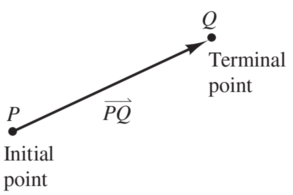
Section 11.1 Vectors in the Plane

- **Write the component form of a vector.**
- **Perform vector operations and interpret the results geometrically.**
- **Write a vector as a linear combination of standard unit vectors.**
- **Use vectors to solve problems involving force or velocity.**

Component Form of a Vector

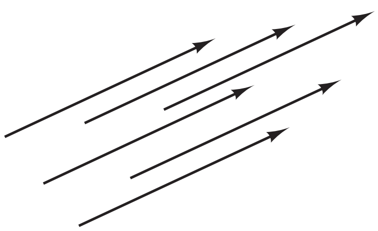


A directed line segment
Figure 11.1

Many quantities in geometry and physics, such as area, volume, temperature, mass, and time, can be characterized by a single real number scaled to appropriate units of measure. These are called **scalar quantities,** and the real number associated with each is called a **scalar.**

Other quantities, such as force, velocity, and acceleration, involve both magnitude and direction and cannot be characterized completely by a single real number. A **directed line segment** is used to represent such a quantity, as shown in Figure 11.1. The directed line segment $\overrightarrow{PQ}$ has **initial point** P and **terminal point** Q, and its **length** (or **magnitude**) is denoted by $\|\overrightarrow{PQ}\|$. Directed line segments that have the same length and direction are **equivalent,** as shown in Figure 11.2. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a **vector in the plane** and is denoted by $\mathbf{v} = \overrightarrow{PQ}$. In typeset material, vectors are usually denoted by lowercase, boldface letters such as **u**, **v**, and **w**. When written by hand, however, vectors are often denoted by letters with arrows above them, such as $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Be sure you see that a vector in the plane can be represented by many different directed line segments—all pointing in the same direction and all of the same length.



Equivalent directed line segments
Figure 11.2

EXAMPLE 1 Vector Representation by Directed Line Segments

Let **v** be represented by the directed line segment from (0, 0) to (3, 2), and let **u** be represented by the directed line segment from (1, 2) to (4, 4). Show that **v** and **u** are equivalent.

Solution Let $P(0, 0)$ and $Q(3, 2)$ be the initial and terminal points of **v**, and let $R(1, 2)$ and $S(4, 4)$ be the initial and terminal points of **u**, as shown in Figure 11.3. You can use the Distance Formula to show that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the *same length.*

$$\|\overrightarrow{PQ}\| = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13} \qquad \text{Length of } \overrightarrow{PQ}$$
$$\|\overrightarrow{RS}\| = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{13} \qquad \text{Length of } \overrightarrow{RS}$$

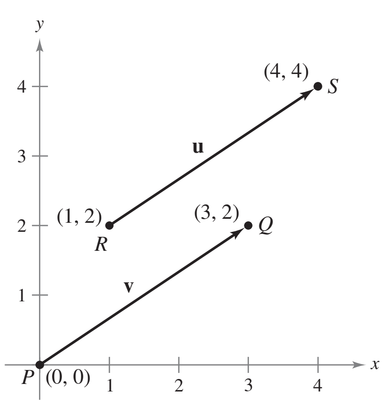
Both line segments have the *same direction*, because they both are directed toward the upper right on lines having the same slope.

$$\text{Slope of } \overrightarrow{PQ} = \frac{2-0}{3-0} = \frac{2}{3}$$

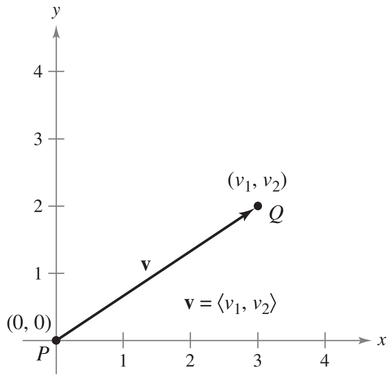
and

$$\text{Slope of } \overrightarrow{RS} = \frac{4-2}{4-1} = \frac{2}{3}$$

Because $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same length and direction, you can conclude that the two vectors are equivalent. That is, **v** and **u** are equivalent.



The vectors **u** and **v** are equivalent.
Figure 11.3



The standard position of a vector
Figure 11.4

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments such as those shown in Figure 11.3. This representation of $\mathbf{v}$ is said to be in **standard position.** A directed line segment whose initial point is the origin can be uniquely represented by the coordinates of its terminal point $Q(v_1, v_2)$, as shown in Figure 11.4.

Definition of Component Form of a Vector in the Plane

If $\mathbf{v}$ is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **component form of v** is given by

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

The coordinates v_1 and v_2 are called the **components of v.** If both the initial point and the terminal point lie at the origin, then $\mathbf{v}$ is called the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

This definition implies that two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

The following procedures can be used to convert directed line segments to component form or vice versa.

NOTE It is important to understand that a vector represents a *set* of directed line segments (each having the same length and direction). In practice, however, it is common not to distinguish between a vector and one of its representatives.

1. If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are the initial and terminal points of a directed line segment, the component form of the vector $\mathbf{v}$ represented by $\overrightarrow{PQ}$ is $\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$. Moreover, the **length** (or **magnitude**) **of v** is

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

Length of a vector

2. If $\mathbf{v} = \langle v_1, v_2 \rangle$, $\mathbf{v}$ can be represented by the directed line segment, in standard position, from $P(0, 0)$ to $Q(v_1, v_2)$.

The length of $\mathbf{v}$ is also called the **norm of v**. If $\|\mathbf{v}\| = 1$, $\mathbf{v}$ is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v}$ is the zero vector $\mathbf{0}$.

EXAMPLE 2 Finding the Component Form and Length of a Vector

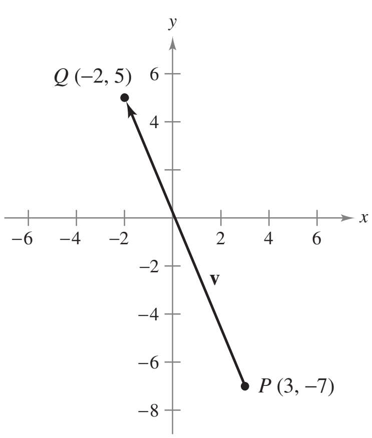
Find the component form and length of the vector $\mathbf{v}$ that has initial point $(3, -7)$ and terminal point $(-2, 5)$.

Solution Let $P(3, -7) = (p_1, p_2)$ and $Q(-2, 5) = (q_1, q_2)$. Then the components of $\mathbf{v} = \langle v_1, v_2 \rangle$ are

$$v_1 = q_1 - p_1 = -2 - 3 = -5$$
$$v_2 = q_2 - p_2 = 5 - (-7) = 12.$$

So, as shown in Figure 11.5, $\mathbf{v} = \langle -5, 12 \rangle$, and the length of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13.$$



Component form of $\mathbf{v}$: $\mathbf{v} = \langle -5, 12 \rangle$
Figure 11.5

Vector Operations

Definitions of Vector Addition and Scalar Multiplication

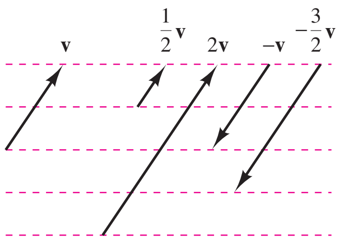
Let $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ be vectors and let c be a scalar.

1. The **vector sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle$.
2. The **scalar multiple** of c and $\mathbf{u}$ is the vector $c\mathbf{u} = \langle cu_1, cu_2\rangle$.
3. The **negative** of $\mathbf{v}$ is the vector

$$-\mathbf{v} = (-1)\mathbf{v} = \langle -v_1, -v_2\rangle.$$

4. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2\rangle.$$



The scalar multiplication of **v**
Figure 11.6

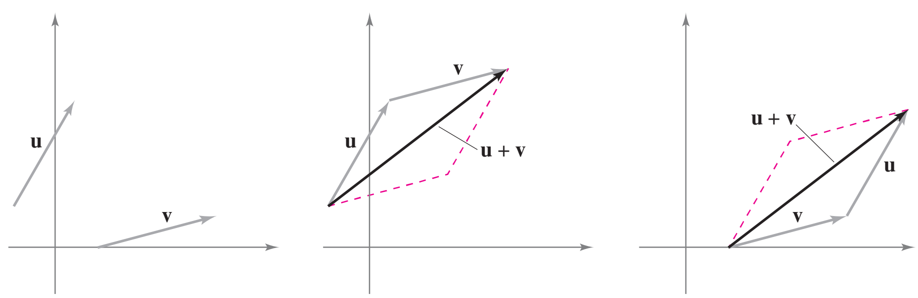
Geometrically, the scalar multiple of a vector $\mathbf{v}$ and a scalar c is the vector that is $|c|$ times as long as $\mathbf{v}$, as shown in Figure 11.6. If c is positive, $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If c is negative, $c\mathbf{v}$ has the opposite direction.

The sum of two vectors can be represented geometrically by positioning the vectors (without changing their magnitudes or directions) so that the initial point of one coincides with the terminal point of the other, as shown in Figure 11.7. The vector $\mathbf{u} + \mathbf{v}$, called the **resultant vector,** is the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides.


The Granger Collection

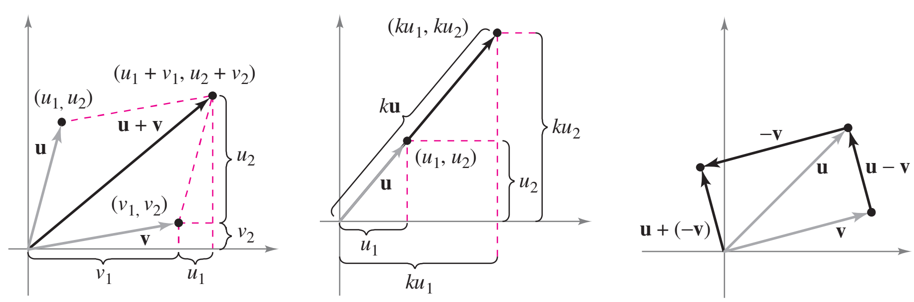
Isaac William Rowan Hamilton (1805–1865)

Some of the earliest work with vectors was done by the Irish mathematician William Rowan Hamilton. Hamilton spent many years developing a system of vector-like quantities called *quaternions*. Although Hamilton was convinced of the benefits of quaternions, the operations he defined did not produce good models for physical phenomena. It wasn't until the latter half of the nineteenth century that the Scottish physicist James Maxwell (1831–1879) restructured Hamilton's quaternions in a form useful for representing physical quantities such as force, velocity, and acceleration.



To find $\mathbf{u} + \mathbf{v}$, (1) move the initial point of **v** to the terminal point of **u**, or (2) move the initial point of **u** to the terminal point of **v**.
Figure 11.7

Figure 11.8 shows the equivalence of the geometric and algebraic definitions of vector addition and scalar multiplication, and presents (at far right) a geometric interpretation of $\mathbf{u} - \mathbf{v}$.



Vector addition Scalar multiplication Vector subtraction
Figure 11.8

EXAMPLE 3 Vector Operations

Given $\mathbf{v} = \langle -2, 5\rangle$ and $\mathbf{w} = \langle 3, 4\rangle$, find each of the vectors.

a. $\frac{1}{2}\mathbf{v}$ **b.** $\mathbf{w} - \mathbf{v}$ **c.** $\mathbf{v} + 2\mathbf{w}$

Solution

a. $\frac{1}{2}\mathbf{v} = \left\langle \frac{1}{2}(-2), \frac{1}{2}(5)\right\rangle = \left\langle -1, \frac{5}{2}\right\rangle$

b. $\mathbf{w} - \mathbf{v} = \langle w_1 - v_1, w_2 - v_2\rangle = \langle 3 - (-2), 4 - 5\rangle = \langle 5, -1\rangle$

c. Using $2\mathbf{w} = \langle 6, 8\rangle$, you have

$$\begin{aligned}\mathbf{v} + 2\mathbf{w} &= \langle -2, 5\rangle + \langle 6, 8\rangle\\ &= \langle -2 + 6, 5 + 8\rangle\\ &= \langle 4, 13\rangle.\end{aligned}$$

Vector addition and scalar multiplication share many properties of ordinary arithmetic, as shown in the following theorem.

THEOREM 11.1 Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutative Property
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ Associative Property
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ Additive Identity Property
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ Additive Inverse Property
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ Distributive Property
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributive Property
8. $1(\mathbf{u}) = \mathbf{u}$, $0(\mathbf{u}) = \mathbf{0}$

Proof The proof of the *Associative Property* of vector addition uses the Associative Property of addition of real numbers.

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= [\langle u_1, u_2\rangle + \langle v_1, v_2\rangle] + \langle w_1, w_2\rangle\\ &= \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle\\ &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2\rangle\\ &= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\rangle\\ &= \langle u_1, u_2\rangle + \langle v_1 + w_1, v_2 + w_2\rangle = \mathbf{u} + (\mathbf{v} + \mathbf{w})\end{aligned}$$

Similarly, the proof of the *Distributive Property* of vectors depends on the Distributive Property of real numbers.

$$\begin{aligned}(c + d)\mathbf{u} &= (c + d)\langle u_1, u_2\rangle\\ &= \langle (c + d)u_1, (c + d)u_2\rangle\\ &= \langle cu_1 + du_1, cu_2 + du_2\rangle\\ &= \langle cu_1, cu_2\rangle + \langle du_1, du_2\rangle = c\mathbf{u} + d\mathbf{u}\end{aligned}$$

The other properties can be proved in a similar manner.

The Granger Collection

Emmy Noether (1882–1935)

One person who contributed to our knowledge of axiomatic systems was the German mathematician Emmy Noether. Noether is generally recognized as the leading woman mathematician in recent history.

Any set of vectors (with an accompanying set of scalars) that satisfies the eight properties given in Theorem 11.1 is a **vector space.*** The eight properties are the *vector space axioms*. So, this theorem states that the set of vectors in the plane (with the set of real numbers) forms a vector space.

THEOREM 11.2 Length of a Scalar Multiple

Let $\mathbf{v}$ be a vector and let c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|.$$ $|c|$ is the absolute value of c.

Proof Because $c\mathbf{v} = \langle cv_1, cv_2\rangle$, it follows that

$$\begin{aligned}\|c\mathbf{v}\| = \|\langle cv_1, cv_2\rangle\| &= \sqrt{(cv_1)^2 + (cv_2)^2}\\ &= \sqrt{c^2v_1^2 + c^2v_2^2}\\ &= \sqrt{c^2(v_1^2 + v_2^2)}\\ &= |c|\sqrt{v_1^2 + v_2^2}\\ &= |c|\,\|\mathbf{v}\|.\end{aligned}$$

FOR FURTHER INFORMATION For more information on Emmy Noether, see the article "Emmy Noether, Greatest Woman Mathematician" by Clark Kimberling in *The Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given vector. The following theorem gives a procedure for doing this.

THEOREM 11.3 Unit Vector in the Direction of v

If $\mathbf{v}$ is a nonzero vector in the plane, then the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

has length 1 and the same direction as $\mathbf{v}$.

Proof Because $1/\|\mathbf{v}\|$ is positive and $\mathbf{u} = (1/\|\mathbf{v}\|)\mathbf{v}$, you can conclude that $\mathbf{u}$ has the same direction as $\mathbf{v}$. To see that $\|\mathbf{u}\| = 1$, note that

$$\begin{aligned}\|\mathbf{u}\| &= \left\|\left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}\right\|\\ &= \left|\frac{1}{\|\mathbf{v}\|}\right|\,\|\mathbf{v}\|\\ &= \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\|\\ &= 1.\end{aligned}$$

So, $\mathbf{u}$ has length 1 and the same direction as $\mathbf{v}$.

In Theorem 11.3, $\mathbf{u}$ is called a **unit vector in the direction of v.** The process of multiplying $\mathbf{v}$ by $1/\|\mathbf{v}\|$ to get a unit vector is called **normalization of v.**

* *For more information about vector spaces, see* Elementary Linear Algebra, *Fifth Edition, by Larson, Edwards, and Falvo (Boston: Houghton Mifflin Company, 2004).*

EXAMPLE 4 Finding a Unit Vector

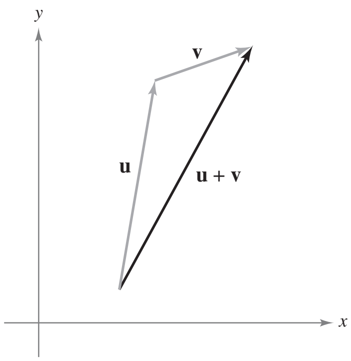
Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5\rangle$ and verify that it has length 1.

Solution From Theorem 11.3, the unit vector in the direction of $\mathbf{v}$ is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5\rangle}{\sqrt{(-2)^2 + (5)^2}} = \frac{1}{\sqrt{29}}\langle -2, 5\rangle = \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.$$

This vector has length 1, because

$$\sqrt{\left(\frac{-2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{\frac{4}{29} + \frac{25}{29}} = \sqrt{\frac{29}{29}} = 1.$$



Triangle inequality
Figure 11.9

Generally, the length of the sum of two vectors is not equal to the sum of their lengths. To see this, consider the vectors $\mathbf{u}$ and $\mathbf{v}$ as shown in Figure 11.9. By considering $\mathbf{u}$ and $\mathbf{v}$ as two sides of a triangle, you can see that the length of the third side is $\|\mathbf{u} + \mathbf{v}\|$, and you have

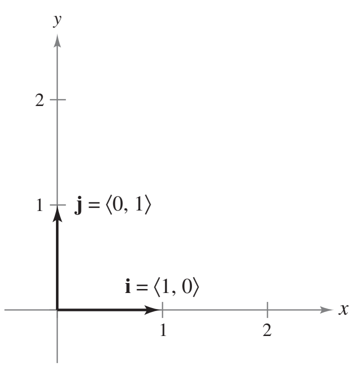
$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Equality occurs only if the vectors $\mathbf{u}$ and $\mathbf{v}$ have the *same direction*. This result is called the **triangle inequality** for vectors. (You are asked to prove this in Exercise 89, Section 11.3.)

