Part 1 MCQ (30%)

($6pts$) **Problem 1**

If f is an integrable function, then

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}f\left(1+\frac{6i}{n}\right)=$$

(a) $\int_1^7 f(x)\,dx$

(b) $\int_1^6 f(x)\,dx$

(c) $\int_1^5 6f(x)\,dx$

(d) $\int_1^7 6f(x)\,dx$

(e) $\int_1^6 6f(x)\,dx$

Solution

$$a = 1, \quad b-a=6 \Rightarrow b=7.$$

Answer is (a)

$(6pts)$ **Problem 2**

If $\int_1^{10} f(x)\,dx = 5$ and $\int_2^{1} f(x)\,dx = 3$, then $\int_2^{10} (1 + f(x))\,dx =$

(a) 15

(b) 16

(c) 14

(d) 12

(e) 10

Solution

$$\begin{aligned}
\int_2^{10} (1 + f(x))\,dx &= (10-2) + \int_2^{10} f(x)dx \\
&= 8 + \left[\int_1^{10} f(x)dx + \int_2^{1} f(x)dx\right] \\
&= 8 + 5 + 3 = 16.
\end{aligned}$$

Answer is (b)

$(6pts)$ **Problem 3**

$$f(x) = \begin{cases} x+2 & \text{if} \quad -3 \le x \le -1 \\ x^2 & \text{if} \quad -1 < x \le 3 \end{cases} \text{, then } \int_{-2}^{0} f(x)\,dx =$$

(a) $\frac{5}{6}$

(b) $\frac{7}{6}$

(c) $\frac{11}{6}$

(d) $\frac{13}{6}$

(e) $\frac{1}{6}$

Solution

$$\begin{aligned} \int_{-2}^{0} f(x)dx &= \int_{-2}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx \\ &= \int_{-2}^{-1} (x+2)\,dx + \int_{-1}^{0} x^2 dx \\ &= \frac{5}{6} \end{aligned}$$

Answer is (a)

($6pts$) **Problem 4**

A particle moves along a line so that its velocity at time t is $v(t) = t - t^3$ (measured in meters per second). The distance traveled, in meters, during the time period $0 \leq t \leq 2$ is equal to

$(a)\quad \dfrac{3}{2}$

$(b)\quad \dfrac{1}{2}$

$(c)\quad \dfrac{7}{2}$

$(d)\quad \dfrac{9}{2}$

$(e)\quad \dfrac{5}{2}$

Solution

$$\text{Total distance} \quad = \quad \int_0^2 \left| t - t^3 \right| dt = \frac{5}{2}.$$

Answer is (e)

(6pts) **Problem 5**

$$\frac{d}{dx}\left[\int_{e^{-x}}^{e^x} \ln\, t\, dt\right] =$$

(a) $x\,(e^x - e^{-x})$

(b) 0

(c) $2x\,e^x$

(d) $e^x + e^{-x}$

(e) $x\,e^x - 1$

Solution

$$\begin{aligned}\frac{d}{dx}\int_{e^{-x}}^{e^x} \ln t dt &= e^x \ln e^x + e^{-x} \ln e^{-x}\\ &= xe^x - xe^{-x}\\ &= x\left(e^x - e^{-x}\right).\\ &\quad \textbf{Answer is } (a)\end{aligned}$$

Part 2 Written Questions (70%)

$(15pts)$**Problem 1**
Evaluate

$$1. \int \frac{\sin^3 x}{\sqrt{\cos x}} dx \qquad\qquad 2. \int_0^1 \left(x\sqrt{x} + \sqrt[3]{x} \right) dx$$

Solution
1.

$$\begin{aligned}
\int \frac{\sin^3 x}{\sqrt{\cos x}} dx &= \int \frac{\sin^2 x}{\sqrt{\cos x}} \sin x dx \\
&= \int \frac{1 - \cos^2 x}{\sqrt{\cos x}} \sin x dx \qquad (4pts)
\end{aligned}$$

Put $u = \cos x, \quad du = -\sin x dx$. The integral becomes

$$\begin{aligned}
\int \frac{1 - \cos^2 x}{\sqrt{\cos x}} \sin x dx &= -\int \frac{1 - u^2}{\sqrt{u}} du \\
&= \frac{2}{5} u^{\frac{5}{2}} - 2\sqrt{u} + C \\
&= \frac{2}{5} \cos^{\frac{5}{2}} x - 2\sqrt{\cos x} + C \qquad (4pts)
\end{aligned}$$

2.

$$\begin{aligned}
\int_0^1 \left(x\sqrt{x} + \sqrt[3]{x} dx \right) &= \int_0^1 \left(x^{3/2} + x^{1/3} \right) dx \\
&= \left. \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} \right|_0^1 \\
&= \frac{23}{20} = 1.15 \qquad (7pts)
\end{aligned}$$

$(10pts)$**Problem 2**
Use area of circles to find the average value of the function $f(x)=\sqrt{\pi^2-x^2}$ on the interval $[-\pi,\ \pi]$.
Solution

$$\begin{aligned} f_{ave} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{\pi^2-x^2}\,dx=\frac{1}{2\pi}\left(\frac{\pi\cdot\pi^2}{2}\right) \\ &= \frac{1}{4}\pi^2=2.4674 \qquad\qquad (10pts) \end{aligned}$$

