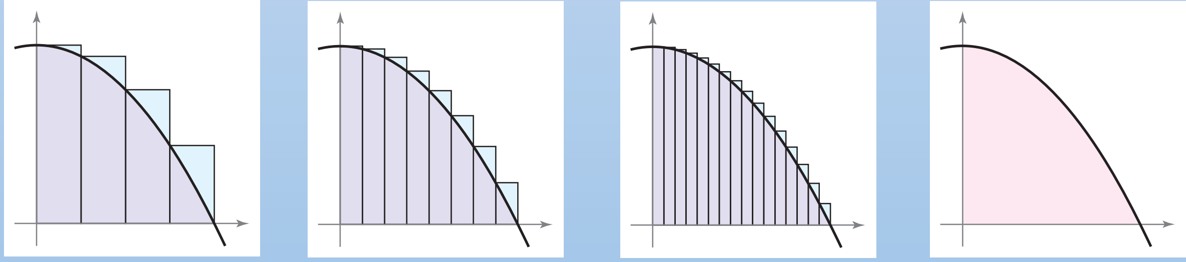
4 Integration

In this chapter, you will study an important process of calculus that is closely related to differentiation–*integration*. You will learn new methods and rules for solving definite and indefinite integrals, including the Fundamental Theorem of Calculus. Then you will apply these rules to find such things as the position function for an object and the average value of a function.

In this chapter, you should learn the following.

- How to evaluate indefinite integrals using basic integration rules. (4.1)
- How to evaluate a sum and approximate the area of a plane region. (4.2)
- How to evaluate a definite integral using a limit. (4.3)
- How to evaluate a definite integral using the Fundamental Theorem of Calculus. (4.4)
- How to evaluate different types of definite and indefinite integrals using a variety of methods. (4.5)
- How to approximate a definite integral using the Trapezoidal Rule and Simpson's Rule. (4.6)

Although its official nickname is the Emerald City, Seattle is sometimes called the Rainy City due to its weather. But there are several cities, including New York and Boston, that typically get more annual precipitation. How could you use integration to calculate the normal annual precipitation for the Seattle area? (See Section 4.5, Exercise 117.)



The area of a parabolic region can be approximated as the sum of the areas of rectangles. As you increase the number of rectangles, the approximation tends to become more and more accurate. In Section 4.2, you will learn how the limit process can be used to find areas of a wide variety of regions.

4.1 Antiderivatives and Indefinite Integration

- Write the general solution of a differential equation.
- Use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Find a particular solution of a differential equation.

Antiderivatives

Suppose you were asked to find a function F whose derivative is $f(x) = 3x^2$. From your knowledge of derivatives, you would probably say that

$$F(x) = x^3 \text{ because } \frac{d}{dx}[x^3] = 3x^2.$$

The function F is an *antiderivative* of f.

> **EXPLORATION**
>
> ***Finding Antiderivatives*** For each derivative, describe the original function F.
>
> **a.** $F'(x) = 2x$ **b.** $F'(x) = x$
>
> **c.** $F'(x) = x^2$ **d.** $F'(x) = \dfrac{1}{x^2}$
>
> **e.** $F'(x) = \dfrac{1}{x^3}$ **f.** $F'(x) = \cos x$
>
> What strategy did you use to find F?

DEFINITION OF ANTIDERIVATIVE

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Note that F is called *an* antiderivative of f, rather than *the* antiderivative of f. To see why, observe that

$$F_1(x) = x^3, \quad F_2(x) = x^3 - 5, \quad \text{and} \quad F_3(x) = x^3 + 97$$

are all antiderivatives of $f(x) = 3x^2$. In fact, for any constant C, the function given by $F(x) = x^3 + C$ is an antiderivative of f.

THEOREM 4.1 REPRESENTATION OF ANTIDERIVATIVES

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.

PROOF The proof of Theorem 4.1 in one direction is straightforward. That is, if $G(x) = F(x) + C$, $F'(x) = f(x)$, and C is a constant, then

$$G'(x) = \frac{d}{dx}[F(x) + C] = F'(x) + 0 = f(x).$$

To prove this theorem in the other direction, assume that G is an antiderivative of f. Define a function H such that

$$H(x) = G(x) - F(x).$$

For any two points a and b $(a < b)$ in the interval, H is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem,

$$H'(c) = \frac{H(b) - H(a)}{b - a}$$

for some c in (a, b). However, $H'(c) = 0$, so $H(a) = H(b)$. Because a and b are arbitrary points in the interval, you know that H is a constant function C. So, $G(x) - F(x) = C$ and it follows that $G(x) = F(x) + C$. ■

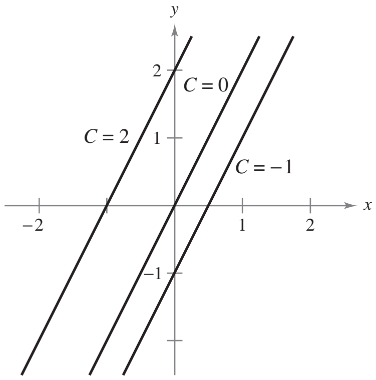
Using Theorem 4.1, you can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative. For example, knowing that $D_x[x^2] = 2x$, you can represent the family of *all* antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C \qquad \text{Family of all antiderivatives of } f(x) = 2x$$

where C is a constant. The constant C is called the **constant of integration.** The family of functions represented by G is the **general antiderivative** of f, and $G(x) = x^2 + C$ is the **general solution** of the *differential equation*

$$G'(x) = 2x. \qquad \text{Differential equation}$$

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y. For instance, $y' = 3x$ and $y' = x^2 + 1$ are examples of differential equations.



Functions of the form $y = 2x + C$
Figure 4.1

EXAMPLE 1 Solving a Differential Equation

Find the general solution of the differential equation $y' = 2$.

Solution To begin, you need to find a function whose derivative is 2. One such function is

$$y = 2x. \qquad 2x \text{ is } \textit{an} \text{ antiderivative of } 2.$$

Now, you can use Theorem 4.1 to conclude that the general solution of the differential equation is

$$y = 2x + C. \qquad \text{General solution}$$

The graphs of several functions of the form $y = 2x + C$ are shown in Figure 4.1. ■

Notation for Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\,dx.$$

The operation of finding all solutions of this equation is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign $\int$. The general solution is denoted by

$$y = \int \underbrace{f(x)}_{\text{Integrand}}\, \overbrace{dx}^{\text{Variable of integration}} = \underbrace{F(x)}_{\text{An antiderivative of } f(x)} + \overbrace{C}^{\text{Constant of integration}}.$$

NOTE In this text, the notation $\int f(x)\,dx = F(x) + C$ means that F is an antiderivative of f *on an interval*.

The expression $\int f(x)\,dx$ is read as the *antiderivative of f with respect to x*. So, the differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

Moreover, if $\int f(x)\,dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.

These two equations allow you to obtain integration formulas directly from differentiation formulas, as shown in the following summary.

BASIC INTEGRATION RULES

Differentiation Formula	*Integration Formula*	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$	
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$	
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$	

NOTE Note that the Power Rule for Integration has the restriction that $n \neq -1$. The evaluation of $\int 1/x\,dx$ must wait until the introduction of the natural logarithmic function in Chapter 5. ■

EXAMPLE 2 Applying the Basic Integration Rules

Describe the antiderivatives of $3x$.

Solution

$$\begin{aligned}\int 3x\,dx &= 3\int x\,dx && \text{Constant Multiple Rule}\\ &= 3\int x^1\,dx && \text{Rewrite } x \text{ as } x^1.\\ &= 3\left(\frac{x^2}{2}\right) + C && \text{Power Rule } (n = 1)\\ &= \frac{3}{2}x^2 + C && \text{Simplify.}\end{aligned}$$

So, the antiderivatives of $3x$ are of the form $\frac{3}{2}x^2 + C$, where C is any constant. ■

When indefinite integrals are evaluated, a strict application of the basic integration rules tends to produce complicated constants of integration. For instance, in Example 2, you could have written

$$\int 3x\,dx = 3\int x\,dx = 3\left(\frac{x^2}{2} + C\right) = \frac{3}{2}x^2 + 3C.$$

However, because C represents *any* constant, it is both cumbersome and unnecessary to write $3C$ as the constant of integration. So, $\frac{3}{2}x^2 + 3C$ is written in the simpler form, $\frac{3}{2}x^2 + C$.

In Example 2, note that the general pattern of integration is similar to that of differentiation.

Original integral ⇨ Rewrite ⇨ Integrate ⇨ Simplify

EXAMPLE 3 Rewriting Before Integrating

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
a.	$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b.	$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
c.	$\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$

■

TECHNOLOGY Some software programs, such as *Maple*, *Mathematica*, and the *TI-89*, are capable of performing integration symbolically. If you have access to such a symbolic integration utility, try using it to evaluate the indefinite integrals in Example 3.

Remember that you can check your answer to an antidifferentiation problem by differentiating. For instance, in Example 3(b), you can check that $\frac{2}{3}x^{3/2} + C$ is the correct antiderivative by differentiating the answer to obtain

$$D_x\left[\frac{2}{3}x^{3/2} + C\right] = \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)x^{1/2} = \sqrt{x}. \qquad \text{Use differentiation to check antiderivative.}$$

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

The basic integration rules listed on page 250 allow you to integrate any polynomial function, as shown in Example 4.

EXAMPLE 4 Integrating Polynomial Functions

a. $\int dx = \int 1\, dx$ — Integrand is understood to be 1.

$= x + C$ — Integrate.

b. $\int (x + 2)\, dx = \int x\, dx + \int 2\, dx$

$= \dfrac{x^2}{2} + C_1 + 2x + C_2$ — Integrate.

$= \dfrac{x^2}{2} + 2x + C$ — $C = C_1 + C_2$

The second line in the solution is usually omitted.

c. $\int (3x^4 - 5x^2 + x)\, dx = 3\left(\dfrac{x^5}{5}\right) - 5\left(\dfrac{x^3}{3}\right) + \dfrac{x^2}{2} + C$ — Integrate.

$= \dfrac{3}{5}x^5 - \dfrac{5}{3}x^3 + \dfrac{1}{2}x^2 + C$ — Simplify.

EXAMPLE 5 Rewriting Before Integrating

$\int \dfrac{x+1}{\sqrt{x}}\, dx = \int \left(\dfrac{x}{\sqrt{x}} + \dfrac{1}{\sqrt{x}}\right) dx$ — Rewrite as two fractions.

$= \int (x^{1/2} + x^{-1/2})\, dx$ — Rewrite with fractional exponents.

$= \dfrac{x^{3/2}}{3/2} + \dfrac{x^{1/2}}{1/2} + C$ — Integrate.

$= \dfrac{2}{3}x^{3/2} + 2x^{1/2} + C$ — Simplify.

$= \dfrac{2}{3}\sqrt{x}(x + 3) + C$ ■

STUDY TIP Remember that you can check your answer by differentiating.

NOTE When integrating quotients, do not integrate the numerator and denominator separately. This is no more valid in integration than it is in differentiation. For instance, in Example 5, be sure you understand that

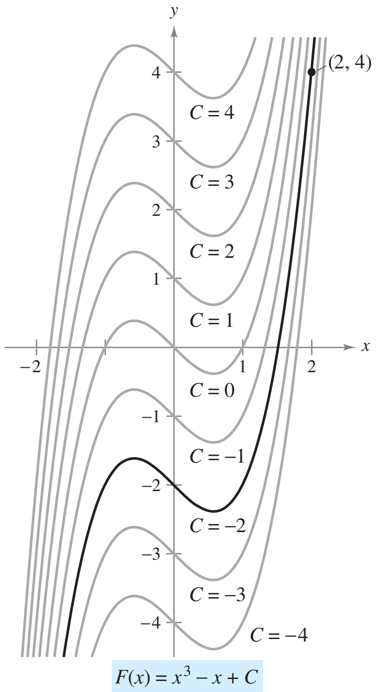
$$\int \frac{x+1}{\sqrt{x}}\, dx = \frac{2}{3}\sqrt{x}(x+3) + C \text{ is not the same as } \frac{\int (x+1)\, dx}{\int \sqrt{x}\, dx} = \frac{\frac{1}{2}x^2 + x + C_1}{\frac{2}{3}x\sqrt{x} + C_2}.$$ ■

EXAMPLE 6 Rewriting Before Integrating

$\int \dfrac{\sin x}{\cos^2 x}\, dx = \int \left(\dfrac{1}{\cos x}\right)\left(\dfrac{\sin x}{\cos x}\right) dx$ — Rewrite as a product.

$= \int \sec x \tan x\, dx$ — Rewrite using trigonometric identities.

$= \sec x + C$ — Integrate. ■



The particular solution that satisfies the initial condition $F(2) = 4$ is $F(x) = x^3 - x - 2$.
Figure 4.2

Initial Conditions and Particular Solutions

You have already seen that the equation $y = \int f(x)\,dx$ has many solutions (each differing from the others by a constant). This means that the graphs of any two antiderivatives of f are vertical translations of each other. For example, Figure 4.2 shows the graphs of several antiderivatives of the form

$$y = \int (3x^2 - 1)\,dx = x^3 - x + C \qquad \text{General solution}$$

for various integer values of C. Each of these antiderivatives is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2 - 1.$$

In many applications of integration, you are given enough information to determine a **particular solution.** To do this, you need only know the value of $y = F(x)$ for one value of x. This information is called an **initial condition.** For example, in Figure 4.2, only one curve passes through the point $(2, 4)$. To find this curve, you can use the following information.

$$F(x) = x^3 - x + C \qquad \text{General solution}$$
$$F(2) = 4 \qquad \text{Initial condition}$$

By using the initial condition in the general solution, you can determine that $F(2) = 8 - 2 + C = 4$, which implies that $C = -2$. So, you obtain

$$F(x) = x^3 - x - 2. \qquad \text{Particular solution}$$

EXAMPLE 7 Finding a Particular Solution

Find the general solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

and find the particular solution that satisfies the initial condition $F(1) = 0$.

Solution To find the general solution, integrate to obtain

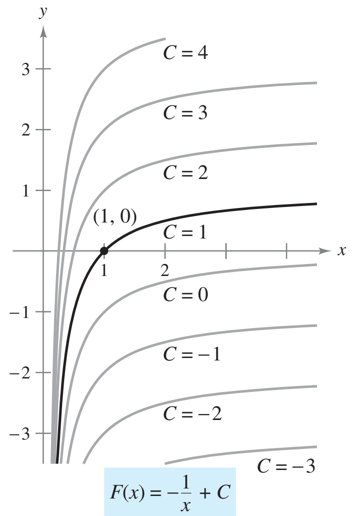
$$F(x) = \int \frac{1}{x^2}\,dx \qquad F(x) = \int F'(x)\,dx$$
$$= \int x^{-2}\,dx \qquad \text{Rewrite as a power.}$$
$$= \frac{x^{-1}}{-1} + C \qquad \text{Integrate.}$$
$$= -\frac{1}{x} + C, \quad x > 0. \qquad \text{General solution}$$

Using the initial condition $F(1) = 0$, you can solve for C as follows.

$$F(1) = -\frac{1}{1} + C = 0 \quad \Longrightarrow \quad C = 1$$

So, the particular solution, as shown in Figure 4.3, is

$$F(x) = -\frac{1}{x} + 1, \quad x > 0. \qquad \text{Particular solution}$$



The particular solution that satisfies the initial condition $F(1) = 0$ is $F(x) = -(1/x) + 1$, $x > 0$.
Figure 4.3

So far in this section you have been using x as the variable of integration. In applications, it is often convenient to use a different variable. For instance, in the following example involving *time*, the variable of integration is t.

EXAMPLE 8 Solving a Vertical Motion Problem

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet.

a. Find the position function giving the height s as a function of the time t.

b. When does the ball hit the ground?

Solution

a. Let $t = 0$ represent the initial time. The two given initial conditions can be written as follows.

$$s(0) = 80 \qquad \text{Initial height is 80 feet.}$$
$$s'(0) = 64 \qquad \text{Initial velocity is 64 feet per second.}$$

Using -32 feet per second per second as the acceleration due to gravity, you can write

$$s''(t) = -32$$
$$s'(t) = \int s''(t)\,dt = \int -32\,dt = -32t + C_1.$$

Using the initial velocity, you obtain $s'(0) = 64 = -32(0) + C_1$, which implies that $C_1 = 64$. Next, by integrating $s'(t)$, you obtain

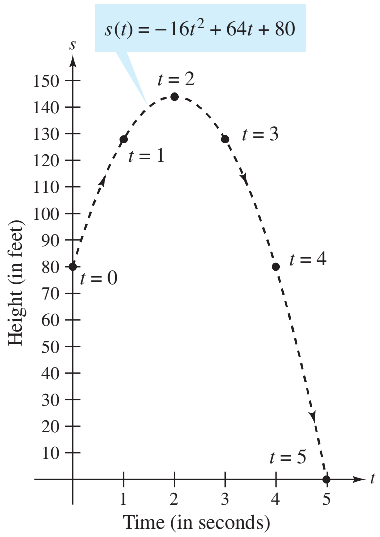
$$s(t) = \int s'(t)\,dt = \int (-32t + 64)\,dt = -16t^2 + 64t + C_2.$$

Using the initial height, you obtain

$$s(0) = 80 = -16(0^2) + 64(0) + C_2$$

which implies that $C_2 = 80$. So, the position function is

$$s(t) = -16t^2 + 64t + 80. \qquad \text{See Figure 4.4.}$$



Height of a ball at time t
Figure 4.4

b. Using the position function found in part (a), you can find the time at which the ball hits the ground by solving the equation $s(t) = 0$.

$$s(t) = -16t^2 + 64t + 80 = 0$$
$$-16(t + 1)(t - 5) = 0$$
$$t = -1, 5$$

Because t must be positive, you can conclude that the ball hits the ground 5 seconds after it was thrown. ■

NOTE In Example 8, note that the position function has the form

$$s(t) = \tfrac{1}{2}gt^2 + v_0 t + s_0$$

where $g = -32$, v_0 is the initial velocity, and s_0 is the initial height, as presented in Section 2.2.

Example 8 shows how to use calculus to analyze vertical motion problems in which the acceleration is determined by a gravitational force. You can use a similar strategy to analyze other linear motion problems (vertical or horizontal) in which the acceleration (or deceleration) is the result of some other force, as you will see in Exercises 81–89.

Before you begin the exercise set, be sure you realize that one of the most important steps in integration is *rewriting the integrand* in a form that fits the basic integration rules. To illustrate this point further, here are some additional examples.

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int \frac{2}{\sqrt{x}}\,dx$	$2\int x^{-1/2}\,dx$	$2\left(\frac{x^{1/2}}{1/2}\right) + C$	$4x^{1/2} + C$
$\int (t^2+1)^2\,dt$	$\int (t^4 + 2t^2 + 1)\,dt$	$\frac{t^5}{5} + 2\left(\frac{t^3}{3}\right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
$\int \frac{x^3+3}{x^2}\,dx$	$\int (x + 3x^{-2})\,dx$	$\frac{x^2}{2} + 3\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
$\int \sqrt[3]{x}(x-4)\,dx$	$\int (x^{4/3} - 4x^{1/3})\,dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$	$\frac{3}{7}x^{7/3} - 3x^{4/3}$

4.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, verify the statement by showing that the derivative of the right side equals the integrand of the left side.

1. $\int \left(-\frac{6}{x^4}\right) dx = \frac{2}{x^3} + C$

2. $\int \left(8x^3 + \frac{1}{2x^2}\right) dx = 2x^4 - \frac{1}{2x} + C$

3. $\int (x-4)(x+4)\,dx = \frac{1}{3}x^3 - 16x + C$

4. $\int \frac{x^2-1}{x^{3/2}}\,dx = \frac{2(x^2+3)}{3\sqrt{x}} + C$

In Exercises 5–8, find the general solution of the differential equation and check the result by differentiation.

5. $\frac{dy}{dt} = 9t^2$

6. $\frac{dr}{d\theta} = \pi$

7. $\frac{dy}{dx} = x^{3/2}$

8. $\frac{dy}{dx} = 2x^{-3}$

In Exercises 9–14, complete the table.

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
9.	$\int \sqrt[3]{x}\,dx$			
10.	$\int \frac{1}{4x^2}\,dx$			
11.	$\int \frac{1}{x\sqrt{x}}\,dx$			
12.	$\int x(x^3+1)\,dx$			
13.	$\int \frac{1}{2x^3}\,dx$			
14.	$\int \frac{1}{(3x)^2}\,dx$			

In Exercises 15–34, find the indefinite integral and check the result by differentiation.

15. $\int (x+7)\,dx$

16. $\int (13-x)\,dx$

17. $\int (2x-3x^2)\,dx$

18. $\int (8x^3 - 9x^2 + 4)\,dx$

19. $\int (x^5+1)\,dx$

20. $\int (x^3 - 10x - 3)\,dx$

21. $\int (x^{3/2} + 2x + 1)\,dx$

22. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$

23. $\int \sqrt[3]{x^2}\,dx$

24. $\int \left(\sqrt[4]{x^3} + 1\right) dx$

25. $\int \frac{1}{x^5}\,dx$

26. $\int \frac{1}{x^6}\,dx$

27. $\int \frac{x+6}{\sqrt{x}}\,dx$

28. $\int \frac{x^2+2x-3}{x^4}\,dx$

29. $\int (x+1)(3x-2)\,dx$

30. $\int (2t^2-1)^2\,dt$

31. $\int y^2\sqrt{y}\,dy$

32. $\int (1+3t)t^2\,dt$

33. $\int dx$

34. $\int 14\,dt$

In Exercises 35–44, find the indefinite integral and check the result by differentiation.

35. $\int (5\cos x + 4\sin x)\,dx$

36. $\int (t^2 - \cos t)\,dt$

37. $\int (1 - \csc t\cot t)\,dt$

38. $\int (\theta^2 + \sec^2\theta)\,d\theta$

39. $\int (\sec^2\theta - \sin\theta)\,d\theta$

40. $\int \sec y(\tan y - \sec y)\,dy$

41. $\int (\tan^2 y + 1)\, dy$

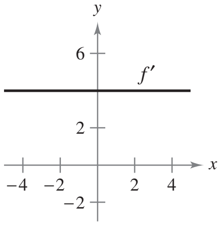
42. $\int (4x - \csc^2 x)\, dx$

43. $\int \frac{\cos x}{1 - \cos^2 x}\, dx$

44. $\int \frac{\sin x}{1 - \sin^2 x}\, dx$

In Exercises 45–48, the graph of the derivative of a function is given. Sketch the graphs of *two* functions that have the given derivative. (There is more than one correct answer.) To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

45.



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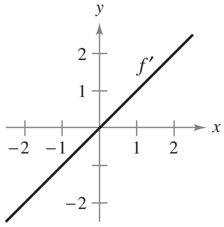

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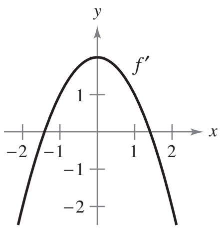

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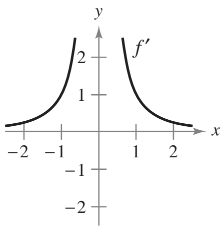
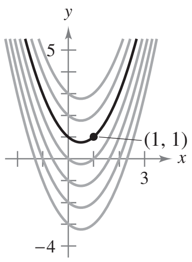
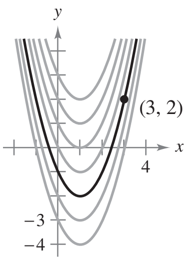


In Exercises 49 and 50, find the equation of y, given the derivative and the indicated point on the curve.

