(**10pts**)**Problem 1**
Write the following complex numbers in the form $re^{i\theta}$ where $r = |z|$ and $\theta = \arg z$.

1. $z = \dfrac{5 + 11i\sqrt{3}}{7 - 4i\sqrt{3}}$, 2. $z = \dfrac{\sqrt{2}}{1 - i}$ (Show your detailed work)

Solution
1.

$$\begin{aligned} z &= \frac{5 + 11i\sqrt{3}}{7 - 4i\sqrt{3}} = \frac{\left(5 + 11i\sqrt{3}\right)\left(7 + 4i\sqrt{3}\right)}{\left(7 - 4i\sqrt{3}\right)\left(7 + 4i\sqrt{3}\right)} \\ &= \frac{97i\sqrt{3} - 97}{97} = -1 + i\sqrt{3} \qquad (\mathbf{2pts}) \end{aligned}$$

$$r = \sqrt{1 + 3} = 2, \qquad \tan\theta = -\sqrt{3} \Rightarrow \theta = \frac{2\pi}{3}.$$

Thus,

$$z = \frac{5 + 11i\sqrt{3}}{7 - 4i\sqrt{3}} = 2e^{i\frac{2\pi}{3}} \qquad (\mathbf{3pts}).$$

2.

$$\begin{aligned} z &= \frac{\sqrt{2}}{1 - i} = \frac{\sqrt{2}\,(1 + i)}{(1 - i)\,(1 + i)} \\ &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \qquad (\mathbf{2pts}) \end{aligned}$$

$$r = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1, \qquad \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$z = \frac{\sqrt{2}}{1 - i} = e^{i\frac{\pi}{4}} \qquad (\mathbf{3pts})$$

(**10pts**)**Problem 2**
Solve the quadratic equation

$$z^2 + (3i - 4)z + 1 - 7i = 0. \qquad \text{(Show your detailed work)}$$

Solution

$$\begin{aligned} z &= \frac{-(3i-4) \pm \sqrt{(3i-4)^2 - (4)(1)(1-7i)}}{2} \\ &= \frac{-(3i-4) \pm \sqrt{3+4i}}{2} \qquad (\mathbf{4pts}) \end{aligned}$$

To find $\sqrt{3+4i}$, put

$$\sqrt{3+4i} = a + ib$$

$$a^2 - b^2 + 2iab = 3 + 4i$$

$$\begin{cases} a^2 - b^2 = 3 \\ ab = 2 \end{cases}$$

$$a = 2 \text{ and } b = 1 \ \text{ or } a = -2 \text{ and } b = -1.$$

Using the convention, we have

$$\sqrt{3+4i} = 2 + i. \qquad (\mathbf{4pts})$$

The solutions are

$$z = \frac{-(3i-4) \pm (2+i)}{2}$$

$$z = \frac{-(3i-4) + (2+i)}{2} = 3 - i \qquad (\mathbf{1pt})$$

or

$$z = \frac{-(3i-4) - (2+i)}{2} = 1 - 2i. \qquad (\mathbf{1pt})$$

(10pts)Problem 3

Consider the surface whose equation is given by

$$x^2 + y^2 + z^2 - 4x - 12y - 8z = m.$$

For what value (s) of m will the surface be a sphere. (Justify and show your work).

Solution

$$x^2 + y^2 + z^2 - 4x - 12y - 8z = m.$$

After gouping and completing the square, we have

$$(x-2)^2 - 4 + (y-6)^2 - 36 + (z-4)^2 - 16 = m.$$

$$(x-2)^2 + (y-6)^2 + (z-4)^2 = m + 56. \qquad \textbf{(6pts)}$$

This equation is a sphere if and only if $m + 56 > 0 \Leftrightarrow m > -56$. **(4pts)**

(10pts)Problem 4

Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be two vectors such that $\|\overrightarrow{a}\| = 3$ and $\left\|\overrightarrow{b}\right\| = 4$. If $\theta = \dfrac{2\pi}{3}$ is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$, find

1. $\left\|\overrightarrow{a} - \overrightarrow{b}\right\|$ 2. $\left(3\overrightarrow{a} - 2\overrightarrow{b}\right) \cdot \left(\overrightarrow{a} + 2\overrightarrow{b}\right)$ (Show your detailed work)

Solution

1.