Standard Unit Vectors

The unit vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are called the **standard unit vectors** in the plane and are denoted by

$$\mathbf{i} = \langle 1, 0\rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1\rangle$$ Standard unit vectors



Standard unit vectors **i** and **j**
Figure 11.10

as shown in Figure 11.10. These vectors can be used to represent any vector uniquely, as follows.

$$\mathbf{v} = \langle v_1, v_2\rangle = \langle v_1, 0\rangle + \langle 0, v_2\rangle = v_1\langle 1, 0\rangle + v_2\langle 0, 1\rangle = v_1\mathbf{i} + v_2\mathbf{j}$$

The vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is called a **linear combination** of $\mathbf{i}$ and $\mathbf{j}$. The scalars v_1 and v_2 are called the **horizontal** and **vertical components of v.**

EXAMPLE 5 Writing a Linear Combination of Unit Vectors

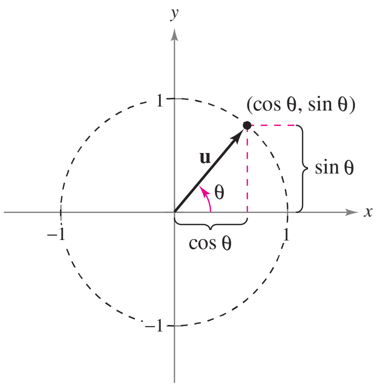
Let $\mathbf{u}$ be the vector with initial point $(2, -5)$ and terminal point $(-1, 3)$, and let $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Write each vector as a linear combination of $\mathbf{i}$ and $\mathbf{j}$.

a. $\mathbf{u}$ **b.** $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$

Solution

a. $\mathbf{u} = \langle q_1 - p_1, q_2 - p_2\rangle$
$= \langle -1 - 2, 3 - (-5)\rangle$
$= \langle -3, 8\rangle = -3\mathbf{i} + 8\mathbf{j}$

b. $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v} = 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j})$
$= -6\mathbf{i} + 16\mathbf{j} - 6\mathbf{i} + 3\mathbf{j}$
$= -12\mathbf{i} + 19\mathbf{j}$



The angle θ from the positive x-axis to the vector **u**

Figure 11.11

If **u** is a unit vector and θ is the angle (measured counterclockwise) from the positive x-axis to **u**, then the terminal point of **u** lies on the unit circle, and you have

$$\mathbf{u} = \langle \cos \theta, \sin \theta \rangle = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \qquad \text{Unit vector}$$

as shown in Figure 11.11. Moreover, it follows that any other nonzero vector **v** making an angle θ with the positive x-axis has the same direction as **u**, and you can write

$$\mathbf{v} = \|\mathbf{v}\|\langle \cos \theta, \sin \theta \rangle = \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j}.$$

EXAMPLE 6 Writing a Vector of Given Magnitude and Direction

The vector **v** has a magnitude of 3 and makes an angle of $30° = \pi/6$ with the positive x-axis. Write **v** as a linear combination of the unit vectors **i** and **j**.

Solution Because the angle between **v** and the positive x-axis is $\theta = \pi/6$, you can write the following.

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j} \\ &= 3 \cos \frac{\pi}{6}\mathbf{i} + 3 \sin \frac{\pi}{6}\mathbf{j} \\ &= \frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\end{aligned}$$

Applications of Vectors

Vectors have many applications in physics and engineering. One example is force. A vector can be used to represent force because force has both magnitude and direction. If two or more forces are acting on an object, then the **resultant force** on the object is the vector sum of the vector forces.

EXAMPLE 7 Finding the Resultant Force

Two tugboats are pushing an ocean liner, as shown in Figure 11.12. Each boat is exerting a force of 400 pounds. What is the resultant force on the ocean liner?

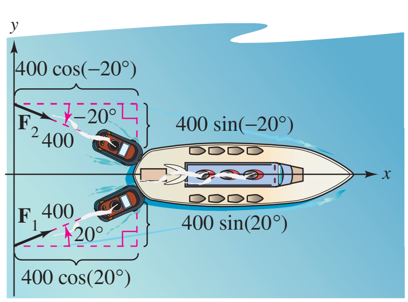
Solution Using Figure 11.12, you can represent the forces exerted by the first and second tugboats as

$$\begin{aligned}\mathbf{F}_1 &= 400\langle \cos 20°, \sin 20° \rangle \\ &= 400 \cos(20°)\mathbf{i} + 400 \sin(20°)\mathbf{j} \\ \mathbf{F}_2 &= 400\langle \cos(-20°), \sin(-20°) \rangle \\ &= 400 \cos(20°)\mathbf{i} - 400 \sin(20°)\mathbf{j}.\end{aligned}$$

The resultant force on the ocean liner is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 \\ &= [400 \cos(20°)\mathbf{i} + 400 \sin(20°)\mathbf{j}] + [400 \cos(20°)\mathbf{i} - 400 \sin(20°)\mathbf{j}] \\ &= 800 \cos(20°)\mathbf{i} \\ &\approx 752\mathbf{i}.\end{aligned}$$

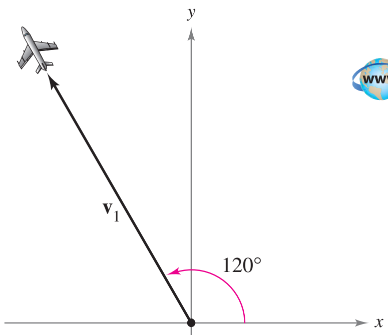
So, the resultant force on the ocean liner is approximately 752 pounds in the direction of the positive x-axis.



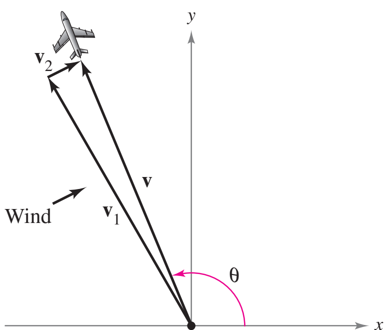
The resultant force on the ocean liner that is exerted by the two tugboats.

Figure 11.12

In surveying and navigation, a bearing is a direction that measures the acute angle that a path or line of sight makes with a fixed north-south line. In air navigation, bearings are measured in degrees clockwise from north.



(a) Direction without wind



(b) Direction with wind
Figure 11.13

EXAMPLE 8 Finding a Velocity

An airplane is traveling at a fixed altitude with a negligible wind factor. The airplane is traveling at a speed of 500 miles per hour with a bearing of 330°, as shown in Figure 11.13(a). As the airplane reaches a certain point, it encounters wind with a velocity of 70 miles per hour in the direction N 45° E (45° east of north), as shown in Figure 11.13(b). What are the resultant speed and direction of the airplane?

Solution Using Figure 11.13(a), represent the velocity of the airplane (alone) as

$$\mathbf{v}_1 = 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j}.$$

The velocity of the wind is represented by the vector

$$\mathbf{v}_2 = 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}.$$

The resultant velocity of the airplane (in the wind) is

$$\begin{aligned}\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 &= 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j} + 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}\\ &\approx -200.5\mathbf{i} + 482.5\mathbf{j}.\end{aligned}$$

To find the resultant speed and direction, write $\mathbf{v} = \|\mathbf{v}\|(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$. Because $\|\mathbf{v}\| \approx \sqrt{(-200.5)^2 + (482.5)^2} \approx 522.5$, you can write

$$\mathbf{v} \approx 522.5\left(\frac{-200.5}{522.5}\mathbf{i} + \frac{482.5}{522.5}\mathbf{j}\right) \approx 522.5\,[\cos(112.6°)\mathbf{i} + \sin(112.6°)\mathbf{j}].$$

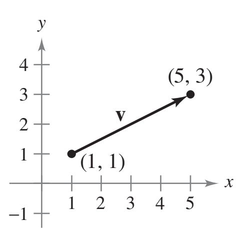
The new speed of the airplane, as altered by the wind, is approximately 522.5 miles per hour in a path that makes an angle of 112.6° with the positive x-axis.

Exercises for Section 11.1

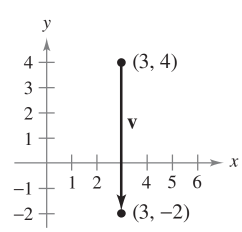
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, (a) find the component form of the vector v and (b) sketch the vector with its initial point at the origin.

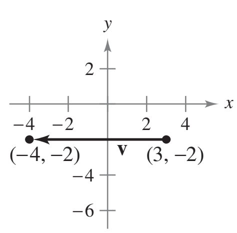
1.



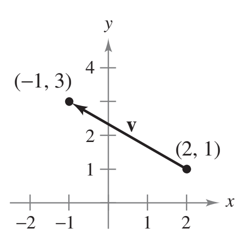
2.



3.



4.



In Exercises 5–8, find the vectors u and v whose initial and terminal points are given. Show that u and v are equivalent.

5. $\mathbf{u}$: $(3, 2)$, $(5, 6)$
$\mathbf{v}$: $(-1, 4)$, $(1, 8)$

6. $\mathbf{u}$: $(-4, 0)$, $(1, 8)$
$\mathbf{v}$: $(2, -1)$, $(7, 7)$

7. $\mathbf{u}$: $(0, 3)$, $(6, -2)$
$\mathbf{v}$: $(3, 10)$, $(9, 5)$

8. $\mathbf{u}$: $(-4, -1)$, $(11, -4)$
$\mathbf{v}$: $(10, 13)$, $(25, 10)$

In Exercises 9–16, the initial and terminal points of a vector v are given. (a) Sketch the given directed line segment, (b) write the vector in component form, and (c) sketch the vector with its initial point at the origin.

	Initial Point	*Terminal Point*		*Initial Point*	*Terminal Point*
9.	$(1, 2)$	$(5, 5)$	**10.**	$(2, -6)$	$(3, 6)$
11.	$(10, 2)$	$(6, -1)$	**12.**	$(0, -4)$	$(-5, -1)$

indicates that in the HM mathSpace® *CD-ROM and the online* Eduspace® *system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

	Initial Point	Terminal Point		Initial Point	Terminal Point
13.	$(6, 2)$	$(6, 6)$	**14.**	$(7, -1)$	$(-3, -1)$
15.	$\left(\frac{3}{2}, \frac{4}{3}\right)$	$\left(\frac{1}{2}, 3\right)$	**16.**	$(0.12, 0.60)$	$(0.84, 1.25)$

In Exercises 17 and 18, sketch each scalar multiple of v.

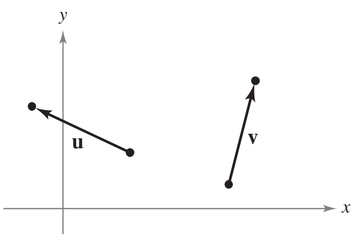
17. $\mathbf{v} = \langle 2, 3 \rangle$

(a) $2\mathbf{v}$ (b) $-3\mathbf{v}$ (c) $\frac{7}{2}\mathbf{v}$ (d) $\frac{2}{3}\mathbf{v}$

18. $\mathbf{v} = \langle -1, 5 \rangle$

(a) $4\mathbf{v}$ (b) $-\frac{1}{2}\mathbf{v}$ (c) $0\mathbf{v}$ (d) $-6\mathbf{v}$

In Exercises 19–22, use the figure to sketch a graph of the vector. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



19. $-\mathbf{u}$ **20.** $2\mathbf{u}$

21. $\mathbf{u} - \mathbf{v}$ **22.** $\mathbf{u} + 2\mathbf{v}$

In Exercises 23 and 24, find (a) $\frac{2}{3}\mathbf{u}$, (b) $\mathbf{v} - \mathbf{u}$, and (c) $2\mathbf{u} + 5\mathbf{v}$.

23. $\mathbf{u} = \langle 4, 9 \rangle$, $\mathbf{v} = \langle 2, -5 \rangle$

24. $\mathbf{u} = \langle -3, -8 \rangle$, $\mathbf{v} = \langle 8, 25 \rangle$

In Exercises 25–28, find the vector v where $\mathbf{u} = \langle 2, -1 \rangle$ and $\mathbf{w} = \langle 1, 2 \rangle$. Illustrate the vector operations geometrically.

25. $\mathbf{v} = \frac{3}{2}\mathbf{u}$ **26.** $\mathbf{v} = \mathbf{u} + \mathbf{w}$

27. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$ **28.** $\mathbf{v} = 5\mathbf{u} - 3\mathbf{w}$

In Exercises 29 and 30, the vector v and its initial point are given. Find the terminal point.

29. $\mathbf{v} = \langle -1, 3 \rangle$; Initial point: $(4, 2)$

30. $\mathbf{v} = \langle 4, -9 \rangle$; Initial point: $(3, 2)$

In Exercises 31–36, find the magnitude of v.

31. $\mathbf{v} = \langle 4, 3 \rangle$ **32.** $\mathbf{v} = \langle 12, -5 \rangle$

33. $\mathbf{v} = 6\mathbf{i} - 5\mathbf{j}$ **34.** $\mathbf{v} = -10\mathbf{i} + 3\mathbf{j}$

35. $\mathbf{v} = 4\mathbf{j}$ **36.** $\mathbf{v} = \mathbf{i} - \mathbf{j}$

In Exercises 37–40, find the unit vector in the direction of u and verify that it has length 1.

37. $\mathbf{u} = \langle 3, 12 \rangle$ **38.** $\mathbf{u} = \langle 5, 15 \rangle$

39. $\mathbf{u} = \left\langle \frac{3}{2}, \frac{5}{2} \right\rangle$ **40.** $\mathbf{u} = \langle -6.2, 3.4 \rangle$

In Exercises 41–44, find the following.

(a) $\|\mathbf{u}\|$ **(b)** $\|\mathbf{v}\|$ **(c)** $\|\mathbf{u} + \mathbf{v}\|$

(d) $\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\|$ **(e)** $\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\|$ **(f)** $\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\|$

41. $\mathbf{u} = \langle 1, -1 \rangle$, $\mathbf{v} = \langle -1, 2 \rangle$

42. $\mathbf{u} = \langle 0, 1 \rangle$, $\mathbf{v} = \langle 3, -3 \rangle$

43. $\mathbf{u} = \left\langle 1, \frac{1}{2} \right\rangle$, $\mathbf{v} = \langle 2, 3 \rangle$

44. $\mathbf{u} = \langle 2, -4 \rangle$, $\mathbf{v} = \langle 5, 5 \rangle$

In Exercises 45 and 46, sketch a graph of u, v, and u + v. Then demonstrate the triangle inequality using the vectors u and v.

45. $\mathbf{u} = \langle 2, 1 \rangle$, $\mathbf{v} = \langle 5, 4 \rangle$ **46.** $\mathbf{u} = \langle -3, 2 \rangle$, $\mathbf{v} = \langle 1, -2 \rangle$

In Exercises 47–50, find the vector v with the given magnitude and the same direction as u.

	Magnitude	Direction
47.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle 1, 1 \rangle$
48.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle -1, 1 \rangle$
49.	$\|\mathbf{v}\| = 2$	$\mathbf{u} = \langle \sqrt{3}, 3 \rangle$
50.	$\|\mathbf{v}\| = 3$	$\mathbf{u} = \langle 0, 3 \rangle$

In Exercises 51–54, find the component form of v given its magnitude and the angle it makes with the positive *x*-axis.

51. $\|\mathbf{v}\| = 3$, $\theta = 0°$ **52.** $\|\mathbf{v}\| = 5$, $\theta = 120°$

53. $\|\mathbf{v}\| = 2$, $\theta = 150°$ **54.** $\|\mathbf{v}\| = 1$, $\theta = 3.5°$

In Exercises 55–58, find the component form of u + v given the lengths of u and v and the angles that u and v make with the positive *x*-axis.

55. $\|\mathbf{u}\| = 1$, $\theta_{\mathbf{u}} = 0°$; $\|\mathbf{v}\| = 3$, $\theta_{\mathbf{v}} = 45°$

56. $\|\mathbf{u}\| = 4$, $\theta_{\mathbf{u}} = 0°$; $\|\mathbf{v}\| = 2$, $\theta_{\mathbf{v}} = 60°$

57. $\|\mathbf{u}\| = 2$, $\theta_{\mathbf{u}} = 4$; $\|\mathbf{v}\| = 1$, $\theta_{\mathbf{v}} = 2$

58. $\|\mathbf{u}\| = 5$, $\theta_{\mathbf{u}} = -0.5$; $\|\mathbf{v}\| = 5$, $\theta_{\mathbf{v}} = 0.5$

Writing About Concepts

59. In your own words, state the difference between a scalar and a vector. Give examples of each.

60. Give geometric descriptions of the operations of addition of vectors and multiplication of a vector by a scalar.

61. Identify the quantity as a scalar or as a vector. Explain your reasoning.

(a) The muzzle velocity of a gun

(b) The price of a company's stock

62. Identify the quantity as a scalar or as a vector. Explain your reasoning.

(a) The air temperature in a room

(b) The weight of a car

In Exercises 63–68, find a and b such that $\mathbf{v} = a\mathbf{u} + b\mathbf{w}$, where $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 1, -1 \rangle$.

63. $\mathbf{v} = \langle 2, 1 \rangle$ **64.** $\mathbf{v} = \langle 0, 3 \rangle$

65. $\mathbf{v} = \langle 3, 0 \rangle$ **66.** $\mathbf{v} = \langle 3, 3 \rangle$

67. $\mathbf{v} = \langle 1, 1 \rangle$ **68.** $\mathbf{v} = \langle -1, 7 \rangle$

In Exercises 69–74, find a unit vector (a) parallel to and (b) normal to the graph of $f(x)$ at the given point. Then sketch a graph of the vectors and the function.

