Exam Solutions

(10pts)Problem 1.
Evaluate the following limits

1. $\lim_{x\to 3} \frac{\sqrt{2x+3}-3}{x-3}$ 2. $\lim_{x\to -1^+} \frac{2|x|-2}{x+1}$ 3. $\lim_{x\to -\infty} \frac{3|x|-1}{2x+7}$

Solution of Problem 1
1.

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{2x+3}-3}{x-3} &= \lim_{x\to 3} \frac{\left(\sqrt{2x+3}-3\right)\left(\sqrt{2x+3}+3\right)}{(x-3)\left(\sqrt{2x+3}+3\right)} \\
&= \lim_{x\to 3} \frac{2x-6}{(x-3)\left(\sqrt{2x+3}+3\right)} \\
&= \lim_{x\to 3} \frac{2(x-3)}{(x-3)\left(\sqrt{2x+3}+3\right)} \\
&= \lim_{x\to 3} \frac{2}{\left(\sqrt{2x+3}+3\right)} = \frac{2}{6} = \frac{1}{3} = 0.333\,33 \qquad (\mathbf{3pts}) \\
&\text{(You may also use the l'Hospital's rule)} \\
\lim_{x\to 3} \frac{\sqrt{2x+3}-3}{x-3} &= \lim_{x\to 3} \frac{\frac{1}{\sqrt{2x+3}}}{1} = \frac{1}{3}
\end{aligned}$$

2.

$$\lim_{x\to -1^+} \frac{2|x|-2}{x+1}$$

When $x \to -1^+$, then $x < 0$ and $|x| = -x$. Thus,

$$\begin{aligned}
\lim_{x\to -1^+} \frac{2|x|-2}{x+1} &= \lim_{x\to -1^+} \frac{-2x-2}{x+1} \\
&= \lim_{x\to -1^+} \frac{-2(x+1)}{x+1} = -2 \qquad (\mathbf{4pts})
\end{aligned}$$

3.

$$\begin{aligned}
\lim_{x\to -\infty} \frac{3|x|-1}{2x+7} &= \lim_{x\to -\infty} \frac{-3x-1}{2x+7} \\
&= \lim_{x\to -\infty} \frac{-3\cancel{x}}{2\cancel{x}} = \frac{-3}{2} = -1.5 \qquad (\mathbf{3pts})
\end{aligned}$$

(10pts)Problem 2.
Given that $f(2)=3$, find the values of a and b for which the function

$$f(x)=\begin{cases} \ln x & \text{if } 0<x\le 1 \\ ax^2+b & \text{if } 1<x\le 5 \end{cases}$$

is continuous at $x=1$.

Solution of Problem 2

$$\begin{aligned} f(2) &= 3 \Rightarrow \\ 4a+b &= 3 \qquad\qquad (\textbf{3pts}) \end{aligned}$$

For the function to be continuous at $x=1$, we must have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x).$$

Equivalently,

$$\ln 1 = a+b \Leftrightarrow a+b=0. \qquad\qquad (\textbf{3pts})$$

We have the system

$$\begin{cases} 4a+b=3 \\ a+b=0 \end{cases}.$$

Solution is:

$$a=1, \qquad b=-1 \qquad\qquad (\textbf{4pts})$$

(10pts)Problem 3

Find the equation of the tangent line to the graph of $f(x) = \dfrac{1+\ln x}{x^2+1}$ at $x = 1$.

Solution of Problem 3

$$
\begin{aligned}
f\ '(x) &= \frac{\left(\frac{1}{x}\right)(x^2+1) - 2x\,(1+\ln x)}{(x^2+1)^2} \\
&= \frac{\frac{1}{x}\,(2x^2\ln x + x^2 - 1)}{(x^2+1)^2}. \qquad (\mathbf{4pts})
\end{aligned}
$$

The slope of the tangent line is

$$f\ '(1) = 0. \qquad (\mathbf{2pts})$$

$$f(1) = \frac{1}{2}.$$

The equation of the tangent line is

$$y = 0\,(x-1) + \frac{1}{2}$$

$\Leftrightarrow$

$$y = \frac{1}{2} \qquad (\mathbf{4pts})$$

(10pts)Problem 4.

