EXAMINATION COVERSHEET

Autumn 2022 Final Examination

UNIVERSITY OF WOLLONGONG IN DUBAI

THIS EXAMINATION CONTENT IS STRICTLY CONFIDENTIAL Students must comply with requirements stated in the Examination Policy & Procedures	
Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	**14/12/2022**
Subject Code:	**MATH 141**
Subject Title:	**Foundation of Engineering Mathematics**
Time Permitted to Write Exam:	2 Hours
Total Number of Questions:	**11 (6 MCQ's + 5 written questions)**
Total Number of Pages *(including this page)*:	**9**

Part 1 MCQ 30% (circle your choice)

(5pts)Problem 1
Find the values of x and y in the equation

$$x\,(1+i)^2 + y\,(2-i)^2 = 3 + 10i.$$

$x + y$ is equal to

(a) 10 $\quad$ **(b) 8** $\quad$ (c) 7 $\quad$ (d) 11 $\quad$ (e) 13

Solution

$$x\,(1+i)^2 + y\,(2-i)^2 = 3 + 10i.$$

After expanding, we obtain

$$2ix + 3y - 4iy = 3 + 10i$$

$\Leftrightarrow$

$$3y + (2x - 4y)\,i = 3 + 10i.$$

Now we identify to get

$$\begin{cases} 3y = 3 \\ 2x - 4y = 10 \end{cases}.$$

Solution is: $[x = 7,\ y = 1]$. Thus $x + y = 8$ and the answer is (b).

....

(5pts)Problem 2
The complex conjugate of $z = x + iy$ is denoted by $\bar{z} = x - iy$. Solve the equation

$$z - 8 = i\,(7 - 2\bar{z})$$

The modulus of the solution z is equal to

(a) $|z| = \sqrt{19}$ $\quad$ (b) $|z| = 13\sqrt{7}$ $\quad$ (c) $|z| = \sqrt{7}$ $\quad$ **(d) $|\mathbf{z}| = \sqrt{13}$** $\quad$ (e) $|z| = \sqrt{23}$

Solution

$$x + iy - 8 = i\,(7 - 2\,(x - iy))$$
$$x - 8 + iy = -2y + (7 - 2x)\,i.$$

Hence,

$$\begin{cases} x - 8 = -2y \\ y = 7 - 2x \end{cases} \Leftrightarrow \begin{cases} x + 2y = 8 \\ 2x + y = 7 \end{cases}.$$

Solution is: $[x = 2, y = 3]$

$$z = 2 + 3i \text{ and } |z| = \sqrt{4+9} = \sqrt{13}\ .\ \text{Answer is } (d)$$

(5pts)Problem 3

The point of intersection of the line whose parametric equations are

$$\begin{cases} x = 2 - 2t \\ y = 3t \\ z = 1 + t \end{cases}$$

and the plane $x + 2y - z = 7$ is (a, b, c).

$a + b + c =$

$\underline{(\mathbf{a})\ \ \mathbf{7}}$ $\quad (b)\ \ 6 \quad (c)\ \ 5 \quad (d)\ \ 4 \quad (e)\ \ 0$

Solution

We have

$$(2 - 2t) + 6t - (1 + t) = 7.$$

Now solve for t to get

$$t = 2$$

$$\begin{cases} x = 2 - 2(2) = -2 = a \\ y = 3(2) = 6 = b \\ z = 1 + (2) = 3 = c \end{cases}$$

$$a + b + c = -2 + 6 + 3 = 7. \text{ Answer is } (a).$$

....

(5pts)Problem 4

If the angle between the vectors $\langle 2,\ 1,\ -1\rangle$ and $\langle 1,\ x,\ 0\rangle$ is $\theta = \dfrac{\pi}{4}$, then $x =$

$(a)\ \ 1\ \pm\sqrt{6} \quad (b)\ 2\ \pm\sqrt{6} \quad (c)\ \ 1\ \pm\sqrt{3} \quad (d)\ \ 1\ \pm\frac{\sqrt{3}}{2}$ $\quad \underline{(\mathbf{e})\ \ \mathbf{1} \pm \boldsymbol{\frac{\sqrt{6}}{2}}}$

Solution

$$\cos\frac{\pi}{4} = \frac{\langle 2,\ 1,\ -1\rangle \cdot \langle 1,\ x,\ 0\rangle}{\|\langle 2,\ 1,\ -1\rangle\|\,\|\langle 1,\ x,\ 0\rangle\|}.$$

Equivalently,

$$\frac{2 + x}{\sqrt{6}\sqrt{1 + x^2}} = \frac{1}{\sqrt{2}}.$$

$\Leftrightarrow$

$$\sqrt{6\left(1 + x^2\right)} = \sqrt{2}\,(2 + x)$$

$\Leftrightarrow$

$$6 + 6x^2 = 2\,(2 + x)^2$$

$\Leftrightarrow$

$$3 + 3x^2 = x^2 + 4x + 4$$

$\Leftrightarrow$

$$2x^2 - 4x - 1 = 0$$

Solution is: $1\ \pm\dfrac{\sqrt{6}}{2}$. Answer is (e)