	Function	*Point*
69.	$f(x) = x^2$	$(3, 9)$
70.	$f(x) = -x^2 + 5$	$(1, 4)$
71.	$f(x) = x^3$	$(1, 1)$
72.	$f(x) = x^3$	$(-2, -8)$
73.	$f(x) = \sqrt{25 - x^2}$	$(3, 4)$
74.	$f(x) = \tan x$	$\left(\frac{\pi}{4}, 1\right)$

In Exercises 75 and 76, find the component form of v given the magnitudes of u and u + v and the angles that u and u + v make with the positive x-axis.

75. $\|\mathbf{u}\| = 1, \theta = 45°$
$\|\mathbf{u} + \mathbf{v}\| = \sqrt{2}, \theta = 90°$

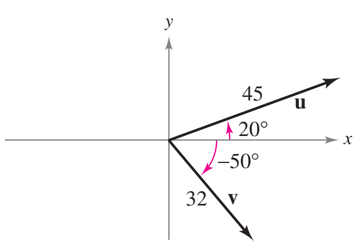
76. $\|\mathbf{u}\| = 4, \theta = 30°$
$\|\mathbf{u} + \mathbf{v}\| = 6, \theta = 120°$

77. ***Programming*** You are given the magnitudes of $\mathbf{u}$ and $\mathbf{v}$ and the angles $\mathbf{u}$ and $\mathbf{v}$ make with the positive x-axis. Write a program for a graphing utility in which the output is the following.

(a) $\mathbf{u} + \mathbf{v}$ (b) $\|\mathbf{u} + \mathbf{v}\|$

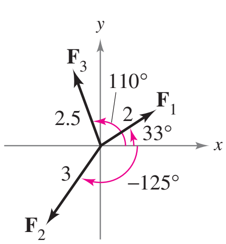
(c) The angle $\mathbf{u} + \mathbf{v}$ makes with the positive x-axis

78. ***Programming*** Use the program you wrote in Exercise 77 to find the magnitude and direction of the resultant of the vectors shown.

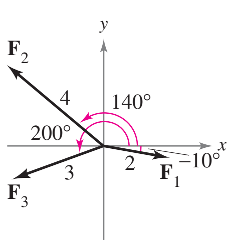


In Exercises 79 and 80, use a graphing utility to find the magnitude and direction of the resultant of the vectors.

79.



80.



81. ***Numerical and Graphical Analysis*** Forces with magnitudes of 180 newtons and 275 newtons act on a hook (see figure). The angle between the two forces is θ degrees.

(a) If $\theta = 30°$, find the direction and magnitude of the resultant force.

(b) Write the magnitude M and direction α of the resultant force as functions of θ, where $0° \le \theta \le 180°$.

(c) Use a graphing utility to complete the table.

θ	0°	30°	60°	90°	120°	150°	180°
M							
α							

(d) Use a graphing utility to graph the two functions M and α.

(e) Explain why one of the functions decreases for increasing values of θ whereas the other does not.

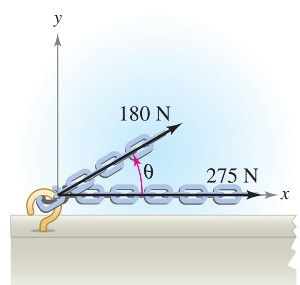


Figure for 81

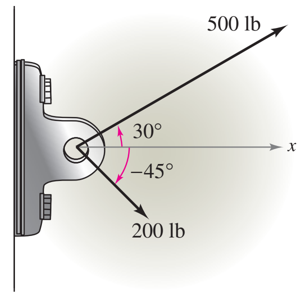


Figure for 82

82. ***Resultant Force*** Forces with magnitudes of 500 pounds and 200 pounds act on a machine part at angles of 30° and $-45°$, respectively, with the x-axis (see figure). Find the direction and magnitude of the resultant force.

83. ***Resultant Force*** Three forces with magnitudes of 75 pounds, 100 pounds, and 125 pounds act on an object at angles of 30°, 45°, and 120°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

84. ***Resultant Force*** Three forces with magnitudes of 400 newtons, 280 newtons, and 350 newtons act on an object at angles of $-30°$, 45°, and 135°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

85. ***Think About It*** Consider two forces of equal magnitude acting on a point.

(a) If the magnitude of the resultant is the sum of the magnitudes of the two forces, make a conjecture about the angle between the forces.

(b) If the resultant of the forces is $\mathbf{0}$, make a conjecture about the angle between the forces.

(c) Can the magnitude of the resultant be greater than the sum of the magnitudes of the two forces? Explain.

86. ***Graphical Reasoning*** Consider two forces $\mathbf{F}_1 = \langle 20, 0 \rangle$ and $\mathbf{F}_2 = 10\langle \cos\theta, \sin\theta \rangle$.

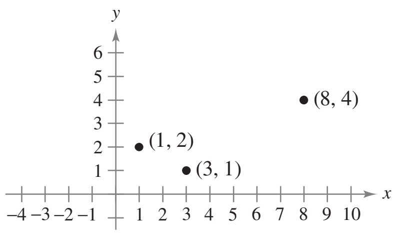
(a) Find $\|\mathbf{F}_1 + \mathbf{F}_2\|$.

(b) Determine the magnitude of the resultant as a function of θ. Use a graphing utility to graph the function for $0 \le \theta < 2\pi$.

(c) Use the graph in part (b) to determine the range of the function. What is its maximum and for what value of θ does it occur? What is its minimum and for what value of θ does it occur?

(d) Explain why the magnitude of the resultant is never 0.

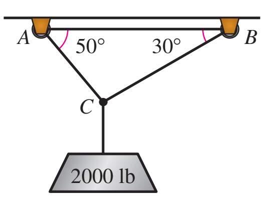
87. Three vertices of a parallelogram are $(1, 2)$, $(3, 1)$, and $(8, 4)$. Find the three possible fourth vertices (see figure).



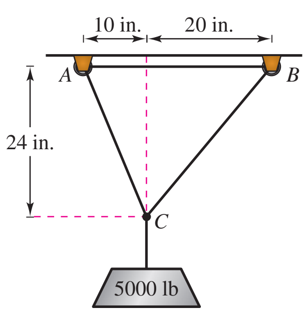
88. Use vectors to find the points of trisection of the line segment with endpoints $(1, 2)$ and $(7, 5)$.

Cable Tension **In Exercises 89 and 90, use the figure to determine the tension in each cable supporting the given load.**

89.



90.



91. ***Projectile Motion*** A gun with a muzzle velocity of 1200 feet per second is fired at an angle of 6° above the horizontal. Find the vertical and horizontal components of the velocity.

92. ***Shared Load*** To carry a 100-pound cylindrical weight, two workers lift on the ends of short ropes tied to an eyelet on the top center of the cylinder. One rope makes a 20° angle away from the vertical and the other makes a 30° angle (see figure).

(a) Find each rope's tension if the resultant force is vertical.

(b) Find the vertical component of each worker's force.

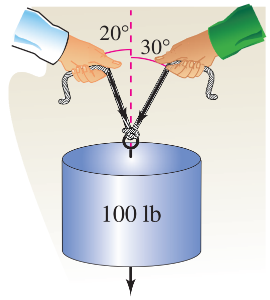


Figure for 92

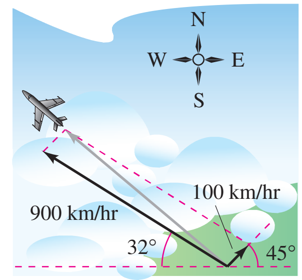


Figure for 93

93. ***Navigation*** A plane is flying in the direction 302°. Its speed with respect to the air is 900 kilometers per hour. The wind at the plane's altitude is from the southwest at 100 kilometers per hour (see figure). What is the true direction of the plane, and what is its speed with respect to the ground?

94. ***Navigation*** A plane flies at a constant groundspeed of 400 miles per hour due east and encounters a 50-mile-per-hour wind from the northwest. Find the airspeed and compass direction that will allow the plane to maintain its groundspeed and eastward direction.

True or False? **In Exercises 95–100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

95. If $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude and direction, then $\mathbf{u}$ and $\mathbf{v}$ are equivalent.

96. If $\mathbf{u}$ is a unit vector in the direction of $\mathbf{v}$, then $\mathbf{v} = \|\mathbf{v}\|\,\mathbf{u}$.

97. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $a^2 + b^2 = 1$.

98. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \mathbf{0}$, then $a = -b$.

99. If $a = b$, then $\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}a$.

100. If $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude but opposite directions, then $\mathbf{u} + \mathbf{v} = \mathbf{0}$.

101. Prove that $\mathbf{u} = (\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j}$ and $\mathbf{v} = (\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ are unit vectors for any angle θ.

102. ***Geometry*** Using vectors, prove that the line segment joining the midpoints of two sides of a triangle is parallel to, and one-half the length of, the third side.

103. ***Geometry*** Using vectors, prove that the diagonals of a parallelogram bisect each other.

104. Prove that the vector $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

105. Consider the vector $\mathbf{u} = \langle x, y \rangle$. Describe the set of all points (x, y) such that $\|\mathbf{u}\| = 5$.

Putnam Exam Challenge

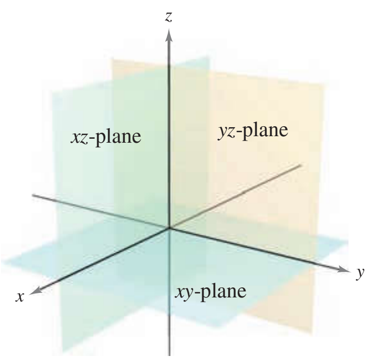
106. A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If air resistance is neglected and the muzzle velocity is constant ($= v_0$), determine the set H of points in the plane and above the horizontal which can be hit.

This problem was composed by the Committee on the Putnam Prize Competition.

Section 11.2 Space Coordinates and Vectors in Space

- **Understand the three-dimensional rectangular coordinate system.**
- **Analyze vectors in space.**
- **Use three-dimensional vectors to solve real-life problems.**

Coordinates in Space



The three-dimensional coordinate system
Figure 11.14

Up to this point in the text, you have been primarily concerned with the two-dimensional coordinate system. Much of the remaining part of your study of calculus will involve the three-dimensional coordinate system.

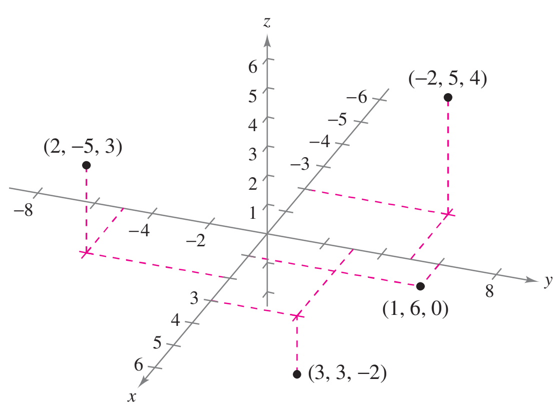
Before extending the concept of a vector to three dimensions, you must be able to identify points in the **three-dimensional coordinate system.** You can construct this system by passing a z-axis perpendicular to both the x- and y-axes at the origin. Figure 11.14 shows the positive portion of each coordinate axis. Taken as pairs, the axes determine three **coordinate planes:** the ***xy*-plane,** the ***xz*-plane,** and the ***yz*-plane.** These three coordinate planes separate three-space into eight **octants.** The first octant is the one for which all three coordinates are positive. In this three-dimensional system, a point P in space is determined by an ordered triple (x, y, z) where x, y, and z are as follows.

$x =$ directed distance from yz-plane to P

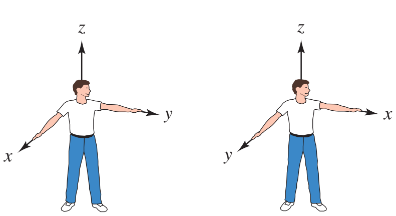
$y =$ directed distance from xz-plane to P

$z =$ directed distance from xy-plane to P

Several points are shown in Figure 11.15.



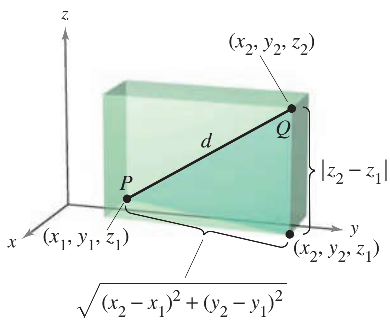
Points in the three-dimensional coordinate system are represented by ordered triples.
Figure 11.15



Right-handed system Left-handed system
Figure 11.16

A three-dimensional coordinate system can have either a **left-handed** or a **right-handed** orientation. To determine the orientation of a system, imagine that you are standing at the origin, with your arms pointing in the direction of the positive x- and y-axes, and with the z-axis pointing up, as shown in Figure 11.16. The system is right-handed or left-handed depending on which hand points along the x-axis. In this text, you will work exclusively with the right-handed system.

NOTE The three-dimensional rotatable graphs that are available in the *HM mathSpace*® CD-ROM and the online *Eduspace*® system for this text will help you visualize points or objects in a three-dimensional coordinate system.



The distance between two points in space
Figure 11.17

Many of the formulas established for the two-dimensional coordinate system can be extended to three dimensions. For example, to find the distance between two points in space, you can use the Pythagorean Theorem twice, as shown in Figure 11.17. By doing this, you will obtain the formula for the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$ Distance Formula

EXAMPLE 1 Finding the Distance Between Two Points in Space

The distance between the points $(2, -1, 3)$ and $(1, 0, -2)$ is

$$\begin{aligned} d &= \sqrt{(1 - 2)^2 + (0 + 1)^2 + (-2 - 3)^2} && \text{Distance Formula} \\ &= \sqrt{1 + 1 + 25} \\ &= \sqrt{27} \\ &= 3\sqrt{3}. \end{aligned}$$

A **sphere** with center at (x_0, y_0, z_0) and radius r is defined to be the set of all points (x, y, z) such that the distance between (x, y, z) and (x_0, y_0, z_0) is r. You can use the Distance Formula to find the **standard equation of a sphere** of radius r, centered at (x_0, y_0, z_0). If (x, y, z) is an arbitrary point on the sphere, the equation of the sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$ Equation of sphere

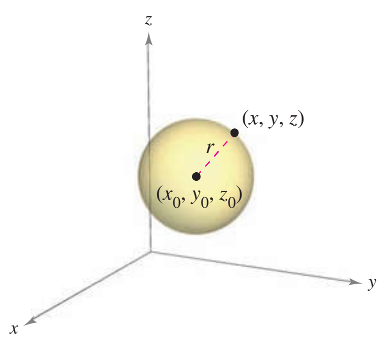


Figure 11.18

as shown in Figure 11.18. Moreover, the midpoint of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$ Midpoint Rule

EXAMPLE 2 Finding the Equation of a Sphere

Find the standard equation of the sphere that has the points $(5, -2, 3)$ and $(0, 4, -3)$ as endpoints of a diameter.

Solution By the Midpoint Rule, the center of the sphere is

$$\left(\frac{5 + 0}{2}, \frac{-2 + 4}{2}, \frac{3 - 3}{2}\right) = \left(\frac{5}{2}, 1, 0\right).$$ Midpoint Rule

By the Distance Formula, the radius is

$$r = \sqrt{\left(0 - \frac{5}{2}\right)^2 + (4 - 1)^2 + (-3 - 0)^2} = \sqrt{\frac{97}{4}} = \frac{\sqrt{97}}{2}.$$

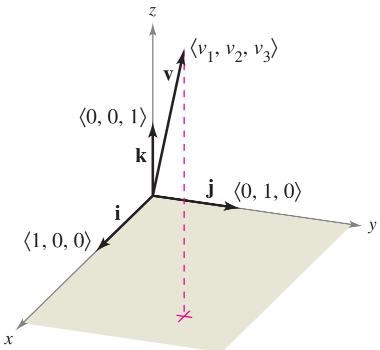
Therefore, the standard equation of the sphere is

$$\left(x - \frac{5}{2}\right)^2 + (y - 1)^2 + z^2 = \frac{97}{4}.$$ Equation of sphere

Vectors in Space

In space, vectors are denoted by ordered triples $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. The **zero vector** is denoted by $\mathbf{0} = \langle 0, 0, 0 \rangle$. Using the unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ in the direction of the positive z-axis, the **standard unit vector notation** for $\mathbf{v}$ is

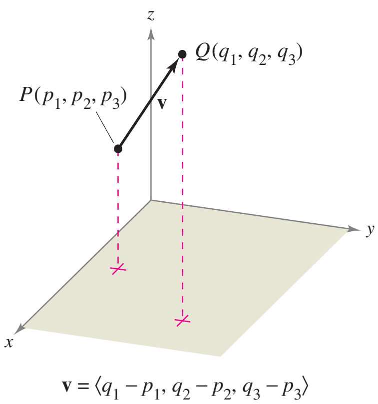
$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$



The standard unit vectors in space
Figure 11.19

as shown in Figure 11.19. If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, as shown in Figure 11.20, the component form of $\mathbf{v}$ is given by subtracting the coordinates of the initial point from the coordinates of the terminal point, as follows.

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$$



$\mathbf{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$

Figure 11.20

Vectors in Space

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in space and let c be a scalar.

1. *Equality of Vectors:* $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.
2. *Component Form:* If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, then
$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle.$$
3. *Length:* $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
4. *Unit Vector in the Direction of* $\mathbf{v}$: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\dfrac{1}{\|\mathbf{v}\|}\right)\langle v_1, v_2, v_3 \rangle, \quad \mathbf{v} \neq \mathbf{0}$
5. *Vector Addition:* $\mathbf{v} + \mathbf{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$
6. *Scalar Multiplication:* $c\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle$

NOTE The properties of vector addition and scalar multiplication given in Theorem 11.1 are also valid for vectors in space.

