Practice Problems on Limits

I.1. Find the limit

$$\lim_{x\to 2} \frac{x^2-4}{x^2+x-6}.$$

Solution: First we evaluate the fraction at $x = 2$. We get $0/0$. Therefore, in order to find the limit, we need to simplify the fraction. Here to simplify our fraction we factor the top and bottom and cancel a similar factor, if there is one. We have

$$\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x+3)} = \lim_{x\to 2} \frac{x+2}{x+3} = \frac{2+2}{2+3} = \frac{4}{5},$$

where to find the last limit we evaluated the continuous function $\frac{x+2}{x+3}$ at $x = 2$.

I.2. Find the limit

$$\lim_{h\to 0} \frac{(1+h)^2-1}{h}.$$

Solution: The solution is similar to the one above. First we evaluate the fraction at $h = 0$. We get $0/0$. Therefore, in order to find the limit, we need to simplify the fraction. We have

$$\lim_{h\to 0} \frac{(1+h)^2-1}{h} = \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2+0 = 2.$$

I.3. Find the limit

$$\lim_{x\to 1} \frac{x^2-x}{x^2+2x-3}.$$

Solution: As before, when $x = 1$, the fraction takes the "value" $0/0$. However, we see that we can factor and cancel $(x-1)$, which will solve the problem of dividing by zero, i.e.,

$$\lim_{x\to 1} \frac{x^2-x}{x^2+2x-3} = \lim_{x\to 1} \frac{x(x-1)}{(x+3)(x-1)} = \lim_{x\to 1} \frac{x}{x+3} = \frac{1}{1+3} = \frac{1}{4}.$$

II.1. Find the limit

$$\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}.$$

Solution: When $x = 0$ the fraction becomes $0/0$. Here we will simplify the fraction by multiplying by the conjugate. This means that we multiply our expression by

$$\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}.$$

Then the numerator of the fraction simplify to x which will cancel with the x on the bottom, and the problem of dividing by 0 will disappear. In other words we have

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} &= \lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\\
&= \lim_{x\to 0}\frac{(\sqrt{x+2})^2-(\sqrt{2})^2}{x(\sqrt{x+2}+\sqrt{2})}\\
&= \lim_{x\to 0}\frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}\\
&= \lim_{x\to 0}\frac{x}{x(\sqrt{x+2}+\sqrt{2})}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}}\\
&= \frac{1}{\sqrt{0+2}+\sqrt{2}}\\
&= \frac{1}{2\sqrt{2}}.
\end{aligned}$$

II.2. Find the limit

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}.$$

Solution: We argue as in the previous example. We multiply by the conjugate, which is

$$\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2}.$$

We obtain

$$\begin{aligned}
\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2} &= \lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}\cdot\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2}\\
&= \lim_{x\to 1}\frac{(x-1)(\sqrt{x^2+3}+2)}{(x^2+3)-2^2}\\
&= \lim_{x\to 1}\frac{(x-1)(\sqrt{x^2+3}+2)}{x^2-1}\\
&= \lim_{x\to 1}\frac{(x-1)(\sqrt{x^2+3}+2)}{(x-1)(x+1)}\\
&= \lim_{x\to 1}\frac{\sqrt{x^2+3}+2}{x+1}\\
&= \frac{\sqrt{1^2+3}+2}{1+1}\\
&= 2.
\end{aligned}$$

III.1. Find the limit
$$\lim_{x\to 1^+} \frac{x-2}{x-1}.$$

Solution: We first evaluate the fraction $(x-2)/(x-1)$ at $x=1$. We get $-1/0$. This tells us that the limit $\lim_{x\to 1^+} \frac{x-2}{x-1}$ is either $+\infty$, $-\infty$, or it does not exist. We are approaching 1 from the right, which means that $x>1$. In this case $x-1$ is a positive number. Thus the bottom of the fraction is positive, and the top is close to -1, hence negative. Together the fraction is negative, which gives
$$\lim_{x\to 1^+} \frac{x-2}{x-1} = -\infty.$$

