EXAMINATION COVERSHEET

Autumn 2023 Midterm

UNIVERSITY OF WOLLONGONG IN DUBAI

THIS EXAMINATION CONTENT IS STRICTLY CONFIDENTIAL Students must comply with requirements stated in the Examination Policy & Procedures	
Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	**10/27/2023**
Subject Code:	**Math 141**
Subject Title:	**Foundation of Engineering Mathematics**
Time Permitted to Write Exam:	2 Hours
Total Number of Questions:	**6 MCQs + 5 written questions = 11**
Total Number of Pages *(including this page):*	**9**

INSTRUCTIONS TO STUDENTS FOR THE EXAM

1. Please note that subject lecturer/tutor will be unavailable during exams. *If there is a doubt in any of the exam questions i.e. problem solving etc. students should proceed by assuming values etc. Students should mention their assumption on the question paper.*
2. Answers must be written (and drawn) in black or blue ink
3. Any mistakes must be crossed out. Whitener and ink erasers must not be used.
4. Part A (MCQ): Answer ALL/ 6 questions. The marks for each question are shown next to each question. The total for Part A is 30 marks.
5. Part B (Written): Answer ALL/ 5 questions. The marks for each question are shown next to each question. The total for Part B is 70 marks.)
6. Total marks: 100. This Exam is worth 30% of your final marks for MATH 141.

Part 1 MCQ 30% (circle your choice)

(5pts)Problem 1
Find the value (s) of k for which the function

$$f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$$

is continuous at $x = \dfrac{\pi}{2}$.

(A) 2

(B) π

(C) 2π

(D) 6

(E) 3

Solution

Since $f(x)$ is continuous at $x = \dfrac{\pi}{2}$

so, $\lim_{x \to \frac{\pi}{2}} f(x) = f\left(\dfrac{\pi}{2}\right) = 3$

$\Rightarrow \quad \lim_{x \to \frac{\pi}{2}} \left(\dfrac{k\cos x}{\pi - 2x}\right) = 3$

$\Rightarrow \quad \lim_{x \to \frac{\pi}{2}} \left(\dfrac{-k\sin x}{-2}\right) = 3$

$\Rightarrow \quad \dfrac{k}{2} = 3$

$\Rightarrow \quad k = 6$

Hence, the value of k is 6.

Answer is (D)

(5pts)Problem 2

$$\lim_{x\to-\infty}\frac{3x^5+7}{\sqrt{4x^{10}+x^7+11}}$$

$(A)\quad \frac{3}{2}$

$(B)\quad -\frac{3}{2}$

$(C)\quad -\frac{7}{3}$

$(D)\quad \frac{3}{4}$

$(E)\quad \frac{7}{3}$

Solution

$$\begin{aligned}
\lim_{x\to-\infty}\frac{3x^5+7}{\sqrt{4x^{10}+x^7+11}} &= \lim_{x\to-\infty}\frac{3x^5}{\sqrt{4x^{10}}}\\
&= \lim_{x\to-\infty}\frac{3x^5}{2\left|x^5\right|}\\
&= \lim_{x\to-\infty}\frac{3\cancel{x^5}}{-2\cancel{x^5}} = -\frac{3}{2}\\
&\ \text{Answer is } (B)
\end{aligned}$$

:

(5pts)Problem 3

Let f be a function for which $f'(x) = \dfrac{1}{x^2+1}$. If $g(x) = f(3x-1)$, find $g'(0)$,

$(A)\quad \frac{3}{2}$

$(B)\quad 3$

$(C)\quad 1$

$(D)\quad -1$

$(E)\quad \frac{1}{2}$

Solution

$$\begin{aligned}\text{Given}\qquad g(x) &= f(3x-1)\\ g'(x) &= 3f'(3x-1)\\ &= 3 \times \frac{1}{(3x-1)^2+1}\\ &= \frac{3}{9x^2-6x+2}\end{aligned}$$

$g'(0) \quad = \quad \dfrac{3}{2}.$ Answer is (A).

(5pts)Problem 4

If $F(x) = \displaystyle\int_{x^3}^{3x} \frac{1}{1+2t}dt$, then $F'(-1)$ is equal to

(A) 6

(B) $\frac{12}{5}$

(C) $\frac{2}{5}$

(D) 6

(E) -4

Solution

$$F'(x) = 3\frac{1}{1+6x} - 3x^2\frac{1}{1+2x^3}$$

$$F'(-1) = 3\frac{1}{1+6(-1)} - 3(-1)^2\frac{1}{1+2(-1)^3} = \frac{12}{5}$$

Answer is (B).

(5pts)Problem 5

A particle moves along the curve $y = 2x^3 - 3x^2 + 4$. At a certain moment, when $x = 2$, the particle's x-coordinate is increasing at the rate of 0.5 unit per second. How fast is its y-coordinate changing at that moment?

