Part 1 MCQ (30%)

Directions: Circle the letter that corresponds to the correct answer. There is only one correct answer for each question. You do not need to show your work.

$(5pts)$**Problem 1**

$$\lim_{x\to 2}\frac{x^2-4}{x^2+4} \quad =$$

$(A)\ 1 \quad (B)\ \ 0 \quad (C)\ \ \frac{-1}{2} \quad (D)\ -1 \quad (E)\ \infty$

Solution

$$\lim_{x\to 2}\frac{x^2-4}{x^2+4} = \frac{0}{8} = 0 \qquad \boxed{\text{Answer is B}}$$

$(5pts)$**Problem 2**

$$\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16} \quad =$$

$(A)\ 0 \quad (B)\ \ \frac{16}{3} \quad (C)\ \ -\frac{1}{8} \quad (D)\ \frac{-1}{16} \quad (E)\ 1$

Solution

$$\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16} = -\frac{1}{8} \qquad \boxed{\text{Answer is C}}$$

$(5pts)$**Problem 3**

$$\lim_{x\to 0}\frac{\sin x}{x^2+3x} \quad =$$

$(A)\ 1 \quad (B)\ \ \frac{1}{3} \quad (C)\ \ 3 \quad (D)\ \infty \quad (E)\ \frac{1}{4}$

Solution

$$\lim_{x\to 0}\frac{\sin x}{x^2+3x} = \frac{1}{3} \qquad \boxed{\text{Answer is B}}$$

$(5pts)$**Problem 4**

If $\begin{cases} f(x) = \dfrac{x^2 - x}{2x} \text{ for } x \neq 0, \\ f(0) = k, \end{cases}$

and if f is continuous at $x = 0$, then $k =$

(A) -1 **(B)** $-\frac{1}{2}$ **(C)** 0 **(D)** $\frac{1}{2}$ **(E)** 1

Solution

$$\lim_{x \to 0} f(x) = f(0) = k$$

$$\lim_{x \to 0} \frac{x^2 - x}{2x} = k$$

$$k = -\frac{1}{2} \qquad \boxed{\text{Answer is B}}$$

(5*pts*)**Problem 5**
If

$$y = 2\sqrt{x} - \frac{1}{2\sqrt{x}}$$

The $\dfrac{dy}{dx}$ is equal to

(A) $x + \dfrac{1}{x\sqrt{x}}$ **(B)** $x^{-1/2} + x^{-3/2}$ **(C)** $\dfrac{4x-1}{4x\sqrt{x}}$

(D) $\dfrac{1}{\sqrt{x}} + \dfrac{1}{4x\sqrt{x}}$ **(E)** $\dfrac{4}{\sqrt{x}} + \dfrac{1}{x\sqrt{x}}$

Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} + \frac{1}{4x^{\frac{3}{2}}} = \frac{1}{\sqrt{x}} + \frac{1}{4x\sqrt{x}}$$

$\boxed{\text{Answer is D}}$

(5*pts*)**Problem 6**
If

$$y = \frac{2-x}{3x+1}$$

The $\dfrac{dy}{dx}$ is equal to

(A) $-\dfrac{7}{(3x+1)^2}$ **(B)** $\dfrac{6x-5}{(3x+1)^2}$ **(C)** $-\dfrac{9}{(3x+1)^2}$

(D) $\dfrac{7}{(3x+1)^2}$ **(E)** $\dfrac{7-6x}{(3x+1)^2}$

Solution

$$\frac{dy}{dx} = -\frac{7}{(3x+1)^2}$$

$\boxed{\text{Answer is A}}$

Part 2 Written Questions (70%)

$(15pts)$**Problem 1**
Find the equation of te tangent line to the graph $f(x) = e^x \ln x$ at $x = 1$.

Solution

$$f\ '(x) = e^x \ln x + \frac{e^x}{x}. \qquad (6pts)$$

The slope of the tangent line is

$$f\ '(1) = e. \qquad (2pts)$$

$$f(1) = 0 \qquad (2pts)$$

The equation of the tangent line is

$$\begin{aligned} y &= e\,(x-1) + 0 \\ &= ex - e \qquad (5pts) \end{aligned}$$

$(15pts)$**Problem 2**
Find the slope of the curve

$$x^3 - xy + y^3 = 1$$

at the point $(1, 1)$.
Solution
Using implicit differentiation, we get

$$3x^2 - y - xy' + 3y^2y' = 0$$

$$-xy' + 3y^2y' = -3x^2 + y$$

$$y'\left(-x + 3y^2\right) = -3x^2 + y$$

$$y' = \frac{-3x^2 + y}{-x + 3y^2}. \qquad (10pts)$$

The slope at $(1, 1)$ is

$$m = \frac{-3 + 1}{-1 + 3} = -1 \qquad (5pts)$$

$(10pts)$**Problem 3**

Find $\dfrac{dy}{dx}$ if

$$\sin x - \cos y - 2 = 0$$

Solution

$$\cos x + y' \sin y = 0 \qquad (5pts)$$

$$y' = \frac{-\cos x}{\sin y} \qquad (5pts)$$

$(15pts)$**Problem 4**

The volume of a cylinder is given by the formula $V = \pi r^2 h$ where r is the base radius and h is the height.
(a) Find the rate of change of V with respect to h if r remains constant.
(b) Find the rate of change of V with respect to r if h remains constant.
(c) Find the rate of change of h with respect to r if V remains constant.

Solution

$$V = \pi r^2 h$$

(a)

$$\frac{dV}{dh} = \pi r^2 \qquad (5pts)$$

(b)

$$\frac{dV}{dr} = 2\pi r h \qquad (5pts)$$

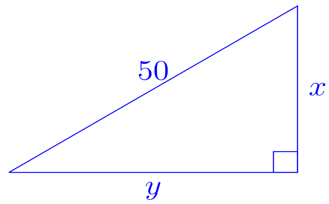
(c)

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi} r^{-2}$$

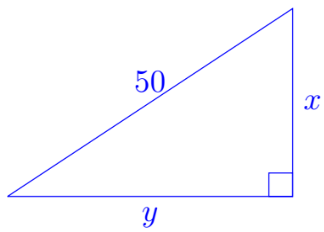
$$\begin{aligned} \frac{dh}{dr} &= -2\frac{V}{\pi} r^{-3} \\ &= \frac{-2V}{\pi r^3} \qquad (5pts) \end{aligned}$$

(15pts)**Problem 5**

A 50ft ladder is placed against a large building. The base of the ladder is resting on an oil spill, and it slips at the rate of 3 ft. per minute. Find the rate of change of the height of the top of the ladder above the ground at the instant when the base of the ladder is 30 ft. from the base of the building.



Solution



Organizing information:

- $\dfrac{dy}{dt} = 3$
- Goal: Find $\dfrac{dx}{dt}$ when $y = 30$.

We use Pythagorean Theorem again:

$$x^2 + 30^2 = 50^2 \Longrightarrow x = 40.$$

And differentiating (notice how the hypotenuse is constant):

$$2xx' + 2yy' = 0x' \qquad = \frac{-2yy'}{2x} = \frac{-yy'}{x}$$

Plugging in, $x' = -30 \cdot 3 \div 40 = -2.25$.

Note: x' is negative, that means the distance x is decreasing—the ladder is slipping down the building.

(15pts)