III.2. Find the limit
$$\lim_{x\to 2^-} \frac{x-1}{x^2-3x+2}.$$

Solution: The argument is similar to the one above. We first evaluate the fraction $(x-1)/(x^2-3x+2)$ at $x=2$. We get $1/0$. This tells us that the limit $\lim_{x\to 2^-} \frac{x-1}{x^2-3x+2}$ is either $+\infty$, $-\infty$, or it does not exist. We are approaching 2 from the left, which means that $x<2$, but really close to 2. In this case x^2-3x+2 is a negative number, since x^2-3x+2 is a parabola that opens up an has x-intercepts at the points 1 and 2. Thus the bottom of the fraction is negative, and the top is close to 1, hence positive. Together the fraction is negative, which gives
$$\lim_{x\to 2^-} \frac{x-1}{x^2-3x+2} = -\infty.$$

Remark We could first simplify the fraction $(x-1)/(x^2-3x+2)$ to $1/(x-2)$.

III.3. Find the limit
$$\lim_{x\to -1} \frac{x}{(x+1)^2}.$$

Solution: We first evaluate the fraction $x/(x+1)^2$ at $x=-1$. We get $-1/0$. Thus the limit $\lim_{x\to -1} \frac{x}{(x+1)^2}$ is either $+\infty$, $-\infty$, or it does not exist. Since the bottom of the fraction is always positive, as we are raising to the second power, and the top is negative, the fraction is negative. Hence
$$\lim_{x\to -1} \frac{x}{(x+1)^2} = -\infty.$$

III.4. Find the limit
$$\lim_{x\to -3} \frac{x+2}{x+3}.$$

Solution: Arguing as we did above we can show that
$$\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty \quad \text{and} \quad \lim_{x\to -3^-} \frac{x+2}{x+3} = +\infty.$$
Since the limit from the right is different then the limit from the left the full limit $\lim_{x\to -3} \frac{x+2}{x+3}$ does not exist.

IV.1. Find the limit

$$\lim_{x\to-\infty} \frac{x^3-2x^2+1}{x^4-2}.$$

Solution: When we deal with limits at infinity of a rational function (polynomial divided by a polynomial) we first look at the degrees of polynomials on top and bottom of the fraction. If the degree of the polynomial in the numerator is smaller than the degree of the polynomial in the denominator, the limit is zero. Hence

$$\lim_{x\to-\infty} \frac{x^3-2x^2+1}{x^4-2} = 0,$$

since the degree of the polynomial of top is 3 and on the bottom 4.

IV.2. Find the limit

$$\lim_{x\to-\infty} \frac{-x^5-x^3+x-3}{2x^3+3x-2}.$$

Solution: As before, since we have a limit of a rational function at infinity, we look at the degrees of the polynomials defining our rational function. The degree of the polynomial in the numerator is 5, and the degree of the polynomial in the denominator is 3. Since the degree on top is bigger than on the bottom, the limit is either $=\infty$ or $-\infty$. To check which "infinity" it is, we drop the lower order terms, simplify the fraction, and hopefully are able to determine what the answer is. We have

$$\begin{aligned}\lim_{x\to-\infty} \frac{-x^5-x^3+x-3}{2x^3+3x-2} &= \lim_{x\to-\infty} \frac{-x^5}{2x^3}\\ &= \lim_{x\to-\infty} -\frac{1}{2}x^2\\ &= -\infty.\end{aligned}$$

IV.3. Find the limit

$$\lim_{x\to\infty} \frac{x^4-5x^2+x-1}{3x^4+x-1}.$$

Solution: When the degree of the polynomial in the numerator is the same as the degree of the polynomial in the denominator, to find the limit at infinity, we just divide the coefficients in front of the highest powers of the polynomials, i.e.,

$$\lim_{x\to\infty} \frac{x^4-5x^2+x-1}{3x^4+x-1} = \frac{1}{3}.$$

Remark. When trying to find the limit at infinity of a polynomial divided by a polynomial the argument of forgetting the lower order terms always works, no matter what the degrees of the polynomials are.