49. $\frac{dy}{dx} = 2x - 1$

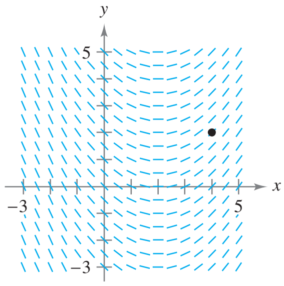


50. $\frac{dy}{dx} = 2(x - 1)$

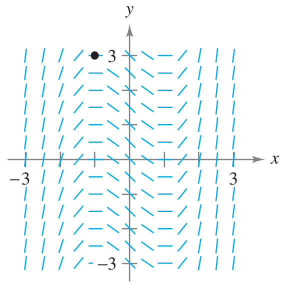


Slope Fields **In Exercises 51–54, a differential equation, a point, and a slope field are given. A *slope field* (or *direction field*) consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the slopes of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

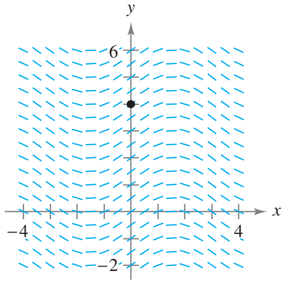
51. $\frac{dy}{dx} = \frac{1}{2}x - 1,\ (4, 2)$



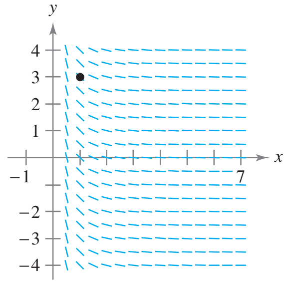
52. $\frac{dy}{dx} = x^2 - 1,\ (-1, 3)$



53. $\frac{dy}{dx} = \cos x,\ (0, 4)$



54. $\frac{dy}{dx} = -\frac{1}{x^2},\ x > 0,\ (1, 3)$



Slope Fields **In Exercises 55 and 56, (a) use a graphing utility to graph a slope field for the differential equation, (b) use integration and the given point to find the particular solution of the differential equation, and (c) graph the solution and the slope field in the same viewing window.**

55. $\frac{dy}{dx} = 2x,\ (-2, -2)$

56. $\frac{dy}{dx} = 2\sqrt{x},\ (4, 12)$

In Exercises 57–64, solve the differential equation.

57. $f'(x) = 6x,\ f(0) = 8$

58. $g'(x) = 6x^2,\ g(0) = -1$

59. $h'(t) = 8t^3 + 5,\ h(1) = -4$

60. $f'(s) = 10s - 12s^3,\ f(3) = 2$

61. $f''(x) = 2,\ f'(2) = 5,\ f(2) = 10$

62. $f''(x) = x^2,\ f'(0) = 8,\ f(0) = 4$

63. $f''(x) = x^{-3/2},\ f'(4) = 2,\ f(0) = 0$

64. $f''(x) = \sin x,\ f'(0) = 1,\ f(0) = 6$

65. ***Tree Growth*** An evergreen nursery usually sells a certain type of shrub after 6 years of growth and shaping. The growth rate during those 6 years is approximated by $dh/dt = 1.5t + 5$, where t is the time in years and h is the height in centimeters. The seedlings are 12 centimeters tall when planted $(t = 0)$.

(a) Find the height after t years.

(b) How tall are the shrubs when they are sold?

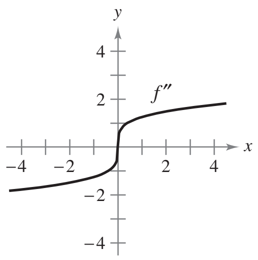
66. ***Population Growth*** The rate of growth dP/dt of a population of bacteria is proportional to the square root of t, where P is the population size and t is the time in days $(0 \le t \le 10)$. That is, $dP/dt = k\sqrt{t}$. The initial size of the population is 500. After 1 day the population has grown to 600. Estimate the population after 7 days.

WRITING ABOUT CONCEPTS

67. What is the difference, if any, between finding the antiderivative of $f(x)$ and evaluating the integral $\int f(x)\,dx$?

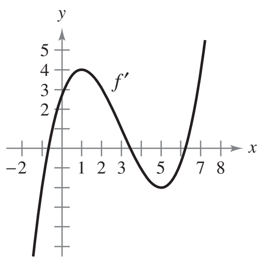
68. Consider $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$. What do you notice about the derivatives of $f(x)$ and $g(x)$? What can you conclude about the relationship between $f(x)$ and $g(x)$?

69. The graphs of f and f' each pass through the origin. Use the graph of f'' shown in the figure to sketch the graphs of f and f'. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



CAPSTONE

70. Use the graph of f' shown in the figure to answer the following, given that $f(0) = -4$.



(a) Approximate the slope of f at $x = 4$. Explain.

(b) Is it possible that $f(2) = -1$? Explain.

(c) Is $f(5) - f(4) > 0$? Explain.

(d) Approximate the value of x where f is maximum. Explain.

(e) Approximate any intervals in which the graph of f is concave upward and any intervals in which it is concave downward. Approximate the x-coordinates of any points of inflection.

(f) Approximate the x-coordinate of the minimum of $f''(x)$.

(g) Sketch an approximate graph of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

Vertical Motion **In Exercises 71–74, use $a(t) = -32$ feet per second per second as the acceleration due to gravity. (Neglect air resistance.)**

71. A ball is thrown vertically upward from a height of 6 feet with an initial velocity of 60 feet per second. How high will the ball go?

72. Show that the height above the ground of an object thrown upward from a point s_0 feet above the ground with an initial velocity of v_0 feet per second is given by the function

$$f(t) = -16t^2 + v_0 t + s_0.$$

73. With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approximately 550 feet)?

74. A balloon, rising vertically with a velocity of 16 feet per second, releases a sandbag at the instant it is 64 feet above the ground.

(a) How many seconds after its release will the bag strike the ground?

(b) At what velocity will it hit the ground?

Vertical Motion **In Exercises 75–78, use $a(t) = -9.8$ meters per second per second as the acceleration due to gravity. (Neglect air resistance.)**

75. Show that the height above the ground of an object thrown upward from a point s_0 meters above the ground with an initial velocity of v_0 meters per second is given by the function

$$f(t) = -4.9t^2 + v_0 t + s_0.$$

76. The Grand Canyon is 1800 meters deep at its deepest point. A rock is dropped from the rim above this point. Write the height of the rock as a function of the time t in seconds. How long will it take the rock to hit the canyon floor?

77. A baseball is thrown upward from a height of 2 meters with an initial velocity of 10 meters per second. Determine its maximum height.

78. With what initial velocity must an object be thrown upward (from a height of 2 meters) to reach a maximum height of 200 meters?

79. ***Lunar Gravity*** On the moon, the acceleration due to gravity is -1.6 meters per second per second. A stone is dropped from a cliff on the moon and hits the surface of the moon 20 seconds later. How far did it fall? What was its velocity at impact?

80. ***Escape Velocity*** The minimum velocity required for an object to escape Earth's gravitational pull is obtained from the solution of the equation

$$\int v\,dv = -GM\int \frac{1}{y^2}\,dy$$

where v is the velocity of the object projected from Earth, y is the distance from the center of Earth, G is the gravitational constant, and M is the mass of Earth. Show that v and y are related by the equation

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

where v_0 is the initial velocity of the object and R is the radius of Earth.

Rectilinear Motion **In Exercises 81–84, consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $x''(t)$ is its acceleration.**

81. $x(t) = t^3 - 6t^2 + 9t - 2, \quad 0 \le t \le 5$

(a) Find the velocity and acceleration of the particle.

(b) Find the open t-intervals on which the particle is moving to the right.

(c) Find the velocity of the particle when the acceleration is 0.

82. Repeat Exercise 81 for the position function

$$x(t) = (t - 1)(t - 3)^2, \quad 0 \le t \le 5$$

83. A particle moves along the x-axis at a velocity of $v(t) = 1/\sqrt{t}$, $t > 0$. At time $t = 1$, its position is $x = 4$. Find the acceleration and position functions for the particle.

84. A particle, initially at rest, moves along the x-axis such that its acceleration at time $t > 0$ is given by $a(t) = \cos t$. At the time $t = 0$, its position is $x = 3$.

(a) Find the velocity and position functions for the particle.

(b) Find the values of t for which the particle is at rest.

85. ***Acceleration*** The maker of an automobile advertises that it takes 13 seconds to accelerate from 25 kilometers per hour to 80 kilometers per hour. Assuming constant acceleration, compute the following.

(a) The acceleration in meters per second per second

(b) The distance the car travels during the 13 seconds

86. ***Deceleration*** A car traveling at 45 miles per hour is brought to a stop, at constant deceleration, 132 feet from where the brakes are applied.

(a) How far has the car moved when its speed has been reduced to 30 miles per hour?

(b) How far has the car moved when its speed has been reduced to 15 miles per hour?

(c) Draw the real number line from 0 to 132, and plot the points found in parts (a) and (b). What can you conclude?

87. ***Acceleration*** At the instant the traffic light turns green, a car that has been waiting at an intersection starts with a constant acceleration of 6 feet per second per second. At the same instant, a truck traveling with a constant velocity of 30 feet per second passes the car.

(a) How far beyond its starting point will the car pass the truck?

(b) How fast will the car be traveling when it passes the truck?

88. ***Acceleration*** Assume that a fully loaded plane starting from rest has a constant acceleration while moving down a runway. The plane requires 0.7 mile of runway and a speed of 160 miles per hour in order to lift off. What is the plane's acceleration?

89. ***Airplane Separation*** Two airplanes are in a straight-line landing pattern and, according to FAA regulations, must keep at least a three-mile separation. Airplane A is 10 miles from touchdown and is gradually decreasing its speed from 150 miles per hour to a landing speed of 100 miles per hour. Airplane B is 17 miles from touchdown and is gradually decreasing its speed from 250 miles per hour to a landing speed of 115 miles per hour.

(a) Assuming the deceleration of each airplane is constant, find the position functions s_A and s_B for airplane A and airplane B. Let $t = 0$ represent the times when the airplanes are 10 and 17 miles from the airport.

(b) Use a graphing utility to graph the position functions.

(c) Find a formula for the magnitude of the distance d between the two airplanes as a function of t. Use a graphing utility to graph d. Is $d < 3$ for some time prior to the landing of airplane A? If so, find that time.

True or False? **In Exercises 90–95, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

90. Each antiderivative of an nth-degree polynomial function is an $(n + 1)$th-degree polynomial function.

91. If $p(x)$ is a polynomial function, then p has exactly one antiderivative whose graph contains the origin.

92. If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x) = G(x) + C$.

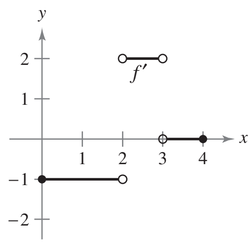
93. If $f'(x) = g(x)$, then $\int g(x)\,dx = f(x) + C$.

94. $\int f(x)g(x)\,dx = \int f(x)\,dx \int g(x)\,dx$

95. The antiderivative of $f(x)$ is unique.

96. Find a function f such that the graph of f has a horizontal tangent at $(2, 0)$ and $f''(x) = 2x$.

97. The graph of f' is shown. Sketch the graph of f given that f is continuous and $f(0) = 1$.



98. If $f'(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3x, & 2 \le x \le 5 \end{cases}$, f is continuous, and $f(1) = 3$, find f. Is f differentiable at $x = 2$?

99. Let $s(x)$ and $c(x)$ be two functions satisfying $s'(x) = c(x)$ and $c'(x) = -s(x)$ for all x. If $s(0) = 0$ and $c(0) = 1$, prove that $[s(x)]^2 + [c(x)]^2 = 1$.

PUTNAM EXAM CHALLENGE

100. Suppose f and g are nonconstant, differentiable, real-valued functions on R. Furthermore, suppose that for each pair of real numbers x and y, $f(x + y) = f(x)f(y) - g(x)g(y)$ and $g(x + y) = f(x)g(y) + g(x)f(y)$. If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

4.2 Area

- Use sigma notation to write and evaluate a sum.
- Understand the concept of area.
- Approximate the area of a plane region.
- Find the area of a plane region using limits.

Sigma Notation

In the preceding section, you studied antidifferentiation. In this section, you will look further into a problem introduced in Section 1.1—that of finding the area of a region in the plane. At first glance, these two ideas may seem unrelated, but you will discover in Section 4.4 that they are closely related by an extremely important theorem called the Fundamental Theorem of Calculus.

This section begins by introducing a concise notation for sums. This notation is called **sigma notation** because it uses the uppercase Greek letter sigma, written as Σ.

SIGMA NOTATION

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation,** a_i is the ***i*th term** of the sum, and the **upper and lower bounds of summation** are n and 1.

NOTE The upper and lower bounds must be constant with respect to the index of summation. However, the lower bound doesn't have to be 1. Any integer less than or equal to the upper bound is legitimate.

EXAMPLE 1 Examples of Sigma Notation

a. $\displaystyle\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

b. $\displaystyle\sum_{i=0}^{5} (i + 1) = 1 + 2 + 3 + 4 + 5 + 6$

c. $\displaystyle\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d. $\displaystyle\sum_{k=1}^{n} \frac{1}{n}(k^2 + 1) = \frac{1}{n}(1^2 + 1) + \frac{1}{n}(2^2 + 1) + \cdots + \frac{1}{n}(n^2 + 1)$

e. $\displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$

From parts (a) and (b), notice that the same sum can be represented in different ways using sigma notation.

■ **FOR FURTHER INFORMATION** For a geometric interpretation of summation formulas, see the article, "Looking at $\displaystyle\sum_{k=1}^{n} k$ and $\displaystyle\sum_{k=1}^{n} k^2$ Geometrically" by Eric Hegblom in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

Although any variable can be used as the index of summation i, j, and k are often used. Notice in Example 1 that the index of summation does not appear in the terms of the expanded sum.

THE SUM OF THE FIRST 100 INTEGERS

A teacher of Carl Friedrich Gauss (1777–1855) asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{rcrcrcccr} 1 & + & 2 & + & 3 & + & \cdots & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

$$\frac{100 \times 101}{2} = 5050$$

This is generalized by Theorem 4.2, where

$$\sum_{i=1}^{100} i = \frac{100(101)}{2} = 5050.$$

The following properties of summation can be derived using the associative and commutative properties of addition and the distributive property of addition over multiplication. (In the first property, k is a constant.)

1. $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

2. $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

The next theorem lists some useful formulas for sums of powers. A proof of this theorem is given in Appendix A.

THEOREM 4.2 SUMMATION FORMULAS

1. $\displaystyle\sum_{i=1}^{n} c = cn$

2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

EXAMPLE 2 Evaluating a Sum

Evaluate $\displaystyle\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000$, and $10{,}000$.

Solution Applying Theorem 4.2, you can write

$$\begin{aligned} \sum_{i=1}^{n} \frac{i+1}{n^2} &= \frac{1}{n^2}\sum_{i=1}^{n}(i+1) && \text{Factor the constant } 1/n^2 \text{ out of sum.} \\ &= \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) && \text{Write as two sums.} \\ &= \frac{1}{n^2}\left[\frac{n(n+1)}{2} + n\right] && \text{Apply Theorem 4.2.} \\ &= \frac{1}{n^2}\left[\frac{n^2+3n}{2}\right] && \text{Simplify.} \\ &= \frac{n+3}{2n}. && \text{Simplify.} \end{aligned}$$

Now you can evaluate the sum by substituting the appropriate values of n, as shown in the table at the left. ■

n	$\displaystyle\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{n+3}{2n}$
10	0.65000
100	0.51500
1,000	0.50150
10,000	0.50015

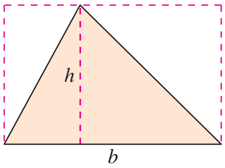
In the table, note that the sum appears to approach a limit as n increases. Although the discussion of limits at infinity in Section 3.5 applies to a variable x, where x can be any real number, many of the same results hold true for limits involving the variable n, where n is restricted to positive integer values. So, to find the limit of $(n+3)/2n$ as n approaches infinity, you can write

$$\lim_{n\to\infty} \frac{n+3}{2n} = \lim_{n\to\infty}\left(\frac{n}{2n} + \frac{3}{2n}\right) = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{3}{2n}\right) = \frac{1}{2} + 0 = \frac{1}{2}.$$

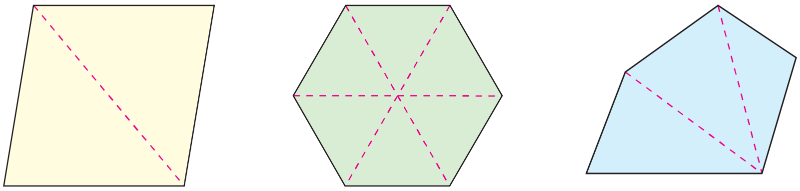
Area

In Euclidean geometry, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is $A = bh$, it is actually more proper to say that this is the *definition* of the **area of a rectangle.**

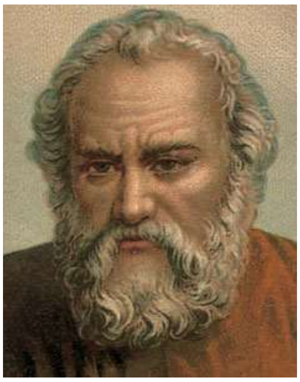
From this definition, you can develop formulas for the areas of many other plane regions. For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle, as shown in Figure 4.5. Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown in Figure 4.6.



Triangle: $A = \frac{1}{2}bh$
Figure 4.5



Parallelogram   Hexagon   Polygon
Figure 4.6



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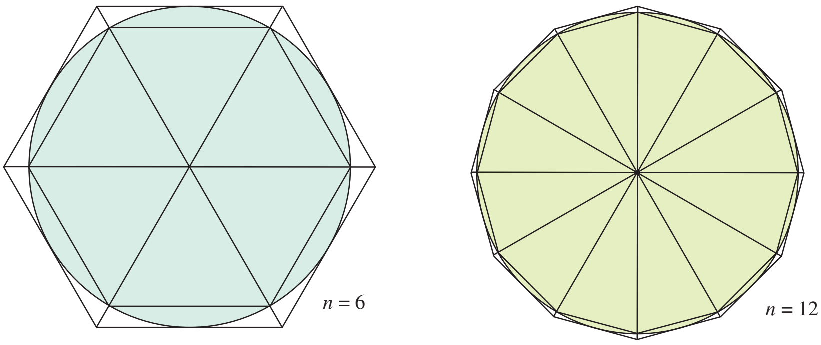
ARCHIMEDES (287–212 B.C.)

Archimedes used the method of exhaustion to derive formulas for the areas of ellipses, parabolic segments, and sectors of a spiral. He is considered to have been the greatest applied mathematician of antiquity.

Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions (principally those bounded by conics) by the *exhaustion* method. The clearest description of this method was given by Archimedes. Essentially, the method is a limiting process in which the area is squeezed between two polygons—one inscribed in the region and one circumscribed about the region.

For instance, in Figure 4.7 the area of a circular region is approximated by an n-sided inscribed polygon and an n-sided circumscribed polygon. For each value of n, the area of the inscribed polygon is less than the area of the circle, and the area of the circumscribed polygon is greater than the area of the circle. Moreover, as n increases, the areas of both polygons become better and better approximations of the area of the circle.



The exhaustion method for finding the area of a circular region
Figure 4.7

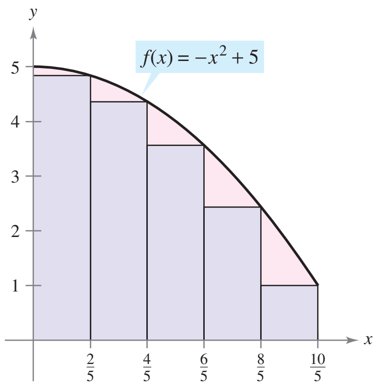
A process that is similar to that used by Archimedes to determine the area of a plane region is used in the remaining examples in this section.

■ **FOR FURTHER INFORMATION** For an alternative development of the formula for the area of a circle, see the article "Proof Without Words: Area of a Disk is πR^2" by Russell Jay Hendel in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

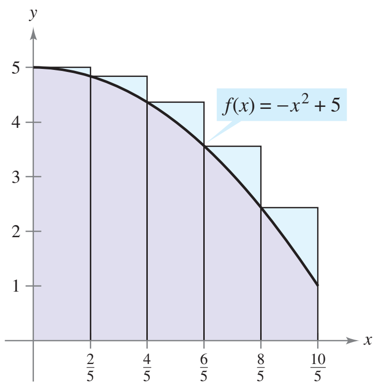
The Area of a Plane Region

Recall from Section 1.1 that the origins of calculus are connected to two classic problems: the tangent line problem and the area problem. Example 3 begins the investigation of the area problem.

EXAMPLE 3 Approximating the Area of a Plane Region



(a) The area of the parabolic region is greater than the area of the rectangles.



(b) The area of the parabolic region is less than the area of the rectangles.

Figure 4.8

Use the five rectangles in Figure 4.8(a) and (b) to find *two* approximations of the area of the region lying between the graph of

$$f(x) = -x^2 + 5$$

and the x-axis between $x = 0$ and $x = 2$.

Solution

a. The right endpoints of the five intervals are $\frac{2}{5}i$, where $i = 1, 2, 3, 4, 5$. The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the right endpoint of each interval.

$$\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \left[\frac{6}{5}, \frac{8}{5}\right], \left[\frac{8}{5}, \frac{10}{5}\right]$$

Evaluate f at the right endpoints of these intervals.

The sum of the areas of the five rectangles is

$$\sum_{i=1}^{5} \underbrace{f\left(\frac{2i}{5}\right)}_{\text{Height}} \underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48.$$

Because each of the five rectangles lies inside the parabolic region, you can conclude that the area of the parabolic region is greater than 6.48.

b. The left endpoints of the five intervals are $\frac{2}{5}(i - 1)$, where $i = 1, 2, 3, 4, 5$. The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the left endpoint of each interval. So, the sum is

$$\sum_{i=1}^{5} \underbrace{f\left(\frac{2i - 2}{5}\right)}_{\text{Height}} \underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i - 2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08.$$

Because the parabolic region lies within the union of the five rectangular regions, you can conclude that the area of the parabolic region is less than 8.08.

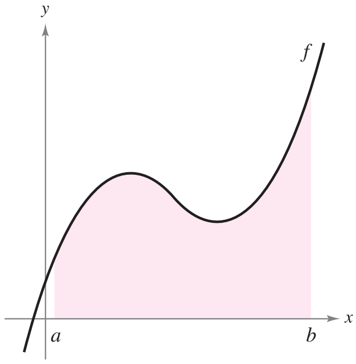
By combining the results in parts (a) and (b), you can conclude that

$$6.48 < (\text{Area of region}) < 8.08.$$

NOTE By increasing the number of rectangles used in Example 3, you can obtain closer and closer approximations of the area of the region. For instance, using 25 rectangles of width $\frac{2}{25}$ each, you can conclude that

$$7.17 < (\text{Area of region}) < 7.49.$$

Upper and Lower Sums



The region under a curve
Figure 4.9

The procedure used in Example 3 can be generalized as follows. Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$, as shown in Figure 4.9. The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.

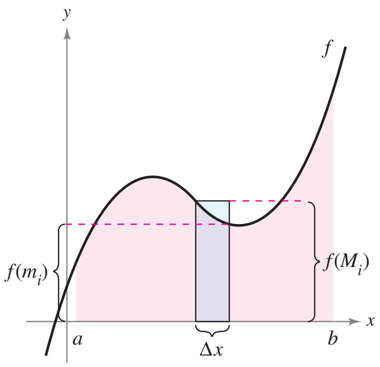
To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, as shown in Figure 4.10. The endpoints of the intervals are as follows.

$$\overbrace{a + 0(\Delta x)}^{a = x_0} < \overbrace{a + 1(\Delta x)}^{x_1} < \overbrace{a + 2(\Delta x)}^{x_2} < \cdot \cdot \cdot < \overbrace{a + n(\Delta x)}^{x_n = b}$$

Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in *each* subinterval.

