($5pts$) **Problem 1**

Evaluate the following limits

$$(a)\quad \lim_{x\to 0}\frac{e^x-1}{2x} \qquad\qquad (b)\quad \lim_{x\to -\infty}\frac{\sqrt{4x^2-7x+1}}{3x+2}$$

Solution

(a)

$$\lim_{x\to 0}\frac{e^x-1}{2x} = \text{"}\frac{0}{0}\text{"}$$

We use the l'Hospital's rule.

$$\lim_{x\to 0}\frac{e^x-1}{2x} = \lim_{x\to 0}\frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}. \qquad\qquad (\mathbf{2pts})$$

(b)

$$\begin{aligned}
\lim_{x\to -\infty}\frac{\sqrt{4x^2-7x+1}}{3x+2} &= \lim_{x\to -\infty}\frac{\sqrt{4x^2}}{3x} \\
&= \lim_{x\to -\infty}\frac{2\,|x|}{3x} \\
&= \lim_{x\to -\infty}\frac{2\,(-\cancel{x})}{3\cancel{x}} \qquad (\text{ when } x\to -\infty,\ \text{ then } x<0 \Rightarrow |x| = -x\ ) \\
&= \lim_{x\to -\infty}\frac{-2}{3} = \frac{-2}{3}. \qquad\qquad (\mathbf{3pts})
\end{aligned}$$

$(5pts)$ **Problem 2**

Find the value (s) of the constant p for which the function f continuous at $x = 0$

$$f(x) = \begin{cases} \dfrac{\sin 2px}{3x}, & \text{if } x > 0 \\ x^2 + x + 1 & \text{if } x \leq 0 \end{cases}$$

Solution

f continuous at $x = 0$ if and only if

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \qquad \textbf{(2pts)}$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \frac{\sin 2px}{3x} \\ &= \lim_{x \to 0^+} \frac{1}{3} \cdot \frac{\sin 2px}{x} \\ &= \lim_{x \to 0^+} \frac{2p}{3} \cdot \frac{\sin 2px}{2px} \\ &= \frac{2p}{3}. \end{aligned}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x^2 + x + 1\right) = 1 = f(0).$$

Thus, the condition of continuity gives the equation

$$\frac{2p}{3} = 1$$

$$p = \frac{3}{2}. \qquad \textbf{(3pts)}$$

$(5pts)$**Problem 3**

Find $\dfrac{dy}{dx}$ for

$$(a) \quad y = \ln\left[\frac{2x+1}{(x-6)^5}\right] \qquad\qquad (b) \quad x^2y + y^3\sec x = 3$$

Solution

(a)

$$\begin{aligned} y &= \ln\left[\frac{2x+1}{(x-6)^5}\right] = \ln(2x+1) - \ln(x-6)^5 \\ &= \ln(2x+1) - 5\ln(x-6) \end{aligned}$$

Now we use the formula

$$\frac{d}{dx}\ln u(x) = \frac{u'(x)}{u(x)} \quad \text{to get}$$

$$\frac{dy}{dx} = \frac{2}{2x+1} - \frac{5}{x-6} \qquad (\mathbf{2pts})$$

(b) $x^2y + y^3\sec x = 3$

We first differentiate both sides to get

$$2xy + x^2y' + 3y'y^2\sec x + y^3\sec x\tan x = 0$$

Now keep the terms with y' on the left and move all other terms to the right.

$$x^2y' + 3y'y^2\sec x = -2xy - y^3\sec x\tan x.$$

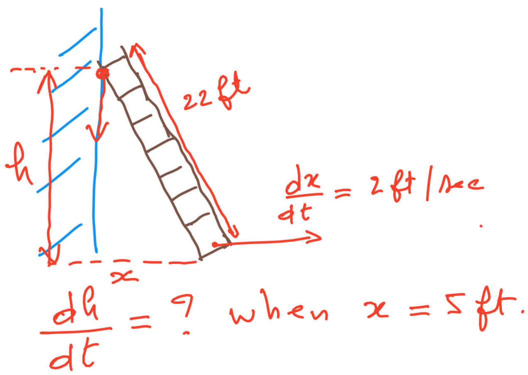
Now factor y' and solve to get

$$y' = \frac{dy}{dx} = \frac{-2xy - y^3\sec x\tan x}{x^2 + 3y^2\sec x} \qquad (\mathbf{3pts})$$

$(5pts)$ **Problem 4**

A 22-foot ladder is leaning against the wall of a house. The base of the ladder is being pulled away from the wall at a rate of 2 feet per second. How fast is the top of the ladder moving down the wall when the base is 5 feet from the wall?

