Cross Product

Let $\vec{a}$ and $\vec{b}$ be two vectors in the space
The cross product of $\vec{a}$ and $\vec{b}$ given by

is the vector with the following characteristics

Direction

$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

Orientation

The orientation of $\vec{a} \times \vec{b}$ is given by the right hand rule

Curl the fingers of your right hand in the direction of

Magnitude

$\|\vec{a} \times \vec{b}\|$ = Area of the parallelogram with adjacent side $\vec{a}$ and $\vec{b}$

Some geometric properties

1. If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$
2. $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = 0$

Algebraic definition of cross product

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \langle a_2 b_3 - b_2 a_3, -(a_1 b_3 - b_1 a_3), a_1 b_2 - b_1 a_2 \rangle$$

Example

Let $\vec{a} = \langle -2, -1, 1 \rangle$

$\vec{b} = \langle 0, -3, 2 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 1 \\ 0 & -3 & 2 \end{vmatrix} = \langle -2+3, -(-4-0), 6-0 \rangle$$

$$= \langle 1, 4, 6 \rangle$$

$$\langle 1, 4, 6 \rangle \cdot \langle -2, -1, 1 \rangle = -2 - 4 + 6 = 0$$

$$\langle 1, 4, 6 \rangle \cdot \langle 0, -3, 2 \rangle = 0 - 12 + 12 = 0$$

Algebraic Properties of Cross Product

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$\lambda (\vec{a} \times \vec{b}) = \lambda \vec{a} \times \vec{b} = \vec{a} \times (\lambda \vec{b})$

$\vec{a} \times \vec{a} = \vec{0}$

Example

Find a vector perpendicular to the plane containing $A(-1,2,0)$ $B(1,1,-1)$ and $C(2,0,-3)$

Find the area of the triangle ABC

$\vec{AB} = \langle 2, -2, -1 \rangle$

$\vec{AC} = \langle 3, -2, -3 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & -1 \\ 3 & -2 & -3 \end{vmatrix} = \langle 4, 3, 2 \rangle$$

$$\text{Area of } \Delta = \frac{1}{2} \| \vec{AB} \times \vec{AC} \|$$

$$= \frac{1}{2} \sqrt{16+9+4}$$

$$= \frac{\sqrt{29}}{2}$$

Equation of lines in the space

In the three dimensional space, the direction of a line is given by a vector called the direction vector of the line.

$\vec{u} = \langle a, b, c \rangle$

$\vec{PP_0}$ is parallel to $\vec{u}$

$\vec{PP_0} = t\vec{u}$

$\langle x-x_0, y-y_0, z-z_0 \rangle = t \langle a, b, c \rangle$

$x - x_0 = at$ $\quad y - y_0 = bt$ $\quad z - z_0 = ct$

$x = x_0 + at$ $\quad y = y_0 + bt$ $\quad z = z_0 + ct$ $\Rightarrow$ Parametric Equations

These equations are called the parametric equations of the line passing through the point $P(x_0, y_0, z_0)$ in the direction of the vector $\vec{u}$ with components a, b, c

$$\left.\begin{aligned} t &= \frac{x - x_0}{a} \\ t &= \frac{y - y_0}{b} \\ t &= \frac{z - z_0}{c} \end{aligned}\right\} \quad a, b, c \neq 0$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \Rightarrow$$ Symmetric equations of the line

Example

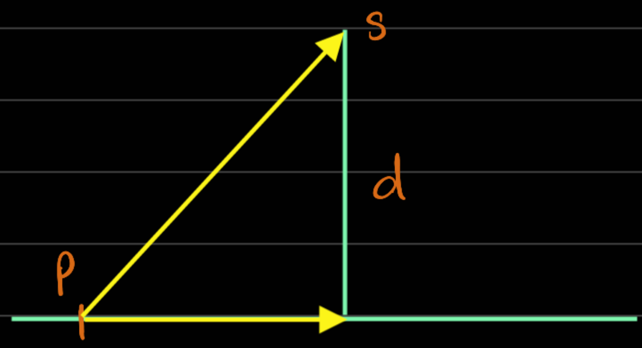
Find the parametric equation of the line containing the point $(2, -1, 3)$ and $B(5, 1, 0)$

$$\vec{u} = \vec{AB} = \langle 3, 2, -3 \rangle$$

The parametric equations of the line are

$$\begin{aligned} x &= x_0 + at & \qquad x &= 2 + 3t \\ y &= y_0 + bt & \qquad y &= -1 + 2t \\ z &= z_0 + ct & \qquad z &= 3 - 3t \end{aligned}$$

Distance from a point to a line



$$\sin\theta = \frac{d}{\|\vec{PS}\|}$$

$$\|\vec{PS} \times \vec{u}\| = \|\vec{PS}\| \, \|\vec{u}\| \sin\theta$$

$$\sin\theta = \frac{\|\vec{PS} \times \vec{u}\|}{\|\vec{PS}\| \, \|\vec{u}\|}$$

$$\frac{d}{\cancel{\|\vec{PS}\|}} = \frac{\|\vec{PS} \times \vec{u}\|}{\cancel{\|\vec{PS}\|} \, \|\vec{u}\|}$$

$$d = \frac{\|\vec{PS} \times \vec{u}\|}{\|\vec{u}\|}$$

Example

Find the distance from $S(-1,2,3)$ to the line with parametric equations

$$\begin{cases} x = 1 - t \\ y = 2 + 3t \\ z = 4t \end{cases}$$

$P = (1,2,0)$

$\vec{u} = \langle -1, 3, 4 \rangle$

$$\vec{PS} = \langle -2, 0, 3 \rangle$$

$$\vec{PS} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & 3 \\ -1 & 3 & 4 \end{vmatrix} = \langle -9, 5, -6 \rangle$$

$$\|\vec{PS} \times \vec{u}\| = \sqrt{81 + 25 + 36} = \sqrt{142}$$

$$\|\vec{u}\| = \sqrt{1+9+16} = \sqrt{26}$$

$$d = \sqrt{\frac{142}{26}} = \sqrt{\frac{71}{13}}$$