IV.4. Find the limit

$$\lim_{x\to-\infty} \frac{x^3+\sqrt{4x^6+4}}{5x^3+2x}.$$

Solution: We want to divide by the fastest growing power of x in the denominator, so when we take the limit when $x\to-\infty$, the bottom of the fraction will converge. The fastest growing term is x^3. We will divide the top and bottom of the fraction by x^3. Before we do that we need to simplify the term with the square root. Since x approaches $-\infty$, we have $x < 0$. Then $|x| = -x$ and

$$\sqrt{x^6} = \sqrt{(x^3)^2} = |x^3| = |x|^3 = (-x)^3 = -x^3.$$

Hence

$$\begin{aligned}\sqrt{4x^6+4} &= \sqrt{4x^6(1+1/x^6)} \\ &= \sqrt{4x^6}\sqrt{1+1/x^6} \\ &- 2x^3\sqrt{1+1/x^6}.\end{aligned}$$

Coming back to our limit, we now have

$$\begin{aligned}\lim_{x\to-\infty} \frac{x^3+\sqrt{4x^6+4}}{5x^3+2x} &= \lim_{x\to-\infty} \frac{x^3-2x^3\sqrt{1+1/x^6}}{5x^3-2x} \\ &= \lim_{x\to-\infty} \frac{1-2\sqrt{1+1/x^6}}{5-2/x^2} \\ &= \frac{1-2\sqrt{1}}{5} \\ &= -\frac{1}{5}.\end{aligned}$$

Remark. We could first forget about the lower order terms, and then argue as above. Forgetting the lower order terms gives us

$$\lim_{x\to-\infty} \frac{x^3+\sqrt{4x^6+4}}{5x^3+2x} = \lim_{x\to-\infty} \frac{x^3+\sqrt{4x^6}}{5x^3}.$$

Now, using the fact that $\sqrt{x^6} = -x^3$ for $x < 0$, we obtain

$$\begin{aligned}\lim_{x\to-\infty} \frac{x^3+\sqrt{4x^6+4}}{5x^3+2x} &= \lim_{x\to-\infty} \frac{x^3+\sqrt{4x^6}}{5x^3} \\ &= \lim_{x\to-\infty} \frac{x^3-2x^3}{5x^3} \\ &= \lim_{x\to-\infty} \frac{-1}{5} \\ &= -\frac{1}{5}.\end{aligned}$$

V.1. Find the limit

$$\lim_{x\to-\infty} \frac{2x+\sqrt{x^2-x-1}}{x-5}.$$

Solution: Here one of the possible arguments is to drop the lower order terms. Moreover, since we are taking the limit when x goes to $-\infty$, we may assume that our function is defined at negative numbers only. Then $\sqrt{x^2} = -x$. Therefore

$$\begin{aligned}
\lim_{x\to-\infty} \frac{2x + \sqrt{x^2 - x - 1}}{x - 5} &= \lim_{x\to-\infty} \frac{2x + \sqrt{x^2}}{x} \\
&= \lim_{x\to-\infty} \frac{2x - x}{x} \\
&= \lim_{x\to\infty 1} \\
&= 1.
\end{aligned}$$

V.2. Find the limit

$$\lim_{x\to\infty} \frac{e^x + e^{-x}}{2 + e^x}.$$

Solution: Here the fastest growing term is e^x. We substitute $z = e^x$. Since $e^x \to \infty$ when $x \to \infty$, we deduce that $z \to \infty$. Now we need to write the fraction $(e^x + e^{-x})/(2 + e^x)$ in terms of z only. Since $z = e^x$ we have

$$e^{-x} = \frac{1}{e^x} = \frac{1}{z}.$$

Our limit problem becomes

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^x + e^{-x}}{2 + e^x} &= \lim_{z\to\infty} \frac{z + 1/z}{2 + z} \\
&= \lim_{z\to\infty} \frac{1 + 1/z^2}{2/z + 1} \\
&= 1,
\end{aligned}$$

where to get from the second to the third limit we divided the top and bottom of the fraction by z.