EXAMPLE 3 Finding the Component Form of a Vector in Space

Find the component form and magnitude of the vector $\mathbf{v}$ having initial point $(-2, 3, 1)$ and terminal point $(0, -4, 4)$. Then find a unit vector in the direction of $\mathbf{v}$.

Solution The component form of $\mathbf{v}$ is

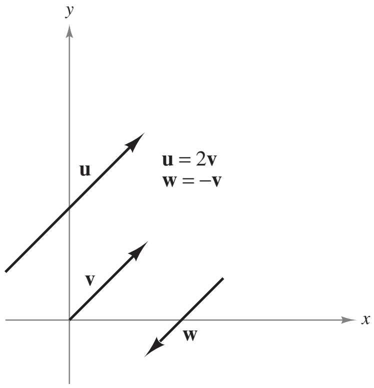
$$\begin{aligned}\mathbf{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle &= \langle 0 - (-2), -4 - 3, 4 - 1 \rangle \\ &= \langle 2, -7, 3 \rangle\end{aligned}$$

which implies that its magnitude is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-7)^2 + 3^2} = \sqrt{62}.$$

The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{62}}\langle 2, -7, 3 \rangle.$$



Parallel vectors
Figure 11.21

Recall from the definition of scalar multiplication that positive scalar multiples of a nonzero vector **v** have the same direction as **v**, whereas negative multiples have the direction opposite of **v**. In general, two nonzero vectors **u** and **v** are **parallel** if there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.

Definition of Parallel Vectors

Two nonzero vectors **u** and **v** are **parallel** if there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.

For example, in Figure 11.21, the vectors **u**, **v**, and **w** are parallel because $\mathbf{u} = 2\mathbf{v}$ and $\mathbf{w} = -\mathbf{v}$.

EXAMPLE 4 Parallel Vectors

Vector **w** has initial point $(2, -1, 3)$ and terminal point $(-4, 7, 5)$. Which of the following vectors is parallel to **w**?

a. $\mathbf{u} = \langle 3, -4, -1 \rangle$

b. $\mathbf{v} = \langle 12, -16, 4 \rangle$

Solution Begin by writing **w** in component form.

$$\mathbf{w} = \langle -4 - 2, 7 - (-1), 5 - 3 \rangle = \langle -6, 8, 2 \rangle$$

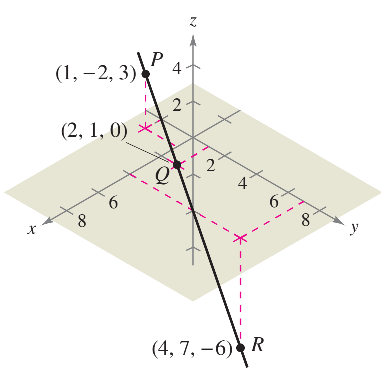
a. Because $\mathbf{u} = \langle 3, -4, -1 \rangle = -\frac{1}{2}\langle -6, 8, 2 \rangle = -\frac{1}{2}\mathbf{w}$, you can conclude that **u** is parallel to **w**.

b. In this case, you want to find a scalar c such that

$$\langle 12, -16, 4 \rangle = c\langle -6, 8, 2 \rangle.$$

$$\begin{aligned} 12 &= -6c \rightarrow c = -2 \\ -16 &= 8c \rightarrow c = -2 \\ 4 &= 2c \rightarrow c = 2 \end{aligned}$$

Because there is no c for which the equation has a solution, the vectors are not parallel.



The points P, Q, and R lie on the same line.
Figure 11.22

EXAMPLE 5 Using Vectors to Determine Collinear Points

Determine whether the points $P(1, -2, 3)$, $Q(2, 1, 0)$, and $R(4, 7, -6)$ are collinear.

Solution The component forms of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \langle 2 - 1, 1 - (-2), 0 - 3 \rangle = \langle 1, 3, -3 \rangle$$

and

$$\overrightarrow{PR} = \langle 4 - 1, 7 - (-2), -6 - 3 \rangle = \langle 3, 9, -9 \rangle.$$

These two vectors have a common initial point. So, P, Q, and R lie on the same line if and only if $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel—which they are because $\overrightarrow{PR} = 3\,\overrightarrow{PQ}$, as shown in Figure 11.22.

EXAMPLE 6 Standard Unit Vector Notation

a. Write the vector $\mathbf{v} = 4\mathbf{i} - 5\mathbf{k}$ in component form.

b. Find the terminal point of the vector $\mathbf{v} = 7\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, given that the initial point is $P(-2, 3, 5)$.

Solution

a. Because $\mathbf{j}$ is missing, its component is 0 and

$$\mathbf{v} = 4\mathbf{i} - 5\mathbf{k} = \langle 4, 0, -5\rangle.$$

b. You need to find $Q(q_1, q_2, q_3)$ such that $\mathbf{v} = \overrightarrow{PQ} = 7\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. This implies that $q_1 - (-2) = 7$, $q_2 - 3 = -1$, and $q_3 - 5 = 3$. The solution of these three equations is $q_1 = 5$, $q_2 = 2$, and $q_3 = 8$. Therefore, Q is $(5, 2, 8)$.

Application

EXAMPLE 7 Measuring Force

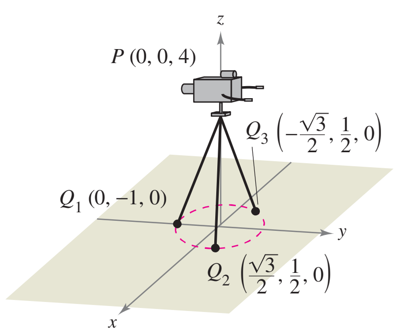


Figure 11.23

A television camera weighing 120 pounds is supported by a tripod, as shown in Figure 11.23. Represent the force exerted on each leg of the tripod as a vector.

Solution Let the vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ represent the forces exerted on the three legs. From Figure 11.23, you can determine the directions of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ to be as follows.

$$\overrightarrow{PQ}_1 = \langle 0 - 0, -1 - 0, 0 - 4\rangle = \langle 0, -1, -4\rangle$$
$$\overrightarrow{PQ}_2 = \left\langle \frac{\sqrt{3}}{2} - 0, \frac{1}{2} - 0, 0 - 4\right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle$$
$$\overrightarrow{PQ}_3 = \left\langle -\frac{\sqrt{3}}{2} - 0, \frac{1}{2} - 0, 0 - 4\right\rangle = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle$$

Because each leg has the same length, and the total force is distributed equally among the three legs, you know that $\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = \|\mathbf{F}_3\|$. So, there exists a constant c such that

$$\mathbf{F}_1 = c\langle 0, -1, -4\rangle, \quad \mathbf{F}_2 = c\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle, \quad \text{and} \quad \mathbf{F}_3 = c\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle.$$

Let the total force exerted by the object be given by $\mathbf{F} = -120\mathbf{k}$. Then, using the fact that

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

you can conclude that $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ all have a vertical component of -40. This implies that $c(-4) = -40$ and $c = 10$. Therefore, the forces exerted on the legs can be represented by

$$\mathbf{F}_1 = \langle 0, -10, -40\rangle$$
$$\mathbf{F}_2 = \left\langle 5\sqrt{3}, 5, -40\right\rangle$$
$$\mathbf{F}_3 = \left\langle -5\sqrt{3}, 5, -40\right\rangle.$$

Exercises for Section 11.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, plot the points on the same three-dimensional coordinate system.

1. (a) $(2, 1, 3)$ (b) $(-1, 2, 1)$

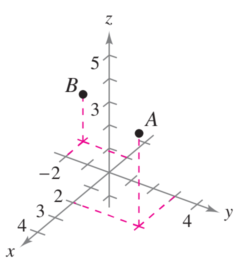
2. (a) $(3, -2, 5)$ (b) $\left(\frac{3}{2}, 4, -2\right)$

3. (a) $(5, -2, 2)$ (b) $(5, -2, -2)$

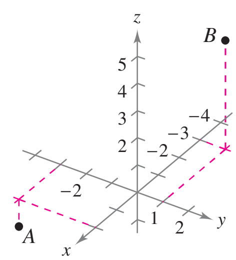
4. (a) $(0, 4, -5)$ (b) $(4, 0, 5)$

In Exercises 5 and 6, approximate the coordinates of the points.

5.



6.



In Exercises 7–10, find the coordinates of the point.

7. The point is located three units behind the yz-plane, four units to the right of the xz-plane, and five units above the xy-plane.

8. The point is located seven units in front of the yz-plane, two units to the left of the xz-plane, and one unit below the xy-plane.

9. The point is located on the x-axis, 10 units in front of the yz-plane.

10. The point is located in the yz-plane, three units to the right of the xz-plane, and two units above the xy-plane.

11. ***Think About It*** What is the z-coordinate of any point in the xy-plane?

12. ***Think About It*** What is the x-coordinate of any point in the yz-plane?

In Exercises 13–24, determine the location of a point (x, y, z) that satisfies the condition(s).

13. $z = 6$

14. $y = 2$

15. $x = 4$

16. $z = -3$

17. $y < 0$

18. $x < 0$

19. $|y| \le 3$

20. $|x| > 4$

21. $xy > 0, \quad z = -3$

22. $xy < 0, \quad z = 4$

23. $xyz < 0$

24. $xyz > 0$

In Exercises 25–28, find the distance between the points.

25. $(0, 0, 0), \quad (5, 2, 6)$

26. $(-2, 3, 2), \quad (2, -5, -2)$

27. $(1, -2, 4), \quad (6, -2, -2)$

28. $(2, 2, 3), \quad (4, -5, 6)$

In Exercises 29–32, find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.

29. $(0, 0, 0), (2, 2, 1), (2, -4, 4)$

30. $(5, 3, 4), (7, 1, 3), (3, 5, 3)$

31. $(1, -3, -2), (5, -1, 2), (-1, 1, 2)$

32. $(5, 0, 0), (0, 2, 0), (0, 0, -3)$

33. ***Think About It*** The triangle in Exercise 29 is translated five units upward along the z-axis. Determine the coordinates of the translated triangle.

34. ***Think About It*** The triangle in Exercise 30 is translated three units to the right along the y-axis. Determine the coordinates of the translated triangle.

In Exercises 35 and 36, find the coordinates of the midpoint of the line segment joining the points.

35. $(5, -9, 7), (-2, 3, 3)$

36. $(4, 0, -6), (8, 8, 20)$

In Exercises 37–40, find the standard equation of the sphere.

37. Center: $(0, 2, 5)$
Radius: 2

38. Center: $(4, -1, 1)$
Radius: 5

39. Endpoints of a diameter: $(2, 0, 0), (0, 6, 0)$

40. Center: $(-3, 2, 4)$, tangent to the yz-plane

In Exercises 41–44, complete the square to write the equation of the sphere in standard form. Find the center and radius.

41. $x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$

42. $x^2 + y^2 + z^2 + 9x - 2y + 10z + 19 = 0$

43. $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

44. $4x^2 + 4y^2 + 4z^2 - 4x - 32y + 8z + 33 = 0$

In Exercises 45–48, describe the solid satisfying the condition.

45. $x^2 + y^2 + z^2 \le 36$

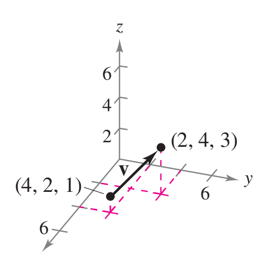
46. $x^2 + y^2 + z^2 > 4$

47. $x^2 + y^2 + z^2 < 4x - 6y + 8z - 13$

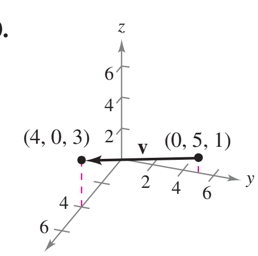
48. $x^2 + y^2 + z^2 > -4x + 6y - 8z - 13$

In Exercises 49–52, (a) find the component form of the vector v and (b) sketch the vector with its initial point at the origin.

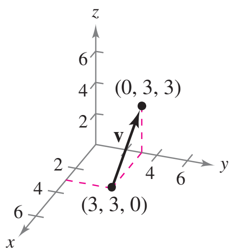
49.



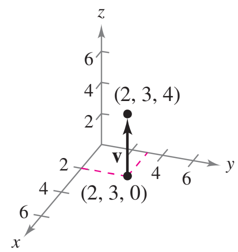
50.



51.



52.



In Exercises 53–56, find the component form and magnitude of the vector u with the given initial and terminal points. Then find a unit vector in the direction of u.

	Initial Point	*Terminal Point*
53.	$(3, 2, 0)$	$(4, 1, 6)$
54.	$(4, -5, 2)$	$(-1, 7, -3)$
55.	$(-4, 3, 1)$	$(-5, 3, 0)$
56.	$(1, -2, 4)$	$(2, 4, -2)$

In Exercises 57 and 58, the initial and terminal points of a vector v are given. (a) Sketch the directed line segment, (b) find the component form of the vector, and (c) sketch the vector with its initial point at the origin.

57. Initial point: $(-1, 2, 3)$
Terminal point: $(3, 3, 4)$

58. Initial point: $(2, -1, -2)$
Terminal point: $(-4, 3, 7)$

In Exercises 59 and 60, the vector v and its initial point are given. Find the terminal point.

59. $\mathbf{v} = \langle 3, -5, 6 \rangle$
Initial point: $(0, 6, 2)$

60. $\mathbf{v} = \left\langle 1, -\frac{2}{3}, \frac{1}{2} \right\rangle$
Initial point: $\left(0, 2, \frac{5}{2}\right)$

In Exercises 61 and 62, find each scalar multiple of v and sketch its graph.

61. $\mathbf{v} = \langle 1, 2, 2 \rangle$
(a) $2\mathbf{v}$ (b) $-\mathbf{v}$
(c) $\frac{3}{2}\mathbf{v}$ (d) $0\mathbf{v}$

62. $\mathbf{v} = \langle 2, -2, 1 \rangle$
(a) $-\mathbf{v}$ (b) $2\mathbf{v}$
(c) $\frac{1}{2}\mathbf{v}$ (d) $\frac{5}{2}\mathbf{v}$

In Exercises 63–68, find the vector z, given that $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 2, 2, -1 \rangle$, and $\mathbf{w} = \langle 4, 0, -4 \rangle$.

63. $\mathbf{z} = \mathbf{u} - \mathbf{v}$

64. $\mathbf{z} = \mathbf{u} - \mathbf{v} + 2\mathbf{w}$

65. $\mathbf{z} = 2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$

66. $\mathbf{z} = 5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$

67. $2\mathbf{z} - 3\mathbf{u} = \mathbf{w}$

68. $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$

In Exercises 69–72, determine which of the vectors is (are) parallel to z. Use a graphing utility to confirm your results.

69. $\mathbf{z} = \langle 3, 2, -5 \rangle$
(a) $\langle -6, -4, 10 \rangle$
(b) $\left\langle 2, \frac{4}{3}, -\frac{10}{3} \right\rangle$
(c) $\langle 6, 4, 10 \rangle$
(d) $\langle 1, -4, 2 \rangle$

70. $\mathbf{z} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}$
(a) $6\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}$
(b) $-\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{3}{2}\mathbf{k}$
(c) $12\mathbf{i} + 9\mathbf{k}$
(d) $\frac{3}{4}\mathbf{i} - \mathbf{j} + \frac{9}{8}\mathbf{k}$

71. $\mathbf{z}$ has initial point $(1, -1, 3)$ and terminal point $(-2, 3, 5)$.
(a) $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$ (b) $4\mathbf{j} + 2\mathbf{k}$

72. $\mathbf{z}$ has initial point $(5, 4, 1)$ and terminal point $(-2, -4, 4)$.
(a) $\langle 7, 6, 2 \rangle$ (b) $\langle 14, 16, -6 \rangle$

In Exercises 73–76, use vectors to determine whether the points are collinear.

73. $(0, -2, -5), (3, 4, 4), (2, 2, 1)$

74. $(4, -2, 7), (-2, 0, 3), (7, -3, 9)$

75. $(1, 2, 4), (2, 5, 0), (0, 1, 5)$

76. $(0, 0, 0), (1, 3, -2), (2, -6, 4)$

In Exercises 77 and 78, use vectors to show that the points form the vertices of a parallelogram.

77. $(2, 9, 1), (3, 11, 4), (0, 10, 2), (1, 12, 5)$

78. $(1, 1, 3), (9, -1, -2), (11, 2, -9), (3, 4, -4)$

In Exercises 79–84, find the magnitude of v.

79. $\mathbf{v} = \langle 0, 0, 0 \rangle$

80. $\mathbf{v} = \langle 1, 0, 3 \rangle$

81. $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

82. $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$

83. Initial point of $\mathbf{v}$: $(1, -3, 4)$
Terminal point of $\mathbf{v}$: $(1, 0, -1)$

84. Initial point of $\mathbf{v}$: $(0, -1, 0)$
Terminal point of $\mathbf{v}$: $(1, 2, -2)$

In Exercises 85–88, find a unit vector (a) in the direction of u and (b) in the direction opposite of u.

85. $\mathbf{u} = \langle 2, -1, 2 \rangle$

86. $\mathbf{u} = \langle 6, 0, 8 \rangle$

87. $\mathbf{u} = \langle 3, 2, -5 \rangle$

88. $\mathbf{u} = \langle 8, 0, 0 \rangle$

89. ***Programming*** You are given the component forms of the vectors $\mathbf{u}$ and $\mathbf{v}$. Write a program for a graphing utility in which the output is (a) the component form of $\mathbf{u} + \mathbf{v}$, (b) $\|\mathbf{u} + \mathbf{v}\|$, (c) $\|\mathbf{u}\|$, and (d) $\|\mathbf{v}\|$.