(5pts)**Problem 5**

Find the value of x for which the matrix A does not have an inverse.

$$A = \begin{pmatrix} 1 & -1 & -x \\ 0 & 1 & 3 \\ x & 0 & 0 \end{pmatrix}.$$

A does not have an inverseThe sum of all possible values of x is

(a) 4 $\quad$ (b) -6 $\quad$ <u>**(c) 3**</u> $\quad$ (d) 0 $\quad$ (e) 5

Solution

A does not have an inverse if and only if $\det A = 0$.

$$\begin{aligned} \det A &= x^2 - 3x \\ &= x(x-3) \\ &= 0 \end{aligned}$$

$x = 0$ or $x = 3$.

$0 + 3 = 3$. The answer is (c).

(5pts)**Problem 6**

Suppose two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ satisfy

$$\overrightarrow{a} \cdot \overrightarrow{b} = \sqrt{15} \text{ and } \overrightarrow{a} \times \overrightarrow{b} = \langle -2,\ 0,\ -1 \rangle,$$

then the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is

<u>**(a)** $\boldsymbol{\frac{\pi}{6}}$</u> $\quad$ (b) $\frac{\pi}{3}$ $\quad$ (c) $\frac{\pi}{2}$ $\quad$ (d) $\frac{\pi}{4}$ $\quad$ (e) $\frac{2\pi}{5}$

Solution

$$\cos\theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{\|\overrightarrow{a}\| \left\|\overrightarrow{b}\right\|}, \quad \sin\theta = \frac{\left\|\overrightarrow{a} \times \overrightarrow{b}\right\|}{\|\overrightarrow{a}\| \left\|\overrightarrow{b}\right\|}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\left\|\overrightarrow{a} \times \overrightarrow{b}\right\|}{\|\overrightarrow{a}\| \|\overrightarrow{b}\|}}{\frac{\overrightarrow{a} \cdot \overrightarrow{b}}{\|\overrightarrow{a}\| \|\overrightarrow{b}\|}} = \frac{\left\|\overrightarrow{a} \times \overrightarrow{b}\right\|}{\overrightarrow{a} \cdot \overrightarrow{b}} = \frac{\sqrt{5}}{\sqrt{15}} = \frac{1}{\sqrt{3}}$$

$$\theta = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

Part 2 Written 70%

(15pts)**Problem 1**
Consider the points $A\,(2,\ 0,\ -1)$, $B\,(1,\ -1,\ 1)$, $C\,(0,\ 3,\ -2)$ and $D\,(5, -2,\ -1)$.
(a) Find the equation of the plane $(\mathcal{P})$ containing the points $A,\ B,$ and C.
(b) Find the distance from D to $(\mathcal{P})$.
Solution
(a)

$$\overrightarrow{AB} = \langle -1, -1, 2\rangle, \qquad \overrightarrow{AC} = \langle -2, 3, -1\rangle. \qquad (\mathbf{4pts})$$

The normal vector is given by

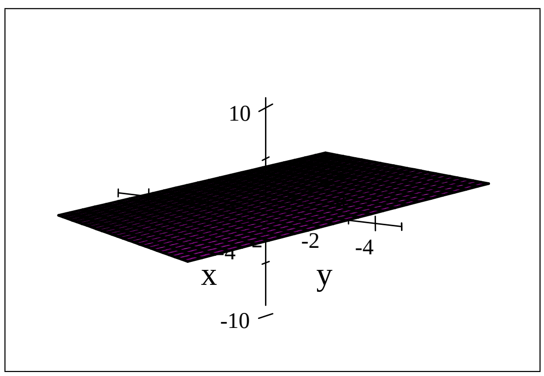
$$\overrightarrow{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -1 & -1 & 2 \\ -2 & 3 & -1 \end{vmatrix} = \langle -5, -5, -5\rangle. \quad (\mathbf{3pts})$$

The equation of the plane $(\mathcal{P})$ is

$$-5\,(x-2) - 5\,(y-0) - 5\,(z+1) = 0$$

$$(x-2) + (y-0) + (z+1) = 0$$

$$x + y + z = 1. \qquad (\mathbf{4pts})$$



(b)

$$\begin{aligned} \text{Distance} &= \frac{\left|\overrightarrow{AD}\cdot\overrightarrow{n}\right|}{\|\overrightarrow{n}\|} = \frac{|\langle 3, -2, 0\rangle \cdot \langle -5, -5, -5\rangle|}{5\sqrt{3}} \\ &= \frac{|-15+10|}{5\sqrt{3}} = \frac{1}{\sqrt{3}}. \qquad (\mathbf{4pts}) \end{aligned}$$