$$\begin{aligned}
\left\|\overrightarrow{a} - \overrightarrow{b}\right\|^2 &= \left(\overrightarrow{a} - \overrightarrow{b}\right) \cdot \left(\overrightarrow{a} - \overrightarrow{b}\right) \\
&= \|\overrightarrow{a}\|^2 + \left\|\overrightarrow{b}\right\|^2 - 2\overrightarrow{a} \cdot \overrightarrow{b} \qquad (\textbf{2pts}) \\
&= \|\overrightarrow{a}\|^2 + \left\|\overrightarrow{b}\right\|^2 - 2\|\overrightarrow{a}\|\left\|\overrightarrow{b}\right\| \cos\frac{2\pi}{3} \\
&= 9 + 16 - 2(3)(4)\left(\frac{-1}{2}\right) = 37
\end{aligned}$$

$$\left\|\overrightarrow{a} - \overrightarrow{b}\right\| = \sqrt{37} \qquad (\textbf{3pts})$$

2.

$$\begin{aligned}
\left(3\overrightarrow{a} - 2\overrightarrow{b}\right) \cdot \left(\overrightarrow{a} + 2\overrightarrow{b}\right) &= 3\|\overrightarrow{a}\|^2 + 6\overrightarrow{a} \cdot \overrightarrow{b} - 2\overrightarrow{a} \cdot \overrightarrow{b} - 4\left\|\overrightarrow{b}\right\|^2 \\
&= 3\|\overrightarrow{a}\|^2 + 4\overrightarrow{a} \cdot \overrightarrow{b} - 4\left\|\overrightarrow{b}\right\|^2 \qquad (\textbf{2pts}) \\
&= 3\|\overrightarrow{a}\|^2 + 4\|\overrightarrow{a}\|\left\|\overrightarrow{b}\right\| \cos\frac{2\pi}{3} - 4\left\|\overrightarrow{b}\right\|^2 \\
&= 3(9) + 4(3)(4)\left(\frac{-1}{2}\right) - 4(4)^2 \\
&= -61 \qquad (\textbf{3pts})
\end{aligned}$$

(**10pts**)**Problem 5**

Find the parametric equations of the line passing through the point $A(2, 3, 5)$ and parallel to the line of intersection of the two planes $3x - y + z = 0$ and $x - y + z = 0$. (Show your detailed work)

Solution

Let

$$\overrightarrow{n_1} = \langle 3,\ -1,\ 1 \rangle \qquad \text{and} \quad \overrightarrow{n_2} = \langle 1,\ -1,\ 1 \rangle\,. \qquad (\mathbf{2pts})$$

A direction vector of the line is given by

$$\overrightarrow{u} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 3 & -1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \langle 0,\ -2,\ -2 \rangle \qquad (\mathbf{5pts})$$

The parametric equation of the line are given by

$$\begin{cases} x = 2 + 0t \\ y = 3 - 2t \\ z = 5 - 2t \end{cases} \qquad (\mathbf{3pts})$$

(10pt$s$)Problem 6
(a) Find the distance from the point $Q(3,1,0)$ to the line with parametric equations

$$\begin{cases} x = 2+3t \\ y = -1+2t \\ z = 5+t \end{cases}$$

(b) Find the distance from the point $Q(3,1,0)$ to the plane $x+2y-5z=1$. (Show your detailed work)

Solution
(a)

$$\overrightarrow{u} = \langle 3,\ 2,\ 1\rangle$$

is a direction vector of the line.

$$d = \frac{\left\|\overrightarrow{PQ}\times\overrightarrow{u}\right\|}{\|\overrightarrow{u}\|}$$

$$\overrightarrow{PQ} = \langle 1,\ 2,\ -5\rangle$$

$$\overrightarrow{PQ}\times\overrightarrow{u} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 2 & -5 \\ 3 & 2 & 1 \end{vmatrix} = \langle 12,\ -16,\ -4\rangle \qquad (\mathbf{2pts})$$

$$\begin{aligned} d &= \frac{\left\|\overrightarrow{PQ}\times\overrightarrow{u}\right\|}{\|\overrightarrow{u}\|} = \frac{\sqrt{(12)^2+(-16)^2+(-4)^2}}{\sqrt{9+4+1}} \\ &= \frac{4}{7}\sqrt{91} = 5.4511. \qquad (\mathbf{3pts}) \end{aligned}$$

