1.4 Continuity and One-Sided Limits

- Determine continuity at a point and continuity on an open interval.
- Determine one-sided limits and continuity on a closed interval.
- Use properties of continuity.
- Understand and use the Intermediate Value Theorem.

EXPLORATION

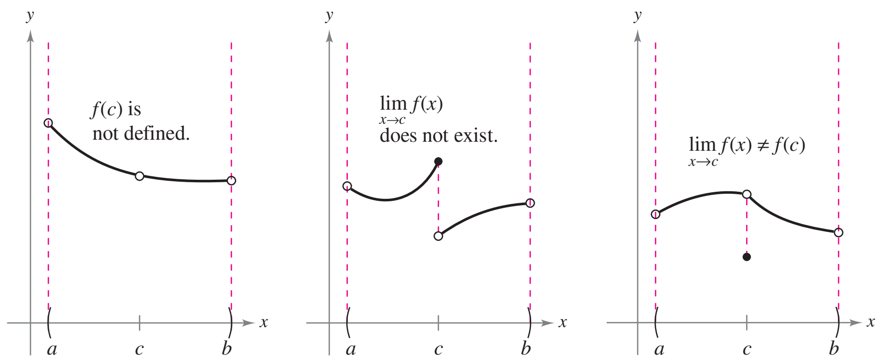
Informally, you might say that a function is *continuous* on an open interval if its graph can be drawn with a pencil without lifting the pencil from the paper. Use a graphing utility to graph each function on the given interval. From the graphs, which functions would you say are continuous on the interval? Do you think you can trust the results you obtained graphically? Explain your reasoning.

	Function	*Interval*
a.	$y = x^2 + 1$	$(-3, 3)$
b.	$y = \dfrac{1}{x - 2}$	$(-3, 3)$
c.	$y = \dfrac{\sin x}{x}$	$(-\pi, \pi)$
d.	$y = \dfrac{x^2 - 4}{x + 2}$	$(-3, 3)$
e.	$y = \begin{cases} 2x - 4, & x \le 0 \\ x + 1, & x > 0 \end{cases}$	$(-3, 3)$

FOR FURTHER INFORMATION For more information on the concept of continuity, see the article "Leibniz and the Spell of the Continuous" by Hardy Grant in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Continuity at a Point and on an Open Interval

In mathematics, the term *continuous* has much the same meaning as it has in everyday usage. Informally, to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of f at c. That is, its graph is unbroken at c and there are no holes, jumps, or gaps. Figure 1.25 identifies three values of x at which the graph of f is *not* continuous. At all other points in the interval (a, b), the graph of f is uninterrupted and **continuous.**



Three conditions exist for which the graph of f is not continuous at $x = c$.
Figure 1.25

In Figure 1.25, it appears that continuity at $x = c$ can be destroyed by any one of the following conditions.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

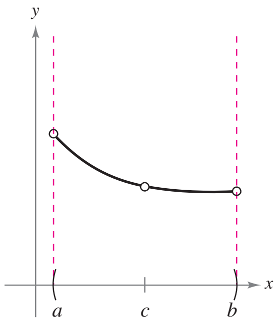
If *none* of the three conditions above is true, the function f is called **continuous at c,** as indicated in the following important definition.

DEFINITION OF CONTINUITY

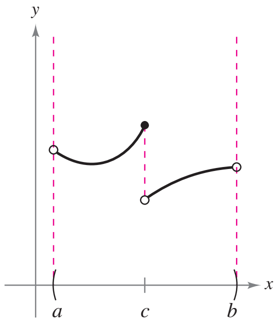
Continuity at a Point: A function f is **continuous at c** if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$

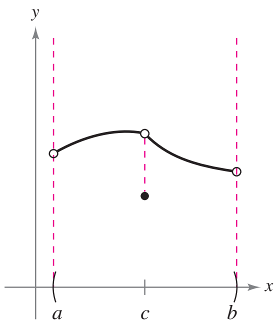
Continuity on an Open Interval: A function is **continuous on an open interval (a, b)** if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**



(a) Removable discontinuity



(b) Nonremovable discontinuity



(c) Removable discontinuity

Figure 1.26

Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a **discontinuity** at c. Discontinuities fall into two categories: **removable** and **nonremovable.** A discontinuity at c is called removable if f can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the functions shown in Figures 1.26(a) and (c) have removable discontinuities at c and the function shown in Figure 1.26(b) has a nonremovable discontinuity at c.

EXAMPLE 1 Continuity of a Function

Discuss the continuity of each function.

a. $f(x) = \dfrac{1}{x}$ **b.** $g(x) = \dfrac{x^2 - 1}{x - 1}$ **c.** $h(x) = \begin{cases} x + 1, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$ **d.** $y = \sin x$

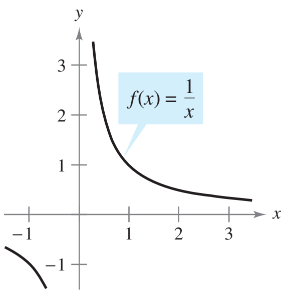
Solution

a. The domain of f is all nonzero real numbers. From Theorem 1.3, you can conclude that f is continuous at every x-value in its domain. At $x = 0$, f has a nonremovable discontinuity, as shown in Figure 1.27(a). In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

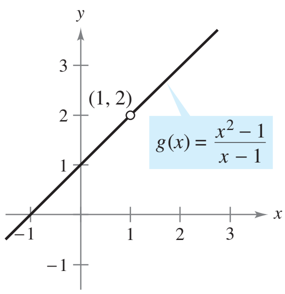
b. The domain of g is all real numbers except $x = 1$. From Theorem 1.3, you can conclude that g is continuous at every x-value in its domain. At $x = 1$, the function has a removable discontinuity, as shown in Figure 1.27(b). If $g(1)$ is defined as 2, the "newly defined" function is continuous for all real numbers.

c. The domain of h is all real numbers. The function h is continuous on $(-\infty, 0)$ and $(0, \infty)$, and, because $\lim_{x \to 0} h(x) = 1$, h is continuous on the entire real line, as shown in Figure 1.27(c).

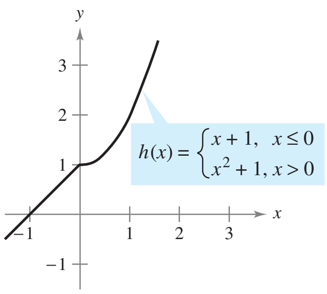
d. The domain of y is all real numbers. From Theorem 1.6, you can conclude that the function is continuous on its entire domain, $(-\infty, \infty)$, as shown in Figure 1.27(d).



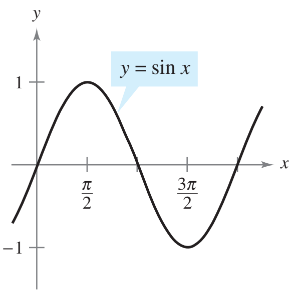
(a) Nonremovable discontinuity at $x = 0$



(b) Removable discontinuity at $x = 1$



(c) Continuous on entire real line



(d) Continuous on entire real line

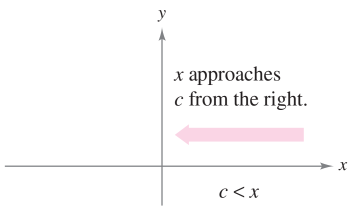
Figure 1.27

STUDY TIP Some people may refer to the function in Example 1(a) as "discontinuous." We have found that this terminology can be confusing. Rather than saying that the function is discontinuous, we prefer to say that it has a discontinuity at $x = 0$.

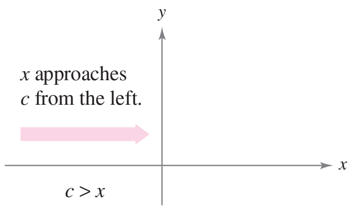
One-Sided Limits and Continuity on a Closed Interval

To understand continuity on a closed interval, you first need to look at a different type of limit called a **one-sided limit.** For example, the **limit from the right** (or right-hand limit) means that x approaches c from values greater than c [see Figure 1.28(a)]. This limit is denoted as

$$\lim_{x \to c^+} f(x) = L. \qquad \text{Limit from the right}$$



(a) Limit from right



(b) Limit from left
Figure 1.28

Similarly, the **limit from the left** (or left-hand limit) means that x approaches c from values less than c [see Figure 1.28(b)]. This limit is denoted as

$$\lim_{x \to c^-} f(x) = L. \qquad \text{Limit from the left}$$

One-sided limits are useful in taking limits of functions involving radicals. For instance, if n is an even integer,

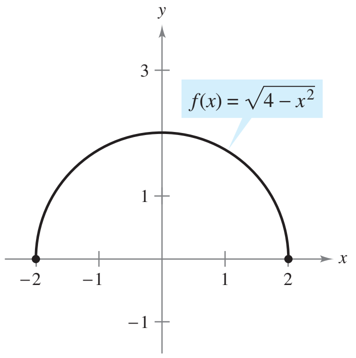
$$\lim_{x \to 0^+} \sqrt[n]{x} = 0.$$

EXAMPLE 2 A One-Sided Limit

Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches -2 from the right.

Solution As shown in Figure 1.29, the limit as x approaches -2 from the right is

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0.$$



The limit of $f(x)$ as x approaches -2 from the right is 0.
Figure 1.29

One-sided limits can be used to investigate the behavior of **step functions.** One common type of step function is the **greatest integer function** $[\![x]\!]$, defined by

$$[\![x]\!] = \text{greatest integer } n \text{ such that } n \leq x. \qquad \text{Greatest integer function}$$

For instance, $[\![2.5]\!] = 2$ and $[\![-2.5]\!] = -3$.

EXAMPLE 3 The Greatest Integer Function

Find the limit of the greatest integer function $f(x) = [\![x]\!]$ as x approaches 0 from the left and from the right.

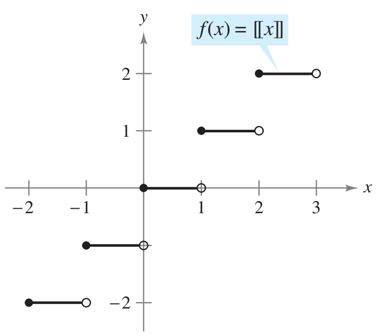
Solution As shown in Figure 1.30, the limit as x approaches 0 *from the left* is given by

$$\lim_{x \to 0^-} [\![x]\!] = -1$$

and the limit as x approaches 0 *from the right* is given by

$$\lim_{x \to 0^+} [\![x]\!] = 0.$$

The greatest integer function has a discontinuity at zero because the left and right limits at zero are different. By similar reasoning, you can see that the greatest integer function has a discontinuity at any integer n.



