Part 1 MCQ 30% (circle your choice)

(6pts)Problem 1
If β is the angle between the two planes

$$x - 2y + z = 1441 \quad \text{and} \quad 2x + y - z = 2019$$

then $\cos\beta$ is equal to

$$(a)\ \frac{-1}{2} \qquad (b)\ \frac{-1}{6} \qquad (c)\ \frac{-2}{\sqrt{3}} \qquad (d)\ \frac{-1}{\sqrt{3}} \qquad (e)\ -1$$

Solution

$$Ans:\ (b)$$

(6pts)Problem 2
Which of the following statement (s) is / are **TRUE.**
(i) For any vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, $\left\|\overrightarrow{a} + \overrightarrow{b}\right\| = \|\overrightarrow{a}\| + \left\|\overrightarrow{b}\right\|$.
(ii) For any vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, $\left(\overrightarrow{a} \times \overrightarrow{b}\right) \cdot \overrightarrow{a} = 0$.
(iii) If $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$, then $\overrightarrow{a} = \overrightarrow{0}$ or $\overrightarrow{b} = \overrightarrow{0}$.

(a) (i) and (ii) $\qquad (b)$ (i) only $\qquad (c)$ (ii) and (iii) $\qquad (d)$ (iii) only $\qquad (e)$ (ii) only

Solution

$$Ans:\ (e)$$

(6pts)Problem 3
The sum of all the values of x for which the two vectors $\overrightarrow{a} = \langle 3,\ 2,\ x\rangle$ and $\overrightarrow{b} = \langle 2x,\ 4,\ x\rangle$ are orthogonal is

$$(a)\ -6 \qquad (b)\ 6 \qquad (c)\ 4 \qquad (d)\ -9 \qquad (e)\ -8$$

Solution

$$Ans:\ (a)$$

(6pts)Problem 4

If $\langle a,\ b,\ c\rangle$ is a unit vector orthogonal to both vectors $\langle 1,\ -1,\ 1\rangle$, $\langle 0,\ 4,\ 4\rangle$, then $|a|+|b|+|c|$ is equal to

$$(a)\ \frac{2}{\sqrt{6}} \qquad (b)\ \frac{8}{\sqrt{6}} \qquad (c)\ \frac{10}{\sqrt{6}} \qquad (d)\ \frac{4}{\sqrt{6}} \qquad (e)\ \frac{1}{\sqrt{6}}$$

Solution

$$Ans:\ (d)$$

(6pts)Problem 5

The distance between the planes $(x-1)+(y-2)+2\,(z-5)=0$ and $x+y+2z=4$ is equal to

$$(a)\ \frac{9}{\sqrt{6}} \qquad (b)\ \frac{4}{\sqrt{6}} \qquad (c)\ \frac{2}{\sqrt{6}} \qquad (d)\ \frac{13}{\sqrt{6}} \qquad (e)\ 9$$

Solution

$$Ans:\ (a)$$

Part 2 Written 70%

$(10pts)$**Problem 1**
Find the point where the line that passes through the points $(1,\ 0,\ 1)$ and $(4,\ -2,\ 2)$ intersect the plane $2x+3y-4z=6$.

Solution
A direction vector of the line is given by

$$\begin{aligned}\overrightarrow{u} &= \langle 4-1,\ -2-0,\ 2-1\rangle \\ &= \langle 3,\ -2,\ 1\rangle . \qquad (\mathbf{2pts})\end{aligned}$$

A set of parametric equations of the line is

$$\begin{cases} x = 1+3t \\ y = 0-2t \\ z = 1+t \end{cases}. \qquad (\mathbf{3pts})$$

To find the point of intersection we do

$$2\,(1+3t)+3\,(-2t)-4\,(1+t)=6$$

$\Leftrightarrow$

$$-4t-2=6 \Rightarrow t=-2. \qquad (\mathbf{2pts})$$

The point of intersection is given by

$$\begin{cases} x = 1+3\,(-2) \\ y = -2\,(-2) \\ z = 1+(-2) \end{cases} \Leftrightarrow \begin{cases} x = -5 \\ y = 4 \\ z = -1 \end{cases}. \qquad (\mathbf{3pts})$$

