Problem ① Use Gauss elimination method to solve the linear system.

$$x_1 + x_2 - 2x_3 = 9$$

$$2x_1 + 4x_2 - 3x_3 = 1$$

$$3x_1 + 6x_2 - 5x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow[(-3)R_1+R_3]{(-2)R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 9 \\ 0 & 2 & 1 & -17 \\ 0 & 3 & 1 & -27 \end{array}\right] \xrightarrow{(\frac{1}{2})R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 9 \\ 0 & 1 & \frac{1}{2} & -\frac{17}{2} \\ 0 & 3 & 1 & -27 \end{array}\right] \xrightarrow{(-3)R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 9 \\ 0 & 1 & \frac{1}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right] \xrightarrow{(-2)R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 9 \\ 0 & 1 & \frac{1}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$x_1 + x_2 - 2x_3 = 9$$

$$x_2 + \frac{1}{2}x_3 = -\frac{17}{2}$$

$$x_3 = 3$$

solve for leading variables

$$x_1 = 9 - x_2 + 2x_3$$

$$x_2 = -\frac{17}{2} - \frac{1}{2}x_3$$

$$x_3 = 3$$

Back - substitution $\quad x_2 = -\frac{17}{2} - \frac{1}{2}(3)$

$$= -10$$

$$x_1 = 9 + 1 + 2(-15)$$

$$= 25$$

Problem 2

Let $A = \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -6 & 5 \end{bmatrix}$

and $C = \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix}$

1. $A + B = \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ -2 & -6 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 6 & -2 \\ 4 & -3 & 5 \end{bmatrix}$$

2. $C + B^T = \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & -6 \\ 1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 5 & 2 \\ 6 & -6 \\ -1 & 7 \end{bmatrix}$$

3. $-2AC = -2 \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix}$

$2 \times 3 \qquad 3 \times 2$

$$= -2 \begin{bmatrix} (1)(2) + (4)(4) + (-3)(-2) & (1)(4) + (4)(0) + (-3)(2) \\ (6)(4) + (3)(0) + (0)(-2) & (6)(4) + (3)(0) + (0)(2) \end{bmatrix}$$

$$= -2 \begin{bmatrix} 24 & -2 \\ 24 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} -48 & 4 \\ -48 & -48 \end{bmatrix}$$

Problem 3

Solve the given vector equation for x, or explain why no solution exists.

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 8 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -3 & -6 \\ 0 & -3 & -3x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & 4-3x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}$$

$4 - 3x = -2$

$-3x = -6$

$\boxed{x = 2}$

Problem 4

Show that the system

$$2x_1 + 3x_2 - x_3 - 9x_4 = -16$$

$$x_1 + 2x_2 + x_3 = 0$$

$$-x_1 + 2x_2 + 3x_3 + 4x_4 = 8$$

has infinitely many solution and find the solution set.

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & -9 & -16 \\ 1 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 4 & 8 \end{array}\right] \xrightarrow{\text{swap } R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & -9 & -16 \\ -1 & 2 & 3 & 4 & 8 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{(-2)R_1 + R_2} \\ \xrightarrow{(1)\,R_1 + R_3} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -3 & -9 & -16 \\ 0 & 4 & 4 & 4 & 8 \end{array}\right] \xrightarrow{(-1)\,R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 9 & 16 \\ 0 & 4 & 4 & 4 & 8 \end{array}\right] \xrightarrow{(-4)R_2 + R_3} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 9 & 16 \\ 0 & 0 & -8 & -32 & -56 \end{array}\right] \xrightarrow{(\frac{1}{-8})R}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 9 & 16 \\ 0 & 0 & 1 & 4 & 7 \end{array}\right]$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_2 + 3x_3 + 9x_4 = 16$$

$$x_3 + 4x_4 = 7$$

Solve for leading variables

$$x_1 = -2x_2 - x_3$$

$$x_2 = 16 - 3x_3 - 9x_4$$

$$x_3 = 7 - 4x_4$$

we have x_4 is a free variable. We parametrize x_4

let $x_4 = t$ where $t \in \mathbb{R}$.

Back-substitution

$x_3 = 7 - 4t$

$$\begin{aligned} x_2 &= 16 - 3(7-4t) - 9t \\ &= 16 - 21 + 12t - 9t \\ &= 3t - 5 \end{aligned}$$

and

$$\begin{aligned} x_1 &= -2(3t-5) - (7-4t) \\ &= -6t + 10 - 7 + 4t \\ &= -2t + 3 \end{aligned}$$

solution set $= \{(-2t+3, 3t-5, 7-4t, t) : t \in \mathbb{R}\}$

clearly the system has infinitely many solutions.