90. ***Programming*** Run the program you wrote in Exercise 89 for the vectors $\mathbf{u} = \langle -1, 3, 4 \rangle$ and $\mathbf{v} = \langle 5, 4.5, -6 \rangle$.

In Exercises 91 and 92, determine the values of *c* that satisfy the equation. Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

91. $\|c\mathbf{v}\| = 5$

92. $\|c\mathbf{u}\| = 3$

In Exercises 93–96, find the vector v with the given magnitude and direction u.

	Magnitude	*Direction*
93.	10	$\mathbf{u} = \langle 0, 3, 3 \rangle$
94.	3	$\mathbf{u} = \langle 1, 1, 1 \rangle$
95.	$\frac{3}{2}$	$\mathbf{u} = \langle 2, -2, 1 \rangle$
96.	$\sqrt{5}$	$\mathbf{u} = \langle -4, 6, 2 \rangle$

In Exercises 97 and 98, sketch the vector v and write its component form.

97. $\mathbf{v}$ lies in the yz-plane, has magnitude 2, and makes an angle of 30° with the positive y-axis.

98. $\mathbf{v}$ lies in the xz-plane, has magnitude 5, and makes an angle of 45° with the positive z-axis.

In Exercises 99 and 100, use vectors to find the point that lies two-thirds of the way from *P* to *Q*.

99. $P(4, 3, 0)$, $Q(1, -3, 3)$ **100.** $P(1, 2, 5)$, $Q(6, 8, 2)$

101. Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$.

(a) Sketch $\mathbf{u}$ and $\mathbf{v}$.

(b) If $\mathbf{w} = \mathbf{0}$, show that a and b must both be zero.

(c) Find a and b such that $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(d) Show that no choice of a and b yields $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

102. ***Writing*** The initial and terminal points of the vector $\mathbf{v}$ are (x_1, y_1, z_1) and (x, y, z). Describe the set of all points (x, y, z) such that $\|\mathbf{v}\| = 4$.

Writing About Concepts

103. A point in the three-dimensional coordinate system has coordinates (x_0, y_0, z_0). Describe what each coordinate measures.

104. Give the formula for the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

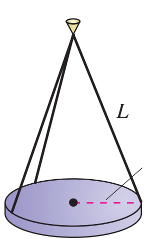
105. Give the standard equation of a sphere of radius r, centered at (x_0, y_0, z_0).

106. State the definition of parallel vectors.

107. Let A, B, and C be vertices of a triangle. Find $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

108. Let $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle 1, 1, 1\rangle$. Describe the set of all points (x, y, z) such that $\|\mathbf{r} - \mathbf{r}_0\| = 2$.

109. ***Numerical, Graphical, and Analytic Analysis*** The lights in an auditorium are 24-pound discs of radius 18 inches. Each disc is supported by three equally spaced cables that are L inches long (see figure).



(a) Write the tension T in each cable as a function of L. Determine the domain of the function.

(b) Use a graphing utility and the function in part (a) to complete the table.

L	20	25	30	35	40	45	50
T							

(c) Use a graphing utility to graph the function in part (a). Determine the asymptotes of the graph.

(d) Confirm the asymptotes of the graph in part (c) analytically.

(e) Determine the minimum length of each cable if a cable is designed to carry a maximum load of 10 pounds.

110. ***Think About It*** Suppose the length of each cable in Exercise 109 has a fixed length $L = a$, and the radius of each disc is r_0 inches. Make a conjecture about the limit $\lim_{r_0 \to a^-} T$ and give a reason for your answer.

111. ***Diagonal of a Cube*** Find the component form of the unit vector $\mathbf{v}$ in the direction of the diagonal of the cube shown in the figure.

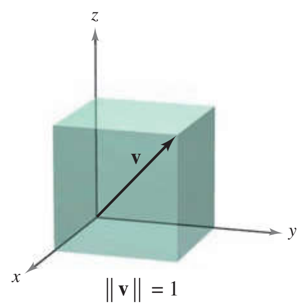


Figure for 111

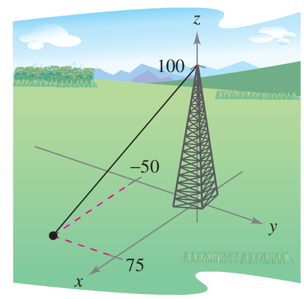


Figure for 112

112. ***Tower Guy Wire*** The guy wire to a 100-foot tower has a tension of 550 pounds. Using the distances shown in the figure, write the component form of the vector $\mathbf{F}$ representing the tension in the wire.

113. ***Load Supports*** Find the tension in each of the supporting cables in the figure if the weight of the crate is 500 newtons.

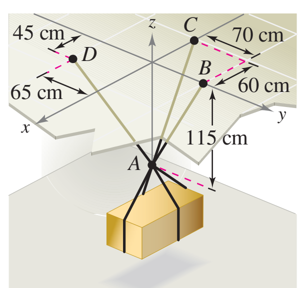


Figure for 113

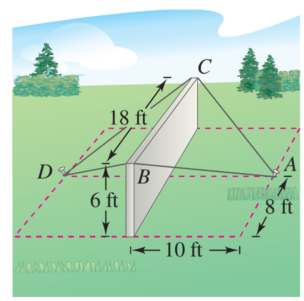


Figure for 114

114. ***Construction*** A precast concrete wall is temporarily kept in its vertical position by ropes (see figure). Find the total force exerted on the pin at position A. The tensions in AB and AC are 420 pounds and 650 pounds.

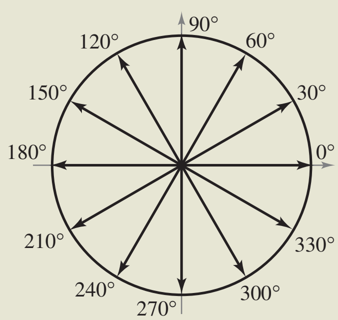
115. Write an equation whose graph consists of the set of points $P(x, y, z)$ that are twice as far from $A(0, -1, 1)$ as from $B(1, 2, 0)$.

Section 11.3 The Dot Product of Two Vectors

- **Use properties of the dot product of two vectors.**
- **Find the angle between two vectors using the dot product.**
- **Find the direction cosines of a vector in space.**
- **Find the projection of a vector onto another vector.**
- **Use vectors to find the work done by a constant force.**

EXPLORATION

Interpreting a Dot Product Several vectors are shown below on the unit circle. Find the dot products of several pairs of vectors. Then find the angle between each pair that you used. Make a conjecture about the relationship between the dot product of two vectors and the angle between the vectors.



The Dot Product

So far you have studied two operations with vectors—vector addition and multiplication by a scalar—each of which yields another vector. In this section you will study a third vector operation, called the **dot product.** This product yields a scalar, rather than a vector.

Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ is

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2.$$

The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

NOTE Because the dot product of two vectors yields a scalar, it is also called the **inner product** (or **scalar product**) of the two vectors.

THEOREM 11.4 Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ Commutative Property
2. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$ Distributive Property
3. $c(\mathbf{u}\cdot\mathbf{v}) = c\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot c\mathbf{v}$
4. $\mathbf{0}\cdot\mathbf{v} = 0$
5. $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$

Proof To prove the first property, let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3\\ &= v_1u_1 + v_2u_2 + v_3u_3\\ &= \mathbf{v}\cdot\mathbf{u}.\end{aligned}$$

For the fifth property, let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\begin{aligned}\mathbf{v}\cdot\mathbf{v} &= v_1^2 + v_2^2 + v_3^2\\ &= \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2\\ &= \|\mathbf{v}\|^2.\end{aligned}$$

Proofs of the other properties are left to you.

EXAMPLE 1 Finding Dot Products

Given $\mathbf{u} = \langle 2, -2 \rangle$, $\mathbf{v} = \langle 5, 8 \rangle$, and $\mathbf{w} = \langle -4, 3 \rangle$, find each of the following.

a. $\mathbf{u} \cdot \mathbf{v}$ **b.** $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

c. $\mathbf{u} \cdot (2\mathbf{v})$ **d.** $\|\mathbf{w}\|^2$

Solution

a. $\mathbf{u} \cdot \mathbf{v} = \langle 2, -2 \rangle \cdot \langle 5, 8 \rangle = 2(5) + (-2)(8) = -6$

b. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -6\langle -4, 3 \rangle = \langle 24, -18 \rangle$

c. $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-6) = -12$ Theorem 11.4

d. $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ Theorem 11.4

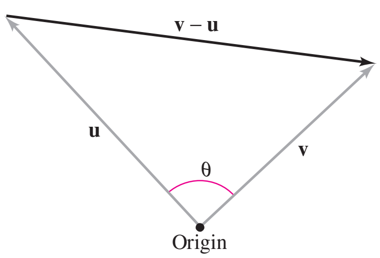
$= \langle -4, 3 \rangle \cdot \langle -4, 3 \rangle$ Substitute $\langle -4, 3 \rangle$ for $\mathbf{w}$.

$= (-4)(-4) + (3)(3)$ Definition of dot product

$= 25$ Simplify.

Notice that the result of part (b) is a *vector* quantity, whereas the results of the other three parts are *scalar* quantities.

Angle Between Two Vectors



The angle between two vectors
Figure 11.24

The **angle between two nonzero vectors** is the angle θ, $0 \le \theta \le \pi$, between their respective standard position vectors, as shown in Figure 11.24. The next theorem shows how to find this angle using the dot product. (Note that the angle between the zero vector and another vector is not defined here.)

THEOREM 11.5 Angle Between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Proof Consider the triangle determined by vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} - \mathbf{u}$, as shown in Figure 11.24. By the Law of Cosines, you can write

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

Using the properties of the dot product, the left side can be rewritten as

$$\begin{aligned}
\|\mathbf{v} - \mathbf{u}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \\
&= (\mathbf{v} - \mathbf{u}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{u}) \cdot \mathbf{u} \\
&= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} \\
&= \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2
\end{aligned}$$

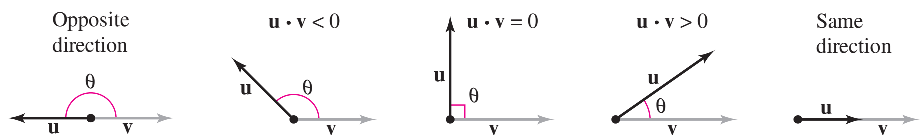
and substitution back into the Law of Cosines yields

$$\begin{aligned}
\|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\
-2\mathbf{u} \cdot \mathbf{v} &= -2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\
\cos\theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.
\end{aligned}$$

If the angle between two vectors is known, rewriting Theorem 11.5 in the form

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta \qquad \text{Alternative form of dot product}$$

produces an alternative way to calculate the dot product. From this form, you can see that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u} \cdot \mathbf{v}$ and $\cos \theta$ will always have the same sign. Figure 11.25 shows the possible orientations of two vectors.



$\theta = \pi$	$\pi/2 < \theta < \pi$	$\theta = \pi/2$	$0 < \theta < \pi/2$	$\theta = 0$
$\cos \theta = -1$	$-1 < \cos \theta < 0$	$\cos \theta = 0$	$0 < \cos \theta < 1$	$\cos \theta = 1$

Figure 11.25

From Theorem 11.5, you can see that two nonzero vectors meet at a right angle if and only if their dot product is zero. Two such vectors are said to be **orthogonal.**

Definition of Orthogonal Vectors

The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

NOTE The terms "perpendicular," "orthogonal," and "normal" all mean essentially the same thing—meeting at right angles. However, it is common to say that two vectors are *orthogonal*, two lines or planes are *perpendicular*, and a vector is *normal* to a given line or plane.

From this definition, it follows that the zero vector is orthogonal to every vector $\mathbf{u}$, because $\mathbf{0} \cdot \mathbf{u} = 0$. Moreover, for $0 \leq \theta \leq \pi$, you know that $\cos \theta = 0$ if and only if $\theta = \pi/2$. So, you can use Theorem 11.5 to conclude that two *nonzero* vectors are orthogonal if and only if the angle between them is $\pi/2$.

EXAMPLE 2 Finding the Angle Between Two Vectors

For $\mathbf{u} = \langle 3, -1, 2 \rangle$, $\mathbf{v} = \langle -4, 0, 2 \rangle$, $\mathbf{w} = \langle 1, -1, -2 \rangle$, and $\mathbf{z} = \langle 2, 0, -1 \rangle$, find the angle between each pair of vectors.

a. $\mathbf{u}$ and $\mathbf{v}$ **b.** $\mathbf{u}$ and $\mathbf{w}$ **c.** $\mathbf{v}$ and $\mathbf{z}$

Solution

a. $\cos \theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \dfrac{-12 + 0 + 4}{\sqrt{14}\sqrt{20}} = \dfrac{-8}{2\sqrt{14}\sqrt{5}} = \dfrac{-4}{\sqrt{70}}$

Because $\mathbf{u} \cdot \mathbf{v} < 0$, $\theta = \arccos \dfrac{-4}{\sqrt{70}} \approx 2.069$ radians.

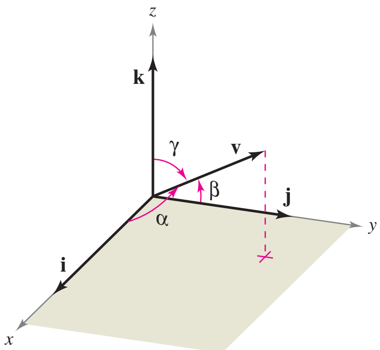
b. $\cos \theta = \dfrac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\| \, \|\mathbf{w}\|} = \dfrac{3 + 1 - 4}{\sqrt{14}\sqrt{6}} = \dfrac{0}{\sqrt{84}} = 0$

Because $\mathbf{u} \cdot \mathbf{w} = 0$, $\mathbf{u}$ and $\mathbf{w}$ are *orthogonal*. So, $\theta = \pi/2$.

c. $\cos \theta = \dfrac{\mathbf{v} \cdot \mathbf{z}}{\|\mathbf{v}\| \, \|\mathbf{z}\|} = \dfrac{-8 + 0 - 2}{\sqrt{20}\sqrt{5}} = \dfrac{-10}{\sqrt{100}} = -1$

Consequently, $\theta = \pi$. Note that $\mathbf{v}$ and $\mathbf{z}$ are parallel, with $\mathbf{v} = -2\mathbf{z}$.

Direction Cosines



Direction angles
Figure 11.26

For a vector in the plane, you have seen that it is convenient to measure direction in terms of the angle, measured counterclockwise, *from* the positive x-axis *to* the vector. In space it is more convenient to measure direction in terms of the angles *between* the nonzero vector $\mathbf{v}$ and the three unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, as shown in Figure 11.26. The angles α, β, and γ are the **direction angles of v,** and $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the **direction cosines of v.** Because

$$\mathbf{v}\cdot\mathbf{i} = \|\mathbf{v}\|\,\|\mathbf{i}\|\cos\alpha = \|\mathbf{v}\|\cos\alpha$$

and

$$\mathbf{v}\cdot\mathbf{i} = \langle v_1, v_2, v_3\rangle \cdot \langle 1, 0, 0\rangle = v_1$$

it follows that $\cos\alpha = v_1/\|\mathbf{v}\|$. By similar reasoning with the unit vectors $\mathbf{j}$ and $\mathbf{k}$, you have

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}$$ α is the angle between $\mathbf{v}$ and $\mathbf{i}$.

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|}$$ β is the angle between $\mathbf{v}$ and $\mathbf{j}$.

$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|}.$$ γ is the angle between $\mathbf{v}$ and $\mathbf{k}$.

Consequently, any nonzero vector $\mathbf{v}$ in space has the normalized form

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|}\mathbf{i} + \frac{v_2}{\|\mathbf{v}\|}\mathbf{j} + \frac{v_3}{\|\mathbf{v}\|}\mathbf{k} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

and because $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector, it follows that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

EXAMPLE 3 Finding Direction Angles

Find the direction cosines and angles for the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Solution Because $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$, you can write the following.

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|} = \frac{2}{\sqrt{29}} \Rightarrow \alpha \approx 68.2^\circ$$ Angle between $\mathbf{v}$ and $\mathbf{i}$

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|} = \frac{3}{\sqrt{29}} \Rightarrow \beta \approx 56.1^\circ$$ Angle between $\mathbf{v}$ and $\mathbf{j}$

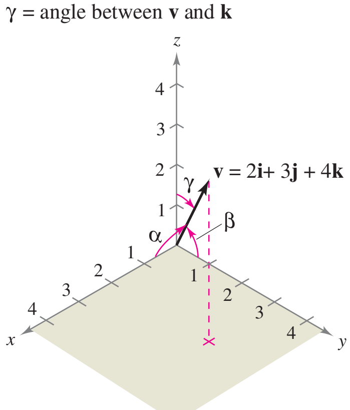
$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|} = \frac{4}{\sqrt{29}} \Rightarrow \gamma \approx 42.0^\circ$$ Angle between $\mathbf{v}$ and $\mathbf{k}$

Furthermore, the sum of the squares of the direction cosines is

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \frac{4}{29} + \frac{9}{29} + \frac{16}{29}\\ &= \frac{29}{29}\\ &= 1.\end{aligned}$$

See Figure 11.27.

α = angle between $\mathbf{v}$ and $\mathbf{i}$
β = angle between $\mathbf{v}$ and $\mathbf{j}$
γ = angle between $\mathbf{v}$ and $\mathbf{k}$



The direction angles of $\mathbf{v}$
Figure 11.27

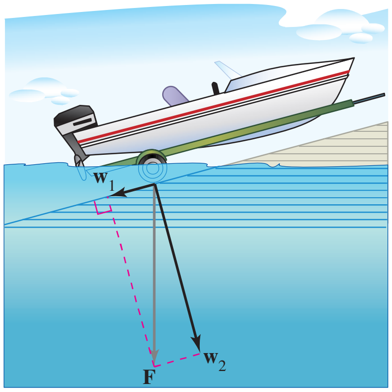
Projections and Vector Components

You have already seen applications in which two vectors are added to produce a resultant vector. Many applications in physics and engineering pose the reverse problem—decomposing a given vector into the sum of two **vector components.** The following physical example enables you to see the usefulness of this procedure.