$(12pts)$**Problem 2**

A) Find a direction vector of the line of intersection of the two planes

$$2x - y + 3z = 1 \quad \text{and} \quad -x + 3y + 3z = 5$$

Solution

$$\overrightarrow{n_1} = \langle 2,\ -1,\ 3 \rangle \quad \text{and} \quad \overrightarrow{n_2} = \langle -1,\ 3,\ 3 \rangle . \qquad (\mathbf{2pts})$$

A direction vector of the line of interection is given by

$$\overrightarrow{u} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & -1 & 3 \\ -1 & 3 & 3 \end{vmatrix} = \langle -12,\ -9,\ 5 \rangle \qquad (\mathbf{4pts})$$

B) The angle between a unit vector $\overrightarrow{a}$ and a vector $\overrightarrow{b}$ is $\frac{\pi}{3}$. If $\left\|\overrightarrow{b}\right\| = 2$, then find the magnitude of the vector $\overrightarrow{a} + \overrightarrow{b}$.

Solution

$$\begin{aligned} \left\|\overrightarrow{a} + \overrightarrow{b}\right\|^2 &= \|\overrightarrow{a}\|^2 + \left\|\overrightarrow{b}\right\|^2 + 2\,\overrightarrow{a} \cdot \overrightarrow{b} && (\mathbf{2pts}) \\ &= \|\overrightarrow{a}\|^2 + \left\|\overrightarrow{b}\right\|^2 + 2\,\|\overrightarrow{a}\|\,\left\|\overrightarrow{b}\right\| \cos\frac{\pi}{3} && (\mathbf{2pts}) \\ &= 1 + 4 + 2\,(1)\,(2)\left(\frac{1}{2}\right) = 7 \end{aligned}$$

$$\left\|\overrightarrow{a} + \overrightarrow{b}\right\| = \sqrt{7}. \qquad (\mathbf{2pts})$$

(12pts)**Problem 3**

Use the **Gauss elimination** method to solve the linear system

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases} . \quad \text{(Show your work)}$$

Solution

Augmented Matrix

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third row by -2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(**6pts**)

$$\begin{aligned} x \quad\quad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

(**6pts**)

(12pts)**Problem 4**
Consider the following matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

(a) Use the **Gauss Jordan method** to find C^{-1}. (Do NOT use the formula and show your work)
(b) Compute the following matrices, where possible.

1. $A + 2B^T$, 2. AC

Solution

(a) Put

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

and use the Gauss Jordan Method to obtain

$$C^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \quad (\mathbf{4pts})$$

(b)
1.

$$A + 2B^T = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 4 \\ 5 & 11 \end{bmatrix} \quad (\mathbf{4pts})$$

2.

$$AC = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 5 & -2 \\ 4 & -1 \end{bmatrix} \quad (\mathbf{4pts})$$

(12pts)**Problem 5**
Use the **cofactor expansion method** to find the determinant of the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{bmatrix}. \qquad \text{(Show your work)}$$

Solution

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \qquad \boxed{\text{Cofactor expansion along the first column}}$$

(**6pts**)

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \qquad \boxed{\text{We added the first row to the third row.}}$$

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} \qquad \boxed{\text{Cofactor expansion along the first column}}$$

$$= -18$$

(**6pts**)

(12pts)**Problem 6**

Given that $\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -6$, find

1. $\det\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$, 2. $\det\begin{bmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{bmatrix}$, 3. $\det\begin{bmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{bmatrix}$

Solution

1. The matrix $\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$ is obtained from $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ by swapping the first and second rows and the second and third row. You pickup a negative sign in each swapping. The two negative signs cancel to give

$$\det\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} = \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -6 \qquad (\mathbf{4pts})$$

2.

$$\begin{aligned} \det\begin{bmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{bmatrix} &= (3)(-1)(4)\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= (3)(-1)(4)(-6) \\ &= 72 \qquad (\mathbf{4pts}) \end{aligned}$$

3.

$$\det\begin{bmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{bmatrix} = \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -6 \qquad (\mathbf{4pts})$$