u - Substitution

$$\int g'(t)\, f(g(t))\, dt$$

Put $u = g(t)$

$$\frac{du}{dt} = g'(t)$$

$$du = g'(t)\, dt$$

$$\int f(u)\, du$$

Integrate

Substitute $u = g(t)$

Example

Use u-substitution to evaluate the following

$$\int 3x\sqrt{1-2x^2}\, dx$$

$$\int 3x\sqrt{1-2x^2}\, dx$$

$u = 1-2x^2$ $\qquad\qquad$ $u = g(x)$

$du = -4x\, dx \implies x\, dx = -\frac{du}{4}$ $\qquad\qquad$ $du = g'(x)\, dx$

$$\int 3(1-2x^2)^{\frac{1}{2}}\, x\, dx$$

$$= \int 3u^{1/2}\left(-\frac{du}{4}\right)$$

$$= -\frac{3}{4}\int u^{1/2}\, du$$

$$= -\frac{\cancel{3}}{\cancel{4}_2} \times \frac{\cancel{2}}{\cancel{3}}\, u^{3/2} + C$$

$$= -\frac{u^{3/2}}{2} + C = \frac{-(1-2x^2)^{3/2}}{2} + C$$

$$\int x^3 \sqrt{5x^4 - 18} \; dx$$

$u = 5x^4 - 18$

$du = 20x^3 \; dx$

$x^3 dx = \frac{du}{20}$

$$\int u^{1/2} \frac{du}{20}$$

$= \frac{1}{\cancel{20}_{10}} \times \frac{\cancel{2}}{3} u^{3/2} + C$

$= \frac{u^{3/2}}{30} + C$

$= \frac{(5x^4 - 18)^{3/2}}{30} + C$

$$\int x^5 \sqrt{1 + x^2} \; dx$$

$u = x^2 + 1 \implies x^2 = u - 1 \quad \therefore x^4 = (u-1)^2$

$du = 2x \; dx$

$x \, dx = \frac{du}{2}$

$$\int x^4 u^{1/2} \frac{du}{2}$$

$= \frac{1}{2} \int x^4 u^{1/2} du$

$= \frac{1}{2} \int (u-1)^2 u^{1/2} du$

$= \frac{1}{2} \int (u^2 - 2u + 1) u^{1/2} du$

$= \frac{1}{2} \int \left[u^{5/2} - 2u^{3/2} + u^{1/2} \right] du$

$$= \frac{1}{2}\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + C$$

$$= \frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} + \frac{u^{3/2}}{3} + C$$

$$= \frac{(1+x^2)^{7/2}}{7} - \frac{2(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} + C$$

$$\int x^3\sqrt{5x^4-18}\,dx$$

$$= \int x^3(5x^4-18)^{1/2}\,dx$$

$$= \frac{1}{20}\int 20x^3(5x^4-18)^{1/2}\,dx$$

$$= \frac{2}{20}\frac{(5x^4-18)^{3/2}}{3} + C$$

$$= \frac{(5x^4-18)^{3/2}}{30} + C$$

$$\int u'(x)\,u^n(x)\,dx = \frac{u^{n+1}(x)}{n+1} + C$$

$$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$$

$$\int u'(x)\,e^{u(x)}\,dx = e^{u(x)} + C$$

$$\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx$$

$$F(x) = \int \frac{x}{\sqrt{9-x^2}}\,dx$$

$$F(4) - F(0)$$

$$\int_0^4 x(9+x^2)^{-1/2}\,dx$$

$$\int_a^b u'(x)\,u^n(x)\,dx = \left.\frac{u^{n+1}(x)}{n+1}\right|_a^b$$

$u = 9+x^2 \Rightarrow$ @ $x=0$, $u=9$

$du = 2x\,dx$ @ $x=4$, $u=25$

$x\,dx = \frac{du}{2}$

$$\int_0^4 (9+x^2)^{-1/2}\,x\,dx = \int_9^{25} u^{-1/2}\frac{du}{2}$$

$$= \frac{1}{2}\left[2u^{1/2}\right]_9^{25}$$

$$= \frac{1}{2} \left[2(25)^{1/2} - 2(9)^{1/2} \right]$$

$$= \frac{1}{2} [10 - 6]$$

$$= \frac{2}{2}$$

Trigonometric Integrals

In this section we will apply u-substitution to evaluate integrals of the form

$$\int \sin^m x \cos^n x \, dx \quad \text{and} \quad \int \sec^n x \tan^m x \, dx$$

$$\int \sin^m x . \cos^n x \, dx$$

Rule 1 :

