$(8pts)$ **Problem 1**
Evaluate the following indefinite integrals

1. $\displaystyle\int \left(x\sqrt{x} + 3x - 1\right) dx$ 2. $\displaystyle\int_1^2 \left(\frac{2}{x} - \frac{3}{x^2}\right) dx.$

Solution
1.

$$\begin{aligned}\int \left(x\sqrt{x} + 3x - 1\right) dx &= \int \left(x^{\frac{3}{2}} + 3x - 1\right) dx \\ &= \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{2}x^2 - x + C \qquad [\mathbf{4pts}]\end{aligned}$$

2.

$$\begin{aligned}\int_1^2 \left(\frac{2}{x} - \frac{3}{x^2}\right) dx &= 2\ln|x| + \frac{3}{x}\bigg|_1^2 \\ &= 2\ln 2 - \frac{3}{2} = -0.113\,71 \qquad [\mathbf{4pts}]\end{aligned}$$

($8pts$) **Problem 2**

Use u-substitution to evaluate the following integrals

$$1. \int_0^1 x^3\sqrt{x^4+1}dx \qquad\qquad 2. \int \sin^3 x\sqrt{\cos x}dx.$$

Solution

1.

$$\int_0^1 x^3\sqrt{x^4+1}dx = ?$$

Put $u = x^4+1$, $\qquad du = 4x^3dx \Rightarrow x^3dx = \dfrac{du}{4}$

$$\begin{aligned} x &= 0 \Rightarrow u = 1 \\ x &= 1 \Rightarrow u = 2 \end{aligned}$$

$$\begin{aligned} \int_0^1 x^3\sqrt{x^4+1}dx &= \int_0^1 \sqrt{x^4+1}x^3dx \\ &= \frac{1}{4}\int_1^2 u^{1/2}du \\ &= \frac{1}{3}\sqrt{2} - \frac{1}{6} = 0.304\,74 \qquad [\mathbf{4pts}] \end{aligned}$$

2.

$$\int \sin^3 x\sqrt{\cos x}dx?$$

$$\begin{aligned} \int \sin^3 x\sqrt{\cos x}dx &= \int \sin^2 x\sqrt{\cos x}\sin x dx \\ &= \int \left(1-\cos^2 x\right)\sqrt{\cos x}\sin x dx. \end{aligned}$$

Now put $u = \cos x$. $du = -\sin x dx$. The integral becomes

$$\begin{aligned} \int \sin^3 x\sqrt{\cos x}dx &= \int \left(u^2-1\right)\sqrt{u}du \\ &= \int \left(u^{5/2}-u^{1/2}\right)du \\ &= \frac{2}{7}u^{\frac{7}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{7}\cos^{\frac{7}{2}} x - \frac{2}{3}\cos^{\frac{3}{2}} x + C \qquad [\mathbf{4pts}] \end{aligned}$$

$(6pts)$ **Problem 3**
Use definite integrals to evaluate

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\sqrt{1+\frac{3i}{n}}.$$

Solution

$$\lim_{n\to\infty}\sum_{i=1}^{n}2\sqrt{1+\frac{3i}{n}}\frac{3}{n} = \int_a^b f(x)dx$$

$$a = 1 \quad \text{and} \quad \frac{b-1}{\cancel{n}} = \frac{3}{\cancel{n}} \Rightarrow b = 4 \qquad [\textbf{2pts}]$$

$$f(x) = 2\sqrt{x}$$

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}2\sqrt{1+\frac{3i}{n}}\frac{3}{n} &= \int_1^4 2\sqrt{x}dx \qquad [\textbf{2pts}]\\ &= 2\left(\frac{2}{3}\right)x^{3/2}\Big|_1^4\\ &= \frac{28}{3} = 9.3333 \qquad [\textbf{2pts}]\end{aligned}$$

