$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \frac{1/2\sqrt{x}}{1} = \frac{1}{2\sqrt{x}}$$

Example

$$\lim_{x \to 0} x \ln x$$

$$x \ln = \frac{\ln x}{1/x}$$

$$\lim_{x \to 0} \frac{\ln x}{1/x} = \frac{\infty}{1/0} = \frac{-\infty}{\infty}$$

$$\lim_{x \to 0} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0} \frac{1}{\cancel{x}} \times -x^{\cancel{2}}$$

$$= \lim_{x \to 0} (-x)$$

$$= 0$$

$$\lim_{\theta \to 0} \frac{\sin 2\theta}{\tan \theta}$$

$$= \lim_{\theta \to 0} \frac{2 \cos 2\theta}{\sec^2 \theta}$$

$$= \lim_{\theta \to 0} \frac{2 \cos 2(0)}{\sec^2 (0)}$$

$$= 2$$

Evaluate the following limit

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$

$$= \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$$

$$= \lim_{x\to 0} \frac{-\sin x}{6x}$$

$$= -\frac{1}{6}$$

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x}$$

$$= \lim_{x\to 0} \frac{2e^{2x}}{1}$$

$$= \lim_{x\to 0} 2e^{2x}$$

$$= 2e^{0}$$

$$= 2$$

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x}$$

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x} = \boxed{e^{\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)}}$$

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x} = e$$

$$\lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right)$$

$$e^{0 \ln \square} = e^{\ln \square^{0}} = \square^{0}$$

$$= \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{\cancel{-1/x^2}}{1 + 1/x}}{\cancel{-1/x^2}} = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}} = 1$$

$$\therefore e^{1} = e$$

$$\lim_{x \to 1^-} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

$$= \lim_{x \to 1^+} \left(\frac{x-1 - \ln x}{(x-1)\ln x} \right)$$

$$= \lim_{x \to 1^-} \frac{1 - 1/x}{\ln x + \left(\frac{x-1}{x}\right)}$$

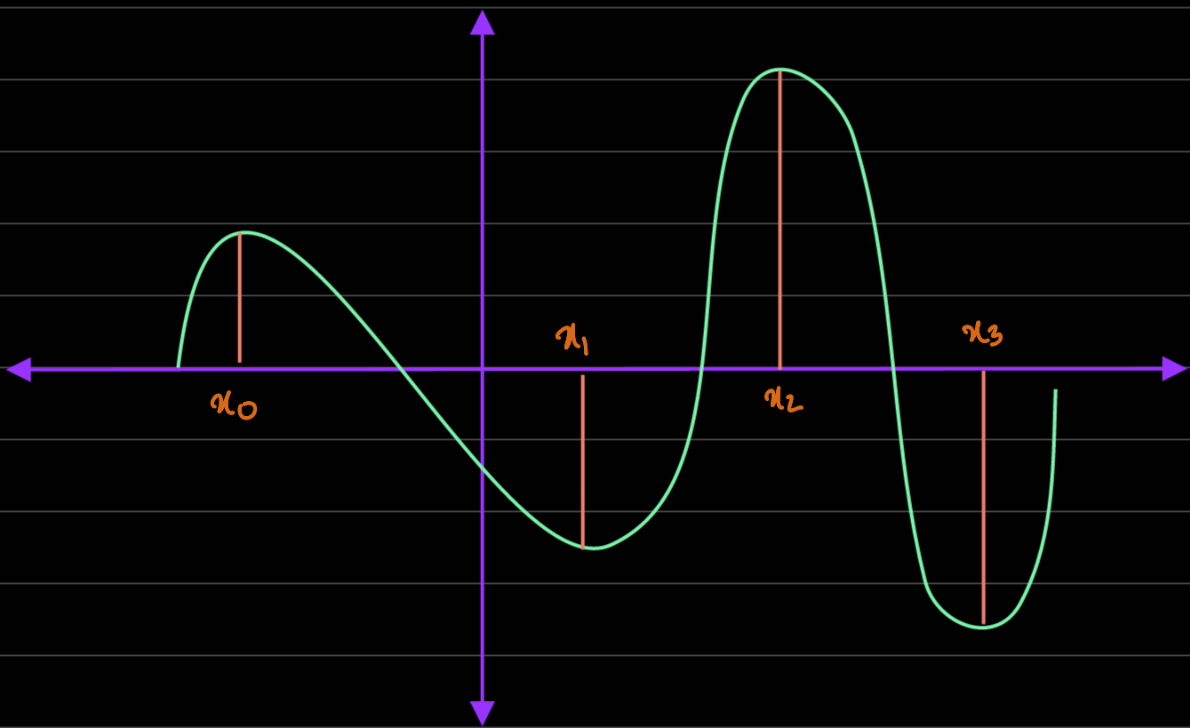
$$= \lim_{x \to 1^-} \frac{1 - 1/x}{\ln x + 1 - 1/x}$$

$$= \lim_{x \to 1^+} \frac{1/x^2}{\frac{1}{x} + \frac{1}{x^2}}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Extreme Values of Functions



$f(x_2) \geq f(x)\ \forall\ x$ in the domain

$f(x_2)$ is the absolute maximum of f.

