Math 141 Final Exam review

Problem 1.
Solve the equation

$$x^2 - 2x + 3 = 0.$$

Solution

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm 2\sqrt{-2}}{2} = 1 \pm \sqrt{2}\, i.$$

Problem 2

$$\text{Find } x, y \text{ if } (3 + 4i)^2 - 2(x - iy) = x + iy.$$

Solution

$$\begin{aligned} \text{Left hand side (LHS)} &= 9 - 16 + 24i - 2x + i2y \\ &= -7 - 2x + i(24 + 2y) \end{aligned}$$

$$\begin{aligned} \therefore -7 - 2x &= x \\ 3x &= -7 \\ x &= -\tfrac{7}{3} \\ \& \ 24 + 2y &= y \\ y &= -24 \quad \square \end{aligned}$$

Problem 3

$$\text{Find } x, y \text{ if } \frac{x}{1+i} + \frac{y}{2-i} = 2+4i.$$

Solution

$$\begin{aligned}
\text{LHS} &= \frac{x}{1+i} + \frac{y}{2-i} \\
&= \frac{x}{1+i} \times \frac{1-i}{1-i} + \frac{y}{2-i} \times \frac{2+i}{2+i} \\
&= \frac{x(1-i)}{1+1} + \frac{y(2+i)}{4+1} \\
&= \frac{x(1-i)}{2} + \frac{y(2+i)}{5}
\end{aligned}$$

$$\text{Now } \frac{x(1-i)}{2} + \frac{y(2+i)}{5} = 2+4i.$$

$$\begin{aligned}
\therefore 5x(1-i) + 2y(2+i) &= 20 + 40i \\
5x - i5x + 4y + i2y &= 20 + 40i \\
5x + 4y + i(-5x + 2y) &= 20 + 40i
\end{aligned}$$

Equating real and imaginary part,

$$\begin{aligned}
5x + 4y &= 20 \\
-5x + 2y &= 40 \\
\hline
6y &= 60 \\
y &= 10 \\
\& \ \therefore x &= -4. \quad \square
\end{aligned}$$

Solving simultaneously,

Problem 4
Find the square root of
$$z = 35 - 12i$$

Solution

$$\begin{aligned}
\text{Let } \sqrt{35-12i} &= a+ib :- \text{ square both sides.}\\
35-12i &= (a+ib)^2\\
&= a^2-b^2+i(2ab)\\
\therefore a^2-b^2 &= 35\\
\text{and } 2ab &= -12\\
ab &= -6.
\end{aligned}$$

By inspection, solutions are $a = 6\&\ b = -1$ or $a = -6$ or $b = 1$.

or

$$\begin{aligned}
a^2-b^2 &= 35\\
ab &= -6\\
b &= -\frac{6}{a}.\\
\therefore a^2 - \left(-\frac{6}{a}\right)^2 &= 35\\
a^2 - \frac{36}{a^2} &= 35.\\
a^4 - 36 &= 35a^2\\
a^4 - 35a^2 - 36 &= 0.\\
(a^2-36)(a^2+1) &= 0\\
a^2 &= 36 \qquad \&\ a^2+1=0 \Rightarrow a \notin \mathbb{R}\\
\therefore a &= \pm 6\ \&\ \therefore b = \pm 1.\\
\&\ \therefore \sqrt{35-12i} &= 6-i. \quad \square \text{ (By convention, } \operatorname{sign}(\Re(\sqrt{z})) = \operatorname{sign}(\Re(z)))
\end{aligned}$$

Problem 5
Find the root of

$$z^2 - (1-i)\,z + 7i - 4 = 0$$

in the form $a + ib$.
Solution

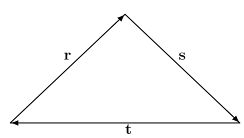
$$\begin{aligned}
z &= \frac{(1-i) \pm \sqrt{(1-i)^2 - 4(1)(7i-4)}}{2} \\
&= \frac{(1-i) \pm \sqrt{1-1-2i-28i+16}}{2} \\
&= \frac{(1-i) \pm \sqrt{16-30i}}{2}
\end{aligned}$$

From beside,

$$\begin{aligned}
&= \frac{(1-i) \pm (5-3i)}{2} \\
&= \frac{1-i+5-3i}{2} \text{ or } \frac{1-i-(5-3i)}{2} \\
&= 3-2i \text{ or } -2+i. \quad \square
\end{aligned}$$

$$\begin{aligned}
\sqrt{16-30i} &= (a+ib) \\
16-30i &= a^2 - b^2 + i(2ab) \\
a^2 - b^2 &= 16 \\
2ab &= -30 \\
ab &= -15 \\
a = 5 \ \& \ b &= -3 \\
\text{or } a = -5 \ \& \ b &= 3 \\
\& \therefore \sqrt{16-30i} &= 5-3i \\
\because \ \text{sign}(16) = \text{sign}(5) &= +
\end{aligned}$$

Problem 6
Use the diagram below to construct a vector equation for $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$.