V.3. Find the limit

$$\lim_{x\to-\infty} (x^5 - x^4 + 3x).$$

Solution: This is a limit at infinite of a polynomial function. Hence we can drop the lower order terms to find the limit. We get

$$\begin{aligned}
\lim_{x\to-\infty} (x^5 - x^4 + 3x) &= \lim_{x\to-\infty} x^5 \\
&= (-\infty)^5 \\
&= -\infty.
\end{aligned}$$

V.4. Find the limit

$$\lim_{x\to\infty} \frac{2x^{5/2} - 2x^{3/2} + x}{3x + x^{5/2} - 4}.$$

Solution: The function in the limit is not a rational function, but it only involves powers of x, so we know which terms are growing the fastest. Since the limit is at infinity, we drop all but the fastest growing terms. In other words we drop the lower oder terms. We get

$$\begin{aligned}\lim_{x\to\infty}\frac{2x^{5/2}-2x^{3/2}+x}{3x+x^{5/2}-4} &= \lim_{x\to\infty}\frac{2x^{5/2}}{x^{5/2}}\\ &= 2.\end{aligned}$$

VI.1. Find the limit

$$\lim_{x\to-\infty}\tan^{-1}(x^2+1).$$

Solution: We first notice that $x^2+1\to\infty$ when $x\to-\infty$. Therefore, if we substitute $z=x^2+1$, our limit takes the form

$$\begin{aligned}\lim_{x\to-\infty}\tan^{-1}(x^2+1) &= \lim_{z\to\infty}\tan^{-1}(z)\\ &= \frac{\pi}{2}.\end{aligned}$$

VI.2. Find the limit

$$\lim_{x\to\infty}\ln\left(\frac{1}{x}\right).$$

Solution: We substitute $z=1/x$. Since $1/x\to 0$ and $1/x>0$ when $x\to\infty$, we get that $z\to 0^+$. Therefore

$$\lim_{x\to\infty}\ln\left(\frac{1}{x}\right)=\lim_{z\to 0^+}\ln z=-\infty.$$

VII.1. Use the Squeeze Theorem to find the limit

$$\lim_{x\to\infty}\frac{\cos x}{x-1}.$$

Solution: We see that $1/(x-1)$ converges to 0 when $x\to\infty$ and $\cos x$ is a bounded function. This tells us that the limit $\lim_{x\to\infty}\frac{\cos x}{x-1}=0$. If we want to be precise we need to use the Squeeze Theorem. Since

$$-1\le\cos x\le 1,$$

we multiply the above double inequality by $1/(x-1)$, which is positive when x is "close" to $+\infty$, to get that

$$-\frac{1}{x-1}\le\frac{\cos x}{x-1}\le\frac{1}{x-1}. \tag{1}$$

Moreover

$$\lim_{x\to\infty}-\frac{1}{x-1}=0 \quad\text{and}\quad \lim_{x\to\infty}\frac{1}{x-1}=0. \tag{2}$$

Therefore (1) and (2) together with the Squeeze Theorem give

$$\lim_{x\to\infty}\frac{\cos x}{x-1}=0.$$

VII.2. Use the Squeeze Theorem to find the limit

$$\lim_{x\to 0} x^2(1+\sin(1/x)).$$

Solution: As above, we see that $x^2 \to 0$ when $x \to 0$, hence if we can prove that the function $1+\sin(1/x)$ is bounded, the Squeeze Theorem will imply that

$$\lim_{x\to 0} x^2(1+\sin(1/x)) = 0.$$

Since

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

adding 1 gives

$$0 \le 1+\sin\left(\frac{1}{x}\right) \le 2.$$

Thus the function $1+\sin(1/x)$ is bounded between 0 and 2.

VIII. Use the Intermediate Value Theorem to show that the equation

$$x^3 = 3x - 1$$

has a solution in the interval $(-2, 0)$.

Solution: The equation $x^3 = 3x - 1$ is equivalent to $x^3 - 3x + 1 = 0$. If we put $f(x) = x^3 - 3x + 1$, we need to show that there is a point x in the interval $(-2, 0)$ at which our function f takes the value 0, i.e., $f(x) = 0$. We first compute the values of f at the end-points of the intevral $(-2, 0)$. We have

$$f(-2) = (-2)^3 - 3(-2) + 1 = -1$$
$$f(0) = 0^3 - 3\cdot 0 + 1 = 1.$$

By the Intermediate Value Theorem f, as a continuous function, takes all values between $f(-2) = -1$ and $f(0) = 1$ on the inteval $(-2, 0)$, in particular it takes value 0. This is what we wanted to show.