$f(m_i) =$ Minimum value of $f(x)$ in ith subinterval

$f(M_i) =$ Maximum value of $f(x)$ in ith subinterval



The interval $[a, b]$ is divided into n subintervals of width $\Delta x = \dfrac{b - a}{n}$.
Figure 4.10

Next, define an **inscribed rectangle** lying *inside* the ith subregion and a **circumscribed rectangle** extending *outside* the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$. For *each* i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\left(\begin{matrix}\text{Area of inscribed} \\ \text{rectangle}\end{matrix}\right) = f(m_i)\,\Delta x \le f(M_i)\,\Delta x = \left(\begin{matrix}\text{Area of circumscribed} \\ \text{rectangle}\end{matrix}\right)$$

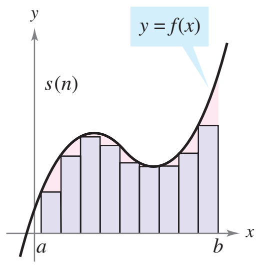
The sum of the areas of the inscribed rectangles is called a **lower sum,** and the sum of the areas of the circumscribed rectangles is called an **upper sum.**

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x \qquad \text{Area of inscribed rectangles}$$

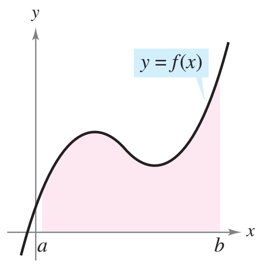
$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x \qquad \text{Area of circumscribed rectangles}$$

From Figure 4.11, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

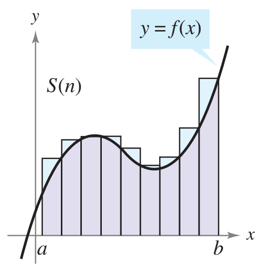
$$s(n) \le (\text{Area of region}) \le S(n)$$



Area of inscribed rectangles is less than area of region.

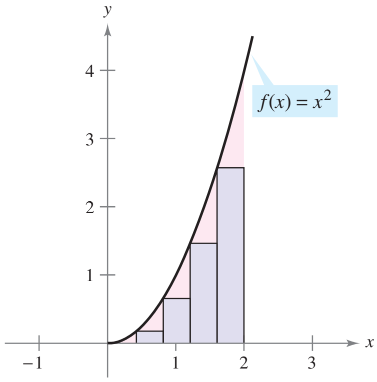


Area of region

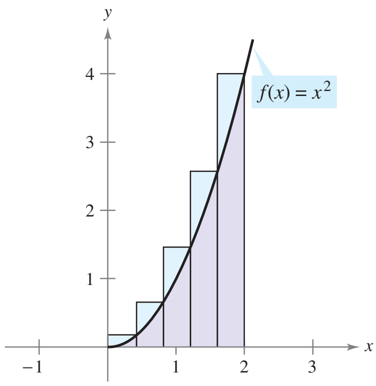


Area of circumscribed rectangles is greater than area of region.

Figure 4.11



Inscribed rectangles



Circumscribed rectangles
Figure 4.12

EXAMPLE 4 Finding Upper and Lower Sums for a Region

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 2$.

Solution To begin, partition the interval $[0, 2]$ into n subintervals, each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$

Figure 4.12 shows the endpoints of the subintervals and several inscribed and circumscribed rectangles. Because f is increasing on the interval $[0, 2]$, the minimum value on each subinterval occurs at the left endpoint, and the maximum value occurs at the right endpoint.

Left Endpoints	*Right Endpoints*
$m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$	$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

Using the left endpoints, the lower sum is

$$\begin{aligned}
s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x &= \sum_{i=1}^{n} f\left[\frac{2(i-1)}{n}\right]\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left[\frac{2(i-1)}{n}\right]^2\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right)(i^2 - 2i + 1) \\
&= \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) \\
&= \frac{8}{n^3}\left\{\frac{n(n+1)(2n+1)}{6} - 2\left[\frac{n(n+1)}{2}\right] + n\right\} \\
&= \frac{4}{3n^3}(2n^3 - 3n^2 + n) \\
&= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}. \qquad \text{Lower sum}
\end{aligned}$$

Using the right endpoints, the upper sum is

$$\begin{aligned}
S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right) i^2 \\
&= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{4}{3n^3}(2n^3 + 3n^2 + n) \\
&= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}. \qquad \text{Upper sum}
\end{aligned}$$

■

EXPLORATION

For the region given in Example 4, evaluate the lower sum

$$s(n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}$$

and the upper sum

$$S(n) = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

for $n = 10$, 100, and 1000. Use your results to determine the area of the region.

Example 4 illustrates some important things about lower and upper sums. First, notice that for any value of n, the lower sum is less than (or equal to) the upper sum.

$$s(n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} < \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} = S(n)$$

Second, the difference between these two sums lessens as n increases. In fact, if you take the limits as $n \to \infty$, both the upper sum and the lower sum approach $\frac{8}{3}$.

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\left(\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3} \qquad \text{Lower sum limit}$$

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3} \qquad \text{Upper sum limit}$$

The next theorem shows that the equivalence of the limits (as $n \to \infty$) of the upper and lower sums is not mere coincidence. It is true for all functions that are continuous and nonnegative on the closed interval $[a, b]$. The proof of this theorem is best left to a course in advanced calculus.

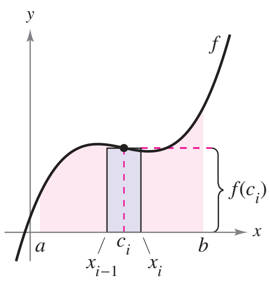
THEOREM 4.3 LIMITS OF THE LOWER AND UPPER SUMS

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\,\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\,\Delta x \\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

Because the same limit is attained for both the minimum value $f(m_i)$ and the maximum value $f(M_i)$, it follows from the Squeeze Theorem (Theorem 1.8) that the choice of x in the ith subinterval does not affect the limit. This means that you are free to choose an *arbitrary* x-value in the ith subinterval, as in the following *definition of the area of a region in the plane*.



The width of the ith subinterval is $\Delta x = x_i - x_{i-1}$.
Figure 4.13

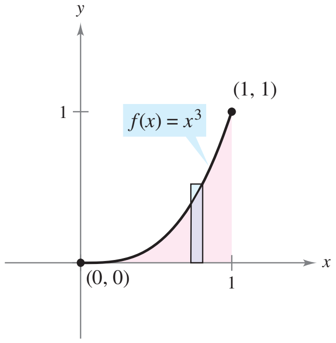
DEFINITION OF THE AREA OF A REGION IN THE PLANE

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$ (see Figure 4.13).

EXAMPLE 5 Finding Area by the Limit Definition



The area of the region bounded by the graph of f, the x-axis, $x = 0$, and $x = 1$ is $\frac{1}{4}$.
Figure 4.14

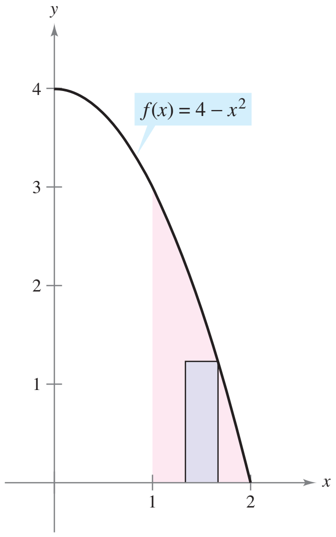
Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$, as shown in Figure 4.14.

Solution Begin by noting that f is continuous and nonnegative on the interval $[0, 1]$. Next, partition the interval $[0, 1]$ into n subintervals, each of width $\Delta x = 1/n$. According to the definition of area, you can choose any x-value in the ith subinterval. For this example, the right endpoints $c_i = i/n$ are convenient.

$$\begin{aligned}
\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right) && \text{Right endpoints: } c_i = \frac{i}{n} \\
&= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 \\
&= \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4}\right] \\
&= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) \\
&= \frac{1}{4}
\end{aligned}$$

The area of the region is $\frac{1}{4}$.

EXAMPLE 6 Finding Area by the Limit Definition



The area of the region bounded by the graph of f, the x-axis, $x = 1$, and $x = 2$ is $\frac{5}{3}$.
Figure 4.15

Find the area of the region bounded by the graph of $f(x) = 4 - x^2$, the x-axis, and the vertical lines $x = 1$ and $x = 2$, as shown in Figure 4.15.

Solution The function f is continuous and nonnegative on the interval $[1, 2]$, and so begin by partitioning the interval into n subintervals, each of width $\Delta x = 1/n$. Choosing the right endpoint

$$c_i = a + i\Delta x = 1 + \frac{i}{n} \qquad \text{Right endpoints}$$

of each subinterval, you obtain

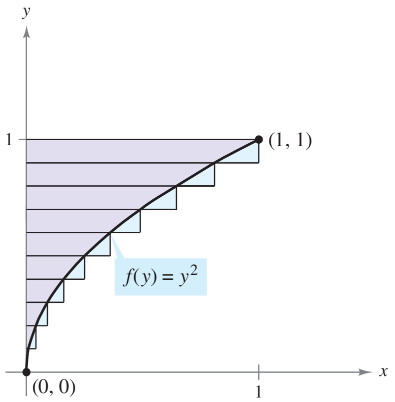
$$\begin{aligned}
\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[4 - \left(1 + \frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3 - \frac{2i}{n} - \frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} 3 - \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left[3 - \left(1 + \frac{1}{n}\right) - \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\right] \\
&= 3 - 1 - \frac{1}{3} \\
&= \frac{5}{3}.
\end{aligned}$$

The area of the region is $\frac{5}{3}$. ■

The last example in this section looks at a region that is bounded by the y-axis (rather than by the x-axis).

EXAMPLE 7 A Region Bounded by the y-axis

Find the area of the region bounded by the graph of $f(y) = y^2$ and the y-axis for $0 \le y \le 1$, as shown in Figure 4.16.



The area of the region bounded by the graph of f and the y-axis for $0 \le y \le 1$ is $\frac{1}{3}$.
Figure 4.16

Solution When f is a continuous, nonnegative function of y, you still can use the same basic procedure shown in Examples 5 and 6. Begin by partitioning the interval $[0, 1]$ into n subintervals, each of width $\Delta y = 1/n$. Then, using the upper endpoints $c_i = i/n$, you obtain

$$\begin{aligned}
\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta y &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \qquad \text{Upper endpoints: } c_i = \frac{i}{n}\\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2\\
&= \lim_{n\to\infty} \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6}\right]\\
&= \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\\
&= \frac{1}{3}.
\end{aligned}$$

The area of the region is $\frac{1}{3}$. ■

4.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the sum. Use the summation capabilities of a graphing utility to verify your result.

1. $\displaystyle\sum_{i=1}^{6} (3i + 2)$
2. $\displaystyle\sum_{k=5}^{8} k(k - 4)$
3. $\displaystyle\sum_{k=0}^{4} \frac{1}{k^2 + 1}$
4. $\displaystyle\sum_{j=4}^{7} \frac{2}{j}$
5. $\displaystyle\sum_{k=1}^{4} c$
6. $\displaystyle\sum_{i=1}^{4} [(i - 1)^2 + (i + 1)^3]$

In Exercises 7–14, use sigma notation to write the sum.

7. $\dfrac{1}{5(1)} + \dfrac{1}{5(2)} + \dfrac{1}{5(3)} + \cdots + \dfrac{1}{5(11)}$
8. $\dfrac{9}{1 + 1} + \dfrac{9}{1 + 2} + \dfrac{9}{1 + 3} + \cdots + \dfrac{9}{1 + 14}$
9. $\left[7\left(\frac{1}{6}\right) + 5\right] + \left[7\left(\frac{2}{6}\right) + 5\right] + \cdots + \left[7\left(\frac{6}{6}\right) + 5\right]$
10. $\left[1 - \left(\frac{1}{4}\right)^2\right] + \left[1 - \left(\frac{2}{4}\right)^2\right] + \cdots + \left[1 - \left(\frac{4}{4}\right)^2\right]$
11. $\left[\left(\frac{2}{n}\right)^3 - \frac{2}{n}\right]\left(\frac{2}{n}\right) + \cdots + \left[\left(\frac{2n}{n}\right)^3 - \frac{2n}{n}\right]\left(\frac{2}{n}\right)$
12. $\left[1 - \left(\frac{2}{n} - 1\right)^2\right]\left(\frac{2}{n}\right) + \cdots + \left[1 - \left(\frac{2n}{n} - 1\right)^2\right]\left(\frac{2}{n}\right)$
13. $\left[2\left(1 + \frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) + \cdots + \left[2\left(1 + \frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$
14. $\left(\frac{1}{n}\right)\sqrt{1 - \left(\frac{0}{n}\right)^2} + \cdots + \left(\frac{1}{n}\right)\sqrt{1 - \left(\frac{n-1}{n}\right)^2}$

In Exercises 15–22, use the properties of summation and Theorem 4.2 to evaluate the sum. Use the summation capabilities of a graphing utility to verify your result.

15. $\displaystyle\sum_{i=1}^{12} 7$
16. $\displaystyle\sum_{i=1}^{30} -18$
17. $\displaystyle\sum_{i=1}^{24} 4i$
18. $\displaystyle\sum_{i=1}^{16} (5i - 4)$
19. $\displaystyle\sum_{i=1}^{20} (i - 1)^2$
20. $\displaystyle\sum_{i=1}^{10} (i^2 - 1)$
21. $\displaystyle\sum_{i=1}^{15} i(i - 1)^2$
22. $\displaystyle\sum_{i=1}^{10} i(i^2 + 1)$

In Exercises 23 and 24, use the summation capabilities of a graphing utility to evaluate the sum. Then use the properties of summation and Theorem 4.2 to verify the sum.

23. $\displaystyle\sum_{i=1}^{20} (i^2 + 3)$
24. $\displaystyle\sum_{i=1}^{15} (i^3 - 2i)$

25. Consider the function $f(x) = 3x + 2$.

(a) Estimate the area between the graph of f and the x-axis between $x = 0$ and $x = 3$ using six rectangles and right endpoints. Sketch the graph and the rectangles.

(b) Repeat part (a) using left endpoints.

26. Consider the function $g(x) = x^2 + x - 4$.

(a) Estimate the area between the graph of g and the x-axis between $x = 2$ and $x = 4$ using four rectangles and right endpoints. Sketch the graph and the rectangles.

(b) Repeat part (a) using left endpoints.

In Exercises 27–32, use left and right endpoints and the given number of rectangles to find two approximations of the area of the region between the graph of the function and the *x*-axis over the given interval.

27. $f(x) = 2x + 5$, $[0, 2]$, 4 rectangles

28. $f(x) = 9 - x$, $[2, 4]$, 6 rectangles

29. $g(x) = 2x^2 - x - 1$, $[2, 5]$, 6 rectangles

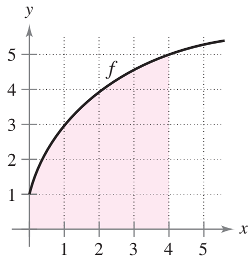
30. $g(x) = x^2 + 1$, $[1, 3]$, 8 rectangles

31. $f(x) = \cos x$, $\left[0, \dfrac{\pi}{2}\right]$, 4 rectangles

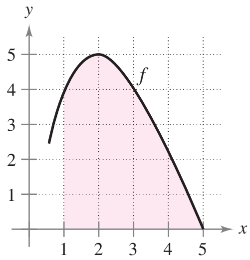
32. $g(x) = \sin x$, $[0, \pi]$, 6 rectangles

In Exercises 33–36, bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

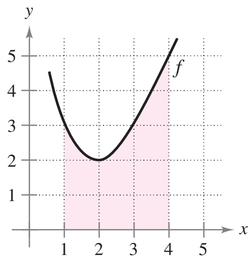
33.



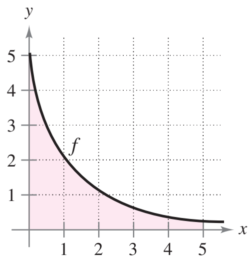
34.



35.



36.



In Exercises 37–40, find the limit of $s(n)$ as $n \to \infty$.

37. $s(n) = \dfrac{81}{n^4}\left[\dfrac{n^2(n+1)^2}{4}\right]$

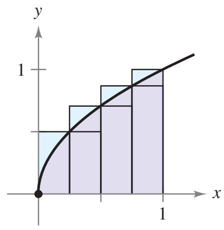
38. $s(n) = \dfrac{64}{n^3}\left[\dfrac{n(n+1)(2n+1)}{6}\right]$

39. $s(n) = \dfrac{18}{n^2}\left[\dfrac{n(n+1)}{2}\right]$

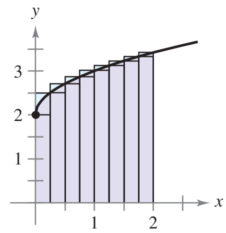
40. $s(n) = \dfrac{1}{n^2}\left[\dfrac{n(n+1)}{2}\right]$

In Exercises 41–44, use upper and lower sums to approximate the area of the region using the given number of subintervals (of equal width).

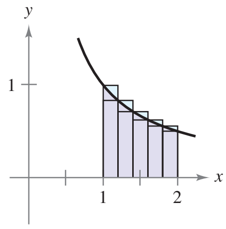
41. $y = \sqrt{x}$



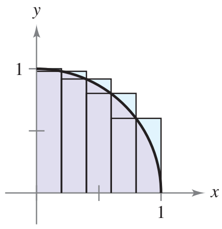
42. $y = \sqrt{x} + 2$



43. $y = \dfrac{1}{x}$



44. $y = \sqrt{1 - x^2}$



In Exercises 45–48, use the summation formulas to rewrite the expression without the summation notation. Use the result to find the sums for $n = 10$, 100, 1000, and 10,000.

45. $\displaystyle\sum_{i=1}^{n} \frac{2i+1}{n^2}$

46. $\displaystyle\sum_{j=1}^{n} \frac{4j+3}{n^2}$

47. $\displaystyle\sum_{k=1}^{n} \frac{6k(k-1)}{n^3}$

48. $\displaystyle\sum_{i=1}^{n} \frac{4i^2(i-1)}{n^4}$

In Exercises 49–54, find a formula for the sum of n terms. Use the formula to find the limit as $n \to \infty$.

49. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{24i}{n^2}$

50. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

51. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$

52. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

53. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right)$

54. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$

55. ***Numerical Reasoning*** Consider a triangle of area 2 bounded by the graphs of $y = x$, $y = 0$, and $x = 2$.

(a) Sketch the region.

(b) Divide the interval $[0, 2]$ into n subintervals of equal width and show that the endpoints are

$$0 < 1\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \displaystyle\sum_{i=1}^{n} \left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \displaystyle\sum_{i=1}^{n} \left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 2$.

56. ***Numerical Reasoning*** Consider a trapezoid of area 4 bounded by the graphs of $y = x$, $y = 0$, $x = 1$, and $x = 3$.

(a) Sketch the region.

(b) Divide the interval $[1, 3]$ into n subintervals of equal width and show that the endpoints are

$$1 < 1 + 1\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 4$.

In Exercises 57–66, use the limit process to find the area of the region between the graph of the function and the x-axis over the given interval. Sketch the region.

57. $y = -4x + 5, \quad [0, 1]$

58. $y = 3x - 2, \quad [2, 5]$

59. $y = x^2 + 2, \quad [0, 1]$

60. $y = x^2 + 1, \quad [0, 3]$

61. $y = 25 - x^2, \quad [1, 4]$

62. $y = 4 - x^2, \quad [-2, 2]$

63. $y = 27 - x^3, \quad [1, 3]$

64. $y = 2x - x^3, \quad [0, 1]$

65. $y = x^2 - x^3, \quad [-1, 1]$

66. $y = x^2 - x^3, \quad [-1, 0]$

In Exercises 67–72, use the limit process to find the area of the region between the graph of the function and the y-axis over the given y-interval. Sketch the region.

67. $f(y) = 4y, 0 \le y \le 2$

68. $g(y) = \frac{1}{2}y, 2 \le y \le 4$

69. $f(y) = y^2, 0 \le y \le 5$

70. $f(y) = 4y - y^2, 1 \le y \le 2$

71. $g(y) = 4y^2 - y^3, 1 \le y \le 3$

72. $h(y) = y^3 + 1, 1 \le y \le 2$

In Exercises 73–76, use the *Midpoint Rule*

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$$

with $n = 4$ to approximate the area of the region bounded by the graph of the function and the x-axis over the given interval.

73. $f(x) = x^2 + 3, \quad [0, 2]$

74. $f(x) = x^2 + 4x, \quad [0, 4]$

75. $f(x) = \tan x, \quad \left[0, \frac{\pi}{4}\right]$

76. $f(x) = \sin x, \quad \left[0, \frac{\pi}{2}\right]$

Programming **Write a program for a graphing utility to approximate areas by using the Midpoint Rule. Assume that the function is positive over the given interval and that the subintervals are of equal width. In Exercises 77–80, use the program to approximate the area of the region between the graph of the function and the x-axis over the given interval, and complete the table.**

n	4	8	12	16	20
Approximate Area					

77. $f(x) = \sqrt{x}, \quad [0, 4]$

78. $f(x) = \dfrac{8}{x^2 + 1}, \quad [2, 6]$

79. $f(x) = \tan\left(\dfrac{\pi x}{8}\right), \quad [1, 3]$

80. $f(x) = \cos\sqrt{x}, \quad [0, 2]$

WRITING ABOUT CONCEPTS

Approximation **In Exercises 81 and 82, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region and not by performing calculations.)**

81. $f(x) = 4 - x^2, \quad [0, 2]$

(a) -2 (b) 6 (c) 10 (d) 3 (e) 8

82. $f(x) = \sin\dfrac{\pi x}{4}, \quad [0, 4]$

(a) 3 (b) 1 (c) -2 (d) 8 (e) 6

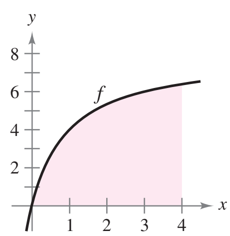
83. In your own words and using appropriate figures, describe the methods of upper sums and lower sums in approximating the area of a region.

84. Give the definition of the area of a region in the plane.

85. ***Graphical Reasoning*** Consider the region bounded by the graphs of $f(x) = 8x/(x + 1)$, $x = 0$, $x = 4$, and $y = 0$, as shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) Redraw the figure, and complete and shade the rectangles representing the lower sum when $n = 4$. Find this lower sum.

(b) Redraw the figure, and complete and shade the rectangles representing the upper sum when $n = 4$. Find this upper sum.



(c) Redraw the figure, and complete and shade the rectangles whose heights are determined by the functional values at the midpoint of each subinterval when $n = 4$. Find this sum using the Midpoint Rule.

(d) Verify the following formulas for approximating the area of the region using n subintervals of equal width.

Lower sum: $s(n) = \sum_{i=1}^{n} f\left[(i-1)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Upper sum: $S(n) = \sum_{i=1}^{n} f\left[(i)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Midpoint Rule: $M(n) = \sum_{i=1}^{n} f\left[\left(i-\frac{1}{2}\right)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

(e) Use a graphing utility and the formulas in part (d) to complete the table.

n	4	8	20	100	200
$s(n)$					
$S(n)$					
$M(n)$					

(f) Explain why $s(n)$ increases and $S(n)$ decreases for increasing values of n, as shown in the table in part (e).

CAPSTONE

86. Consider a function $f(x)$ that is increasing on the interval $[1, 4]$. The interval $[1, 4]$ is divided into 12 subintervals.

(a) What are the left endpoints of the first and last subintervals?

(b) What are the right endpoints of the first two subintervals?

(c) When using the right endpoints, will the rectangles lie above or below the graph of $f(x)$? Use a graph to explain your answer.

(d) What can you conclude about the heights of the rectangles if a function is constant on the given interval?

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. The sum of the first n positive integers is $n(n+1)/2$.

88. If f is continuous and nonnegative on $[a, b]$, then the limits as $n \to \infty$ of its lower sum $s(n)$ and upper sum $S(n)$ both exist and are equal.

89. ***Writing*** Use the figure to write a short paragraph explaining why the formula $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ is valid for all positive integers n.

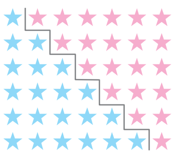


Figure for 89

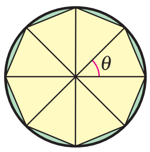


Figure for 90

90. ***Graphical Reasoning*** Consider an n-sided regular polygon inscribed in a circle of radius r. Join the vertices of the polygon to the center of the circle, forming n congruent triangles (see figure).

