(10pts)Problem 1.

Evaluate the following limits

1. $\lim\limits_{x\to 1}\dfrac{1-\sqrt{8x-7}}{x-1}$ 2. $\lim\limits_{x\to -2^+}\dfrac{4+x\,|x|}{x+2}$

Solution

1.

$$\lim_{x\to 1}\frac{1-\sqrt{8x-7}}{x-1}=?$$

Method 1: multiplication by conjugate

$$\begin{aligned}
\lim_{x\to 1}\frac{1-\sqrt{8x-7}}{x-1} &= \lim_{x\to 1}\frac{\left(1-\sqrt{8x-7}\right)\left(1+\sqrt{8x-7}\right)}{(x-1)\left(1+\sqrt{8x-7}\right)}\\
&= \lim_{x\to 1}\frac{8-8x}{(x-1)\left(1+\sqrt{8x-7}\right)}\\
&= \lim_{x\to 1}\frac{-8\,(x-1)}{(x-1)\left(1+\sqrt{8x-7}\right)}\\
&= \lim_{x\to 1}\frac{-8}{\left(1+\sqrt{8x-7}\right)}\\
&= \frac{-8}{2}=-4 \qquad\qquad (\mathbf{5pts})
\end{aligned}$$

Method 2: L'Hospital's rule

$$\begin{aligned}
\lim_{x\to 1}\frac{1-\sqrt{8x-7}}{x-1} &= \lim_{x\to 1}\frac{\frac{d}{dx}\left[1-\sqrt{8x-7}\right]}{\frac{d}{dx}\left[x-1\right]}\\
&= \lim_{x\to 1}\frac{-\frac{4}{\sqrt{8x-7}}}{1}\\
&= \frac{-\frac{4}{1}}{1}=-4.
\end{aligned}$$

2.

$$\lim_{x\to -2^+}\frac{4+x\,|x|}{x+2}=?$$

When $x \to -2^+$, then $x < 0$ and $|x| = -x$. Hence, (**2pts**)

$$\begin{aligned}
\lim_{x \to -2^+} \frac{4 + x\,|x|}{x + 2} &= \lim_{x \to -2^+} \frac{4 + x\,(-x)}{x + 2} \\
&= \lim_{x \to -2^+} \frac{4 - x^2}{x + 2} \\
&= \lim_{x \to -2^+} \frac{-(x-2)(x+2)}{x + 2} \\
&= \lim_{x \to -2^+} \frac{-(x-2)}{1} \\
&= 4 \qquad (\mathbf{3pts})
\end{aligned}$$

(10pts)Problem 2.
Find the values of a and b for which the function

$$f(x) = \begin{cases} 3x^2 - a & \text{if } x > 1 \\ -a + b & \text{if } x = 1 \\ x - 2b & \text{if } x < 1 \end{cases}$$

is continuous at $x = 1$.
Solution
The condition of continuity at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(3x^2 - a\right) = 3 - a \qquad (\mathbf{2pts})$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - 2b) = 1 - 2b \qquad (\mathbf{2pts})$$

$$f(1) = -a + b. \qquad (\mathbf{2pts})$$

We have

$$\begin{cases} 3 - a = -a + b \\ 1 - 2b = -a + b \end{cases} \qquad (\mathbf{2pts})$$

$$3 - a = -a + b \Leftrightarrow b = 3$$

$$1 - 2b = -a + b \Leftrightarrow 1 - 2(3) = -a + 3 \Rightarrow a = 8$$

$$a = 8 \text{ and } b = 3 \qquad (\mathbf{2pts})$$

(10pts)Problem 3

Find the equation of the tangent line to the graph of $f(x) = \dfrac{xe^x}{x+1}$ at $x = 0$.

Solution

Using the quotient rule, we get

$$\begin{aligned} f\,'(x) &= \frac{(e^x + xe^x)(x+1) - (1)(xe^x)}{(x+1)^2} \\ &= \frac{e^x}{(x+1)^2}\left(x^2 + x + 1\right) \qquad (\mathbf{5pts}) \end{aligned}$$

$$\begin{aligned} f\,'(0) &= \frac{e^0}{(0+1)^2}\left(0^2 + 0 + 1\right) \\ &= 1. \qquad (\mathbf{2pts}) \end{aligned}$$

$$f(0) = 0$$

The equation of the tangent line is

$$y = f\,'(0)(x - 0) + f(0)$$

$$y = x \qquad (\mathbf{3pts}).$$

(10pts)Problem 4.