IX. Use the Intermediate Value Theorem to show that the equation

$$\sin(5x) = x^2 - 1$$

has a solution in the interval $(0, 2)$.

Solution: As before, the equation $\sin(5x) = x^2 - 1$ is equivalent to the equation $x^2 - 1 - \sin(5x) = 0$. We define $f(x) = x^2 - 1 - \sin(5x)$. Hence we need to show that there is a point x in the interval $(0, 2)$ at which our function f takes the value 0, i.e., $f(x) = 0$. We compute the values of f at the end-points of the interval $(0, 2)$. We have

$$f(0) = 0^2 - 1 - \sin(0) = -1$$
$$f(2) = 2^2 - 1 - \sin(10) = 3 - \sin(10) \ge 2 > 0.$$

By the Intermediate Value Theorem f, as a continuous function, takes all values between $f(0) = -1$ and $f(2) = 3 - \sin(10) > 0$ on the interval $(-2, 0)$, in particular it takes value 0.

XI. Prove, using the ϵ-δ definition of a limit, that

$$\lim_{x\to 2}(8-3x)=2.$$

Solution: We need to show that for any $\epsilon>0$ there exists $\delta>0$ such that

$$\text{if}\quad |x-2|<\delta\quad\text{then}\quad |(8-3x)-2|<\epsilon.$$

Before we write th formal proof we need to simplify the inequality $|(8-3x)-2|<\epsilon$ to determine δ needs to be as a function of ϵ. We see that $|(8-3x)-2|<\epsilon$ is equivalent to the inequality

$$|6-3x|<\epsilon,$$

which simplifies to

$$|-3(x-2)|<\epsilon,$$

and when we factor $|-3|=3$ we have

$$3|x-2|<\epsilon.$$

We divide by 3 to get that $|x-2|<\epsilon/3$. This tells us that we need δ to be $\epsilon/3$. We are ready to write the proof.

Let $\epsilon>0$. Choose $\delta=\epsilon/3$. We need to show that

$$\text{if}\quad |x-2|<\delta\quad\text{then}\quad |(8-3x)-2|<\epsilon.$$

Therefore, suppose that $|x-2|<\delta$. Since $\delta=\epsilon/3$ we have

$$|x-2|<\frac{\epsilon}{3}.$$

(Now we "reverse the direction" in the argument where we were trying to determine δ) We multiply by 3 to get that

$$3|x-2|<\epsilon.$$

We put $3=|-3|$ under the absolute values

$$|-3(x-2)|<\epsilon.$$

Therefore

$$|6-3x|<\epsilon.$$

And finally we get by writing $6=8-2$ that

$$|(8-3x)-2|<\epsilon,$$

which is what we wanted to show.

XII. Use the definition of the derivative of a function to find $f'(2)$, where $f(x)=x^2-x+1$.

Solution: Recall that the derivative $f'(a)$ of a function f at a point a, if it exists, is given by the limit

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Therefore, we need to find the limit

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}.$$

We compute

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{[(2+h)^2-(2+h)+1]-[2^2-2+1]}{h} \\
&= \lim_{h\to 0} \frac{4+4h-h^2-2-h+1-3}{h} \\
&= \lim_{h\to 0} \frac{4h-h^2-h}{h} \\
&= \lim_{h\to 0} \frac{3h-h^2}{h} \\
&= \lim_{h\to 0} 3-h \\
&= 3.
\end{aligned}$$

Hence $f'(2)=3$.

XIII. Use the definition of the derivative of a function to find $f'(0)$, where $f(x)=\dfrac{2x+3}{x+1}$.

Solution: As above, we need to find the limit

$$\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0} \frac{f(h)-f(0)}{h}.
\end{aligned}$$

We compute

$$\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{2h+3}{h+1}-\frac{2\cdot 0+3}{0+1}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2h+3}{h+1}-3}{h} \\
&= \lim_{h\to 0} \frac{2h+3-3(h+1)}{h(h+1)} \\
&= \lim_{h\to 0} \frac{-h)}{h(h+1)} \\
&= \lim_{h\to 0} \frac{-1)}{h+1} \\
&= -1.
\end{aligned}$$

Hence $f'(0)=-1$.