(10pts)Problem 1
Evaluate the following limits

$$(1)\ \lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right), \qquad (2)\ \lim_{x\to -\infty}\frac{9x^2-4x+1}{\sqrt{3x^4+5x^3-x^2+2}}.$$

Solution of Problem 1

(1)

$$\begin{aligned}
&\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)\\
=&\lim_{x\to 2}\left(\frac{x+2}{x^2-4}-\frac{4}{x^2-4}\right)\\
=&\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right)\\
=&\lim_{x\to 2}\frac{(x-2)}{(x-2)(x+2)}\\
=&\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}=0.25. \qquad [\mathbf{5\ points}]
\end{aligned}$$

(2)

$$\begin{aligned}
\lim_{x\to -\infty}\frac{9x^2-4x+1}{\sqrt{3x^4+5x^3-x^2+2}} &= \lim_{x\to -\infty}\frac{9x^2}{\sqrt{3x^4}}\\
&= \lim_{x\to -\infty}\frac{9\cancel{x}^2}{\sqrt{3}\cancel{x}^2}\\
&= \lim_{x\to -\infty}\frac{9}{\sqrt{3}}\\
&= \frac{9}{\sqrt{3}} = 3\sqrt{3} = 5.196\,2 \qquad [\mathbf{5\ points}]
\end{aligned}$$

(**10pts**)**Problem 2**
Let $g(x)$ be the function such that

$$g(x) = \begin{cases} \frac{x^2-m^2}{x-m}, \text{ if } x \neq m \\ \\ m^2+1, \text{ if } x = m \end{cases}.$$

Find the value of m for which the function $g(x)$ is continuous at $x = m$.

Solution of Problem 2
The function $g(x)$ is continuous at $x = m$ if and only if

$$\lim_{x \to m} g(x) = g(m). \qquad [3 \textbf{ points}]$$

Note that when x is approaching m, then $x \neq m \Rightarrow$

$$g(x) = \frac{x^2 - m^2}{x - m} = \frac{(x-m)(x+m)}{x-m} = x + m.$$

Hence,

$$\lim_{x \to m} g(x) = \lim_{x \to m} (x + m) = 2m. \qquad [4 \textbf{ points}]$$

Since $g(m) = m^2 + 1$, the condition of continuity gives

$$\begin{aligned} m^2 + 1 &= 2m \Leftrightarrow m^2 - 2m + 1 = 0 \\ &\Leftrightarrow (m-1)^2 = 0 \Leftrightarrow m = 1 \qquad [3 \textbf{ points}] \end{aligned}$$

(**10pts**)**Problem 3**

Find the slope of the tangent line to the graph of the functions at the indicated points

$$(1) \quad f(x) = (7x^3 - 4x^2 + 2)^{1/3} \quad \text{at } x = 1$$

$$(2) \quad g(x) = \ln\left(\tfrac{x+1}{\sqrt{x+2}}\right) \quad \text{at } x = 0.$$

Solution of Problem 3

(1)

$$\begin{aligned}
f(x) &= \left(7x^3 - 4x^2 + 2\right)^{1/3} \\
f\,'(x) &= \left(\frac{1}{3}\right)\left(21x^2 - 8x\right)\left(7x^3 - 4x^2 + 2\right)^{\frac{1}{3}-1} && [3 \textbf{ points}] \\
&= \left(\frac{1}{3}\right)\left(21x^2 - 8x\right)\left(7x^3 - 4x^2 + 2\right)^{\frac{-2}{3}} \\
&= \frac{1}{3}x\frac{21x - 8}{(7x^3 - 4x^2 + 2)^{\frac{2}{3}}}
\end{aligned}$$

$$\begin{aligned}
f\,'(1) &= \frac{1}{3}\frac{21 - 8}{(7 - 4 + 2)^{\frac{2}{3}}} \\
&= \frac{13}{15}\sqrt[3]{5} = 1.4820 && [2 \textbf{ points}]
\end{aligned}$$

(2)

$$\begin{aligned}
g(x) &= \ln\left(\frac{x+1}{\sqrt{x+2}}\right) = \ln(x+1) - \frac{1}{2}\ln(x+2) \\
g'(x) &= \frac{1}{x+1} - \frac{1}{2(x+2)} && [3 \textbf{ points}] \\
g'(0) &= \frac{1}{0+1} - \frac{1}{2(0+2)} = \frac{3}{4} = 0.75. && [2 \textbf{ points}]
\end{aligned}$$

(**10pts**)**Problem 4**
Find the equation of the tangent line to the graph of $f(x) = e^x \cos x + \tan x$ at $x = 0$.

