Curve Sketching

Example 1

Sketch the following curve

$2x^5 - 5x^2 + 1$

a) Domain $= (-\infty, \infty)$

b) Derivative & Critical numbers

$f'(x) = 10x^4 - 10x$

$= 10x(x^3 - 1)$

$= 10x(x-1)(x^2+x+1)$

$(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$

$f'(x) = 0$

$10x = 0$ $\quad$ $x - 1 = 0$ $\quad$ $x^2 + x + 1 = 0$

$x = 0$ $\quad$ $x = 1$ $\quad$ Unsolvable

Always positive

Critical numbers $= 0, 1$

Table of variation

x	$-\infty$	0	1	$+\infty$
$10x$	$-$	$+$	$+$	
$x-1$	$-$	$-$	$+$	
x^2+x+1	$+$	$+$	$+$	
$f'(x)$	$+$	$-$	$+$	
$f(x)$	$\uparrow$	$\downarrow$	$\uparrow$	

Local max $= f(0) = 1$

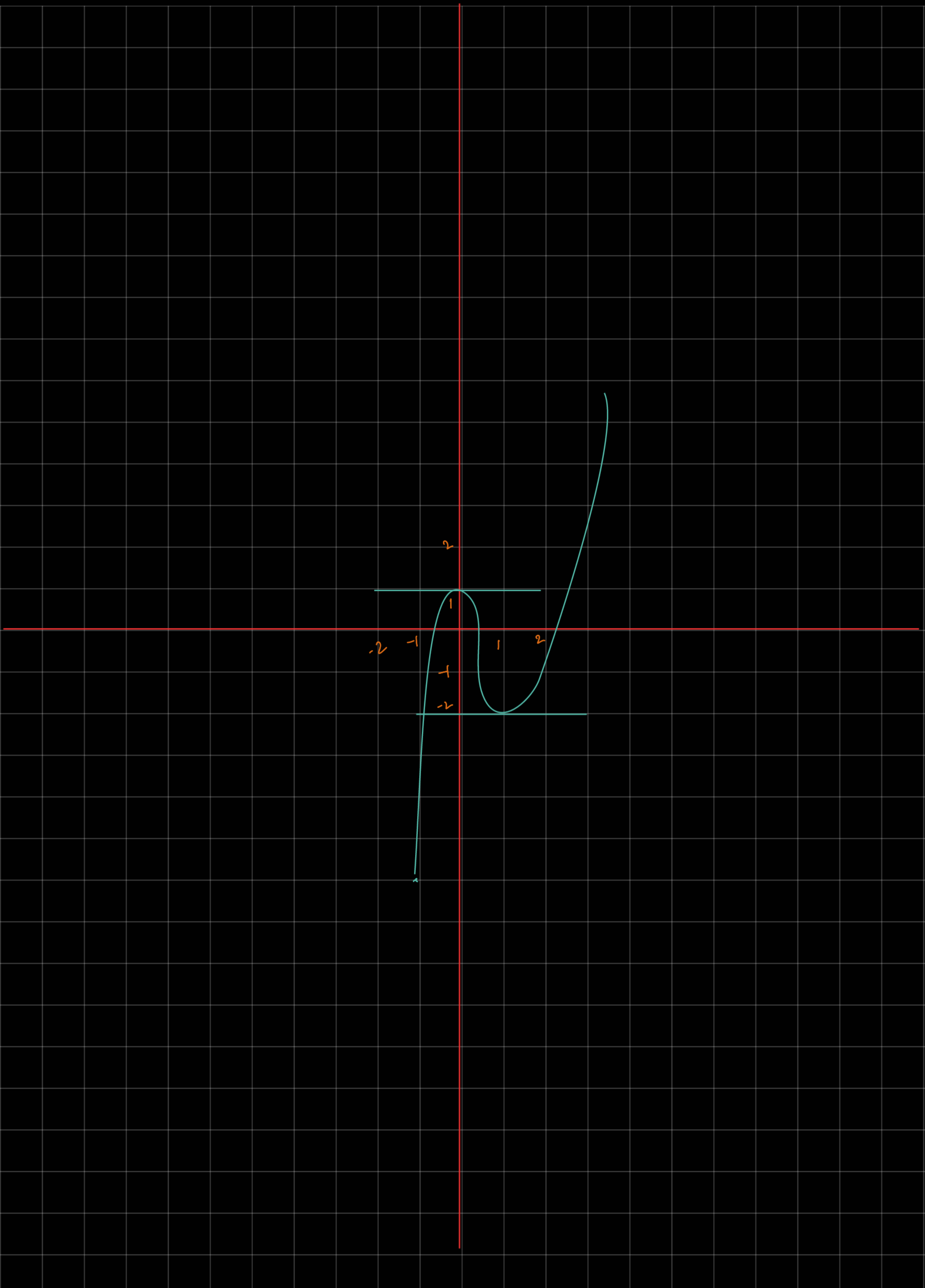
Local min $= f(1) = -2$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 2x^5 - 5x^2 + 1$$