Consider a boat on an inclined ramp, as shown in Figure 11.28. The force $\mathbf{F}$ due to gravity pulls the boat *down* the ramp and *against* the ramp. These two forces, $\mathbf{w}_1$ and $\mathbf{w}_2$, are orthogonal—they are called the vector components of $\mathbf{F}$.

$$\mathbf{F} = \mathbf{w}_1 + \mathbf{w}_2 \qquad \text{Vector components of } \mathbf{F}$$

The forces $\mathbf{w}_1$ and $\mathbf{w}_2$ help you analyze the effect of gravity on the boat. For example, $\mathbf{w}_1$ indicates the force necessary to keep the boat from rolling down the ramp, whereas $\mathbf{w}_2$ indicates the force that the tires must withstand.

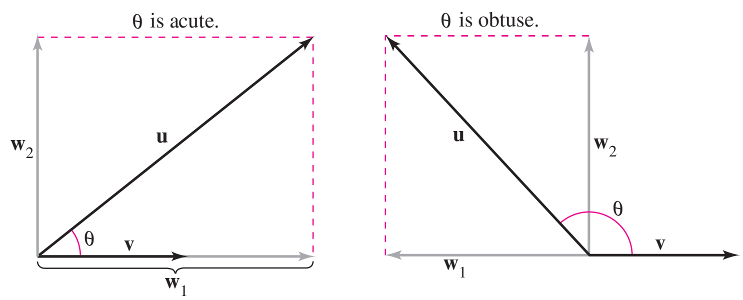


The force due to gravity pulls the boat against the ramp and down the ramp.
Figure 11.28

Definition of Projection and Vector Components

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. Moreover, let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure 11.29.

1. $\mathbf{w}_1$ is called the **projection of u onto v** or the **vector component of u along v,** and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.
2. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ is called the **vector component of u orthogonal to v.**



$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} =$ projection of $\mathbf{u}$ onto $\mathbf{v} =$ vector component of $\mathbf{u}$ along $\mathbf{v}$
$\mathbf{w}_2 =$ vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$
Figure 11.29

EXAMPLE 4 Finding a Vector Component of u Orthogonal to v

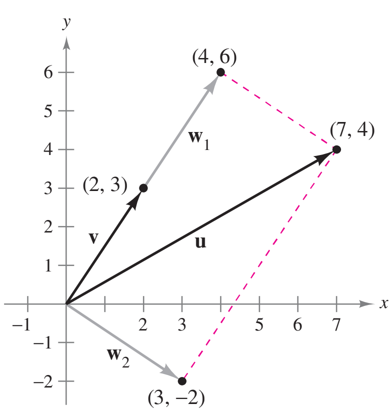
Find the vector component of $\mathbf{u} = \langle 7, 4\rangle$ that is orthogonal to $\mathbf{v} = \langle 2, 3\rangle$, given that $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 4, 6\rangle$ and

$$\mathbf{u} = \langle 7, 4\rangle = \mathbf{w}_1 + \mathbf{w}_2.$$

Solution Because $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$, it follows that $\mathbf{w}_2$ is the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$. So, you have

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{u} - \mathbf{w}_1\\ &= \langle 7, 4\rangle - \langle 4, 6\rangle\\ &= \langle 3, -2\rangle.\end{aligned}$$

Check to see that $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure 11.30.



$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$
Figure 11.30

From Example 4, you can see that it is easy to find the vector component $\mathbf{w}_2$ once you have found the projection, $\mathbf{w}_1$, of $\mathbf{u}$ onto $\mathbf{v}$. To find this projection, use the dot product given in the theorem below, which you will prove in Exercise 90.

NOTE Note the distinction between the terms "component" and "vector component." For example, using the standard unit vectors with $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, u_1 is the *component* of $\mathbf{u}$ in the direction of $\mathbf{i}$ and $u_1\mathbf{i}$ is the *vector component* in the direction of $\mathbf{i}$.

THEOREM 11.6 Projection Using the Dot Product

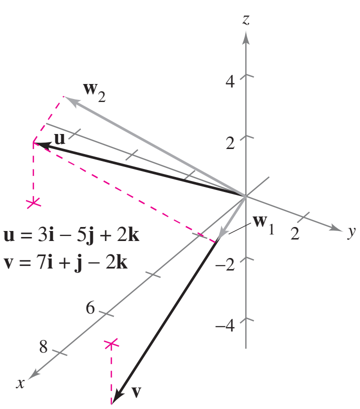
If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

The projection of $\mathbf{u}$ onto $\mathbf{v}$ can be written as a scalar multiple of a unit vector in the direction of $\mathbf{v}$. That is,

$$\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} = (k)\frac{\mathbf{v}}{\|\mathbf{v}\|} \quad\Longrightarrow\quad k = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\|\cos\theta.$$

The scalar k is called the **component of u in the direction of v.**



$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$
Figure 11.31

EXAMPLE 5 Decomposing a Vector into Vector Components

Find the projection of $\mathbf{u}$ onto $\mathbf{v}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ for the vectors $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 7\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ shown in Figure 11.31.

Solution The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\mathbf{w}_1 = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{12}{54}\right)(7\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{14}{9}\mathbf{i} + \frac{2}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}.$$

The vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ is the vector

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}) - \left(\frac{14}{9}\mathbf{i} + \frac{2}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}\right) = \frac{13}{9}\mathbf{i} - \frac{47}{9}\mathbf{j} + \frac{22}{9}\mathbf{k}.$$

EXAMPLE 6 Finding a Force

A 600-pound boat sits on a ramp inclined at 30°, as shown in Figure 11.32. What force is required to keep the boat from rolling down the ramp?

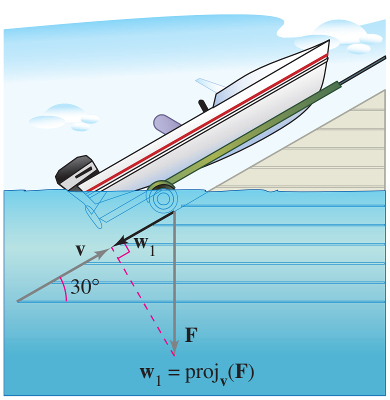


Figure 11.32

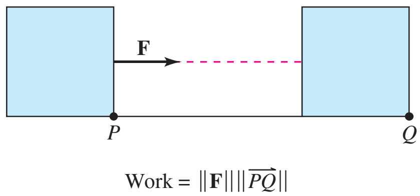
Solution Because the force due to gravity is vertical and downward, you can represent the gravitational force by the vector $\mathbf{F} = -600\mathbf{j}$. To find the force required to keep the boat from rolling down the ramp, project $\mathbf{F}$ onto a unit vector $\mathbf{v}$ in the direction of the ramp, as follows.

$$\mathbf{v} = \cos 30°\mathbf{i} + \sin 30°\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \qquad \text{Unit vector along ramp}$$

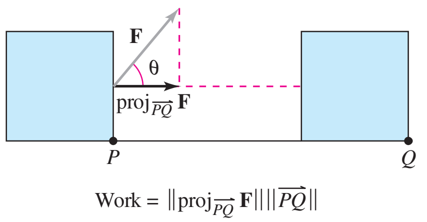
Therefore, the projection of $\mathbf{F}$ onto $\mathbf{v}$ is given by

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = (\mathbf{F}\cdot\mathbf{v})\mathbf{v} = (-600)\left(\frac{1}{2}\right)\mathbf{v} = -300\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right).$$

The magnitude of this force is 300, and therefore a force of 300 pounds is required to keep the boat from rolling down the ramp.



(a) Force acts along the line of motion.



(b) Force acts at angle θ with the line of motion.

Figure 11.33

Work

The work W done by the constant force $\mathbf{F}$ acting along the line of motion of an object is given by

$$W = (\text{magnitude of force})(\text{distance}) = \|\mathbf{F}\|\,\|\overrightarrow{PQ}\|$$

as shown in Figure 11.33(a). If the constant force $\mathbf{F}$ is not directed along the line of motion, you can see from Figure 11.33(b) that the work W done by the force is

$$W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\| = (\cos\theta)\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| = \mathbf{F}\cdot\overrightarrow{PQ}.$$

This notion of work is summarized in the following definition.

Definition of Work

The work W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following.

1. $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\|$ Projection form

2. $W = \mathbf{F}\cdot\overrightarrow{PQ}$ Dot product form

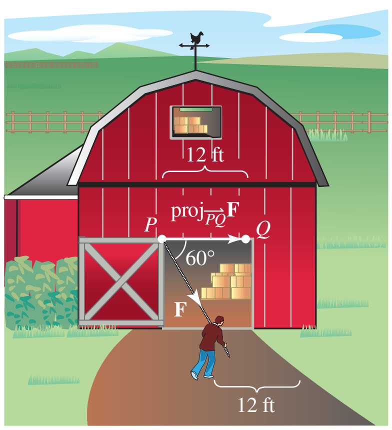


Figure 11.34

EXAMPLE 7 Finding Work

To close a sliding door, a person pulls on a rope with a constant force of 50 pounds at a constant angle of 60°, as shown in Figure 11.34. Find the work done in moving the door 12 feet to its closed position.

Solution Using a projection, you can calculate the work as follows.

$$\begin{aligned} W &= \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\| && \text{Projection form for work} \\ &= \cos(60^\circ)\,\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| \\ &= \frac{1}{2}(50)(12) \\ &= 300 \text{ foot-pounds} \end{aligned}$$

Exercises for Section 11.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find (a) $\mathbf{u}\cdot\mathbf{v}$, (b) $\mathbf{u}\cdot\mathbf{u}$, (c) $\|\mathbf{u}\|^2$, (d) $(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$, and (e) $\mathbf{u}\cdot(2\mathbf{v})$.

1. $\mathbf{u} = \langle 3, 4\rangle,\quad \mathbf{v} = \langle 2, -3\rangle$

2. $\mathbf{u} = \langle 4, 10\rangle,\quad \mathbf{v} = \langle -2, 3\rangle$

3. $\mathbf{u} = \langle 5, -1\rangle,\quad \mathbf{v} = \langle -3, 2\rangle$

4. $\mathbf{u} = \langle -4, 8\rangle,\quad \mathbf{v} = \langle 6, 3\rangle$

5. $\mathbf{u} = \langle 2, -3, 4\rangle,\quad \mathbf{v} = \langle 0, 6, 5\rangle$

6. $\mathbf{u} = \mathbf{i},\quad \mathbf{v} = \mathbf{i}$

7. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$
$\mathbf{v} = \mathbf{i} - \mathbf{k}$

8. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
$\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

In Exercises 9 and 10, find $\mathbf{u}\cdot\mathbf{v}$.

9. $\|\mathbf{u}\| = 8$, $\|\mathbf{v}\| = 5$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/3$.

10. $\|\mathbf{u}\| = 40$, $\|\mathbf{v}\| = 25$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $5\pi/6$.

In Exercises 11–18, find the angle θ between the vectors.

11. $\mathbf{u} = \langle 1, 1\rangle, \mathbf{v} = \langle 2, -2\rangle$

12. $\mathbf{u} = \langle 3, 1\rangle, \mathbf{v} = \langle 2, -1\rangle$

13. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}, \mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$

14. $\mathbf{u} = \cos\left(\dfrac{\pi}{6}\right)\mathbf{i} + \sin\left(\dfrac{\pi}{6}\right)\mathbf{j}$
$\mathbf{v} = \cos\left(\dfrac{3\pi}{4}\right)\mathbf{i} + \sin\left(\dfrac{3\pi}{4}\right)\mathbf{j}$

15. $\mathbf{u} = \langle 1, 1, 1\rangle$
$\mathbf{v} = \langle 2, 1, -1\rangle$

16. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
$\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

17. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$
$\mathbf{v} = -2\mathbf{j} + 3\mathbf{k}$

18. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
$\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

In Exercises 19–26, determine whether u and v are orthogonal, parallel, or neither.

19. $\mathbf{u} = \langle 4, 0\rangle,\quad \mathbf{v} = \langle 1, 1\rangle$

20. $\mathbf{u} = \langle 2, 18\rangle,\quad \mathbf{v} = \left\langle \frac{3}{2}, -\frac{1}{6}\right\rangle$

21. $\mathbf{u} = \langle 4, 3 \rangle$
$\mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3} \right\rangle$

22. $\mathbf{u} = -\frac{1}{3}(\mathbf{i} - 2\mathbf{j})$
$\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

23. $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$
$\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

24. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

25. $\mathbf{u} = \langle 2, -3, 1 \rangle$
$\mathbf{v} = \langle -1, -1, -1 \rangle$

26. $\mathbf{u} = \langle \cos\theta, \sin\theta, -1 \rangle$
$\mathbf{v} = \langle \sin\theta, -\cos\theta, 0 \rangle$

In Exercises 27–30, the vertices of a triangle are given. Determine whether the triangle is an acute triangle, an obtuse triangle, or a right triangle. Explain your reasoning.

27. $(1, 2\ 0), (0, 0, 0), (-2, 1, 0)$

28. $(-3, 0, 0), (0, 0, 0), (1, 2, 3)$

29. $(2, -3, 4), (0, 1, 2), (-1, 2, 0)$

30. $(2, -7, 3), (-1, 5, 8), (4, 6, -1)$

In Exercises 31–34, find the direction cosines of u and demonstrate that the sum of the squares of the direction cosines is 1.

31. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

32. $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

33. $\mathbf{u} = \langle 0, 6, -4 \rangle$

34. $\mathbf{u} = \langle a, b, c \rangle$

In Exercises 35–38, find the direction angles of the vector.

35. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

36. $\mathbf{u} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

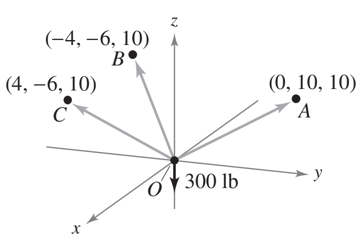
37. $\mathbf{u} = \langle -1, 5, 2 \rangle$

38. $\mathbf{u} = \langle -2, 6, 1 \rangle$

In Exercises 39 and 40, use a graphing utility to find the magnitude and direction angles of the resultant of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with initial points at the origin. The magnitude and terminal point of each vector are given.

	Vector	*Magnitude*	*Terminal Point*
39.	$\mathbf{F}_1$	50 lb	$(10, 5, 3)$
	$\mathbf{F}_2$	80 lb	$(12, 7, -5)$
40.	$\mathbf{F}_1$	300 N	$(-20, -10, 5)$
	$\mathbf{F}_2$	100 N	$(5, 15, 0)$

41. ***Load-Supporting Cables*** A load is supported by three cables, as shown in the figure. Find the direction angles of the load-supporting cable OA.



42. ***Load-Supporting Cables*** The tension in the cable OA in Exercise 41 is 200 newtons. Determine the weight of the load.

In Exercises 43–46, find the component of u that is orthogonal to v, given $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.

43. $\mathbf{u} = \langle 6, 7 \rangle, \quad \mathbf{v} = \langle 1, 4 \rangle, \quad \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 2, 8 \rangle$

44. $\mathbf{u} = \langle 9, 7 \rangle, \quad \mathbf{v} = \langle 1, 3 \rangle, \quad \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 3, 9 \rangle$

45. $\mathbf{u} = \langle 0, 3, 3 \rangle, \quad \mathbf{v} = \langle -1, 1, 1 \rangle, \quad \text{proj}_{\mathbf{v}}\mathbf{u} = \langle -2, 2, 2 \rangle$

46. $\mathbf{u} = \langle 8, 2, 0 \rangle, \quad \mathbf{v} = \langle 2, 1, -1 \rangle, \quad \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 6, 3, -3 \rangle$

In Exercises 47–50, (a) find the projection of u onto v, and (b) find the vector component of u orthogonal to v.

47. $\mathbf{u} = \langle 2, 3 \rangle, \quad \mathbf{v} = \langle 5, 1 \rangle$

48. $\mathbf{u} = \langle 2, -3 \rangle, \quad \mathbf{v} = \langle 3, 2 \rangle$

49. $\mathbf{u} = \langle 2, 1, 2 \rangle$
$\mathbf{v} = \langle 0, 3, 4 \rangle$

50. $\mathbf{u} = \langle 1, 0, 4 \rangle$
$\mathbf{v} = \langle 3, 0, 2 \rangle$

Writing About Concepts

51. Define the dot product of vectors $\mathbf{u}$ and $\mathbf{v}$.

52. State the definition of orthogonal vectors. If vectors are neither parallel nor orthogonal, how do you find the angle between them? Explain.

53. What is known about θ, the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, if

(a) $\mathbf{u} \cdot \mathbf{v} = 0$? (b) $\mathbf{u} \cdot \mathbf{v} > 0$? (c) $\mathbf{u} \cdot \mathbf{v} < 0$?

54. Determine which of the following are defined for nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Explain your reasoning.

(a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ (b) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
(c) $\mathbf{u} \cdot \mathbf{v} + \mathbf{w}$ (d) $\|\mathbf{u}\| \cdot (\mathbf{v} + \mathbf{w})$

55. Describe direction cosines and direction angles of a vector $\mathbf{v}$.

56. Give a geometric description of the projection of $\mathbf{u}$ onto $\mathbf{v}$.

57. What can be said about the vectors $\mathbf{u}$ and $\mathbf{v}$ if (a) the projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{u}$ and (b) the projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{0}$?

58. If the projection of $\mathbf{u}$ onto $\mathbf{v}$ has the same magnitude as the projection of $\mathbf{v}$ onto $\mathbf{u}$, can you conclude that $\|\mathbf{u}\| = \|\mathbf{v}\|$? Explain.