Only one distinct absolute maximum in a function.

$f(x_3) \leq f(x)\ \forall\ x$ in the domain

$f(x_3)$ is the absolute minimum of f.

Only one distinct absolute minimum in a function.

Critical Numbers

Let f be defined at c. If $f'(c) = 0$ or $f'(c)$ is undefined then c is called a critical number of f.

$f(x) = \sqrt{x}$

Domain $[0, \infty)$

$f(0) = \sqrt{0} = 0$

$f'(0) = \frac{1}{2\sqrt{x}} \Rightarrow$ undefined

$f(x) = \sqrt[3]{x}$

Domain $[0, \infty)$

$f(0) = \sqrt[3]{0} = 0$

$f'(0) = \frac{1}{3\sqrt[3]{x^2}} \Rightarrow$ undefined

Example Find if exist, the critical numbers

1. $f(x) = 5x^3 - 3x^5 + 4$

$f'(x) = 15x^2 - 15x^4$

$= 15x^2(1 - x^2)$

$f'(x) = 0$

$15x^2(1-x^2) = 0$ $\qquad$ $1 - x^2 = 0$

$15x^2 = 0 \quad \therefore x = 0$ $\qquad$ $x^2 = \pm 1$

$\therefore$ Critical numbers are $-1, 0, 1$

2. $g(x) = 2x - 3x^{2/3} + 7$

$$g'(x) = 2 - 3 \cdot \frac{2}{3} x^{-1/3}$$

$$= 2 - \frac{2}{\sqrt[3]{x}}$$

$$= 2\left(1 - \frac{1}{\sqrt[3]{x}}\right)$$

$g(0) = 7$

$g'(0) \Rightarrow$ undefined

$\therefore$ 0 is a critical number

$g'(x) = 0$

$$2\left(1 - \frac{1}{\sqrt[3]{x}}\right) = 0$$

$$\frac{1}{\sqrt[3]{x}} = 1$$

$$\sqrt[3]{x} = 1$$

$\therefore x = 1$

$\therefore$ 1 is a critical number

3. $h(x) = x \ln x$

$h'(x) = \ln x + 1$

$h'(x) = 0$

$$\ln x + 1 = 0$$

$$\ln x = -1$$

$$e^{\ln x} = e^{-1}$$

$$x = \frac{1}{e}$$

$\therefore$ $1/e$ is a critical point

Finding Absolute Extremes

The Extreme Value Theorem

If f is continuous on a closed interval $[a,b]$, then f has an absolute maximum and an absolute minimum.

Steps To find absolute extrema, do the following:

1. Find all the critical numbers of f.
2. Evaluate function at all critical numbers that are in the interval $[a,b]$ and at the endpoints a & b [find y coordinate]
3. Largest is absolute maximum, smallest is absolute minimum

Example PDF Page 46 4.4 Q46

$$f(x) = \frac{x^3}{3} - x^2 - 3x + 1 \qquad x \in [-1, 2]$$

$$f'(x) = x^2 - 2x - 3$$

$$f'(x) = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x+1)(x-3) = 0$$

$x = -1$ OR $x = 3$

$\downarrow$

not within the interval

$$f(-1) = \frac{-1}{3} - 1 + 3 + 1$$

$$= \frac{8}{3} \Rightarrow \text{Absolute Maximum}$$

$$f(2) = \frac{8}{3} - 4 - 6 + 1$$

$$= \frac{-19}{3} \Rightarrow \text{Absolute Minimum}$$

PDF Page 46 4.4 Q51

$f(x) = 1 - x^{2/3}$ $[-1, 1]$

$f'(x) = \dfrac{-2}{3\sqrt[3]{x^2}}$

$f(0) = 1$

$f'(0) \Rightarrow$ undefined

$\therefore$ 0 is a critical point

$f(0) = 1 \Rightarrow$ Absolute Maximum

$\left.\begin{array}{l} f(1) = 0 \\ f(-1) = 0 \end{array}\right\}$ Absolute Minimum