MATH 141 Tutorial 9 Solution

Problem 1

Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

Solution

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix},$$

the equation $\det(A - \lambda I) = 0$ becomes

$$-\lambda(5-\lambda) + 6 = 0 \quad \Longrightarrow \quad \lambda^2 - 5\lambda + 6 = 0$$

Factor:

$$(\lambda - 2)(\lambda - 3) = 0.$$

So the eigenvalues are 2 and 3.

Problem 2

Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & -5-\lambda & 0 \\ 1 & 8 & 1-\lambda \end{vmatrix}$$

$$= (-5-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (-5-\lambda)\left[(1-\lambda)^2 - 1\right]$$

$$= (-5-\lambda)\left[1 - 2\lambda + \lambda^2 - 1\right] = -(5+\lambda)\lambda[-2+\lambda] = 0$$

$$\Rightarrow \lambda = -5, 0, 2$$

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Problem 3

Let

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \textit{ and } \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(1) Are **u** and **v** eigenvectors of A?
(2) Show that 1 is an eigenvalue of A.

Solution

(1)

$$A\mathbf{u} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 5\mathbf{u}.$$

$$A\mathbf{v} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \neq \lambda\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus **u** is an eigenvector corresponding to an eigenvalue (5), but **v** is not an eigenvector of A, because $A\mathbf{v}$ is not a multiple of **v**. □

(2) The scalar 1 is an eigenvalue of A if and only if the equation

$$Ax = 1 \cdot x$$

has a nontrivial solution. The equation is equivalent to $Ax - x = 0$, or

$$(A - I)\, x = 0$$

To solve this homogeneous equation, form the matrix

$$(A - I) = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

The columns of $A - I$ are obviously linearly dependent, so $(A - I)\, x$ has nontrivial solutions. Thus 1 is an eigenvalue of A.
To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The general solution has the form $x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 1$. □

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