{aligned}\overrightarrow{AD} \cdot \overrightarrow{n} &= \langle 0,\ 1,\ -1\rangle \cdot \langle -7,\ 3,\ -1\rangle \\ &= 3 + 1 = 4 \qquad (\mathbf{3pts})\end{aligned}$$

$$\begin{aligned}D &= \frac{4}{\sqrt{49+9+1}} \\ &= \frac{4}{\sqrt{59}} = 0.520\,76 \qquad (\mathbf{4pts})\end{aligned}$$

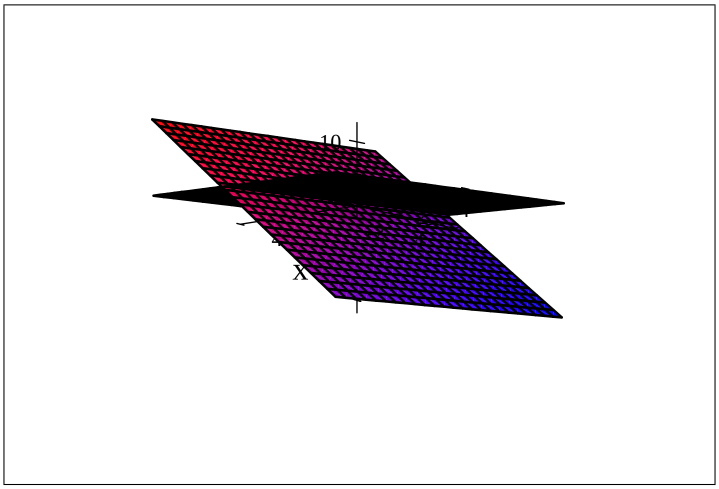
$(14pts)$**Problem 2**

Find the parametric equations of the line of intersection of the planes

$$2x + y - z = 2 \quad \text{and} \quad x - y + z = 1.$$

Solution

$$2x + y - z = 2 \quad \text{and} \quad x - y + z = 1.$$



$$\overrightarrow{n_1} = \langle 2,\ 1,\ -1 \rangle\,, \qquad \overrightarrow{n_2} = \langle 1,\ -1,\ 1 \rangle\,.$$

The direction vector of the line of intersection is given by

$$\overrightarrow{u} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \langle 0,\ -3,\ -3 \rangle\,. \quad (\mathbf{6pts})$$

Now put $z = 0$ and solve for x and y to get a point P on the line of intersection.

$$\begin{cases} 2x + y = 2 \\ x - y = 1 \end{cases}$$

Solution is: $[x = 1, y = 0]$. P has coordinates $(1, 0, 0)$. (**4pts**)

The parametric equations of the line are

$$\begin{cases} x = 1 + 0t \\ y = 0 - 3t \\ z = 0 - 3t \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = -3t \\ z = -3t \end{cases} \qquad (\mathbf{4pts})$$

$(12pts)$**Problem 3**

Use the **Gauss elimination** method to solve the linear system

$$\begin{cases} x - 3y - 2z = 6 \\ 2x - 4y - 3z = 8 \\ -3x + 6y + 8z = -5 \end{cases} . \quad (\text{ Show your work})$$

Solution
The augmented matrix is

$$A = \begin{pmatrix} 1 & -3 & -2 & 6 \\ 2 & -4 & -3 & 8 \\ -3 & 6 & 8 & -5 \end{pmatrix} \qquad (\mathbf{2pts})$$

After reducing in row echelon form by elementary operation we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}. \qquad (\mathbf{7pts})$$

Now we do the back substitution to get

$$x = 1, \quad y = -3, \quad z = 2 \qquad (\mathbf{3pts})$$

(15pts)**Problem 4**

Consider the following matrices

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 5 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} -7 & 1 & 12 \\ 2 & 0 & 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -1 & 1 \\ 0 & 5 \end{bmatrix}$$

(a) Use the **Gauss Jordan method** to find C^{-1}. (Do NOT use the formula and show your work)

(b) Compute the following matrices, where possible.

$$1.\ A + 2B^T, \qquad 2.\ AC$$

Solution

(a)

$$\left[\begin{array}{cccc} -1 & 1 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{array}\right].$$

After perfoming the elementary operations, we obtain

$$\left[\begin{array}{cccc} 1 & 0 & -1 & \frac{1}{5} \\ 0 & 1 & 0 & \frac{1}{5} \end{array}\right].$$

$$C^{-1} = \begin{bmatrix} -1 & \frac{1}{5} \\ 0 & \frac{1}{5} \end{bmatrix}. \qquad (\mathbf{5pts})$$

(b)

1.

$$A + 2B^T = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 5 & -2 \end{bmatrix} + 2\begin{bmatrix} -7 & 2 \\ 1 & 0 \\ 12 & 1 \end{bmatrix} = \begin{bmatrix} -12 & 3 \\ 6 & 3 \\ 29 & 0 \end{bmatrix} \qquad (\mathbf{5pts})$$

2.

$$AC = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -4 & 19 \\ -5 & -5 \end{bmatrix} \qquad (\mathbf{5pts})$$

(15pts)**Problem 5**
Use the **cofactor expansion method** to find the determinant of the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 2 & -1 & 1 \\ 2 & 0 & 0 & 3 \\ 0 & 1 & 6 & 0 \end{bmatrix}. \qquad \text{(Show your work)}$$

Solution
We choose an expansion along the first column.

$$\begin{aligned}
\det A &= (1)\,(-1)^{1+1} \det \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 3 \\ 1 & 6 & 0 \end{pmatrix} + 2\,(-1)^{3+3} \det \begin{pmatrix} -1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 6 & 0 \end{pmatrix} \qquad (\mathbf{5pts}) \\
&= -3 \det \begin{pmatrix} 2 & -1 \\ 1 & 6 \end{pmatrix} + 2 \det \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} - 12 \det \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \qquad (\mathbf{5pts}) \\
&= (-3)\,(13) + (2)\,(1) - (12)\,(-5) \\
&= 23 \qquad (\mathbf{5pts})
\end{aligned}$$