$A(BC) = (AB)C$

$A(B+C) = AB + AC$

$c(AB) = (cA)B = A(cB)$

Inverse of a matrix

$AA^{-1} = A^{-1}A = I$

Matrix Size	Identity
1×1	$I_1 = [1]$
2×2	$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
3×3	$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
4×4	$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$n \times n$	$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 1 \end{bmatrix}$

Let A be an $n \times n$ matrix

If there exists an $n \times n$ matrix B satisfying $AB = BA = I$, then the matrix B is called the inverse of A and is denoted by

$$B = A^{-1}$$

Finding the inverse of a matrix

The Gauss - Jordan method

$$(A \,\vdots\, I_n) \longrightarrow (I_n \,\vdots\, B)$$

Use elementary options to transform LHS to identity matrix

Example

Find inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{bmatrix}$

$$(A | I_3) = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Transpose of matrix

Columns are interchanged with rows

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Works on any and all matrices

Square matrix is said to be symmetric if $A^T = A$

$$(A^T)^T = A$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 10 & -6 \\ 3 & -6 & 11 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 10 & -6 \\ 3 & -6 & 11 \end{bmatrix}$$

$$(MN)^T = N^T M^T$$

Determinants

A number that tells if the inverse of the matrix exists.

$$A = [a]$$

$$A^{-1} = \left[\frac{1}{a}\right]$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$ad - bc \neq 0$ for inverse to exist

$\det A = ad - bc$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A square matrix A has an inverse if and only if its determinant is not zero

Example

Which of the following matrices have an inverse?

$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & -1 \\ 5 & -5/4 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$

$|A| = -6$ $\quad |B| = 0$ $\quad |C| = -1$

The Cofactor Expansion Method

To find the determinant for $n \times n$ matrices, where $n \geq 3$ we use the cofactor expansion method.

Consider $A = \begin{bmatrix} 5 & 2 & 3 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \leftarrow$

$$(5)(-1)^{1+1} \det \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} + 2(-1)^{1+2} \det \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + 3(-1)^{1+3} \det \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= 5(2) - 2(-1-1) + 3(2-0)$$

$$= 10 + 4 + 6 = \underline{20} = \det(A)$$

$$(2)(-1)^{1+2} \det \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + 0 + (-2)(-1)^{3+2} \det \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$$

$$-2(-1-1) + 2(5+3)$$

$$= 4 + 16$$

$$= \underline{20} = \det(A)$$

Example

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 4 & 2 & 5 \\ -1 & -2 & 0 & 0 \\ 4 & 2 & -1 & 0 \end{bmatrix}$$

$$(-1)(-1)^{3+1} \det \begin{bmatrix} -1 & 0 & 3 \\ 4 & 2 & 5 \\ 2 & -1 & 0 \end{bmatrix} + (-2)(-1)^{3+2} \det \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & -1 & 0 \end{bmatrix} + 0 + 0$$

$$= -1\left[-1(5) + 3(-4-4)\right] + 2\left[1(5) + 3(-8)\right]$$

$$= -1\left[-5 - 24\right] + 2\left[5 - 24\right]$$

$$= 29 + 2\left[-19\right]$$

$$= 29 - 38$$

$$= \underline{-9}$$

Effect of Elementary Row Operations on Determinant

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \qquad \det A = 3$$

$$\det \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & k \end{bmatrix} = -3$$

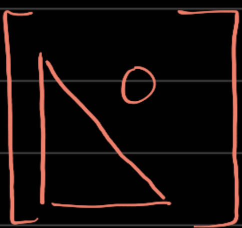
Every time a row is changed, the determinant picks a negative sign

Adding multiple of row to another row (pivoting) does not change determinant value.

Multiplying one row with a constant, the value of determinant is multiplied by the same.

$$|cA| = c^n |A| \quad \text{where } n \text{ is size of matrix}$$

$$\det(A^T) = \det(A)$$
$$\det(AB) = \det(A) \cdot \det(B)$$
$$\det(A^{-1}) = \frac{1}{\det(A)}$$

A = 

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 10 & -2 & 0 & 0 \\ 6 & 100 & 3 & 0 \\ 1000 & 500 & 400 & 5 \end{bmatrix}$$

↳ Lower triangular matrix

det (A) = Product of diagonal elements = $(1)(-2)(3)(5) = \boxed{-30}$

$$A = \begin{bmatrix} 5 & 6 & 10 & 11 \\ 0 & 2 & 9 & 8 \\ 0 & 0 & -4 & 100 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\det(A) = 5 \times 2 \times 2 \times (-4)$$
$$= -80$$

The determinant of any triangular matrix of size $n \times n$ is the product of the diagonal entries.

Cramer's Rule

$$\begin{cases} 2x_1 - 3x_2 + 4x_3 = -1 \\ x_2 - 4x_3 = 9 \\ 3x_1 + 6x_3 = 5 \end{cases}$$

Coefficient Matrix : $A = \begin{bmatrix} 2 & -3 & 4 \\ 0 & 1 & -4 \\ 3 & 0 & 6 \end{bmatrix}$

$$\det(A) = 2(6-0) + 3(-12-0) + 4(-3)$$
$$= 12 - 12 - 12$$
$$= -12$$

The solution of the linear system is given by

$$\begin{cases} x_1 = \dfrac{\det \begin{bmatrix} -1 & -3 & 4 \\ 9 & 1 & -4 \\ 5 & 0 & 6 \end{bmatrix}}{\det A} \\ \vdots \end{cases}$$

Replace the n^{th} column with the right hand side and find determinant

e.g. $x_1 \Rightarrow$ Replace col 1

$x_2 \Rightarrow$ Replace col 2

$\vdots$