Exam 1 Review Solution
Spring 2019-UOWD

Problem 1 Find the following limits

1. $\lim_{x\to 4} \frac{x^2-16}{x^2+x-20}$ 2. $\lim_{x\to 9} \frac{9-x}{3-\sqrt{x}}$ 3. $\lim_{x\to 0^+} \left[\frac{1}{x}-\frac{1}{|x|}\right]$ 4. $\lim_{x\to -\infty} \frac{x}{\sqrt{x^2+1}}$

Solution

1.

$$\begin{aligned}
\lim_{x\to 4} \frac{x^2-16}{x^2+x-20} &= \lim_{x\to 4} \frac{(x-4)(x+4)}{(x-4)(x+5)} \\
&= \lim_{x\to 4} \frac{(x+4)}{(x+5)} \\
&= \frac{8}{9}
\end{aligned}$$

2.

$$\begin{aligned}
\lim_{x\to 9} \frac{9-x}{3-\sqrt{x}} &= \lim_{x\to 9} \frac{(9-x)(3+\sqrt{x})}{9-x} \\
&= \lim_{x\to 9} \left(3+\sqrt{x}\right) \\
&= 6
\end{aligned}$$

3. When $x \to 0^+$, then $x > 0$ and $|x| = x$. Hence

$$\begin{aligned}
\lim_{x\to 0^+} \left[\frac{1}{x}-\frac{1}{|x|}\right] &= \lim_{x\to 0^+} \left[\frac{1}{x}-\frac{1}{x}\right] \\
&= \lim_{x\to 0^+} 0 \\
&= 0
\end{aligned}$$

4.

$$\begin{aligned}
\lim_{x\to -\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to -\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} \\
&= \lim_{x\to -\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}} \\
&= \lim_{x\to -\infty} \frac{x}{-x\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to -\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}} \\
&= -1
\end{aligned}$$

Problem 2
Find the value (s) of a that makes the function

$$f(x) = \begin{cases} a - x, & \text{if } x \le -1 \\ x + 1 & \text{if } x > -1 \end{cases}$$

continuous at $x = -1$.

Solution
For f to be continuous at $x = -1$, we must have

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1).$$

This leads to the equation

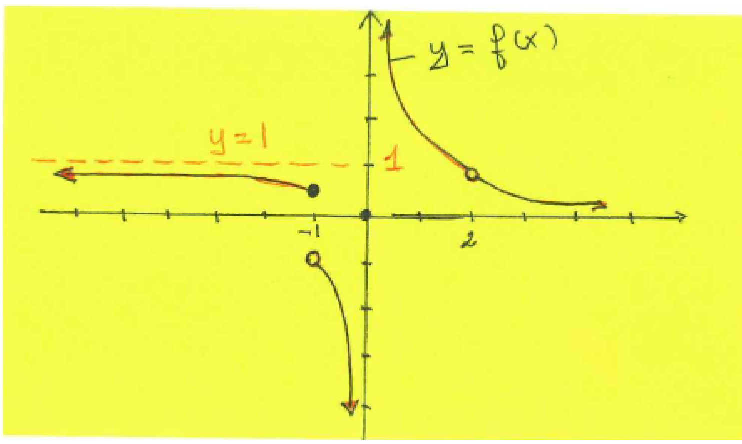
$$a + 1 = 0 \Longrightarrow a = -1.$$

(10pts)**Problem 3**

Sketch the graph of a function f that satisfies the following conditions

1. $f(0) = 0$
2. $\lim_{x \to -\infty} f(x) = 1$
3. f has a jump discontinuity at $x = -1$
4. $\lim_{x \to 0^-} f(x) = -\infty$
5. $\lim_{x \to 0^+} f(x) = \infty$
6. f has a removable discontinuity at $x = 2$
7. $\lim_{x \to \infty} f(x) = 0$

Solution



Rk. 1 point is deducted if your graph does not pass the vertical line test.