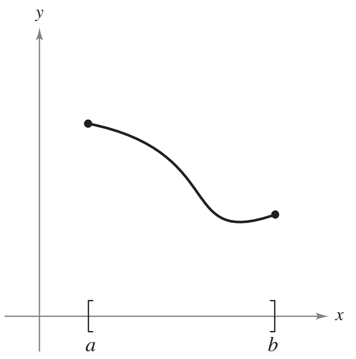
Greatest integer function
Figure 1.30

When the limit from the left is not equal to the limit from the right, the (two-sided) limit *does not exist*. The next theorem makes this more explicit. The proof of this theorem follows directly from the definition of a one-sided limit.

THEOREM 1.10 THE EXISTENCE OF A LIMIT

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$



Continuous function on a closed interval
Figure 1.31

The concept of a one-sided limit allows you to extend the definition of continuity to closed intervals. Basically, a function is continuous on a closed interval if it is continuous in the interior of the interval and exhibits one-sided continuity at the endpoints. This is stated formally as follows.

DEFINITION OF CONTINUITY ON A CLOSED INTERVAL

A function f is **continuous on the closed interval $[a, b]$** if it is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31).

Similar definitions can be made to cover continuity on intervals of the form $(a, b]$ and $[a, b)$ that are neither open nor closed, or on infinite intervals. For example, the function

$$f(x) = \sqrt{x}$$

is continuous on the infinite interval $[0, \infty)$, and the function

$$g(x) = \sqrt{2 - x}$$

is continuous on the infinite interval $(-\infty, 2]$.

EXAMPLE 4 Continuity on a Closed Interval

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

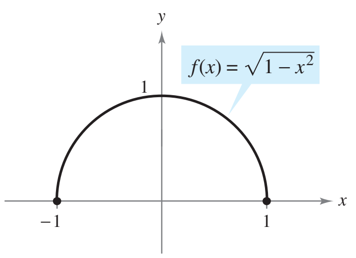
Solution The domain of f is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the continuity of f follows from Theorems 1.4 and 1.5. Moreover, because

$$\lim_{x\to -1^+} \sqrt{1 - x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x\to 1^-} \sqrt{1 - x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$

you can conclude that f is continuous on the closed interval $[-1, 1]$, as shown in Figure 1.32. ■



f is continuous on $[-1, 1]$.
Figure 1.32

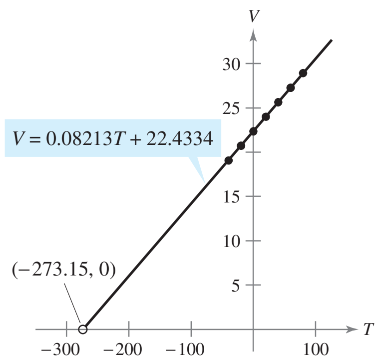
The next example shows how a one-sided limit can be used to determine the value of absolute zero on the Kelvin scale.

EXAMPLE 5 Charles's Law and Absolute Zero

On the Kelvin scale, *absolute zero* is the temperature 0 K. Although temperatures very close to 0 K have been produced in laboratories, absolute zero has never been attained. In fact, evidence suggests that absolute zero *cannot* be attained. How did scientists determine that 0 K is the "lower limit" of the temperature of matter? What is absolute zero on the Celsius scale?

Solution The determination of absolute zero stems from the work of the French physicist Jacques Charles (1746–1823). Charles discovered that the volume of gas at a constant pressure increases linearly with the temperature of the gas. The table illustrates this relationship between volume and temperature. To generate the values in the table, one mole of hydrogen is held at a constant pressure of one atmosphere. The volume V is approximated and is measured in liters, and the temperature T is measured in degrees Celsius.

T	-40	-20	0	20	40	60	80
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038



The volume of hydrogen gas depends on its temperature.
Figure 1.33

The points represented by the table are shown in Figure 1.33. Moreover, by using the points in the table, you can determine that T and V are related by the linear equation

$$V = 0.08213T + 22.4334 \qquad \text{or} \qquad T = \frac{V - 22.4334}{0.08213}.$$

By reasoning that the volume of the gas can approach 0 (but can never equal or go below 0), you can determine that the "least possible temperature" is given by

$$\begin{aligned}\lim_{V\to 0^+} T &= \lim_{V\to 0^+} \frac{V - 22.4334}{0.08213} \\ &= \frac{0 - 22.4334}{0.08213} \qquad \text{Use direct substitution.} \\ &\approx -273.15.\end{aligned}$$

So, absolute zero on the Kelvin scale (0 K) is approximately $-273.15°$ on the Celsius scale. ■

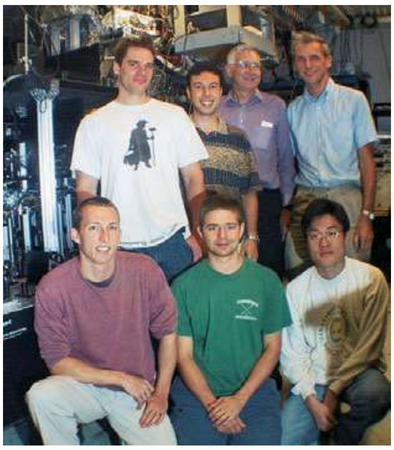


Photo courtesy of W. Ketterle, MIT

In 2003, researchers at the Massachusetts Institute of Technology used lasers and evaporation to produce a supercold gas in which atoms overlap. This gas is called a Bose-Einstein condensate. They measured a temperature of about 450 pK (picokelvin), or approximately -273.14999999955°C. (*Source:* Science *magazine, September 12, 2003*)

The following table shows the temperatures in Example 5 converted to the Fahrenheit scale. Try repeating the solution shown in Example 5 using these temperatures and volumes. Use the result to find the value of absolute zero on the Fahrenheit scale.

T	-40	-4	32	68	104	140	176
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

NOTE Charles's Law for gases (assuming constant pressure) can be stated as

$$V = RT \qquad \text{Charles's Law}$$

where V is volume, R is a constant, and T is temperature. In the statement of this law, what property must the temperature scale have? ■

AUGUSTIN-LOUIS CAUCHY (1789–1857)

The concept of a continuous function was first introduced by Augustin-Louis Cauchy in 1821. The definition given in his text *Cours d'Analyse* stated that indefinite small changes in y were the result of indefinite small changes in x. "…$f(x)$ will be called a *continuous* function if … the numerical values of the difference $f(x + \alpha) - f(x)$ decrease indefinitely with those of α …."

Properties of Continuity

In Section 1.3, you studied several properties of limits. Each of those properties yields a corresponding property pertaining to the continuity of a function. For instance, Theorem 1.11 follows directly from Theorem 1.2. (A proof of Theorem 1.11 is given in Appendix A.)

THEOREM 1.11 PROPERTIES OF CONTINUITY

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$
2. Rational: $r(x) = \dfrac{p(x)}{q(x)}, \quad q(x) \neq 0$
3. Radical: $f(x) = \sqrt[n]{x}$
4. Trigonometric: $\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$

By combining Theorem 1.11 with this summary, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

EXAMPLE 6 Applying Properties of Continuity

By Theorem 1.11, it follows that each of the functions below is continuous at every point in its domain.

$$f(x) = x + \sin x, \qquad f(x) = 3 \tan x, \qquad f(x) = \frac{x^2 + 1}{\cos x}$$ ■

The next theorem, which is a consequence of Theorem 1.5, allows you to determine the continuity of *composite* functions such as

$$f(x) = \sin 3x, \qquad f(x) = \sqrt{x^2 + 1}, \qquad f(x) = \tan \frac{1}{x}.$$

THEOREM 1.12 CONTINUITY OF A COMPOSITE FUNCTION

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

PROOF By the definition of continuity, $\lim_{x \to c} g(x) = g(c)$ and $\lim_{x \to g(c)} f(x) = f(g(c))$. Apply Theorem 1.5 with $L = g(c)$ to obtain $\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(g(c))$. So, $(f \circ g) = f(g(x))$ is continuous at c. ■

NOTE One consequence of Theorem 1.12 is that if f and g satisfy the given conditions, you can determine the limit of $f(g(x))$ as x approaches c to be

$$\lim_{x \to c} f(g(x)) = f(g(c)).$$

EXAMPLE 7 Testing for Continuity

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$ **b.** $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **c.** $h(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution

a. The tangent function $f(x) = \tan x$ is undefined at

$$x = \frac{\pi}{2} + n\pi, \qquad n \text{ is an integer.}$$

At all other points it is continuous. So, $f(x) = \tan x$ is continuous on the open intervals

$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

as shown in Figure 1.34(a).

b. Because $y = 1/x$ is continuous except at $x = 0$ and the sine function is continuous for all real values of x, it follows that $y = \sin(1/x)$ is continuous at all real values except $x = 0$. At $x = 0$, the limit of $g(x)$ does not exist (see Example 5, Section 1.2). So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$, as shown in Figure 1.34(b).

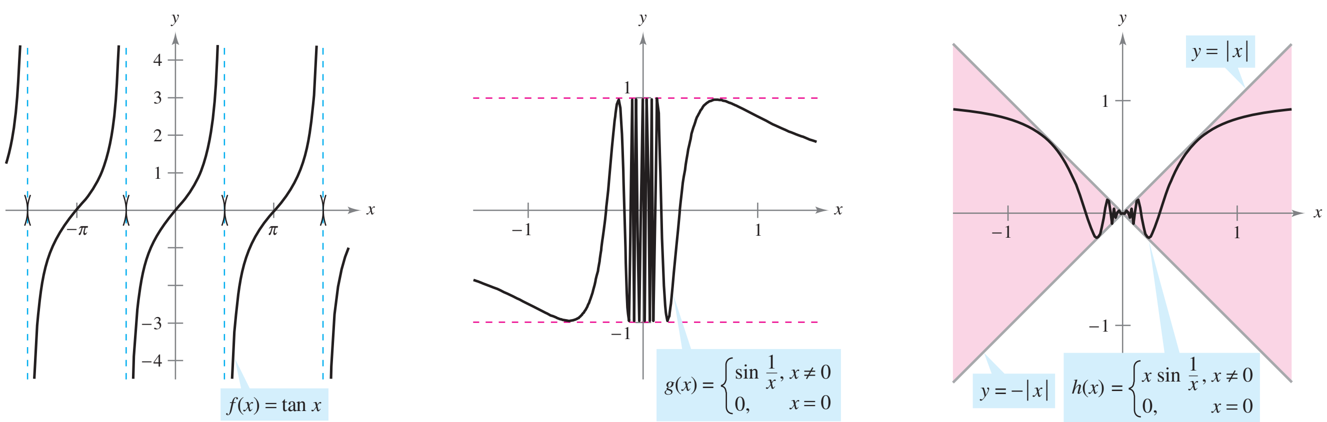
c. This function is similar to the function in part (b) except that the oscillations are damped by the factor x. Using the Squeeze Theorem, you obtain

$$-|x| \leq x \sin \frac{1}{x} \leq |x|, \quad x \neq 0$$

and you can conclude that

$$\lim_{x \to 0} h(x) = 0.$$

So, h is continuous on the entire real line, as shown in Figure 1.34(c).