If the power of $\sin x$ is odd and +ve then save one $\sin x$ next to the dx, convert the rest into $\cos x$ and do the substitution $u = \cos x$

$$\int \frac{\sin^3 x}{\sqrt{\cos x}} \, dx$$

$$= \int \frac{\sin^2 x}{\sqrt{\cos x}} \sin x \, dx$$

$$= \int \frac{1 - \cos^2 x}{\sqrt{\cos x}} \sin x \, dx$$

$u = \cos x$

$du = - \sin x \, dx$

$\sin x \, dx = - du$

$$-1\int \frac{1-u^2}{\sqrt{u}}\,du$$

$$= \int u^{3/2} - u^{-1/2}\,du$$

$$= \frac{2}{5}u^{5/2} - 2u^{1/2} + C$$

$$= \frac{2\cos^{5/2} x}{5} - 2\sqrt{\cos x} + C$$

Rule 2

If the power of $\cos x$ is odd and +ve then save one $\cos x$ next to the dx, convert the rest into $\sin x$ and do the substitution $u = \sin x$

$$\int \frac{\cos^5 x}{\sin x}\,dx$$

$$= \int \frac{\cos^4 x}{\sin x}\cos x\,dx$$

$$= \int \frac{(1-\sin^2 x)^2}{\sin x}\cos x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$\int \frac{(1-u^2)^2}{u}\,du$$

$$= \int \frac{1+u^4-2u^2}{u}\,du$$

$$= \int \left(u^3 - 2u + \frac{1}{u}\right) du$$

$$= \frac{u^4}{4} - u^2 + \ln|u| + C$$

$$= \frac{\sin^4 x}{4} - \sin^2 x + \ln|\sin x| + c$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

Rule 3

If the powers of $\sin x$ and $\cos x$ are both even and positive, then use trigonometric identities

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

expand and use Rule 1 or Rule 2 if necessary

$$\int \sin^2 x \cos^2 x \, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{4}\int 1-\cos^2 2x \, dx$$

$$= \frac{1}{4}\int 1-\left[\frac{1+\cos 4x}{2}\right] dx$$

$$= \frac{1}{4}\int 1-\frac{1}{2}+\frac{\cos 4x}{2} \, dx$$

$$= \frac{1}{4}\int \frac{1}{2}+\frac{\cos 4x}{2} \, dx$$

$$* \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$$

$$= \frac{1}{8}\int (1-\cos 4x)\, dx$$

$$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right]$$

$$* \quad \int \sin(ax+b)\,dx = -\frac{1}{a}\cos(a\ldots$$

$$\int \sec^n x \tan^m x \, dx$$

Rule 1

If the power of sec x is even and positive, then save one $\sec^2 x$ next to the dx, convert the rest into tan x and do the substitution $u = \tan x$

$$\int \sec^4 x \tan x \, dx$$

$$\int \sec^2 x \tan x \sec^2 x \, dx$$

$$\sec^2 x = 1 + \tan^2 x$$

$$= \int (1+\tan^2 x) \tan x \sec^2 x \, dx$$

$$u = \tan x$$

$$du = \sec^2 x$$

$$\int (1+u^2)\, u \, du$$

$$= \int u + u^3 \, du$$

$$= \frac{u^2}{2} + \frac{u^4}{4} + C$$

$$= \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C$$

Rule 2

If the power of tan x is odd and positive, then save one $\sec x \tan x$ next to the dx, convert the rest into $\sec x$ and do the substitution $u = \sec x$

$$\int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx$$

$$= \int \frac{\tan^2 x}{(\sec x)^{3/2}} \sec x \tan x \, dx$$

$$= \int \frac{(\sec^2 x - 1)}{(\sec x)^{3/2}} \sec x \tan x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int \frac{(u^2 - 1)}{u^{3/2}} \, du$$

$$= \int \left[u^{1/2} - u^{-3/2} \right] du$$

$$= \frac{2}{3} u^{3/2} + 2u^{-1/2} + c$$

$$= \frac{2}{3} \sec^{3/2} x + \frac{2}{\sqrt{\sec x}} + c$$

Complex Numbers

$$(-1) = (-1)^1 = (-1)^{2/2} = (-1)^{2\cdot\frac{1}{2}} = \left[(-1)^{1/2}\right]^2$$

The number whose square is equal to -1 is denoted by i and is called an imaginary number

$$i^2 = -1$$
$$i = \sqrt{-1}$$

Complex Number

Any number of the form $z = a + ib$ where a and b are real numbers and $i^2 = -1$ is called a complex number.

Example

Solve for x, the equation $x^2 - 2x + 3 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 4(1)(3)}}{2}$$

$$= \frac{2 \pm \sqrt{-8}}{2}$$

$$= \frac{2 + 2\sqrt{2}i}{2} \quad \text{OR} \quad \frac{2 - 2\sqrt{2}i}{2}$$

$$= 1 + i\sqrt{2} \quad \text{OR} \quad 1 - i\sqrt{2}$$

Operation of complex numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers

Consider the complex number

$$z = \underbrace{a}_{\text{Real part}} + i\underbrace{\boxed{b}}_{\text{Imaginary part}}$$

$Re(z) = a$

	Complex	Real	Imaginary
$\frac{1}{2} - \frac{i}{3}$	Yes	$1/2$	$-1/3$
$2i$	Yes	0	2
2	Yes	2	0
0	Yes	0	0
$-1 + i\sqrt{2}$	Yes	-1	$\sqrt{2}$
$3/2$	Yes	$3/2$	0

All Real Numbers are Complex Numbers

Conjugate of Complex Numbers

If $z = a + ib$ then the conjugate of z, denoted by $\bar{z}$, is given by

$$\bar{z} = a - ib$$

$$\overline{2 - i} = 2 + i$$

$$\overline{-i} = i$$

$$\overline{3/2} = 3/2$$