(a) Determine the central angle θ in terms of n.

(b) Show that the area of each triangle is $\frac{1}{2}r^2 \sin \theta$.

(c) Let A_n be the sum of the areas of the n triangles. Find $\lim_{n\to\infty} A_n$.

91. ***Modeling Data*** The table lists the measurements of a lot bounded by a stream and two straight roads that meet at right angles, where x and y are measured in feet (see figure).

x	0	50	100	150	200	250	300
y	450	362	305	268	245	156	0

(a) Use the regression capabilities of a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model in part (a) to estimate the area of the lot.

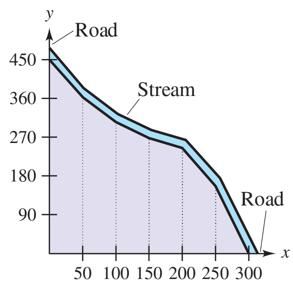
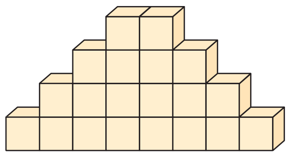


Figure for 91



n is even.

Figure for 92

92. ***Building Blocks*** A child places n cubic building blocks in a row to form the base of a triangular design (see figure). Each successive row contains two fewer blocks than the preceding row. Find a formula for the number of blocks used in the design. (*Hint:* The number of building blocks in the design depends on whether n is odd or even.)

93. Prove each formula by mathematical induction. (You may need to review the method of proof by induction from a precalculus text.)

(a) $\sum_{i=1}^{n} 2i = n(n+1)$ (b) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

PUTNAM EXAM CHALLENGE

94. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Write your answer in the form $(a\sqrt{b} + c)/d$, where a, b, c, and d are positive integers.

This problem was composed by the Committee on the Putnam Prize Competition.

4.3 Riemann Sums and Definite Integrals

- Understand the definition of a Riemann sum.
- Evaluate a definite integral using limits.
- Evaluate a definite integral using properties of definite integrals.

Riemann Sums

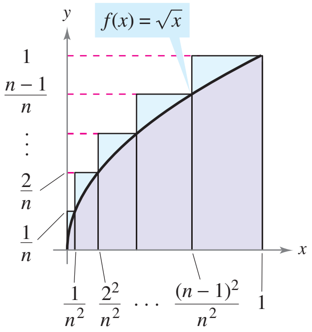
In the definition of area given in Section 4.2, the partitions have subintervals of *equal width*. This was done only for computational convenience. The following example shows that it is not necessary to have subintervals of equal width.

EXAMPLE 1 A Partition with Subintervals of Unequal Widths

Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \le x \le 1$, as shown in Figure 4.17. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

where c_i is the right endpoint of the partition given by $c_i = i^2/n^2$ and Δx_i is the width of the ith interval.



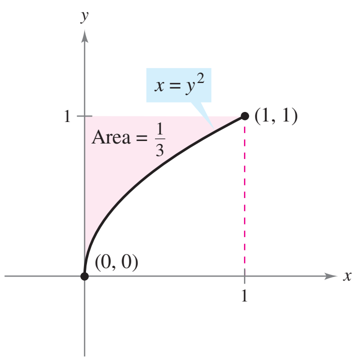
The subintervals do not have equal widths.
Figure 4.17

Solution The width of the ith interval is given by

$$\begin{aligned}\Delta x_i &= \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2}\\ &= \frac{i^2 - i^2 + 2i - 1}{n^2}\\ &= \frac{2i-1}{n^2}.\end{aligned}$$

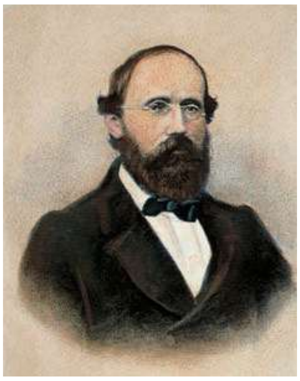
So, the limit is

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}}\left(\frac{2i-1}{n^2}\right)\\ &= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - i)\\ &= \lim_{n\to\infty} \frac{1}{n^3}\left[2\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty} \frac{4n^3 + 3n^2 - n}{6n^3}\\ &= \frac{2}{3}.\end{aligned}$$



The area of the region bounded by the graph of $x = y^2$ and the y-axis for $0 \le y \le 1$ is $\frac{1}{3}$.
Figure 4.18

From Example 7 in Section 4.2, you know that the region shown in Figure 4.18 has an area of $\frac{1}{3}$. Because the square bounded by $0 \le x \le 1$ and $0 \le y \le 1$ has an area of 1, you can conclude that the area of the region shown in Figure 4.17 has an area of $\frac{2}{3}$. This agrees with the limit found in Example 1, even though that example used a partition having subintervals of unequal widths. The reason this particular partition gave the proper area is that as n increases, the *width of the largest subinterval approaches zero*. This is a key feature of the development of definite integrals.



The Granger Collection

GEORG FRIEDRICH BERNHARD RIEMANN (1826–1866)

German mathematician Riemann did his most famous work in the areas of non-Euclidean geometry, differential equations, and number theory. It was Riemann's results in physics and mathematics that formed the structure on which Einstein's General Theory of Relativity is based.

In the preceding section, the limit of a sum was used to define the area of a region in the plane. Finding area by this means is only one of *many* applications involving the limit of a sum. A similar approach can be used to determine quantities as diverse as arc lengths, average values, centroids, volumes, work, and surface areas. The following definition is named after Georg Friedrich Bernhard Riemann. Although the definite integral had been defined and used long before the time of Riemann, he generalized the concept to cover a broader category of functions.

In the following definition of a Riemann sum, note that the function f has no restrictions other than being defined on the interval $[a, b]$. (In the preceding section, the function f was assumed to be continuous and nonnegative because we were dealing with the area under a curve.)

DEFINITION OF RIEMANN SUM

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

NOTE The sums in Section 4.2 are examples of Riemann sums, but there are more general Riemann sums than those covered there. ■

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal width, the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}. \qquad \text{Regular partition}$$

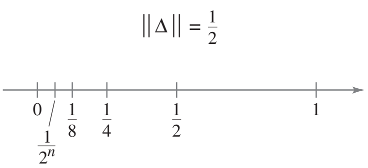
For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \le n \qquad \text{General partition}$$

So, the number of subintervals in a partition approaches infinity as the norm of the partition approaches 0. That is, $\|\Delta\| \to 0$ implies that $n \to \infty$.

The converse of this statement is not true. For example, let Δ_n be the partition of the interval $[0, 1]$ given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdot \cdot \cdot < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1.$$



$n \to \infty$ does not imply that $\|\Delta\| \to 0$.
Figure 4.19

As shown in Figure 4.19, for any positive value of n, the norm of the partition Δ_n is $\frac{1}{2}$. So, letting n approach infinity does not force $\|\Delta\|$ to approach 0. In a regular partition, however, the statements $\|\Delta\| \to 0$ and $n \to \infty$ are equivalent.

Definite Integrals

To define the definite integral, consider the following limit.

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = L$$

To say that this limit exists means there exists a real number L such that for each $\varepsilon > 0$ there exists a $\delta > 0$ so that for every partition with $\|\Delta\| < \delta$ it follows that

$$\left| L - \sum_{i=1}^{n} f(c_i)\,\Delta x_i \right| < \varepsilon$$

regardless of the choice of c_i in the ith subinterval of each partition Δ.

■ **FOR FURTHER INFORMATION** For insight into the history of the definite integral, see the article "The Evolution of Integration" by A. Shenitzer and J Steprāns in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

DEFINITION OF DEFINITE INTEGRAL

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

It is not a coincidence that the notation for definite integrals is similar to that used for indefinite integrals. You will see why in the next section when the Fundamental Theorem of Calculus is introduced. For now it is important to see that definite integrals and indefinite integrals are different concepts. A definite integral is a *number*, whereas an indefinite integral is a *family of functions*.

Though Riemann sums were defined for functions with very few restrictions, a sufficient condition for a function f to be integrable on $[a, b]$ is that it is continuous on $[a, b]$. A proof of this theorem is beyond the scope of this text.

STUDY TIP Later in this chapter, you will learn convenient methods for calculating $\int_a^b f(x)\,dx$ for continuous functions. For now, you must use the limit definition.

THEOREM 4.4 CONTINUITY IMPLIES INTEGRABILITY

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

EXPLORATION

The Converse of Theorem 4.4 Is the converse of Theorem 4.4 true? That is, if a function is integrable, does it have to be continuous? Explain your reasoning and give examples.

Describe the relationships among continuity, differentiability, and integrability. Which is the strongest condition? Which is the weakest? Which conditions imply other conditions?

EXAMPLE 2 Evaluating a Definite Integral as a Limit

Evaluate the definite integral $\displaystyle\int_{-2}^{1} 2x\,dx$.

Solution The function $f(x) = 2x$ is integrable on the interval $[-2, 1]$ because it is continuous on $[-2, 1]$. Moreover, the definition of integrability implies that any partition whose norm approaches 0 can be used to determine the limit. For computational convenience, define Δ by subdividing $[-2, 1]$ into n subintervals of equal width

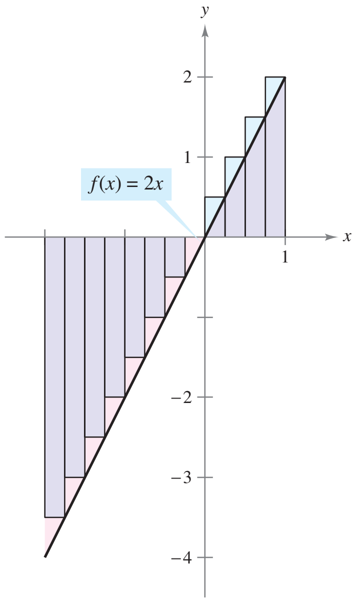
$$\Delta x_i = \Delta x = \frac{b-a}{n} = \frac{3}{n}.$$

Choosing c_i as the right endpoint of each subinterval produces

$$c_i = a + i(\Delta x) = -2 + \frac{3i}{n}.$$

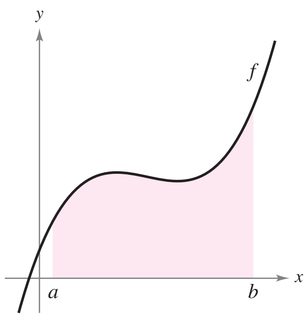
So, the definite integral is given by

$$\begin{aligned}
\int_{-2}^{1} 2x\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left(-2 + \frac{3i}{n}\right) \\
&= \lim_{n\to\infty} \frac{6}{n}\left\{-2n + \frac{3}{n}\left[\frac{n(n+1)}{2}\right]\right\} \\
&= \lim_{n\to\infty} \left(-12 + 9 + \frac{9}{n}\right) \\
&= -3.
\end{aligned}$$



Because the definite integral is negative, it does not represent the area of the region.
Figure 4.20

Because the definite integral in Example 2 is negative, it *does not* represent the area of the region shown in Figure 4.20. Definite integrals can be positive, negative, or zero. For a definite integral to be interpreted as an area (as defined in Section 4.2), the function f must be continuous and nonnegative on $[a, b]$, as stated in the following theorem. The proof of this theorem is straightforward—you simply use the definition of area given in Section 4.2, because it is a Riemann sum.



You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.
Figure 4.21

THEOREM 4.5 THE DEFINITE INTEGRAL AS THE AREA OF A REGION

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 4.21.)

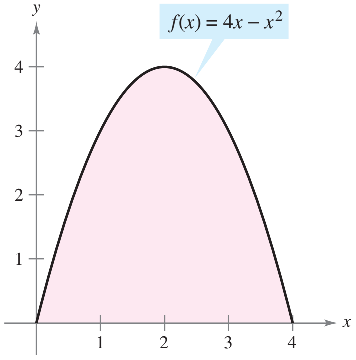
As an example of Theorem 4.5, consider the region bounded by the graph of

$$f(x) = 4x - x^2$$

and the x-axis, as shown in Figure 4.22. Because f is continuous and nonnegative on the closed interval $[0, 4]$, the area of the region is

$$\text{Area} = \int_0^4 (4x - x^2)\,dx.$$

A straightforward technique for evaluating a definite integral such as this will be discussed in Section 4.4. For now, however, you can evaluate a definite integral in two ways—you can use the limit definition *or* you can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.



Area $= \int_0^4 (4x - x^2)\,dx$
Figure 4.22

EXAMPLE 3 Areas of Common Geometric Figures

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. $\int_1^3 4\,dx$ **b.** $\int_0^3 (x + 2)\,dx$ **c.** $\int_{-2}^2 \sqrt{4 - x^2}\,dx$

Solution A sketch of each region is shown in Figure 4.23.

a. This region is a rectangle of height 4 and width 2.

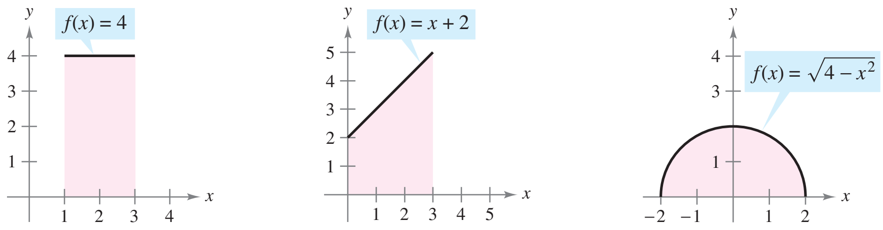
$$\int_1^3 4\,dx = (\text{Area of rectangle}) = 4(2) = 8$$

b. This region is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$.

$$\int_0^3 (x + 2)\,dx = (\text{Area of trapezoid}) = \frac{1}{2}(3)(2 + 5) = \frac{21}{2}$$

c. This region is a semicircle of radius 2. The formula for the area of a semicircle is $\frac{1}{2}\pi r^2$.

$$\int_{-2}^2 \sqrt{4 - x^2}\,dx = (\text{Area of semicircle}) = \frac{1}{2}\pi(2^2) = 2\pi$$



(a) (b) (c)
Figure 4.23

NOTE The variable of integration in a definite integral is sometimes called a *dummy variable* because it can be replaced by any other variable without changing the value of the integral. For instance, the definite integrals

$$\int_0^3 (x + 2)\,dx$$

and

$$\int_0^3 (t + 2)\,dt$$

have the same value.

Properties of Definite Integrals

The definition of the definite integral of f on the interval $[a, b]$ specifies that $a < b$. Now, however, it is convenient to extend the definition to cover cases in which $a = b$ or $a > b$. Geometrically, the following two definitions seem reasonable. For instance, it makes sense to define the area of a region of zero width and finite height to be 0.

DEFINITIONS OF TWO SPECIAL DEFINITE INTEGRALS

1. If f is defined at $x = a$, then we define $\displaystyle\int_a^a f(x)\,dx = 0$.

2. If f is integrable on $[a, b]$, then we define $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

EXAMPLE 4 Evaluating Definite Integrals

a. Because the sine function is defined at $x = \pi$, and the upper and lower limits of integration are equal, you can write

$$\int_\pi^\pi \sin x\,dx = 0.$$

b. The integral $\int_3^0 (x + 2)\,dx$ is the same as that given in Example 3(b) except that the upper and lower limits are interchanged. Because the integral in Example 3(b) has a value of $\frac{21}{2}$, you can write

$$\int_3^0 (x + 2)\,dx = -\int_0^3 (x + 2)\,dx = -\frac{21}{2}.$$

In Figure 4.24, the larger region can be divided at $x = c$ into two subregions whose intersection is a line segment. Because the line segment has zero area, it follows that the area of the larger region is equal to the sum of the areas of the two smaller regions.

THEOREM 4.6 ADDITIVE INTERVAL PROPERTY

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

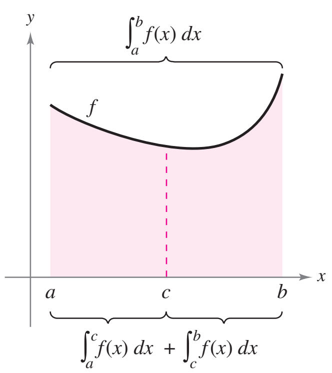


Figure 4.24

EXAMPLE 5 Using the Additive Interval Property

$$\begin{aligned}\int_{-1}^1 |x|\,dx &= \int_{-1}^0 -x\,dx + \int_0^1 x\,dx && \text{Theorem 4.6}\\ &= \frac{1}{2} + \frac{1}{2} && \text{Area of a triangle}\\ &= 1\end{aligned}$$

Because the definite integral is defined as the limit of a sum, it inherits the properties of summation given at the top of page 260.

THEOREM 4.7 PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

Note that Property 2 of Theorem 4.7 can be extended to cover any finite number of functions. For example,

$$\int_a^b [f(x) + g(x) + h(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \int_a^b h(x)\,dx.$$

EXAMPLE 6 Evaluation of a Definite Integral

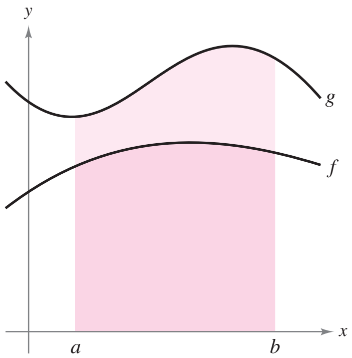
Evaluate $\displaystyle\int_1^3 (-x^2 + 4x - 3)\,dx$ using each of the following values.

$$\int_1^3 x^2\,dx = \frac{26}{3}, \qquad \int_1^3 x\,dx = 4, \qquad \int_1^3 dx = 2$$

Solution

$$\begin{aligned}
\int_1^3 (-x^2 + 4x - 3)\,dx &= \int_1^3 (-x^2)\,dx + \int_1^3 4x\,dx + \int_1^3 (-3)\,dx \\
&= -\int_1^3 x^2\,dx + 4\int_1^3 x\,dx - 3\int_1^3 dx \\
&= -\left(\frac{26}{3}\right) + 4(4) - 3(2) \\
&= \frac{4}{3}
\end{aligned}$$

■



$\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Figure 4.25

If f and g are continuous on the closed interval $[a, b]$ and

$0 \le f(x) \le g(x)$

for $a \le x \le b$, the following properties are true. First, the area of the region bounded by the graph of f and the x-axis (between a and b) must be nonnegative. Second, this area must be less than or equal to the area of the region bounded by the graph of g and the x-axis (between a and b), as shown in Figure 4.25. These two properties are generalized in Theorem 4.8. (A proof of this theorem is given in Appendix A.)

THEOREM 4.8 PRESERVATION OF INEQUALITY

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)\, dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

4.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use Example 1 as a model to evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

over the region bounded by the graphs of the equations.

1. $f(x) = \sqrt{x}, \quad y = 0, \quad x = 0, \quad x = 3$
   (*Hint:* Let $c_i = 3i^2/n^2$.)
2. $f(x) = \sqrt[3]{x}, \quad y = 0, \quad x = 0, \quad x = 1$
   (*Hint:* Let $c_i = i^3/n^3$.)

In Exercises 3–8, evaluate the definite integral by the limit definition.

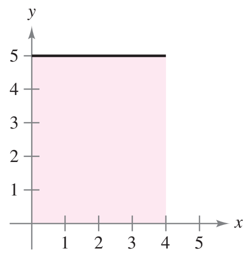
3. $\displaystyle\int_2^6 8\, dx$
4. $\displaystyle\int_{-2}^3 x\, dx$
5. $\displaystyle\int_{-1}^1 x^3\, dx$
6. $\displaystyle\int_1^4 4x^2\, dx$
7. $\displaystyle\int_1^2 (x^2 + 1)\, dx$
8. $\displaystyle\int_{-2}^1 (2x^2 + 3)\, dx$

In Exercises 9–12, write the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

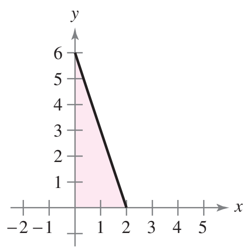
	Limit	*Interval*
9.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i$	$[-1, 5]$
10.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i$	$[0, 4]$
11.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i$	$[0, 3]$
12.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i$	$[1, 3]$

In Exercises 13–22, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

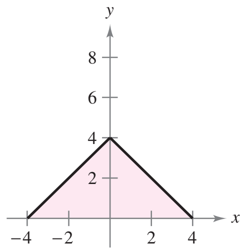
13. $f(x) = 5$



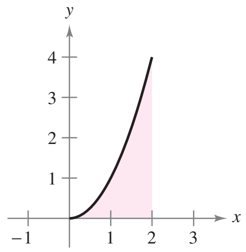
14. $f(x) = 6 - 3x$



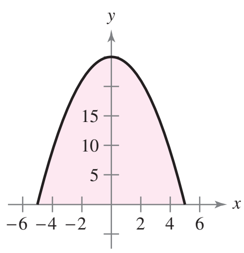
15. $f(x) = 4 - |x|$



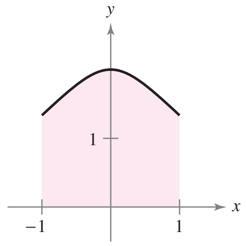
16. $f(x) = x^2$



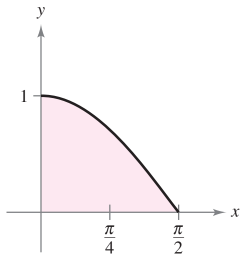
17. $f(x) = 25 - x^2$



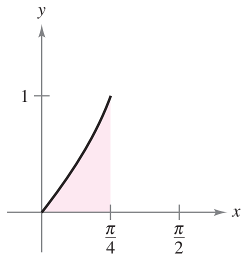
18. $f(x) = \dfrac{4}{x^2 + 2}$



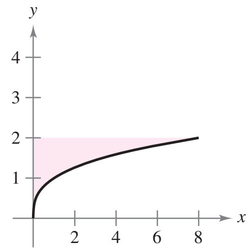
19. $f(x) = \cos x$



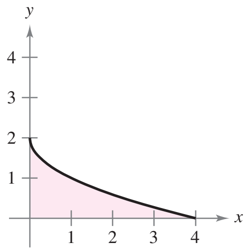
20. $f(x) = \tan x$



21. $g(y) = y^3$



22. $f(y) = (y - 2)^2$



In Exercises 23–32, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral ($a > 0, r > 0$)

23. $\int_0^3 4\,dx$

24. $\int_{-a}^{a} 4\,dx$

25. $\int_0^4 x\,dx$

26. $\int_0^4 \frac{x}{2}\,dx$

27. $\int_0^2 (3x + 4)\,dx$

28. $\int_0^6 (6 - x)\,dx$

29. $\int_{-1}^{1} (1 - |x|)\,dx$

30. $\int_{-a}^{a} (a - |x|)\,dx$

31. $\int_{-7}^{7} \sqrt{49 - x^2}\,dx$

32. $\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$

In Exercises 33–40, evaluate the integral using the following values.

$$\int_2^4 x^3\,dx = 60, \qquad \int_2^4 x\,dx = 6, \qquad \int_2^4 dx = 2$$

33. $\int_4^2 x\,dx$

34. $\int_2^2 x^3\,dx$

35. $\int_2^4 8x\,dx$

36. $\int_2^4 25\,dx$

37. $\int_2^4 (x - 9)\,dx$

38. $\int_2^4 (x^3 + 4)\,dx$

39. $\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right)\,dx$

40. $\int_2^4 (10 + 4x - 3x^3)\,dx$

41. Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, evaluate

(a) $\int_0^7 f(x)\,dx.$ (b) $\int_5^0 f(x)\,dx.$

(c) $\int_5^5 f(x)\,dx.$ (d) $\int_0^5 3f(x)\,dx.$

42. Given $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$, evaluate

(a) $\int_0^6 f(x)\,dx.$ (b) $\int_6^3 f(x)\,dx.$

(c) $\int_3^3 f(x)\,dx.$ (d) $\int_3^6 -5f(x)\,dx.$

43. Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, evaluate

(a) $\int_2^6 [f(x) + g(x)]\,dx.$ (b) $\int_2^6 [g(x) - f(x)]\,dx.$

(c) $\int_2^6 2g(x)\,dx.$ (d) $\int_2^6 3f(x)\,dx.$

44. Given $\int_{-1}^{1} f(x)\,dx = 0$ and $\int_0^1 f(x)\,dx = 5$, evaluate

(a) $\int_{-1}^{0} f(x)\,dx.$ (b) $\int_0^1 f(x)\,dx - \int_{-1}^{0} f(x)\,dx.$

(c) $\int_{-1}^{1} 3f(x)\,dx.$ (d) $\int_0^1 3f(x)\,dx.$

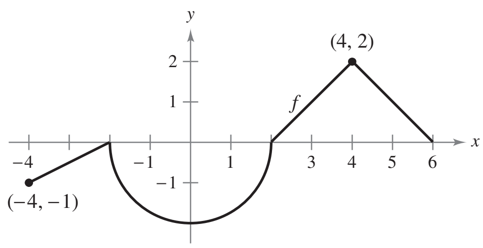
45. Use the table of values to find lower and upper estimates of $\int_0^{10} f(x)\,dx$. Assume that f is a decreasing function.

x	0	2	4	6	8	10
$f(x)$	32	24	12	-4	-20	-36

46. Use the table of values to estimate $\int_0^6 f(x)\,dx$. Use three equal subintervals and the (a) left endpoints, (b) right endpoints, and (c) midpoints. If f is an increasing function, how does each estimate compare with the actual value? Explain your reasoning.

x	0	1	2	3	4	5	6
$f(x)$	-6	0	8	18	30	50	80

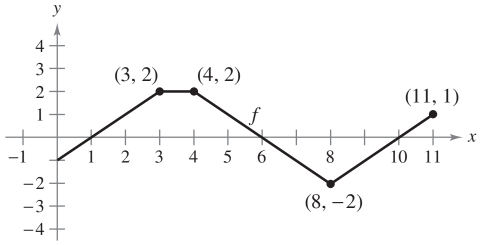
47. ***Think About It*** The graph of f consists of line segments and a semicircle, as shown in the figure. Evaluate each definite integral by using geometric formulas.