59. ***Revenue*** The vector $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$ gives the numbers of hamburgers, chicken sandwiches, and cheeseburgers, respectively, sold at a fast-food restaurant in one week. The vector $\mathbf{v} = \langle 1.35, 2.65, 1.85 \rangle$ gives the prices (in dollars) per unit for the three food items. Find the dot product $\mathbf{u} \cdot \mathbf{v}$, and explain what information it gives.

60. ***Revenue*** Repeat Exercise 59 after increasing prices by 4%. Identify the vector operation used to increase prices by 4%.

61. ***Programming*** Given vectors $\mathbf{u}$ and $\mathbf{v}$ in component form, write a program for a graphing utility in which the output is (a) $\|\mathbf{u}\|$, (b) $\|\mathbf{v}\|$, and (c) the angle between $\mathbf{u}$ and $\mathbf{v}$.

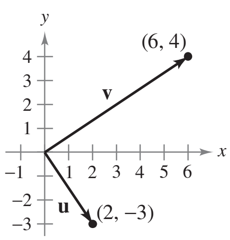
62. ***Programming*** Use the program you wrote in Exercise 61 to find the angle between the vectors $\mathbf{u} = \langle 8, -4, 2 \rangle$ and $\mathbf{v} = \langle 2, 5, 2 \rangle$.

63. ***Programming*** Given vectors $\mathbf{u}$ and $\mathbf{v}$ in component form, write a program for a graphing utility in which the output is the component form of the projection of $\mathbf{u}$ onto $\mathbf{v}$.

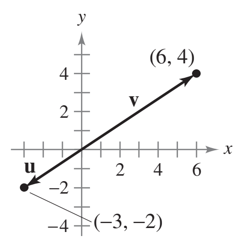
64. ***Programming*** Use the program you wrote in Exercise 63 to find the projection of $\mathbf{u}$ onto $\mathbf{v}$ for $\mathbf{u} = \langle 5, 6, 2 \rangle$ and $\mathbf{v} = \langle -1, 3, 4 \rangle$.

Think About It **In Exercises 65 and 66, use the figure to determine mentally the projection of u onto v. (The coordinates of the terminal points of the vectors in standard position are given.) Verify your results analytically.**

65.



66.



In Exercises 67–70, find two vectors in opposite directions that are orthogonal to the vector u. (The answers are not unique.)

67. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j}$

68. $\mathbf{u} = -8\mathbf{i} + 3\mathbf{j}$

69. $\mathbf{u} = \langle 3, 1, -2 \rangle$

70. $\mathbf{u} = \langle 0, -3, 6 \rangle$

71. ***Braking Load*** A 48,000-pound truck is parked on a 10° slope (see figure). Assume the only force to overcome is that due to gravity. Find (a) the force required to keep the truck from rolling down the hill and (b) the force perpendicular to the hill.

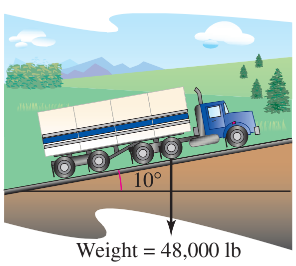


Figure for 71

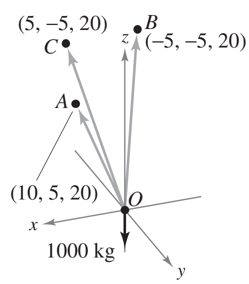


Figure for 72

72. ***Load-Supporting Cables*** Find the magnitude of the projection of the load-supporting cable OA onto the positive z-axis as shown in the figure.

73. ***Work*** An object is pulled 10 feet across a floor, using a force of 85 pounds. The direction of the force is 60° above the horizontal (see figure). Find the work done.

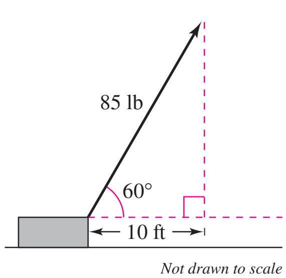


Figure for 73

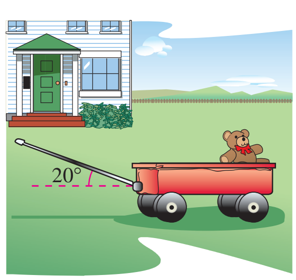


Figure for 74

74. ***Work*** A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal (see figure in left column). Find the work done in pulling the wagon 50 feet.

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{v} = \mathbf{w}$.

76. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to $\mathbf{w}$, then $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{w}$.

77. Find the angle between a cube's diagonal and one of its edges.

78. Find the angle between the diagonal of a cube and the diagonal of one of its sides.

In Exercises 79–82, (a) find the unit tangent vectors to each curve at their points of intersection and (b) find the angles $(0 \leq \theta \leq 90°)$ between the curves at their points of intersection.

79. $y = x^2, \quad y = x^{1/3}$

80. $y = x^3, \quad y = x^{1/3}$

81. $y = 1 - x^2, \quad y = x^2 - 1$

82. $(y + 1)^2 = x, \quad y = x^3 - 1$

83. Use vectors to prove that the diagonals of a rhombus are perpendicular.

84. Use vectors to prove that a parallelogram is a rectangle if and only if its diagonals are equal in length.

85. ***Bond Angle*** Consider a regular tetrahedron with vertices $(0, 0, 0)$, $(k, k, 0)$, $(k, 0, k)$, and $(0, k, k)$, where k is a positive real number.

(a) Sketch the graph of the tetrahedron.

(b) Find the length of each edge.

(c) Find the angle between any two edges.

(d) Find the angle between the line segments from the centroid $(k/2, k/2, k/2)$ to two vertices. This is the bond angle for a molecule such as CH_4 or $PbCl_4$, where the structure of the molecule is a tetrahedron.

86. Consider the vectors

$$\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle$$

and

$$\mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$$

where $\alpha > \beta$. Find the dot product of the vectors and use the result to prove the identity

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

87. Prove that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$.

88. Prove the **Cauchy-Schwarz Inequality** $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$.

89. Prove the triangle inequality $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

90. Prove Theorem 11.6.

Section 11.4 The Cross Product of Two Vectors in Space

- **Find the cross product of two vectors in space.**
- **Use the triple scalar product of three vectors in space.**

The Cross Product

Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors. In this section you will study a product that will yield such a vector. It is called the **cross product,** and it is most conveniently defined and calculated using the standard unit vector form. Because the cross product yields a vector, it is also called the **vector product.**

Definition of Cross Product of Two Vectors in Space

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

NOTE Be sure you see that this definition applies only to three-dimensional vectors. The cross product is not defined for two-dimensional vectors.

A convenient way to calculate $\mathbf{u} \times \mathbf{v}$ is to use the following *determinant form* with cofactor expansion. (This 3×3 determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because the entries of the corresponding matrix are not all real numbers.)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{array}{l} \\ \leftarrow \text{Put "}\mathbf{u}\text{" in Row 2.} \\ \leftarrow \text{Put "}\mathbf{v}\text{" in Row 3.} \end{array}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Note the minus sign in front of the **j**-component. Each of the three 2×2 determinants can be evaluated by using the following diagonal pattern.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Here are a couple of examples.

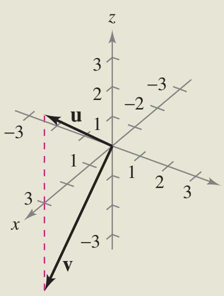
$$\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} = (2)(-1) - (4)(3) = -2 - 12 = -14$$

$$\begin{vmatrix} 4 & 0 \\ -6 & 3 \end{vmatrix} = (4)(3) - (0)(-6) = 12$$

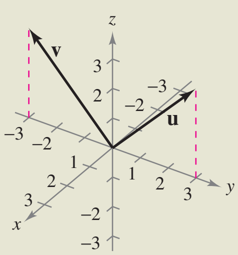
EXPLORATION

Geometric Property of the Cross Product Three pairs of vectors are shown below. Use the definition to find the cross product of each pair. Sketch all three vectors in a three-dimensional system. Describe any relationships among the three vectors. Use your description to write a conjecture about **u**, **v**, and **u** × **v**.

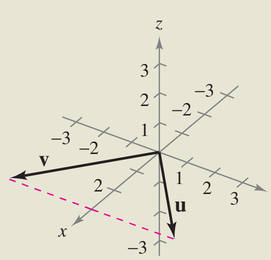
a. $\mathbf{u} = \langle 3, 0, 3\rangle$, $\mathbf{v} = \langle 3, 0, -3\rangle$



b. $\mathbf{u} = \langle 0, 3, 3\rangle$, $\mathbf{v} = \langle 0, -3, 3\rangle$



c. $\mathbf{u} = \langle 3, 3, 0\rangle$, $\mathbf{v} = \langle 3, -3, 0\rangle$



Notation for Dot and Cross Products

The notation for the dot product and cross product of vectors was first introduced by the American physicist Josiah Willard Gibbs (1839–1903). In the early 1880s, Gibbs built a system to represent physical quantities called "vector analysis." The system was a departure from Hamilton's theory of quaternions.

EXAMPLE 1 Finding the Cross Product

Given $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find each of the following.

a. $\mathbf{u} \times \mathbf{v}$ **b.** $\mathbf{v} \times \mathbf{u}$ **c.** $\mathbf{v} \times \mathbf{v}$

Solution

$$\textbf{a. } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$
$$= (4 - 1)\mathbf{i} - (-2 - 3)\mathbf{j} + (1 + 6)\mathbf{k}$$
$$= 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

$$\textbf{b. } \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$
$$= (1 - 4)\mathbf{i} - (3 + 2)\mathbf{j} + (-6 - 1)\mathbf{k}$$
$$= -3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

Note that this result is the negative of that in part (a).

$$\textbf{c. } \mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

The results obtained in Example 1 suggest some interesting *algebraic* properties of the cross product. For instance, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$, and $\mathbf{v} \times \mathbf{v} = \mathbf{0}$. These properties, and several others, are summarized in the following theorem.

THEOREM 11.7 Algebraic Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Proof To prove Property 1, let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then,

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

and

$$\mathbf{v} \times \mathbf{u} = (v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k}$$

which implies that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. Proofs of Properties 2, 3, 5, and 6 are left as exercises (see Exercises 57–60).

Note that Property 1 of Theorem 11.7 indicates that the cross product is *not commutative*. In particular, this property indicates that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The following theorem lists some other *geometric* properties of the cross product of two vectors.

NOTE It follows from Properties 1 and 2 in Theorem 11.8 that if $\mathbf{n}$ is a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \times \mathbf{v} = \pm(\|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta)\mathbf{n}.$$

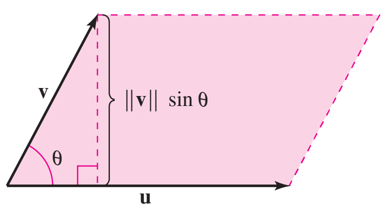
THEOREM 11.8 Geometric Properties of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
4. $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

Proof To prove Property 2, note because $\cos\theta = (\mathbf{u} \cdot \mathbf{v})/(\|\mathbf{u}\|\,\|\mathbf{v}\|)$, it follows that

$$\begin{aligned}
\|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta &= \|\mathbf{u}\|\ \|\mathbf{v}\|\sqrt{1 - \cos^2\theta} \\
&= \|\mathbf{u}\|\ \|\mathbf{v}\|\sqrt{1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \\
&= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\
&= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2} \\
&= \sqrt{(u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2} \\
&= \|\mathbf{u} \times \mathbf{v}\|.
\end{aligned}$$



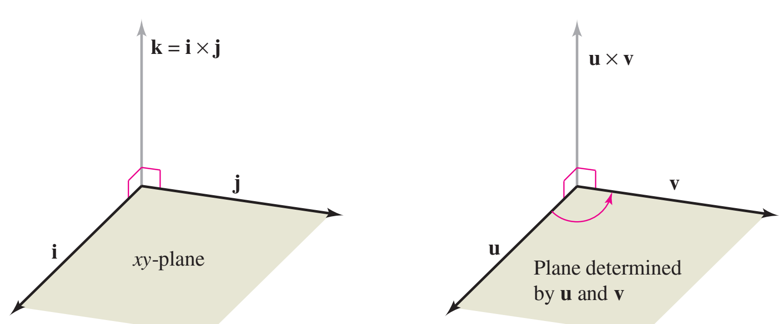
The vectors $\mathbf{u}$ and $\mathbf{v}$ form adjacent sides of a parallelogram.
Figure 11.35

To prove Property 4, refer to Figure 11.35, which is a parallelogram having $\mathbf{v}$ and $\mathbf{u}$ as adjacent sides. Because the height of the parallelogram is $\|\mathbf{v}\| \sin\theta$, the area is

$$\begin{aligned}
\text{Area} &= (\text{base})(\text{height}) \\
&= \|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta \\
&= \|\mathbf{u} \times \mathbf{v}\|.
\end{aligned}$$

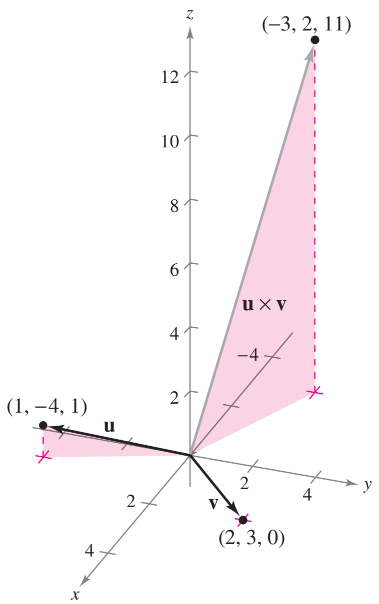
Proofs of Properties 1 and 3 are left as exercises (see Exercises 61 and 62).

Both $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are perpendicular to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. One way to remember the orientations of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is to compare them with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, as shown in Figure 11.36. The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system*, whereas the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} \times \mathbf{u}$ form a *left-handed system*.



Right-handed systems
Figure 11.36

EXAMPLE 2 Using the Cross Product



The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
Figure 11.37

Find a unit vector that is orthogonal to both

$$\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{v} = 2\mathbf{i} + 3\mathbf{j}.$$

Solution The cross product $\mathbf{u} \times \mathbf{v}$, as shown in Figure 11.37, is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix} \qquad \text{Cross product}$$

$$= -3\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}$$

Because

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 11^2} = \sqrt{134}$$

a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

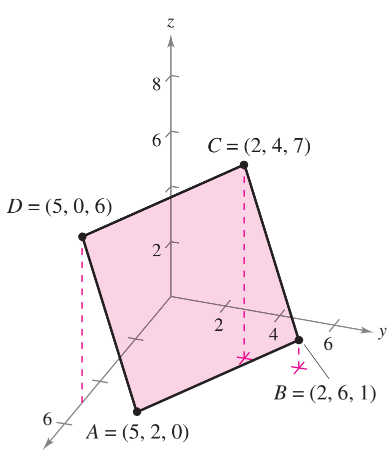
$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{3}{\sqrt{134}}\mathbf{i} + \frac{2}{\sqrt{134}}\mathbf{j} + \frac{11}{\sqrt{134}}\mathbf{k}.$$

NOTE In Example 2, note that you could have used the cross product $\mathbf{v} \times \mathbf{u}$ to form a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. With that choice, you would have obtained the negative of the unit vector found in the example.

EXAMPLE 3 Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram, and find its area.

$$A = (5, 2, 0) \qquad B = (2, 6, 1)$$
$$C = (2, 4, 7) \qquad D = (5, 0, 6)$$



The area of the parallelogram is approximately 32.19.
Figure 11.38

Solution From Figure 11.38 you can see that the sides of the quadrilateral correspond to the following four vectors.

$$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} \qquad \overrightarrow{CD} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB}$$
$$\overrightarrow{AD} = 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} \qquad \overrightarrow{CB} = 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD}$$

So, $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$, and you can conclude that the quadrilateral is a parallelogram with $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as adjacent sides. Moreover, because

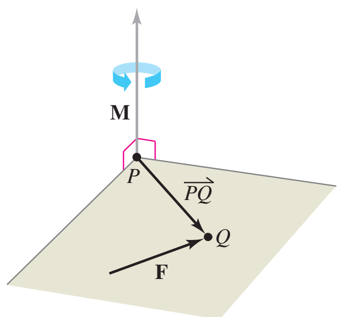
$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} \qquad \text{Cross product}$$

$$= 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$

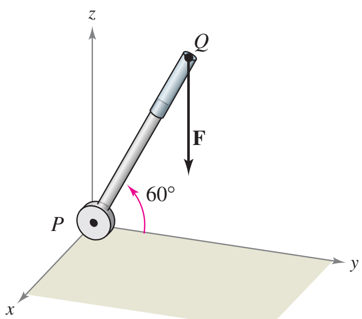
the area of the parallelogram is

$$\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{1036} \approx 32.19.$$

Is the parallelogram a rectangle? You can determine whether it is by finding the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.



The moment of **F** about P
Figure 11.39



A vertical force of 50 pounds is applied at point Q.
Figure 11.40

In physics, the cross product can be used to measure **torque**—the **moment M of a force F about a point *P*,** as shown in Figure 11.39. If the point of application of the force is Q, the moment of **F** about P is given by

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F}.$$

Moment of **F** about P

The magnitude of the moment **M** measures the tendency of the vector $\overrightarrow{PQ}$ to rotate counterclockwise (using the right-hand rule) about an axis directed along the vector **M**.

EXAMPLE 4 An Application of the Cross Product

A vertical force of 50 pounds is applied to the end of a one-foot lever that is attached to an axle at point P, as shown in Figure 11.40. Find the moment of this force about the point P when $\theta = 60°$.

Solution If you represent the 50-pound force as $\mathbf{F} = -50\mathbf{k}$ and the lever as

$$\overrightarrow{PQ} = \cos(60°)\mathbf{j} + \sin(60°)\mathbf{k} = \frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$$

the moment of **F** about P is given by

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -50 \end{vmatrix} = -25\mathbf{i}.$$

Moment of **F** about P

The magnitude of this moment is 25 foot-pounds.

