($8pts$) **Problem 1**
Evaluate the following limits

$$(a)\quad \lim_{x\to 1}\frac{7-9x^3}{x+1} \qquad (b)\quad \lim_{x\to -\infty}\frac{2\left|x^3\right|+12x+100}{1-9x}$$

$$(c)\quad \lim_{x\to 6}\frac{\sqrt{10+x}-4}{x-6} \qquad (d)\quad \lim_{x\to e}\frac{\ln x^3-3}{x^2-e^2}$$

Solution
(a)

$$\lim_{x\to 1}\frac{7-9x^3}{x+1}=\frac{7-9}{1+1}=-1 \qquad (\textbf{2pts})$$

(b)

$$\begin{aligned}\lim_{x\to -\infty}\frac{2\left|x^3\right|+12x+100}{1-9x} &= \lim_{x\to -\infty}\frac{-2x^3+12x+100}{1-9x}\\ &= \lim_{x\to -\infty}\frac{-2x^3}{-9x}\\ &= \lim_{x\to -\infty}\frac{2x^2}{9}=+\infty \qquad (\textbf{2pts})\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 6}\frac{\sqrt{10+x}-4}{x-6} &= \lim_{x\to 6}\frac{\left(\sqrt{10+x}-4\right)\left(\sqrt{10+x}+4\right)}{(x-6)\left(\sqrt{10+x}+4\right)}\\ &= \lim_{x\to 6}\frac{10+x-16}{(x-6)\left(\sqrt{10+x}+4\right)}\\ &= \lim_{x\to 6}\frac{x-6}{(x-6)\left(\sqrt{10+x}+4\right)}\\ &= \lim_{x\to 6}\frac{1}{\sqrt{10+x}+4}=\frac{1}{8} \qquad (\textbf{2pts})\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to e}\frac{\ln x^3-3}{x^2-e^2} &= \lim_{x\to e}\frac{\frac{d}{dx}\left(\ln x^3-3\right)}{\frac{d}{dx}\left(x^2-e^2\right)}\\ &= \lim_{x\to e}\frac{\frac{3}{x}}{2x}\\ &= \lim_{x\to e}\frac{3}{2x^2}=\frac{3}{2e^2}=0.203 \qquad (\textbf{2pts})\end{aligned}$$

$(8pts)$ **Problem 2**

For what value (s) of the constants p and q that make f continuous at $x = 2$.

$$f(x) = \begin{cases} p + qx, & \text{if } x > 2 \\ 3, & \text{if } x = 2 \\ q - px^2 & \text{if } x < 2 \end{cases}$$

Solution

$$\begin{aligned}
\lim_{x \to 2^-} f(x) &= \lim_{x \to 2^+} f(x) = f(2) = 3 \Leftrightarrow \qquad (\mathbf{2pts}) \\
\begin{cases} p + 2q = 3 \\ q - 4p = 3 \end{cases} &\Leftrightarrow \begin{cases} p + 2q = 3 \\ -4p + q = 3 \end{cases} \qquad (\mathbf{4pts}) \\
q &= \frac{5}{3} \text{ and } p = \frac{-1}{3} \qquad (\mathbf{2pts})
\end{aligned}$$

$(8pts)$**Problem 3**

Find $\dfrac{dy}{dx}$ for

$(a)\quad y = \left(3^x + \log_4 \sqrt{x}\right)(\tan x + \sin x)$ $\qquad (b)\ \ y = \ln \sqrt[9]{x-1}$

$(c)\ y = e^{\cos x + \sec x}$ $\qquad (d)\ xy^3 + e^x y = \ln x$

Solution

$(a)\quad y = \left(3^x + \log_4 \sqrt{x}\right)(\tan x + \sin x)$ Use the product rule

$$\frac{dy}{dx} = \left(3^x \ln 3 + \frac{1}{2x \ln 4}\right)(\tan x + \sin x) + \left(3^x + \log_4 \sqrt{x}\right)\left(\sec^2 x + \cos x\right) \qquad (\mathbf{2pts})$$

