(11*pts*) **Problem 1**
Solve for $z = x + iy$ the equation

$$2z - i\bar{z} = 3\,(3 - 5i)\,.$$

Solution

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy.$$

The equation becomes

$$2\,(x + iy) - i\,(x - iy) = 3\,(3 - 5i)\,.$$

$$(2x - y) + i\,(2y - x) = 9 - 15i. \qquad [5\ points]$$

After identifying, you obtain the system

$$\begin{cases} 2x - y = 9 \\ 2y - x = -15 \end{cases}$$

Solution is: $[x = 1, y = -7]$.

$$z = 1 - 7i \qquad [6\ points]$$

$(12pts)$ **Problem 2**
Solve for $z = a + ib$ the quadratic equation

$$z^2 - 6z + 10 + (z - 6)\, i = 0.$$

Solution
The equation is equivalent to

$$z^2 - (6 - i)\, z + 10 - 6i = 0.$$

Now applying the quadratic formula, we obtain

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{(6-i) \pm \sqrt{(6-i)^2 - 4\,(10-6i)}}{2} \\ &= \frac{(6-i) \pm \sqrt{-5+12i}}{2}. \qquad [5\ \mathit{points}] \end{aligned}$$

Now put

$$\begin{gathered} \sqrt{-5+12i} = a + ib \\ (a+ib)^2 = -5 + 12i \\ a^2 - b^2 + 2iab = -5 + 12i \\ \begin{cases} a^2 - b^2 = -5 \\ ab = 6 \end{cases} \end{gathered}$$

With the sign convention,

$$a = -2 \text{ and } b = -3.$$

The solutions are

$$\begin{aligned} z &= \frac{(6-i) \pm (2+3i)}{2} \\ &= 4 + i \quad \text{or} \quad 2 - 2i. \qquad [7\ \mathit{points}] \end{aligned}$$

$(11pts)$**Problem 3**
If

$$x^2 + y^2 + z^2 + 4x - 6y + 8z = -4$$

represents a sphere, find its center and radius.

Solution

$$x^2 + 4x + y^2 - 6y + z^2 + 8z = -4$$

$$(x+2)^2 - 4 + (y-3)^2 - 9 + (z+4)^2 - 16 = -4$$

$$(x+2)^2 + (y-3)^2 + (z+4)^2 = -4 + 4 + 9 + 16$$

$$(x+2)^2 + (y-3)^2 + (z+4)^2 = 25 \qquad [6\ points]$$

$$\text{Center } = (-2,\ 3,\ -4)\,, \qquad \text{Radius } R = 5 \qquad [5\ points]$$

$(12pts)$ **Problem 4**
Find the point where the line passing through the points $A\,(1,\ 0,\ 1)$ and $B\,(4,\ -2,\ 2)$ intersects the plane $2x + 3y - 4z = 6$.

Solution
A direction vector of the line passing through the points $A\,(1,\ 0,\ 1)$ and $B\,(4,\ -2,\ 2)$ is given by

$$\overrightarrow{AB} = \langle 3,\ -2,\ 1 \rangle\,.$$

The parametric equations of the line are

$$\left\{ \begin{array}{c} x = 1 + 3t \\ y = 0 - 2t \\ z = 1 + t \end{array} \right. .$$

Now substituting these in the equation of the plane, we obtain

$$2\,(1 + 3t) + 3\,(-2t) - 4\,(1 + t) = 6. \qquad [6\ points]$$

Solving for t, we get

$$t = -2.$$

The coordinates of the point of intersection are obtain by substituting the value of t in the equation

$$\left\{ \begin{array}{c} x = 1 + 3t \\ y = 0 - 2t \\ z = 1 + t \end{array} \right. .$$

Thus,

$$\left\{ \begin{array}{c} x = -5 \\ y = 4 \\ z = -1 \end{array} \right. . \qquad [6\ points]$$

The point of intersection is $(-5,\ 4,\ -1)\,.$

($12pts$) **Problem 5**

Consider the points $A\,(1,\ 0,\ 1)\,,\quad B\,(2,\ 1,\ 1)$ and $C\,(-1,\ -1,\ 2)$ in the 3-dimensional space.