$(13pts)$**Problem 2**

Find the equation of the plane that contains the two lines

$$L_1 : \quad \begin{cases} x = 1 + t \\ y = 1 - t \\ z = 2t \end{cases} \quad \text{and} \quad L_2 : \quad \begin{cases} x = 2 - t \\ y = t \\ z = 2 \end{cases} .$$

Solution

Denote by $\overrightarrow{u_1}$ the direction vector of L_1 and $\overrightarrow{u_2}$ the direction vector of L_2.

$$\overrightarrow{u_1} = \langle 1, \ -1, \ 2 \rangle \ \text{and} \ \overrightarrow{u_2} = \langle -1, \ 1, \ 0 \rangle \qquad (\mathbf{4pts})$$

The normal vector of the plane is given by

$$\begin{aligned} \overrightarrow{n} &= \overrightarrow{u_1} \times \overrightarrow{u_2} \qquad (\mathbf{2pts}) \\ &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = \langle -2, \ -2, \ 0 \rangle . \qquad (\mathbf{3pts}) \end{aligned}$$

Now using the point $(1, \ 1, \ 0)$, the equation of the plane is

$$-2(x-1) - 2(y-1) = 0$$

$$x + y - 2 = 0. \qquad (\mathbf{4pts})$$

$(13pts)$**Problem 3**

Use the **Gauss elimination** method to solve the linear system

$$\begin{cases} 3x - y - 5z = 3 \\ 4x - 4y - 3z = -4 \\ x - 5z = 2 \end{cases} . \quad \text{(Show your work)}$$

Solution

The augmented matrix is

$$A = \begin{bmatrix} 3 & -1 & -5 & 3 \\ 4 & -4 & -3 & -4 \\ 1 & 0 & -5 & 2 \end{bmatrix} . \quad (\mathbf{2pts})$$

After reducing in echelon form, we obtain

$$\text{Row echelon form} : \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \quad (\mathbf{5pts})$$

Now the back substitution gives

$$[x = 2, \; y = 3, \; z = 0] \qquad (\mathbf{2pts} + \mathbf{2pts} + \mathbf{2pts})$$

(15pts)**Problem 4**

Consider the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 6 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 1 & 2 \\ 4 & 0 & -5 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 2 & 0 \\ -3 & -1 \end{bmatrix}$$

(a) Compute the following matrices, where possible.

1. $A + B^T$, 2. AC

(b) Find the matrix X such that

$$\frac{3}{2}X + C = \begin{bmatrix} 3 & -4 \\ 5 & 4 \end{bmatrix}.$$

Solution

(a)

$$\begin{aligned} A + B^T &= \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 6 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 1 & 0 \\ 2 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 \\ 3 & -1 \\ 8 & -8 \end{bmatrix}. \qquad (\textbf{5pts}) \end{aligned}$$

$$\begin{aligned} AC &= \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 7 & 1 \\ 21 & 3 \end{bmatrix}. \qquad (\textbf{5pts}) \end{aligned}$$

(b)

$$\begin{aligned} \frac{3}{2}X + C &= \begin{bmatrix} 3 & -4 \\ 5 & 4 \end{bmatrix} \Leftrightarrow \\ \frac{3}{2}X + \begin{bmatrix} 2 & 0 \\ -3 & -1 \end{bmatrix} &= \begin{bmatrix} 3 & -4 \\ 5 & 4 \end{bmatrix} \\ \frac{3}{2}X &= \begin{bmatrix} 3 & -4 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -4 \\ 8 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= \frac{2}{3}\begin{bmatrix} 1 & -4 \\ 8 & 5 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & -\frac{8}{3} \\ \frac{16}{3} & \frac{10}{3} \end{bmatrix}. \qquad (\textbf{5pts}) \end{aligned}$$

(14*pts*)**Problem 5**

Use the **cofactor expansion method** to find the determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}. \qquad \text{(Show your work)}$$

Solution

Choosing an expansion along the first column, we obtain

$$\begin{aligned}
\det A &= (2)\,(-1)^{1+1}\det\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} + (1)\,(-1)^{2+1}\det\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (\mathbf{5pts}) \\
&= 2\det\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} - \det\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\
&= (2)\left[(2)\,(-1)^{1+1}\det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + (1)\,(-1)^{2+1}\det\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\right] - (1)\,(-1)^{1+1}\det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (\mathbf{5pts}) \\
&= (2)\,(2)\,(-1)^{1+1}\,(3) + (2)\,(1)\,(-1)^{2+1}\,(2) - (1)\,(-1)^{1+1}\,(3) \\
&= 5 \qquad (\mathbf{4pts})
\end{aligned}$$