(A) 2 unit per second

(B) 4 unit per second

(C) 6 unit per second

(D) 8 unit per second

(E) 10 unit per second

Solution

We have

$$\frac{dx}{dt} = 0.5 \text{ when } x = 2.$$

$$\begin{aligned}\frac{dy}{dt} &= 6\frac{dx}{dt}x^2 - 6\frac{dx}{dt}x \\ &= 6\left(\frac{1}{2}\right)(4) - 6\left(\frac{1}{2}\right)(2) \\ &= 6 \\ &\quad \text{Answer is } (C)\end{aligned}$$

(5pts)Problem 6
What is the slope of the tangent line to the graph of

$$f(x) = \sin^2 x \text{ at } x = \frac{\pi}{3}?$$

$(A)\quad 2\sqrt{3}$

$(B)\quad -\frac{3}{4}$

$(C)\quad \sqrt{3}$

$(D)\quad \frac{1}{2}$

$(E)\quad \frac{\sqrt{3}}{2}$

Solution

$$\begin{aligned} f'(x) &= 2\cos x \sin x \\ Slope &= f'(\frac{\pi}{3}) = 2\cos\frac{\pi}{3}\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \\ &\quad \text{Answer is } (E). \end{aligned}$$

Part 2 Written 70%

$(14pts)$**Problem 1**
Find the equation of the tangent line to the curve

$$y \ln x + y^2 + 1 = 2e^{x-1} \text{ at the point } (1,1).$$

Solution
We first take derivative with respect to x on both sides (keeping in mind that y is a function of x).

$$y' \ln x + \frac{y}{x} + 2yy' = 2e^{x-1} \qquad (\mathbf{4pts})$$

$$y' \ln x + 2yy' = 2e^{x-1} - \frac{y}{x}$$

$$y' (\ln x + 2y) = 2e^{x-1} - \frac{y}{x}$$

$$y' = \frac{2e^{x-1} - \frac{y}{x}}{\ln x + 2y}. \qquad (\mathbf{4pts})$$

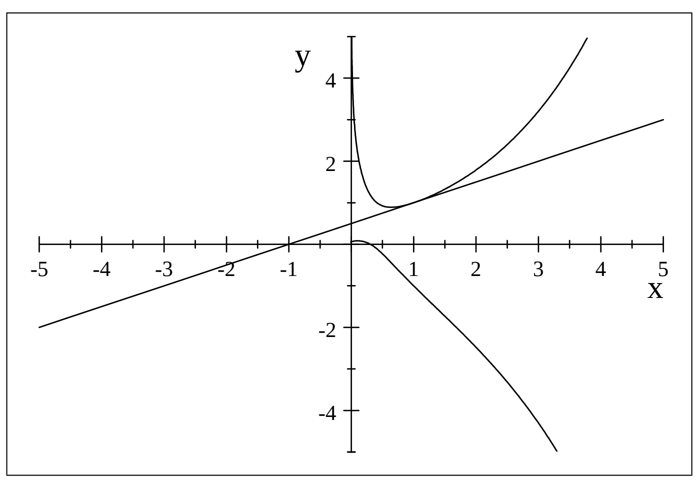
At the point $(1,1)$, the slope is

$$m = \frac{2-1}{2} = \frac{1}{2}. \qquad (\mathbf{2pts})$$

The equation of the tangent line is

$$y = \frac{1}{2}(x-1) + 1$$

$$y = \frac{1}{2}x + \frac{1}{2} \qquad (\mathbf{4pts})$$



The graph of $y \ln x + y^2 + 1 = 2e^{x-1}$ and the corresponding tangent line $y = \frac{1}{2}x + \frac{1}{2}$

$(14pts)$**Problem 2**

A civil engineer is tasked with designing a bridge to span a river in a remote area. The bridge must be at least 50 meters long and no longer than 200 meters.The objective is to minimize construction costs while ensuring the bridge meets safety standards. The bridge design is described by the following equation:

$$C(x) = 2x + \frac{10000}{x}, \qquad 50 \leq x \leq 200.$$

Where:

$C(x)$ represents the total cost (in thousands of dollars) of building the bridge and x represents the length of the bridge (in meters).

Determine the optimal length for the bridge that minimizes construction costs while meeting the safety and design constraints.

Solution

The problem comes down to finding the absolute minimum of the cost function on the closed interval $[50,\ 200]$.

$$C'(x) = 2 - \frac{10000}{x^2} \qquad \textbf{(4pts)}$$

$$C'(x) = 0 \Leftrightarrow 2 - \frac{10000}{x^2} = 0$$

$$x = -50\sqrt{2} \quad \text{or} \quad x = 50\sqrt{2} \qquad \textbf{(2pts)}$$

$$C(50) = 2\,(50) + \frac{10000}{50} = 300 \qquad \textbf{(2pts)}$$

$$\begin{aligned} C(50\sqrt{2}) &= 2\left(50\sqrt{2}\right) + \frac{10000}{50\sqrt{2}} = 282.84 && \textbf{(2pts)} \\ C(200) &= 2\,(200) + \frac{10000}{200} = 450.0 && \textbf{(2pts)} \end{aligned}$$

The optimal length for the bridge that minimizes construction cost is $x = 50\sqrt{2}$. (**2pts**)

(14pts)**Problem 3**
Find the critical points and the local extrema of the function

$$f(x) = \int_0^x \left(t^2 - t - 2\right) dt.$$

Hint: $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$.