NOTE In Example 4, note that the moment (the tendency of the lever to rotate about its axle) is dependent on the angle θ. When $\theta = \pi/2$, the moment is **0**. The moment is greatest when $\theta = 0$.

The Triple Scalar Product

For vectors **u**, **v**, and **w** in space, the dot product of **u** and $\mathbf{v} \times \mathbf{w}$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the **triple scalar product,** as defined in Theorem 11.9. The proof of this theorem is left as an exercise (see Exercise 56).

FOR FURTHER INFORMATION To see how the cross product is used to model the torque of the robot arm of a space shuttle, see the article "The Long Arm of Calculus" by Ethan Berkove and Rich Marchand in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

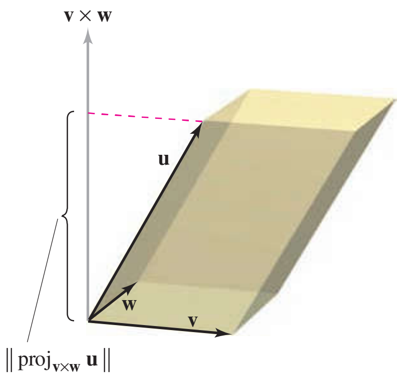
THEOREM 11.9 The Triple Scalar Product

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the triple scalar product is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

NOTE The value of a determinant is multiplied by -1 if two rows are interchanged. After two such interchanges, the value of the determinant will be unchanged. So, the following triple scalar products are equivalent.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$



Area of base $= \|\mathbf{v} \times \mathbf{w}\|$
Volume of parallelepiped $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$
Figure 11.41

If the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ do not lie in the same plane, the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ can be used to determine the volume of the parallelepiped (a polyhedron, all of whose faces are parallelograms) with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges, as shown in Figure 11.41. This is established in the following theorem.

THEOREM 11.10 Geometric Property of Triple Scalar Product

The volume V of a parallelepiped with vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

Proof In Figure 11.41, note that

$$\|\mathbf{v} \times \mathbf{w}\| = \text{area of base}$$

and

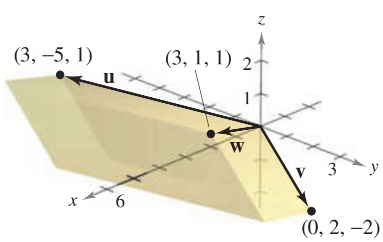
$$\|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| = \text{height of parallelepiped.}$$

Therefore, the volume is

$$\begin{aligned} V = (\text{height})(\text{area of base}) &= \|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|\|\mathbf{v} \times \mathbf{w}\| \\ &= \left|\frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{\|\mathbf{v} \times \mathbf{w}\|}\right|\|\mathbf{v} \times \mathbf{w}\| \\ &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|. \end{aligned}$$

EXAMPLE 5 Volume by the Triple Scalar Product

Find the volume of the parallelepiped shown in Figure 11.42 having $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.



The parallelepiped has a volume of 36.
Figure 11.42

Solution By Theorem 11.10, you have

$$\begin{aligned} V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| && \text{Triple scalar product} \\ &= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 3(4) + 5(6) + 1(-6) \\ &= 36. \end{aligned}$$

A natural consequence of Theorem 11.10 is that the volume of the parallelepiped is 0 if and only if the three vectors are coplanar. That is, if the vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ have the same initial point, they lie in the same plane if and only if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

Exercises for Section 11.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the cross product of the unit vectors and sketch your result.

1. $\mathbf{j} \times \mathbf{i}$ **2.** $\mathbf{i} \times \mathbf{j}$

3. $\mathbf{j} \times \mathbf{k}$ **4.** $\mathbf{k} \times \mathbf{j}$

5. $\mathbf{i} \times \mathbf{k}$ **6.** $\mathbf{k} \times \mathbf{i}$

In Exercises 7–10, find (a) $\mathbf{u} \times \mathbf{v}$, (b) $\mathbf{v} \times \mathbf{u}$, and (c) $\mathbf{v} \times \mathbf{v}$.

7. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

8. $\mathbf{u} = 3\mathbf{i} + 5\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

9. $\mathbf{u} = \langle 7, 3, 2 \rangle$, $\mathbf{v} = \langle 1, -1, 5 \rangle$

10. $\mathbf{u} = \langle 3, -2, -2 \rangle$, $\mathbf{v} = \langle 1, 5, 1 \rangle$

In Exercises 11–16, find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

11. $\mathbf{u} = \langle 2, -3, 1 \rangle$, $\mathbf{v} = \langle 1, -2, 1 \rangle$

12. $\mathbf{u} = \langle -1, 1, 2 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$

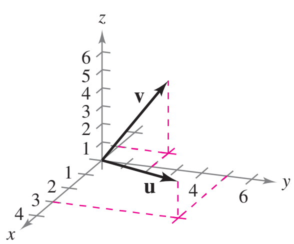
13. $\mathbf{u} = \langle 12, -3, 0 \rangle$, $\mathbf{v} = \langle -2, 5, 0 \rangle$

14. $\mathbf{u} = \langle -10, 0, 6 \rangle$, $\mathbf{v} = \langle 7, 0, 0 \rangle$

15. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

16. $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

Think About It **In Exercises 17–20, use the vectors $\mathbf{u}$ and $\mathbf{v}$ shown in the figure to sketch a vector in the direction of the indicated cross product in a right-handed system.**



17. $\mathbf{u} \times \mathbf{v}$ **18.** $\mathbf{v} \times \mathbf{u}$

19. $(-\mathbf{v}) \times \mathbf{u}$ **20.** $\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$

In Exercises 21–24, use a computer algebra system to find $\mathbf{u} \times \mathbf{v}$ and a unit vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$.

21. $\mathbf{u} = \langle 4, -3.5, 7 \rangle$, $\mathbf{v} = \langle -1, 8, 4 \rangle$

22. $\mathbf{u} = \langle -8, -6, 4 \rangle$, $\mathbf{v} = \langle 10, -12, -2 \rangle$

23. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{10}\mathbf{k}$

24. $\mathbf{u} = \frac{2}{3}\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} + 6\mathbf{k}$

25. ***Programming*** Given the vectors $\mathbf{u}$ and $\mathbf{v}$ in component form, write a program for a graphing utility in which the output is $\mathbf{u} \times \mathbf{v}$ and $\|\mathbf{u} \times \mathbf{v}\|$.

26. ***Programming*** Use the program you wrote in Exercise 25 to find $\mathbf{u} \times \mathbf{v}$ and $\|\mathbf{u} \times \mathbf{v}\|$ for $\mathbf{u} = \langle -2, 6, 10 \rangle$ and $\mathbf{v} = \langle 3, 8, 5 \rangle$.

Area **In Exercises 27–30, find the area of the parallelogram that has the given vectors as adjacent sides. Use a computer algebra system or a graphing utility to verify your result.**

27. $\mathbf{u} = \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$

28. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$

29. $\mathbf{u} = \langle 3, 2, -1 \rangle$, $\mathbf{v} = \langle 1, 2, 3 \rangle$

30. $\mathbf{u} = \langle 2, -1, 0 \rangle$, $\mathbf{v} = \langle -1, 2, 0 \rangle$

Area **In Exercises 31 and 32, verify that the points are the vertices of a parallelogram, and find its area.**

31. $(1, 1, 1)$, $(2, 3, 4)$, $(6, 5, 2)$, $(7, 7, 5)$

32. $(2, -3, 1)$, $(6, 5, -1)$, $(3, -6, 4)$, $(7, 2, 2)$

Area **In Exercises 33–36, find the area of the triangle with the given vertices. (*Hint:* $\frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|$ is the area of the triangle having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.)**

33. $(0, 0, 0)$, $(1, 2, 3)$, $(-3, 0, 0)$

34. $(2, -3, 4)$, $(0, 1, 2)$, $(-1, 2, 0)$

35. $(2, -7, 3)$, $(-1, 5, 8)$, $(4, 6, -1)$

36. $(1, 2, 0)$, $(-2, 1, 0)$, $(0, 0, 0)$

37. ***Torque*** A child applies the brakes on a bicycle by applying a downward force of 20 pounds on the pedal when the crank makes a $40°$ angle with the horizontal (see figure). The crank is 6 inches in length. Find the torque at P.

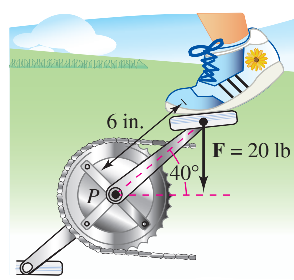


Figure for 37

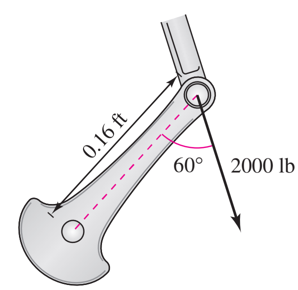


Figure for 38

38. ***Torque*** Both the magnitude and the direction of the force on a crankshaft change as the crankshaft rotates. Find the torque on the crankshaft using the position and data shown in the figure.

39. ***Optimization*** A force of 60 pounds acts on the pipe wrench shown in the figure on the next page.

(a) Find the magnitude of the moment about O by evaluating $\|\overrightarrow{OA} \times \mathbf{F}\|$. Use a graphing utility to graph the resulting function of θ.

(b) Use the result of part (a) to determine the magnitude of the moment when $\theta = 45°$.

(c) Use the result of part (a) to determine the angle θ when the magnitude of the moment is maximum. Is the answer what you expected? Why or why not?

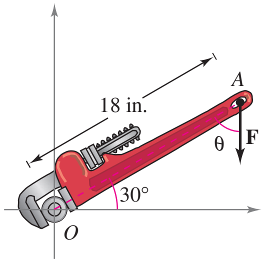


Figure for 39

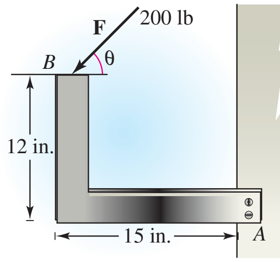


Figure for 40

40. ***Optimization*** A force of 200 pounds acts on the bracket shown in the figure.

(a) Determine the vector $\overrightarrow{AB}$ and the vector $\mathbf{F}$ representing the force. ($\mathbf{F}$ will be in terms of θ.)

(b) Find the magnitude of the moment about A by evaluating $\|\overrightarrow{AB} \times \mathbf{F}\|$.

(c) Use the result of part (b) to determine the magnitude of the moment when $\theta = 30°$.

(d) Use the result of part (b) to determine the angle θ when the magnitude of the moment is maximum. At that angle, what is the relationship between the vectors $\mathbf{F}$ and $\overrightarrow{AB}$? Is it what you expected? Why or why not?

(e) Use a graphing utility to graph the function for the magnitude of the moment about A for $0° \le \theta \le 180°$. Find the zero of the function in the given domain. Interpret the meaning of the zero in the context of the problem.

In Exercises 41–44, find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

41. $\mathbf{u} = \mathbf{i}$
$\mathbf{v} = \mathbf{j}$
$\mathbf{w} = \mathbf{k}$

42. $\mathbf{u} = \langle 1, 1, 1 \rangle$
$\mathbf{v} = \langle 2, 1, 0 \rangle$
$\mathbf{w} = \langle 0, 0, 1 \rangle$

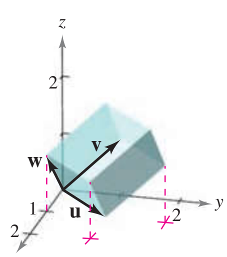
43. $\mathbf{u} = \langle 2, 0, 1 \rangle$
$\mathbf{v} = \langle 0, 3, 0 \rangle$
$\mathbf{w} = \langle 0, 0, 1 \rangle$

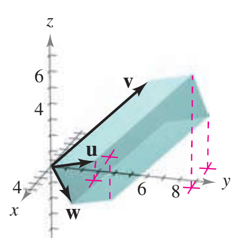
44. $\mathbf{u} = \langle 2, 0, 0 \rangle$
$\mathbf{v} = \langle 1, 1, 1 \rangle$
$\mathbf{w} = \langle 0, 2, 2 \rangle$

Volume **In Exercises 45 and 46, use the triple scalar product to find the volume of the parallelepiped having adjacent edges u, v, and w.**

45. $\mathbf{u} = \mathbf{i} + \mathbf{j}$
$\mathbf{v} = \mathbf{j} + \mathbf{k}$
$\mathbf{w} = \mathbf{i} + \mathbf{k}$

46. $\mathbf{u} = \langle 1, 3, 1 \rangle$
$\mathbf{v} = \langle 0, 6, 6 \rangle$
$\mathbf{w} = \langle -4, 0, -4 \rangle$





Volume **In Exercises 47 and 48, find the volume of the parallelepiped with the given vertices (see figures).**

47. (0, 0, 0), (3, 0, 0), (0, 5, 1), (3, 5, 1)
(2, 0, 5), (5, 0, 5), (2, 5, 6), (5, 5, 6)

48. (0, 0, 0), (1, 1, 0), (1, 0, 2), (0, 1, 1)
(2, 1, 2), (1, 1, 3), (1, 2, 1), (2, 2, 3)

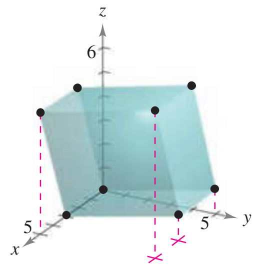


Figure for 47

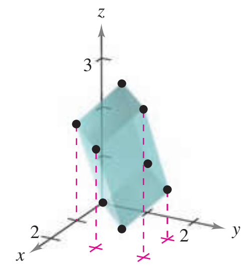


Figure for 48

Writing About Concepts

49. Define the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$.

50. State the geometric properties of the cross product.

51. If the magnitudes of two vectors are doubled, how will the magnitude of the cross product of the vectors change? Explain.

52. The vertices of a triangle in space are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3). Explain how to find a vector perpendicular to the triangle.

True or False? **In Exercises 53–55, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

53. It is possible to find the cross product of two vectors in a two-dimensional coordinate system.

54. If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

55. If $\mathbf{u} \neq \mathbf{0}$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

56. Prove Theorem 11.9.

In Exercises 57–62, prove the property of the cross product.

57. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

58. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$

59. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

60. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

61. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

62. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.

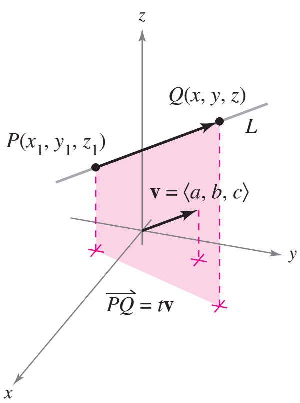
63. Prove $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\|$ if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

64. Prove $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Section 11.5 Lines and Planes in Space

- **Write a set of parametric equations for a line in space.**
- **Write a linear equation to represent a plane in space.**
- **Sketch the plane given by a linear equation.**
- **Find the distances between points, planes, and lines in space.**

Lines in Space



Line L and its direction vector $\mathbf{v}$
Figure 11.43

In the plane, *slope* is used to determine an equation of a line. In space, it is more convenient to use *vectors* to determine the equation of a line.

In Figure 11.43, consider the line L through the point $P(x_1, y_1, z_1)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$. The vector $\mathbf{v}$ is a **direction vector** for the line L, and a, b, and c are **direction numbers.** One way of describing the line L is to say that it consists of all points $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$. This means that $\overrightarrow{PQ}$ is a scalar multiple of $\mathbf{v}$, and you can write $\overrightarrow{PQ} = t\mathbf{v}$, where t is a scalar (a real number).

$$\overrightarrow{PQ} = \langle x - x_1, y - y_1, z - z_1\rangle = \langle at, bt, ct\rangle = t\mathbf{v}$$

By equating corresponding components, you can obtain **parametric equations** of a line in space.

THEOREM 11.11 Parametric Equations of a Line in Space

A line L parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the **parametric equations**

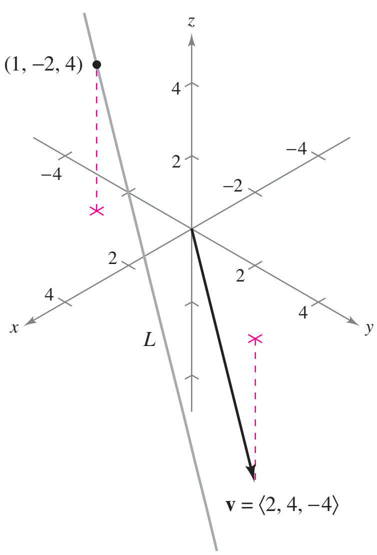
$$x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.$$

If the direction numbers a, b, and c are all nonzero, you can eliminate the parameter t to obtain **symmetric equations** of the line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \text{Symmetric equations}$$

EXAMPLE 1 Finding Parametric and Symmetric Equations

Find parametric and symmetric equations of the line L that passes through the point $(1, -2, 4)$ and is parallel to $\mathbf{v} = \langle 2, 4, -4\rangle$.



The vector $\mathbf{v}$ is parallel to the line L.
Figure 11.44

Solution To find a set of parametric equations of the line, use the coordinates $x_1 = 1$, $y_1 = -2$, and $z_1 = 4$ and direction numbers $a = 2$, $b = 4$, and $c = -4$ (see Figure 11.44).

$$x = 1 + 2t, \quad y = -2 + 4t, \quad z = 4 - 4t \qquad \text{Parametric equations}$$

Because a, b, and c are all nonzero, a set of symmetric equations is

$$\frac{x - 1}{2} = \frac{y + 2}{4} = \frac{z - 4}{-4}. \qquad \text{Symmetric equations}$$