Solution



(**2pts**)

The related equation is

$$x^2 + h^2 = 22^2.$$

The related rate equation is

$$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$$

$$\frac{dx}{dt} = 2 \text{ and when } x = 5,\ h = \sqrt{22^2 - 5^2} = 21.424.$$

We have

$$(2)(5)(2) + (2)(21.424)\frac{dh}{dt} = 0$$

$$\frac{dh}{dt} = \frac{-(2)(5)(2)}{(2)(21.424)} = -0.46677\ ft/s. \qquad \textbf{(3pts)}$$

$(5pts)$ **Problem 5**

Find the absolute extrema of the function $f(x) = 2x - 3x^{\frac{2}{3}}$ on the interval $[-1,\ 3]$.

Solution

$$f(x) = 2x - 3x^{\frac{2}{3}}$$

$$\begin{aligned} f\,'(x) &= 2 - 2x^{\frac{-1}{3}} \\ &= 2\left(1 - \frac{1}{\sqrt[3]{x}}\right). \end{aligned}$$

The critical numbers are 0 and 1. (**2pts**)

$$f(0) = 0$$

$$f(1) = -1$$

$$f(-1) = -5$$

$$f(3) = (2)(3) - (3)(3)^{\frac{2}{3}} = -0.240\,25$$

$$\text{Absolute Max } = 0 \text{ and Absolute Min } = -5 \qquad (\textbf{3pts})$$

$(5pts)$**Problem 6**
Use definite integrals to evaluate

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{5}{n}\left(1+\frac{3i}{n}\right)^3$$

Solution
Here we use the formula

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(a+i\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) = \int_a^b f(x)dx.$$

$$a = 1 \text{ and } \frac{b-a}{\cancel{n}} = \frac{3}{\cancel{n}} \Rightarrow b = 4$$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{5}{n}\left(1+\frac{3i}{n}\right)^3 &= \lim_{n\to\infty}\sum_{i=1}^{n} 5\left(1+\frac{3i}{n}\right)^3\frac{1}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{5}{3}\left(1+\frac{3i}{n}\right)^3\frac{3}{n} \Rightarrow f(x) = \frac{5}{3}x^3. \quad (\mathbf{3pts})
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{5}{n}\left(1+\frac{3i}{n}\right)^3 &= \int_1^4 \frac{5}{3}x^3dx \\
&= \frac{5}{3}\frac{x^4}{4}\bigg|_1^4 = \frac{425}{4} = 106.25 \qquad (\mathbf{2pts})
\end{aligned}$$

$(5pts)$**Problem 7**
Find the critical numbers of the function

$$F(x) = \int_1^x \left(2t^2 + 5t + 3\right) dt$$

Solution
Here we use the first fundamental theorem

$$\frac{d}{dx} \int_a^x f(t)dt = f(x).$$

$$\begin{aligned} F'(x) &= 2x^2 + 5x + 3 \\ &= (2x+3)(x+1) \qquad \textbf{(3pts)} \end{aligned}$$

The critical numbers are

$$x = \frac{-3}{2} \text{ and } x = -1. \qquad \textbf{(2pts)}$$

$(5pts)$**Problem 8**
Find the average value of the function $f(x) = x\sqrt{x^2+3}$ on the interval $[1,6]$.
Solution

$$f_{ave} = \frac{1}{6-1}\int_1^6 x\sqrt{x^2+3}dx.$$

Now we use the u-substitution to evaluate $\int_1^6 x\sqrt{x^2+3}dx$.
Put

$$u = x^2+3, \qquad du = 2xdx \Rightarrow xdx = \frac{du}{2}$$
$$x = 1 \Rightarrow u = 4 \qquad \text{and } x = 6 \Rightarrow u = 39$$

$$\begin{aligned} f_{ave} &= \frac{1}{5}\int_1^6 \sqrt{x^2+3}xdx = \frac{1}{5}\int_4^{39} u^{1/2}\frac{du}{2} \\ &= \frac{1}{10}\int_4^{39} u^{1/2}du \qquad (\mathbf{3pts}) \\ &= \frac{1}{10}\frac{2}{3}u^{3/2}\Big|_4^{39} \\ &= \frac{13}{5}\sqrt{39} - \frac{8}{15} \\ &= 15.704. \qquad (\mathbf{2pts}) \end{aligned}$$