$$= \lim_{x \to +\infty} 2x^5 = -\infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} 2x^5 - 5x^2 + 1$$

$$= \lim_{x \to +\infty} 2x^5 = +\infty$$


2
1
-2
-1
1
2
-1
-2

$$f(x) = \frac{x}{x^2 - 9}$$

Domain $= \mathbb{R} - \{-3, 3\}$

$= (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Derivative & Critical Numbers

$$f'(x) = \frac{(x^2 - 9) - x(2x)}{(x^2 - 9)^2}$$

$$= \frac{x^2 - 9 - 2x^2}{(x^2 - 9)^2}$$

$$= \frac{-x^2 - 9}{(x^2 - 9)^2}$$

$$= \frac{-(x^2 + 9)}{(x^2 - 9)^2}$$

There are no critical numbers

Table of variation

x	$-\infty$	-3		3	$+\infty$
$f'(x)$	$-$	$\Vert$	$-$	$\Vert$	$-$
$f(x)$	$\searrow$	$\Vert$	$\searrow$	$\Vert$	$\searrow$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x}{x^2 - 9}$$

$$= \lim_{x \to +\infty} \frac{x}{x^2}$$

$$= \lim_{x \to \pm\infty} \frac{1}{x}$$

$$= 0$$

$\therefore$ $y = 0$ is a horizontal asymptote

x	$-\infty$	-3		0		3		$+\infty$
x	$-$	|	$-$	|	$+$	|	$+$	
$x^2 - 9$	$+$	|	$-$	|	$-$	|	$+$	
$f(x)$	$-$	|	$+$	|	$-$	|	$+$	

$$\lim_{x \to -3^-} = -\infty \qquad \lim_{x \to +3^-} = -\infty$$

$$\lim_{x \to -3^+} = +\infty \qquad \lim_{x \to +3^+} = +\infty$$

The graph has vertical asymptote at $x = -3$ & $x = 3$


-∞
-3
3
∞

$$f(x) = (x^2 - 4)^{2/3}$$

Domain = $(-\infty, \infty)$

Derivative & Critical Numbers

$$f'(x) = \frac{2}{3}(2x)(x^2-4)^{-1/3}$$

$$= \frac{4x}{3} \times \frac{1}{\sqrt[3]{x^2-4}}$$

$$= \frac{4}{3}\frac{x}{\sqrt[3]{x^2-4}}$$

$$f'(x) = 0$$

$$\frac{x}{\sqrt[3]{x^2-4}} = 0$$

$$x = 0$$

$\therefore$ 0 is a critical number

Critical numbers are : $-2, 0, 2$

Table of Variation

x	$-\infty$	-2		0		2	∞
x	$-$	‖	$-$		$+$	‖	$+$
$\sqrt[3]{x^2-4}$	$+$	‖	$-$		$-$	‖	$+$
$f'(x)$	$-$	‖	$+$		$-$	‖	$+$
$f(x)$	$\downarrow$		$\uparrow$		$\downarrow$		$\uparrow$

