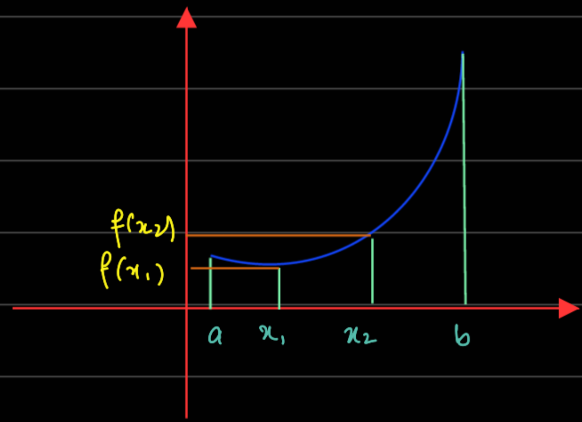
Increasing & Decreasing Functions

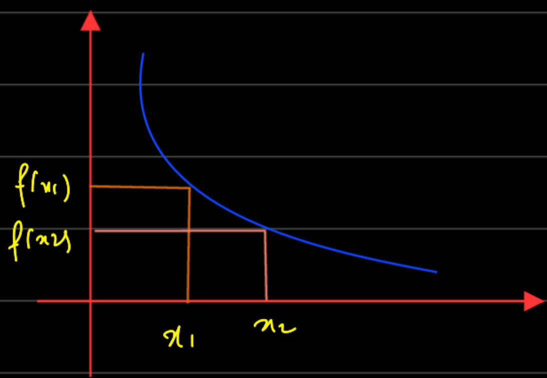
Increasing

The function f is increasing on an open interval I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all x_1, x_2 in I



f is increasing in I if and only if $f'(x) > 0$ in I

The function f is decreasing on an open interval I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all x_1, x_2 in I



f is decreasing in I if and only if $f'(x) < 0$ in I

Example

Find the open interval(s) where the function $f(x) = x^3 - \frac{3}{2}x + 1$ is increasing or decreasing

$x^2 + x - 1 \Rightarrow$ No zeroes

x	$-\infty$					∞
x^2+x-1	+	+	+	+	+	+

ALWAYS THE SIGN OF a

$$f'(x) = 3x^2 - 3x$$
$$= 3x(x-1)$$

x	$-\infty$		0		1		$+\infty$
$3x$		−	0	+		+	
$x-1$		−		−	0	+	
$f'(x)$		+		−		+	
$f(x)$		↑		↓		↑	

$\therefore f$ is increasing in the interval $(-\infty, 0) \cup (1, \infty)$ and decreasing in $(0, 1)$

REMARK

Consider a quadratic function $f(x) = ax^2 + bx + c$

If $b^2 - 4ac < 0$, $ax^2 + bx + c$ has the sign of 'a'

If $b^2 - 4 > 0$, then the quadratic function has two roots

$$x' = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x'' = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$x' < x''$

Inside

Always opp sign of a

x	$-\infty$		x'		x''		∞
ax^2+bx+c		Sign of a	0	opp sign of a	0	Sign of a	

Outside

Always sign of a

Redo of previous example $f'(x) = 3x^2 - 3x$

x	$-\infty$		0		1		∞
$3x^2 - 3x$		+	0	−	0	+	
$f(x)$		Inc		Dec		Inc	

First Derivative Test

Let c be a critical number of f.

→ If $f'(x)$ changes from +ve to −ve at c, then $f(c)$ is a local maximum

→ If $f'(x)$ changes from −ve to +ve at c, then $f(c)$ is a local minimum

Example

Find the local extrema for the following functions

PDF Page 42 Q 29 $f(x) = 3x^5 - 5x^4$

$$f'(x) = 15x^4 - 20x^3$$
$$= 5x^3(3x - 4)$$

$$f'(x) = 0$$

$5x^3 = 0$ $3x - 4 = 0$

$x = 0$ $x = 4/3$

Table of Variation

x	$-\infty$	0		$4/3$	$+\infty$
$5x^3$	$-$	0	$+$		$+$
$3x-4$	$-$		$-$	0	$+$
$f'(x)$	$+$		$-$		$+$
$f(x)$	$\uparrow$		$\downarrow$		$\uparrow$

$f(0) = 0$ is a local maximum

$f(4/3) = 3 \times \left(\frac{4}{3}\right)^5 - 5\left(\frac{4}{3}\right)^4 = -3.16$ is a local minimum

$$f(x) = (x^2 - 4)^{2/3}$$
$$f'(x) = \frac{2}{3}(2x)(x^2 - 4)^{-1/3}$$
$$= \frac{4x}{3}(x^2 - 4)^{-1/3}$$
$$= \frac{4}{3} \frac{x}{\sqrt[3]{x^2 - 4}}$$

Critical points $\Rightarrow$ $-2, 0, 2$

x	$-\infty$		-2		0		2		∞
x		$-$		$-$	0	$+$		$+$	
$\sqrt[3]{x^2-4}$		$+$	0	$-$		$-$	0	$+$	
$f'(x)$		$-$		$+$	0	$-$		$+$	
$f(x)$		$\downarrow$		$\uparrow$		$\downarrow$		$\uparrow$	