(a) f is continuous on each open interval in its domain.

(b) g is continuous on $(-\infty, 0)$ and $(0, \infty)$.

(c) h is continuous on the entire real line

Figure 1.34

■

The Intermediate Value Theorem

Theorem 1.13 is an important theorem concerning the behavior of functions that are continuous on a closed interval.

THEOREM 1.13 INTERMEDIATE VALUE THEOREM

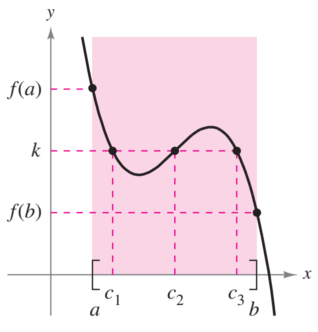
If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

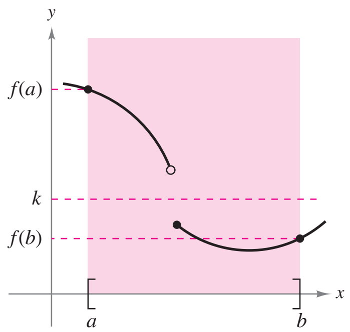
NOTE The Intermediate Value Theorem tells you that at least one number c exists, but it does not provide a method for finding c. Such theorems are called **existence theorems.** By referring to a text on advanced calculus, you will find that a proof of this theorem is based on a property of real numbers called *completeness*. The Intermediate Value Theorem states that for a continuous function f, if x takes on all values between a and b, $f(x)$ must take on all values between $f(a)$ and $f(b)$.

As a simple example of the application of this theorem, consider a person's height. Suppose that a girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 5 feet 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

The Intermediate Value Theorem guarantees the existence of *at least one* number c in the closed interval $[a, b]$. There may, of course, be more than one number c such that $f(c) = k$, as shown in Figure 1.35. A function that is not continuous does not necessarily exhibit the intermediate value property. For example, the graph of the function shown in Figure 1.36 jumps over the horizontal line given by $y = k$, and for this function there is no value of c in $[a, b]$ such that $f(c) = k$.

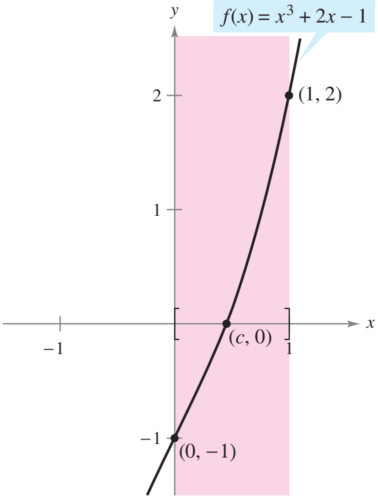


f is continuous on $[a, b]$.
[There exist three c's such that $f(c) = k$.]
Figure 1.35



f is not continuous on $[a, b]$.
[There are no c's such that $f(c) = k$.]
Figure 1.36

The Intermediate Value Theorem often can be used to locate the zeros of a function that is continuous on a closed interval. Specifically, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.



f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$.

Figure 1.37

EXAMPLE 8 An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0, 1]$.

Solution Note that f is continuous on the closed interval $[0, 1]$. Because

$$f(0) = 0^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in $[0, 1]$ such that

$$f(c) = 0 \qquad f \text{ has a zero in the closed interval } [0, 1].$$

as shown in Figure 1.37. ■

The **bisection method** for approximating the real zeros of a continuous function is similar to the method used in Example 8. If you know that a zero exists in the closed interval $[a, b]$, the zero must lie in the interval $[a, (a + b)/2]$ or $[(a + b)/2, b]$. From the sign of $f([a + b]/2)$, you can determine which interval contains the zero. By repeatedly bisecting the interval, you can "close in" on the zero of the function.

TECHNOLOGY You can also use the *zoom* feature of a graphing utility to approximate the real zeros of a continuous function. By repeatedly zooming in on the point where the graph crosses the x-axis, and adjusting the x-axis scale, you can approximate the zero of the function to any desired accuracy. The zero of $x^3 + 2x - 1$ is approximately 0.453, as shown in Figure 1.38.

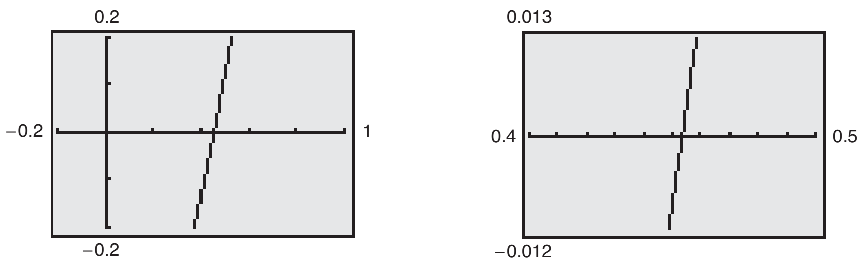


Figure 1.38 Zooming in on the zero of $f(x) = x^3 + 2x - 1$

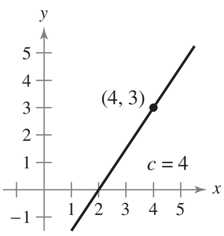
1.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

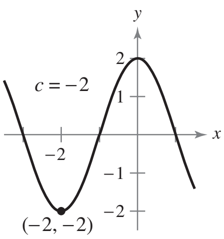
In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim\limits_{x\to c^+} f(x)$ **(b)** $\lim\limits_{x\to c^-} f(x)$ **(c)** $\lim\limits_{x\to c} f(x)$

1.



2.



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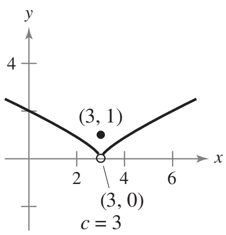

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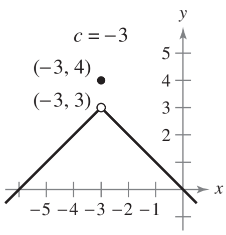

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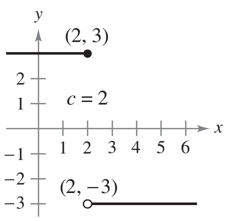

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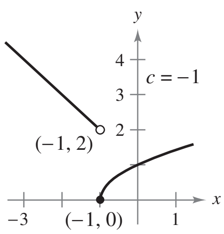


In Exercises 7–26, find the limit (if it exists). If it does not exist, explain why.

7. $\lim_{x\to 8^+} \dfrac{1}{x+8}$

8. $\lim_{x\to 5^-} \dfrac{-3}{x+5}$

9. $\lim_{x\to 5^+} \dfrac{x-5}{x^2-25}$

10. $\lim_{x\to 2^+} \dfrac{2-x}{x^2-4}$

11. $\lim_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$

12. $\lim_{x\to 9^-} \dfrac{\sqrt{x}-3}{x-9}$

13. $\lim_{x\to 0^-} \dfrac{|x|}{x}$

14. $\lim_{x\to 10^+} \dfrac{|x-10|}{x-10}$

15. $\lim_{\Delta x\to 0^-} \dfrac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x}$

16. $\lim_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2+x+\Delta x-(x^2+x)}{\Delta x}$

17. $\lim_{x\to 3^-} f(x)$, where $f(x)=\begin{cases}\dfrac{x+2}{2}, & x\le 3\\[4pt] \dfrac{12-2x}{3}, & x>3\end{cases}$

18. $\lim_{x\to 2} f(x)$, where $f(x)=\begin{cases}x^2-4x+6, & x<2\\ -x^2+4x-2, & x\ge 2\end{cases}$

19. $\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases}x^3+1, & x<1\\ x+1, & x\ge 1\end{cases}$

20. $\lim_{x\to 1^+} f(x)$, where $f(x)=\begin{cases}x, & x\le 1\\ 1-x, & x>1\end{cases}$

21. $\lim_{x\to\pi} \cot x$

22. $\lim_{x\to\pi/2} \sec x$

23. $\lim_{x\to 4^-} (5[\![x]\!]-7)$

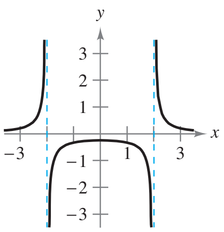
24. $\lim_{x\to 2^+} (2x-[\![x]\!])$

25. $\lim_{x\to 3} (2-[\![-x]\!])$

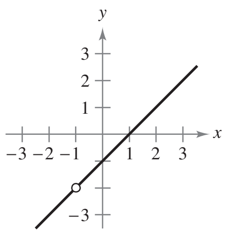
26. $\lim_{x\to 1} \left(1-\left[\!\!\left[-\dfrac{x}{2}\right]\!\!\right]\right)$

In Exercises 27–30, discuss the continuity of each function.

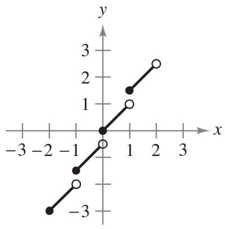
27. $f(x)=\dfrac{1}{x^2-4}$



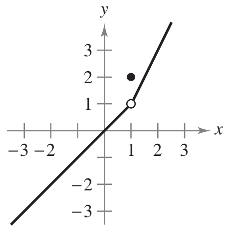
28. $f(x)=\dfrac{x^2-1}{x+1}$



29. $f(x)=\frac{1}{2}[\![x]\!]+x$



30. $f(x)=\begin{cases}x, & x<1\\ 2, & x=1\\ 2x-1, & x>1\end{cases}$



In Exercises 31–34, discuss the continuity of the function on the closed interval.

Function	*Interval*
31. $g(x)=\sqrt{49-x^2}$	$[-7, 7]$
32. $f(t)=3-\sqrt{9-t^2}$	$[-3, 3]$
33. $f(x)=\begin{cases}3-x, & x\le 0\\ 3+\frac{1}{2}x, & x>0\end{cases}$	$[-1, 4]$
34. $g(x)=\dfrac{1}{x^2-4}$	$[-1, 2]$

In Exercises 35–60, find the *x*-values (if any) at which *f* is not continuous. Which of the discontinuities are removable?

