MATH141-Determinant

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Let us start with 1×1 matrices, of the form

$$A = (a)$$

.

Note here that $\mathsf{I}_1 = (\ 1\ )$.

If $a \neq 0$, then clearly the matrix A is invertible, with inverse matrix

$$A^{-1} = (\frac{1}{a})$$

On the other hand, if $a = 0$, then clearly no matrix B can satisfy $AB = BA = I_1$, so that the matrix A is not invertible.
We therefore conclude that the value a is a good "determinant" to determine whether the 1×1 matrix A is invertible, since the matrix A is invertible if and only if $a \neq 0$.
Let us then agree on the following definition.
Suppose that

$$A = (a)$$

is a 1×1 matrix. We write

$$\det A = a$$

and call this the determinant of the matrix A.

Next, let us turn to 2×2 matrices, of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We shall use elementary row operations to find out when the matrix A is invertible. So we consider the array

$$(A|I_2) = \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}, \qquad (1)$$

and try to use elementary row operations to reduce the left hand half of the array to I_2.
Suppose first of all that $a = c = 0$. Then the array becomes

$$\begin{pmatrix} 0 & b & 1 & 0 \\ 0 & d & 0 & 1 \end{pmatrix}.$$

and so it is impossible to reduce the left hand half of the array by elementary row operations to the matrix I_2.
Consider next the case $a \neq 0$. Multiplying row 2 of the array (1) by a, we obtain

$$\begin{pmatrix} a & b & 1 & 0 \\ ac & ad & 0 & a \end{pmatrix}.$$

Adding $-c$ times row 1 to row 2, we obtain

$$\begin{pmatrix} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{pmatrix}. \tag{2}$$

If $D = ad - bc = 0$, then this becomes

$$\begin{pmatrix} a & b & 1 & 0 \\ 0 & 0 & -c & a \end{pmatrix},$$

and so it is impossible to reduce the left hand half of the array by elementary row operations to the matrix I_2.

On the other hand, if $D = ad - bc \neq 0$, then the array (2) can be reduced by elementary row operations to

$$\begin{pmatrix} 1 & 0 & d/D & -b/D \\ 0 & 1 & -c/D & a/D \end{pmatrix},$$

so that

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Consider finally the case $c \neq 0$. Interchanging rows 1 and 2 of the array (1), we obtain

$$\begin{pmatrix} c & d & 0 & 1 \\ a & b & 1 & 0 \end{pmatrix}.$$

Multiplying row 2 of the array by c, we obtain

$$\begin{pmatrix} c & d & 0 & 1 \\ ac & bc & c & 0 \end{pmatrix}.$$

Adding $-a$ times row 1 to row 2, we obtain

$$\begin{pmatrix} c & d & 0 & 1 \\ 0 & bc - ad & c & -a \end{pmatrix}.$$

Multiplying row 2 by -1, we obtain

$$\begin{pmatrix} c & d & 0 & 1 \\ 0 & ad - bc & -c & a \end{pmatrix}. \tag{3}$$

Again, if $D = ad - bc = 0$, then this becomes

$$\begin{pmatrix} c & d & 0 & 1 \\ 0 & 0 & -c & a \end{pmatrix};$$

and so it is impossible to reduce the left hand half of the array by elementary row operations to the matrix I_2.
On the other hand, if $D = ad - bc \neq 0$, then the array (3) can be reduced by elementary row operations to

$$\begin{pmatrix} 1 & 0 & d/D & -b/D \\ 0 & 1 & -c/D & a/D \end{pmatrix},$$

So that

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Finally, note that $a = c = 0$ is a special case of $ad - bc = 0$.
We therefore conclude that the value $ad - bc$ is a good "determinant" to determine whether the 2 $\times$ 2 matrix A is invertible, since the matrix A is invertible if and only if $ad - bc \neq 0$.
Let us then agree on the following definition.

Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix. We write

$$\det(A) = ad - bc$$

and call this the determinant of the matrix A.

If we attempt to repeat the argument for 2×2 matrices to 3×3 matrices, then it is very likely that we shall end up in a mess with possibly no firm conclusion.
Try the argument on 4×4 matrices if you must.
Those who have their feet firmly on the ground will try a different approach.
Our approach is inductive in nature. In other words, we shall define the determinant of 2×2 matrices in terms of determinants of 1×1 matrices, define the determinant of 3×3 matrices in terms of determinants of 2×2 matrices, define the determinant of 4×4 matrices in terms of determinants of 3×3 matrices,
and so on.

Suppose now that we have defined the determinant of $(n-1) \times (n-1)$ matrices. Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

be an n $\times$ n matrix.

