(10pts) **Problem 1**
Evaluate the following limits

$$(a) \quad \lim_{x\to 0} \frac{\sin x - x}{x^2} \qquad\qquad (b) \quad \lim_{x\to -\infty} \frac{2x\,|x^3| + 4x^2 - 10}{5x^4 - x^3 + x^2 - 1}$$

Solution
(a)

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x - x}{x^2} &= \lim_{x\to 0} \frac{\cos x - 1}{2x} \\ &= \lim_{x\to 0} \frac{-\sin x}{2} = 0 \qquad (\mathbf{5pts}) \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x\to -\infty} \frac{2x\,|x^3| + 4x^2 - 10}{5x^4 - x^3 + x^2 - 1} &= \lim_{x\to -\infty} \frac{2x\,(-x^3) + 4x^2 - 10}{5x^4 - x^3 + x^2 - 1} \\ &= \lim_{x\to -\infty} \frac{-2\cancel{x^4}}{5\cancel{x^4}} = \frac{-2}{5} \qquad (\mathbf{5pts}) \end{aligned}$$

$(10pts)$ **Problem 2**
Find the value (s) of the constant m for which the function f continuous at $x = 9$

$$f(x) = \begin{cases} \dfrac{\sqrt{x}-3}{x-9} + m, & \text{if } x > 9 \\ & \\ mx + 1 & \text{if } x \leq 9 \end{cases}$$

Solution
The function f continuous at $x = 9$ if and only if

$$\lim_{x \to 9^-} f(x) = \lim_{x \to 9^+} f(x) = f(9)$$

$$\begin{aligned} \lim_{x \to 9^+} \left(\frac{\sqrt{x}-3}{x-9} \right) &= \lim_{x \to 9^+} \left(\frac{\sqrt{x}-3}{x-9} \right) \left(\frac{\sqrt{x}+3}{\sqrt{x}+3} \right) \\ &= \lim_{x \to 9^+} \frac{1}{\sqrt{x}+3} = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \lim_{x \to 9^+} f(x) &= \lim_{x \to 9^+} \left(\frac{\sqrt{x}-3}{x-9} + m \right) \\ &= \lim_{x \to 9^+} \left(\frac{\sqrt{x}-3}{x-9} \right) + m \\ &= \frac{1}{6} + m \qquad (\textbf{3pts}) \end{aligned}$$

$$\begin{aligned} \lim_{x \to 9^-} f(x) &= \lim_{x \to 9^-} mx + 1 \\ &= 9m + 1. \qquad (\textbf{3pts}) \end{aligned}$$

The condition of continuity gives

$$\frac{1}{6} + m = 9m + 1.$$

$$m = -\frac{5}{48} \qquad (\textbf{4pts})$$

(10pts)**Problem 3**

Find $\dfrac{dy}{dx}$ for

$$(a) \quad y = \ln\left[\frac{\sqrt{9x+4}}{(3x-1)^4}\right] \qquad\qquad (b) \quad xy^2 + y\tan x = 2x+1$$

Solution

(a)

$$\begin{aligned} y &= \ln\left[\frac{\sqrt{9x+4}}{(3x-1)^4}\right] \\ &= \ln(9x+4)^{1/2} - \ln(3x-1)^4 \\ &= \frac{1}{2}\ln(9x+4) - 4\ln(3x-1) \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}\cdot\frac{9}{9x+4} - 4\cdot\frac{3}{3x-1} \\ &= \frac{9}{2(9x+4)} - \frac{12}{3x-1} \qquad (\mathbf{5pts}) \\ &= -\frac{189x+105}{54x^2+6x-8} \end{aligned}$$

(b)

$$\begin{gathered} xy^2 + y\tan x = 2x+1 \\ y\left(\tan^2 x + 1\right) + (y)^2 + (\tan x)\,y' + 2xyy' = 2 \\ \frac{dy}{dx} = \frac{2 - y - y\tan^2 x - y^2}{\tan x + 2xy} \qquad (\mathbf{5pts}) \end{gathered}$$

