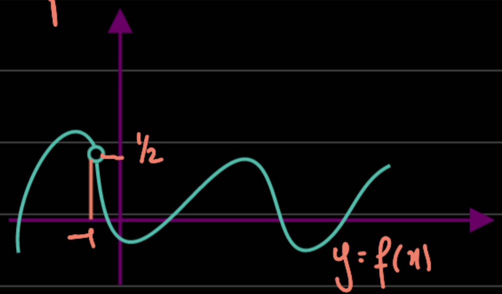
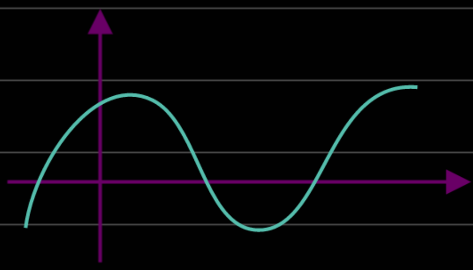
Continuity

Function is a continuous at a point if it does not have a break or gap at that point.



not continuous $f(-1) = 4$



continuous function

$$\lim_{x \to -1} f(x) = \frac{1}{2}$$

Continuity

The function f is continuous at point c if it is defined at point c and

$$\lim_{x \to c} f(x) = f(c)$$

meaning

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

A point where a function is not continuous is called a point of discontinuity.

Example

1. Find the value(s) of 'm' for which the function is continuous at $x = -2$.

$$f(x) = \begin{cases} \dfrac{\sqrt{x^2+1} - \sqrt{5}}{m(x+2)} & \text{if } x \neq -2 \\ 2 & \text{if } x = -2 \end{cases}$$

$$\lim_{x \to -2} f(x) = f(-2)$$

$$f(-2) = 2$$

$$\lim_{x \to -2} \frac{\sqrt{x^2+1} - \sqrt{5}}{m(x+2)}$$

$$= \lim_{x \to -2} \frac{\sqrt{x^2+1} - \sqrt{5}}{m(x+2)} \times \frac{\sqrt{x^2+1} + \sqrt{5}}{\sqrt{x^2+1} + \sqrt{5}}$$

$$= \lim_{x \to -2} \frac{(\sqrt{x^2+1})^2 - (\sqrt{5})^2}{m(x+2)(\sqrt{x^2+1} + \sqrt{5})}$$

$$= \lim_{x \to -2} \frac{x^2+1-5}{m(x+2)(\sqrt{x^2+1} + \sqrt{5})}$$

$$= \lim_{x \to -2} \frac{x^2-4}{m(x+2)(\sqrt{x^2+1} + \sqrt{5})}$$

$$= \lim_{x \to -2} \frac{\cancel{(x+2)}(x-2)}{m\cancel{(x+2)}(\sqrt{x^2+1} + \sqrt{5})}$$

$$= \lim_{x \to -2} \frac{x-2}{m(\sqrt{x^2+1} + \sqrt{5})}$$

$$= \frac{-4}{2m\sqrt{5}}$$

$$\frac{-4}{2m\sqrt{5}} = 2$$

$$-4 = 4m\sqrt{5}$$

$$\frac{-4}{4\sqrt{5}} = m$$

$$m = \frac{-1}{\sqrt{5}}$$

$\therefore$ m should be equal to $\frac{-1}{\sqrt{5}}$ for $f(x)$ to be continuous at $x = -2$.

Classification of Discontinuities

Removable discontinuity   Jump Discontinuity   Infinite Discontinuity

↳ If f is undefined at $x = c$ but $\lim_{x \to c} f(x) = L$ exists. Then c is a removable discontinuity.

f can be extended to a continuous function g given by

$$g(x) = \begin{cases} f(x), & \text{if } x \neq c \\ L, & \text{if } x = c \end{cases}$$

Example

Show that the function $f(x) = \dfrac{\sin 3x}{2x}$ has a removable discontinuity at $x = 0$ and give the expression of the continuous extension.