Solution
Using the operations on vectors in directed line segment, you can see that

$$\mathbf{r} + \mathbf{s} = -\mathbf{t}.$$

Thus

$$\mathbf{r} + \mathbf{s} + \mathbf{t} = \mathbf{0}$$

Problem 7.
Let $P(2,-1,3)$, $Q(3,1,2)$ and $R(2,1,-4)$ be three points in the 3D-space.
(a) Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
(b) Find the vector projection of $\overrightarrow{PQ}$ onto $\overrightarrow{PR}$.
(c) Find the area of the triangle with vertices P, Q and R.
(d) Find an equation of the plane containing the points P, Q and R.
Solution
(a) Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\begin{aligned}\overrightarrow{PQ} &= \langle 1,2,-1\rangle \\ \overrightarrow{PR} &= \langle 0,2,-7\rangle\end{aligned}$$

(b) Find the vector projection of $\overrightarrow{PQ}$ onto $\overrightarrow{PR}$.

$$\mathbf{proj}_{\overrightarrow{PR}}\overrightarrow{PQ} = \left(\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\left\|\overrightarrow{PR}\right\|^2}\right)\overrightarrow{PR}.$$

$$\overrightarrow{PQ}\cdot\overrightarrow{PR} = 11 \text{ and } \left\|\overrightarrow{PR}\right\|^2 = 53.$$

Hence,

$$\mathbf{proj}_{\overrightarrow{PR}}\overrightarrow{PQ} = \frac{11}{53}\langle 0,2,-7\rangle$$

(c) Find the area of the triangle with vertices P, Q and R.

$$\text{Area} = \frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|.$$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 2 & -1 \\ 0 & 2 & -7 \end{vmatrix} = \langle -12,7,2\rangle$$

$$\begin{aligned}\text{Area} &= \frac{1}{2}\left\|\langle -12,7,2\rangle\right\| \\ &= \frac{1}{2}\sqrt{(-12)^2+(7)^2+(2^2)} \\ &= \frac{1}{2}\sqrt{197} = 7.0178\end{aligned}$$

(d) Find an equation of the plane containing the points P, Q and R.
A normal vector to that plane is given by

$$\begin{aligned}\overrightarrow{n} &= \overrightarrow{PQ}\times\overrightarrow{PR} \\ &= \langle -12,7,2\rangle.\end{aligned}$$

Choosing a point, say P, the equation of the plane is

$$-12\,(x-2)+7\,(y+1)+2\,(z-3)=0$$

$$-12x+7y+2z=-25$$

Problem 8.
Find the parametric equations of the line passing through the point $(2,4,1)$ that is perpendicular to the plane

$$3x - y + 5z = 77.$$

__**Solution**__:
A direction vector of the line is given by

$$\overrightarrow{u} = \langle 3, -1, 5 \rangle$$

The parametric equations are

$$\left\{ \begin{array}{c} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} x = 2 + 3t \\ y = 4 - t \\ z = 1 + 5t \end{array} \right.$$

Problem 9.
Find parametric equations of the line of intersection of the planes

$$x - 3y + 2z = -1 \text{ and } \ 4x + y + 7z = 9$$

<u>**Solution**</u>:
Put

$$\overrightarrow{n_1} = \langle 1, -3, 2 \rangle \qquad \text{and} \ \ \overrightarrow{n_2} = \langle 4, 1, 7 \rangle \ \ .$$

The direction vector of the line of intersection is

$$\begin{aligned} \overrightarrow{u} &= \overrightarrow{n_1} \times \overrightarrow{n_2} \\ &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -3 & 2 \\ 4 & 1 & 7 \end{vmatrix} = \langle -23, 1, 13 \rangle \end{aligned}$$

Choose P $\in$ L so the z-coordinate of P is zero. Setting z = 0, we obtain:

$$\begin{aligned} x - 3y &= -1 \\ 4x + y &= 9 \end{aligned}$$

Solution is: $[x = 2, \ y = 1]$. Hence,

$$P = (2, 1, 0) \ \text{ lies on the line.}$$

The parametric equations are:

$$\begin{gathered} x = 2 - 23t \\ y = 1 + t \\ z = 0 + 13t = 13t \end{gathered}$$

Problem 10.

