Operation of Complex Numbers

Addition / Subtraction / Multiplication

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$$
$$= a_1 + a_2 + i(b_1 + b_2)$$

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2)$$
$$= (a_1 - a_2) + i(b_1 - b_2)$$

$$z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2)$$
$$= a_1a_2 + ia_1b_2 + ib_1a_2 + i^2 b_1b_2$$
$$= a_1a_2 + i(a_1b_2 + a_2b_1) - b_1b_2$$

$$\boxed{i^2 = -1}$$

Division

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = A + iB$$

Multiply with conjugate

$$= \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$$

$$= \frac{a_1a_2 + i(a_2b_1 - a_1b_2) + b_1b_2}{a_2^2 + b_2^2}$$

Example

Find x and y if

a) $(3+4i)^2 - 2(x-iy) = x+iy$

$9+24i-16-2x+2iy = x+iy$

$-7-2x+i(24+2y) = x+iy$

$-7-2x = x$

$3x = -7$

$x = -7/3$

$24+2y = y$

$y = -24$

b) $\frac{x}{1+i} + \frac{y}{2-i} = 2+4i$

$\frac{x(2-i)+y(1+i)}{(1+i)(2-i)} = 2+4i$

$\frac{2x-xi+y+yi}{3+i} = 2+4i$

$2x+y+i(y-x) = (2+4i)(3+i)$

$2x+y+i(y-x) = 2+14i$

$2x+y = 2$

$-x+y = 14$

$y = 14+x$

$2x+14+x = 2$

$3x = -12$

$x = -4$

$y = 14+x$

$y = 14-4$

$y = 10$

Square root of complex numbers

Example

Find the square root of $z = 35 - 12i$

We want to find a and b such that

$$\sqrt{35 - 12i} = a + ib$$

$$35 - 12i = (a+ib)^2$$

$$35 - 12i = a^2 - b^2 + 2iab$$

$$a^2 - b^2 = 35 \qquad 2ab = -12$$

$$ab = -6$$

a	6	−6
b	−1	1

In finding the root of a complex number, we shall use the following convention
sign of $Re(z)$ = sign $Re(z)$

$$\sqrt{35 - 12i} = 6 - i$$

Example

Solve for z, the quadratic equation

$$z^2 - (1-i)z + 7i - 4 = 0$$

$$a = 1, \quad b = -(1-i), \quad c = 7i - 4$$

$$z = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

$$z = \frac{(1-i) \pm \sqrt{(1-i)^2 - 4(7i-4)}}{2}$$

$$= \frac{1-i \pm \sqrt{1-2i-1-28i+16}}{2}$$

$$= \frac{1-i \pm \sqrt{16-30i}}{2}$$

$a^2 - 2ab + b^2$

$1^2 - 2(1)(i) + i^2$

$1 - 1 - 2i$

$\sqrt{16-30i} = a + ib$

$16 - 30i = a^2 - b^2 + 2iab$

$a^2 - b^2 = 16$

$2ab = 30$

$ab = 15$

$a = 5, \quad b = -3$

$\sqrt{16-30i} = 5 - 3i$

$$z = \frac{1-i+5-3i}{2} \quad \text{OR} \quad \frac{1-i-5+3i}{2}$$

$$= \frac{6-4i}{2} \quad \text{OR} \quad \frac{-4+2i}{2}$$

$= 3 - 2i \quad$ OR $\quad -2 + i$

Graphical Representation of Complex Numbers

Complex numbers are represented on the xy plane

$z = a + ib$ corresponds to the point P with x coordinate a and y - coordinate b

Modulus and Arguments of Complex Numbers

If x is a real number, then $|x|$ = distance from 0 to x.
If z is a complex number, then its absolute value is called modulus and is equal to the distance from $(0,0)$ to z.

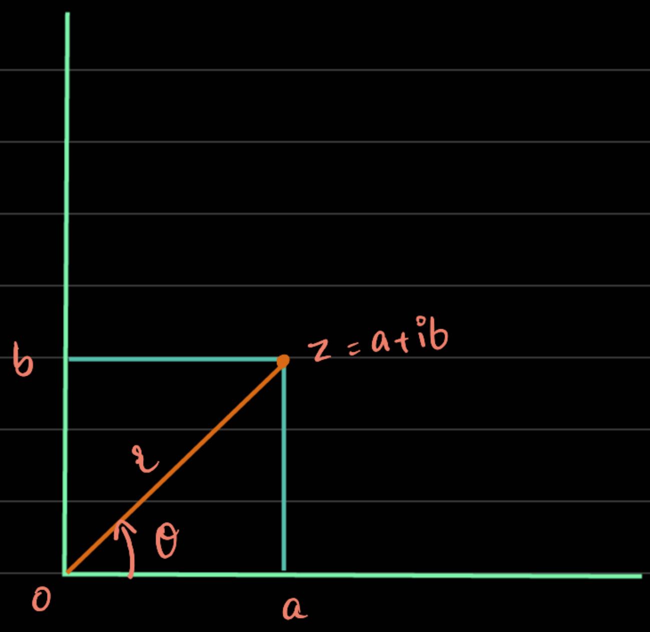
If $z = a + ib$, then

$$|z| = \sqrt{a^2 + b^2}$$

$r = |z|$

$\cos\theta = \frac{a}{r}$ $\quad\quad$ $\sin\theta = \frac{b}{r}$

$a = r\cos\theta$ $\quad\quad$ $b = r\sin\theta$



$$Z = a + ib$$
$$= r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

θ is called the argument of z

Example

Find the modulus and argument of the following complex numbers

a) $z = 2 + 2i$

$|z| = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}$

$\tan\theta = \frac{b}{a}$

$\theta = \tan^{-1}\frac{b}{a}$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$ OR $\frac{5\pi}{4}$

coordinate does not match angle

$\therefore \theta = \pi/4$

b) $z = -1 + i\sqrt{3}$

$|z| = \sqrt{1+3}$

$= 2$

$\tan\theta = \frac{\sqrt{3}}{-1}$

$= -\sqrt{3}$

$\theta = \tan^{-1}(-\sqrt{3})$

$= \frac{2\pi}{3}$ OR $\frac{5\pi}{3}$

$\therefore \theta = 2\pi/3$

Euler Representation of Complex Numbers

If r and θ are the modulus and argument of a complex number z, then

$$z = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

$\cos\theta + i\sin\theta = e^{i\theta}$ $\Rightarrow$ Euler has shown

$$z = re^{i\theta}$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$
$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$