Domain $= \mathbb{R} - \{0\}$

f is not defined at $x = 0$.

$$\lim_{x \to 0} \frac{\sin 3x}{2x}$$

$$= \lim_{x \to 0} \frac{1}{2x} \times \frac{\cancel{\sin} 3x}{1} \times \frac{3x}{\cancel{3x}}$$

$$= \lim_{x \to 0} \frac{3}{2}$$

$$= \frac{3}{2}$$

$\Rightarrow f$ has a removable discontinuity at $x = 0$.

The continuous extension g is given by

$$g(x) = \begin{cases} \dfrac{\sin 3x}{2x} & \text{if } x \neq 0 \\ \dfrac{3}{2} & \text{if } x = 0 \end{cases}$$

Jump Discontinuity

If f is defined at $x = c$ but $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+}$, then c is a jump discontinuity.

Infinite Discontinuity

If $\lim_{\substack{x \to c^+ \\ \text{or } x \to c^-}} f(x) = \pm \infty$ then c is called an infinite discontinuity.

Example PDF Q 82

$f(1) = \underline{3}$

$\lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1^-} x^2$

$= 1^2$

$= \underline{1}$

$\lim_{x \to 1^+} f(x)$

$= \lim_{x \to 1^+} 2x + 1$

$= 2(1) + 1$

$= \underline{3}$

$\therefore$ It is a jump discontinuity

Continuity of Trigonometric functions

All 6 trigonometric functions ($\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$) are continuous at every point in their domain.

Example

Find the value(s) of a for which the function

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x} & \text{if } x \neq 0 \\ m^2 - 3 & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$.

$$\lim_{x \to 0} f(x) = f(0)$$

$$f(x) = m^2 - 3$$

$$\lim_{x \to 0} \frac{1-\cos x}{x} \times \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$\sin^2 x + \cancel{\cos^2 x} = 1 - \cos^2 x$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x} \times \frac{x}{x} \times \frac{1}{1+\cos x}$$

$\dfrac{\sin \square}{\square} = 1$

$$= \lim_{x \to 0} \frac{x}{1+\cos x} \quad 1$$

$$= \frac{0}{2}$$

$$= 0$$

$$\lim_{x\to 0} f(x) = f(0)$$

$$0 = m^2 - 3$$

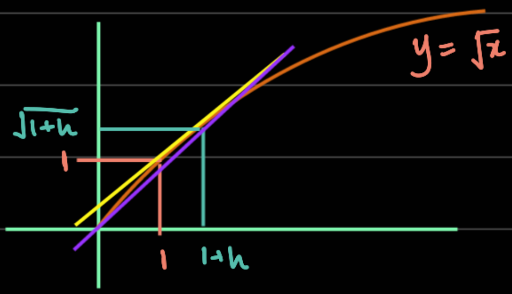
$$m^2 = 3$$

$$m = \pm\sqrt{3}$$

Differentiation

$f(x) = \sqrt{x}$

$x = 1$



yellow line

The slope of the tangent line is

$$\frac{f(1+h) - f(1)}{(1+h) - 1}$$

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{(1+h) - 1} \Rightarrow \text{secant line}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \times \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$$

$$= \lim_{h\to 0} \frac{1+h - 1}{h(\sqrt{1+h} + 1)}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{1+h} + 1)}$$

$$= \frac{1}{2}$$

$m = \frac{1}{2}$ $\quad (1, 1)$

The equation of the tangent line is $y = \frac{1}{2}(x-1) + 1$

If f is defined at $x=c$ then the derivative of f at c denoted by $f'(c)$ is given by

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$ ← Instantaneous rate of change

If the limit exists, then we say that the function f is differentiable at $x = c$

Otherwise the function is not differentiable at $x = c$.

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

= The slope of the tangent line to the graph of f at c

= The rate of change of f at c