(a) $\int_0^2 f(x)\,dx$ (b) $\int_2^6 f(x)\,dx$ (c) $\int_{-4}^{2} f(x)\,dx$

(d) $\int_{-4}^{6} f(x)\,dx$ (e) $\int_{-4}^{6} |f(x)|\,dx$ (f) $\int_{-4}^{6} [f(x) + 2]\,dx$

48. ***Think About It*** The graph of f consists of line segments, as shown in the figure. Evaluate each definite integral by using geometric formulas.



(a) $\int_0^1 -f(x)\,dx$ (b) $\int_3^4 3f(x)\,dx$

(c) $\int_0^7 f(x)\,dx$ (d) $\int_5^{11} f(x)\,dx$

(e) $\int_0^{11} f(x)\,dx$ (f) $\int_4^{10} f(x)\,dx$

49. ***Think About It*** Consider the function f that is continuous on the interval $[-5, 5]$ and for which

$$\int_0^5 f(x)\,dx = 4.$$

Evaluate each integral.

(a) $\int_0^5 [f(x) + 2]\,dx$ (b) $\int_{-2}^3 f(x + 2)\,dx$

(c) $\int_{-5}^5 f(x)\,dx$ (f is even.) (d) $\int_{-5}^5 f(x)\,dx$ (f is odd.)

50. ***Think About It*** A function f is defined below. Use geometric formulas to find $\int_0^8 f(x)\,dx$.

$$f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$$

51. ***Think About It*** A function f is defined below. Use geometric formulas to find $\int_0^{12} f(x)\,dx$.

$$f(x) = \begin{cases} 6, & x > 6 \\ -\frac{1}{2}x + 9, & x \leq 6 \end{cases}$$

CAPSTONE

52. Find possible values of a and b that make the statement true. If possible, use a graph to support your answer. (There may be more than one correct answer.)

(a) $\int_{-2}^1 f(x)\,dx + \int_1^5 f(x)\,dx = \int_a^b f(x)\,dx$

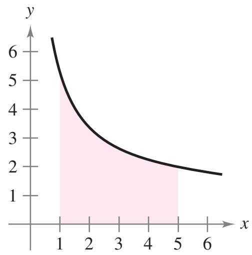
(b) $\int_{-3}^3 f(x)\,dx + \int_3^6 f(x)\,dx - \int_a^b f(x)\,dx = \int_{-1}^6 f(x)\,dx$

(c) $\int_a^b \sin x\,dx < 0$

(d) $\int_a^b \cos x\,dx = 0$

WRITING ABOUT CONCEPTS

In Exercises 53 and 54, use the figure to fill in the blank with the symbol <, >, or =.



53. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the left endpoint of the ith subinterval.

$$\sum_{i=1}^n f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

54. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the right endpoint of the ith subinterval.

$$\sum_{i=1}^n f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

55. Determine whether the function $f(x) = \dfrac{1}{x - 4}$ is integrable on the interval $[3, 5]$. Explain.

56. Give an example of a function that is integrable on the interval $[-1, 1]$, but not continuous on $[-1, 1]$.

In Exercises 57–60, determine which value best approximates the definite integral. Make your selection on the basis of a sketch.

57. $\int_0^4 \sqrt{x}\,dx$

(a) 5 (b) -3 (c) 10 (d) 2 (e) 8

58. $\int_0^{1/2} 4\cos \pi x\,dx$

(a) 4 (b) $\frac{4}{3}$ (c) 16 (d) 2π (e) -6

59. $\int_0^1 2\sin \pi x\,dx$

(a) 6 (b) $\frac{1}{2}$ (c) 4 (d) $\frac{5}{4}$

60. $\int_0^9 (1 + \sqrt{x})\,dx$

(a) -3 (b) 9 (c) 27 (d) 3

Programming **Write a program for your graphing utility to approximate a definite integral using the Riemann sum**

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

where the subintervals are of equal width. The output should give three approximations of the integral, where c_i is the left-hand endpoint $L(n)$, the midpoint $M(n)$, and the right-hand endpoint $R(n)$ of each subinterval. In Exercises 61–64, use the program to approximate the definite integral and complete the table.

n	4	8	12	16	20
$L(n)$					
$M(n)$					
$R(n)$					

61. $\displaystyle\int_0^3 x\sqrt{3-x}\,dx$

62. $\displaystyle\int_0^3 \frac{5}{x^2+1}\,dx$

63. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$

64. $\displaystyle\int_0^3 x\sin x\,dx$

True or False? **In Exercises 65–70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

65. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

66. $\displaystyle\int_a^b f(x)g(x)\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right]$

67. If the norm of a partition approaches zero, then the number of subintervals approaches infinity.

68. If f is increasing on $[a, b]$, then the minimum value of $f(x)$ on $[a, b]$ is $f(a)$.

69. The value of $\int_a^b f(x)\,dx$ must be positive.

70. The value of $\int_2^2 \sin(x^2)\,dx$ is 0.

71. Find the Riemann sum for $f(x) = x^2 + 3x$ over the interval $[0, 8]$, where $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 7$, and $x_4 = 8$, and where $c_1 = 1$, $c_2 = 2$, $c_3 = 5$, and $c_4 = 8$.

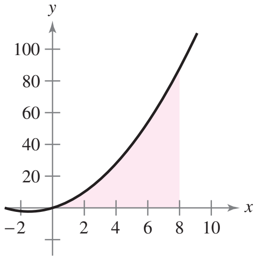


Figure for 71

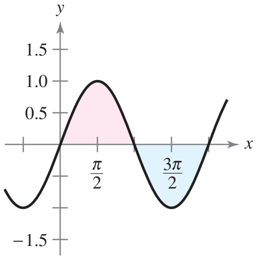


Figure for 72

72. Find the Riemann sum for $f(x) = \sin x$ over the interval $[0, 2\pi]$, where $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/3$, $x_3 = \pi$, and $x_4 = 2\pi$, and where $c_1 = \pi/6$, $c_2 = \pi/3$, $c_3 = 2\pi/3$, and $c_4 = 3\pi/2$.

73. Prove that $\displaystyle\int_a^b x\,dx = \frac{b^2-a^2}{2}$.

74. Prove that $\displaystyle\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$.

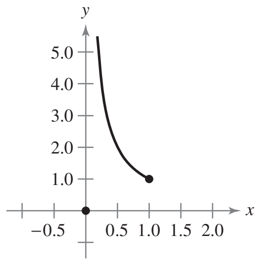
75. ***Think About It*** Determine whether the Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is integrable on the interval $[0, 1]$. Explain.

76. Suppose the function f is defined on $[0, 1]$, as shown in the figure.

$$f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{1}{x}, & 0 < x \le 1 \end{cases}$$



Show that $\int_0^1 f(x)\,dx$ does not exist. Why doesn't this contradict Theorem 4.4?

77. Find the constants a and b that maximize the value of

$$\int_a^b (1-x^2)\,dx.$$

Explain your reasoning.

78. Evaluate, if possible, the integral $\displaystyle\int_0^2 [\![x]\!]\,dx$.

79. Determine

$$\lim_{n\to\infty} \frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2]$$

by using an appropriate Riemann sum.

PUTNAM EXAM CHALLENGE

80. For each continuous function $f\colon [0, 1] \to R$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(x) = \int_0^1 x(f(x))^2\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

4.4 The Fundamental Theorem of Calculus

- Evaluate a definite integral using the Fundamental Theorem of Calculus.
- Understand and use the Mean Value Theorem for Integrals.
- Find the average value of a function over a closed interval.
- Understand and use the Second Fundamental Theorem of Calculus.
- Understand and use the Net Change Theorem.

The Fundamental Theorem of Calculus

EXPLORATION

Integration and Antidifferentiation Throughout this chapter, you have been using the integral sign to denote an antiderivative (a family of functions) and a definite integral (a number).

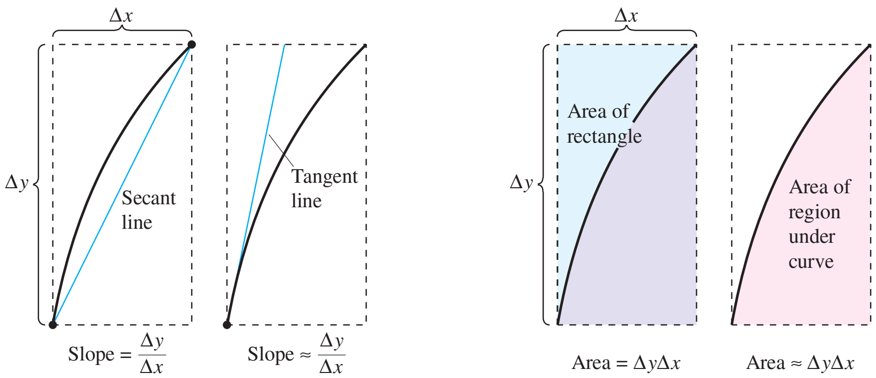
Antidifferentiation: $\int f(x)\,dx$

Definite integration: $\int_a^b f(x)\,dx$

The use of this same symbol for both operations makes it appear that they are related. In the early work with calculus, however, it was not known that the two operations were related. Do you think the symbol $\int$ was first applied to antidifferentiation or to definite integration? Explain your reasoning. (*Hint:* The symbol was first used by Leibniz and was derived from the letter *S*.)

You have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem). At this point, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the **Fundamental Theorem of Calculus.**

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in Figure 4.26. The slope of the tangent line was defined using the *quotient* $\Delta y/\Delta x$ (the slope of the secant line). Similarly, the area of a region under a curve was defined using the *product* $\Delta y \Delta x$ (the area of a rectangle). So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations. The Fundamental Theorem of Calculus states that the limit processes (used to define the derivative and definite integral) preserve this inverse relationship.



(a) Differentiation **(b)** Definite integration

Differentiation and definite integration have an "inverse" relationship.
Figure 4.26

THEOREM 4.9 THE FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF The key to the proof is in writing the difference $F(b) - F(a)$ in a convenient form. Let Δ be any partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

By pairwise subtraction and addition of like terms, you can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdot \cdot \cdot - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

By the Mean Value Theorem, you know that there exists a number c_i in the ith subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Because $F'(c_i) = f(c_i)$, you can let $\Delta x_i = x_i - x_{i-1}$ and obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

This important equation tells you that by repeatedly applying the Mean Value Theorem, you can always find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$ for any partition. Theorem 4.4 guarantees that the limit of Riemann sums over the partition with $\|\Delta\| \to 0$ exists. So, taking the limit (as $\|\Delta\| \to 0$) produces

$$F(b) - F(a) = \int_a^b f(x)\,dx. \qquad \blacksquare$$

The following guidelines can help you understand the use of the Fundamental Theorem of Calculus.

GUIDELINES FOR USING THE FUNDAMENTAL THEOREM OF CALCULUS

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\begin{aligned} \int_a^b f(x)\,dx &= F(x)\Big]_a^b \\ &= F(b) - F(a) \end{aligned}$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\begin{aligned} \int_a^b f(x)\,dx &= \Big[F(x) + C\Big]_a^b \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a). \end{aligned}$$

EXAMPLE 1 Evaluating a Definite Integral

Evaluate each definite integral.

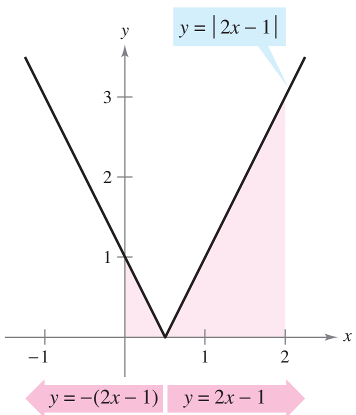
a. $\int_1^2 (x^2 - 3)\, dx$ **b.** $\int_1^4 3\sqrt{x}\, dx$ **c.** $\int_0^{\pi/4} \sec^2 x\, dx$

Solution

a. $\int_1^2 (x^2 - 3)\, dx = \left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = -\frac{2}{3}$

b. $\int_1^4 3\sqrt{x}\, dx = 3\int_1^4 x^{1/2}\, dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$

c. $\int_0^{\pi/4} \sec^2 x\, dx = \tan x\Big]_0^{\pi/4} = 1 - 0 = 1$



The definite integral of y on $[0, 2]$ is $\frac{5}{2}$.
Figure 4.27

EXAMPLE 2 A Definite Integral Involving Absolute Value

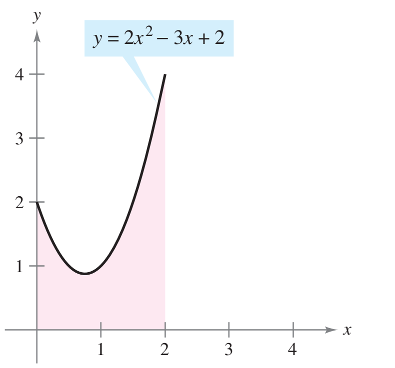
Evaluate $\int_0^2 |2x - 1|\, dx$.

Solution Using Figure 4.27 and the definition of absolute value, you can rewrite the integrand as shown.

$$|2x - 1| = \begin{cases} -(2x - 1), & x < \frac{1}{2} \\ 2x - 1, & x \geq \frac{1}{2} \end{cases}$$

From this, you can rewrite the integral in two parts.

$$\begin{aligned}\int_0^2 |2x - 1|\, dx &= \int_0^{1/2} -(2x - 1)\, dx + \int_{1/2}^2 (2x - 1)\, dx \\ &= \Big[-x^2 + x\Big]_0^{1/2} + \Big[x^2 - x\Big]_{1/2}^2 \\ &= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0 + 0) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{5}{2}\end{aligned}$$



The area of the region bounded by the graph of y, the x-axis, $x = 0$, and $x = 2$ is $\frac{10}{3}$.
Figure 4.28

EXAMPLE 3 Using the Fundamental Theorem to Find Area

Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$, as shown in Figure 4.28.

Solution Note that $y > 0$ on the interval $[0, 2]$.

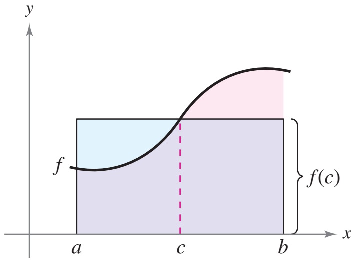
$$\text{Area} = \int_0^2 (2x^2 - 3x + 2)\, dx \qquad \text{Integrate between } x = 0 \text{ and } x = 2.$$

$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2 \qquad \text{Find antiderivative.}$$

$$= \left(\frac{16}{3} - 6 + 4\right) - (0 - 0 + 0) \qquad \text{Apply Fundamental Theorem.}$$

$$= \frac{10}{3} \qquad \text{Simplify.}$$

The Mean Value Theorem for Integrals



Mean value rectangle:

$f(c)(b-a) = \int_a^b f(x)\,dx$

Figure 4.29

In Section 4.2, you saw that the area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure 4.29.

THEOREM 4.10 MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

PROOF

Case 1: If f is constant on the interval $[a, b]$, the theorem is clearly valid because c can be any point in $[a, b]$.

Case 2: If f is not constant on $[a, b]$, then, by the Extreme Value Theorem, you can choose $f(m)$ and $f(M)$ to be the minimum and maximum values of f on $[a, b]$. Because $f(m) \le f(x) \le f(M)$ for all x in $[a, b]$, you can apply Theorem 4.8 to write the following.

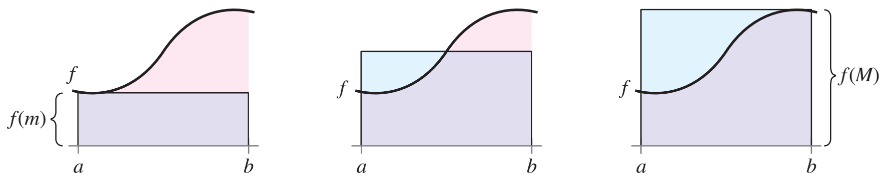
$$\int_a^b f(m)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(M)\,dx \qquad \text{See Figure 4.30.}$$

$$f(m)(b-a) \le \int_a^b f(x)\,dx \le f(M)(b-a)$$

$$f(m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(M)$$

From the third inequality, you can apply the Intermediate Value Theorem to conclude that there exists some c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{or} \quad f(c)(b-a) = \int_a^b f(x)\,dx.$$



Inscribed rectangle (less than actual area) $\int_a^b f(m)\,dx = f(m)(b-a)$

Mean value rectangle (equal to actual area) $\int_a^b f(x)\,dx$

Circumscribed rectangle (greater than actual area) $\int_a^b f(M)\,dx = f(M)(b-a)$

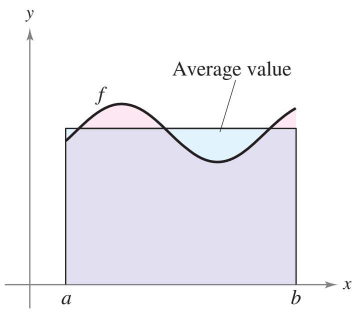
Figure 4.30

■

NOTE Notice that Theorem 4.10 does not specify how to determine c. It merely guarantees the existence of at least one number c in the interval. ■

Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Figure 4.31

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

NOTE Notice in Figure 4.31 that the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value.

To see why the average value of f is defined in this way, suppose that you partition $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. If c_i is any point in the ith subinterval, the arithmetic average (or mean) of the function values at the c_i's is given by

$$a_n = \frac{1}{n}\left[f(c_1) + f(c_2) + \cdot\cdot\cdot + f(c_n)\right]. \qquad \text{Average of } f(c_1), \ldots, f(c_n)$$

By multiplying and dividing by $(b-a)$, you can write the average as

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\,\Delta x. \end{aligned}$$

Finally, taking the limit as $n \to \infty$ produces the average value of f on the interval $[a, b]$, as given in the definition above.

This development of the average value of a function on an interval is only one of many practical uses of definite integrals to represent summation processes. In Chapter 7, you will study other applications, such as volume, arc length, centers of mass, and work.

EXAMPLE 4 Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

Solution The average value is given by

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{4-1}\int_1^4 (3x^2 - 2x)\,dx \\ &= \frac{1}{3}\Big[x^3 - x^2\Big]_1^4 \\ &= \frac{1}{3}[64 - 16 - (1 - 1)] = \frac{48}{3} = 16. \end{aligned}$$

(See Figure 4.32.)

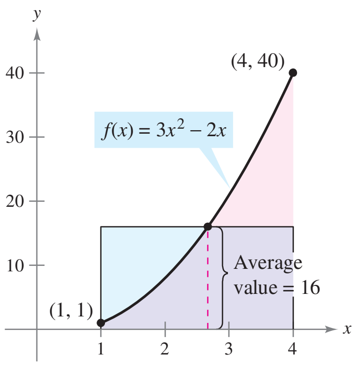


Figure 4.32

George Hall/Corbis

The first person to fly at a speed greater than the speed of sound was Charles Yeager. On October 14, 1947, Yeager was clocked at 295.9 meters per second at an altitude of 12.2 kilometers. If Yeager had been flying at an altitude below 11.275 kilometers, this speed would not have "broken the sound barrier." The photo above shows an F-14 *Tomcat*, a supersonic, twin-engine strike fighter. Currently, the *Tomcat* can reach heights of 15.24 kilometers and speeds up to 2 mach (707.78 meters per second).

EXAMPLE 5 The Speed of Sound

At different altitudes in Earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341, & 0 \le x < 11.5 \\ 295, & 11.5 \le x < 22 \\ \frac{3}{4}x + 278.5, & 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \le x \le 80 \end{cases}$$

where x is the altitude in kilometers (see Figure 4.33). What is the average speed of sound over the interval $[0, 80]$?

Solution Begin by integrating $s(x)$ over the interval $[0, 80]$. To do this, you can break the integral into five parts.

$$\int_0^{11.5} s(x)\,dx = \int_0^{11.5} (-4x + 341)\,dx = \Big[-2x^2 + 341x\Big]_0^{11.5} = 3657$$

$$\int_{11.5}^{22} s(x)\,dx = \int_{11.5}^{22} (295)\,dx = \Big[295x\Big]_{11.5}^{22} = 3097.5$$

$$\int_{22}^{32} s(x)\,dx = \int_{22}^{32} \left(\tfrac{3}{4}x + 278.5\right)dx = \Big[\tfrac{3}{8}x^2 + 278.5x\Big]_{22}^{32} = 2987.5$$

$$\int_{32}^{50} s(x)\,dx = \int_{32}^{50} \left(\tfrac{3}{2}x + 254.5\right)dx = \Big[\tfrac{3}{4}x^2 + 254.5x\Big]_{32}^{50} = 5688$$

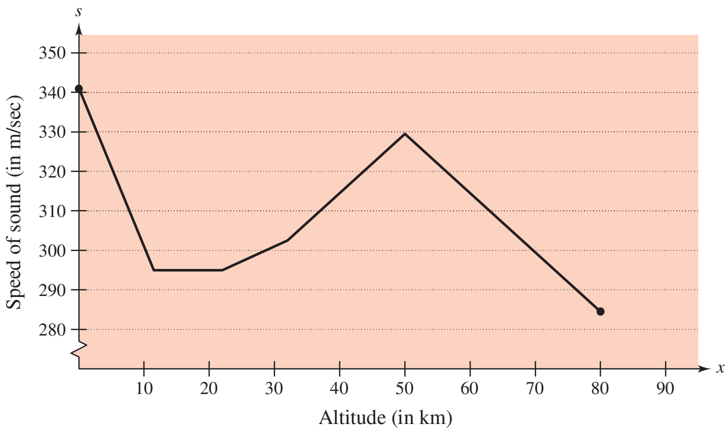
$$\int_{50}^{80} s(x)\,dx = \int_{50}^{80} \left(-\tfrac{3}{2}x + 404.5\right)dx = \Big[-\tfrac{3}{4}x^2 + 404.5x\Big]_{50}^{80} = 9210$$

By adding the values of the five integrals, you have

$$\int_0^{80} s(x)\,dx = 24{,}640.$$

So, the average speed of sound from an altitude of 0 kilometers to an altitude of 80 kilometers is

$$\text{Average speed} = \frac{1}{80}\int_0^{80} s(x)\,dx = \frac{24{,}640}{80} = 308 \text{ meters per second.}$$



Speed of sound depends on altitude.
Figure 4.33

The Second Fundamental Theorem of Calculus

Earlier you saw that the definite integral of f on the interval $[a, b]$ was defined using the constant b as the upper limit of integration and x as the variable of integration. However, a slightly different situation may arise in which the variable x is used in the upper limit of integration. To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration. (Remember that the definite integral is *not* a function of its variable of integration.)

