Tutorial (2) (1)

(1) $f(x) = \begin{cases} \ln x & 0 < x \le 1 \\ ax^2 + b & 1 < x \le 5 \end{cases}$

is cont. and $f(2) = 3$ then $2a + b = ?$

$f(x)$ is cont. at $x = 1 \Rightarrow \lim_{x \to 1} f(x) = f(1)$

or $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

we have $a(1)^2 + b = \ln(1)$

$\boxed{a + b = 0}$ ... (1)

$f(2) = a(2)^2 + b = 3$

$\boxed{4a + b = 3}$ ... (2)

$$\begin{array}{r} a + b = 0 \\ \ominus \quad 4a + b = 3 \\ \hline -3a = -3 \end{array} \Rightarrow a = 1 \text{ and hence } b = -1$$

$2a + b = 2(1) + (-1) = 2 - 1 = 1$

$\therefore \boxed{2a + b = 1}$

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(2) Recall $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (2)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

we have $f'(a) = \lim_{h \to 0} \frac{e^{-3+h} - e^{-3}}{h} \rightarrow$ $f(a+h) = e^{-3+h}$, $f(a) = e^{-3}$

$\therefore$ $f(x) = e^x$ and $a = -3$

(3) $h(x) = f(x)g(x)$

$h'(x) = f'(x)g(x) + g'(x)f(x)$

$h'(1) = f'(1)g(1) + g'(1)f(1)$

Note $h(1) = f(1)g(1)$

$24 = f(1) \cdot 6$

$f(1) = \frac{24}{6} = 4$

$20 = (-2)(6) + g'(1) \cdot 4$

$20 + 12 = 4g'(1)$

$32 = 4g'(1) \Rightarrow \boxed{g'(1) = 8}$

(4) $y = \frac{\sqrt{x} - 2x}{x}$ $x = 4$

$y = \frac{\sqrt{x}}{x} - 2 = \frac{x^{\frac{1}{2}}}{x} - 2 \Rightarrow y = x^{-\frac{1}{2}} - 2$

$y' = -\frac{1}{2}x^{-\frac{3}{2}}$

$y' = \frac{-1}{2x^{\frac{3}{2}}}$

$m = \frac{-1}{2(4)^{\frac{3}{2}}} = -\frac{1}{16}$

$x = 4 \Rightarrow y = \frac{\sqrt{4} - 2(4)}{4}$

$= \frac{2-8}{4}$

$= -\frac{3}{2}$

$m = -\frac{1}{16}$ $(4, -\frac{3}{2}$

$y + \frac{3}{2} = -\frac{1}{16}(x-4)$

$y + \frac{3}{2} = -\frac{1}{16}x + \frac{1}{4}$

$\boxed{y = -\frac{1}{16}x - \frac{5}{4}}$

5 $g(x) = \sqrt{1 + 3f(x)}$ $\quad f(2) = 5 \quad f'(2) = 16$ $\quad$ 3

$$g'(x) = \frac{3f'(x)}{2\sqrt{1+3f(x)}}$$

slope of the tangent line $= g'(2) = \dfrac{3f'(2)}{2\sqrt{1+3f(2)}}$

$$= \frac{3(16)}{2\sqrt{1+3(5)}}$$

$$= \frac{48}{2\sqrt{16}}$$

$$g'(2) = \frac{48}{8}$$

$$= 6$$

slope of normal line $= \dfrac{-1}{6}$

6 Recall $\quad f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\boxed{\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{(alternate Form)} \end{aligned}}$$

f is differentiable at $x = a$

$\Rightarrow \lim \dfrac{f(x) - f(a)}{x - a} = L$ (exists)

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a)$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a)$$

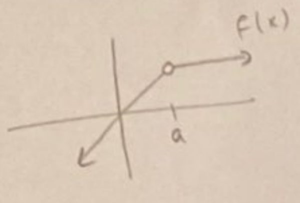
$$= L \cdot 0$$

$$= 0$$

$\therefore \lim\limits_{x \to a} f(x) = f(a)$ $\quad \therefore f$ is cont. at $x = a$

1) True

2) False



$\lim\limits_{x \to a} f(x)$ exists but $f'(a)$ DNE

3) False

4) True, in otherwards if f is differentiable at a, then f is cont at a True

Note the converse is not true.

