Example

1. Evaluate $\lim_{x\to 0} \dfrac{x + 2/x}{x - 3/x}$, if it exists (PDF Q72)

$$\lim_{x\to 0} \frac{x+\frac{2}{x}}{x-\frac{3}{x}}$$

$$= \lim_{x\to 0} \frac{\frac{x^2+2}{\cancel{x}}}{\frac{x^2-3}{\cancel{x}}}$$

$$= \lim_{x\to 0} \frac{x^2+2}{x^2-3}$$

$$= \frac{0+2}{0-3}$$

$$= -\frac{2}{3}$$

Technique 2 — Multiplication by conjugate

Conjugate

$\sqrt{A} + B \xrightarrow{\text{Conj}} \sqrt{A} - B$

$\sqrt{A} - B \xrightarrow{\text{Conj}} \sqrt{A} + B$

$\sqrt{A} - \sqrt{B} \xrightarrow{\text{Conj}} \sqrt{A} + \sqrt{B}$

$\sqrt{A} + \sqrt{B} \xrightarrow{\text{Conj}} \sqrt{A} - \sqrt{B}$

Example (PDF Q30)

Evaluate $\lim_{x\to 1} \dfrac{\sqrt{x^2+4} - \sqrt{5}}{x-1}$

$$\lim_{x\to 1} \frac{\sqrt{x^2+4} - \sqrt{5}}{x-1}$$

$$= \lim_{x\to 1} \frac{\sqrt{x^2+4} - \sqrt{5}}{x-1} \times \frac{\sqrt{x^2+4} + \sqrt{5}}{\sqrt{x^2+4} + \sqrt{5}}$$

$$\lim \quad x^2+4 - 5$$

$$\lim_{x\to 1} \frac{\quad}{(x-1)(\sqrt{x^2+4}+\sqrt{5})}$$

$$= \lim_{x\to 1} \frac{x^2-1}{(x-1)(\sqrt{x^2+4}+\sqrt{5})}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(\sqrt{x^2+4}+\sqrt{5})}$$

$$= \lim_{x\to 1} \frac{x+1}{\sqrt{x^2+4}+\sqrt{5}}$$

$$= \frac{1+1}{\sqrt{1^2+4}+\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}+\sqrt{5}}$$

$$= \frac{2}{2\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}$$

Example – Evaluate $\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+1}-\sqrt{5}}$

$$\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+1}-\sqrt{5}}$$

$$= \lim_{x\to -2} \frac{x+2}{\sqrt{x^2+1}-\sqrt{5}} \times \frac{\sqrt{x^2+1}+\sqrt{5}}{\sqrt{x^2+1}+\sqrt{5}}$$

$$= \lim_{x\to -2} \frac{(x+2)(\sqrt{x^2+1}+\sqrt{5})}{x^2+1-5}$$

$$= \lim_{x\to -2} \frac{(x+2)(\sqrt{x^2+1}+\sqrt{5})}{x^2-4}$$

$$= \lim_{x\to -2} \frac{\cancel{(x+2)}(\sqrt{x^2+1}+\sqrt{5})}{\cancel{(x+2)}(x-2)}$$

$$= \lim_{x\to -2} \frac{\sqrt{x^2+1}+\sqrt{5}}{x-2}$$

$$= \sqrt{(-2)^2+1}+\sqrt{5}$$

$-2-2$

$= \dfrac{\sqrt{5}+\sqrt{5}}{-4}$

$= \dfrac{2\sqrt{5}}{-\cancel{4}_2}$

$= \dfrac{-\sqrt{5}}{2}$

Technique 3 The Squeezing Theorem / The Sandwich Theorem

" If $g(x) \leq f(x) \leq h(x)$ for all x near c and $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} h(x) = L$

then $\boxed{\lim\limits_{x\to c} f(x) = L}$

Example Evaluate $\lim\limits_{x\to 0} 3x^2 \cos \dfrac{4}{x}$

If $\lim = \pm\infty$,
then limit does not exist!

$\lim\limits_{x\to 0} \cos \dfrac{4}{x}$

Limit does not exist for sin and cos!

$\lim\limits_{x\to 0} (-1)^{1/x} \Rightarrow$ Limit does not exist!

Evaluate $\lim\limits_{x\to 0} 3x^2 \cos \dfrac{4}{x}$

cosine of anything is between -1 and 1

$-1 \leq \cos \dfrac{4}{x} \leq 1$

$-3x^2 \leq 3x^2 \cos \dfrac{4}{x} \leq 3x^2$

$\lim\limits_{x\to 0} -3x^2 = 0$

$\lim\limits_{x\to 0} 3x^2 = 0$

$\therefore$ Using Sandwich Th

$\therefore$ Using Sandwich Theorem

$$\lim_{x\to 0} 3x^2 \cos\frac{4}{x} = 0$$

Evaluate $\lim_{x\to 0} x\cos\frac{4}{x}$

$$-1 \leq \cos\frac{4}{x} \leq 1$$

$$\lim_{x\to 0^-} x\cos\frac{4}{x} \qquad x\to 0^-, \; x<0$$

Inequality direction reverses when it is multiplied by a negative!