The Definite Integral as a Number

Constant

$$\int_a^b f(x)\,dx$$

Constant

f is a function of x.

The Definite Integral as a Function of x

F is a function of x.

$$F(x) = \int_a^x f(t)\,dt$$

Constant

f is a function of t.

EXPLORATION

Use a graphing utility to graph the function

$$F(x) = \int_0^x \cos t\,dt$$

for $0 \leq x \leq \pi$. Do you recognize this graph? Explain.

EXAMPLE 6 The Definite Integral as a Function

Evaluate the function

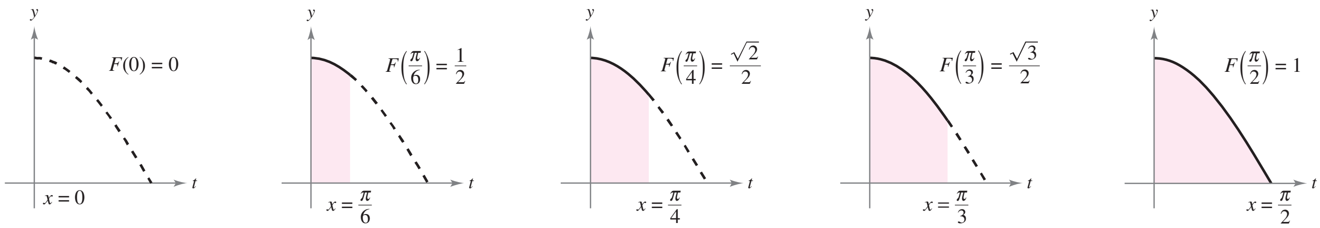
$$F(x) = \int_0^x \cos t\,dt$$

at $x = 0, \pi/6, \pi/4, \pi/3$, and $\pi/2$.

Solution You could evaluate five different definite integrals, one for each of the given upper limits. However, it is much simpler to fix x (as a constant) temporarily to obtain

$$\int_0^x \cos t\,dt = \sin t\Big]_0^x = \sin x - \sin 0 = \sin x.$$

Now, using $F(x) = \sin x$, you can obtain the results shown in Figure 4.34.



$F(x) = \int_0^x \cos t\,dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

Figure 4.34

■

You can think of the function $F(x)$ as *accumulating* the area under the curve $f(t) = \cos t$ from $t = 0$ to $t = x$. For $x = 0$, the area is 0 and $F(0) = 0$. For $x = \pi/2$, $F(\pi/2) = 1$ gives the accumulated area under the cosine curve on the entire interval $[0, \pi/2]$. This interpretation of an integral as an **accumulation function** is used often in applications of integration.

In Example 6, note that the derivative of F is the original integrand (with only the variable changed). That is,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[\sin x] = \frac{d}{dx}\left[\int_0^x \cos t\, dt\right] = \cos x.$$

This result is generalized in the following theorem, called the **Second Fundamental Theorem of Calculus.**

THEOREM 4.11 THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x).$$

PROOF Begin by defining F as

$$F(x) = \int_a^x f(t)\, dt.$$

Then, by the definition of the derivative, you can write

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\, dt - \int_a^x f(t)\, dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\, dt + \int_x^a f(t)\, dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_x^{x+\Delta x} f(t)\, dt\right]. \end{aligned}$$

From the Mean Value Theorem for Integrals (assuming $\Delta x > 0$), you know there exists a number c in the interval $[x, x + \Delta x]$ such that the integral in the expression above is equal to $f(c)\,\Delta x$. Moreover, because $x \le c \le x + \Delta x$, it follows that $c \to x$ as $\Delta x \to 0$. So, you obtain

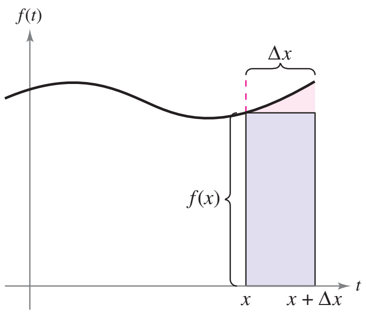
$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x} f(c)\,\Delta x\right] \\ &= \lim_{\Delta x \to 0} f(c) \\ &= f(x). \end{aligned}$$

A similar argument can be made for $\Delta x < 0$. ■

NOTE Using the area model for definite integrals, you can view the approximation

$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\, dt$$

as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equal to the area of the region lying between the graph of f and the x-axis on the interval $[x, x + \Delta x]$, as shown in Figure 4.35. ■



$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\, dt$$

Figure 4.35

Note that the Second Fundamental Theorem of Calculus tells you that if a function is continuous, you can be sure that it has an antiderivative. This antiderivative need not, however, be an elementary function. (Recall the discussion of elementary functions in Section P.3.)

EXAMPLE 7 Using the Second Fundamental Theorem of Calculus

Evaluate $\dfrac{d}{dx}\left[\displaystyle\int_0^x \sqrt{t^2+1}\,dt\right]$.

Solution Note that $f(t) = \sqrt{t^2+1}$ is continuous on the entire real line. So, using the Second Fundamental Theorem of Calculus, you can write

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}.$$

■

The differentiation shown in Example 7 is a straightforward application of the Second Fundamental Theorem of Calculus. The next example shows how this theorem can be combined with the Chain Rule to find the derivative of a function.

EXAMPLE 8 Using the Second Fundamental Theorem of Calculus

Find the derivative of $F(x) = \displaystyle\int_{\pi/2}^{x^3} \cos t\,dt$.

Solution Using $u = x^3$, you can apply the Second Fundamental Theorem of Calculus with the Chain Rule as shown.

$$\begin{aligned}
F'(x) &= \frac{dF}{du}\frac{du}{dx} && \text{Chain Rule}\\
&= \frac{d}{du}[F(x)]\frac{du}{dx} && \text{Definition of } \frac{dF}{du}\\
&= \frac{d}{du}\left[\int_{\pi/2}^{x^3} \cos t\,dt\right]\frac{du}{dx} && \text{Substitute } \int_{\pi/2}^{x^3} \cos t\,dt \text{ for } F(x).\\
&= \frac{d}{du}\left[\int_{\pi/2}^{u} \cos t\,dt\right]\frac{du}{dx} && \text{Substitute } u \text{ for } x^3.\\
&= (\cos u)(3x^2) && \text{Apply Second Fundamental Theorem of Calculus.}\\
&= (\cos x^3)(3x^2) && \text{Rewrite as function of } x.
\end{aligned}$$

■

Because the integrand in Example 8 is easily integrated, you can verify the derivative as follows.

$$F(x) = \int_{\pi/2}^{x^3} \cos t\,dt = \sin t\Big]_{\pi/2}^{x^3} = \sin x^3 - \sin\frac{\pi}{2} = (\sin x^3) - 1$$

In this form, you can apply the Power Rule to verify that the derivative is the same as that obtained in Example 8.

$$F'(x) = (\cos x^3)(3x^2)$$

Net Change Theorem

The Fundamental Theorem of Calculus (Theorem 4.9) states that if f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

But because $F'(x) = f(x)$, this statement can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where the quantity $F(b) - F(a)$ represents the *net change of F* on the interval $[a, b]$.

THEOREM 4.12 THE NET CHANGE THEOREM

The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or **net change,** in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F$$

EXAMPLE 9 Using the Net Change Theorem

A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minute, where $0 \le t \le 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

Solution Let $c(t)$ be the amount of the chemical in the tank at time t. Then $c'(t)$ represents the rate at which the chemical flows into the tank at time t. During the first 20 minutes, the amount that flows into the tank is

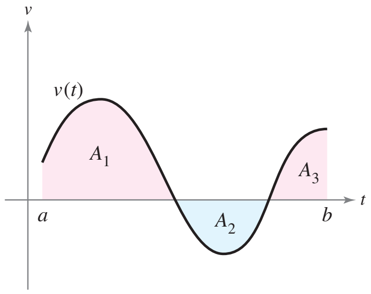
$$\begin{aligned}\int_0^{20} c'(t)\,dt &= \int_0^{20} (180 + 3t)\,dt \\ &= \left[180t + \frac{3}{2}t^2\right]_0^{20} \\ &= 3600 + 600 = 4200.\end{aligned}$$

So, the amount that flows into the tank during the first 20 minutes is 4200 liters. ■

Another way to illustrate the Net Change Theorem is to examine the velocity of a particle moving along a straight line where $s(t)$ is the position at time t. Then its velocity is $v(t) = s'(t)$ and

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

This definite integral represents the net change in position, or **displacement,** of the particle.



A_1, A_2, and A_3 are the areas of the shaded regions.
Figure 4.36

When calculating the *total* distance traveled by the particle, you must consider the intervals where $v(t) \le 0$ and the intervals where $v(t) \ge 0$. When $v(t) \le 0$, the particle moves to the left, and when $v(t) \ge 0$, the particle moves to the right. To calculate the total distance traveled, integrate the absolute value of velocity $|v(t)|$. So, the displacement of a particle and the total distance traveled by a particle over $[a, b]$ can be written as

$$\textbf{Displacement on } [\boldsymbol{a}, \boldsymbol{b}] = \int_a^b v(t)\,dt = A_1 - A_2 + A_3$$

$$\textbf{Total distance traveled on } [\boldsymbol{a}, \boldsymbol{b}] = \int_a^b |v(t)|\,dt = A_1 + A_2 + A_3$$

(see Figure 4.36).

EXAMPLE 10 Solving a Particle Motion Problem

A particle is moving along a line so that its velocity is $v(t) = t^3 - 10t^2 + 29t - 20$ feet per second at time t.

a. What is the displacement of the particle on the time interval $1 \le t \le 5$?

b. What is the total distance traveled by the particle on the time interval $1 \le t \le 5$?

Solution

a. By definition, you know that the displacement is

$$\begin{aligned}
\int_1^5 v(t)\,dt &= \int_1^5 (t^3 - 10t^2 + 29t - 20)\,dt \\
&= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^5 \\
&= \frac{25}{12} - \left(-\frac{103}{12}\right) \\
&= \frac{128}{12} \\
&= \frac{32}{3}.
\end{aligned}$$

So, the particle moves $\frac{32}{3}$ feet to the right.

b. To find the total distance traveled, calculate $\int_1^5 |v(t)|\,dt$. Using Figure 4.37 and the fact that $v(t)$ can be factored as $(t - 1)(t - 4)(t - 5)$, you can determine that $v(t) \ge 0$ on $[1, 4]$ and $v(t) \le 0$ on $[4, 5]$. So, the total distance traveled is

$$\begin{aligned}
\int_1^5 |v(t)|\,dt &= \int_1^4 v(t)\,dt - \int_4^5 v(t)\,dt \\
&= \int_1^4 (t^3 - 10t^2 + 29t - 20)\,dt - \int_4^5 (t^3 - 10t^2 + 29t - 20)\,dt \\
&= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^4 - \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_4^5 \\
&= \frac{45}{4} - \left(-\frac{7}{12}\right) \\
&= \frac{71}{6} \text{ feet.}
\end{aligned}$$

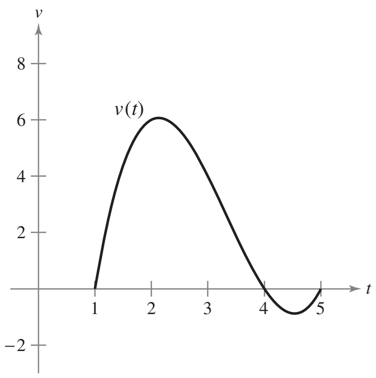


Figure 4.37

4.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the integrand. Use the graph to determine whether the definite integral is positive, negative, or zero.**

1. $\displaystyle\int_0^{\pi} \frac{4}{x^2+1}\,dx$

2. $\displaystyle\int_0^{\pi} \cos x\,dx$

3. $\displaystyle\int_{-2}^{2} x\sqrt{x^2+1}\,dx$

4. $\displaystyle\int_{-2}^{2} x\sqrt{2-x}\,dx$

In Exercises 5–26, evaluate the definite integral of the algebraic function. Use a graphing utility to verify your result.

5. $\displaystyle\int_0^{2} 6x\,dx$

6. $\displaystyle\int_4^{9} 5\,dv$

7. $\displaystyle\int_{-1}^{0} (2x-1)\,dx$

8. $\displaystyle\int_2^{5} (-3v+4)\,dv$

9. $\displaystyle\int_{-1}^{1} (t^2-2)\,dt$

10. $\displaystyle\int_1^{7} (6x^2+2x-3)\,dx$

11. $\displaystyle\int_0^{1} (2t-1)^2\,dt$

12. $\displaystyle\int_{-1}^{1} (t^3-9t)\,dt$

13. $\displaystyle\int_1^{2} \left(\frac{3}{x^2}-1\right)dx$

14. $\displaystyle\int_{-2}^{-1} \left(u-\frac{1}{u^2}\right)du$

15. $\displaystyle\int_1^{4} \frac{u-2}{\sqrt{u}}\,du$

16. $\displaystyle\int_{-3}^{3} v^{1/3}\,dv$

17. $\displaystyle\int_{-1}^{1} \left(\sqrt[3]{t}-2\right)dt$

18. $\displaystyle\int_1^{8} \sqrt{\frac{2}{x}}\,dx$

19. $\displaystyle\int_0^{1} \frac{x-\sqrt{x}}{3}\,dx$

20. $\displaystyle\int_0^{2} (2-t)\sqrt{t}\,dt$

21. $\displaystyle\int_{-1}^{0} \left(t^{1/3}-t^{2/3}\right)dt$

22. $\displaystyle\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}\,dx$

23. $\displaystyle\int_0^{5} |2x-5|\,dx$

24. $\displaystyle\int_1^{4} (3-|x-3|)\,dx$

25. $\displaystyle\int_0^{4} |x^2-9|\,dx$

26. $\displaystyle\int_0^{4} |x^2-4x+3|\,dx$

In Exercises 27–34, evaluate the definite integral of the trigonometric function. Use a graphing utility to verify your result.

27. $\displaystyle\int_0^{\pi} (1+\sin x)\,dx$

28. $\displaystyle\int_0^{\pi} (2+\cos x)\,dx$

29. $\displaystyle\int_0^{\pi/4} \frac{1-\sin^2\theta}{\cos^2\theta}\,d\theta$

30. $\displaystyle\int_0^{\pi/4} \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$

31. $\displaystyle\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx$

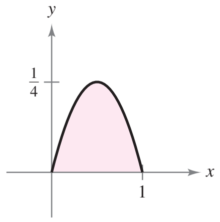
32. $\displaystyle\int_{\pi/4}^{\pi/2} (2-\csc^2 x)\,dx$

33. $\displaystyle\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta$

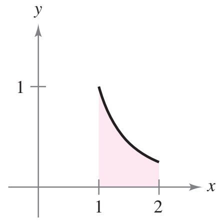
34. $\displaystyle\int_{-\pi/2}^{\pi/2} (2t+\cos t)\,dt$

In Exercises 35–38, determine the area of the given region.

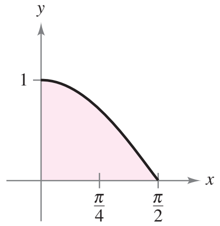
35. $y = x - x^2$



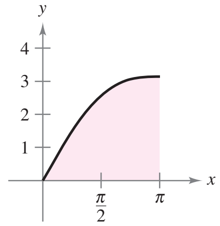
36. $y = \dfrac{1}{x^2}$



37. $y = \cos x$



38. $y = x + \sin x$



In Exercises 39–44, find the area of the region bounded by the graphs of the equations.

39. $y = 5x^2 + 2, \quad x = 0, \quad x = 2, \quad y = 0$

40. $y = x^3 + x, \quad x = 2, \quad y = 0$

41. $y = 1 + \sqrt[3]{x}, \quad x = 0, \quad x = 8, \quad y = 0$

42. $y = (3 - x)\sqrt{x}, \quad y = 0$

43. $y = -x^2 + 4x, \quad y = 0$

44. $y = 1 - x^4, \quad y = 0$

In Exercises 45–50, find the value(s) of *c* guaranteed by the Mean Value Theorem for Integrals for the function over the given interval.

45. $f(x) = x^3, \quad [0, 3]$

46. $f(x) = \dfrac{9}{x^3}, \quad [1, 3]$

47. $f(x) = \sqrt{x}, \quad [4, 9]$

48. $f(x) = x - 2\sqrt{x}, \quad [0, 2]$

49. $f(x) = 2\sec^2 x, \quad [-\pi/4, \pi/4]$

50. $f(x) = \cos x, \quad [-\pi/3, \pi/3]$

In Exercises 51–56, find the average value of the function over the given interval and all values of *x* in the interval for which the function equals its average value.

51. $f(x) = 9 - x^2, \quad [-3, 3]$

52. $f(x) = \dfrac{4(x^2+1)}{x^2}, \quad [1, 3]$

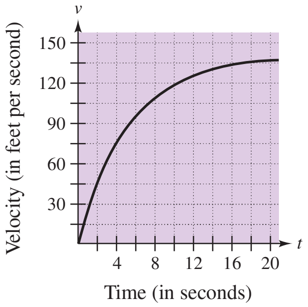
53. $f(x) = x^3, \quad [0, 1]$

54. $f(x) = 4x^3 - 3x^2, \quad [-1, 2]$

55. $f(x) = \sin x, \quad [0, \pi]$

56. $f(x) = \cos x, \quad [0, \pi/2]$

57. ***Velocity*** The graph shows the velocity, in feet per second, of a car accelerating from rest. Use the graph to estimate the distance the car travels in 8 seconds.



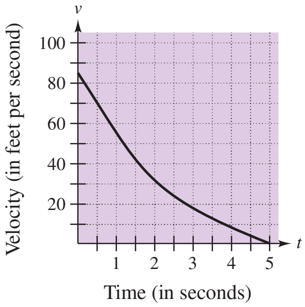
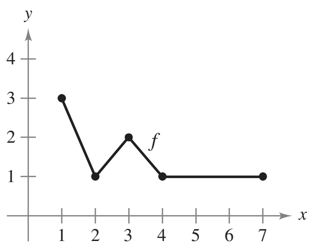


Figure for 57 **Figure for 58**

58. ***Velocity*** The graph shows the velocity, in feet per second, of a decelerating car after the driver applies the brakes. Use the graph to estimate how far the car travels before it comes to a stop.

WRITING ABOUT CONCEPTS

59. The graph of f is shown in the figure.



(a) Evaluate $\int_1^7 f(x)\,dx$.

(b) Determine the average value of f on the interval $[1, 7]$.

(c) Determine the answers to parts (a) and (b) if the graph is translated two units upward.

60. If $r'(t)$ represents the rate of growth of a dog in pounds per year, what does $r(t)$ represent? What does $\int_2^6 r'(t)\,dt$ represent about the dog?

61. ***Force*** The force F (in newtons) of a hydraulic cylinder in a press is proportional to the square of $\sec x$, where x is the distance (in meters) that the cylinder is extended in its cycle. The domain of F is $[0, \pi/3]$, and $F(0) = 500$.

(a) Find F as a function of x.

(b) Find the average force exerted by the press over the interval $[0, \pi/3]$.

62. ***Blood Flow*** The velocity v of the flow of blood at a distance r from the central axis of an artery of radius R is

$$v = k(R^2 - r^2)$$

where k is the constant of proportionality. Find the average rate of flow of blood along a radius of the artery. (Use 0 and R as the limits of integration.)

63. ***Respiratory Cycle*** The volume V, in liters, of air in the lungs during a five-second respiratory cycle is approximated by the model $V = 0.1729t + 0.1522t^2 - 0.0374t^3$, where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle.

64. ***Average Sales*** A company fits a model to the monthly sales data for a seasonal product. The model is

$$S(t) = \frac{t}{4} + 1.8 + 0.5\sin\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

where S is sales (in thousands) and t is time in months.

(a) Use a graphing utility to graph $f(t) = 0.5\sin(\pi t/6)$ for $0 \le t \le 24$. Use the graph to explain why the average value of $f(t)$ is 0 over the interval.

(b) Use a graphing utility to graph $S(t)$ and the line $g(t) = t/4 + 1.8$ in the same viewing window. Use the graph and the result of part (a) to explain why g is called the *trend line*.

65. ***Modeling Data*** An experimental vehicle is tested on a straight track. It starts from rest, and its velocity v (in meters per second) is recorded every 10 seconds for 1 minute (see table).

t	0	10	20	30	40	50	60
v	0	5	21	40	62	78	83

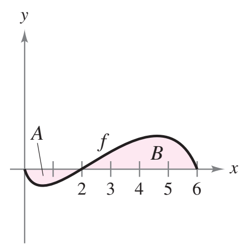
(a) Use a graphing utility to find a model of the form $v = at^3 + bt^2 + ct + d$ for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the Fundamental Theorem of Calculus to approximate the distance traveled by the vehicle during the test.

CAPSTONE

66. The graph of f is shown in the figure. The shaded region A has an area of 1.5, and $\int_0^6 f(x)\,dx = 3.5$. Use this information to fill in the blanks.



(a) $\displaystyle\int_0^2 f(x)\,dx = $ ▭

(b) $\displaystyle\int_2^6 f(x)\,dx = $ ▭

(c) $\displaystyle\int_0^6 |f(x)|\,dx = $ ▭

(d) $\displaystyle\int_0^2 -2f(x)\,dx = $ ▭

(e) $\displaystyle\int_0^6 [2 + f(x)]\,dx = $ ▭

(f) The average value of f over the interval $[0, 6]$ is ▭.

In Exercises 67–72, find F as a function of x and evaluate it at $x = 2$, $x = 5$, and $x = 8$.

67. $\displaystyle F(x) = \int_0^x (4t - 7)\,dt$

68. $\displaystyle F(x) = \int_2^x (t^3 + 2t - 2)\,dt$

69. $F(x) = \int_1^x \frac{20}{v^2}\,dv$ **70.** $F(x) = \int_2^x -\frac{2}{t^3}\,dt$

71. $F(x) = \int_1^x \cos\theta\,d\theta$ **72.** $F(x) = \int_0^x \sin\theta\,d\theta$

73. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown in the figure.

(a) Estimate $g(0)$, $g(2)$, $g(4)$, $g(6)$, and $g(8)$.

(b) Find the largest open interval on which g is increasing. Find the largest open interval on which g is decreasing.

(c) Identify any extrema of g.

(d) Sketch a rough graph of g.

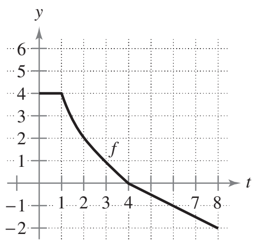


Figure for 73

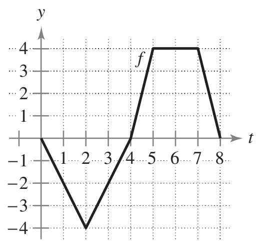


Figure for 74

74. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown in the figure.

(a) Estimate $g(0)$, $g(2)$, $g(4)$, $g(6)$, and $g(8)$.

(b) Find the largest open interval on which g is increasing. Find the largest open interval on which g is decreasing.

(c) Identify any extrema of g.

(d) Sketch a rough graph of g.

In Exercises 75–80, (a) integrate to find F as a function of x and (b) demonstrate the Second Fundamental Theorem of Calculus by differentiating the result in part (a).

75. $F(x) = \int_0^x (t + 2)\,dt$ **76.** $F(x) = \int_0^x t(t^2 + 1)\,dt$

77. $F(x) = \int_8^x \sqrt[3]{t}\,dt$ **78.** $F(x) = \int_4^x \sqrt{t}\,dt$

79. $F(x) = \int_{\pi/4}^x \sec^2 t\,dt$ **80.** $F(x) = \int_{\pi/3}^x \sec t \tan t\,dt$

In Exercises 81–86, use the Second Fundamental Theorem of Calculus to find $F'(x)$.