35. $f(x)=\dfrac{6}{x}$

36. $f(x)=\dfrac{3}{x-2}$

37. $f(x)=x^2-9$

38. $f(x)=x^2-2x+1$

39. $f(x)=\dfrac{1}{4-x^2}$

40. $f(x)=\dfrac{1}{x^2+1}$

41. $f(x)=3x-\cos x$

42. $f(x)=\cos\dfrac{\pi x}{2}$

43. $f(x)=\dfrac{x}{x^2-x}$

44. $f(x)=\dfrac{x}{x^2-1}$

45. $f(x)=\dfrac{x}{x^2+1}$

46. $f(x)=\dfrac{x-6}{x^2-36}$

47. $f(x)=\dfrac{x+2}{x^2-3x-10}$

48. $f(x)=\dfrac{x-1}{x^2+x-2}$

49. $f(x)=\dfrac{|x+7|}{x+7}$

50. $f(x)=\dfrac{|x-8|}{x-8}$

51. $f(x)=\begin{cases}x, & x\le 1\\ x^2, & x>1\end{cases}$

52. $f(x)=\begin{cases}-2x+3, & x<1\\ x^2, & x\ge 1\end{cases}$

53. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

55. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

56. $f(x) = \begin{cases} \csc \frac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

57. $f(x) = \csc 2x$

58. $f(x) = \tan \dfrac{\pi x}{2}$

59. $f(x) = [\![x - 8]\!]$

60. $f(x) = 5 - [\![x]\!]$

In Exercises 61 and 62, use a graphing utility to graph the function. From the graph, estimate

$$\lim_{x \to 0^+} f(x) \quad \text{and} \quad \lim_{x \to 0^-} f(x).$$

Is the function continuous on the entire real line? Explain.

61. $f(x) = \dfrac{|x^2 - 4|x}{x + 2}$

62. $f(x) = \dfrac{|x^2 + 4x|(x + 2)}{x + 4}$

In Exercises 63–68, find the constant *a*, or the constants *a* and *b*, such that the function is continuous on the entire real line.

63. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

64. $f(x) = \begin{cases} 3x^3, & x \le 1 \\ ax + 5, & x > 1 \end{cases}$

65. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

66. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

67. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

68. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

In Exercises 69–72, discuss the continuity of the composite function $h(x) = f(g(x))$.

69. $f(x) = x^2$
$g(x) = x - 1$

70. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

71. $f(x) = \dfrac{1}{x - 6}$
$g(x) = x^2 + 5$

72. $f(x) = \sin x$
$g(x) = x^2$

In Exercises 73–76, use a graphing utility to graph the function. Use the graph to determine any *x*-values at which the function is not continuous.

73. $f(x) = [\![x]\!] - x$

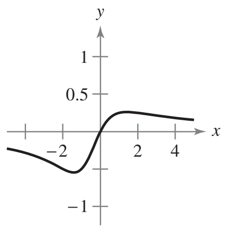
74. $h(x) = \dfrac{1}{x^2 - x - 2}$

75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

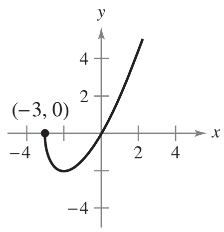
76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

In Exercises 77–80, describe the interval(s) on which the function is continuous.

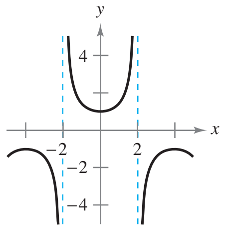
77. $f(x) = \dfrac{x}{x^2 + x + 2}$



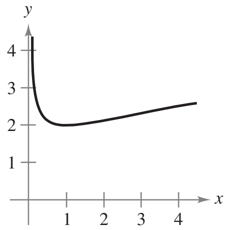
78. $f(x) = x\sqrt{x + 3}$



79. $f(x) = \sec \dfrac{\pi x}{4}$



80. $f(x) = \dfrac{x + 1}{\sqrt{x}}$



Writing **In Exercises 81 and 82, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function appear to be continuous on this interval? Is the function continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

81. $f(x) = \dfrac{\sin x}{x}$

82. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 83–86, explain why the function has a zero in the given interval.**

Function	*Interval*
83. $f(x) = \frac{1}{12}x^4 - x^3 + 4$	$[1, 2]$
84. $f(x) = x^3 + 5x - 3$	$[0, 1]$
85. $f(x) = x^2 - 2 - \cos x$	$[0, \pi]$
86. $f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$	$[1, 4]$

In Exercises 87–90, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval [0, 1]. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the *zero* or *root* feature of the graphing utility to approximate the zero accurate to four decimal places.

87. $f(x) = x^3 + x - 1$

88. $f(x) = x^3 + 5x - 3$

89. $g(t) = 2 \cos t - 3t$

90. $h(\theta) = 1 + \theta - 3 \tan \theta$

In Exercises 91–94, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of *c* guaranteed by the theorem.

91. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

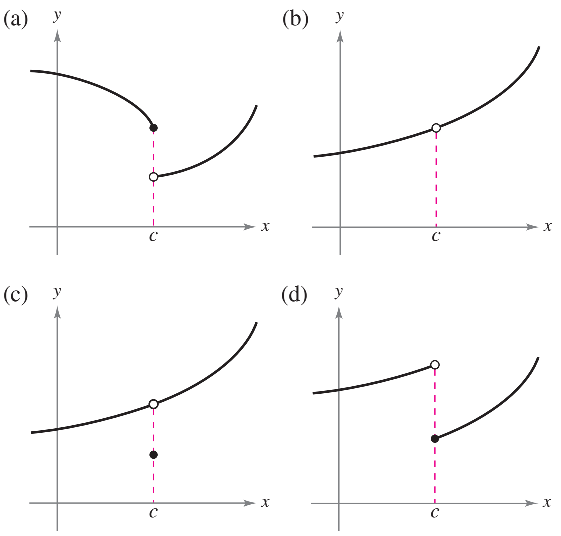
92. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

93. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

94. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

WRITING ABOUT CONCEPTS

95. State how continuity is destroyed at $x = c$ for each of the following graphs.



96. Sketch the graph of any function f such that

$$\lim_{x\to 3^+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 3^-} f(x) = 0.$$

Is the function continuous at $x = 3$? Explain.

97. If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

CAPSTONE

98. Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following descriptions.

(a) A function with a nonremovable discontinuity at $x = 4$

(b) A function with a removable discontinuity at $x = -4$

(c) A function that has both of the characteristics described in parts (a) and (b)

True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

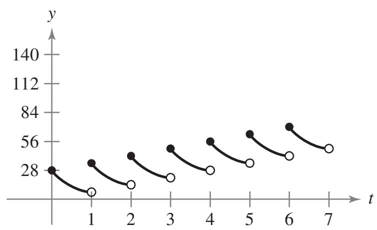
99. If $\lim_{x\to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

100. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

101. A rational function can have infinitely many x-values at which it is not continuous.

102. The function $f(x) = |x - 1|/(x - 1)$ is continuous on $(-\infty, \infty)$.

103. ***Swimming Pool*** Every day you dissolve 28 ounces of chlorine in a swimming pool. The graph shows the amount of chlorine $f(t)$ in the pool after t days.



Estimate and interpret $\lim_{t\to 4^-} f(t)$ and $\lim_{t\to 4^+} f(t)$.

104. ***Think About It*** Describe how the functions

$f(x) = 3 + [\![x]\!]$

and

$g(x) = 3 - [\![-x]\!]$

differ.

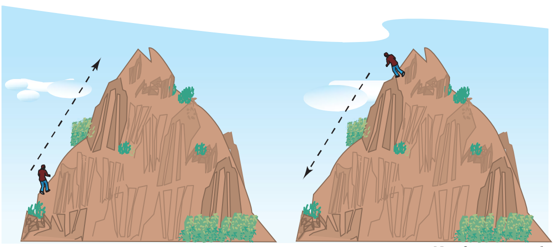
105. ***Telephone Charges*** A long distance phone service charges \$0.40 for the first 10 minutes and \$0.05 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of time t (in minutes). Sketch the graph of this function and discuss its continuity.

106. ***Inventory Management*** The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\left[\frac{t+2}{2}\right]\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

107. ***Déjà Vu*** At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Not drawn to scale

Saturday 8:00 A.M. Sunday 8:00 A.M.

108. ***Volume*** Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[5, 8]$, there is one with a volume of 1500 cubic centimeters.

109. Prove that if f is continuous and has no zeros on $[a, b]$, then either

$$f(x) > 0 \text{ for all } x \text{ in } [a, b] \text{ or } f(x) < 0 \text{ for all } x \text{ in } [a, b].$$

110. Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

111. Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

112. The **signum function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Sketch a graph of $\operatorname{sgn}(x)$ and find the following (if possible).

(a) $\lim_{x\to 0^-} \operatorname{sgn}(x)$ (b) $\lim_{x\to 0^+} \operatorname{sgn}(x)$ (c) $\lim_{x\to 0} \operatorname{sgn}(x)$

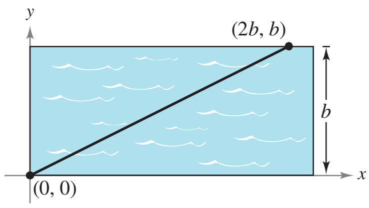
113. ***Modeling Data*** The table lists the speeds S (in feet per second) of a falling object at various times t (in seconds).

t	0	5	10	15	20	25	30
S	0	48.2	53.5	55.2	55.9	56.2	56.3

(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? If there is a limiting speed, identify a possible cause.

114. ***Creating Models*** A swimmer crosses a pool of width b by swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure.)



(a) Let f be a function defined as the y-coordinate of the point on the long side of the pool that is nearest the swimmer at any given time during the swimmer's crossing of the pool. Determine the function f and sketch its graph. Is f continuous? Explain.

(b) Let g be the minimum distance between the swimmer and the long sides of the pool. Determine the function g and sketch its graph. Is g continuous? Explain.

115. Find all values of c such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$$

116. Prove that for any real number y there exists x in $(-\pi/2, \pi/2)$ such that $\tan x = y$.

117. Let $f(x) = \left(\sqrt{x + c^2} - c\right)/x$, $c > 0$. What is the domain of f? How can you define f at $x = 0$ in order for f to be continuous there?

118. Prove that if $\lim_{\Delta x\to 0} f(c + \Delta x) = f(c)$, then f is continuous at c.

119. Discuss the continuity of the function $h(x) = x[\![x]\!]$.

120. (a) Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.

(b) Show that there exists c in $\left[0, \frac{\pi}{2}\right]$ such that $\cos x = x$. Use a graphing utility to approximate c to three decimal places.

PUTNAM EXAM CHALLENGE

121. Prove or disprove: if x and y are real numbers with $y \ge 0$ and $y(y + 1) \le (x + 1)^2$, then $y(y - 1) \le x^2$.

