($6pts$) **Problem 1**
A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Matetial for the base cost \$12 per square meter. Material for the side cost \$5 per square meter. Find the cost of materials for the cheapest such container.
Solution
The cheapest such container will be the one that can hold 10 m^3 with the smallest cost.
Denote by x the length of width and h the height. The length of the base is $2x$. We have

$$V = (2x)(x)h = 2x^2h, \qquad A = 2x^2 + 2xh + 2xh + xh + xh$$

$$2x^2h = 10 \Rightarrow h = \frac{10}{2x^2} = \frac{5}{x^2}$$

Thus the cost of the material with the cheapest such container is

$$\begin{aligned} C &= (12)\left(2x^2\right) + (5)(6x)\left(\frac{5}{x^2}\right) \\ &= 24x^2 + \frac{150}{x} \end{aligned}$$

$$C' = 48x - \frac{150}{x^2}$$

$$C' = 0 \Leftrightarrow 48x - \frac{150}{x^2} = 0 \Leftrightarrow x = \sqrt[3]{\frac{75}{24}}$$

$$C'' = 48 + \frac{300}{x^3} > 0.$$

The cost is minimized at $x = \sqrt[3]{\frac{75}{24}} = 1.462$ and the cost of materials for the cheapest such container

$$C = 24\left(\sqrt[3]{\frac{75}{24}}\right)^2 + \frac{150}{\left(\sqrt[3]{\frac{75}{24}}\right)} = 153.9.$$

(6pts) **Problem 2**
Evaluate the integral

$$\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx$$

Solution

$$\begin{aligned}\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx &= \int_{-3}^{0}(1)\,dx+\int_{-3}^{0}\sqrt{9-x^2}dx\\ &= 3+\int_{-3}^{0}\sqrt{9-x^2}dx\end{aligned}$$

But $\int_{-3}^{0}\sqrt{9-x^2}dx$ is the quarter of the area centered at 0 with radius 3. Thus

$$\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx = 3+\frac{9\pi}{4} = 10.069$$

$(5pts)$**Problem 3**

$$\text{If } g(x) = \int_{\cos\ x}^{\sin\ x} \ln(2+3t)\,dt, \text{ then } g'(0) =$$

Solution

$$g'(x) = (\cos x)(\ln(2+3\sin x)) + (\sin x)(\ln(2+3\cos x))$$
$$g'(0) = \ln 2$$

$(5pts)$ **Problem 4**

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{1}{n}\right)\frac{1}{\sqrt{1-\left(\frac{i}{2n}\right)^2}}\right]=$$

(a) $\displaystyle\int_0^{\frac{1}{2}}\frac{2}{\sqrt{1-x^2}}\,dx$

(b) $\displaystyle\int_0^{\frac{1}{2}}\frac{1}{1-x^2}\,dx$

(c) $\displaystyle\int_{-\frac{1}{2}}^{0}\frac{1}{\sqrt{1-x^2}}\,dx$

(d) $\displaystyle\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{2}}\frac{2}{\sqrt{1-2x^2}}\,dx$

(e) $\displaystyle\int_{-2}^{2}\frac{2}{\sqrt{x}}\,dx$

Solution

$$\frac{b-a}{n}=\frac{1}{2n}.\quad a=0\Rightarrow b=\frac{1}{2}$$

$$\sum_{i=0}^{n}\left(\frac{1}{n}\right)\left(\frac{1}{\sqrt{1-\left(\frac{i}{2n}\right)^2}}\right)=\sum_{i=0}^{n}\left(\frac{2}{\sqrt{1-\left(\frac{i}{2n}\right)^2}}\right)\left(\frac{1}{2n}\right)$$

with

$$x=a+i\left(\frac{b-a}{n}\right)=0+\frac{i}{2n}=\frac{i}{2n},$$

we see that

$$f(x)=\frac{2}{\sqrt{1-x^2}}.$$

So

$$\begin{aligned}\sum_{i=0}^{n}\left(\frac{1}{n}\right)\left(\frac{1}{\sqrt{1-\left(\frac{i}{2n}\right)^2}}\right) &= \int_0^{1/2}\frac{2}{\sqrt{1-x^2}}dx\\ Ans &= (a)\end{aligned}$$

($6pts$) **Problem 5**

A partical is moving in a straight line with velocity $v(t) = 2\sin 2t$ (m/sec). Find the total distance covered in meters by the partical in the time interval $0 \le t \le \frac{3\pi}{4}$.

Solution

$$D = \int_0^{\frac{3\pi}{4}} |2\sin 2t|\, dt.$$

Put

$$u = 2t, \quad du = 2dt \Rightarrow dt = \frac{du}{2}$$

When

$$0 \le t \le \frac{3\pi}{4},$$
$$0 \le 2t \le \frac{3\pi}{2}.$$

$$\begin{aligned} D &= \int_0^{\frac{3\pi}{4}} |2\sin 2t|\, dt = \int_0^{\frac{3\pi}{2}} |2\sin u| \frac{du}{2} \\ &= \int_0^{\frac{3\pi}{2}} |\sin u|\, du = \int_0^{\pi} \sin u du - \int_{\pi}^{\frac{3\pi}{2}} \sin u du \\ &= 2 + 1 = 3. \end{aligned}$$

$(6pts)$**Problem 6**

Evaluate the integral

$$\int \tan^3 x \sqrt[3]{\sec x} dx$$

Solution

$$\begin{aligned}
\int \tan^3 x \sqrt[3]{\sec x} dx &= \int \tan^2 x \frac{\sqrt[3]{\sec x}}{\sec x} (\sec x \tan x)\, dx \\
&= \int \sec^{-2/3} \left(\sec^2 x - 1\right) (\sec x \tan x)\, dx
\end{aligned}$$

Put $u = \sec x$. $du = \sec x \tan x dx$. The integral becomes

$$\begin{aligned}
\int \tan^3 x \sqrt[3]{\sec x} dx &= \int u^{\frac{-2}{3}} \left(u^2 - 1\right) du \\
&= \frac{3}{7} \sqrt[3]{u} \left(u^2 - 7\right) + C \\
&= \frac{3}{7} \sqrt[3]{\sec x} \left(\sec^2 x - 7\right) + C
\end{aligned}$$

(6pts)**Problem 7**

Express the following numbers in the form $a + ib$.

(a) $z_1 = i^4 - 3i^3 + 4i^2 + 2i - 6$

(b) $z_2 = \left(\dfrac{2i}{1+i}\right)^4$.

Solution

(a) $z_1 = i^4 - 3i^3 + 4i^2 + 2i - 6 = -9 + 5i$

(b) $z_2 = \left(\dfrac{2i}{1+i}\right)^4 = -4$