(a) Find the area of the triangle ABC.

(b) Find the equation of the plane containing A, B, and C.

Solution

(a)

$$\overrightarrow{AB} = \langle 1,\ 1,\ 0\rangle \quad \text{and} \quad \overrightarrow{AC} = \langle -2,\ -1,\ 1\rangle$$

$$\text{Area of Triangle } = \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|.$$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{vmatrix} = \langle 1,\ -1,\ 1\rangle$$

$$\begin{aligned} \text{Area of Triangle} &= \frac{1}{2}\left\|\langle 1,\ -1,\ 1\rangle\right\| \\ &= \frac{1}{2}\sqrt{1+1+1} = \\ &= \frac{\sqrt{3}}{2} = 0.866\,03. \qquad [6\ points] \end{aligned}$$

(b)

$$\text{Normal vector } \overrightarrow{n} = \langle 1,\ -1,\ 1\rangle\,.$$

The equation of the plane is

$$(x-1)-(y-0)+(z-1)=0$$

$$x-y+z=2 \qquad [6\ points]$$

$(11pts)$**Problem 6**

Let $\overrightarrow{a}$ be a unit vector and $\overrightarrow{b}$ a vector such that $\left\|\overrightarrow{b}\right\| = 5$. If the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is $\theta = \dfrac{2\pi}{3}$, find the magnitude of the vector $\overrightarrow{a} - \overrightarrow{b}$.

Solution

$$\begin{aligned}
\left\|\overrightarrow{a} - \overrightarrow{b}\right\|^2 &= \left(\overrightarrow{a} - \overrightarrow{b}\right) \cdot \left(\overrightarrow{a} - \overrightarrow{b}\right) \\
&= \overrightarrow{a} \cdot \overrightarrow{a} - \overrightarrow{a} \cdot \overrightarrow{b} - \overrightarrow{b} \cdot \overrightarrow{a} + \overrightarrow{b} \cdot \overrightarrow{b} \\
&= \|\overrightarrow{a}\|^2 + \left\|\overrightarrow{b}\right\|^2 - 2\overrightarrow{a} \cdot \overrightarrow{b} \\
&= \|\overrightarrow{a}\|^2 + \left\|\overrightarrow{b}\right\|^2 - 2\,\|\overrightarrow{a}\|\,\left\|\overrightarrow{b}\right\| \cos\frac{2\pi}{3} && [6\ points] \\
&= 1 + 25 - 2\,(1)\,(5)\left(-\frac{1}{2}\right) \\
&= 31
\end{aligned}$$

$$\left\|\overrightarrow{a} - \overrightarrow{b}\right\| = \sqrt{31} = 5.\,567\,8. \qquad [5\ points]$$

$(11pts)$**Problem 7**
Find the distance between the planes

$$z = x + 2y + 1 \quad \text{and} \quad 3x + 6y - 3z = 4.$$

Solution
First observe that the two planes are parallel.
Putting $y = 0$ and $z = 0$ in the first plane, we get $Q(-1,\ 0,\ 0)$.
$P(4/3,\ 0,\ 0)$ is a point in the second plane whose normal vector is $\overrightarrow{n} = \langle 3,\ 6,\ -3 \rangle$.

$$\begin{aligned} D &= \frac{\left| \overrightarrow{PQ} \cdot \overrightarrow{n} \right|}{\| \overrightarrow{n} \|} \\ &= \frac{\left| \left\langle \frac{-7}{3}, 0, 0 \right\rangle \cdot \langle 3, 6, -3 \rangle \right|}{\| \langle 3, 6, -3 \rangle \|} && [6\ points] \\ &= \frac{7}{\sqrt{9 + 36 + 9}} \\ &= \frac{7}{\sqrt{54}} = 0.952\,58 && [5\ points] \end{aligned}$$