Find the **equation of the plane** containing the lines

$$x = 4 - 4t, \quad y = 3 - t, \quad z = 1 + 5t \quad \text{and}$$

$$x = 4 - t, \quad y = 3 + 2t, \quad z = 1$$

Solution:

- To find the equation of a plane, we need to find its normal $\mathbf{n}$ and a point on it. Setting $t = 0$, we find the point $(4, 3, 1)$ on the first line.
- The part vector $\mathbf{v}_1$ of the first line is $\langle -4, -1, 5 \rangle$ and the vector part $\mathbf{v}_2$ of the second line is $\langle -1, 2, 0 \rangle$.
- Since the vector

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -1 & 5 \\ -1 & 2 & 0 \end{vmatrix} = \langle -10, -5, -9 \rangle,$$

is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, it is the normal to the plane.
- The **equation of the plane** is:

$$\langle -10, -5, -9 \rangle \cdot \langle x - 4, y - 3, z - 1 \rangle$$

$$= -10(x - 4) - 5(y - 3) - 9(z - 1) = 0.$$

□

Problem 11

Determine whether the lines $\mathbf{L}_1$ and $\mathbf{L}_2$ given below are **parallel**, **skew** or **intersecting**. If they intersect, find the point of intersection.

$$\mathbf{L}_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$$

$$\mathbf{L}_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$$

Solution:

- Rewrite these lines as vector equations:

$$\mathbf{L}_1(t) = \langle t, 2t+1, 3t+2 \rangle$$

$$\mathbf{L}_2(s) = \langle -4s+3, -3s+2, 2s+1 \rangle$$

- Equating x and y-coordinates:

$$x = t = -4s+3$$
$$y = 2t+1 = -3s+2.$$

- Solving gives $s = 1$ and $t = -1$.
- $\mathbf{L}_1(-1) = \langle -1, -1, -1 \rangle \neq \langle -1, -1, 3 \rangle = \mathbf{L}_2(1)$. So these lines do **not intersect**.
- Since the lines are clearly **not parallel** (the direction vectors $\langle 1, 2, 3 \rangle$ and $\langle -4, -3, 2 \rangle$ are **not parallel**), the lines are **skew**. $\square$

Problem 12

Use Gauss elimination method to solve the linear system

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

Solution

$$\left[\begin{array}{rrr|r} 3 & -2 & 8 & 9 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1]{R_2 + 2R_1} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & 2 & 19 & 27 \\ 0 & 2 & -12 & -4 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & 2 & -12 & -4 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 31 & 31 \end{array}\right]$$

$$\xrightarrow{\frac{1}{31}R_3} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + 6R_3]{R_1 - 9R_3} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Now we do the back substitution to get

$$x = 3, \quad y = 4 \text{ and } z = 1$$

Problem 13
A) Given

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$$

find the matrix X satisfying the matrix equation

$$2X + B = 3A.$$

Solution

Solution

From the given equation $2X + B = 3A$, we find that

$$\begin{aligned} 2X &= 3A - B \\ &= 3\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 12 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ -2 & 4 \end{bmatrix} \\ X &= \frac{1}{2}\begin{bmatrix} 6 & 10 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} \end{aligned}$$

B) Let

$$A = \begin{bmatrix} 3 & 1 & 4 \\ -1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & -3 \\ 4 & -1 & 2 \\ 2 & 4 & 1 \end{bmatrix}$$

Compute AB.

Solution

$$AB = \begin{bmatrix} 15 & 24 & -3 \\ 13 & 7 & 10 \end{bmatrix}$$

Problem 14
Use the Gauss Jordan method to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

Solution
The augmented matrix is

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

and use the Gauss–Jordan elimination method to reduce it to the form $[I \mid B]$:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 + 3R_1 \\ R_3 + 2R_1}]{-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{-R_2 \\ R_3 - R_2}]{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3]{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

The inverse of A is the matrix

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Problem 15

Compute the determinant of the matrix

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Solution

We use cofactor expansion along the first column of A to get
Thus

$$\det A = 3\det\begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix} - 0\cdot C_{21} - 0\cdot C_{31} - 0\cdot C_{41} - 0\cdot C_{51}$$

$$= 3\cdot 2\det\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$= 6[-(-2)]\det\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$= 12\cdot(-1) = -12$$

Problem 16

Let A be the matrix given by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Assuming that $\det A = -7$, find

(a) $\det(3A)$

(b) $\det(A^{-1})$

(c) $\det\left[(2A)^{-1}\right]$

(d) $\det(A^3)$

(e) $\det\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}$

Solution

First observe that A is a 3×3 matrix.$n = 3$.

(a)

$$\begin{aligned}\det(3A) &= 3^3 \det A \\ &= 27 \cdot (-7) \\ &= -189\end{aligned}$$

(b)

$$\begin{aligned}\det(A^{-1}) &= \frac{1}{\det(A)} \\ &= \frac{-1}{7}\end{aligned}$$

(c)

$$\begin{aligned}\det\left[(2A)^{-1}\right] &= \frac{1}{\det[(2A)]} \\ &= \frac{1}{2^3 \det[(A)]} \\ &= \left(\frac{1}{8}\right) \cdot \left(\frac{-1}{7}\right) \\ &= \frac{-1}{56}\end{aligned}$$

(d)

$$\begin{aligned}\det(A^3) &= [\det(A)]^3 \\ &= (-7)^3 = -343\end{aligned}$$

(e)

$$\begin{aligned}\det\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix} &= \det\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}^T \\ &= \det\begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} \\ &= -\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= -\det(A) = 7\end{aligned}$$