81. $F(x) = \int_{-2}^x (t^2 - 2t)\,dt$ **82.** $F(x) = \int_1^x \frac{t^2}{t^2 + 1}\,dt$

83. $F(x) = \int_{-1}^x \sqrt{t^4 + 1}\,dt$ **84.** $F(x) = \int_1^x \sqrt[4]{t}\,dt$

85. $F(x) = \int_0^x t\cos t\,dt$ **86.** $F(x) = \int_0^x \sec^3 t\,dt$

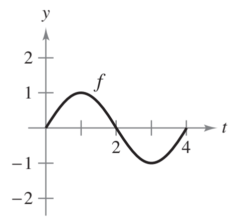
In Exercises 87–92, find $F'(x)$.

87. $F(x) = \int_x^{x+2} (4t + 1)\,dt$ **88.** $F(x) = \int_{-x}^x t^3\,dt$

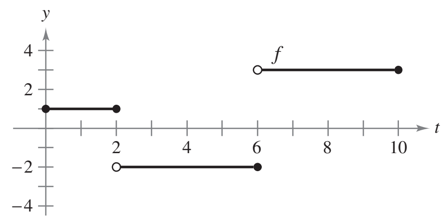
89. $F(x) = \int_0^{\sin x} \sqrt{t}\,dt$ **90.** $F(x) = \int_2^{x^2} \frac{1}{t^3}\,dt$

91. $F(x) = \int_0^{x^3} \sin t^2\,dt$ **92.** $F(x) = \int_0^{x^2} \sin\theta^2\,d\theta$

93. ***Graphical Analysis*** Sketch an approximate graph of g on the interval $0 \le x \le 4$, where $g(x) = \int_0^x f(t)\,dt$. Identify the x-coordinate of an extremum of g. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



94. Use the graph of the function f shown in the figure and the function g defined by $g(x) = \int_0^x f(t)\,dt$.



(a) Complete the table.

x	1	2	3	4	5	6	7	8	9	10
$g(x)$										

(b) Plot the points from the table in part (a) and graph g.

(c) Where does g have its minimum? Explain.

(d) Where does g have a maximum? Explain.

(e) On what interval does g increase at the greatest rate? Explain.

(f) Identify the zeros of g.

95. ***Cost*** The total cost C (in dollars) of purchasing and maintaining a piece of equipment for x years is

$$C(x) = 5000\left(25 + 3\int_0^x t^{1/4}\,dt\right).$$

(a) Perform the integration to write C as a function of x.

(b) Find $C(1)$, $C(5)$, and $C(10)$.

96. ***Area*** The area A between the graph of the function $g(t) = 4 - 4/t^2$ and the t-axis over the interval $[1, x]$ is

$$A(x) = \int_1^x \left(4 - \frac{4}{t^2}\right) dt.$$

(a) Find the horizontal asymptote of the graph of g.

(b) Integrate to find A as a function of x. Does the graph of A have a horizontal asymptote? Explain.

In Exercises 97–102, the velocity function, in feet per second, is given for a particle moving along a straight line. Find (a) the displacement and (b) the total distance that the particle travels over the given interval.

97. $v(t) = 5t - 7, \quad 0 \le t \le 3$

98. $v(t) = t^2 - t - 12, \quad 1 \le t \le 5$

99. $v(t) = t^3 - 10t^2 + 27t - 18, \quad 1 \le t \le 7$

100. $v(t) = t^3 - 8t^2 + 15t, \quad 0 \le t \le 5$

101. $v(t) = \dfrac{1}{\sqrt{t}}, \quad 1 \le t \le 4$ **102.** $v(t) = \cos t, \quad 0 \le t \le 3\pi$

103. A particle is moving along the x-axis. The position of the particle at time t is given by $x(t) = t^3 - 6t^2 + 9t - 2$, $0 \le t \le 5$. Find the total distance the particle travels in 5 units of time.

104. Repeat Exercise 103 for the position function given by $x(t) = (t - 1)(t - 3)^2$, $0 \le t \le 5$.

105. ***Water Flow*** Water flows from a storage tank at a rate of $500 - 5t$ liters per minute. Find the amount of water that flows out of the tank during the first 18 minutes.

106. ***Oil Leak*** At 1:00 P.M., oil begins leaking from a tank at a rate of $4 + 0.75t$ gallons per hour.

(a) How much oil is lost from 1:00 P.M. to 4:00 P.M.?

(b) How much oil is lost from 4:00 P.M. to 7:00 P.M.?

(c) Compare your answers from parts (a) and (b). What do you notice?

In Exercises 107–110, describe why the statement is incorrect.

107. $\cancel{\displaystyle\int_{-1}^{1} x^{-2}\,dx = \left[-x^{-1}\right]_{-1}^{1} = (-1) - 1 = -2}$

108. $\cancel{\displaystyle\int_{-2}^{1} \frac{2}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_{-2}^{1} = -\frac{3}{4}}$

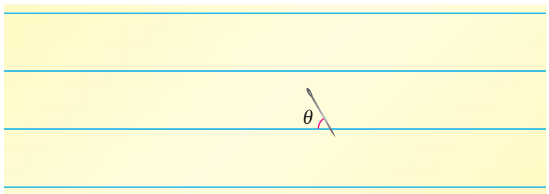
109. $\cancel{\displaystyle\int_{\pi/4}^{3\pi/4} \sec^2 x\,dx = \left[\tan x\right]_{\pi/4}^{3\pi/4} = -2}$

110. $\cancel{\displaystyle\int_{\pi/2}^{3\pi/2} \csc x \cot x\,dx = \left[-\csc x\right]_{\pi/2}^{3\pi/2} = 2}$

111. ***Buffon's Needle Experiment*** A horizontal plane is ruled with parallel lines 2 inches apart. A two-inch needle is tossed randomly onto the plane. The probability that the needle will touch a line is

$$P = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta\,d\theta$$

where θ is the acute angle between the needle and any one of the parallel lines. Find this probability.



112. Prove that $\dfrac{d}{dx}\left[\displaystyle\int_{u(x)}^{v(x)} f(t)\,dt\right] = f(v(x))v'(x) - f(u(x))u'(x)$.

True or False? **In Exercises 113 and 114, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

113. If $F'(x) = G'(x)$ on the interval $[a, b]$, then $F(b) - F(a) = G(b) - G(a)$.

114. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

115. Show that the function

$$f(x) = \int_0^{1/x} \frac{1}{t^2 + 1}\,dt + \int_0^{x} \frac{1}{t^2 + 1}\,dt$$

is constant for $x > 0$.

116. Find the function $f(x)$ and all values of c such that

$$\int_c^x f(t)\,dt = x^2 + x - 2.$$

117. Let $G(x) = \displaystyle\int_0^x \left[s\int_0^s f(t)\,dt\right] ds$, where f is continuous for all real t. Find (a) $G(0)$, (b) $G'(0)$, (c) $G''(x)$, and (d) $G''(0)$.

SECTION PROJECT

Demonstrating the Fundamental Theorem

Use a graphing utility to graph the function $y_1 = \sin^2 t$ on the interval $0 \le t \le \pi$. Let $F(x)$ be the following function of x.

$$F(x) = \int_0^x \sin^2 t\,dt$$

(a) Complete the table. Explain why the values of F are increasing.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$F(x)$							

(b) Use the integration capabilities of a graphing utility to graph F.

(c) Use the differentiation capabilities of a graphing utility to graph $F'(x)$. How is this graph related to the graph in part (b)?

(d) Verify that the derivative of $y = (1/2)t - (\sin 2t)/4$ is $\sin^2 t$. Graph y and write a short paragraph about how this graph is related to those in parts (b) and (c).

4.5 Integration by Substitution

- Use pattern recognition to find an indefinite integral.
- Use a change of variables to find an indefinite integral.
- Use the General Power Rule for Integration to find an indefinite integral.
- Use a change of variables to evaluate a definite integral.
- Evaluate a definite integral involving an even or odd function.

Pattern Recognition

In this section you will study techniques for integrating composite functions. The discussion is split into two parts—*pattern recognition* and *change of variables*. Both techniques involve a ***u*-substitution.** With pattern recognition you perform the substitution mentally, and with change of variables you write the substitution steps.

The role of substitution in integration is comparable to the role of the Chain Rule in differentiation. Recall that for differentiable functions given by $y = F(u)$ and $u = g(x)$, the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

These results are summarized in the following theorem.

NOTE The statement of Theorem 4.13 doesn't tell how to distinguish between $f(g(x))$ and $g'(x)$ in the integrand. As you become more experienced at integration, your skill in doing this will increase. Of course, part of the key is familiarity with derivatives.

THEOREM 4.13 ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Examples 1 and 2 show how to apply Theorem 4.13 *directly*, by recognizing the presence of $f(g(x))$ and $g'(x)$. Note that the composite function in the integrand has an *outside function* f and an *inside function* g. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.

$$\int \overset{\text{Outside function}}{f}(\underset{\text{Inside function}}{g(x)})\underbrace{g'(x)}_{\text{Derivative of inside function}}\,dx = F(g(x)) + C$$

EXAMPLE 1 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int (x^2 + 1)^2(2x)\,dx$.

Solution Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Power Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\,dx = \frac{1}{3}(x^2 + 1)^3 + C.$$

Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2 + 1)^3 + C$ is the integrand of the original integral.

EXAMPLE 2 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int 5 \cos 5x\,dx$.

Solution Letting $g(x) = 5x$, you obtain

$$g'(x) = 5$$

and

$$f(g(x)) = f(5x) = \cos 5x.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Cosine Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(\cos (5x))}^{f(g(x))}\overbrace{(5)}^{g'(x)}\,dx = \sin 5x + C.$$

You can check this by differentiating $\sin 5x + C$ to obtain the original integrand. ■

TECHNOLOGY Try using a computer algebra system, such as *Maple*, *Mathematica*, or the *TI-89*, to solve the integrals given in Examples 1 and 2. Do you obtain the same antiderivatives that are listed in the examples?

EXPLORATION

Recognizing Patterns The integrand in each of the following integrals fits the pattern $f(g(x))g'(x)$. Identify the pattern and use the result to evaluate the integral.

a. $\int 2x(x^2 + 1)^4\,dx$ **b.** $\int 3x^2\sqrt{x^3 + 1}\,dx$ **c.** $\int \sec^2 x(\tan x + 3)\,dx$

The next three integrals are similar to the first three. Show how you can multiply and divide by a constant to evaluate these integrals.

d. $\int x(x^2 + 1)^4\,dx$ **e.** $\int x^2\sqrt{x^3 + 1}\,dx$ **f.** $\int 2\sec^2 x(\tan x + 3)\,dx$

The integrands in Examples 1 and 2 fit the $f(g(x))g'(x)$ pattern exactly—you only had to recognize the pattern. You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple. In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

EXAMPLE 3 Multiplying and Dividing by a Constant

Find $\displaystyle\int x(x^2+1)^2\,dx.$

Solution This is similar to the integral given in Example 1, except that the integrand is missing a factor of 2. Recognizing that $2x$ is the derivative of x^2+1, you can let $g(x) = x^2+1$ and supply the $2x$ as follows.

$$\begin{aligned}
\int x(x^2+1)^2\,dx &= \int (x^2+1)^2\left(\frac{1}{2}\right)(2x)\,dx && \text{Multiply and divide by 2.}\\
&= \frac{1}{2}\int \overbrace{(x^2+1)^2}^{f(g(x))}\,\overbrace{(2x)}^{g'(x)}\,dx && \text{Constant Multiple Rule}\\
&= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C && \text{Integrate.}\\
&= \frac{1}{6}(x^2+1)^3 + C && \text{Simplify.}
\end{aligned}$$

In practice, most people would not write as many steps as are shown in Example 3. For instance, you could evaluate the integral by simply writing

$$\begin{aligned}
\int x(x^2+1)^2\,dx &= \frac{1}{2}\int (x^2+1)^2\,2x\,dx\\
&= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C\\
&= \frac{1}{6}(x^2+1)^3 + C.
\end{aligned}$$

NOTE Be sure you see that the *Constant* Multiple Rule applies only to *constants*. You cannot multiply and divide by a variable and then move the variable outside the integral sign. For instance,

$$\int (x^2+1)^2\,dx \neq \frac{1}{2x}\int (x^2+1)^2(2x)\,dx.$$

After all, if it were legitimate to move variable quantities outside the integral sign, you could move the entire integrand out and simplify the whole process. But the result would be incorrect.

Change of Variables

With a formal **change of variables,** you completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 to 3, it is useful for complicated integrands. The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)\,dx$, and the integral in Theorem 4.13 takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

EXAMPLE 4 Change of Variables

Find $\displaystyle\int \sqrt{2x-1}\,dx$.

Solution First, let u be the inner function, $u = 2x - 1$. Then calculate the differential du to be $du = 2\,dx$. Now, using $\sqrt{2x-1} = \sqrt{u}$ and $dx = du/2$, substitute to obtain

$$\begin{aligned}
\int \sqrt{2x-1}\,dx &= \int \sqrt{u}\left(\frac{du}{2}\right) && \text{Integral in terms of } u\\
&= \frac{1}{2}\int u^{1/2}\,du && \text{Constant Multiple Rule}\\
&= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C && \text{Antiderivative in terms of } u\\
&= \frac{1}{3}u^{3/2} + C && \text{Simplify.}\\
&= \frac{1}{3}(2x-1)^{3/2} + C. && \text{Antiderivative in terms of } x
\end{aligned}$$

STUDY TIP Because integration is usually more difficult than differentiation, you should always check your answer to an integration problem by differentiating. For instance, in Example 4 you should differentiate $\frac{1}{3}(2x-1)^{3/2} + C$ to verify that you obtain the original integrand.

EXAMPLE 5 Change of Variables

Find $\displaystyle\int x\sqrt{2x-1}\,dx$.

Solution As in the previous example, let $u = 2x - 1$ and obtain $dx = du/2$. Because the integrand contains a factor of x, you must also solve for x in terms of u, as shown.

$$u = 2x - 1 \quad \Rightarrow \quad x = (u+1)/2 \qquad \text{Solve for } x \text{ in terms of } u.$$

Now, using substitution, you obtain

$$\begin{aligned}
\int x\sqrt{2x-1}\,dx &= \int \left(\frac{u+1}{2}\right)u^{1/2}\left(\frac{du}{2}\right)\\
&= \frac{1}{4}\int (u^{3/2} + u^{1/2})\,du\\
&= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C\\
&= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.
\end{aligned}$$

■

To complete the change of variables in Example 5, you solved for x in terms of u. Sometimes this is very difficult. Fortunately it is not always necessary, as shown in the next example.

EXAMPLE 6 Change of Variables

Find $\int \sin^2 3x \cos 3x \, dx$.

Solution Because $\sin^2 3x = (\sin 3x)^2$, you can let $u = \sin 3x$. Then

$$du = (\cos 3x)(3)\, dx.$$

Now, because $\cos 3x\, dx$ is part of the original integral, you can write

$$\frac{du}{3} = \cos 3x\, dx.$$

STUDY TIP When making a change of variables, be sure that your answer is written using the same variables as in the original integrand. For instance, in Example 6, you should not leave your answer as

$$\tfrac{1}{9}u^3 + C$$

but rather, replace u by $\sin 3x$.

Substituting u and $du/3$ in the original integral yields

$$\begin{aligned}\int \sin^2 3x \cos 3x\, dx &= \int u^2 \frac{du}{3}\\ &= \frac{1}{3}\int u^2\, du\\ &= \frac{1}{3}\left(\frac{u^3}{3}\right) + C\\ &= \frac{1}{9}\sin^3 3x + C.\end{aligned}$$

You can check this by differentiating.

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{9}\sin^3 3x\right] &= \left(\frac{1}{9}\right)(3)(\sin 3x)^2(\cos 3x)(3)\\ &= \sin^2 3x \cos 3x\end{aligned}$$

Because differentiation produces the original integrand, you know that you have obtained the correct antiderivative. ■

The steps used for integration by substitution are summarized in the following guidelines.

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\, dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. Because of the importance of this type of substitution, it is given a special name~~the~~ **General Power Rule for Integration.** A proof of this rule follows directly from the (simple) Power Rule for Integration, together with Theorem 4.13.

THEOREM 4.14 THE GENERAL POWER RULE FOR INTEGRATION

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

EXAMPLE 7 Substitution and the General Power Rule

a. $$\int 3(3x-1)^4\,dx = \int \overbrace{(3x-1)^4}^{u^4}\overbrace{(3)\,dx}^{du} = \overbrace{\frac{(3x-1)^5}{5}}^{u^5/5} + C$$

b. $$\int (2x+1)(x^2+x)\,dx = \int \overbrace{(x^2+x)^1}^{u^1}\overbrace{(2x+1)\,dx}^{du} = \overbrace{\frac{(x^2+x)^2}{2}}^{u^2/2} + C$$

c. $$\int 3x^2\sqrt{x^3-2}\,dx = \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}}\overbrace{(3x^2)\,dx}^{du} = \overbrace{\frac{(x^3-2)^{3/2}}{3/2}}^{u^{3/2}/(3/2)} + C = \frac{2}{3}(x^3-2)^{3/2} + C$$

d. $$\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}}\overbrace{(-4x)\,dx}^{du} = \overbrace{\frac{(1-2x^2)^{-1}}{-1}}^{u^{-1}/(-1)} + C = -\frac{1}{1-2x^2} + C$$

e. $$\int \cos^2 x \sin x\,dx = -\int \overbrace{(\cos x)^2}^{u^2}\overbrace{(-\sin x)\,dx}^{du} = -\overbrace{\frac{(\cos x)^3}{3}}^{u^3/3} + C$$ ■

Some integrals whose integrands involve quantities raised to powers cannot be found by the General Power Rule. Consider the two integrals

$$\int x(x^2+1)^2\,dx \quad \text{and} \quad \int (x^2+1)^2\,dx.$$

The substitution $u = x^2 + 1$ works in the first integral but not in the second. In the second, the substitution fails because the integrand lacks the factor x needed for du. Fortunately, *for this particular integral*, you can expand the integrand as $(x^2+1)^2 = x^4 + 2x^2 + 1$ and use the (simple) Power Rule to integrate each term.

EXPLORATION

Suppose you were asked to find one of the following integrals. Which one would you choose? Explain your reasoning.

a. $\int \sqrt{x^3+1}\,dx$ or

$\int x^2\sqrt{x^3+1}\,dx$

b. $\int \tan(3x)\sec^2(3x)\,dx$ or

$\int \tan(3x)\,dx$

Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem. The proof follows from Theorem 4.13 combined with the Fundamental Theorem of Calculus.

THEOREM 4.15 CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE 8 Change of Variables

Evaluate $\displaystyle\int_0^1 x(x^2+1)^3\,dx$.

Solution To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \implies du = 2x\,dx.$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	*Upper Limit*
When $x = 0$, $u = 0^2 + 1 = 1$.	When $x = 1$, $u = 1^2 + 1 = 2$.

Now, you can substitute to obtain

$$\begin{aligned}\int_0^1 x(x^2+1)^3\,dx &= \frac{1}{2}\int_0^1 (x^2+1)^3(2x)\,dx && \text{Integration limits for } x\\ &= \frac{1}{2}\int_1^2 u^3\,du && \text{Integration limits for } u\\ &= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2\\ &= \frac{1}{2}\left(4 - \frac{1}{4}\right)\\ &= \frac{15}{8}.\end{aligned}$$

Try rewriting the antiderivative $\frac{1}{2}(u^4/4)$ in terms of the variable x and evaluate the definite integral at the original limits of integration, as shown.

$$\begin{aligned}\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 &= \frac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_0^1\\ &= \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}\end{aligned}$$

Notice that you obtain the same result. ■

EXAMPLE 9 Change of Variables

Evaluate $A = \int_1^5 \frac{x}{\sqrt{2x - 1}}\, dx.$

Solution To evaluate this integral, let $u = \sqrt{2x - 1}$. Then, you obtain

$$\begin{aligned} u^2 &= 2x - 1 \\ u^2 + 1 &= 2x \\ \frac{u^2 + 1}{2} &= x \\ u\, du &= dx. \end{aligned}$$

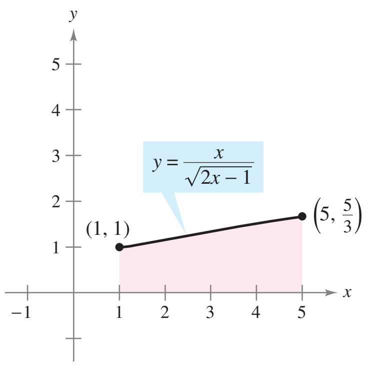
Differentiate each side.

Before substituting, determine the new upper and lower limits of integration.

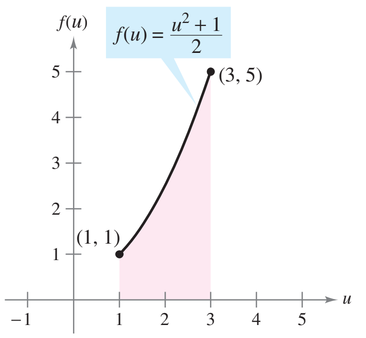
Lower Limit	*Upper Limit*
When $x = 1$, $u = \sqrt{2 - 1} = 1$.	When $x = 5$, $u = \sqrt{10 - 1} = 3$.

Now, substitute to obtain

$$\begin{aligned} \int_1^5 \frac{x}{\sqrt{2x - 1}}\, dx &= \int_1^3 \frac{1}{u}\left(\frac{u^2 + 1}{2}\right) u\, du \\ &= \frac{1}{2}\int_1^3 (u^2 + 1)\, du \\ &= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3 \\ &= \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right) \\ &= \frac{16}{3}. \end{aligned}$$



The region before substitution has an area of $\frac{16}{3}$.
Figure 4.38



The region after substitution has an area of $\frac{16}{3}$.
Figure 4.39

Geometrically, you can interpret the equation

$$\int_1^5 \frac{x}{\sqrt{2x - 1}}\, dx = \int_1^3 \frac{u^2 + 1}{2}\, du$$

to mean that the two *different* regions shown in Figures 4.38 and 4.39 have the *same* area.

When evaluating definite integrals by substitution, it is possible for the upper limit of integration of the u-variable form to be smaller than the lower limit. If this happens, don't rearrange the limits. Simply evaluate as usual. For example, after substituting $u = \sqrt{1 - x}$ in the integral

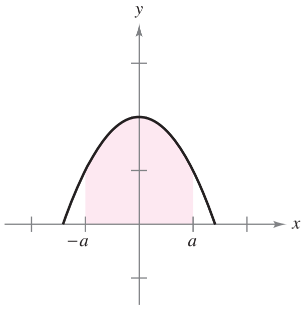
$$\int_0^1 x^2(1 - x)^{1/2}\, dx$$

you obtain $u = \sqrt{1 - 1} = 0$ when $x = 1$, and $u = \sqrt{1 - 0} = 1$ when $x = 0$. So, the correct u-variable form of this integral is

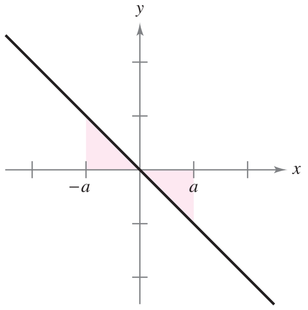
$$-2\int_1^0 (1 - u^2)^2 u^2\, du.$$

Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the y-axis or about the origin by recognizing the integrand to be an even or odd function (see Figure 4.40).



