Problem ①

Tutorial 5 – key

Recall
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{\left(1+\frac{2}{n}i\right)} \cdot \frac{2}{n}$$

$f(x) = \sqrt[3]{x}$ $\qquad \frac{b-a}{n} = \frac{2}{n} \longrightarrow b-a=2$ and $a=1$

$\therefore\ b-1=2$

$b=3$

we have: $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{\left(1+\frac{2}{n}i\right)}\ \frac{2}{n} = \int_1^3 \sqrt[3]{x}\,dx$

$$= \int_1^3 x^{\frac{1}{3}}\,dx$$

$$= \frac{3}{4}\left[x^{\frac{4}{3}}\right]_1^3$$

$$= \frac{3}{4}\left[(3)^{4/3} - (1)^{4/3}\right]$$

$$= \frac{3}{4}\left[3^{4/3} - 1\right]$$

More practice

Use definite integrals to evaluate

a) $$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{2}{n}i\right)^3 \frac{4}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\left(2+\frac{2}{n}i\right)^3 \frac{2}{n}$$

$f(x) = 2x^3 \qquad a = 2 \qquad \frac{b-a}{n} = \frac{2}{n} \to b-a=2$

$b-2=2$

$b=4$

$$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{2}{n}i\right)^3 \frac{4}{n} = \int_2^4 2x^3\,dx$$

$$= 2\left[\frac{x^4}{4}\right]_2^4$$

$$= \frac{1}{2}\left[(4)^4 - (2)^4\right]$$

$$= \frac{1}{2}(240) = 120$$

b) $$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{5}{n} \to \lim_{n\to\infty} \sum_{i=1}^{n} 5(4)\left(i \cdot \frac{1}{n}\right)^2 \frac{1}{n}$$

$f(x) = 20x^2 \qquad a = 0 \qquad \frac{b-a}{n} = \frac{1}{n} \to b-a=1$

$b=1$

$$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} 20\left(\frac{i}{n}\right)^2 \frac{1}{n} = \int_0^1 20x^2\,dx$$

$$= 20\left[\frac{x^3}{3}\right]_0^1 = \frac{20}{3}$$

prob. 2 The average (or the mean) value of $f(x)$ on $[a,b]$ is defined by $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$\bar{f} = \frac{1}{5-1}\int_1^5 f(x)\,dx$$

$$= \frac{1}{4}\int_1^5 f(x)\,dx$$

$$= \frac{1}{4}\left[\frac{1}{2}(1+5)2 - \frac{1}{2}(1)(1)\right]$$

$$= \frac{1}{8}\left[12-1\right]$$

$$= \frac{11}{8}$$

Prob. 3 $\displaystyle\int_0^1 f(x)\,dx = 5 \qquad \int_0^3 f(x)\,dx = 3 \qquad \int_3^7 f(x)\,dx = 1$, Find:

Recall ① $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

② $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

a) $$\int_0^7 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^7 f(x)\,dx$$

$$= 3 + 1$$

$$= 4$$

b) $$\int_1^3 f(x)\,dx = \int_1^0 f(x)\,dx + \int_0^3 f(x)\,dx$$

$$= -5 + 3$$

$$= -2$$

c) $$\int_1^7 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^7 f(x)\,dx$$

$$= -2 + 1$$

$$= -1$$

3)

$$\int_3^0 f(x)\,dx = -\int_0^3 f(x)\,dx$$

$$= -3$$

prob. 4

$$\int \frac{x^2-4}{\sqrt[3]{x^2}}\,dx = \int \frac{x^2-4}{x^{2/3}}\,dx$$

$$= \int \left(\frac{x^2}{x^{2/3}} - \frac{4}{x^{2/3}}\right) dx$$

$$= \int \left(x^{4/3} - 4x^{-2/3}\right) dx$$

$$= \frac{3}{7}x^{7/3} - 4\left(\frac{3}{1}\right)x^{\frac{1}{3}} + C$$

$$= \frac{3}{7}x^{7/3} - 12x^{\frac{1}{3}} + C$$

$$\int (x+1)\sqrt{x}\,dx = \int (x+1)x^{\frac{1}{2}}\,dx$$

$$= \int \left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right) dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C$$

