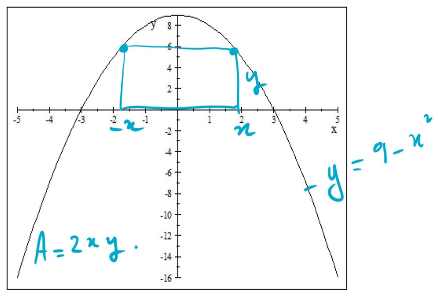
(12pts)**Problem 1**

A) Find the area of the largest rectangle that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 9 - x^2$.

Solution



$$A = 2xy \;\text{ with } y = 9 - x^2, \qquad x > 0.$$
$$A(x) = 2x\left(9 - x^2\right) = 18x - 2x^3 \qquad \textbf{(3pts)}$$
$$A'(x) = 18 - 6x^2 = -6\left(x^2 - 3\right)$$

The critical numbers are $-\sqrt{3}$ and $\sqrt{3}$.

$$A''(x) = -12x. \quad A''(\sqrt{3}) = -12\sqrt{3} < 0 \Rightarrow \text{max is at } \sqrt{3}$$

The maximun area is

$$A(\sqrt{3}) = (18)\left(\sqrt{3}\right) - 2\left(\sqrt{3}\right)^3 = 12\sqrt{3} = 20.785 \qquad \textbf{(3pts)}$$

B) A cylindrical can is to be made to hold 16π cm^3 of laban. If r is the radius and h is the height of the can, then find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution

$$V = \pi r^2 h.$$

We want to minimize the area A of the cone under the constraint $\pi r^2 h = 16\pi$

$$\Rightarrow h = \frac{16}{r^2}, \qquad r > 0$$

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r\left(\frac{16}{r^2}\right) \\ &= 2\pi\left(r^2 + \frac{16}{r}\right) \qquad\qquad (\mathbf{3pts}) \end{aligned}$$

$$A' = 2\pi\left(2r - \frac{16}{r^2}\right)$$

$$A' = 0 \Leftrightarrow \left(2r - \frac{16}{r^2}\right) = 0 \Leftrightarrow r^3 = 8 \Leftrightarrow r = 2$$

$$A''(r) = 2\pi\left(\frac{32}{r^3} + 2\right)$$

$$A''(2) > 0 \Rightarrow \ A \text{ is minimized when } r = 2 \text{ and } h = \frac{16}{4} = 4. \qquad (\mathbf{3pts})$$

$(12pts)$**Problem 2**

Use definite integral to find the limit

$$\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\sqrt{1-\left(-1+\frac{2i}{n}\right)^2}$$

Solution

$$\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\sqrt{1-\left(-1+\frac{2i}{n}\right)^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\sqrt{1-\left(-1+\frac{2i}{n}\right)^2}\right)\frac{2}{n}$$

$$a=-1, \qquad \frac{b-a}{n} = \frac{2}{n} \Rightarrow b=1 \text{ and } f(x)=\sqrt{1-x^2}$$

$$\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\sqrt{1-\left(-1+\frac{2i}{n}\right)^2} = \int_{-1}^{1}\sqrt{1-x^2}dx. \qquad (\mathbf{6pts})$$

The inetgral $\int_{-1}^{1}\sqrt{1-x^2}dx$ is half of the area of the circle centered at (0, 0) with radius 1.

$$\int_{-1}^{1}\sqrt{1-x^2}dx = \frac{\pi}{2}$$

$$\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\sqrt{1-\left(-1+\frac{2i}{n}\right)^2} = \frac{\pi}{2} \quad (\mathbf{6pts})$$

$(12pts)$**Problem 3**

A) If $\int_1^7 f(x)dx = 7$ and $\int_1^3 2f(x)dx = 6$, find $\int_3^7 f(x)dx$.

Solution

$$\int_1^7 f(x)dx = \int_1^3 f(x)dx + \int_3^7 f(x)dx$$

$\Leftrightarrow$

$$7 = 3 + \int_3^7 f(x)dx$$

$$\int_3^7 f(x)dx = 7 - 3 = 4. \quad (\mathbf{6pts})$$

B) If $G(x) = \displaystyle\int_1^{e^x} (\ln t)^2 \, dt$, find $G'(\ln x)$.

Solution

$$G'(x) = e^x \left(\ln e^x\right)^2 - 0 = x^2 e^x$$

$$G'(\ln x) = (\ln x)^2 e^{\ln x} = x(\ln x)^2 \qquad (\mathbf{6pts})$$

(12pts)**Problem 4**

Evaluate the following integrals

$$1.\quad \int \frac{\cos^3 x}{\sin x}dx \qquad\qquad 2.\quad \int_0^1 -2x^3\sqrt{1-x^2}dx$$

Solution

1.

$$\begin{aligned}\int \frac{\cos^3 x}{\sin x}dx &= \int \frac{\cos^2 x}{\sin x}\cos x dx\\ &= \int \left(\frac{1-\sin^2 x}{\sin x}\right)\cos x dx\end{aligned}$$

