Rules of Differentiation

Product Rule

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'(x)\,g(x) + g'(x)\,f(x)$$

Example

Find $f'(-1)$ if $f(x) = (x^2+2x-4)(x+\sqrt[3]{x}-1)$

$$f'(x) = (2x+2)(x+\sqrt{x}-1) + (x^2+2x-4)\left(1+\frac{1}{3x^{2/3}}\right)$$

$$f'(-1) = 0 + (1-2-4)\left(1+\frac{1}{3(-1)^{2/3}}\right)$$

$$\sqrt[3]{x^2} = \sqrt[3]{(-1)^2} = \sqrt[3]{1} = 1$$

$$= 0 + -5\left(\frac{4}{3}\right)$$

$$= \frac{-20}{3}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

Example

Find the equation of the tangent line to the graph of $f(x) = \frac{x^2-4}{2x-7}$ at $x=2$

$$f'(x) = \frac{(2x)(2x-7) - (x^2-4)(2)}{(2x-7)^2}$$

$$f'(2) = \frac{4(4-7) - (4-4)(2)}{(4-7)^2}$$

$$= \frac{-12}{9}$$

$$= -\frac{4}{3}$$

$y = m(x - x_1) + y_1$ $\quad x_1 = 2$

$y = \frac{-4}{3}(x - 2) + 0$ $\quad y_1 = f(2) = 0$

$y = -\frac{4}{3}x + \frac{8}{3}$

Chain Rule

$\frac{d}{dx} g[f(x)] = f'(x) \cdot g'[f(x)]$

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$f(1) = 2$ $\quad \boxed{g(1) = 1}$ $\quad f'(1) = 3$ $\quad \boxed{g'(1) = 2}$ $\quad f'(2) = 0$ $\quad$ Evaluate $(f \cdot g)'(1)$

$(f \circ g)'(1)$

$\frac{d}{dx} f(g(x))$

$= g'(x) \cdot f'[g(x)]$

$(f \circ g)'(1) = g'(1) \cdot f'(g(1))$

$= 2 \cdot 3$

$= 6$

$\frac{d}{dx}$

Simple Power Rule

$$\frac{d}{dx} x^n = nx^{n-1}$$

General Power Rule

$$\frac{d}{dx} [f(x)]^n = f'(x) \cdot n[f(x)]^{n-1}$$

$$= nf'(x) f(x)^{n-1}$$

Example

Evaluate $f'(x)$

1. $f(x) = \left(\frac{1}{x} + \frac{1}{x^2}\right)^4$

1. $\frac{d}{dx} [u(x)]^n = n u'(x) u(x)^{n-1}$

$$\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{x^2}\right)^4 = 4\left(-x^{-2} - 2x^{-3}\right)\left(x^{-1} + x^{-2}\right)^3$$

$$= 4\left(\frac{-1}{x^2} - \frac{2}{x^3}\right)\left(\frac{1}{x} + \frac{1}{x^2}\right)^3$$

2. $f(x) = \frac{1}{\sqrt{3x^2 - 4x + 5}}$

2. $\frac{d}{dx}\left[3x^2 - 4x + 5\right]^{-1/2}$

$$= \frac{-1}{2}\left[3x - 4\right]\left[3x^2 - 4x + 5\right]^{-3/2}$$

Derivative of Trigonometric functions

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x = \frac{-1}{\sin^2 x}$

Example

Find $f'(x)$

1. $f(x) = x^2 \cos x + \tan x$
2. $f(x) = \sec^3 x$

1. $-x^2 \sin x + 2x \cos x + \sec^2 x$
2. $3 \sec x \tan x \sec^2 x$

$= 3 \sec^3 x \tan x$

Derivatives of Exponents & Logarithmic Functions

Exponential Functions

Let $a > 0$. The function $f(x) = a^x$ is called an exponential function of base a.

Its domain is $(-\infty, \infty)$

Its range is $(0, \infty)$

$\lim_{x \to -\infty} a^x = 0$ $\quad \lim_{x \to +\infty} a^x = +\infty$

If $a = e$, then e^x is called the natural exponential function

Logarithmic Function

The function $f(x) = a^x$ has an inverse and its inverse is denoted by $\log_a x$ and is called the logarithmic function of base a.