A) Find $\frac{dy}{dx}$ if

$$y^2 \ln x + x\sqrt{y} = 2.$$

Solution

$$y^2 \ln x + xy^{1/2} = 2.$$

We first differentiate both sides to get

$$2yy' \ln x + \frac{y^2}{x} + (1)y^{1/2} + (x)\left(\frac{1}{2}\right)y'y^{-1/2} = 0. \qquad (\mathbf{3pts})$$

Next, we keep the term with y' on the left and move all other terms to the right.

$$2yy' \ln x + \frac{x}{2}y'y^{-1/2} = -\frac{y^2}{x} - y^{1/2}.$$

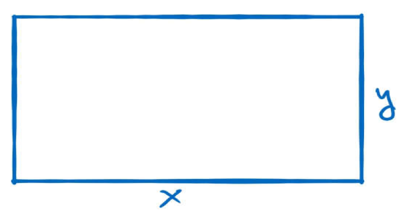
Now we factor y' and solve

$$y'\left(2y\ln x + \frac{x}{2}y^{-1/2}\right) = -\frac{y^2}{x} - y^{1/2}$$

$$y' = \frac{-\frac{y^2}{x} - y^{1/2}}{2y\ln x + \frac{x}{2}y^{-1/2}} = \frac{-\frac{y^2}{x} - \sqrt{y}}{2y\ln x + \frac{x}{2\sqrt{y}}} \qquad (\mathbf{2pts})$$

B) One side of a rectangle is increasing at a rate of 3 cm/sec and the other side is decreasing at a rate of 4 cm/sec. How fast is the area of the rectangle changing when the increasing side is 12 cm long and the decreasing side is 10 cm long?

Solution



$$\frac{dx}{dt} = 3cm/\sec, \qquad \frac{dy}{dt} = -4cm/\sec, \qquad (\mathbf{2pts})$$

We want to find $\frac{dA}{dt}$ when $x = 12$ and $y = 10$. $(\mathbf{1pt})$

$$A = xy$$

$$\begin{aligned} \frac{dA}{dt} &= x\frac{dy}{dt} + y\frac{dx}{dt} \\ &= (12)(-4) + (10)(3) \\ &= -18cm^2/\sec \qquad (\mathbf{2pts}) \end{aligned}$$

(10pts)Problem 5.
Find all the critical numbers of the function

$$f(x) = \sqrt[3]{2x - x^2}.$$

Solution

$$f(x) = \left(2x - x^2\right)^{\frac{1}{3}}.$$

$$\begin{aligned} f'(x) &= \left(\frac{1}{3}\right)(2 - 2x)\left(2x - x^2\right)^{-\frac{2}{3}} \\ &= \frac{2}{3}\frac{1 - x}{\left(\sqrt[3]{x(2 - x)}\right)^2} \qquad (\mathbf{4pts}) \end{aligned}$$

$$f'(x) = 0 \Rightarrow x = 1. \qquad (\mathbf{3pts})$$

$x = 1$ is a critical number.
Since the function is defined at 0 and 2 and the derivative is undefined at 0 and 2, $x = 0$ and $x = 2$ are also critical numbers. (**3pts**)
The critical numbers are

$$0, \quad 1, \quad \text{and } 2.$$

(10pts)Problem 6.
Find the absolute extrema of the function $g(x) = e^{x^4-2x^2}$ on $[-1,\ 1]$.
Solution

$$\begin{aligned} g'(x) &= \left(4x^3-4x\right)e^{x^4-2x^2} \qquad (\mathbf{2pts}) \\ &= 4x\left(x^2-1\right)e^{x^4-2x^2}. \end{aligned}$$

The critical numbers are

$$-1, \quad 0, \quad \text{and } 1. \qquad (\mathbf{2pts})$$

$$g(-1) = e^{1-2} = e^{-1} = \frac{1}{e} = 0.367\,88$$

$$g(0) = e^0 = 1$$

$$g(1) = e^{1-2} = e^{-1} = \frac{1}{e} = 0.367\,88$$

$$\text{The absolute Maximum} = 1 \qquad (\mathbf{3pts})$$

$$\text{The absolute Minimu} = \frac{1}{e} = 0.367\,88 \qquad (\mathbf{3pts})$$

(10pts)Problem 7.
Find the open intervals on which the function $f(x) = 1 + 2x + 6x^2 - x^4$ is concave up or down.

