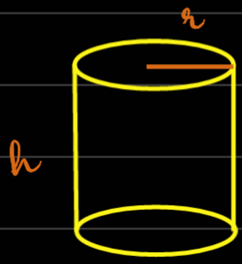
Example 3

A canning company wishes to design a can of a volume of 100 cm^3 using the least amount of material possible. Find the dimensions that the canning company should use.

$V = \pi r^2 h$ $A = 2\pi r h + 2\pi r^2$

$V = 100 \, cm^3$



$100 = \pi r^2 h$

$h = \frac{100}{\pi r^2}$

$A = 2\pi r h + 2\pi r^2$

$= \frac{2\cancel{\pi}\cancel{r} \times 100}{\cancel{\pi} r^{\cancel{2}}} + 2\pi r^2$

$= \frac{200}{r} + 2\pi r^2$

$= 200 r^{-1} + 2\pi r^2$

$A' = \frac{-200}{r^2} + 4\pi r$

$A' = 0$

$\frac{-200}{r^2} + 4\pi r = 0$

$4\pi r = \frac{200}{r^2}$

$4\pi r^3 = 200$

$\pi r^3 = 50$

$r^3 = 15.92$

$r = 2.515 \, cm$

$A' = 4\pi r - \frac{200}{r^2}$

$A'' = \left(4\pi + \frac{200}{r^3}\right) > 0$

$\therefore$ A will have a minimum at $r = 2.515 \, cm$

$h = \frac{100}{\pi r^2}$

$h = 5.03 \, cm$

Integration

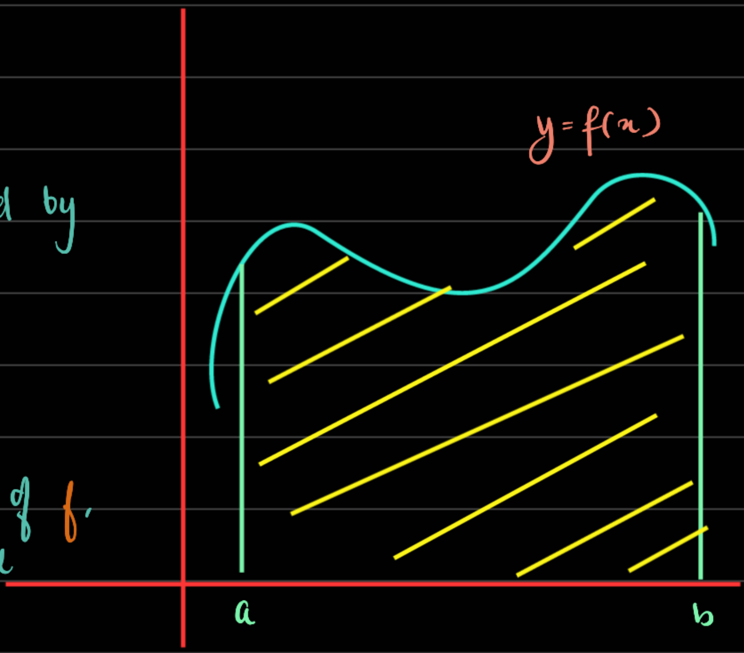
Definite Integral

Let f be continuous on $[a,b]$

The definite integral of f on $[a,b]$ denoted by

$$\int_b^a f(x)\,dx$$

is denoted to be the area under the graph of f, under the x axis and between the vertical lines $x=a$ and $x=b$



Example

Evaluate the following definite integral

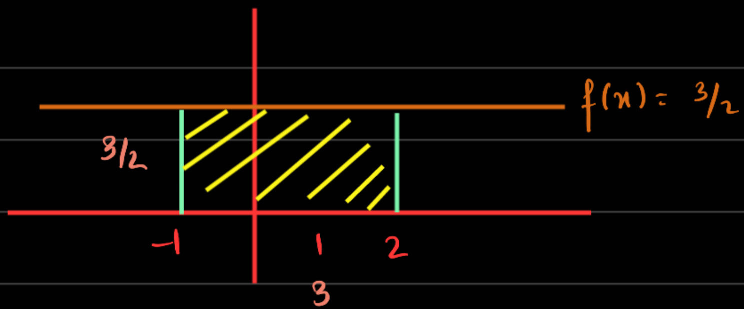
$$\int_{-1}^{2} \frac{3}{2}\,dx$$

$$\int_{-1}^{2} \frac{3}{2}\,dx$$

$$= \frac{3}{2}\left[x\right]_{-1}^{2}$$

$$= \frac{3}{2}\left(2-(-1)\right)$$

$$= \frac{9}{2}$$



$$\int_{-1}^{2} \frac{3}{2}\,dx = 3 \cdot \frac{3}{2} = \frac{9}{2}$$

$$\int_0^1 \frac{x}{2}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{4}\left(1^2 - 0^2\right)$$