$(10pts)$**Problem 3**
Evaluate the integrals

1. $\int \cos 2x \sin 3x dx$ 2. $\int \sin 5x \sin 4x dx$

Solution
Use the corresponding product to sum formulas and integrate.
1.
$$\int \cos 2x \sin 3x dx = -\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C \qquad (5pts)$$
2.
$$\int \sin 5x \sin 4x dx = \frac{1}{2}\sin x - \frac{1}{18}\sin 9x + C \qquad (5pts)$$

(15*pts*)**Problem 4**

(a) Find the modulus and the argument of the complex number

$$w = \frac{-9+3i}{1-2i}.$$

(b) Solve the equation

$$2z^2 - 2iz - 5 = 0$$

Solution

(a)

$$\left|\frac{-9+3i}{1-2i}\right| = \sqrt{18} \quad \text{and} \quad \arg\left(\frac{-9+3i}{1-2i}\right) = -\frac{3}{4}\pi \qquad (4pts + 4pts)$$

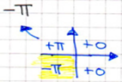
METHOD A

$$w = \frac{-9+3i}{1-2i} = \frac{(-9+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{-9-18i+3i-6}{1+\cancel{2i}-\cancel{2i}+4}$$

$$= \frac{-15-15i}{5} = -3-3i$$

- $|w| = |-3-3i| = \sqrt{(-3)^2+(-3)^2} = \sqrt{18} = 3\sqrt{2}$
- $\arg w = \arg(-3-3i) = \arctan\left(\frac{-3}{-3}\right) - \pi$

$$= \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$



METHOD B

- $|w| = \left|\frac{-9+3i}{1-2i}\right| = \frac{|-9+3i|}{|1-2i|} = \frac{\sqrt{81+9}}{\sqrt{1+4}} = \frac{\sqrt{90}}{\sqrt{5}}$

$$= \frac{\cancel{\sqrt{5}}\sqrt{2}\times\sqrt{9}}{\cancel{\sqrt{5}}} = 3\sqrt{2}$$

- $\arg w = \arg\left[\frac{-9+3i}{1-2i}\right] = \arg(-9+3i) - \arg(1-2i)$

$$= \left[\arctan\left(\frac{3}{-9}\right)+\pi\right] - \left[\arctan\left(\frac{-2}{1}\right)\right] \quad \text{(SEE ABOVE DIAGRAM)}$$

$$= \pi - \arctan\frac{1}{3} + \arctan 2$$

$$= \frac{5}{4}\pi \qquad -2\pi \text{ TO GET IN RANGE}$$

$$= -\frac{3\pi}{4}$$

(b)

$$2z^2 - 2iz - 5 = 0$$

Using the quadratic formula, we get

$$z_1 = -\frac{3}{2} + \frac{1}{2}i \quad \text{and} \quad z_2 = \frac{3}{2} + \frac{1}{2}i \qquad (7pts)$$

$2z^2 - 2iz - 5 = 0$

BY QUADRATIC FORMULA

$$z = \frac{2i \pm \sqrt{(-2i)^2 - 4\times 2\times(-5)}}{2\times 2} = \frac{2i \pm \sqrt{-4+40}}{4}$$

$$z = \frac{2i \pm 6}{4} = \frac{1}{2}i \pm \frac{3}{2} = \pm\frac{3}{2} + \frac{1}{2}i$$

(10pts)**Problem 5**
Find x and y given that

$$(a) \quad (x+iy)(2+i)=0 \qquad\qquad (b) \quad x(1+i)^2+y(2-i)^2=3+10i$$

Solution
(a)

$$(x+iy)(2+i)=0$$

$$(2x-y)+i(x+2y)=0 \Rightarrow$$

$$\left\{\begin{array}{l} 2x-y=0 \\ x+2y=0 \end{array}\right. \Rightarrow x=y=0 \qquad (5pts)$$

(b)

$$x(1+i)^2+y(2-i)^2=3+10i$$

$$x=7 \;\text{ and } y=1 \qquad (5pts)$$

$$x(1+i)^2+y(2-i)^2=3+10i$$
$$\Rightarrow x(1+2i-1)+y(4-4i-1)=3+10i$$
$$\Rightarrow 2xi+3y-4yi=3+10i$$
$$\Rightarrow (3y)+i(2x-4y)=3+10i$$
$$\Rightarrow \left(\begin{array}{l} 3y=3 \\ 2x-4y=10 \end{array}\right) \Longrightarrow \left.\begin{array}{l} y=1 \\ 2x-4=10 \end{array}\right\} \Rightarrow \begin{array}{l} y=1 \\ x=7 \end{array}$$

($10pts$)**Problem 6**

Write the complex number in the form $a+ib$.

$$(a)\ z_1=(2-i)^2+\frac{7-4i}{2+i}-8 \qquad\qquad (b)\quad z_2=(1+i)^{10}$$

Solution

(a)

$$\begin{aligned} z_1 &= (2-i)^2+\frac{7-4i}{2+i}-8 \\ &= -3-7i \qquad (5pts). \end{aligned}$$

(b)

$$\begin{aligned} z_2 &= (1+i)^{10} \\ &= \left(\sqrt{2}\right)^{10}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{10} \\ &= \left(\sqrt{2}\right)^{10}\left(\cos 10\frac{\pi}{4}+i\sin 10\frac{\pi}{4}\right) \\ &= \left(\sqrt{2}\right)^{10}\left(\cos\frac{5\pi}{2}+i\sin\frac{5\pi}{2}\right) \\ &= 32i \qquad (5pts). \end{aligned}$$

: $32i$: $\frac{1}{32}$