7. $f(x) = x + 72^x \qquad f'(1) =$

$f'(x) = 1 + \ln(72)\,72^x$

$f'(1) = 1 + 72\ln(72)$

8. $$\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{3}+h\right)\cos\left(\frac{\pi}{3}+h\right) - \frac{\sqrt{3}}{4}}{h} = \frac{\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}}{0}$$

$$= \frac{\frac{\sqrt{3}}{2}\cdot\frac{1}{2} - \frac{\sqrt{3}}{4}}{0}$$

Note: $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$$= \frac{0}{0}$$

$$\lim_{h\to 0} \frac{\frac{1}{2}\sin 2\left(\frac{\pi}{3}+h\right) - \frac{\sqrt{3}}{4}}{h} = \lim_{h\to 0} \frac{\frac{1}{2}\sin\left(\frac{2\pi}{3}+2h\right) - \frac{\sqrt{3}}{4}}{h} \qquad \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}\cdot 2\cos\left(\frac{2\pi}{3}+2h\right)}{1}$$

$$= \lim_{h\to 0} \cos\left(\frac{2\pi}{3}+2h\right) \qquad \cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}$$

$$= \cos\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2}$$

9 $f(x) = (2x^2+3x-1)e^x$

$f'(x) = (4x+3)e^x + e^x(2x^2+3x-1)$

$= e^x(4x+3+2x^2+3x-1)$

$= e^x(2x^2+7x+2)$

10 $y = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5) \quad x>0$

$y = uvw$

$y' = u'vw + uv'w + uvw'$

$\frac{dy}{dx} = (1)(1+x^2)(1+x^3)(1+x^4)(1+x^5)$ 16

$+ (1+x)(2x)(1+x^3)(1+x^4)(1+x^5)$ 32

$+ (1+x)(1+x^2)(3x^2)(1+x^4)(1+x^5)$ 48

$+ (1+x)(1+x^2)(1+x^3)(4x^3)(1+x^5)$ 64

$+ (1+x)(1+x^2)(1+x^3)(1+x^4)(5x^4)$ 80

$\left.\frac{dy}{dx}\right|_{x=1} = 240$

11 $\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} = \frac{1-0-1}{1-1} = \frac{0}{0}$

L'Hopital's Rule $\Rightarrow \lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{e^x - 1}{-\sin x} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{e^x}{-\cos x}$

$= \frac{e^0}{-\cos(0)}$

$= -1$

$$\lim_{x \to a} \frac{x-a}{\ln x - \ln a} = \frac{0}{0} \quad \text{Ind. Form}$$

$$\text{L'Hopital Rule} \Rightarrow \lim_{x \to a} \frac{1}{\frac{1}{x}} = \lim_{x \to a} (x)$$

$$= a$$

$$\lim_{x \to 0} \left(\frac{e^x}{e^x - 1} - \frac{1}{x} \right) = \infty - \infty \quad \text{Ind. Form}$$

$$\lim_{x \to 0} \left(\frac{xe^x - e^x + 1}{x(e^x - 1)} \right) = \frac{0}{0} \quad \text{Ind. Form (L'Hopital Form)}$$

$$\lim_{x \to 0} \frac{(1)e^x + xe^x - e^x}{(1)(e^x - 1) + e^x \cdot x} = \lim_{x \to 0} \frac{xe^x}{e^x - 1 + xe^x} \quad \left(\frac{0}{0}\right)$$

$$\text{(L'Hop.)} \quad = \lim_{x \to 0} \frac{e^x + xe^x}{e^x + e^x + xe^x}$$

$$= \lim_{x \to 0} \frac{e^x(x+1)}{e^x(2+x)}$$

$$= \frac{1}{2}$$

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$$2x^2 - 3xy + 3y^2 = 2$$

$$4x - 3y - 3xy' + 6yy' = 0$$

$$-3xy' + 6yy' = -4x + 3y$$

$$y'(-3x + 6y) = -4x + 3y$$

$$y' = \frac{-4x + 3y}{-3x + 6y} \qquad P(-1, -1)$$

$$\left.\frac{dy}{dx}\right|_{x=-1,\, y=-1} = -\frac{1}{3}$$

$$\text{Eq. of tangent} \quad y + 1 = -\frac{1}{3}(x + 1)$$

$$y = -\frac{1}{3}x - \frac{4}{3}$$

$$\text{Eq. of normal} \quad y = 3x + 2$$