$$= \frac{1}{4}$$

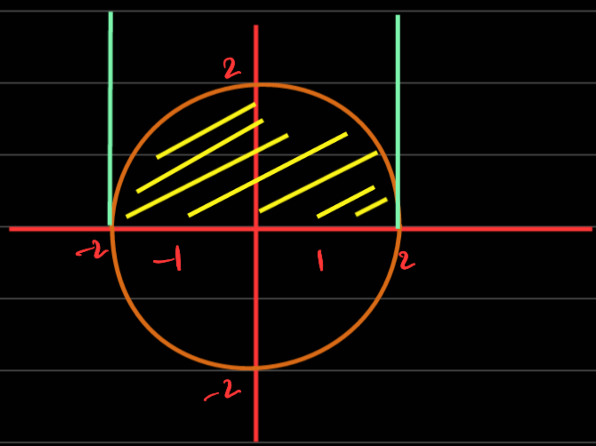
$$\int_0^1 \frac{x}{2}\,dx = \frac{1}{2}\times 1 \times \frac{1}{2} = \frac{1}{4}$$

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$y = \sqrt{4-x^2}$$

$$y^2 = 4 - x^2$$

$$x^2 + y^2 = 4$$



$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{\pi r^2}{2} = \frac{\pi \times 2^2}{2} = 2\pi$$

Sigma Notation Summation Formula

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

sigma notation

Example

Write out the terms of the following sums

$$\sum_{i=1}^{6} (2i+1)^2$$

$$= (2(1)+1)^2 + (2(2)+1)^2 + (2(3)+1)^2 + (2(4)+1)^2 + (2(5)+1)^2 + (2(6)+1)^2$$

$$= 9 + 25 + 49 + 81 + 121 + 169$$

$$= 454$$

$$\sum_{i=1}^{k}\left(\frac{i^2}{k}-3\right)$$

$$\left(\frac{1^2}{k}-3\right)+\left(\frac{2^2}{k}-3\right)+\cdots\cdots+\left(\frac{k^2}{k}-3\right)$$

Summation Formula

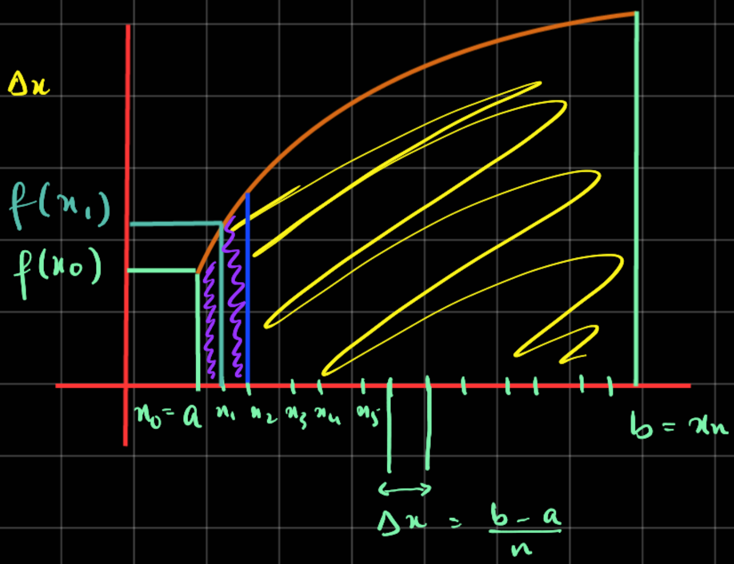
$$\sum_{i=1}^{n} c = nc \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\int_a^b f(x)\,dx \approx f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$

$$= \sum_{i=1}^{n-1} f(x_i)\Delta x$$

$$= \sum_{i=1}^{n-1} f(x_i)\frac{b-a}{n}$$



$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n}$$

More precisely we have the following

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left[a+(i-1)\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)$$

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\underbrace{\sum_{i=1}^{n} f\left[a+i\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)}_{\text{Riemann Sum}}}$$

Example

Use the limit of Riemann Sum to evaluate

$$\int_0^1 (x^2+2)\,dx$$

$f(x) = x^2+2 \qquad a=0 \qquad b=1$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left[a + i\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)$$

$$\int_0^1 (x^2+2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(0 + i\left(\frac{1-0}{n}\right)\right]\left(\frac{1-0}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^2 + 2\right]\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{i^2}{n^2} + 2\right]\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[\frac{i^2}{n^2} + 2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \frac{i^2}{n^2} + \sum_{i=1}^{n} 2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^2}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^{\cancel{2}}} \times \frac{\cancel{n}(n+1)(2n+1)}{6} + 2n\right]$$

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \left[\frac{(n+1)(2n+1)}{6n^2} + 2\right]$$

$$= \lim_{n\to\infty} \frac{2n^2}{6n^2} + 2$$

$$= \frac{1}{3} + 2 = \frac{7}{3}$$