$(b)\ \ y = \ln \sqrt[9]{x-1} = \frac{1}{6}\ln(x-1)$

$$\frac{dy}{dx} = \frac{1}{9(x-1)} \qquad (\mathbf{2pts})$$

$(c)\ y = e^{\cos x + \sec x}$ We apply the general exponential rule to get

$$\begin{aligned}\frac{dy}{dx} &= (-\sin x + \sec x \csc x)\, e^{\cos x + \sec x} \\ &= -e^{\cos x + \frac{1}{\cos x}}\left(\sin x - \frac{1}{\cos^2 x}\sin x\right) \qquad (\mathbf{2pts})\end{aligned}$$

$(d)\ xy^3 + e^x y = \ln x$ Perform an implicit differentiation to get

$$\begin{aligned}(y)^3 + 3xy'(y)^2 + e^x y + e^x y' &= \frac{1}{x} \\ 3xy'(y)^2 + e^x y' &= \frac{1}{x} - y^3 - e^x y \\ y'\left(3xy^2 + e^x\right) &= \frac{1}{x} - y^3 - e^x y \\ \frac{dy}{dx} &= y' = \frac{\frac{1}{x} - y^3 - e^x y}{3xy^2 + e^x} \qquad (\mathbf{2pts})\end{aligned}$$

($6pts$) **Problem 4**

Let f be a differentiable function such that $f(2)=2,\ f(4)=1,\ f\,'(2)=3$ and $f\,'(4)=-1$. If $G(x)=f(2x)\cdot f(x)$, find $G'(2)$ and the equation of the tangent line to the graph of G at $x=2$.

Solution

$$G'(x) = 2f\,'(2x)\cdot f(x) + f(2x)\cdot f\,'(x) \qquad (\mathbf{2pts})$$

$$\begin{aligned} G'(2) &= 2f\,'(4)\cdot f(2) + f(4)\cdot f\,'(2) \\ &= 2\,(-1)\,(2) + (1)\,(3) \\ &= -1 \qquad (\mathbf{2pts}) \end{aligned}$$

$$G(2) = f(4)\cdot f(2) = 2$$

The equation of the tangent line is

$$y = G'(2)\,(x-2) + G(2)$$

$$\begin{aligned} y &= (-1)\,(x-2) + 2 \\ &= -x + 4 \qquad (\mathbf{2pts}) \end{aligned}$$

($5pts$) **Problem 5**

A particle is moving along the hyperbola $xy = 16$. As it reaches the point $(8, 2)$, the y-coordinate is decreasing at a rate of $3cm/s$. How fast is the x-coordinate of the point changing at that instant?

Solution

We have

$$\frac{dy}{dt} = -3 \text{ when } x = 8 \text{ and } y = 2.$$

We want to find $\frac{dx}{dt}$ when $x = 8$ and $y = 2$.

The related equation is

$$xy = 16$$

$$\frac{dx}{dt}y + x\frac{dy}{dt} = 0 \qquad \textbf{(2pts)}$$

Now using

$$\frac{dy}{dt} = -3 \text{ when } x = 8 \text{ and } y = 2,$$

we get

$$2\frac{dx}{dt} + (8)(-3) = 0$$

$$\frac{dx}{dt} = 12. \quad \textbf{(3pts)}$$

(5pts)**Problem 6**
Find the open interval (s) over which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 9$ is decreasing.

Solution

$$\begin{aligned} f\,'(x) &= 12x^3 - 12x^2 - 24x \\ &= 12x\,(x+1)\,(x-2) \end{aligned}$$

The critical numbers are 0 -1 and 2. (**2pts**)

x	$-\infty$		-1		0		2		$+\infty$
$12x$		$-$		$-$	0	$+$		$+$	
$x+1$		$-$	0	$+$		$+$		$+$	
$x-2$		$-$		$-$		$-$	0	$+$	
$f'(x)$		$-$		$+$		$-$		$+$	
$f(x)$		$\searrow$		$\nearrow$		$\searrow$		$\nearrow$	

* Increasing on $(-1, 0) \cup (2, \infty)$
* Decreasing on $(-\infty, -1) \cup (0, 2)$

(**3pts**)