Let A be a $n \times n$ matrix
If there is a non-zero vector v such that

$$Av = \lambda v \quad , \text{ where } \lambda \text{ is a number}$$

then λ is called an eigenvalue of A
v is called the eigenvector associated with the eigenvalue A.

An eigenvector cannot have more than one eigenvalue.

$$\begin{tikzcd} & \lambda_1 \\ v_1 \arrow[ur] \arrow[dr] & \\ & \lambda_2 \end{tikzcd} \qquad \lambda_1 = \lambda_2$$

$$AV_1 = \lambda_1 V_1$$
$$AV_2 = \lambda_2 V_1$$

$$\lambda_1 v_1 = \lambda_2 v_1$$
$$\lambda_1 v_1 - \lambda_2 v_1 = 0$$
$$v_1(\lambda_1 - \lambda_2) = 0 \qquad v_1 \neq 0$$
$$\lambda_1 - \lambda_2 = 0$$
$$\lambda_1 = \lambda_2$$

Finding eigenvalues and eigenvectors

Let v be an eigenvector of A that is associated with eigenvalue λ.

$Av = \lambda v$ where $v \neq 0$

$Av = \lambda I v$

$Av - \lambda I v = 0$

$v(A - \lambda I) = 0$

$\det(A - \lambda I) = 0$

$Bv = 0$

contradicts

$v \neq 0$

$\therefore$ B cannot have an inverse

$\underbrace{B^{-1}B}v = 0$

$Iv = 0$

$v = 0$

If A is an $n \times n$ matrix then

$p(\lambda) = \det(A - \lambda I)$

is a polynomial in λ of degree

and is called the characteristic polynomial of A.

Example

$$A = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$$

$$\det(A - \lambda) = \det \begin{bmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{bmatrix}$$

$$= (3-\lambda)(5-\lambda) - 3$$

$$= 15 - 3\lambda - 5\lambda + \lambda^2 - 3$$

$$= 12 - 8\lambda + \lambda^2$$

$$= \lambda^2 - 8\lambda + 12$$

$$\lambda^2 - 8\lambda + 12 = 0$$

$$\lambda^2 - 2\lambda - 6\lambda + 12 = 0$$

$$\lambda(\lambda - 2) - 6(\lambda - 2) = 0$$

$$(\lambda - 6)(\lambda - 2) = 0$$

$$\lambda = 2 \quad \text{OR} \quad \lambda = 6$$

Eigenvalues $= 2$ OR 6

Example

$$A = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$$

$\lambda = 2$

$$(A - 2I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 + 3x_2 = 0$$

$$x_1 = -3x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -3 \\ 1 \end{bmatrix}}_{v}$$

The eigenspace $= \left\{ t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \ t \in \mathbb{R} \right\}$

Eigenvector for $\lambda = 2$ is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$$\text{LHS} = Av = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \lambda v = \text{RHS}$$

$$A = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$$

$$(A - 6I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-3x_1 + 3x_2 = 0$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvector for $\lambda = 6$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -1-\lambda & 6 & -12 \\ 0 & -13-\lambda & 30 \\ 0 & -9 & 20-\lambda \end{vmatrix} = 0$$

$$-1-\lambda\left((-13-\lambda)(20-\lambda) + 270 \right) = 0$$

$$-1-\lambda\left(-260 + 13\lambda - 20\lambda + \lambda^2 + 270\right) = 0$$

$$-1-\lambda\left(\lambda^2 - 7\lambda + 10 \right) = 0$$

$$-(1+\lambda) = 0$$

$$\lambda = -1$$

$$\lambda^2 - 7\lambda + 10 = 0$$

$$\lambda^2 - 2\lambda - 5\lambda + 10 = 0$$

$$\lambda(\lambda-2) - 5(\lambda-2) = 0$$

$$(\lambda-5)(\lambda-2) = 0$$

$$\lambda = -1, 2, 5$$

$$\lambda = 2$$

$$(A - 2I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & 6 & -12 \\ 0 & -15 & 30 \\ 0 & -9 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-3x_1 + 6x_2 - 12x_3 = 0$$

$$-15x_2 + 30x_3 = 0$$

$$-9x_2 + 18x_3 = 0$$

$$x_2 = 2x_3$$

$$-3x_1 + 12x_3 - 12x_3 = 0$$

$$-3x_1 = 0$$

$$x_1 = 0$$

$$\begin{bmatrix} 0 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Eigenvector : $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ at $\lambda = 2$