$$x \leq x\cos\frac{4}{x} \leq -x$$

$$\lim_{x\to 0} x = 0$$

$$\lim_{x\to 0} -x = 0$$

$\therefore$ Using sandwich theorem

$$\lim_{x\to 0} x\cos\frac{4}{x} = 0$$

Technique 4 Limits involving trigonometric functions

$\sin x$, $\cos x$, $\tan x$, $\cot x$, $\operatorname{cosec} x$, $\sec x$

FORMULAS

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \Bigg| \quad \lim_{\square\to 0}\frac{\sin\square}{\square} = 1 \quad \Bigg| \quad \lim_{x\to 0}\frac{\tan x}{x} = 1$$

Example Evaluate $\lim_{x\to 0}\frac{\sin x/2}{3x}$

$$\lim_{x\to 0}\frac{\sin x/2}{3x}$$

$$\lim \frac{1}{} \sin x/2$$

$$\lim_{x\to 0} \frac{1}{3} \cdots \frac{\quad}{x}$$

$$= \lim_{x\to 0} \frac{1}{3} \times \frac{1}{2} \times \frac{\sin x/2}{x/2}$$

$$= \lim_{x\to 0} \frac{1}{6} \frac{\sin x/2}{x/2}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \frac{1}{6} \times 1$$

$$= \frac{1}{6}$$

Example (PDF Q 103)

103. Evaluate $\lim_{\theta\to 0} \frac{\sin 2\theta}{\tan \theta}$

$$\lim_{\theta\to 0} \frac{\sin 2\theta}{\tan \theta}$$

$$= \lim_{\theta\to 0} \frac{2 \cancel{\sin\theta} \cos\theta}{\frac{\cancel{\sin\theta}}{\cos\theta}}$$

$$= \lim_{\theta\to 0} 2\cos^2\theta$$

$$= 2\cos^2(0)$$

$$= \underline{2}$$

$$\lim_{\theta\to 0} \frac{\sin 2\theta}{\tan \theta}$$

$$= \lim_{\theta\to 0} \frac{\sin 2\theta}{\tan \theta} \times \frac{2\theta}{2\theta}$$

$$= \lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \times \frac{2\theta}{\tan \theta}$$

$$= \lim_{\theta\to 0} \frac{2\theta}{\tan\theta}$$

$$= \lim_{\theta\to 0} \frac{2}{\frac{\tan\theta}{\theta}}$$

$$= \lim_{\theta\to 0} 2$$

$$= \underline{2}$$

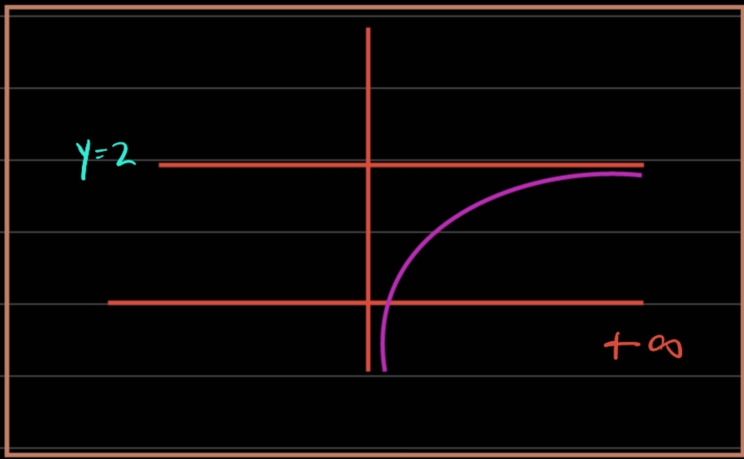
$$\frac{2\theta}{\tan\theta} = \frac{2}{\tan\theta} \times \theta \quad \frac{1}{1/\theta}$$

$$= \frac{2}{\tan\theta} \times \frac{1}{\frac{1}{\theta}} = \frac{2}{\frac{\tan\theta}{\theta}}$$

$$\frac{\tan\theta}{\theta} \to 1$$

Limit at Infinity $\lim_{x\to +\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$

If $\lim_{x \to \pm\infty} f(x) = L$ then the horizontal line $y = L$ is a horizontal asymptote to the graph of f.



Evaluate the following limit

1. $\lim_{x \to +\infty} \frac{3x^4 - 6x^2 + 7}{x - 5x^4}$

2. $\lim_{x \to -\infty} \frac{\sqrt{x^2 + x + 1}}{4x + 1}$

3. $\lim_{x \to -\infty} \frac{3x^3 - 5}{1 - 2|x|^3}$

4. $\lim_{x \to +\infty} \frac{3x^2 + 4x - 1}{5x^3 + 4}$

FORMULAS

1. $\frac{\text{Constant}}{\pm\infty} = 0$

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1. $\lim_{x \to +\infty} \frac{3x^4 - 6x^2 + 7}{x - 5x^4}$

Long rigorous way
Do not be stupid!