For every $i, j = 1, ..., n$, let us delete row i and column j of A to obtain the $(n-1) \times (n-1)$ matrix

$$A_{ij} = \begin{pmatrix} a_{11} & \cdots & a_{1(j-1)} & \bullet & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & \bullet & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ \bullet & \cdots & \bullet & \bullet & \bullet & \cdots & \bullet \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & \bullet & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & \bullet & a_{n(j+1)} & \cdots & a_{nn} \end{pmatrix}.$$

Here $\bullet$ denotes that the entry has been deleted.

The number $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is called the cofactor of the entry a_{ij} of A. In other words, the cofactor of the entry a_{ij} is obtained from A by first deleting the row and the column containing the entry a_{ij}, then calculating the determinant of the resulting $(n-1) \times (n-1)$ matrix, and finally multiplying by a sign $(-1)^{i+j}$.

Note that the entries of A in row i are given by

$$\begin{pmatrix} a_{i1} & \dots & a_{in} \end{pmatrix}.$$

By the cofactor expansion of A by row i, we mean the expression

$$\sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + \dots + a_{in}C_{in}. \tag{4}$$

Note that the entries of A in column j are given by

$$\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}.$$

By the cofactor expansion of A by column j, we mean the expression

$$\sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + \ldots + a_{nj}C_{nj}. \tag{5}$$

We have the following important Theorem
Suppose that $A = (a_{ij})$ is an n $\times$ n matrix . Then the expressions (4) and (5) are all equal and independent of the row or column chosen.

Suppose that $A = (a_{ij})$ is an n $\times$ n matrix. We call the common value in (4) and (5) the determinant of the matrix A, denoted by $\det$(A).

Let us check whether this agrees with our earlier definition of the determinant of a 2×2 matrix. Writing

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we have

$$C_{11} = a_{22}, \qquad C_{12} = -a_{21}, \qquad C_{21} = -a_{12}, \qquad C_{22} = a_{11}.$$

It follows that

$$\begin{aligned}
\text{by row } 1: \quad & a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21}, \\
\text{by row } 2: \quad & a_{21}C_{21} + a_{22}C_{22} = -a_{21}a_{12} + a_{22}a_{11}, \\
\text{by column } 1: \quad & a_{11}C_{11} + a_{21}C_{21} = a_{11}a_{22} - a_{21}a_{12}, \\
\text{by column } 2: \quad & a_{12}C_{12} + a_{22}C_{22} = -a_{12}a_{21} + a_{22}a_{11}.
\end{aligned}$$

The four values are clearly equal, and of the form $ad - bc$ as before.

Example 1
Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 2 & 1 & 5 \end{pmatrix}.$$

Let us use cofactor expansion by row 1.

Then

$$C_{11} = (-1)^{1+1} \det \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix} = (-1)^2(20-2) = 18,$$

$$C_{12} = (-1)^{1+2} \det \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = (-1)^3(5-4) = -1,$$

$$C_{13} = (-1)^{1+3} \det \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} = (-1)^4(1-8) = -7,$$

so that

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 36 - 3 - 35 = -2.$$

Alternatively, let us use cofactor expansion by column 2. Then

$$C_{12} = (-1)^{1+2} \det \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = (-1)^3(5-4) = -1,$$

$$C_{22} = (-1)^{2+2} \det \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} = (-1)^4(10-10) = 0,$$

$$C_{32} = (-1)^{3+2} \det \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix} = (-1)^5(4-5) = 1,$$

so that

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = -3 + 0 + 1 = -2.$$

Important remark: When using cofactor expansion, we should choose a row or column with as few non-zero entries as possible in order to minimize the calculations.

Example 2. Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 0 & 5 \\ 1 & 4 & 0 & 2 \\ 5 & 4 & 8 & 5 \\ 2 & 1 & 0 & 5 \end{pmatrix}.$$

Here it is convenient to use cofactor expansion by column 3, since then

$$\det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} = 8C_{33} = 8(-1)^{3+3}\det\begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 2 & 1 & 5 \end{pmatrix} = -16,$$

Some Simple Observations

In this section, we shall describe two simple observations which follow immediately from the definition of the determinant by cofactor expansion.
Suppose that a square matrix A has a zero row or has a zero column. Then $\det(A) = 0$.
Proof. We simply use cofactor expansion by the zero row or zero column.

Consider an n $\times$ n matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

If $a_{ij} = 0$ whenever $i > j$, then A is called an upper triangular matrix. If $a_{ij} = 0$ whenever $i < j$, then A is called a lower triangular matrix. We also say that A is a triangular matrix if it is upper triangular or lower triangular.

Example 3 The matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

is upper triangular

Example 4 A diagonal matrix is both upper triangular and lower triangular.

Suppose that the n $\times$ n matrix $A = (a_{ij})$ is upper triangular. Then $\det(A) = a_{11}a_{22}...a_{nn}$, the product of the diagonal entries.