Solution of Problem 4

$$f(x) = e^x \cos x + \tan x$$
$$f\,'(x) = e^x \cos x - e^x \sin x + \sec^2 x \qquad [4\ \textbf{points}]$$

$$\begin{aligned} f\,'(0) &= e^0 \cos 0 - e^0 \sin 0 + \sec^2 0 \\ &= 2. \qquad [3\ \textbf{points}] \end{aligned}$$

$$f(0) = 1$$

The equation of the tangent line is given by

$$\begin{aligned} y &= 2\,(x - 0) + 1 \\ &= 2x + 1 \qquad [3\ \textbf{points}] \end{aligned}$$

(10pts)Problem 5
Find the slope of the tangent line to the graph of

$$x^2 - xy + y^2 = 3 \text{ at the point } (1, 2).$$

Solution of Problem 5
Here, we find the derivative by implicit differentiation.
Taking derivative with respect to x keeping in mind that y is a function of x, we get

$$2x - y - xy' + 2yy' = 0 \qquad [3 \textbf{ points}]$$

$$\begin{aligned} -xy' + 2yy' &= y - 2x \\ y'(-x + 2y) &= y - 2x \\ \frac{dy}{dx} &= y' = \frac{y - 2x}{-x + 2y} \qquad [4 \textbf{ points}] \end{aligned}$$

The slope at the point $(1, 2)$ is

$$m = \frac{2 - 2}{-1 + 4} = 0. \qquad [3 \textbf{ points}]$$

(10pts)Problem 6
A meteor (spherical in shape) enters earth's atmosphere and starts burning up in such a way that its surface area decreases at a constant rate of 100 cm^2/s. Find the rate at which the diameter is changing when the radius is $5m$.
Hint: The surface area of a sphere of radius r is $4\pi r^2$.

Solution of Problem 6

- Given:
$$\frac{dS}{dt} = -100cm^2/s. \qquad [2 \textbf{ points}]$$
- Want:
$$\frac{dD}{dt} \text{ when } r = 5m = 500cm \quad (D = 2r \text{ is the diameter}). \qquad [2 \textbf{ points}]$$
- Related equation:
$$S = 4\pi r^2$$
- Related rate equation:
$$\frac{dS}{dt} = (4\pi)(2r)\frac{dr}{dt} \qquad [2 \textbf{ points}]$$
-
$$\begin{aligned} -100 &= 4000\pi\frac{dr}{dt} \Rightarrow \\ \frac{dr}{dt} &= \frac{-100}{4000\pi} = \\ &= -\frac{1}{40\pi} \qquad [2 \textbf{ points}] \end{aligned}$$

$$\begin{aligned} D &= 2r \Rightarrow \frac{dD}{dt} = 2\frac{dr}{dt} \\ &= -\frac{1}{20\pi} = 1.5915 \times 10^{-2} \qquad [2 \textbf{ points}] \end{aligned}$$

(10pts)Problem 7
Find the absolute maximum and minimum of $f(x) = 4x^3 - 8x^2 + 1$ on the closed interval $[-1, 1]$.

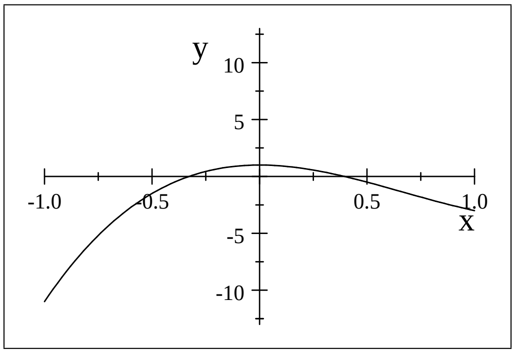
Solution of Problem 7

$$\begin{aligned} f'(x) &= 12x^2 - 16x \\ &= 4x(3x-4). \qquad [2 \textbf{ points}] \end{aligned}$$

The critical numbers are 0 and $\frac{4}{3}$. [2 **points**]
Since $\frac{4}{3} = 1.3333 \notin [-1, 1]$, we will only consider 0. [2 **points**]

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 4 - 8 + 1 = -3 \\ f(-1) &= -4 - 8 + 1 = -11. \end{aligned}$$

- Absolute Maximum is equal to 1 occurring at $x = 0$. [2 **points**]
- Absolute Minimum is equal to -11 occurring at $x = -1$. [2 **points**]



