Tutorial 6

1. $\int \sin^4 x \cos x \, dx = \int (\sin x)^4 \cos x \, dx$

let $u = \sin x$

$du = \cos x \, dx$

$= \int u^4 \, du$

$= \frac{u^5}{5} + C$

$= \frac{(\sin x)^5}{5} + C$

$= \frac{\sin^5 x}{5} + C$

2. $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cdot \cos x \, dx$

$= \int \sin^2 x \left(1 - \sin^2 x\right) \cos x \, dx$

let $u = \sin x$

$du = \cos x \, dx$

$= \int u^2 (1 - u^2) \, du$

$= \int (u^2 - u^4) \, du$

$= \frac{u^3}{3} - \frac{u^5}{5} + C$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

3 $\int \sin^2 x \cos^2 x \, dx = \int (\sin x \cos x)^2 \, dx$

Note $\sin 2x = 2 \sin x \cos x$

$\sin x \cos x = \frac{1}{2} \sin 2x$

$= \int \left(\frac{1}{2} \sin 2x\right)^2 dx$

$= \frac{1}{4} \int \sin^2 2x \, dx$

Note $\sin^2 x = \frac{1 - \cos 2x}{2}$

$\sin^2 2x = \frac{1 - \cos 4x}{2}$

$= \frac{1}{8} \int (1 - \cos 4x) \, dx$

$= \frac{1}{8} \left(x - \frac{\sin 4x}{4}\right) + C$

$= \frac{x}{8} - \frac{\sin 4x}{32} + C$

2nd way

$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) dx$

Note $\cos^2 x = \frac{1 + \cos 2x}{2}$

$\cos^2 2x = \frac{1 + \cos 4x}{2}$

$= \frac{1}{4} \int (1 - \cos 2x)(1 + \cos 2x) \, dx$

$= \frac{1}{4} \int (1 - \cos^2 2x) \, dx$

$= \frac{1}{4} \int 1 - \left(\frac{1 + \cos 4x}{2}\right) dx$

$= \frac{1}{4} \int \left(1 - \frac{1}{2} - \frac{1}{2} \cos 4x\right) dx$

$= \frac{1}{4} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4x\right) dx$

$= \frac{1}{4} \left(\frac{1}{2} x - \frac{1}{2} \frac{\sin 4x}{4}\right) + C$

$= \frac{1}{8} x - \frac{1}{32} \sin 4x + C$

4 $\int \tan^3 x \sec^2 x \, dx = \int (\tan x)^3 \sec^2 x \, dx$

let $u = \tan x$ $\quad = \int u^3 \, du$

$du = \sec^2 x \, dx$

$$= \frac{u^4}{4} + C$$

$$= \frac{(\tan x)^4}{4} + C$$

6 $\int \sqrt{\sec x} \tan x \, dx$

let $u = \sec x \Rightarrow du = \sec x \tan x \, dx$

$du = u \tan x \, dx$

$\tan x \, dx = \frac{du}{u}$

$$\int \sqrt{u} \, \frac{du}{u} = \int u^{\frac{1}{2}} \frac{du}{u}$$

$$= \int u^{-\frac{1}{2}} du$$

$$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2u^{\frac{1}{2}} + C$$

$$= 2(\sec x)^{\frac{1}{2}} + C$$

$$= 2\sqrt{\sec x} + C$$

Complex Numbers

1. $z = \frac{1+4i}{3+i} = \frac{1+4i}{3+i} \cdot \frac{3-i}{3-i}$

$$= \frac{(1+4i)(3-i)}{(3+i)(3-i)}$$

$$= \frac{3-i+12i-4i^2}{(3)^2-(i)^2} \qquad i^2=-1$$

$$= \frac{7+11i}{9+1}$$

$$= \frac{7}{10} + \frac{11}{10}i$$