122. Determine all polynomials $P(x)$ such that

$$P(x^2 + 1) = (P(x))^2 + 1 \text{ and } P(0) = 0.$$

1.5 Infinite Limits

- **Determine infinite limits from the left and from the right.**
- **Find and sketch the vertical asymptotes of the graph of a function.**

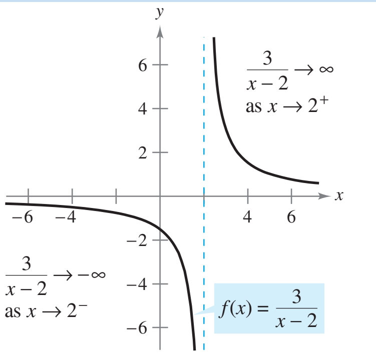
Infinite Limits

Let f be the function given by $3/(x - 2)$. From Figure 1.39 and the table, you can see that $f(x)$ *decreases without bound* as x approaches 2 from the left, and $f(x)$ *increases without bound* as x approaches 2 from the right. This behavior is denoted as

$$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty$$ $f(x)$ decreases without bound as x approaches 2 from the left.

and

$$\lim_{x\to 2^+} \frac{3}{x-2} = \infty$$ $f(x)$ increases without bound as x approaches 2 from the right.



$f(x)$ increases and decreases without bound as x approaches 2.
Figure 1.39

x approaches 2 from the left. | x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	-6	-30	-300	-3000	?	3000	300	30	6

$f(x)$ decreases without bound. | $f(x)$ increases without bound.

A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an **infinite limit.**

DEFINITION OF INFINITE LIMITS

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

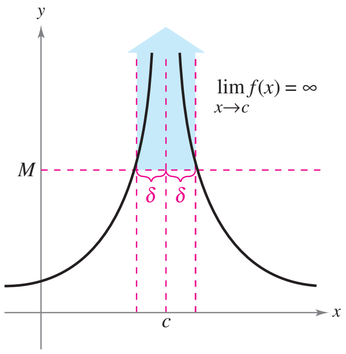
$$\lim_{x\to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.40). Similarly, the statement

$$\lim_{x\to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$.

To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.



Infinite limits
Figure 1.40

Be sure you see that the equal sign in the statement $\lim f(x) = \infty$ does not mean that the limit exists! On the contrary, it tells you how the limit *fails to exist* by denoting the unbounded behavior of $f(x)$ as x approaches c.

EXPLORATION

Use a graphing utility to graph each function. For each function, analytically find the single real number c that is not in the domain. Then graphically find the limit (if it exists) of $f(x)$ as x approaches c from the left and from the right.

a. $f(x) = \dfrac{3}{x - 4}$

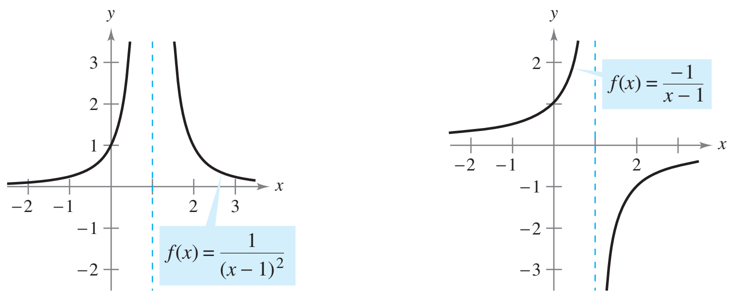
b. $f(x) = \dfrac{1}{2 - x}$

c. $f(x) = \dfrac{2}{(x - 3)^2}$

d. $f(x) = \dfrac{-3}{(x + 2)^2}$

EXAMPLE 1 Determining Infinite Limits from a Graph

Determine the limit of each function shown in Figure 1.41 as x approaches 1 from the left and from the right.



(a) (b)

Each graph has an asymptote at $x = 1$.

Figure 1.41

Solution

a. When x approaches 1 from the left or the right, $(x - 1)^2$ is a small positive number. Thus, the quotient $1/(x - 1)^2$ is a large positive number and $f(x)$ approaches infinity from each side of $x = 1$. So, you can conclude that

$$\lim_{x \to 1} \frac{1}{(x - 1)^2} = \infty. \qquad \text{Limit from each side is infinity.}$$

Figure 1.41(a) confirms this analysis.

b. When x approaches 1 from the left, $x - 1$ is a small negative number. Thus, the quotient $-1/(x - 1)$ is a large positive number and $f(x)$ approaches infinity from the left of $x = 1$. So, you can conclude that

$$\lim_{x \to 1^-} \frac{-1}{x - 1} = \infty. \qquad \text{Limit from the left side is infinity.}$$

When x approaches 1 from the right, $x - 1$ is a small positive number. Thus, the quotient $-1/(x - 1)$ is a large negative number and $f(x)$ approaches negative infinity from the right of $x = 1$. So, you can conclude that

$$\lim_{x \to 1^+} \frac{-1}{x - 1} = -\infty. \qquad \text{Limit from the right side is negative infinity.}$$

Figure 1.41(b) confirms this analysis. ■

Vertical Asymptotes

If it were possible to extend the graphs in Figure 1.41 toward positive and negative infinity, you would see that each graph becomes arbitrarily close to the vertical line $x = 1$. This line is a **vertical asymptote** of the graph of f. (You will study other types of asymptotes in Sections 3.5 and 3.6.)

NOTE If the graph of a function f has a vertical asymptote at $x = c$, then f is *not continuous* at c.

DEFINITION OF VERTICAL ASYMPTOTE

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

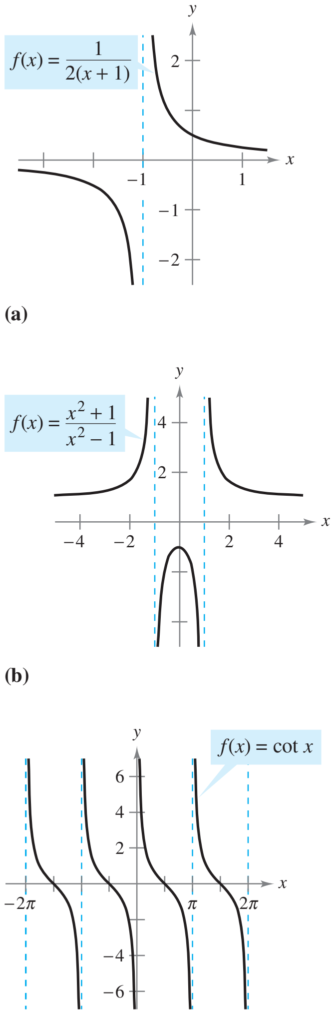
In Example 1, note that each of the functions is a *quotient* and that the vertical asymptote occurs at a number at which the denominator is 0 (and the numerator is not 0). The next theorem generalizes this observation. (A proof of this theorem is given in Appendix A.)

THEOREM 1.14 VERTICAL ASYMPTOTES

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.



(a) (b) (c)
Functions with vertical asymptotes
Figure 1.42

EXAMPLE 2 Finding Vertical Asymptotes

Determine all vertical asymptotes of the graph of each function.

a. $f(x) = \dfrac{1}{2(x+1)}$ **b.** $f(x) = \dfrac{x^2+1}{x^2-1}$ **c.** $f(x) = \cot x$

Solution

a. When $x = -1$, the denominator of

$$f(x) = \frac{1}{2(x+1)}$$

is 0 and the numerator is not 0. So, by Theorem 1.14, you can conclude that $x = -1$ is a vertical asymptote, as shown in Figure 1.42(a).

b. By factoring the denominator as

$$f(x) = \frac{x^2+1}{x^2-1} = \frac{x^2+1}{(x-1)(x+1)}$$

you can see that the denominator is 0 at $x = -1$ and $x = 1$. Moreover, because the numerator is not 0 at these two points, you can apply Theorem 1.14 to conclude that the graph of f has two vertical asymptotes, as shown in Figure 1.42(b).

c. By writing the cotangent function in the form

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

you can apply Theorem 1.14 to conclude that vertical asymptotes occur at all values of x such that $\sin x = 0$ and $\cos x \neq 0$, as shown in Figure 1.42(c). So, the graph of this function has infinitely many vertical asymptotes. These asymptotes occur at $x = n\pi$, where n is an integer. ■

Theorem 1.14 requires that the value of the numerator at $x = c$ be nonzero. If both the numerator and the denominator are 0 at $x = c$, you obtain the *indeterminate form* $0/0$, and you cannot determine the limit behavior at $x = c$ without further investigation, as illustrated in Example 3.

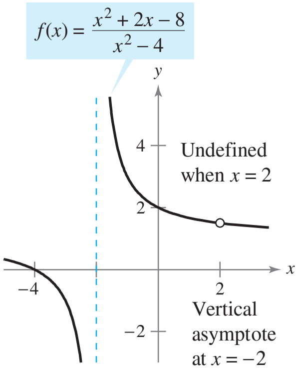
EXAMPLE 3 A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Solution Begin by simplifying the expression, as shown.

$$\begin{aligned} f(x) &= \frac{x^2 + 2x - 8}{x^2 - 4} \\ &= \frac{(x + 4)\cancel{(x - 2)}}{(x + 2)\cancel{(x - 2)}} \\ &= \frac{x + 4}{x + 2}, \quad x \neq 2 \end{aligned}$$



$f(x)$ increases and decreases without bound as x approaches -2.
Figure 1.43

At all x-values other than $x = 2$, the graph of f coincides with the graph of $g(x) = (x + 4)/(x + 2)$. So, you can apply Theorem 1.14 to g to conclude that there is a vertical asymptote at $x = -2$, as shown in Figure 1.43. From the graph, you can see that

$$\lim_{x\to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x\to -2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty.$$

Note that $x = 2$ is *not* a vertical asymptote.

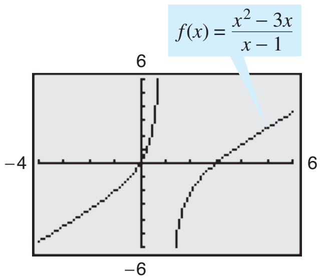
EXAMPLE 4 Determining Infinite Limits

Find each limit.

$$\lim_{x\to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{and} \quad \lim_{x\to 1^+} \frac{x^2 - 3x}{x - 1}$$

Solution Because the denominator is 0 when $x = 1$ (and the numerator is not zero), you know that the graph of

$$f(x) = \frac{x^2 - 3x}{x - 1}$$



f has a vertical asymptote at $x = 1$.
Figure 1.44

has a vertical asymptote at $x = 1$. This means that each of the given limits is either ∞ or $-\infty$. You can determine the result by analyzing f at values of x close to 1, or by using a graphing utility. From the graph of f shown in Figure 1.44, you can see that the graph approaches ∞ from the left of $x = 1$ and approaches $-\infty$ from the right of $x = 1$. So, you can conclude that

$$\lim_{x\to 1^-} \frac{x^2 - 3x}{x - 1} = \infty \qquad \text{The limit from the left is infinity.}$$

and

$$\lim_{x\to 1^+} \frac{x^2 - 3x}{x - 1} = -\infty. \qquad \text{The limit from the right is negative infinity.}$$

TECHNOLOGY PITFALL When using a graphing calculator or graphing software, be careful to interpret correctly the graph of a function with a vertical asymptote—graphing utilities often have difficulty drawing this type of graph.