Solution

$$f(x) = \int_0^x \left(t^2 - t - 2\right) dt.$$

$$\begin{aligned} f'(x) &= x^2 - x - 2 \\ &= x + 1)(x-2) \qquad \textbf{(2pts)} \end{aligned}$$

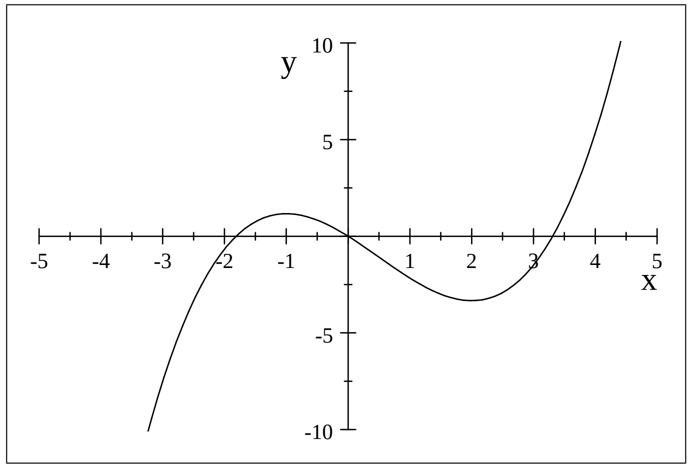
The critical points are $x = -1$ and $x = 2$. **(2pts)**

x	$-\infty$	-1		2	$+\infty$
$x^2 - x - 2$	$+$	0	$-$	0	$+$
$f'(x)$	$+$		$-$		$+$
$f(x)$	$\nearrow$	$f(-1)$	$\searrow$	$f(2)$	$\nearrow$

(2pts)

$$\begin{aligned} \text{Local Maximum} &= f(-1) = \int_0^{-1} \left(t^2 - t - 2\right) dt = \frac{t^3}{3} - \frac{t^2}{2} - 2t\Big|_0^{-1} \\ &= \frac{-1}{3} - \frac{1}{2} + 2 = \frac{7}{6} = 1.1667 \qquad \textbf{(4pts)} \end{aligned}$$

$$\begin{aligned} \text{Local Minimum} &= f(2) = \int_0^{2} \left(t^2 - t - 2\right) dt \\ &= \frac{t^3}{3} - \frac{t^2}{2} - 2t\Big|_0^{2} \\ &= -\frac{10}{3} = -3.3333 \qquad \textbf{(4pts)} \end{aligned}$$



Graph of $f(x) \;=\; \int_0^x \left(t^2 - t - 2\right) dt.$

(14pts)**Problem 4**
Find the open interval(s) where the function is concave up or down.

$$f(x) = x^3 + 15x^2 + 6x + 1.$$

Solution

$$f'(x) = 3x^2 + 30x + 6 \quad \textbf{(2pts)}$$

$$\begin{aligned} f''(x) &= 6x + 30 \\ &= (x+5) \qquad \textbf{(4pts)} \end{aligned}$$

Thus $f''(x) > 0$ when $x > -5$ and $f''(x) < 0$ when $x < -5$

$$\begin{array}{ll} \text{Concave up on } (-\infty,\ -5) & \textbf{(4pts)} \\ \text{Concave down on } (-5,\ +\infty) & \textbf{(4pts)} \end{array}$$

$(14pts)$**Problem 5**

Recall that the work done by the force F from $x = a$ to $x = b$ is given by the formula

$$W = \int_a^b F(x)dx.$$

A vertical pump is used to lift water from a certain depth in a reservoir to an elevated storage tank located 36 meters above. The force exerted by the pump varies with the depth due to changing water pressure and can be represented by the function

$$F(x) = 300\sqrt{x} - 10x\sqrt{x},$$

where x represents the depth. Calculate the total work done by the pump to transport the water from its initial position $x = 0$ to the elevated tank.

Solution

$$\begin{aligned}
W &= \int_0^{36} \left(300\sqrt{x} - 10x\sqrt{x}\right) dx \qquad (\textbf{4pts}) \\
&= \int_0^{36} \left(300x^{1/2} - 10x^{3/2}\right) dx \\
&= \frac{2}{3}(300)\, x^{3/2} - \frac{2}{5}(10)\, x^{5/2}\Big|_0^{36} \qquad (\textbf{6pts}) \\
&= \frac{2}{3}(300)\,(36)^{3/2} - \frac{2}{5}(10)\, 36^{5/2} \\
&= 12\,096 \qquad (\textbf{4pts})
\end{aligned}$$