Complex Numbers

$i = \sqrt{-1}$

Addition / Subtraction / Multiplication - Perform operations on real and imaginary parts separately

Division

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2}$$

$$= \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$$

$$= \frac{a_1 a_2 - i(a_1 b_2 - a_2 b_1) + b_1 b_2}{a_2^2 - b_2^2}$$

Square root

$$\sqrt{x + iy} = a + ib$$

$$x + iy = (a + ib)^2$$

$$= a^2 + 2aib + i^2 b^2$$

$$= a^2 - b^2 + 2aib$$

$a^2 - b^2 = x$ $\quad$ $2aib = iy$

$2ab = y$

Get two values for ab

Sign of x = Sign of a

$|z| = \sqrt{a^2+b^2}$

↳ Distance from (0,0) to z

$a+ib = (a,b)$

$z = a + ib$

$= r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

↳ $r = |z| = \sqrt{a^2+b^2}$

θ is the "argument" of z.

$\cos\theta = \frac{a}{r}$

$\sin\theta = \frac{b}{r}$

$\tan\theta = \frac{b}{a}$

When finding angle, find which quadrant it belongs to and use unit circle in formula sheet to find exact angle.

Euler Representation

$z = re^{i\theta}$

$e^{i\pi} + 1 = 0$

Three Dimensional Space

Distance between two points

$$D = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$P(x_1, y_1, z_1)$

$Q(x_2, y_2, z_2)$

Midpoint of two points

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Equation of Sphere

$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$ where centre (a,b,c) and radius (x,y,z)

Completing the square

$$x^2 + kx = \left(x + \frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2$$

Vectors

$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ where $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

$\vec{a} \pm \vec{b} = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle$ where $\vec{a} = \langle a_1, a_2, a_3 \rangle$
and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\lambda \vec{a} = \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle$

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Unit Vector $\vec{u} = \dfrac{\vec{a}}{\|\vec{a}\|}$

Dot Product

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ where $\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$

$\vec{a} \perp \vec{b}$ if $\vec{a} \cdot \vec{b} = 0$

Projection $\quad \text{proj}_{\vec{b}} \vec{a} = \left(\dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \right) \vec{b}$

Cross Product

Let $\vec{m} = \vec{a} \times \vec{b}$

$\vec{m} \perp \vec{a}$ & $\vec{m} \perp \vec{b}$

$\|\vec{m}\|$ = Area of parallelogram with adjacent side $\vec{a}$ and $\vec{b}$

$\vec{a} \parallel \vec{b}$ if $\vec{a} \times \vec{b} = 0$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \langle bf - ce, -(af - cd), ae - bd \rangle$$

where $\vec{a} = \langle a, b, c \rangle$ & $\vec{b} = \langle d, e, f \rangle$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$
$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$
$$\vec{a} \times \vec{a} = 0$$

Area of $\Delta = \frac{1}{2} \|\vec{a} \times \vec{b}\|$

Equation of line in space

$\vec{u} = \langle a, b, c \rangle \Rightarrow$ Direction Vector

Let $\vec{PP_0}$ be a line

$\vec{PP_0} \parallel \vec{u}$

$\vec{PP_0} = t\vec{u}$

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t \langle a, b, c \rangle$$

$x - x_0 = at \Rightarrow x = x_0 + at$

$y - y_0 = bt \Rightarrow y = y_0 + bt$

$z - z_0 = ct \Rightarrow z = z_0 + ct$

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad \text{where } a, b, c \neq 0$$

$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \Rightarrow$ Symmetric Equations

Distance from point to line

$$\vec{d} = \frac{\| \vec{PS} \times \vec{u} \|}{\| \vec{u} \|}$$

Equation of plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Normal Vector

$\vec{n} = \langle a, b, c \rangle$

Angle between two planes

Angle between two planes = Angle between their normal vectors

$$\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\| \, \|\vec{n}_2\| \cos\theta$$

Distance between point and plane

$$d = \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}$$

where P is point on plane
Q is point in space
$\vec{n}$ = normal vector

NOTE
If P is not given, take equation of plane, take ANY two variables as 0, and get the value for the last variable.

E.g. $6x - 2y + z = 5$
$x = 0, \; y = 0$
$0 + 0 + z = 5$
$z = 5$
$P\,(0, 0, 5)$

Linear Algebra (Matrices)

Elementary Operations

The solution sets of two systems remain the same if

Example

$\downarrow$

→ an equation is swapped with another — $R_1 \leftrightarrow R_2$

→ an equation is multiplied, on both sides, by a non zero constant — $R_1 \to 2R_1$

→ an equation is replaced by the sum of itself and a multiple of another (pivoting) — $R_1 \to R_1 - 3R_2$

While solving, eliminate the first variable starting from 2^{nd} equation (inclusive), second variable from third equation (inclusive) and so on.

Example

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_2 - 7x_3 = -17 \\ 3x_2 - 11x_3 = -27 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_2 - 7x_3 = -17 \\ x_3 = 3 \end{cases}$$

$x_3 = 3$

Substitute in R_2

$x_2 = 2$

Substitute x_3 and x_2 in R_1

$x_1 = 1$

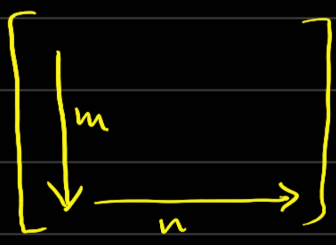
Solution = $(1, 2, 3)$ ⇒ IMPORTANT

If no. of variables > no. of equations
then NO UNIQUE SOLUTION

Matrix

Rectangular array of numbers

Size = rows x columns



$m \times n$

Example

$$
\begin{aligned}
x_1 + 2x_2 \quad\quad &= 4 \\
x_2 - x_3 &= 0 \\
x_1 \quad\quad + 2x_3 &= 4
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}
\qquad\qquad
\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right]
$$

Coefficient Matrix

Augmented Matrix

Steps to reach Reduced Echelon Form

1. Write augmented matrix
2. Use elementary operations (Gauss Elimination same same) to reduce matrix
3. Back substitution (just like Gauss Elimination)

$A(BC) = (AB)C$

$A(B+C) = AB + AC$

$k(AB) = (kA)B = A(kB)$

Inverse of a Matrix

$AA^{-1} = A^{-1}A = I$

Gauss-Jordan Method

$(A \mid I) \longrightarrow (I \mid B)$

↳ Elementary Operations

Transpose

1^{st} row = 1^{st} column

2^{nd} row = 2^{nd} column

$\vdots \qquad \vdots$

Square matrix is symmetric if $A^T = A$

$(A^T)^T = A$

$(MN)^T = N^T M^T$