Problem 4
A) Find an equation for the tangent line to the graph of $f(x) = \ln(x+2)$ at $x = 0$.

Solution

$$f\,'(x) = \frac{1}{x+2}.$$

The slope of the tangent line is

$$f\,'(0) = \frac{1}{2}.$$

The equation of the tangent line is

$$y = f\,'(0)(x-0) + f(0)$$

$\Leftrightarrow$

$$y = \frac{1}{2}x + \ln 2$$

B) Find the slope of the normal line to the graph of

$$x^2y + y^2 = 6$$

at the point $(\sqrt{5}, 1)$.

Solution
Taking derivatives on both sides we get

$$x^2y' + 2xy + 2yy' = 0$$

$\Leftrightarrow$

$$x^2y' + 2yy' = -2xy$$

$\Leftrightarrow$

$$\left(x^2 + 2y\right) y' = -2xy$$

$\Longrightarrow$

$$y' = \frac{-2xy}{x^2 + 2y}$$

The slope at the point $(\sqrt{5}, 1)$ is

$$m = \frac{(-2)\sqrt{5}}{5+2} = \frac{-2\sqrt{5}}{7}$$

The slope of the normal line is

$$\frac{-1}{m} = \frac{-1}{\frac{-2\sqrt{5}}{7}} = \frac{7}{10}\sqrt{5} = 1.\,565\,2$$

Problem 5

Find $\dfrac{dy}{dx}$ if

1. $y = e^{x \ln x}$ 2. $y = \log_2 \sqrt[3]{x+1}$ 3. $y = \cos^2(2x)$ 4. $y = \dfrac{x-2}{x+2}$

Solution

1.

$$\begin{aligned}\frac{dy}{dx} &= \left[(1)\ln x + x\left(\frac{1}{x}\right)\right] e^{x \ln x} \\ &= (\ln x + 1)\, e^{x \ln x}\end{aligned}$$

2.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \log_2 \sqrt[3]{x+1} \\ &= \frac{d}{dx} \frac{1}{3} \log_2 (x+1) \\ &= \frac{1}{3} \cdot \frac{1}{\ln 2} \cdot \frac{1}{x+1}\end{aligned}$$

3.

$$\frac{dy}{dx} = (2)(-2) \sin 2x \;\; \cos 2x$$

4.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1)(x+2) - (1)(x-2)}{(x+2)^2} \\ &= \frac{4}{(x+2)^2}\end{aligned}$$

Problem 6

Sand is falling into a conical pile so that the radius of the base of the pile is always equal to one-half of its altitude. If the sand is falling at a rate of 10 cubic feet per minute, how fast is the altitude of the pile increasing when the pile is 4 feet deep? $V = \frac{1}{3}\pi r^2 h$.

Solution

$$V = \frac{1}{3}\pi r^2 h \text{ and } r = \frac{h}{2}.$$

We would like to find

$$\frac{dh}{dt} \text{ when } h = 4 \text{ and } \frac{dV}{dt} = 10.$$

Hence

$$V = \frac{1}{3}\pi\frac{h^3}{4}.$$

The related rate equation is

$$\frac{dV}{dt} = \frac{1}{3}\cdot\pi\cdot 3\frac{h^2}{4}\cdot\frac{dh}{dt}$$

when $h = 4$ and $\frac{dV}{dt} = 10$, we get

$$\begin{aligned}\frac{dh}{dt} &= \frac{\frac{dV}{dt}}{\frac{1}{3}\cdot\pi\cdot 3\frac{h^2}{4}} = \frac{10}{\left(\frac{1}{3}\right)(\pi)(3)\frac{(4)^2}{4}} \\ &= \frac{5}{2\pi} = 0.795\,77.\end{aligned}$$