Put $u = \sin x$, $\quad du = \cos x dx$. The integral becomes

$$\begin{aligned}\int \left(\frac{1-\sin^2 x}{\sin x}\right)\cos x dx &= \int \left(\frac{1-u^2}{u}\right)du\\ &= \int \left(\frac{1}{u}-u\right)du\\ &= \ln|u| - \frac{1}{2}u^2 + C\\ &= \ln|\sin x| - \frac{1}{2}\sin^2 x + C \qquad \textbf{(6pts)}\end{aligned}$$

2.

$$\int_0^1 -2x^3\sqrt{1-x^2}dx = \int_0^1 x^2\sqrt{1-x^2}\,(-2x)\,dx$$

Put $u = 1-x^2$, $\quad du = -2xdx$ and $x^2 = 1-u$.

$$x = 0 \rightarrow u = 1 \text{ and } x = 1 \rightarrow u = 0$$

The integral becomes

$$\int_0^1 -2x^3\sqrt{1-x^2}dx = \int_1^0 (1-u)\sqrt{u}du = -\frac{4}{15} = -0.266\,67. \qquad \textbf{(6pts)}$$

(10pts)**Problem 5**

A) If f is continuous on $[0\ ,\ 3]$ and $\int_0^3 f(t)dt = 5$, find $\displaystyle\int_0^3 f(3-x)dx$.

Solution

Put $t = 3 - x, \quad dt = -dx \Rightarrow dx = -dt$. The integral becomes.

$$\int_0^3 f(3-x)dx = -\int_3^0 f(t)dt = \int_0^3 f(t)dt = 5 \qquad (\mathbf{5pts})$$

B) Find the average value of $f(x) = 2|x| + 1$ on the interval $[-2,\ 2]$.

Solution

$$\begin{aligned} f_{ave} &= \frac{1}{2-(-2)} \int_{-2}^{2} (2|x|+1)\, dx \\ &= \frac{1}{4} \int_{-2}^{2} (2|x|+1)\, dx \\ &= \left(\frac{1}{4}\right)(12) = 3 \qquad (\mathbf{5pts}) \end{aligned}$$

(10pts)**Problem 6**
A particle moves along a line so that its velocity at time t is $v(t) = t - t^2$. Find the distance traveled by the particle during the time period $0 \leq t \leq 2$.

Solution

$$
\begin{aligned}
D &= \int_0^2 \left| t - t^2 \right| dt \qquad\qquad (\mathbf{3pts}) \\
&= \int_0^2 \left| t\,(1-t) \right| dt \\
&= \int_0^1 \left(t - t^2 \right) dt - \int_1^2 \left(t - t^2 \right) dt \\
&= \frac{1}{6} - \left(\frac{-5}{6} \right) = 1 \qquad (\mathbf{7pts})
\end{aligned}
$$

(10pts)**Problem 7**
Find the absolute extrema of the function

$$F(x) = \int_1^x t^3\,(2t+1)\,dt \quad \text{on the interval} \quad [-1,\ 2].$$

Solution

$$F'(x) = x^3\,(2x+1)$$

The critical numbers are 0 and $\dfrac{-1}{2}$. (**4pts**)

$$F(0) = \int_1^0 t^3\,(2t+1)\,dt \;= -\frac{13}{20} = -0.65$$

$$F(\frac{-1}{2}) = \int_1^{\frac{-1}{2}} t^3\,(2t+1)\,dt = -\frac{207}{320} = -0.646\,88$$

$$F(-1) = \int_1^{-1} t^3\,(2t+1)\,dt = -\frac{4}{5} = -0.8$$

$$F(2) = \int_1^2 t^3\,(2t+1)\,dt = \frac{323}{20} = 16.\,15$$

Absolute max $= 16.15$ and Absolute min $= -0.8$ (**6pts**)

$(10pts)$**Problem 8**
Write the following complex numbers in the form $a + ib$ and find the modulus of each number. (Show your work)

$$z_1 = \left(\frac{1+i}{2-i}\right)^2 \qquad\qquad z_2 = \frac{2+5i}{1-i} + \frac{2-5i}{1+i}$$

Solution

$$z_1 = \left(\frac{1+i}{2-i}\right)^2 = -\frac{8}{25} + \frac{6}{25}i \quad \text{and} \quad |z_1| = \left|\left(\frac{1+i}{2-i}\right)^2\right| = \frac{1}{25}\sqrt{8^2+6^2} = \frac{2}{5} \qquad \textbf{(5pts)}$$

$$z_2 = \frac{2+5i}{1-i} + \frac{2-5i}{1+i} = -3, \qquad |z_2| = 3 \qquad \textbf{(5pts)}$$

$(12pts)$**Problem 9**
Solve the equation

$$2z^2 + (2+3i)\,z + 2i - 1 = 0$$

Solution

$$\Delta = b^2 - 4ac = (2+3i)^2 - 4\,(2)\,(2i-1) = 3 - 4i$$
$$\sqrt{3-4i} = 2 - i \qquad (\mathbf{6pts})$$

The solutions are

$$z = \frac{-\,(2+3i) \pm (2-i)}{4}$$
$$z = -i \;\text{ or }\; z = -1 - \frac{i}{2} \qquad (\mathbf{6pts})$$