$f'(x)$ is undefined at double lines

$f(-2) = 0$ is a local minimum

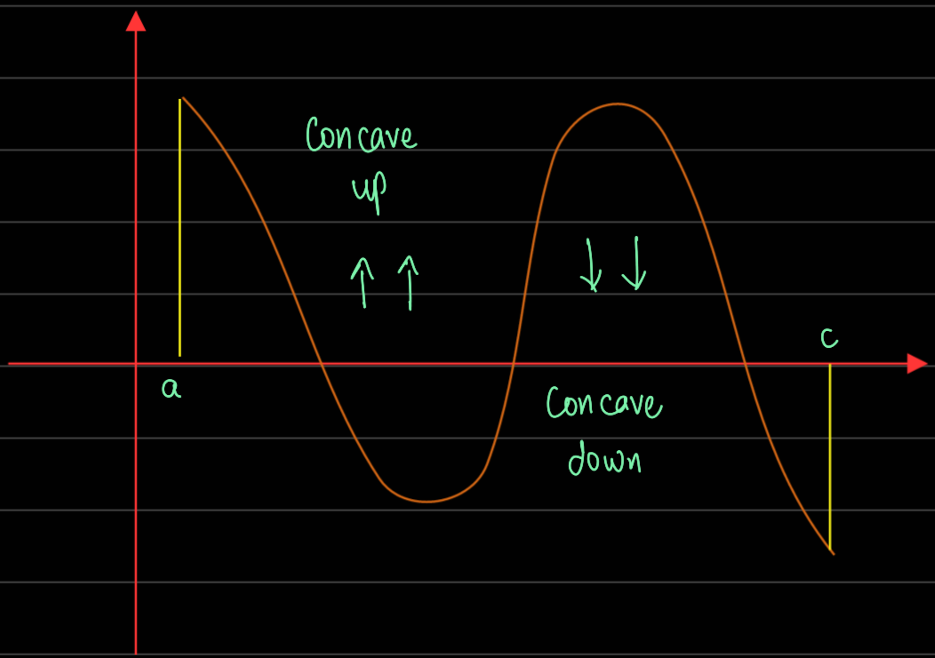
$f(0) = (-4)^{2/3}$ is a local maximum

$f(2) = 0$ is a local minimum

Concavity & the Second Derivative Test

A function is concave upwards on an open interval if the graph is open up on I

A function is concave downward on an open interval if the graph is open down on I



f is concave upwards on an open interval I if and only if $f''(x) > 0$ on I

f is concave downwards on an open interval I if and only if $f''(x) < 0$ on I

Inflection point

An inflection point is a point where the graph turns from concave up to concave down or vice versa.

Inflection points are the critical value of the derivative.

Example

Determine the open intervals where the function $f(x) = e^{\frac{-x^2}{2}}$ is concave upward or downward.

$$f'(x) = -xe^{\frac{-x^2}{2}}$$

$$f''(x) = -e^{\frac{-x^2}{2}} + x^2 e^{\frac{-x^2}{2}}$$

$$= e^{\frac{-x^2}{2}}(x^2 - 1)$$

$e^{\text{anything}} \Rightarrow +ve$

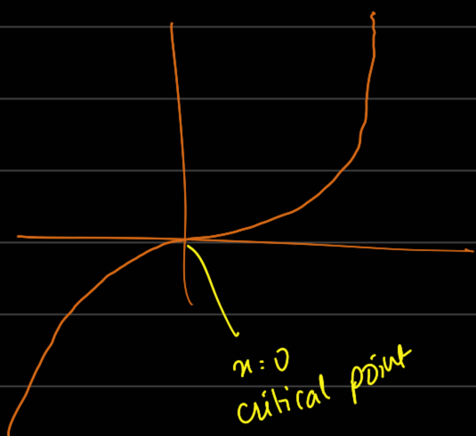
x	$-\infty$ -1	1	∞
$x^2 - 1$	$+$	$-$	$+$
$f''(x)$	$+$	$-$	$+$
$f(x)$	Concave Up	Down	Up

Concave up on $(-\infty, -1) \cup (1, \infty)$

Concave down on $(-1, 1)$

$f(x) = x^3$

$f'(x) = 3x^2$



Local maximum & minimum happen at critical points

But critical point isn't always local maximum / minimum

Second Derivative Test

Let f be such that $f'(c) = 0$

If $f''(c) > 0$ then $f(c)$ is a local minimum

If $f''(c) < 0$ then $f(c)$ is a local maximum

If $f''(c) = 0$ then the test is not conclusive

Example

Use the second derivative test to find the local extrema $f(x) = -3x^5 + 5x^3 - 9$

$$f'(x) = -15x^4 + 15x^2$$

$$= -15x^2(x^2 - 1)$$

$$f'(x) = 0$$

$x^2 - 1 = 0$ $\qquad$ $-15x^2 = 0$

$x = 1, -1$ $\qquad$ $x = 0$

$$x = -1, 0, 1$$

$$f''(x) = -60x^3 + 30x$$

$f''(-1) = 60 - 30 = 30$ $\qquad$ $f(-1) = -11$ is a local minimu

$f''(0) = 0 \Rightarrow$ Test is not condusive

$f''(1) = -60 + 30 = -30$ $\qquad$ $f(1) = -7$ is a local maximum