Problem Set: Complex Numbers

1. Evaluate the following, expressing your answer in Cartesian form ($a + bi$):

   (a) $(1 + 2i)(4 - 6i)^2$

   $$(1+2i)\underbrace{(4-6i)^2}_{4^2-48i+36i^2} = (1+2i)(-20-48i) = -20-48i-40i-96i^2 = \boxed{76-88i}$$

   (b) $(1 - 3i)^3$

   $$(1-3i)^3 = (1-3i)\underbrace{(1-3i)^2}_{1-6i+9i^2} = (1-3i)(-8-6i) = -8-6i+24i+18i^2 = \boxed{-26+18i}$$

   (c) $i(1 + 7i) - 3i(4 + 2i)$

   $$i + 7i^2 - 12i - 6i^2 = i - 7 - 12i + 6 = \boxed{-1 - 11i}$$

2. Solve the following using the quadratic formula, and check your answers:

   (a) $z^2 + 2z + 2 = 0$

   $$z = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = \boxed{-1 \pm i}$$

   (b) $z^2 - z + 1 = 0$

   $$z = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2} = \boxed{\frac{1}{2} \pm i\frac{\sqrt{3}}{2}}$$

3. Evaluate the following, expressing your answer in Cartesian form ($a + bi$):

   (a) $\dfrac{i}{1+i}$

   $$\frac{i}{1+i} \cdot \frac{1-i}{1-i} = \frac{i+1}{1^2+1^2} = \boxed{\frac{1}{2} + \frac{1}{2}i}$$

   (b) $\dfrac{2}{(1-i)(3+i)}$

   $$\frac{2}{3+i-3i+1} = \frac{2}{4-2i} = \frac{2}{4-2i} \cdot \frac{4+2i}{4+2i} = \frac{8+4i}{4^2+2^2} = \boxed{\frac{2}{5} + \frac{1}{5}i}$$

(c) $\dfrac{1-2i}{3+4i} - \dfrac{2+i}{5i}$

$$\frac{1-2i}{3+4i}\cdot\frac{3-4i}{3-4i} - \frac{2+i}{5i}\cdot\frac{-i}{-i} = \frac{-5-10i}{3^2+4^2} - \frac{1-2i}{5} = \left(-\frac{1}{5}-\frac{2}{5}i\right) - \left(\frac{1}{5}-\frac{2}{5}i\right) = \boxed{-\frac{2}{5}}$$

(d) $(1/i)^{2509}$

$$(1/i)^{2509} = \frac{1}{i^{2509}} = \frac{1}{i\cdot i^{2508}} = \frac{1}{i\cdot(i^4)^{627}} = \frac{1}{i\cdot 1^{627}} = \frac{1}{i} = \boxed{-i}$$

4. Solve the following systems of linear equations:

(a) $\begin{cases} ix_1 - ix_2 = -2 \\ 2x_1 + x_2 = i \end{cases}$

You could use Gaussian elimination. Or just use a matrix inverse:

$$\begin{bmatrix} i & -i \\ 2 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} -2 \\ i \end{bmatrix} \implies \mathbf{x} = \begin{bmatrix} i & -i \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -2 \\ i \end{bmatrix} = \frac{1}{3i}\begin{bmatrix} 1 & i \\ -2 & i \end{bmatrix}\begin{bmatrix} -2 \\ i \end{bmatrix} = -\frac{i}{3}\begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

$$\implies \boxed{x_1 = i,\ x_2 = -i}$$

(b) $\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 2i \end{cases}$

You could use a matrix inverse as above. Or use Gaussian elimination:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 2i \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 2i-2 \end{bmatrix}$$

$$\implies \begin{cases} x_2 = \frac{2i-2}{-2} = 1-i \\ x_1 = 2 - x_1 = 2-(1-i) = 1+i \end{cases}$$

5. Evaluate the following by first converting to polar form ($Re^{i\theta}$). Express your answer in Cartesian form ($a+bi$):

(a) $(1+i)^{12}$

$$(1+i)^{12} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{12} = (\sqrt{2})^{12}e^{i3\pi} = 2^6\cdot(-1) = \boxed{-64}$$

(b) $(i)^{1/3}$

$$i^{1/3} = \left(e^{i\frac{\pi}{2}}\right)^{1/3} = e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{2} + \frac{1}{2}i}$$

6. Find every complex root of the following. Express your answer in Cartesian form $(a+bi)$:

(a) $z^3 = i$

$$z^3 = e^{i(\frac{\pi}{2}+n2\pi)} \implies z = e^{i(\frac{\pi}{2}+n2\pi)/3} = e^{i(\frac{\pi}{6}+n\frac{2\pi}{3})}$$

$$n=0: \quad z = e^{i\frac{\pi}{6}} = \cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6} = \boxed{\frac{\sqrt{3}}{2} + \frac{1}{2}i}$$

$$n=1: \quad z = e^{i\frac{5\pi}{6}} = \cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6} = \boxed{-\frac{\sqrt{3}}{2} + \frac{1}{2}i}$$

$$n=2: \quad z = e^{i\frac{3\pi}{2}} = \cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2} = \boxed{-i}$$

(b) $z^3 = -27$

$$z^3 = 27e^{i(\pi+n2\pi)} \implies z = 27^{1/3}e^{i(\pi+n2\pi)/3} = 3e^{i(\frac{\pi}{3}+n\frac{2\pi}{3})}$$

$$n=0: \quad z = 3e^{i\frac{\pi}{3}} = 3(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}) = \boxed{\frac{3}{2} + i\frac{3\sqrt{3}}{2}}$$

$$n=1: \quad z = 3e^{i\pi} = \boxed{-3}$$

$$n=2: \quad z = 3e^{i\frac{5\pi}{3}} = 3(\cos\tfrac{5\pi}{3} + i\sin\tfrac{5\pi}{3}) = \boxed{\frac{3}{2} - i\frac{3\sqrt{3}}{2}}$$