(b) $Q(3,1,0)$. Take $P(1,0,0)$ a point on the plane. $\overrightarrow{n} = \langle 1,\ 2,\ -5\rangle$

$$\overrightarrow{PQ} = \langle 2,\ 1,\ 0\rangle$$

$$\begin{aligned} D &= \frac{\left|\overrightarrow{PQ}\cdot\overrightarrow{n}\right|}{\|\overrightarrow{n}\|} \qquad (\mathbf{2pts}) \\ &= \frac{|\langle 2,\ 1,\ 0\rangle\cdot\langle 1,\ 2,\ -5\rangle|}{\sqrt{1+4+25}} \\ &= \frac{4}{\sqrt{30}} = 0.73030. \qquad (\mathbf{3pts}) \end{aligned}$$

(**10pt**s)**Problem 7**
Use the Gauss elimination method to solve the linear system

$$\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 - x_2 - x_3 = 1 \\ x_1 + x_3 = 3 \end{cases}$$ (Show your detailed work including the elementary operations)

Solution
The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 2 & 5 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix} \qquad (\mathbf{2pts})$$

After reducing in row echelon form, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \qquad (\mathbf{5pts})$$

Next, we do the back substitution to obtain

$$\begin{cases} x_1 = 3 \\ x_2 = 2 \\ x_3 = 0 \end{cases} \qquad (\mathbf{3pts})$$

(10pt$s$)Problem 8
Consider the matrices

$$A = \begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 9 \\ 1 & 1 \end{pmatrix}$$

(a) Find $A^2 + 2AB + B^2$
(b) Find $(A + B)^2$.
(Show your detailed work)

Solution
(a)

$$\begin{aligned} A^2 + 2AB + B^2 &= \begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix} + 2\begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 9 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 9 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 9 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 82 & 172 \\ 20 & 44 \end{pmatrix} \qquad (\mathbf{5pts}) \end{aligned}$$

(b)

$$A + B = \begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 9 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 17 \\ 2 & 3 \end{pmatrix}$$

$$(A + B)^2 = \begin{pmatrix} 7 & 17 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 7 & 17 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 83 & 170 \\ 20 & 43 \end{pmatrix} \qquad (\mathbf{5pts})$$

$$(A + B)^2 \neq A^2 + 2AB + B^2 \qquad \text{for matrices.}$$

(**10pt**s)**Problem 9**
Find all the value (s) of x for which the matrix A does not have an inverse.

$$A = \begin{pmatrix} 1 & 3 & x \\ 4 & 5 & -1 \\ 2 & -1 & 5 \end{pmatrix}$$

Solution

$$\det \begin{pmatrix} 1 & 3 & x \\ 4 & 5 & -1 \\ 2 & -1 & 5 \end{pmatrix} = -14x - 42 \qquad (\textbf{5pts})$$

A does not have an inverse if and only if $\det A = 0$. Thus

$$-14x - 42 = 0 \Rightarrow x = -3. \qquad (\textbf{5pts})$$

(10pt$s$)Problem 10
Use elementary operations to find the determinant of the matrix in terms of x and y.

$$A = \begin{pmatrix} 1 & 0 & x & x^2 \\ 0 & 1 & y & y^2 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 9 \end{pmatrix}$$

Solution
We do the expansion along the first column

$$\begin{aligned} \det A &= (1)\,(-1)^{1+1} \det \begin{pmatrix} 1 & y & y^2 \\ 0 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} + (1)\,(-1)^{3+1} \det \begin{pmatrix} 0 & x & x^2 \\ 1 & y & y^2 \\ 1 & 3 & 9 \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & y & y^2 \\ 0 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} + \det \begin{pmatrix} 0 & x & x^2 \\ 1 & y & y^2 \\ 1 & 3 & 9 \end{pmatrix} \qquad (\mathbf{5pts}) \\ &= \det \begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix} + \det \begin{pmatrix} y & y^2 \\ 2 & 4 \end{pmatrix} - \det \begin{pmatrix} x & x^2 \\ 3 & 9 \end{pmatrix} + \det \begin{pmatrix} x & x^2 \\ y & y^2 \end{pmatrix} \\ &= 6 + 4y - 2y^2 - \left(9x - 3x^2\right) + xy^2 - x^2 y \\ &= -x^2 y + 3x^2 + xy^2 - 9x - 2y^2 + 4y + 6 \qquad (\mathbf{5pts}) \end{aligned}$$