A) Find $\frac{dy}{dx}$ if

$$y \sin x + y^3 = 2x + 1.$$

Solution of Problem 4

Here, we use implicit differentiation

$$3y' (y)^2 + (\cos x) y + (\sin x) y' = 2$$

$\Leftrightarrow$

$$3y' (y)^2 + (\sin x) y' = 2 - (\cos x) y$$

Now solving for y', we get

$$y' = \frac{2 - y \cos x}{3y^2 + \sin x} \qquad (\mathbf{5pts})$$

B) The radius of a cylinder is increasing at a rate of 3 cm / sec and the height is increasing at a rate of 2 cm / sec. How fast is the volume changing when the radius is 1 cm and the height is 4 cm? (The volume of a cylinder is $V = \pi r^2 h$).

Solution

We have

$$\frac{dr}{dt} = 3 \; cm \; / \; \text{sec and} \; \frac{dh}{dt} = 2 \; cm \; / \; \text{sec}.$$

We need to find $\frac{dV}{dt}$ when $r = 1$ and $h = 4$.

The related rate equation is

$$\begin{aligned} \frac{dV}{dt} &= 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt} \\ &= 2\pi (1)(3)(4) + \pi (1)^2 (2) \\ &= 26\pi \; cm^3 / \text{sec} \\ &= 81.681 \; cm^3 / \text{sec}. \qquad (\mathbf{5pts}) \end{aligned}$$

(10pts)Problem 5.
Find the absolute extrema of the function $g(x) = \sqrt{x}\,(x-3)$ on $[0,\ 4]$.

Solution of Problem 5

$$g(x) = \sqrt{x}\,(x-3) = x^{3/2} - 3x^{1/2}$$

$$\begin{aligned} g'(x) &= \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2} \\ &= \frac{3}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) \\ &= \frac{3}{2\sqrt{x}}\,(x-1). \qquad \textbf{(3pts)} \end{aligned}$$

The critical numbers are 0 and 1. They are both in the interval $[0,\ 4]$ (**3pts**)

$$\begin{gathered} f(0) = 0 \\ f(1) = -2 \\ f(4) = 2. \end{gathered}$$

Thus,

The absolute Maximum is equal to 2 (**2pts**)
and
The absolute Minimum is equal to -2. (2**pts**)

(10pts)Problem 6.
Find the smallest possible perimeter of a rectangle of area 50 cm^2.

Solution of Problem 6
Denote by x and y the length and the width of the rectangle respectively. We have

$$xy = 50. \qquad x > 0,\ y > 0 \qquad\qquad (2\textbf{pts})$$

We want to minimize the perimeter

$$P = 2x + 2y. \qquad\qquad (2\textbf{pts})$$

Using

$$y = \frac{50}{x},$$

We have

$$P = 2x + \frac{100}{x}.$$
$$P' = 2 - \frac{100}{x^2}.$$

$$\begin{aligned} P' &= 0 \Leftrightarrow 2 - \frac{100}{x^2} = 0 \Leftrightarrow x = -5\sqrt{2} \text{ or } x = 5\sqrt{2} \qquad\qquad (3\textbf{pts}) \\ \text{We take } x &= 5\sqrt{2} \end{aligned}$$

$$P'' = \frac{200}{x^3} > 0$$

The minimum of the perimeter is achieved when

$$\begin{aligned} x &= 5\sqrt{2} = 7.071\,1 \text{ and} \\ y &= \frac{50}{5\sqrt{2}} = 5\sqrt{2} = 7.071\,1 \qquad\qquad (3\textbf{pts}) \end{aligned}$$

$$\begin{aligned} \text{The smallest possible perimeter is } P &= 2\left(5\sqrt{2} + 5\sqrt{2}\right) \\ &= 20\sqrt{2} = 28.284 \qquad\qquad (3\textbf{pts}) \end{aligned}$$

(10pts)Problem 7.
Find the open intervals on which the function $f(x) = 2x - 3x^{2/3}$ is concave up or down.