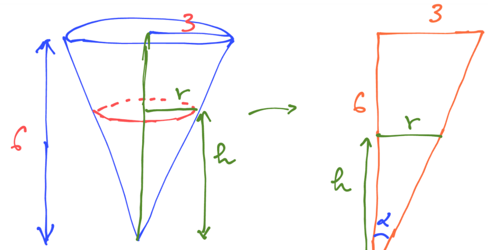
(10*pts*) **Problem 4**

A water tank has the shape of a circular cone with base radius 3 meter and height 6 meter. If water is being pumped into the tank at a rate of 3 m^3/ min, find the rate at which the water levcl is rising when thc water is 4 meter deep. $(V = \frac{1}{3}\pi r^2 h)$

Solution

We have

$$\frac{dV}{dt} = 3m^3/min.$$



$\tan\alpha = \frac{3}{6}$ (From the big Triangle)

$\tan\alpha = \frac{r}{h}$ (From the small Triangle)

$\Rightarrow \frac{r}{h} = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow \boxed{r = \frac{h}{2}}$

$$\frac{r}{h} = \frac{3}{6} \Rightarrow r = \frac{h}{2} \qquad \textbf{(3pts)}$$

We want to find $\frac{dh}{dt}$ when $h = 4$.

The related equation is

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h \\ &= \frac{1}{12}\pi h^3. \qquad \textbf{(4pts)} \end{aligned}$$

The related rate equation is

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$\Longleftrightarrow$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$\Leftrightarrow$

$$\begin{aligned} 3 &= 4\pi \frac{dh}{dt} \Rightarrow \\ \frac{dh}{dt} &= \frac{3}{4\pi} = 0.238\,73 \qquad \textbf{(4pts)} \end{aligned}$$

($10pts$) **Problem 5**
Find the absolute extrema of the function $f(x) = x^3 - 12x + 2$ on the interval $[0,\ 3]$.

Solution

$$
\begin{aligned}
f\ '(x) &= 3x^2 - 12 \\
&= 3\left(x^2 - 4\right).
\end{aligned}
$$

The critical numbers are

$$-2 \text{ and } 2 \qquad (\mathbf{2pts})$$

$$
\begin{aligned}
f(0) &= 2 \\
f(2) &= (2)^3 - 12(2) + 2 = -14 \\
f(3) &= (3)^3 - 12(3) + 2 = -7
\end{aligned}
$$

$$\text{Absolute Max} = 2 \qquad (\mathbf{4pts})$$

$$\text{Absolute Min} = -14 \qquad (\mathbf{4pts})$$

$(10pts)$**Problem 6**
Use definite integrals to evaluate

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(1+\frac{5i}{n}\right)^4$$

Solution

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(1+\frac{5i}{n}\right)^4$$

$$a=1 \qquad (\mathbf{2pts})$$

$$\frac{b-1}{n}=\frac{5}{n}\Rightarrow b=6 \qquad (\mathbf{3pts})$$

$$f(x)=\frac{3}{5}x^4 \qquad (\mathbf{2pts})$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(1+\frac{5i}{n}\right)^4=\int_1^6\frac{3}{5}x^4dx=933. \qquad (\mathbf{3pts})$$

$(10pts)$**Problem 7**
Find the critical numbers of the function

$$F(x) = \int_{11}^{x} \left(-t^2 - t + 6\right) dt$$

Solution

$$\begin{aligned} F'(x) &= -x^2 - x + 6 \\ &= -(x+3)(x-2) \qquad (\mathbf{6pts}) \end{aligned}$$

The critical numbers are -3 and 2. (**4pts**)

$(10pts)$**Problem 8**
Find the average value of the function $f(x) = 2x\sqrt{x^2+2}$ on the interval $\left[0, \sqrt{2}\right]$.
Solution

$$\begin{aligned} f_{ave} &= \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}} 2x\sqrt{x^2+2}dx \\ &= \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}} \sqrt{x^2+2}\,(2xdx) \end{aligned}$$

Put

$$u = x^2 + 2 \Rightarrow du = 2xdx$$

$$\begin{aligned} \text{When } x &= 0, \quad u = 2 \\ \text{When } x &= \sqrt{2}, \quad u = 4 \end{aligned}$$

$$\begin{aligned} f_{ave} &= \frac{1}{\sqrt{2}} \int_2^4 \sqrt{u}du \qquad (\mathbf{6pts}) \\ &= -\frac{1}{2}\sqrt{2}\left(\frac{4}{3}\sqrt{2} - \frac{16}{3}\right) = 2.4379 \qquad (\mathbf{4pts}) \end{aligned}$$