Problem 7

Find the intervals on which

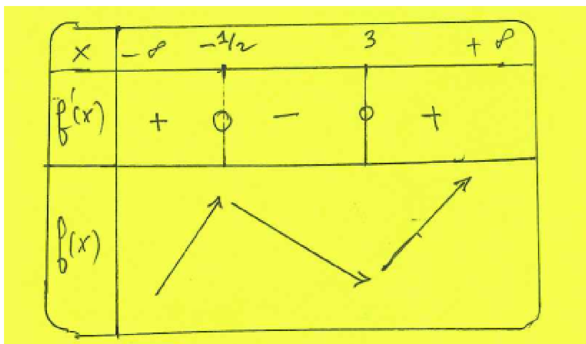
$$f(x) = 4x^3 - 15x^2 - 18x + 10$$

increases and the intervals on which f decreases.

Solution

$$\begin{aligned} f'(x) &= 12x^2 - 30x - 18 \\ &= 6(2x+1)(x-3) \end{aligned}$$

The critical numbers are $-\dfrac{1}{2}$ and 3.



From the table we see that f is decreasing on the interval $\left(-\dfrac{1}{2},\ 3\right)$.

Problem 8

Find the critical numbers and the local extreme values of

$$f(x) = x^{2/3}(5 - x).$$

Solution

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{\frac{-1}{3}}(5-x) - x^{\frac{2}{3}} \\ &= \frac{5}{3\sqrt[3]{x}}(2-x) \end{aligned}$$

The critical numbers are 0 and 2.
The table of variation is given by

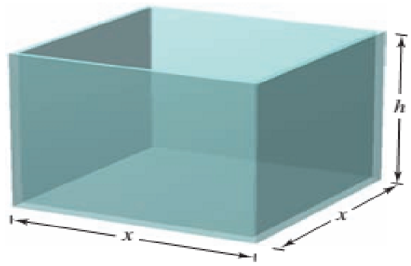
x	$-\infty$		0		2		$+\infty$
$\sqrt[3]{x}$		$-$	0	$+$		$+$	
$2-x$		$+$		$+$	0	$-$	
$f'(x)$		$-$	$\Vert$	$+$	0	$-$	
$f(x)$		$\searrow$	$f(0)$	$\nearrow$	$f(2)$	$\searrow$	

From the table, you see that

- f has a local minimum equal to $f(0) = 0$
- f has a local maximum equal to $f(2) = 2^{2/3}(5-2) = 4.7622$

Problem 9.
If 1200 m^2 of material is used to construct a rectangular box with a square base and an open top. Find the largest possible volume of the box.

Solution



Because the box has a square base, its volume is $V = x^2h$. The area of the box is given by

$$\begin{aligned} S &= \text{area of the base} + \text{area of the four sides} \\ &= x^2 + 4xh = 1200 \end{aligned}$$

Because V is to be maximized, you want to write V as a function of just one variable.
To do this, you can solve the equation $x^2 + 4xh = 1200$ for h in terms of x to obtain $h = \dfrac{1200 - x^2}{4x}$
Substituting into the primary equation produces

$$V = x^2 \left(\frac{1200 - x^2}{4x} \right) = 300x - \frac{x^3}{4}$$

Because $V \geq 0$, the feasible domain is the set of all nonnegative x such that $1200 - x^2 \geq 0$. This set is given by

$$0 \leq x \leq \sqrt{1200}.$$

To maximize find the critical numbers of the volume function by putting $V' = 0$ which is equivalent to

$$300 - \frac{3x^2}{4} = 0 \Longrightarrow x = -20 \text{ or } x = 20.$$

Because -20 is not in the feasible domain, we will only consider $x = 20$.
Now we evaluate the volume at the critical number and at the endpoints.

$$V(0) = 0, \quad V(20) = 4000, \text{ and } V(\sqrt{1200}) = 0.$$

Thus V acchieves a maximum value when $x = 20$. The required dimensions are $x = 20$ and $h = \dfrac{1200 - 20^2}{4\,(20)} = 10.$