THEOREM 1.15 PROPERTIES OF INFINITE LIMITS

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L.$$

1. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = \infty$
2. Product: $\lim_{x\to c} [f(x)g(x)] = \infty, \quad L > 0$
   $\lim_{x\to c} [f(x)g(x)] = -\infty, \quad L < 0$
3. Qotient: $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

PROOF To show that the limit of $f(x) + g(x)$ is infinite, choose $M > 0$. You then need to find $\delta > 0$ such that

$$[f(x) + g(x)] > M$$

whenever $0 < |x - c| < \delta$. For simplicity's sake, you can assume L is positive. Let $M_1 = M + 1$. Because the limit of $f(x)$ is infinite, there exists δ_1 such that $f(x) > M_1$ whenever $0 < |x - c| < \delta_1$. Also, because the limit of $g(x)$ is L, there exists δ_2 such that $|g(x) - L| < 1$ whenever $0 < |x - c| < \delta_2$. By letting δ be the smaller of δ_1 and δ_2, you can conclude that $0 < |x - c| < \delta$ implies $f(x) > M + 1$ and $|g(x) - L| < 1$. The second of these two inequalities implies that $g(x) > L - 1$, and, adding this to the first inequality, you can write

$$f(x) + g(x) > (M + 1) + (L - 1) = M + L > M.$$

So, you can conclude that

$$\lim_{x\to c} [f(x) + g(x)] = \infty.$$

The proofs of the remaining properties are left as exercises (see Exercise 78). ■

EXAMPLE 5 Determining Limits

a. Because $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$, you can write

$$\lim_{x\to 0}\left(1 + \frac{1}{x^2}\right) = \infty. \qquad \text{Property 1, Theorem 1.15}$$

b. Because $\lim_{x\to 1^-} (x^2 + 1) = 2$ and $\lim_{x\to 1^-} (\cot \pi x) = -\infty$, you can write

$$\lim_{x\to 1^-} \frac{x^2 + 1}{\cot \pi x} = 0. \qquad \text{Property 3, Theorem 1.15}$$

c. Because $\lim_{x\to 0^+} 3 = 3$ and $\lim_{x\to 0^+} \cot x = \infty$, you can write

$$\lim_{x\to 0^+} 3 \cot x = \infty. \qquad \text{Property 2, Theorem 1.15}$$

■

1.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.

1. $f(x) = \dfrac{1}{x-4}$

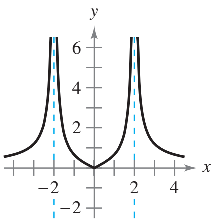
2. $f(x) = \dfrac{-1}{x-4}$

3. $f(x) = \dfrac{1}{(x-4)^2}$

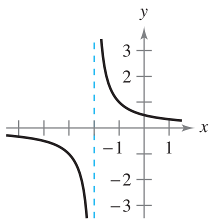
4. $f(x) = \dfrac{-1}{(x-4)^2}$

In Exercises 5–8, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

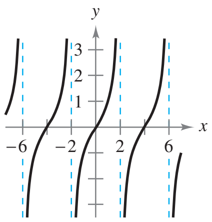
5. $f(x) = 2\left|\dfrac{x}{x^2-4}\right|$



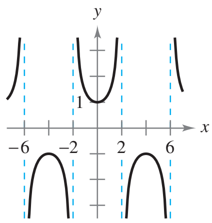
6. $f(x) = \dfrac{1}{x+2}$



7. $f(x) = \tan\dfrac{\pi x}{4}$



8. $f(x) = \sec\dfrac{\pi x}{4}$



Numerical and Graphical Analysis **In Exercises 9–12, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right by completing the table. Use a graphing utility to graph the function to confirm your answer.**

x	-3.5	-3.1	-3.01	-3.001
$f(x)$				

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				

9. $f(x) = \dfrac{1}{x^2-9}$

10. $f(x) = \dfrac{x}{x^2-9}$

11. $f(x) = \dfrac{x^2}{x^2-9}$

12. $f(x) = \sec\dfrac{\pi x}{6}$

In Exercises 13–32, find the vertical asymptotes (if any) of the graph of the function.

13. $f(x) = \dfrac{1}{x^2}$

14. $f(x) = \dfrac{4}{(x-2)^3}$

15. $f(x) = \dfrac{x^2}{x^2-4}$

16. $f(x) = \dfrac{-4x}{x^2+4}$

17. $g(t) = \dfrac{t-1}{t^2+1}$

18. $h(s) = \dfrac{2s-3}{s^2-25}$

19. $h(x) = \dfrac{x^2-2}{x^2-x-2}$

20. $g(x) = \dfrac{2+x}{x^2(1-x)}$

21. $T(t) = 1 - \dfrac{4}{t^2}$

22. $g(x) = \dfrac{\frac{1}{2}x^3 - x^2 - 4x}{3x^2 - 6x - 24}$

23. $f(x) = \dfrac{3}{x^2+x-2}$

24. $f(x) = \dfrac{4x^2+4x-24}{x^4-2x^3-9x^2+18x}$

25. $g(x) = \dfrac{x^3+1}{x+1}$

26. $h(x) = \dfrac{x^2-4}{x^3+2x^2+x+2}$

27. $f(x) = \dfrac{x^2-2x-15}{x^3-5x^2+x-5}$

28. $h(t) = \dfrac{t^2-2t}{t^4-16}$

29. $f(x) = \tan \pi x$

30. $f(x) = \sec \pi x$

31. $s(t) = \dfrac{t}{\sin t}$

32. $g(\theta) = \dfrac{\tan\theta}{\theta}$

In Exercises 33–36, determine whether the graph of the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.

33. $f(x) = \dfrac{x^2-1}{x+1}$

34. $f(x) = \dfrac{x^2-6x-7}{x+1}$

35. $f(x) = \dfrac{x^2+1}{x+1}$

36. $f(x) = \dfrac{\sin(x+1)}{x+1}$

In Exercises 37–54, find the limit (if it exists).

37. $\displaystyle\lim_{x\to -1^+} \frac{1}{x+1}$

38. $\displaystyle\lim_{x\to 1^-} \frac{-1}{(x-1)^2}$

39. $\displaystyle\lim_{x\to 2^+} \frac{x}{x-2}$

40. $\displaystyle\lim_{x\to 1^+} \frac{2+x}{1-x}$

41. $\displaystyle\lim_{x\to 1^+} \frac{x^2}{(x-1)^2}$

42. $\displaystyle\lim_{x\to 4^-} \frac{x^2}{x^2+16}$

43. $\displaystyle\lim_{x\to -3^-} \frac{x+3}{x^2+x-6}$

44. $\displaystyle\lim_{x\to (-1/2)^+} \frac{6x^2+x-1}{4x^2-4x-3}$

45. $\displaystyle\lim_{x\to 1} \frac{x-1}{(x^2+1)(x-1)}$

46. $\displaystyle\lim_{x\to 3} \frac{x-2}{x^2}$

47. $\displaystyle\lim_{x\to 0^-} \left(1 + \frac{1}{x}\right)$

48. $\displaystyle\lim_{x\to 0^-} \left(x^2 - \frac{1}{x}\right)$

49. $\displaystyle\lim_{x\to 0^+} \frac{2}{\sin x}$

50. $\displaystyle\lim_{x\to (\pi/2)^+} \frac{-2}{\cos x}$

51. $\displaystyle\lim_{x\to \pi} \frac{\sqrt{x}}{\csc x}$

52. $\displaystyle\lim_{x\to 0} \frac{x+2}{\cot x}$

53. $\displaystyle\lim_{x\to 1/2} x \sec \pi x$

54. $\displaystyle\lim_{x\to 1/2} x^2 \tan \pi x$

In Exercises 55–58, use a graphing utility to graph the function and determine the one-sided limit.

55. $f(x) = \dfrac{x^2 + x + 1}{x^3 - 1}$

$\displaystyle\lim_{x \to 1^+} f(x)$

56. $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1}$

$\displaystyle\lim_{x \to 1^-} f(x)$

57. $f(x) = \dfrac{1}{x^2 - 25}$

$\displaystyle\lim_{x \to 5^-} f(x)$

58. $f(x) = \sec \dfrac{\pi x}{8}$

$\displaystyle\lim_{x \to 4^+} f(x)$

WRITING ABOUT CONCEPTS

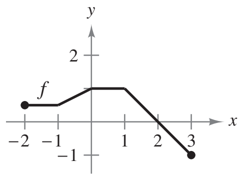
59. In your own words, describe the meaning of an infinite limit. Is ∞ a real number?

60. In your own words, describe what is meant by an asymptote of a graph.

61. Write a rational function with vertical asymptotes at $x = 6$ and $x = -2$, and with a zero at $x = 3$.

62. Does the graph of every rational function have a vertical asymptote? Explain.

63. Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



CAPSTONE

64. Given a polynomial $p(x)$, is it true that the graph of the function given by $f(x) = \dfrac{p(x)}{x - 1}$ has a vertical asymptote at $x = 1$? Why or why not?

65. ***Relativity*** According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c^-.

66. ***Boyle's Law*** For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. Find the limit of P as $V \to 0^+$.

67. ***Rate of Change*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of $\frac{1}{2}$ revolution per second. The rate at which the light beam moves along the wall is $r = 50\pi \sec^2 \theta$ ft/sec.

(a) Find the rate r when θ is $\pi/6$.

(b) Find the rate r when θ is $\pi/3$.

(c) Find the limit of r as $\theta \to (\pi/2)^-$.

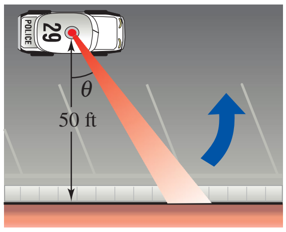


Figure for 67

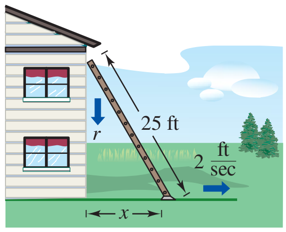


Figure for 68

68. ***Rate of Change*** A 25-foot ladder is leaning against a house (see figure). If the base of the ladder is pulled away from the house at a rate of 2 feet per second, the top will move down the wall at a rate of

$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and the house.

(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as $x \to 25^-$.

69. ***Average Speed*** On a trip of d miles to another city, a truck driver's average speed was x miles per hour. On the return trip the average speed was y miles per hour. The average speed for the round trip was 50 miles per hour.

(a) Verify that $y = \dfrac{25x}{x - 25}$. What is the domain?

(b) Complete the table.

x	30	40	50	60
y				

Are the values of y different than you expected? Explain.

(c) Find the limit of y as $x \to 25^+$ and interpret its meaning.

70. ***Numerical and Graphical Analysis*** Use a graphing utility to complete the table for each function and graph each function to estimate the limit. What is the value of the limit when the power of x in the denominator is greater than 3?