Solution of Problem 7

$$f\ '(x) = 2 - 2x^{-1/3} \qquad (3\textbf{pts})$$

$$f\ ''(x) = \frac{2}{3}x^{-4/3} > 0 \qquad (3\textbf{pts})$$

The function is concave up on $(-\infty,\ \infty)$. $\quad (4\textbf{pts})$

(10pts)Problem 8.
Use definite integrals to evaluate

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2\pi}{n}\cos\left(-\frac{\pi}{2}+\frac{\pi i}{n}\right).$$

Solution of Problem 8
Here we will use the formula

$$\lim_{n\to\infty}\sum_{i=1}^{n}f(a+i\frac{b-a}{n})\left(\frac{b-a}{n}\right)=\int_a^b f(x)dx.$$

$$a=-\frac{\pi}{2}\quad\text{and}\quad\frac{b-a}{\cancel{n}}=\frac{\pi}{\cancel{n}}\qquad(2\textbf{pts})$$

$$b=\pi-\frac{\pi}{2}=\frac{\pi}{2}\qquad(2\textbf{pts})$$

$$f(x)=2\cos x.\qquad(3\textbf{pts})$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2\pi}{n}\cos\left(-\frac{\pi}{2}+\frac{\pi i}{n}\right) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\cos x dx\\ &= [2\sin x]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}=4 \qquad(3\textbf{pts})\end{aligned}$$

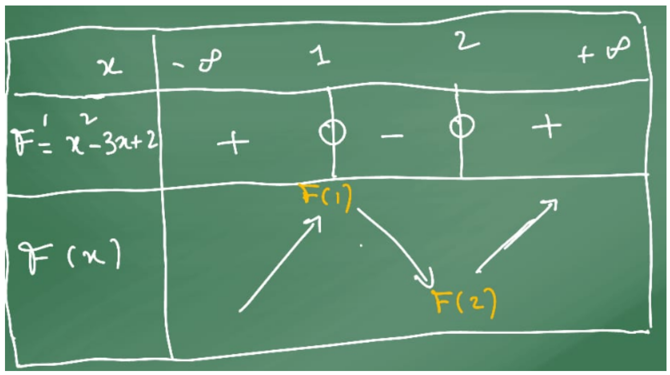
(10pts)Problem 9.
Find the local extrema of the function

$$F(x) = \int_0^x \left(t^2 - 3t + 2\right) dt$$

Solution of Problem 9

$$\begin{aligned} F'(x) &= x^2 - 3x + 2 \\ &= (x-1)(x-2) \qquad (4\textbf{pts}) \end{aligned}$$

The critical numbers are $x = 1$ and $x = 2$.



$$\text{Local Maximum } = F(1) = \int_0^1 \left(t^2 - 3t + 2\right) dt = \frac{5}{6} = 0.833\,33 \qquad (3\textbf{pts})$$

$$\text{Local Minimum } = F(2) = \int_0^2 \left(t^2 - 3t + 2\right) dt = \frac{2}{3} = 0.666\,67 \qquad (3\textbf{pts})$$

(10pts)Problem 10.
Use u-substitution to evaluate

$$\int x^2\left(1-x^3\right)^5 dx$$

Solution of Problem 10

$$\int x^2\left(1-x^3\right)^5 dx=\int\left(1-x^3\right)^5 x^2 dx$$

Put

$$\begin{aligned} u &= 1-x^3. \\ du &= -3x^2dx \Rightarrow x^2dx=\frac{-du}{3} \qquad (4\mathbf{pts}) \end{aligned}$$

$$\begin{aligned} \int x^2\left(1-x^3\right)^5 dx &= \frac{-1}{3}\int u^5du=-\frac{1}{18}u^6+C \qquad (4\mathbf{pts}) \\ &= -\frac{1}{18}\left(1-x^3\right)^6+C. \qquad (2\mathbf{pts}) \end{aligned}$$