$f(-2) = 0$

$f(2) = 0$

$f(0) = 2.52$

$$\lim_{x \to -\infty} (x^2 - 4)^{2/3}$$

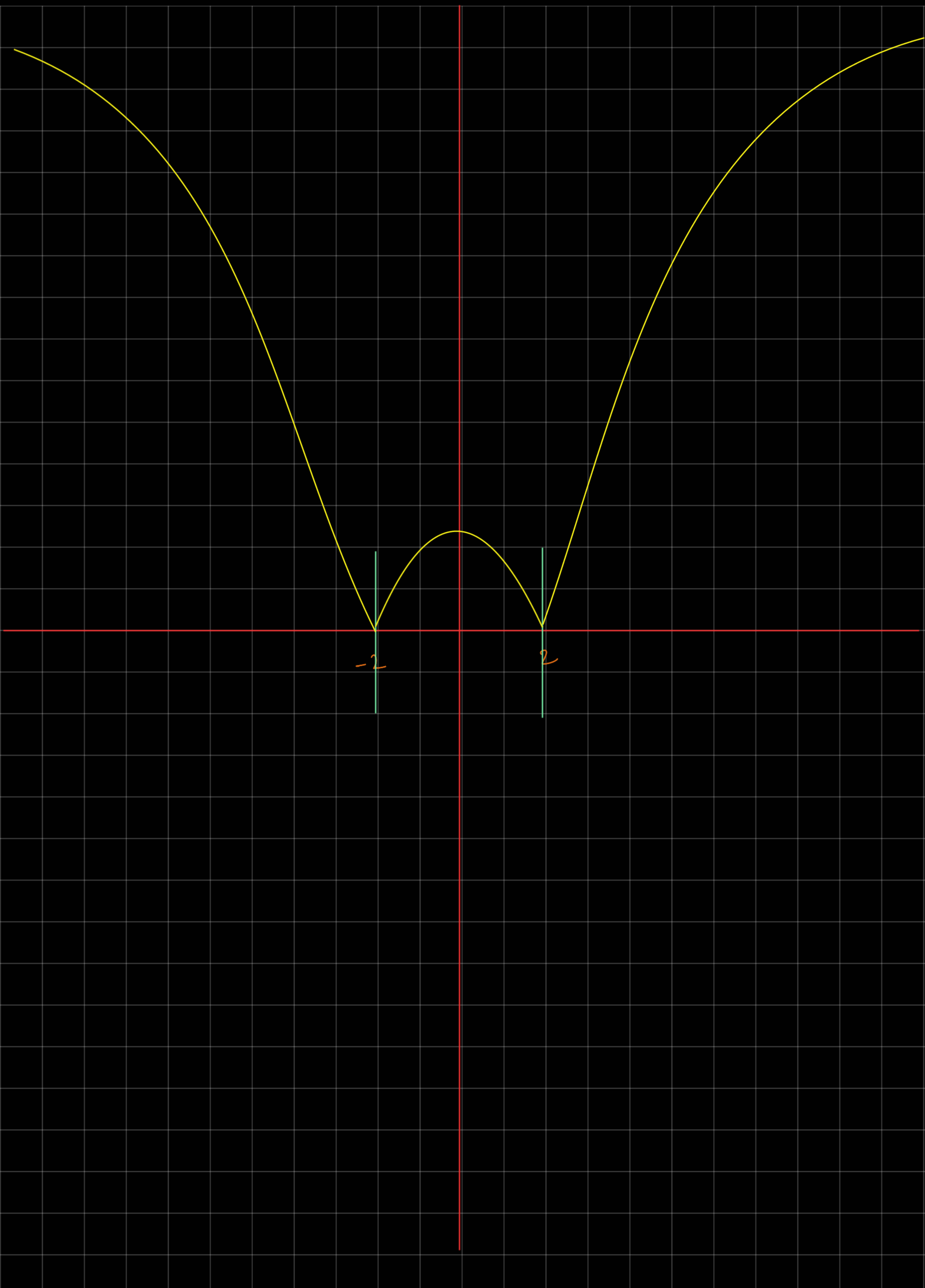
$$= \lim_{x \to -\infty} (x^2)^{2/3}$$

$$= +\infty$$

$$\lim_{x \to +\infty} (x^2 - 4)^{2/3}$$

$$= \lim_{x \to +\infty} (x^2)^{2/3}$$

$$= +\infty$$


-2
2

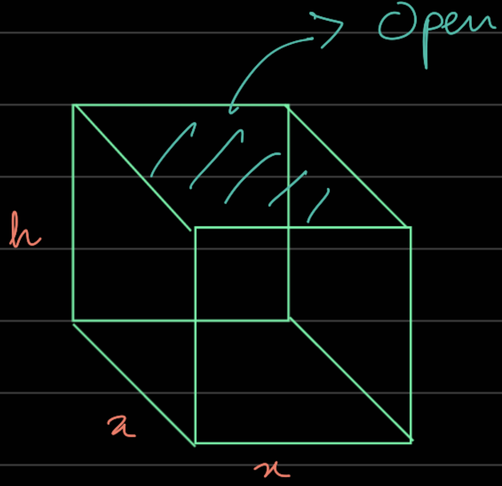
Applied Optimization

Example 1

A manufacturer wants to design an open box having a square base and having a surface area of 108 in^2. What dimension will produce a box with maximum volume?