Solution

$$f\,'(x) = 2 + 12x - 4x^3$$

$$\begin{aligned} f\,''(x) &= 12 - 12x^2 \\ &= 12\left(1 - x^2\right) \qquad (\mathbf{3pts}) \end{aligned}$$

x	$-\infty$		-1		1		$+\infty$
$f''(x)=12(1-x^2)$		$-$	0	$+$	0	$-$	
$f(x)$		$\cap$		$\cup$		$\cap$	

(**3pts**)

Concave down on $(-\infty,\ -1)\ \cup (1,\ \infty)$ (**2pts**)
Concave up on $(-1,\ 1)$ (**2pts**)

(10pts)Problem 8.
Use definite integrals to evaluate

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt[3]{-1+\frac{2i}{n}}.$$

Solution
Here we will use the formula

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[f\left(a+i\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)=\int_a^b f(x)dx$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt[3]{-1+\frac{2i}{n}}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}\sqrt[3]{-1+\frac{2i}{n}}\frac{2}{n}.$$

$$a=-1,\qquad \frac{b-a}{\cancel{n}}=\frac{2}{\cancel{n}}\Leftrightarrow b+1=2\Rightarrow b=1\qquad (\mathbf{6pts})$$

$$f(x)=\frac{1}{2}\sqrt[3]{x}\qquad (\mathbf{2pts})$$

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt[3]{-1+\frac{2i}{n}} &= \int_{-1}^{1}\frac{1}{2}\sqrt[3]{x}dx.\\ &= \frac{1}{2}\int_{-1}^{1}x^{\frac{1}{3}}dx\\ &= \frac{3}{8}\sqrt[3]{-1}+\frac{3}{8}=0\qquad (\mathbf{2pts})\end{aligned}$$

(10pts)Problem 9.
Find the local extrema of the function

$$F(x) = \int_1^x t\,(2-t)\,dt$$

Solution

$$F'(x) = x\,(2-x)\,. \qquad \textbf{(2pts)}$$

The critical numbers are 0 and 2.

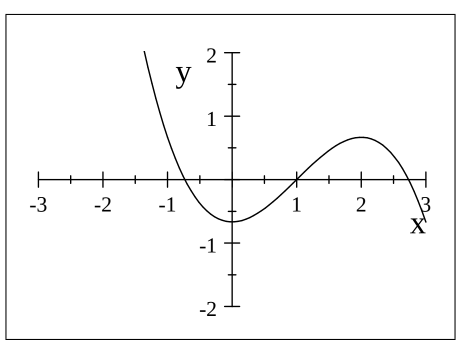
x	$-\infty$		0		2		$+\infty$
$F' = x(2-x)$		$-$		$+$		$-$	
$F(x)$		$\searrow$	$F(0)$	$\nearrow$	$F(1)$	$\searrow$	

(2pts)

The function has a local minimum at 0 and a local maximum at 2.

$$\begin{aligned}
\text{Local Min} &= F(0) = \int_1^0 t\,(2-t)\,dt \\
&= \int_1^0 \left(2t - t^2\right)dt = t^2 - \left.\frac{t^3}{3}\right|_1^0 \\
&= 0 - \left(1 - \frac{1}{3}\right) \\
&= -\frac{2}{3} \qquad \textbf{(3pts)}
\end{aligned}$$

$$\begin{aligned}
\text{Local Max} &= F(2) = \int_1^2 t\,(2-t)\,dt \\
&= t^2 - \left.\frac{t^3}{3}\right|_1^2 \\
&= \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) \\
&= \frac{2}{3} \qquad \textbf{(3pts)}
\end{aligned}$$



(10pts)Problem 10.
Use u-substitution to evaluate

$$\int x^3\sqrt{x^2-10}dx$$

Solution
Put

$$u = x^2 - 10, \qquad du = 2xdx \Rightarrow xdx = \frac{du}{2}$$
$$x^2 = u + 10.$$

The integral becomes

$$\begin{aligned}
\int x^3\sqrt{x^2-10}dx &= \int x^2\sqrt{x^2-10}xdx \\
&= \int (u+10)\sqrt{u}\frac{du}{2} \\
&= \frac{1}{2}\int \left(u^{3/2}+10u^{1/2}\right) du \\
&= \frac{1}{2}\left[\frac{u^{5/2}}{5/2}+10\frac{u^{3/2}}{3/2}\right]+C \\
&= \frac{1}{2}\left[\frac{2}{5}u^{5/2}+\frac{20}{3}u^{3/2}\right]+C \\
&= \frac{1}{5}u^{5/2}+\frac{10}{3}u^{3/2}+C \\
&= \frac{1}{5}\left(x^2-10\right)^{5/2}+\frac{10}{3}\left(x^2-10\right)^{3/2}+C
\end{aligned}$$