Even function



Odd function
Figure 4.40

THEOREM 4.16 INTEGRATION OF EVEN AND ODD FUNCTIONS

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$
2. If f is an *odd* function, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

PROOF Because f is even, you know that $f(x) = f(-x)$. Using Theorem 4.13 with the substitution $u = -x$ produces

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx.$$

Finally, using Theorem 4.6, you obtain

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\\ &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

This proves the first property. The proof of the second property is left to you (see Exercise 137). ■

EXAMPLE 10 Integration of an Odd Function

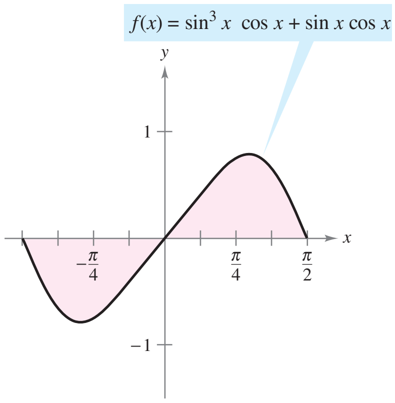
Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx.$

Solution Letting $f(x) = \sin^3 x \cos x + \sin x \cos x$ produces

$$\begin{aligned}f(-x) &= \sin^3(-x)\cos(-x) + \sin(-x)\cos(-x)\\ &= -\sin^3 x \cos x - \sin x \cos x = -f(x).\end{aligned}$$

So, f is an odd function, and because f is symmetric about the origin over $[-\pi/2, \pi/2]$, you can apply Theorem 4.16 to conclude that

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx = 0.$$ ■



Because f is an odd function,
$\displaystyle\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0.$
Figure 4.41

NOTE From Figure 4.41 you can see that the two regions on either side of the y-axis have the same area. However, because one lies below the x-axis and one lies above it, integration produces a cancellation effect. (More will be said about this in Section 7.1.) ■

4.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, complete the table by identifying u and du for the integral.

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (8x^2+1)^2(16x)\,dx$		
2. $\int x^2\sqrt{x^3+1}\,dx$		
3. $\int \frac{x}{\sqrt{x^2+1}}\,dx$		
4. $\int \sec 2x \tan 2x\,dx$		
5. $\int \tan^2 x \sec^2 x\,dx$		
6. $\int \frac{\cos x}{\sin^2 x}\,dx$		

In Exercises 7–10, determine whether it is necessary to use substitution to evaluate the integral. (Do not evaluate the integral.)

7. $\int \sqrt{x}(6-x)\,dx$

8. $\int x\sqrt{x+4}\,dx$

9. $\int x\sqrt[3]{1+x^2}\,dx$

10. $\int x\cos x^2\,dx$

In Exercises 11–38, find the indefinite integral and check the result by differentiation.

11. $\int (1+6x)^4(6)\,dx$

12. $\int (x^2-9)^3(2x)\,dx$

13. $\int \sqrt{25-x^2}\,(-2x)\,dx$

14. $\int \sqrt[3]{3-4x^2}(-8x)\,dx$

15. $\int x^3(x^4+3)^2\,dx$

16. $\int x^2(x^3+5)^4\,dx$

17. $\int x^2(x^3-1)^4\,dx$

18. $\int x(5x^2+4)^3\,dx$

19. $\int t\sqrt{t^2+2}\,dt$

20. $\int t^3\sqrt{t^4+5}\,dt$

21. $\int 5x\sqrt[3]{1-x^2}\,dx$

22. $\int u^2\sqrt{u^3+2}\,du$

23. $\int \frac{x}{(1-x^2)^3}\,dx$

24. $\int \frac{x^3}{(1+x^4)^2}\,dx$

25. $\int \frac{x^2}{(1+x^3)^2}\,dx$

26. $\int \frac{x^2}{(16-x^3)^2}\,dx$

27. $\int \frac{x}{\sqrt{1-x^2}}\,dx$

28. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

29. $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$

30 $\int \left[x^2+\frac{1}{(3x)^2}\right]dx$

31. $\int \frac{1}{\sqrt{2x}}\,dx$

32. $\int \frac{1}{2\sqrt{x}}\,dx$

33. $\int \frac{x^2+5x-8}{\sqrt{x}}\,dx$

34. $\int \frac{t-9t^2}{\sqrt{t}}\,dt$

35. $\int t^2\left(t-\frac{8}{t}\right)dt$

36. $\int \left(\frac{t^3}{3}+\frac{1}{4t^2}\right)dt$

37. $\int (9-y)\sqrt{y}\,dy$

38. $\int 4\pi y(6+y^{3/2})\,dy$

In Exercises 39–42, solve the differential equation.

39. $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16-x^2}}$

40. $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$

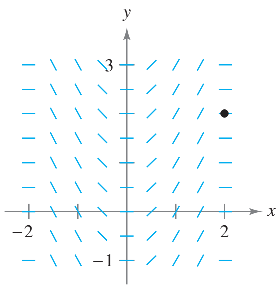
41. $\frac{dy}{dx} = \frac{x+1}{(x^2+2x-3)^2}$

42. $\frac{dy}{dx} = \frac{x-4}{\sqrt{x^2-8x+1}}$

Slope Fields **In Exercises 43–46, a differential equation, a point, and a slope field are given. A *slope field* consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the directions of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

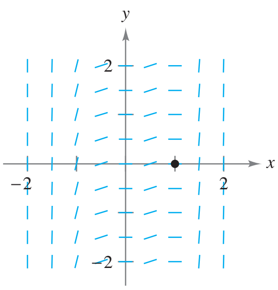
43. $\frac{dy}{dx} = x\sqrt{4-x^2}$

$(2, 2)$



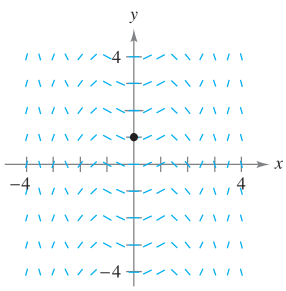
44. $\frac{dy}{dx} = x^2(x^3-1)^2$

$(1, 0)$



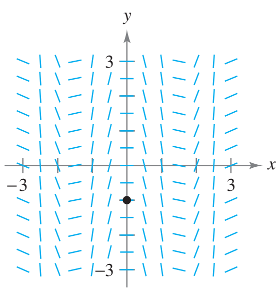
45. $\frac{dy}{dx} = x\cos x^2$

$(0, 1)$



46. $\frac{dy}{dx} = -2\sec(2x)\tan(2x)$

$(0, -1)$



In Exercises 47–60, find the indefinite integral.

47. $\int \pi \sin \pi x \, dx$

48. $\int 4x^3 \sin x^4 \, dx$

49. $\int \sin 4x \, dx$

50. $\int \cos 8x \, dx$

51. $\int \frac{1}{\theta^2} \cos \frac{1}{\theta} \, d\theta$

52. $\int x \sin x^2 \, dx$

53. $\int \sin 2x \cos 2x \, dx$

54. $\int \sec(1 - x) \tan(1 - x) \, dx$

55. $\int \tan^4 x \sec^2 x \, dx$

56. $\int \sqrt{\tan x} \sec^2 x \, dx$

57. $\int \frac{\csc^2 x}{\cot^3 x} \, dx$

58. $\int \frac{\sin x}{\cos^3 x} \, dx$

59. $\int \cot^2 x \, dx$

60. $\int \csc^2\left(\frac{x}{2}\right) dx$

In Exercises 61–66, find an equation for the function f that has the given derivative and whose graph passes through the given point.

Derivative	*Point*
61. $f'(x) = -\sin \frac{x}{2}$	$(0, 6)$
62. $f'(x) = \pi \sec \pi x \tan \pi x$	$\left(\frac{1}{3}, 1\right)$
63. $f'(x) = 2 \sin 4x$	$\left(\frac{\pi}{4}, -\frac{1}{2}\right)$
64. $f'(x) = \sec^2(2x)$	$\left(\frac{\pi}{2}, 2\right)$
65. $f'(x) = 2x(4x^2 - 10)^2$	$(2, 10)$
66. $f'(x) = -2x\sqrt{8 - x^2}$	$(2, 7)$

In Exercises 67–74, find the indefinite integral by the method shown in Example 5.

67. $\int x\sqrt{x + 6} \, dx$

68. $\int x\sqrt{4x + 1} \, dx$

69. $\int x^2\sqrt{1 - x} \, dx$

70. $\int (x + 1)\sqrt{2 - x} \, dx$

71. $\int \frac{x^2 - 1}{\sqrt{2x - 1}} \, dx$

72. $\int \frac{2x + 1}{\sqrt{x + 4}} \, dx$

73. $\int \frac{-x}{(x + 1) - \sqrt{x + 1}} \, dx$

74. $\int t \sqrt[3]{t + 10} \, dt$

In Exercises 75–86, evaluate the definite integral. Use a graphing utility to verify your result.

75. $\int_{-1}^{1} x(x^2 + 1)^3 \, dx$

76. $\int_{-2}^{4} x^2(x^3 + 8)^2 \, dx$

77. $\int_{1}^{2} 2x^2\sqrt{x^3 + 1} \, dx$

78. $\int_{0}^{1} x\sqrt{1 - x^2} \, dx$

79. $\int_{0}^{4} \frac{1}{\sqrt{2x + 1}} \, dx$

80. $\int_{0}^{2} \frac{x}{\sqrt{1 + 2x^2}} \, dx$

81. $\int_{1}^{9} \frac{1}{\sqrt{x}\left(1 + \sqrt{x}\right)^2} \, dx$

82. $\int_{0}^{2} x \sqrt[3]{4 + x^2} \, dx$

83. $\int_{1}^{2} (x - 1)\sqrt{2 - x} \, dx$

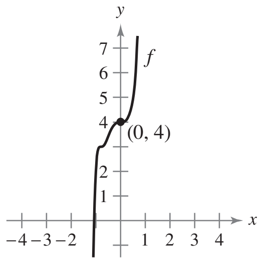
84. $\int_{1}^{5} \frac{x}{\sqrt{2x - 1}} \, dx$

85. $\int_{0}^{\pi/2} \cos\left(\frac{2x}{3}\right) dx$

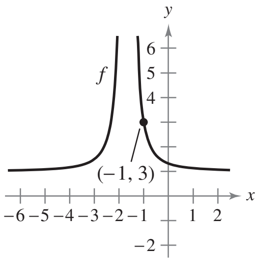
86. $\int_{\pi/3}^{\pi/2} (x + \cos x) \, dx$

Differential Equations **In Exercises 87–90, the graph of a function f is shown. Use the differential equation and the given point to find an equation of the function.**

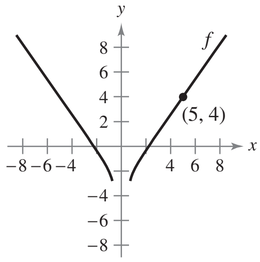
87. $\frac{dy}{dx} = 18x^2(2x^3 + 1)^2$



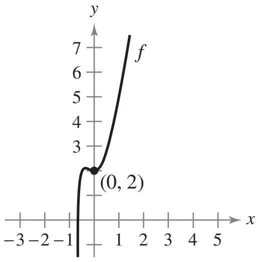
88. $\frac{dy}{dx} = \frac{-48}{(3x + 5)^3}$



89. $\frac{dy}{dx} = \frac{2x}{\sqrt{2x^2 - 1}}$

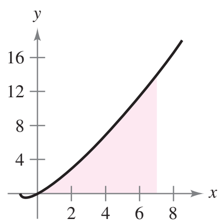


90. $\frac{dy}{dx} = 4x + \frac{9x^2}{(3x^3 + 1)^{(3/2)}}$

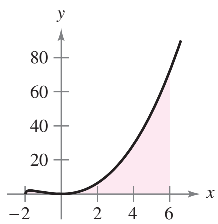


In Exercises 91–96, find the area of the region. Use a graphing utility to verify your result.

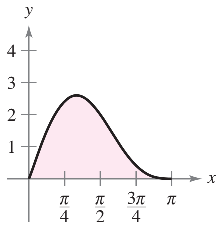
91. $\int_{0}^{7} x \sqrt[3]{x + 1} \, dx$



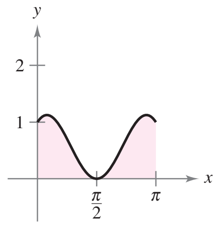
92. $\int_{-2}^{6} x^2 \sqrt[3]{x + 2} \, dx$



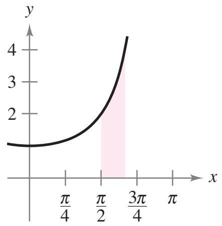
93. $y = 2 \sin x + \sin 2x$



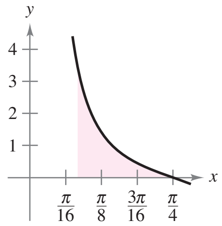
94. $y = \sin x + \cos 2x$



95. $\displaystyle\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx$



96. $\displaystyle\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\, dx$



In Exercises 97–102, use a graphing utility to evaluate the integral. Graph the region whose area is given by the definite integral.

97. $\displaystyle\int_0^6 \frac{x}{\sqrt{4x+1}}\, dx$

98. $\displaystyle\int_0^2 x^3\sqrt{2x+3}\, dx$

99. $\displaystyle\int_3^7 x\sqrt{x-3}\, dx$

100. $\displaystyle\int_1^5 x^2\sqrt{x-1}\, dx$

101. $\displaystyle\int_1^4 \left(\theta + \sin\frac{\theta}{4}\right) d\theta$

102. $\displaystyle\int_0^{\pi/6} \cos 3x\, dx$

In Exercises 103–106, evaluate the integral using the properties of even and odd functions as an aid.

103. $\displaystyle\int_{-2}^{2} x^2(x^2+1)\, dx$

104. $\displaystyle\int_{-2}^{2} x(x^2+1)^3\, dx$

105. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx$

106. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx$

107. Use $\int_0^4 x^2\, dx = \frac{64}{3}$ to evaluate each definite integral without using the Fundamental Theorem of Calculus.

(a) $\displaystyle\int_{-4}^{0} x^2\, dx$ (b) $\displaystyle\int_{-4}^{4} x^2\, dx$

(c) $\displaystyle\int_0^4 -x^2\, dx$ (d) $\displaystyle\int_{-4}^{0} 3x^2\, dx$

108. Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each definite integral.

(a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin x\, dx$ (b) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx$

(c) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\, dx$ (d) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx$

In Exercises 109 and 110, write the integral as the sum of the integral of an odd function and the integral of an even function. Use this simplification to evaluate the integral.

109. $\displaystyle\int_{-3}^{3} (x^3 + 4x^2 - 3x - 6)\, dx$ **110.** $\displaystyle\int_{-\pi/2}^{\pi/2} (\sin 4x + \cos 4x)\, dx$

WRITING ABOUT CONCEPTS

111. Describe why

$$\int x(5-x^2)^3\, dx \neq \int u^3\, du$$

where $u = 5 - x^2$.

112. Without integrating, explain why $\displaystyle\int_{-2}^{2} x(x^2+1)^2\, dx = 0$.

113. If f is continuous and $\displaystyle\int_0^8 f(x)\, dx = 32$, find $\displaystyle\int_0^4 f(2x)\, dx$.

CAPSTONE

114. ***Writing*** Find the indefinite integral in two ways. Explain any difference in the forms of the answers.

(a) $\displaystyle\int (2x-1)^2\, dx$ (b) $\displaystyle\int \sin x \cos x\, dx$

(c) $\displaystyle\int \tan x \sec^2 x\, dx$

115. ***Cash Flow*** The rate of disbursement dQ/dt of a 2 million dollar federal grant is proportional to the square of $100 - t$. Time t is measured in days $(0 \le t \le 100)$, and Q is the amount that remains to be disbursed. Find the amount that remains to be disbursed after 50 days. Assume that all the money will be disbursed in 100 days.

116. ***Depreciation*** The rate of depreciation dV/dt of a machine is inversely proportional to the square of $t + 1$, where V is the value of the machine t years after it was purchased. The initial value of the machine was $500,000, and its value decreased $100,000 in the first year. Estimate its value after 4 years.

117. ***Precipitation*** The normal monthly precipitation at the Seattle-Tacoma airport can be approximated by the model

$R = 2.876 + 2.202 \sin(0.576t + 0.847)$

where R is measured in inches and t is the time in months, with $t = 0$ corresponding to January 1. (*Source: U.S. National Oceanic and Atmospheric Administration*)

(a) Determine the extrema of the function over a one-year period.

(b) Use integration to approximate the normal annual precipitation. (*Hint:* Integrate over the interval $[0, 12]$.)

(c) Approximate the average monthly precipitation during the months of October, November, and December.

118. ***Sales*** The sales S (in thousands of units) of a seasonal product are given by the model

$$S = 74.50 + 43.75 \sin \frac{\pi t}{6}$$

where t is the time in months, with $t = 1$ corresponding to January. Find the average sales for each time period.

(a) The first quarter $(0 \le t \le 3)$

(b) The second quarter $(3 \le t \le 6)$

(c) The entire year $(0 \le t \le 12)$

119. ***Water Supply*** A model for the flow rate of water at a pumping station on a given day is

$$R(t) = 53 + 7 \sin\left(\frac{\pi t}{6} + 3.6\right) + 9 \cos\left(\frac{\pi t}{12} + 8.9\right)$$

where $0 \le t \le 24$. R is the flow rate in thousands of gallons per hour, and t is the time in hours.

(a) Use a graphing utility to graph the rate function and approximate the maximum flow rate at the pumping station.

(b) Approximate the total volume of water pumped in 1 day.

120. ***Electricity*** The oscillating current in an electrical circuit is

$$I = 2 \sin(60\pi t) + \cos(120\pi t)$$

where I is measured in amperes and t is measured in seconds. Find the average current for each time interval.

(a) $0 \le t \le \frac{1}{60}$

(b) $0 \le t \le \frac{1}{240}$

(c) $0 \le t \le \frac{1}{30}$

Probability **In Exercises 121 and 122, the function**

$$f(x) = kx^n(1 - x)^m, \quad 0 \le x \le 1$$

where $n > 0$, $m > 0$, and k is a constant, can be used to represent various probability distributions. If k is chosen such that

$$\int_0^1 f(x)\,dx = 1$$

the probability that x will fall between a and b $(0 \le a \le b \le 1)$ is

$$P_{a,b} = \int_a^b f(x)\,dx.$$

121. The probability that a person will remember between $100a\%$ and $100b\%$ of material learned in an experiment is

$$P_{a,b} = \int_a^b \frac{15}{4} x\sqrt{1 - x}\,dx$$

where x represents the proportion remembered. (See figure.)

(a) For a randomly chosen individual, what is the probability that he or she will recall between 50% and 75% of the material?

(b) What is the median percent recall? That is, for what value of b is it true that the probability of recalling 0 to b is 0.5?

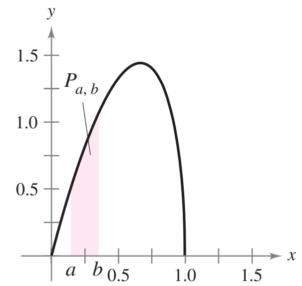


Figure for 121

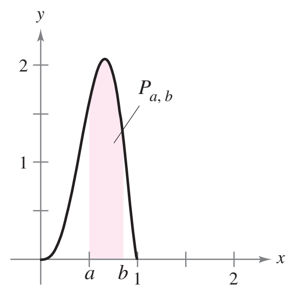
122. The probability that ore samples taken from a region contain between $100a\%$ and $100b\%$ iron is

$$P_{a,b} = \int_a^b \frac{1155}{32} x^3(1 - x)^{3/2}\,dx$$

where x represents the proportion of iron. (See figure.) What is the probability that a sample will contain between

(a) 0% and 25% iron?

(b) 50% and 100% iron?



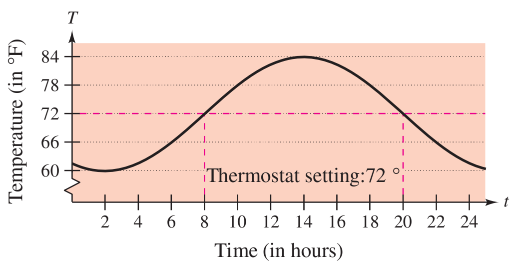
123. ***Temperature*** The temperature in degrees Fahrenheit in a house is

$$T = 72 + 12 \sin\left[\frac{\pi(t - 8)}{12}\right]$$

where t is time in hours, with $t = 0$ representing midnight. The hourly cost of cooling a house is $0.10 per degree.

(a) Find the cost C of cooling the house if its thermostat is set at 72°F by evaluating the integral

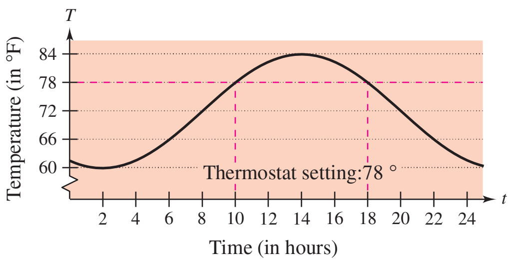
$$C = 0.1\int_8^{20} \left[72 + 12 \sin \frac{\pi(t - 8)}{12} - 72\right] dt. \text{ (See figure.)}$$



(b) Find the savings from resetting the thermostat to 78°F by evaluating the integral

$$C = 0.1\int_{10}^{18}\left[72 + 12\sin\frac{\pi(t-8)}{12} - 78\right]dt.$$

(See figure.)



124. ***Manufacturing*** A manufacturer of fertilizer finds that national sales of fertilizer follow the seasonal pattern

$$F = 100{,}000\left[1 + \sin\frac{2\pi(t-60)}{365}\right]$$

where F is measured in pounds and t represents the time in days, with $t = 1$ corresponding to January 1. The manufacturer wants to set up a schedule to produce a uniform amount of fertilizer each day. What should this amount be?

125. ***Graphical Analysis*** Consider the functions f and g, where

$$f(x) = 6\sin x\cos^2 x \quad \text{and} \quad g(t) = \int_0^t f(x)\,dx.$$

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Explain why g is nonnegative.

(c) Identify the points on the graph of g that correspond to the extrema of f.

(d) Does each of the zeros of f correspond to an extremum of g? Explain.

(e) Consider the function

$$h(t) = \int_{\pi/2}^t f(x)\,dx.$$

Use a graphing utility to graph h. What is the relationship between g and h? Verify your conjecture.

126. Find $\displaystyle\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{\sin(i\pi/n)}{n}$ by evaluating an appropriate definite integral over the interval $[0, 1]$.

127. (a) Show that $\int_0^1 x^2(1-x)^5\,dx = \int_0^1 x^5(1-x)^2\,dx$.

(b) Show that $\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$.

128. (a) Show that $\int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}\cos^2 x\,dx$.

(b) Show that $\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx$, where n is a positive integer.

True or False? **In Exercises 129–134, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

129. $\displaystyle\int (2x+1)^2\,dx = \tfrac{1}{3}(2x+1)^3 + C$

130. $\displaystyle\int x(x^2+1)\,dx = \tfrac{1}{2}x^2\left(\tfrac{1}{3}x^3 + x\right) + C$

131. $\displaystyle\int_{-10}^{10}(ax^3+bx^2+cx+d)\,dx = 2\int_0^{10}(bx^2+d)\,dx$

132. $\displaystyle\int_a^b \sin x\,dx = \int_a^{b+2\pi}\sin x\,dx$

133. $\displaystyle 4\int \sin x\cos x\,dx = -\cos 2x + C$

134. $\displaystyle\int \sin^2 2x\cos 2x\,dx = \tfrac{1}{3}\sin^3 2x + C$

135. Assume that f is continuous everywhere and that c is a constant. Show that

$$\int_{ca}^{cb} f(x)\,dx = c\int_a^b f(cx)\,dx.$$

136. (a) Verify that $\sin u - u\cos u + C = \int u\sin u\,du$.

(b) Use part (a) to show that $\int_0^{\pi^2}\sin\sqrt{x}\,dx = 2\pi$.

137. Complete the proof of Theorem 4.16.

138. Show that if f is continuous on the entire real number line, then

$$\int_a^b f(x+h)\,dx = \int_{a+h}^{b+h} f(x)\,dx.$$

PUTNAM EXAM CHALLENGE

139. If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

show that the equation

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

has at least one real zero.

140. Find all the continuous positive functions $f(x)$, for $0 \le x \le 1$, such that

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 f(x)x\,dx = \alpha$$

$$\int_0^1 f(x)x^2\,dx = \alpha^2$$

where α is a real number.