$$\int (\sqrt{x}+2)^2\,dx = \int (x+4\sqrt{x}+4)\,dx$$

$$= \int \left(x+4x^{\frac{1}{2}}+4\right)dx$$

$$= \frac{x^2}{2} + 4\left(\frac{2}{3}\right)x^{\frac{3}{2}} + 4x + C$$

$$= \frac{x^2}{2} + \frac{8}{3}x^{\frac{3}{2}} + 4x + C$$

prob. 5 Recall

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = v'f(v) - u'f(u)$$

a) Calculate $\frac{d}{dx}\left(\int_{1/x}^{1/\sqrt{x}} \sin t^2\,dt\right)$

let $f(t) = \sin t^2$ $\quad v(x) = \frac{1}{\sqrt{x}}$ $\quad u(x) = \frac{1}{x}$

$$v(x) = x^{-\frac{1}{2}} \rightarrow v'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$= \frac{-1}{2x^{3/2}}$$

$$\text{OR} = \frac{-1}{2x\sqrt{x}}$$

$$u(x) = \frac{1}{x} \rightarrow u'(x) = -\frac{1}{x^2}$$

$$f(v) = f\left(\frac{1}{\sqrt{x}}\right) = \sin\frac{1}{x}$$

$$f(u) = f\left(\frac{1}{x}\right) = \sin\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\int_{1/x}^{\frac{1}{\sqrt{x}}} \sin t^2\,dt\right) = -\frac{1}{2x^{3/2}}\sin\left(\frac{1}{x}\right) + \frac{1}{x^2}\sin\left(\frac{1}{x^2}\right)$$

b) $H(x) = \displaystyle\int_{\sqrt{x}}^{x^2+x} \frac{3}{2+\sqrt{2}t}\,dt \qquad H'(2) = ?$

$$H'(x) = \frac{d}{dx}\left(\int_{\sqrt{x}}^{x^2+x} \frac{2}{2+\sqrt{2}t}\,dt\right)$$

let $f(t) = \dfrac{2}{2+\sqrt{2}t} \qquad u(x) = \sqrt{x} \qquad v(x) = x^2+x$

$$u(x) = \sqrt{x} \rightarrow u'(x) = \frac{1}{2\sqrt{x}}$$

$$v(x) = x^2 + x \longrightarrow v'(x) = 2x+1$$

$$f(v) = f(x^2+x) = \frac{2}{2+\sqrt{2}(x^2+x)}$$

$$f(u) = f(\sqrt{x}) = \frac{2}{2+\sqrt{2}\cdot\sqrt{x}}$$

$$H'(x) = v'f(v) - u'f(u)$$

$$= (2x+1)\cdot\frac{2}{2+\sqrt{2}(x^2+x)} - \frac{1}{2\sqrt{x}}\cdot\frac{2}{2+\sqrt{2}.\sqrt{x}}$$

$$H'(2) = 5\cdot\frac{2}{2+6\sqrt{2}} - \frac{1}{\cancel{2}\sqrt{2}}\cdot\frac{\cancel{2}}{(2+2)}$$

$$= \frac{10}{2+6\sqrt{2}} - \frac{1}{4\sqrt{2}}$$

prob 6.

$$\int_0^{\pi/4} \sec^2 x \, dx = \Big[\tan x\Big]_0^{\pi/4}$$

$$= 1 - 0$$

$$= 1$$

$$\int_0^{\pi/6} \sec x \tan x \, dx = \Big[\sec x\Big]_0^{\pi/6}$$

$$= \sec \frac{\pi}{6} - \sec 0$$

$$= \frac{1}{\cos \pi/6} - \frac{1}{\cos 0}$$

$$= \frac{1}{\sqrt{3}/2} - \frac{1}{1}$$

$$= \frac{2}{\sqrt{3}} - 1$$

$$\int_1^3 f(x)\, dx = \int_1^2 (x+1)^2 dx + \int_2^3 (3 - x^2)\, dx$$

$$= \int_1^2 (x^2 + 2x + 1)\, dx + \int_2^3 (3 - x^2)\, dx$$

$$= \left[\frac{x^3}{3} + x^2 + x\right]_1^2 + \left[3x - \frac{x^3}{3}\right]_2^3$$

$$= \left(\frac{8}{3} + 4 \cancel{+2} - \frac{1}{3} \cancel{-1-1}\right) + \left(\cancel{9} - \cancel{9} - 6 + \frac{8}{3}\right)$$

$$= \frac{19}{3} - \frac{10}{3}$$

$$= 3$$

prob 7 $v(t) = t^2 - t - 2$.