($8pts$) **Problem 4**
Solve the quadratic equation and give the answer in the form $a + ib$

$$z^2 - (4+i)z + (5+5i) = 0.$$

Solution

$$a = 1, \quad b = -(4+i) \;\text{ and }\; c = (5+5i).$$

Using the quadratic formula, the solution is given by

$$\begin{aligned} z &= \frac{(4+i) \pm \sqrt{(4+i)^2 - 4\,(1)\,(5+5i)}}{2} \\ &= \frac{(4+i) \pm \sqrt{-5-12i}}{2}. \qquad \textbf{[2pts]} \end{aligned}$$

Next, we need to find $\sqrt{-5-12i}$. Put

$$\sqrt{-5-12i} = a + ib$$

$$(a+ib)^2 = -5 - 12i$$

$\Leftrightarrow$

$$a^2 - b^2 + 2iab = -5 - 12i \Rightarrow$$
$$\begin{cases} a^2 - b^2 = -5 \\ ab = -6 \end{cases}.$$

By inspection, $a = -2$ and $b = 3$ or $a = 2$ and $b = -3$.
Now using the sign convention

$$\sqrt{-5-12i} = -2 + 3i. \qquad \textbf{[2pts]}$$

$$\begin{aligned} z &= \frac{(4+i) \pm \sqrt{-5-12i}}{2} \\ &= \frac{(4+i) \pm (-2+3i)}{2} \end{aligned}$$

$$z = \frac{(4+i) + (-2+3i)}{2} = 1 + 2i \qquad \textbf{[2pts]}$$
or
$$z = \frac{(4+i) - (-2+3i)}{2} = 3 - i. \qquad \textbf{[2pts]}$$

($10pts$) **Problem 5**
Consider the points $P(1,-1,5)$, $Q(0,1,3)$, $R(4,-1,6)$.
(a) Find $\overrightarrow{PQ}-3\overrightarrow{PR}$ and $\overrightarrow{PQ}\cdot\overrightarrow{PR}$.
(b) Find the parametric equations of the line passing through the points P and Q.
(c) Find the area of the triangle with vertices P, Q and R and the equation of the plane containing the points P, Q and R.

Solution
(a)

$$\overrightarrow{PQ}=\langle -1,\ 2,\ -2\rangle \quad \text{and} \quad \overrightarrow{PR}=\langle 3,\ 0,\ 1\rangle$$

$$\begin{aligned}\overrightarrow{PQ}-3\overrightarrow{PR} &= \langle -1,\ 2,\ -2\rangle - 3\langle 3,\ 0,\ 1\rangle \\ &= \langle -1,\ 2,\ -2\rangle - \langle 9,\ 0,\ 3\rangle \\ &= \langle -10,\ 2,\ -5\rangle \qquad [\mathbf{2pts}]\end{aligned}$$

$$\begin{aligned}\overrightarrow{PQ}\cdot\overrightarrow{PR} &= \langle -1,\ 2,\ -2\rangle\cdot\langle 3,\ 0,\ 1\rangle \\ &= -3-2=-5 \qquad [\mathbf{2pts}]\end{aligned}$$

(b) $\overrightarrow{u}=\overrightarrow{PQ}=\langle -1,\ 2,\ -2\rangle$ is a direction vector of the line. Using the point P, the parametric equations are

$$\left\{\begin{array}{c} x=1-t \\ y=-1+2t \\ z=5-2t \end{array}\right. . \qquad [\mathbf{2pts}]$$

(c)

$$Area=\frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|$$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -1 & 2 & -2 \\ 3 & 0 & 1\end{vmatrix}=\langle 2,\ -5,-6\rangle$$

$$\begin{aligned}Area &= \frac{1}{2}\sqrt{4+25+36} \\ &= \frac{1}{2}\sqrt{65}=4.031\,1 \qquad [\mathbf{2pts}]\end{aligned}$$

The equation of the plane containing P, Q and R is given by

$$\begin{aligned}2(x-1)-5(y+1)-6(z-5) &= 0 \\ 2x-5y-6z+23 &= 0 \qquad [\mathbf{2pts}]\end{aligned}$$