$= \lim_{x \to +\infty} \frac{\cancel{x^4}\left(3 - \frac{6}{x^2} + \frac{7}{x^4}\right)}{\cancel{x^4}\left(\frac{1}{x^3} - 5\right)}$

$= \frac{3 - 0 + 0}{0 - 5}$

$= \frac{3}{-5}$

When degrees are the same, the horizontal asymptote is the ratio of the leading coefficients

Steps:
1. Take the highest degree common
2. Apply (constant/∞) formula to the rest of the equation
3. Apply limits to any remaining variable(s).

1. $\lim_{x \to +\infty} \frac{3x^4 - 6x^2 + 7}{x - 5x^4}$

Shorter faster way
Use your brains

$\lim_{x \to +\infty} \frac{3\cancel{x^4}}{-5\cancel{x^4}}$

$= -\frac{3}{5}$

Steps:
1. Divide the leading term in both the numerator and denominator
2. Apply limit on any remaining variable(s)

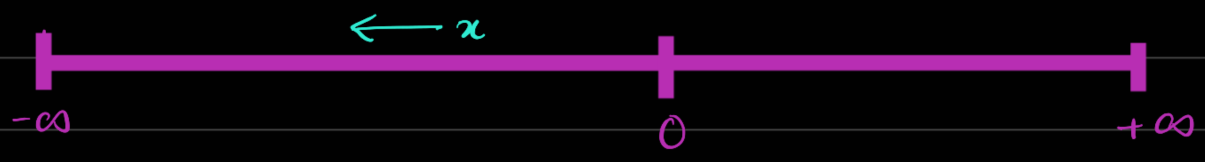
2. $\lim_{x \to -\infty} \frac{\sqrt{x^2 + x + 1}}{4x + 1}$

$\lim_{x \to -\infty} \frac{\sqrt{x^2}}{4x}$

$\boxed{\begin{array}{l} \sqrt{x^2} \neq x \\ \sqrt{x^2} = |x| \end{array}}$

$= \lim_{x \to -\infty} \frac{|x|}{4x}$

$x \to -\infty$



$\therefore$ x MUST be negative

$\lim_{x \to -\infty} \frac{-\cancel{x}}{4\cancel{x}}$

$= \underline{-1/4}$ ↑ Horizontal Asymptote

3. $\lim_{x \to -\infty} \frac{3x^3 - 5}{1 - 2|x|^3}$

$\lim_{x \to -\infty} \frac{3x^3}{-2|x|^3}$

$\lim_{x \to -\infty} \frac{3x^3}{-2(-x)^3}$

$\lim_{x \to -\infty} \frac{3\cancel{x^3}}{2\cancel{x^3}}$

$= 3/2$

4. $\lim_{x \to +\infty} \frac{3x^2 + 4x - 1}{5x^3 + 4}$

$\lim_{x \to +\infty} \frac{3\cancel{x^2}}{5x^{\cancel{3}}}$

$= \lim \frac{3}{}$

$$\lim_{x\to+\infty} \quad 5x$$

$$= \lim_{x\to+\infty} \frac{3}{x} \times \frac{1}{5}$$

$$= \lim_{x\to+\infty} 0 \times \frac{1}{5}$$

$$= \frac{0}{}$$

Infinite Limits

If $\lim_{x\to c^-} f(x) = \pm\infty$ or $\lim_{x\to c^+} f(x) = \pm\infty$ then the vertical line $x = c$ is a vertical asymptote of f.

FORMULAS

$$\frac{\text{constant}}{0} = +\infty \text{ OR } -\infty$$

Example Evaluate $\lim_{x\to 2^-} \frac{x-3}{x^2-4}$

and $\lim_{x\to 2^+} \frac{x-3}{x^2-4}$

$$\frac{x-3}{x^2-4} = \frac{x-3}{(x+2)(x-2)}$$

x	$-\infty$		-2		2		3		$+\infty$
$x-3$		$-$		$-$		$-$	0	$+$	
$x-2$		$-$		$-$	0	$+$		$+$	
$x+2$		$-$	0	$+$		$+$		$+$	
$\frac{x-3}{x^2-4}$		$-$		$+$		$-$		$+$	

Multiply downwards

on the left side
it is $+\infty$

on the right side
it is $-\infty$

Steps:

1. Find zeroes of all equations (except main equation)
2. Draw lines on number line for all zeroes
3. Mark corresponding zero for each expression
4. All signs towards right of marking are positive, all towards left are negative.
5. Multiply each columns' signs (multiply downward), with the final sign being assigned to the main equation.
6. Go back to question and check where the limit approaches.
7. The signs on either side of the line where the limit approaches are the signs for ∞.
8. Sign on left of the line denotes ∞'s sign on LHL. Similarly for right.