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$							

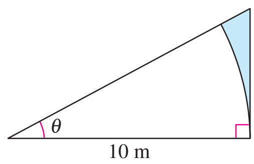
(a) $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x}$

(b) $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x^2}$

(c) $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x^3}$

(d) $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x^4}$

71. ***Numerical and Graphical Analysis*** Consider the shaded region outside the sector of a circle of radius 10 meters and inside a right triangle (see figure).



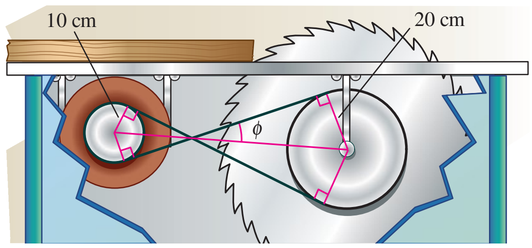
(a) Write the area $A = f(\theta)$ of the region as a function of θ. Determine the domain of the function.

(b) Use a graphing utility to complete the table and graph the function over the appropriate domain.

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$					

(c) Find the limit of A as $\theta \to (\pi/2)^-$.

72. ***Numerical and Graphical Reasoning*** A crossed belt connects a 20-centimeter pulley (10-cm radius) on an electric motor with a 40-centimeter pulley (20-cm radius) on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.



(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of the belt around each pulley.)

(d) Use a graphing utility to complete the table.

ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) Use a graphing utility to graph the function over the appropriate domain.

(f) Find $\displaystyle\lim_{\phi \to (\pi/2)^-} L$. Use a geometric argument as the basis of a second method of finding this limit.

(g) Find $\displaystyle\lim_{\phi \to 0^+} L$.

True or False? **In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

73. The graph of a rational function has at least one vertical asymptote.

74. The graphs of polynomial functions have no vertical asymptotes.

75. The graphs of trigonometric functions have no vertical asymptotes.

76. If f has a vertical asymptote at $x = 0$, then f is undefined at $x = 0$.

77. Find functions f and g such that $\displaystyle\lim_{x \to c} f(x) = \infty$ and $\displaystyle\lim_{x \to c} g(x) = \infty$ but $\displaystyle\lim_{x \to c} [f(x) - g(x)] \neq 0$.

78. Prove the difference, product, and quotient properties in Theorem 1.15.

79. Prove that if $\displaystyle\lim_{x \to c} f(x) = \infty$, then $\displaystyle\lim_{x \to c} \frac{1}{f(x)} = 0$.

80. Prove that if $\displaystyle\lim_{x \to c} \frac{1}{f(x)} = 0$, then $\displaystyle\lim_{x \to c} f(x)$ does not exist.

Infinite Limits **In Exercises 81 and 82, use the ε-δ definition of infinite limits to prove the statement.**

81. $\displaystyle\lim_{x \to 3^+} \frac{1}{x - 3} = \infty$

82. $\displaystyle\lim_{x \to 5^-} \frac{1}{x - 5} = -\infty$

SECTION PROJECT

Graphs and Limits of Trigonometric Functions

Recall from Theorem 1.9 that the limit of $f(x) = (\sin x)/x$ as x approaches 0 is 1.

(a) Use a graphing utility to graph the function f on the interval $-\pi \leq x \leq \pi$. Explain how the graph helps confirm this theorem.

(b) Explain how you could use a table of values to confirm the value of this limit numerically.

(c) Graph $g(x) = \sin x$ by hand. Sketch a tangent line at the point $(0, 0)$ and visually estimate the slope of this tangent line.

(d) Let $(x, \sin x)$ be a point on the graph of g near $(0, 0)$, and write a formula for the slope of the secant line joining $(x, \sin x)$ and $(0, 0)$. Evaluate this formula at $x = 0.1$ and $x = 0.01$. Then find the exact slope of the tangent line to g at the point $(0, 0)$.

(e) Sketch the graph of the cosine function $h(x) = \cos x$. What is the slope of the tangent line at the point $(0, 1)$? Use limits to find this slope analytically.

(f) Find the slope of the tangent line to $k(x) = \tan x$ at $(0, 0)$.

1 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, determine whether the problem can be solved using precalculus or if calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning. Use a graphical or numerical approach to estimate the solution.

1. Find the distance between the points $(1, 1)$ and $(3, 9)$ along the curve $y = x^2$.
2. Find the distance between the points $(1, 1)$ and $(3, 9)$ along the line $y = 4x - 3$.

In Exercises 3 and 4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

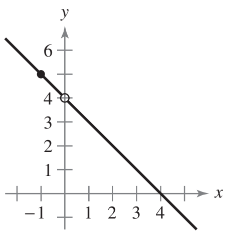
3. $\displaystyle\lim_{x\to 0} \frac{[4/(x+2)] - 2}{x}$
4. $\displaystyle\lim_{x\to 0} \frac{4\left(\sqrt{x+2} - \sqrt{2}\right)}{x}$

In Exercises 5–8, find the limit L. Then use the ε-δ definition to prove that the limit is L.

5. $\displaystyle\lim_{x\to 1} (x + 4)$
6. $\displaystyle\lim_{x\to 9} \sqrt{x}$
7. $\displaystyle\lim_{x\to 2} (1 - x^2)$
8. $\displaystyle\lim_{x\to 5} 9$

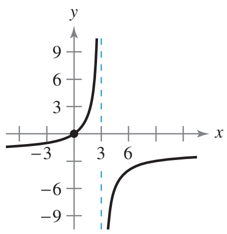
In Exercises 9 and 10, use the graph to determine each limit.

9. $h(x) = \dfrac{4x - x^2}{x}$



(a) $\displaystyle\lim_{x\to 0} h(x)$ (b) $\displaystyle\lim_{x\to -1} h(x)$

10. $g(x) = \dfrac{-2x}{x - 3}$



(a) $\displaystyle\lim_{x\to 3} g(x)$ (b) $\displaystyle\lim_{x\to 0} g(x)$

In Exercises 11–26, find the limit (if it exists).

11. $\displaystyle\lim_{x\to 6} (x - 2)^2$
12. $\displaystyle\lim_{x\to 7} (10 - x)^4$
13. $\displaystyle\lim_{t\to 4} \sqrt{t + 2}$
14. $\displaystyle\lim_{y\to 4} 3|y - 1|$
15. $\displaystyle\lim_{t\to -2} \frac{t + 2}{t^2 - 4}$
16. $\displaystyle\lim_{t\to 3} \frac{t^2 - 9}{t - 3}$
17. $\displaystyle\lim_{x\to 4} \frac{\sqrt{x - 3} - 1}{x - 4}$
18. $\displaystyle\lim_{x\to 0} \frac{\sqrt{4 + x} - 2}{x}$
19. $\displaystyle\lim_{x\to 0} \frac{[1/(x+1)] - 1}{x}$
20. $\displaystyle\lim_{s\to 0} \frac{\left(1/\sqrt{1+s}\right) - 1}{s}$
21. $\displaystyle\lim_{x\to -5} \frac{x^3 + 125}{x + 5}$
22. $\displaystyle\lim_{x\to -2} \frac{x^2 - 4}{x^3 + 8}$
23. $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{\sin x}$
24. $\displaystyle\lim_{x\to \pi/4} \frac{4x}{\tan x}$
25. $\displaystyle\lim_{\Delta x\to 0} \frac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x}$

    [*Hint:* $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$]
26. $\displaystyle\lim_{\Delta x\to 0} \frac{\cos(\pi + \Delta x) + 1}{\Delta x}$

    [*Hint:* $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$]

In Exercises 27–30, evaluate the limit given $\displaystyle\lim_{x\to c} f(x) = -\tfrac{3}{4}$ and $\displaystyle\lim_{x\to c} g(x) = \tfrac{2}{3}$.

27. $\displaystyle\lim_{x\to c} [f(x)g(x)]$
28. $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)}$
29. $\displaystyle\lim_{x\to c} [f(x) + 2g(x)]$
30. $\displaystyle\lim_{x\to c} [f(x)]^2$

Numerical, Graphical, and Analytic Analysis **In Exercises 31 and 32, consider**

$$\lim_{x\to 1^+} f(x).$$

(a) Complete the table to estimate the limit.

(b) Use a graphing utility to graph the function and use the graph to estimate the limit.

(c) Rationalize the numerator to find the exact value of the limit analytically.

x	1.1	1.01	1.001	1.0001
$f(x)$				

31. $f(x) = \dfrac{\sqrt{2x + 1} - \sqrt{3}}{x - 1}$
32. $f(x) = \dfrac{1 - \sqrt[3]{x}}{x - 1}$

    [*Hint:* $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]

Free-Falling Object **In Exercises 33 and 34, use the position function $s(t) = -4.9t^2 + 250$, which gives the height (in meters) of an object that has fallen from a height of 250 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

33. Find the velocity of the object when $t = 4$.
34. At what velocity will the object impact the ground?

In Exercises 35–40, find the limit (if it exists). If the limit does not exist, explain why.

35. $\lim_{x\to 3^-} \dfrac{|x-3|}{x-3}$

36. $\lim_{x\to 4} [\![x-1]\!]$

37. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} (x-2)^2, & x \le 2 \\ 2-x, & x > 2 \end{cases}$

38. $\lim_{x\to 1^+} g(x)$, where $g(x) = \begin{cases} \sqrt{1-x}, & x \le 1 \\ x+1, & x > 1 \end{cases}$

39. $\lim_{t\to 1} h(t)$, where $h(t) = \begin{cases} t^3+1, & t < 1 \\ \frac{1}{2}(t+1), & t \ge 1 \end{cases}$

40. $\lim_{s\to -2} f(s)$, where $f(s) = \begin{cases} -s^2-4s-2, & s \le -2 \\ s^2+4s+6, & s > -2 \end{cases}$

In Exercises 41–52, determine the intervals on which the function is continuous.

41. $f(x) = -3x^2 + 7$

42. $f(x) = x^2 - \dfrac{2}{x}$

43. $f(x) = [\![x+3]\!]$

44. $f(x) = \dfrac{3x^2 - x - 2}{x-1}$

45. $f(x) = \begin{cases} \dfrac{3x^2-x-2}{x-1}, & x \ne 1 \\ 0, & x = 1 \end{cases}$

46. $f(x) = \begin{cases} 5-x, & x \le 2 \\ 2x-3, & x > 2 \end{cases}$

47. $f(x) = \dfrac{1}{(x-2)^2}$

48. $f(x) = \sqrt{\dfrac{x+1}{x}}$

49. $f(x) = \dfrac{3}{x+1}$

50. $f(x) = \dfrac{x+1}{2x+2}$

51. $f(x) = \csc \dfrac{\pi x}{2}$

52. $f(x) = \tan 2x$

53. Determine the value of c such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+3, & x \le 2 \\ cx+6, & x > 2 \end{cases}$$

54. Determine the values of b and c such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2+bx+c, & |x-2| \ge 1 \end{cases}$$

55. Use the Intermediate Value Theorem to show that $f(x) = 2x^3 - 3$ has a zero in the interval $[1, 2]$.

56. ***Delivery Charges*** The cost of sending an overnight package from New York to Atlanta is \$2.80 for the first pound and \$.50 for each additional pound or fraction thereof. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds. Use a graphing utility to graph the function and discuss its continuity.

