Linear Algebra

$a_1x_1 + a_2x_2 + a_3x_3 = d$

Solution (s_1, s_2, s_3)

$a_1s_1 + a_2s_2 + a_3s_3 = d$

$\therefore (s_1, s_2, s_3)$ is a solution to the equation

Linear system → Collection of linear equations

Gauss Elimination Method

Elementary Operations

Two systems have the same set of solutions even if

- → An equation is swapped with another
- → An equation is multiplied on both sides by a non-zero constant
- → An equation is replaced by the sum of itself and a multiple of another

pivoting

Example

Solve the system

$$\begin{cases} x_1 - x_2 = 0 & R_1 \\ 2x_1 - 2x_2 + x_3 + 2x_4 = 4 & R_2 \\ x_2 + x_4 = 0 & R_3 \\ 2x_3 + x_4 = 5 & R_4 \end{cases}$$

$-2R_1 + R_2 \longrightarrow$

$$\begin{cases} x_1 - x_2 = 0 \\ x_3 + 2x_4 = 4 \\ x_2 + x_4 = 0 \\ 2x_3 + x_4 = 5 \end{cases}$$

x_1 MUST BE IN EQ 1

x_2 MUST BE IN EQ 2

and so on

$R_2 \leftrightarrow R_3$

$$\begin{cases} x_1 - x_2 = 0 \\ x_2 + x_4 = 0 \\ x_3 + 2x_4 = 4 \\ 2x_3 + x_4 = 5 \end{cases}$$

$-2R_3 + R_4$

$$\begin{cases} x_1 - x_2 = 0 \\ x_2 + x_4 = 0 \\ x_3 + 2x_4 = 4 \\ -3x_4 = -3 \end{cases}$$

$$-3x_4 = -3$$
$$x_4 = \underline{1}$$
$$x_3 = 4 - 2x_4 = 4 - 2 = \underline{2}$$

$$x_2 + x_4 = 0$$
$$x_2 = -x_4$$
$$= \underline{-1}$$

$$x_1 - x_2 = 0$$
$$x_1 = x_2$$
$$= \underline{-1}$$

Solution $(-1, -1, 2, 1)$

Example

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 & R_1 \\ 2x_1 + 4x_2 - 3x_3 = 1 & R_2 \\ 3x_1 + 6x_2 - 5x_3 = 0 & R_3 \end{cases}$$

$-2R_1 + R_2$ and $-3R_1 + R_3$

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_2 - 7x_3 = -17 \\ 3x_2 - 11x_3 = -27 \end{cases}$$

$3R_2 - 2R_3$

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_2 - 7x_3 = -17 \\ x_3 = 3 \end{cases}$$

$$x_2 = \frac{-17 + 7(3)}{2}$$

$$x_2 = 2$$

$$x_1 + x_2 + 2x_3 = 9$$
$$x_1 + 2 + 6 = 9$$
$$x_1 = 1$$

Solution $(1, 2, 3)$

A linear system has

1. No Solution
2. Unique Solution
3. Infinite Solutions

$\boxed{\text{If no. of variables > no of equations, Then there is no unique solution}}$

Matrix

Rectangular array of numbers

Size of matrix = rows x columns

Example

$$x_1 + 2x_2 = 4$$
$$x_2 - x_3 = 0$$
$$x_1 + 2x_3 = 4$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

Coefficient Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right]$$

Augmented Matrix

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right] & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

STEPS

1 $A = \left[\begin{array}{cccc} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right]$ Write augmented matrix

$-R_1 + R_3$

2 $A = \left[\begin{array}{cccc} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$ Make $x_1 = 0$ below R_1

$2R_2 + R_3$

3 $A = \left[\begin{array}{cccc} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Make $x_2 = 0$ below R_2

Echelon Form

Reduced Matrix

4. $\begin{cases} x_1 + 2x_2 = 4 \rightarrow x_1 = 4 - 2x_2 \\ x_2 - x_3 = 0 \rightarrow x_3 = x_2 \end{cases}$ Back substitution

No of variables > No. of equations

$\therefore$ Infinitely many solutions

Solution is $(4 - 2x_2, x_2, x_2)$, $x_2 \in \mathbb{R}$

Example

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 4 & 5 \\ 1 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -5 \\ 6 & 3 \\ 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -4 & -5 \\ 29 & 2 \\ -4 & -10 \end{bmatrix}$$