Elementary Row Operations

We now study the effect of elementary row operations on determinants.
Recall that the elementary row operations that we consider are:
(1) interchanging two rows;
(2) adding a multiple of one row to another row; and
(3) multiplying one row by a non-zero constant.

Suppose that A is an n $\times$ n matrix

(a) Suppose that the matrix B is obtained from the matrix A by interchanging two rows of A. Then $\det(B) = -\det(A)$.
(b) Suppose that the matrix B is obtained from the matrix A by adding a multiple of one row of A to another row.
Then $\det(B) = \det(A)$.
(c) Suppose that the matrix B is obtained from the matrix A by multiplying one row of A by a non-zero constant c.
Then $\det(B) = c\det(A)$.

In fact, the above operations can also be carried out on the columns of A. More precisely, we have the following result.
Suppose that A is an n $\times$ n matrix

(a) Suppose that the matrix B is obtained from the matrix A by interchanging two columns of A. Then $\det(B) = -\det(A)$.
(b) Suppose that the matrix B is obtained from the matrix A by adding a multiple of one column of A to another column. Then $\det(B) = \det(A)$.
(c) Suppose that the matrix B is obtained from the matrix A by multiplying one column of A by a non-zero constant c. Then $\det(B) = c\det(A)$.

Elementary row and column operations can be combined with cofactor expansion to calculate the determinant of a given matrix.
We shall illustrate this point by the following examples.
Example 5 Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 2 & 5 \\ 1 & 4 & 1 & 2 \\ 5 & 4 & 4 & 5 \\ 2 & 2 & 0 & 4 \end{pmatrix}.$$

Adding -1 times column 3 to column 1, we have

$$\det(A) = \det \begin{pmatrix} 0 & 3 & 2 & 5 \\ 0 & 4 & 1 & 2 \\ 1 & 4 & 4 & 5 \\ 2 & 2 & 0 & 4 \end{pmatrix}.$$

Adding -1/2 times row 4 to row 3, we have

$$\det(A) = \det\begin{pmatrix} 0 & 3 & 2 & 5 \\ 0 & 4 & 1 & 2 \\ 0 & 3 & 4 & 3 \\ 2 & 2 & 0 & 4 \end{pmatrix}.$$

Using cofactor expansion by column 1, we have

$$\det(A) = 2(-1)^{4+1}\det\begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 2 \\ 3 & 4 & 3 \end{pmatrix} = -2\det\begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 2 \\ 3 & 4 & 3 \end{pmatrix}.$$

Adding -1 times row 1 to row 3, we have

$$\det(A) = -2\det\begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 2 \\ 0 & 2 & -2 \end{pmatrix}.$$

Adding 1 times column 2 to column 3, we have

$$\det(A) = -2\det\begin{pmatrix} 3 & 2 & 7 \\ 4 & 1 & 3 \\ 0 & 2 & 0 \end{pmatrix}.$$

Using cofactor expansion by row 3, we have

$$\det(A) = -2 \cdot 2(-1)^{3+2} \det \begin{pmatrix} 3 & 7 \\ 4 & 3 \end{pmatrix} = 4 \det \begin{pmatrix} 3 & 7 \\ 4 & 3 \end{pmatrix}.$$

Using the formula for the determinant of 2×2 matrices, we conclude that $\det(A) = 4(9 - 28) = -76$.

Further Properties of Determinants

For every n $\times$ n matrix A, we have $\det(A^t) = \det(A)$.
For every n $\times$ n matrices A and B, we have $\det(AB) = \det(A)\det(B)$.
Suppose that the n $\times$ n matrix A is invertible. Then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Suppose that A is an n$\times$ n matrix. Then A is invertible if and only if $\det(A) \neq 0$.

CRAMER'S RULE

Next, we turn our attention to systems of n linear equations in n unknowns, of the form

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1, \\ &\vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= b_n, \end{aligned}$$

represented in matrix notation in the form

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

represent the coefficients and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

represents the variables.

For every $j = 1, ..., k$, write

$$A_j(\mathbf{b}) = \begin{pmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{pmatrix};$$

in other words, we replace column j of the matrix A by the column **b**.

Suppose that the matrix A is invertible. Then the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x} = \begin{pmatrix} x_1 = \frac{\det(A_1(\mathbf{b}))}{\det(A)} \\ \vdots \\ x_n = \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{pmatrix}$$

Example 6 Consider the system $A\mathbf{x} = \mathbf{b}$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$\det(A) = -1.$

By Cramer's rule, we have

$$x_1 = \frac{\det\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 3 & 0 & 3 \end{pmatrix}}{\det(A)} = -3, \quad x_2 = \frac{\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix}}{\det(A)} = -4, \quad x_3 = \frac{\det\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix}}{\det(A)} = 3.$$