$V = x^2 h$

$A = x^2 + 4xh$



$x^2 + 4xh = 108$

$$h = \frac{108 - x^2}{4x}$$

$$V = x^2 \left(\frac{108 - x^2}{4x} \right)$$

$$= \frac{x\,(108 - x^2)}{4}$$

$x \geq 0 \qquad V \geq 0 \Rightarrow (108 - x^2) \geq 0$

x	$-\infty$	$-\sqrt{108}$	$\sqrt{108}$	∞
$108 - x^2$	$-$	$+$	$-$	

$V = \frac{x}{4}(108 - x^2)$ where $0 \leq x \leq \sqrt{108}$

$\downarrow$

Feasible Domain

$$V = 27x - \frac{x^3}{4}$$

$$V'(x) = 27 - \frac{3x^2}{4}$$

$$V'(x) = 0$$

$$27 = \frac{3x^2}{4}$$

$$\frac{\overset{9}{\cancel{27}} \times 4}{\cancel{3}} = x^2$$

$$x^2 = 36$$

$$x = 6 \text{ in}$$

$$V(0) = 0$$

$$V(\sqrt{108}) = 0$$

$$V(6) = 108$$

Maximum of $V = 108$ and it occurs at $x = 6$

$$h = \frac{108 - 36}{24}$$

$$= \frac{72}{24}$$

$$= 3 \text{ in}$$

Example 2

Which point on the graph of $y = 4 - x^2$ is closest to the point $(0, 2)$?

$$d = \sqrt{x^2 + (y-2)^2}$$

$$d = \sqrt{x^2 + (4 - x^2 - 2)^2}$$

$$= \sqrt{x^2 + (2 - x^2)^2}$$

$$f(x) = x^2 + 4 + x^4 - 4x^2$$

$$= x^4 - 3x^2 + 4$$

$$f'(x) = 4x^3 - 6x$$

$$f'(x) = 0$$

$$4x^3 - 6x = 0$$

$$2x(2x^2 - 3) = 0$$

$$2x = 0 \qquad\qquad 2x^2 - 3 = 0$$

$$x = 0 \qquad\qquad 2x^2 = 3$$

$$x^2 = \frac{8}{2}$$

$$x = \pm\sqrt{\frac{3}{2}}$$

Critical points are $-\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{3}{2}}$

$$f''(x) = 12x^2 - 6$$

$$= 6(2x^2 - 1)$$

$$f''(0) = 6(2(0)-1)$$
$$= -6 \qquad \therefore f(0) \text{ is a local maximum}$$

$$f''\left(\sqrt{\tfrac{3}{2}}\right) = 6\left(2\left(\sqrt{\tfrac{3}{2}}\right)^2 - 1\right)$$
$$= 6\left(\frac{2\times 3}{2} - 1\right)$$
$$= 6(2)$$
$$= 12 \qquad \therefore f\left(\sqrt{\tfrac{3}{2}}\right) \text{ is a local minimum}$$

$$f''\left(-\sqrt{\tfrac{3}{2}}\right) = 6\left(2\left(-\sqrt{\tfrac{3}{2}}\right)^2 - 1\right)$$
$$= 6\left(\frac{2\times 3}{2} - 1\right)$$
$$= 6(2)$$
$$= 12 \qquad \therefore f\left(-\sqrt{\tfrac{3}{2}}\right) \text{ is a local minimum}$$

$d = \sqrt{f(x)}$ will achieve its minimum @ $x = -\sqrt{\frac{3}{2}}$ and at $x = \sqrt{\frac{3}{2}}$

$$y = 4 - x^2 \qquad\qquad y = 4 - x^2$$
$$= 4 - \left(\sqrt{\tfrac{3}{2}}\right)^2 \qquad\qquad = 4 - \left(-\sqrt{\tfrac{3}{2}}\right)^2$$
$$= 4 - \frac{3}{2} \qquad\qquad = 4 - \frac{3}{2}$$
$$= \frac{5}{2} \qquad\qquad = \frac{5}{2}$$

$\therefore$ The coordinates closest to $(0,2)$ are

$$\left(-\sqrt{\tfrac{3}{2}}, \frac{5}{2}\right) \;\&\; \left(\sqrt{\tfrac{3}{2}}, \frac{5}{2}\right)$$