(**10pts**)**Problem 8**

Find the local extrema of the function

$$f(x) = x\,(x-1)^3\,.$$

Solution of Problem 8

$$\begin{aligned} f\,'(x) &= (x-1)^3 + 3x\,(x-1)^2 \\ &= (x-1)^2\,(x-1+3x) \\ &= (x-1)^2\,(4x-1)\,. \qquad \text{[2 \textbf{points}]} \end{aligned}$$

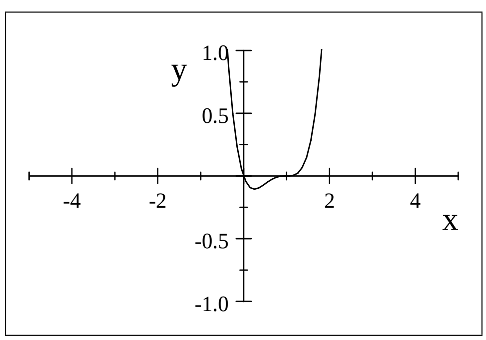
The critical numbers are 1 and $\frac{1}{4}$. [2 **points**]
<u>Table of variation</u>:

x	$-\infty$		$\frac{1}{4}$		1		$+\infty$
$(x-1)^2$		$+$		$+$	0	$+$	
$4x-1$		$-$	0	$+$		$+$	
$f'(x)$		$-$		$+$		$+$	
$f(x)$		$\searrow$	$f(\frac{1}{4}) = -\frac{3}{16}$ Local min	$\nearrow$		$\nearrow$	

[3 **points**]

$$f(\frac{1}{4}) = \left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)^3 = \frac{-27}{256} = -0.105\,47$$

is a local minimum. (the only local extrema) [3 **points**]



(**10pts**)**Problem 9**

Find the open intervals where the function

$$g(x) = x^4 - 18x^2 + 9$$

is concave up or down.

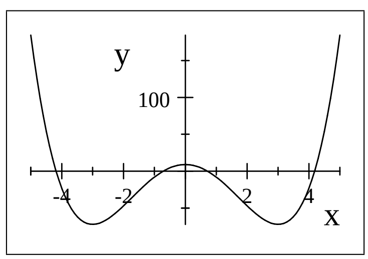
Solution of Problem 9

$$g(x) = x^4 - 18x^2 + 9$$
$$g'(x) = 4x^3 - 36x \qquad [2 \textbf{ points}]$$

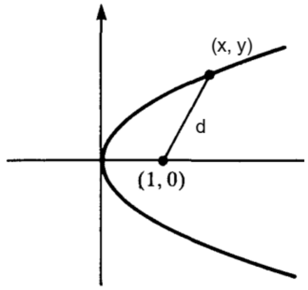
$$\begin{aligned} g''(x) &= 12x^2 - 36 \\ &= 12\left(x^2 - 3\right) \\ &= 12\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right). \qquad [2 \textbf{ points}] \end{aligned}$$

$g''(x)$ as a quadratic function has the sign of $a = 12$ outside of the roots i.e. positive on $(-\infty,\ -\sqrt{3}) \cup (\sqrt{3},\ \infty)$ and the opposite sign inside the roots i.e. negative on $\left(-\sqrt{3},\ \sqrt{3}\right)$. [2 **points**]

- The graph of g is concave upward on $\left(-\infty,\ -\sqrt{3}\right) \cup \left(\sqrt{3},\ \infty\right)$, and [2 **points**]
- concave downward on $\left(-\sqrt{3},\ \sqrt{3}\right)$ [2 **points**]



(10pts)Problem 10
Find the point (s) on the parabola $2x = y^2$ closest to the point $(1, 0)$.



Solution of Problem 10
Let $(x,\ y)$ be an arbitrary point on the curve.
The distance **d** from (x, y) to $(1, 0)$ is

$$\begin{aligned}\mathbf{d} &= \sqrt{(x-1)^2 + y^2} \\ &= \sqrt{(x-1)^2 + 2x}. \qquad [3 \textbf{ points}]\end{aligned}$$

To minimize **d**, it suffices to minimize

$$\begin{aligned}f(x) &= \mathbf{d}^2 = (x-1)^2 + 2x \\ &= x^2 + 1 \qquad [3 \textbf{ points}]\end{aligned}$$

$$f\,'(x) = 2x$$

$x = 0$ is the only critical number. Since

$$f\,''(x) = 2 > 0,$$

$f(x) = \mathbf{d}^2$ achieves a minimum at $x = 0$. Thus $(0, 0)$ is the point on the parabola that is closest to the point $(1, 0)$. [4 **points**]