$$\text{Displacement} = \int_0^3 v(t)\,dt$$

final position − initial position

$$\text{Total distance traveled} = \int_0^3 |v(t)|\,dt$$

$$\text{Displacement} = \int_0^3 v(t)\,dt = s(3) - \overset{\nearrow 0}{s(0)}.$$

$$s(3) = \left[\frac{t^3}{3} - \frac{t^2}{2} - 2t\right]_0^3$$

$$= 9 - \frac{9}{2} - 6$$

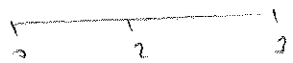
$$= -\frac{3}{2}$$

$$\text{Total distance traveled} = \int_0^3 |t^2 - t - 2|\,dt$$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t = 2 \quad t = -1$



$$= \int_0^2 -(t^2 - t - 2)\,dt + \int_2^3 (t^2 - t - 2)\,dt$$

$$= -\left[\frac{t^3}{3} - \frac{t^2}{2} - 2t\right]_0^2 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 2t\right]_2^3$$

$$= \frac{10}{3} + \frac{11}{6}$$

$$= \frac{31}{6}$$

prob. 8

$$I = \int \frac{x^2}{\sqrt{x+1}}\,dx \qquad \text{let } u = x+1 \to du = dx$$

$$x = u-1$$

$$I = \int \frac{(u-1)^2}{\sqrt{u}}\,du = \int \frac{u^2-2u+1}{u^{1/2}}\,du$$

$$= \int \left(u^{3/2} - 2u^{+1/2} + u^{-1/2}\right) du$$

$$= \frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2} + \frac{2}{1}u^{1/2} + C$$

$$= \frac{2}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + 2(x+1)^{1/2} + C$$

$$I = \int \frac{x-2}{(x^2-4x+4)^2}\,dx \qquad \text{let } u = x^2-4x+4$$

$$du = (2x-4)\,dx$$

$$du = 2(x-2)\,dx$$

$$\frac{du}{2} = (x-2)\,dx$$

$$I = \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\int u^{-2}\,du$$

$$= \frac{1}{2}\,\frac{u^{-1}}{-1} + C$$

$$= \frac{-1}{2u} + C$$

$$= \frac{-1}{2(x^2-4x+4)} + C$$

OR 2nd way $\to$ $\int u'(x)\,[u(x)]^n\,dx = \frac{[u(x)]^{n+1}}{n+1} + C$ $(n \neq -1)$

$$\int \frac{x-2}{(x^2-4x+4)^2}\,dx = \frac{1}{2}\int 2(x-2)(x^2-4x+4)^{-2}\,dx$$

$$= \frac{1}{2}\,\frac{(x^2-4x+4)^{-1}}{-1} + C$$

$$= \frac{-1}{2(x^2-4x+4)} + C$$

$$I = \int_0^1 \frac{dx}{\sqrt{x+1}}$$

$u = x+1 \longrightarrow du = dx$

$x = 0 \longrightarrow u = 1$

$x = 1 \longrightarrow u = 2$

$$I = \int_1^2 \frac{du}{\sqrt{u}} = \int_1^2 u^{-\frac{1}{2}} du$$

$$= \frac{2}{1}\left[u^{\frac{1}{2}}\right]_1^2$$

$$= 2\left(2^{\frac{1}{2}} - 1\right)$$

2nd way

$$\int_0^1 \frac{dx}{\sqrt{x+1}} = \int_0^1 (x+1)^{-\frac{1}{2}} dx$$

$$= 2\left[(x+1)^{\frac{1}{2}}\right]_0^1$$

$$= 2\left[2^{\frac{1}{2}} - 1\right]$$

$$I = \int_1^2 (x^2+1)\sqrt{2x^3+6x}\, dx$$

$u = 2x^3 + 6x$

$du = (6x^2+6)\, dx$

$du = 6(x^2+1)\, dx$

$(x^2+1)\, dx = \frac{1}{6} du$

$x = 1 \longrightarrow u = 8$

$x = 2 \longrightarrow u = 28$

$$I = \frac{1}{6}\int_8^{28} \sqrt{u}\, du$$

$$= \frac{1}{6}\int_8^{28} u^{\frac{1}{2}} du$$

$$= \frac{1}{6} \cdot \frac{2}{3}\left[u^{\frac{3}{2}}\right]_8^{28}$$

$$= \frac{1}{9}\left[(28)^{\frac{3}{2}} - (8)^{\frac{3}{2}}\right]$$

2nd way

$$\int_1^2 (x^2+1)\sqrt{2x^3+6x}\, dx$$

$$= \frac{1}{6}\int_1^2 6(x^2+1)(2x^3+6x)^{\frac{1}{2}} dx$$

$$= \frac{1}{6}\left[\frac{2}{3}(2x^3+6x)^{\frac{3}{2}}\right]_1^2 = \frac{1}{9}\left(28^{\frac{3}{2}} - 8^{\frac{3}{2}}\right)$$