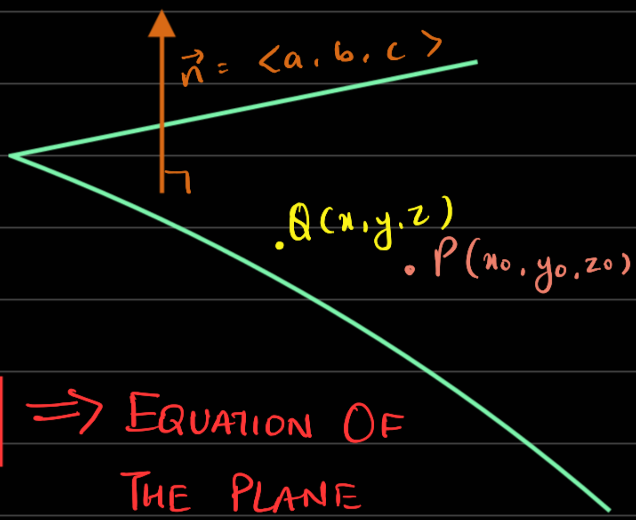
Equation of planes

Normal Vector

A vector that is perpendicular / orthogonal to a plane is called a normal vector.



$\vec{PQ}$ and $\vec{n}$ are perpendicular

$\vec{PQ} = \langle x - x_0, y - y_0, z - z_0 \rangle$

$\vec{n} = \langle a, b, c \rangle$

$\vec{PQ} \cdot \vec{n} = 0$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$ $\Rightarrow$ EQUATION OF THE PLANE

$ax + by + cz = d$ $\Rightarrow$ GENERAL FORM

where $d = ax_0 + by_0 + cz_0$

The equation of a plane containing the point (x_0, y_0, z_0) with normal vector and $\vec{n} = \langle a, b, c \rangle$ is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Example

Find the equation of the plane containing the point $A(-1, 0, 2)$ $B(2, 1, 1)$ and $C(3, -1, 2)$

$\vec{AB} = \langle 3, 1, -1 \rangle$

$\vec{AC} = \langle 4, -1, 0 \rangle$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -1 \\ 4 & -1 & 0 \end{vmatrix} = \langle -1, -4, -7 \rangle$$

Equation of plane

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ Using $A(-1,0,2)$

$-1(x+1) - 4y - 7(z-2) = 0$

$-x-1 - 4y - 7z + 14 = 0$

$-x - 4y - 7z = -13$

$x + 4y + 7z = 13$

$-1(x-3) - 4(y+1) - 7(z-2) = 0$ Using $C(3,-1,2)$

$-x+3 - 4y - 4 - 7z + 14 = 0$

$-x - 4y - 7z = -14 - 3 + 4$

$x + 4y + 7z = 13$

Example

Find a parametric equation for the line through $(1,1,1)$ that is parallel to the line of intersection of the planes $3x - 4y + 2z - 2 = 0$ and $4x - 3y - z - 5 = 0$

$\vec{n}_1 = \langle 3, -4, 2 \rangle$

$\vec{n}_2 = \langle 4, -3, -1 \rangle$

$$\vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 2 \\ 4 & -3 & -1 \end{vmatrix} = \langle 10, 11, 7 \rangle$$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} : \begin{cases} x = 1 + 10t \\ y = 1 + 11t \\ z = 1 + 7t \end{cases}$$

Vector equation: $\langle x, y, z \rangle \cdot \langle 1+10t, 1+11t, 1+7t \rangle$

Angle between two planes

The angle between two planes is equal to the angle between two normal vectors

Example

Find the angle between the planes $5x - 2y + z = 1$ and $-x + 3z = 2$

$\vec{n}_1 = \langle 5, -2, 1 \rangle \qquad \vec{n}_2 = \langle -1, 0, 3 \rangle$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|}$$

$$\cos\theta = \frac{-5 + 0 + 3}{\sqrt{30}\,\sqrt{10}}$$

$$= \frac{-\cancel{2}}{1\cancel{0}\sqrt{3}}$$

$$= \frac{-1}{5\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{-1}{5\sqrt{3}}\right)$$

Example

Find the point P at which the line with parametric equations $x = 1 + 2t$ $y = 4t$ $z = 2 - 3t$ intersects the plane $x + 2y - z + 1 = 0$

$$1 + 2t + 8t - 2 + 3t + 1 = 0$$

$$13t = 0$$

$$t = 0$$

$x = 1, \; y = 0, \; z = 2$

$\therefore \; P(1, 0, 2)$

Distance between a point and a plane

$$D = \left\| \text{Proj}_{\vec{n}} \vec{PQ} \right\|$$

$$D = \left\| \left(\frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|^2} \right) \vec{n} \right\|$$

$$= \left| \vec{PQ} \cdot \vec{n} \right| \cdot \frac{\cancel{\|\vec{n}\|}}{\|\vec{n}\|^{\cancel{2}}}$$

$$\boxed{= \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}}$$

The distance between a point Q and the plane containing P with normal vector $\vec{n}$ is given by

$$\boxed{= \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}}$$

Example

Find the distance from the point $Q(0,3,7)$ to the plane $4x - 3y + z = 1$

$$D = \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}$$

$\vec{n} = \langle 4, -3, 1 \rangle$

$Q = (0, 3, 7)$

$P = (0, 0, 1)$ $\Rightarrow$ Zero any two variables, solve for z – Arbitrary value

$\vec{PQ} = \langle 0, 3, 6 \rangle$

$\vec{PQ} \cdot \vec{n} = 0 - 9 + 6$

$= -3$

$\|\vec{n}\| = \sqrt{16+9+1} = \sqrt{26}$

$D = \dfrac{|-3|}{\sqrt{26}}$

$= \dfrac{3}{\sqrt{26}}$ units

Example

Find the PROBLEM 12 B

Example

Find the distance between the two planes

$P_1 = x - 2y + 4z = 1$

$P_2 = -3x + 6y - 12z = 10$

Let $Q = (1, 0, 0)$

$\vec{n_2} = \langle -3, 6, -12 \rangle$

$P = \left(0, 0, -\frac{5}{6}\right)$

$\vec{PQ} = \langle 1, 0, 5/6 \rangle$

$\vec{PQ} \cdot \vec{n_2} = -3 + 0 - \cancel{12}^{2}\left(\frac{5}{\cancel{6}}\right)$

$= -3 - 10 = -13$

$$\|\vec{n}_2\| = \sqrt{9+36+144} = \sqrt{189}$$

$$D = \frac{|-13|}{\sqrt{189}}$$

$$= \frac{13}{\sqrt{189}} \text{ units}$$