57. Let $f(x) = \dfrac{x^2-4}{|x-2|}$. Find each limit (if possible).

(a) $\lim_{x\to 2^-} f(x)$

(b) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to 2} f(x)$

58. Let $f(x) = \sqrt{x(x-1)}$.

(a) Find the domain of f.

(b) Find $\lim_{x\to 0^-} f(x)$.

(c) Find $\lim_{x\to 1^+} f(x)$.

In Exercises 59–62, find the vertical asymptotes (if any) of the graph of the function.

59. $g(x) = 1 + \dfrac{2}{x}$

60. $h(x) = \dfrac{4x}{4-x^2}$

61. $f(x) = \dfrac{8}{(x-10)^2}$

62. $f(x) = \csc \pi x$

In Exercises 63–74, find the one-sided limit (if it exists).

63. $\lim_{x\to -2^-} \dfrac{2x^2+x+1}{x+2}$

64. $\lim_{x\to (1/2)^+} \dfrac{x}{2x-1}$

65. $\lim_{x\to -1^+} \dfrac{x+1}{x^3+1}$

66. $\lim_{x\to -1^-} \dfrac{x+1}{x^4-1}$

67. $\lim_{x\to 1^-} \dfrac{x^2+2x+1}{x-1}$

68. $\lim_{x\to -1^+} \dfrac{x^2-2x+1}{x+1}$

69. $\lim_{x\to 0^+} \left(x - \dfrac{1}{x^3}\right)$

70. $\lim_{x\to 2^-} \dfrac{1}{\sqrt[3]{x^2-4}}$

71. $\lim_{x\to 0^+} \dfrac{\sin 4x}{5x}$

72. $\lim_{x\to 0^+} \dfrac{\sec x}{x}$

73. $\lim_{x\to 0^+} \dfrac{\csc 2x}{x}$

74. $\lim_{x\to 0^-} \dfrac{\cos^2 x}{x}$

75. ***Environment*** A utility company burns coal to generate electricity. The cost C in dollars of removing $p\%$ of the air pollutants in the stack emissions is

$$C = \frac{80{,}000p}{100-p}, \quad 0 \le p < 100.$$

Find the costs of removing (a) 15%(b) 50%and (c) 90%f the pollutants. (d) Find the limit of C as $p \to 100^-$.

76. The function f is defined as shown.

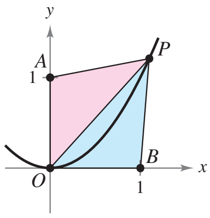
$$f(x) = \frac{\tan 2x}{x}, \quad x \ne 0$$

(a) Find $\lim_{x\to 0} \dfrac{\tan 2x}{x}$ (if it exists).

(b) Can the function f be defined at $x = 0$ such that it is continuous at $x = 0$?

P.S. PROBLEM SOLVING

1. Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the perimeter of each triangle in terms of x.

(b) Let $r(x)$ be the ratio of the perimeters of the two triangles,

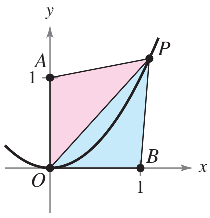
$$r(x) = \frac{\text{Perimeter } \triangle PAO}{\text{Perimeter } \triangle PBO}.$$

Complete the table.

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$					
Perimeter $\triangle PBO$					
$r(x)$					

(c) Calculate $\lim_{x\to 0^+} r(x)$.

2. Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the area of each triangle in terms of x.

(b) Let $a(x)$ be the ratio of the areas of the two triangles,

$$a(x) = \frac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO}.$$

Complete the table.

x	4	2	1	0.1	0.01
Area $\triangle PAO$					
Area $\triangle PBO$					
$a(x)$					

(c) Calculate $\lim_{x\to 0^+} a(x)$.

3. (a) Find the area of a regular hexagon inscribed in a circle of radius 1. How close is this area to that of the circle?

(b) Find the area A_n of an n-sided regular polygon inscribed in a circle of radius 1. Write your answer as a function of n.

(c) Complete the table.

n	6	12	24	48	96
A_n					

(d) What number does A_n approach as n gets larger and larger?

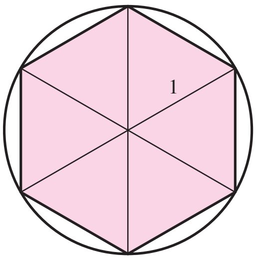


Figure for 3

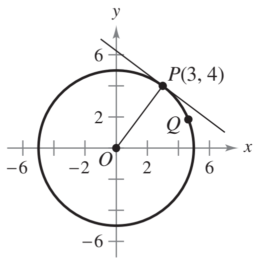


Figure for 4

4. Let $P(3, 4)$ be a point on the circle $x^2 + y^2 = 25$.

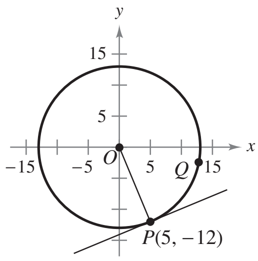
(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the first quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x\to 3} m_x$. How does this number relate to your answer in part (b)?

5. Let $P(5, -12)$ be a point on the circle $x^2 + y^2 = 169$.



(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the fourth quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x\to 5} m_x$. How does this number relate to your answer in part (b)?

6. Find the values of the constants a and b such that

$$\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{3}}{x} = \sqrt{3}.$$

7. Consider the function $f(x) = \dfrac{\sqrt{3+x^{1/3}}-2}{x-1}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph the function.

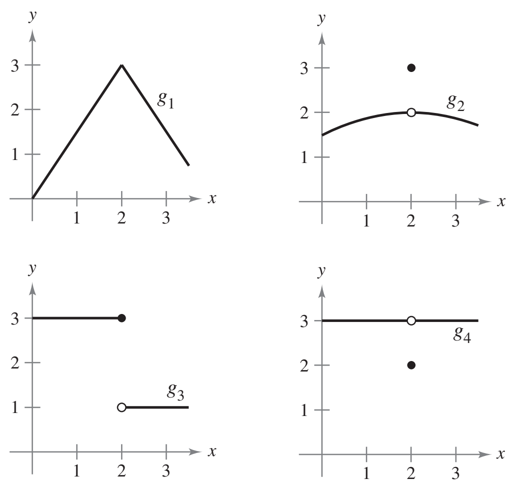
(c) Calculate $\displaystyle\lim_{x\to -27^+} f(x)$.

(d) Calculate $\displaystyle\lim_{x\to 1} f(x)$.

8. Determine all values of the constant a such that the following function is continuous for all real numbers.

$$f(x) = \begin{cases} \dfrac{ax}{\tan x}, & x \ge 0 \\ a^2 - 2, & x < 0 \end{cases}$$

9. Consider the graphs of the four functions g_1, g_2, g_3, and g_4.



For each given condition of the function f, which of the graphs could be the graph of f?

(a) $\displaystyle\lim_{x\to 2} f(x) = 3$

(b) f is continuous at 2.

(c) $\displaystyle\lim_{x\to 2^-} f(x) = 3$

10. Sketch the graph of the function $f(x) = \left[\!\left[\dfrac{1}{x} \right]\!\right]$.

(a) Evaluate $f\left(\frac{1}{4}\right)$, $f(3)$, and $f(1)$.

(b) Evaluate the limits $\displaystyle\lim_{x\to 1^-} f(x)$, $\displaystyle\lim_{x\to 1^+} f(x)$, $\displaystyle\lim_{x\to 0^-} f(x)$, and $\displaystyle\lim_{x\to 0^+} f(x)$.

(c) Discuss the continuity of the function.

11. Sketch the graph of the function $f(x) = [\![x]\!] + [\![-x]\!]$.

(a) Evaluate $f(1)$, $f(0)$, $f\left(\frac{1}{2}\right)$, and $f(-2.7)$.

(b) Evaluate the limits $\displaystyle\lim_{x\to 1^-} f(x)$, $\displaystyle\lim_{x\to 1^+} f(x)$, and $\displaystyle\lim_{x\to \frac{1}{2}} f(x)$.

(c) Discuss the continuity of the function.

12. To escape Earth's gravitational field, a rocket must be launched with an initial velocity called the **escape velocity.** A rocket launched from the surface of Earth has velocity v (in miles per second) given by

$$v = \sqrt{\frac{2GM}{r} + v_0^2 - \frac{2GM}{R}} \approx \sqrt{\frac{192{,}000}{r} + v_0^2 - 48}$$

where v_0 is the initial velocity, r is the distance from the rocket to the center of Earth, G is the gravitational constant, M is the mass of Earth, and R is the radius of Earth (approximately 4000 miles).

(a) Find the value of v_0 for which you obtain an infinite limit for r as v approaches zero. This value of v_0 is the escape velocity for Earth.

(b) A rocket launched from the surface of the moon has velocity v (in miles per second) given by

$$v = \sqrt{\frac{1920}{r} + v_0^2 - 2.17}.$$

Find the escape velocity for the moon.

(c) A rocket launched from the surface of a planet has velocity v (in miles per second) given by

$$v = \sqrt{\frac{10{,}600}{r} + v_0^2 - 6.99}.$$

Find the escape velocity for this planet. Is the mass of this planet larger or smaller than that of Earth? (Assume that the mean density of this planet is the same as that of Earth.)

13. For positive numbers $a < b$, the **pulse function** is defined as

$$P_{a,b}(x) = H(x-a) - H(x-b) = \begin{cases} 0, & x < a \\ 1, & a \le x < b \\ 0, & x \ge b \end{cases}$$

where $H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$ is the Heaviside function.

(a) Sketch the graph of the pulse function.

(b) Find the following limits:

(i) $\displaystyle\lim_{x\to a^+} P_{a,b}(x)$ (ii) $\displaystyle\lim_{x\to a^-} P_{a,b}(x)$

(iii) $\displaystyle\lim_{x\to b^+} P_{a,b}(x)$ (iv) $\displaystyle\lim_{x\to b^-} P_{a,b}(x)$

(c) Discuss the continuity of the pulse function.

(d) Why is

$$U(x) = \frac{1}{b-a} P_{a,b}(x)$$

called the **unit** pulse function?

14. Let a be a nonzero constant. Prove that if $\displaystyle\lim_{x\to 0} f(x) = L$, then $\displaystyle\lim_{x\to 0} f(ax) = L$. Show by means of an example that a must be nonzero.