Domain of $\log_a x$ is $(0, \infty)$

Range of $\log_a x$ is $(-\infty, \infty)$

$\lim_{x \to 0^+} \log_a x = -\infty$ $\quad$ $\lim_{x \to +\infty} \log_a x = \infty$

$\log_a a^x = x \quad \forall x$

$a^{\log_a x} = x \quad$ for $x > 0$

$\log_a (m \cdot n) = \log_a m + \log_a n \qquad m, n > 0$

$\log_a \left(\frac{m}{n}\right) = \frac{\log_a m}{\log_a n}$

$\log_a m^r = r \log_a m$

$\log_{10} = \log$

If $a = e$ $\quad \log_e = \ln$ → Natural Logarithmic Function

$e^{\ln x} = x \quad$ for $x > 0$

$\ln e^x = x \quad \forall x$

$\ln e = 1$

$\ln 1 = 0$

Change of base rule

$\log_a x = \frac{\ln x}{\ln a}$

Derivatives of Exponents & Logarithmic Functions

$$\frac{d}{dx} a^x = a^x \cdot \ln a$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} a^{u^x} = u'(x) . a^{u^x} \cdot \log(a)$$

$$\frac{d}{dx} e^{u^x} = u'(x)\ e^{u^x}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \ln(u(x)) = \frac{u'(x)}{u(x)}$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx} \log_a (u(x)) = \frac{u'(x)}{u(x) \ln a}$$

Examples

Find $f'(0)$ if

1. $f(x) = x \cdot 4^x - \log_3 (x+1)$

2. $f(x) = x^2 \ln(x+2)$

3. $f(x) = \log\left[\dfrac{(x-1)\sqrt[3]{x+2}}{(x^2+x+1)(x+7)}\right]$

1. $f'(x) = 4^x + x \cdot 4^x \cdot \ln 4 - \dfrac{1}{(x+1)\ln 3}$

$f'(0) = 1 + 0 - \dfrac{1}{\ln 3}$

$= 1 - \dfrac{1}{\ln 3}$

2. $f(x) = x^2 \ln(x+2)$

$f'(x) = 2x \ln(x+2) + \dfrac{x^2}{x+2}$

$f'(0) = 0 + 0$

$= 0$

3. $f(x) = \log\left[\dfrac{(x-1)\sqrt[3]{x+2}}{(x^2+x+1)(x+7)}\right]$

$\log \dfrac{m}{n} = \log m - \log n$

$f(x) = \log\left[(x-1)\sqrt[3]{x+2}\right] - \log\left[(x^2+x+1)(x+7)\right]$

$\log mn = \log m + \log n$

$= \log(x-1) + \log(x+2)^{1/3} - \log(x^2+x+1) - \log(x+7)$

$\log m^2 = 2 \log m$

$= \log(x-1) + \dfrac{1}{3}\log(x+2) - \log(x^2+x+1) - \log(x+7)$

$f'(x) = \dfrac{1}{\ln 10\,(x-1)} + \dfrac{1}{\ln 10\, 3(x+2)} - \dfrac{2x+1}{\ln 10\,(x^2+x+1)} - \dfrac{1}{\ln 10\,(x+7)}$

$f'(0) = \dfrac{1}{\ln 10}\left[-1 + \dfrac{1}{6} - 1 - \dfrac{1}{7}\right]$

$= \dfrac{1}{\ln 10}\left[\dfrac{-42+7-42-6}{42}\right]$

$= \dfrac{1}{\ln 10} \quad \dfrac{-83}{42}$

$= \dfrac{-83}{42 \ln 10}$

Higher Order Derivative

$$\frac{d}{dx}\left(f(x)\right) = f'(x) \Rightarrow \text{First Order Derivative}$$

$$\frac{d}{dx}\left(f'(x)\right) = f''(x) = \frac{d^2y}{dx^2} = y'' \Rightarrow \text{Second Order Derivative}$$

$$\frac{d}{dx}\left(f''(x)\right) = f'''(x) = \frac{d^3y}{dx^3} = y''' \Rightarrow \text{Third Order Derivative}$$

$$\frac{d}{dx}\left[f^{(n-1)}(x)\right] = f^{(n)}(x) = \frac{d^ny}{dx^n} = y^{(n)} \Rightarrow n^{th} \text{ Order Derivative}$$

$\cos x \Rightarrow -\sin x$

$-\sin x \Rightarrow -\cos x$

$-\cos x \Rightarrow \sin x$

$\sin x \Rightarrow \cos x$

Example

Evaluate

$$\frac{d^{999}}{dx^{999}}(\cos x)$$