($8pts$) **Problem 1**
Evaluate the following limits

$$(a) \quad \lim_{x\to 0} \frac{\cos^2 x - 1}{\sin x} \qquad\qquad (b) \quad \lim_{x\to -\infty} \frac{\sqrt{x^2+x+1}}{3x-10}$$

Solution
$a)$

$$\begin{aligned} \lim_{x\to 0} \frac{\cos^2 x - 1}{\sin x} &= \lim_{x\to 0} \frac{-2\sin x \cos x}{\cos x} \\ &= \lim_{x\to 0} -2\sin x = 0 \qquad (4pts) \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x\to -\infty} \frac{\sqrt{x^2+x+1}}{3x-10} &= \lim_{x\to -\infty} \frac{\sqrt{x^2}}{3x} \\ &= \lim_{x\to -\infty} \frac{\sqrt{|x|}}{3x} \\ &= \lim_{x\to -\infty} \frac{-\cancel{x}}{3\cancel{x}} = \frac{-1}{3} \qquad (4pts) \end{aligned}$$

$(8pts)$ **Problem 2**

Find the value (s) of the constant m for which the function f continuous at $x = 0$

$$f(x) = \begin{cases} \dfrac{\dfrac{1}{x+7} - \dfrac{1}{7}}{7x} + 7m, & \text{if } x > 0 \\ x^2 - 2m, & \text{if } x \leq 0 \end{cases}$$

Solution

f is continuous at $x = 0$ if and only if

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \qquad (2pts)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x^2 - 2m\right) = -2m \qquad (2pts)$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left(\frac{\dfrac{1}{x+7} - \dfrac{1}{7}}{7x} + 7m \right) \\ &= \lim_{x \to 0^+} \left(\frac{\dfrac{1}{x+7} - \dfrac{1}{7}}{7x} \right) + 7m \\ &= \lim_{x \to 0^+} \left(\frac{1}{49x + 343} \right) + 7m \\ &= \frac{1}{343} + 7m. \qquad (2pts) \end{aligned}$$

The condition of continuity gives

$$\frac{1}{343} + 7m = -2m \Rightarrow m = \frac{1}{3087}. \qquad (2pts)$$

$(8pts)$**Problem 3**

Find $\dfrac{dy}{dx}$ for

$$(a) \quad y = \ln\left[\frac{xe^{x^2}}{\sqrt{x+1}}\right] \qquad\qquad (b) \quad xy^3 + y\ln x = y + 1$$

Solution

(a)

$$\begin{aligned} y &= \ln\left[\frac{xe^{x^2}}{\sqrt{x+1}}\right] = \ln\left(xe^{x^2}\right) - \ln\sqrt{x+1} \\ &= \ln x + \ln e^{x^2} - \frac{1}{2}\ln(x+1) \\ &= \ln x + x^2 - \frac{1}{2}\ln(x+1) \end{aligned}$$

$$\frac{dy}{dx} = \frac{1}{x} + 2x - \frac{1}{2(x+1)} \qquad\qquad (4pts)$$

(b)

$$xy^3 + y\ln x = y + 1$$

Taking derivative on both sides (keeping in mind that y is a function of x), we obtain

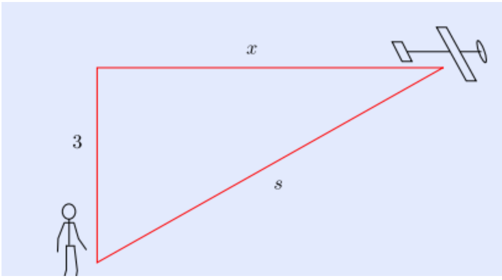
$$y^3 + 3xy'y^2 + y'\ln x + \frac{y}{x} = y'.$$

Now solving for y', we get

$$y' = \frac{\frac{-y}{x} - y^3}{\ln x + 3xy^2 - 1} \qquad\qquad (4pts)$$

($8pts$) **Problem 4**

An airplane flying horizontally at an altitude of 3000 m and a speed of 480 k/hr passes directly above an observer on the ground. How fast is the distance from the observer to the airplane increasing 30 s later?



Solution

The units in the statement of the problem are mixed. In the diagram, we have converted the 3000 m altitude to 3 km. Converting 30 s to 1/120 hr below leaves us with consistent units (kilometers and hours), and we can safely forget about units until the end.

- Given: $\dfrac{dx}{dt} = 480$
- Want: $\dfrac{ds}{dt}$ when $t = 1/120$. $\qquad$ ($2pts$)
- The related equation is
$$s^2 = 3^2 + x^2.$$
- The related rate equation is
$$\cancel{2}\frac{ds}{dt}s = \cancel{2}\frac{dx}{dt}x$$
$$\frac{ds}{dt} = \frac{480x}{s} \qquad (2pts)$$
The final step is to evaluate $\dfrac{ds}{dt}$ at $t = 1/120$. The formula we obtained requires that we find x and s corresponding to this particular time. The plane, moving at a constant rate of $480k/hr$, travels 4 k in $1/120hr$, so
$$x = vt = \frac{480}{120} = 4. \qquad (2pts)$$
.
- The corresponding s is 5 as can be seen from the relationship. Therefore,
$$\left.\frac{ds}{dt}\right|_{t=\frac{1}{120}} = \left.\frac{ds}{dt}\right|_{x=4,\ s=5} = \frac{480\,(4)}{5} = 384k/hr. \qquad (2pts)$$

(8*pts*) **Problem 5**

Find the local extrema of the function

$$f(x) = 2x^3 + 3x^2 - 12x + 5$$

Solution

$$\begin{aligned} f\,'(x) &= 6x^2 + 6x - 12 \\ &= 6\,(x+2)\,(x-1) \end{aligned}$$

(2pts)

The critical numbers are 1 and -2.

x	$-\infty$		-2		1		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$		$\nearrow$	$f(-2)=25$	$\searrow$	$f(1)=-2$	$\nearrow$	

(2pts)

* Local Max $= 25$ at $x = -2$ (2pts)

* Local Min $= -2$ at $x = 1$ (2pts)