$$Re(z) = \frac{7}{10}$$

$$Im(z) = \frac{11}{10}$$

2. Find the square root of $21-20i$

let $\sqrt{21-20i} = a+ib$

$$21-20i = (a+ib)^2$$

$$21-20i = a^2-b^2+i\,2ab$$

$\boxed{a^2-b^2=21}$ and $2ab=-20 \rightarrow \boxed{ab=-10}$

Guess! $a=5$ and $b=-2$ OR $a=-5$ and $b=2$

$\therefore$ the square roots of $21-20i$ are

$5-2i$ and $-5+2i$

$$\sqrt{21-20i} = \pm(5-2i)$$

$a^2 - b^2 = 21 \quad \dots ①$

$ab = -10 \quad \dots ②$

$ab = -10 \rightarrow b = -\frac{10}{a}$

sub. in ① : $a^2 - \left(-\frac{10}{a}\right)^2 = 21$

$$a^2 - \frac{100}{a^2} = 21$$

$$a^4 - 100 = 21a^2$$

$$a^4 - 21a^2 - 100 = 0$$

$$(a^2 - 25)(a^2 + 4) = 0$$

$$a^2 = 25 \quad \text{or} \quad a^2 = -4$$

$a^2 = 25 \rightarrow a = -5 \quad \text{or} \quad a = 5$

$a = -5 \rightarrow b = \frac{-10}{-5} = 2$

$a = 5 \rightarrow b = -\frac{10}{5} = -2$

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3) The multiplicative inverse of $4+3i$ is $\frac{1}{4+3i}$

$$\frac{1}{4+3i} = \frac{1}{4+3i} \cdot \frac{4-3i}{4-3i}$$

$$= \frac{4-3i}{(4)^2-(3i)^2}$$

$$= \frac{4-3i}{16+9}$$

$$= \frac{4-3i}{25}$$

$$= \frac{4}{25} - \frac{3}{25}i$$

4) i) $x+3i+3 = 5+yi$

$(x+3)+3i = 5+yi$

$\therefore$ $x+3=5$ and $\boxed{y=3}$

$x+3=5 \rightarrow \boxed{x=2}$

ii) $x+2yi = ix+y+1$

$x+2yi = (y+1)+xi$

$\therefore$ $x = y+1$ ... ①

and $2y = x$ ... ②

sub. ① in ② $\Rightarrow$ $2y = y+1$

$y=1 \Rightarrow x=2$

iii) $(x,y)(1,2)=(-1,8)$

$(x+iy)(1+2i)=-1+8i$

$x+2xi+iy-2y=-1+8i$

$(x-2y)+(2x+y)i=-1+8i$

$x-2y=-1 \quad \ldots 1$

$2x+y=8 \quad \ldots 2$

$$\begin{array}{r} -2x+4y=2 \\ + \quad 2x+y=8 \\ \hline 5y=10 \rightarrow y=2 \end{array}$$

$\boxed{y=2} \rightarrow x-2(2)=-1$

$x-4=-1$

$\boxed{x=3}$

iv) $(x,-y)(3,-4)=(3,-29)$

$(x-yi)(3-4i)=3-29i$

$3x-4xi-3yi-4y=3-29i$

$(3x-4y)+(-4x-3y)i=3-29i$

$3x-4y=3 \xrightarrow{(4)}$ $\quad \cancel{12x}-16y=12$

$-4x-3y=-29 \xrightarrow{(3)}$ $\quad \cancel{-12x}-9y=-87$

$-25y=-75$

$y=3$

$3x-4y=3$

$3x-4(3)=3$

$3x-12=3$

$3x=15$

$x=5$

i)

5. $z = -2+3i$

$\overline{z} = -2-3i$ ($\overline{z}$ is the conjugate of z)

ii) $(1+i)(-2-i) = -2-i-2i+1$

$= -1-3i$

$\overline{-1-3i} = -1+3i$

iii) $-3i(2+5i) = -6i+15$

$= 15-6i$

$\overline{15-6i} = 15+6i$

iv) $(-5+3i)(2-3i) = -10+15i+6i+9$

$= -1+21i$

$\overline{-1+21i} = -1-21i$

6. Let $z = a+i.b$ the modulus of z is $|z| = \sqrt{a^2+b^2}$

i) $z = -2 = a+i.b$

$a=-2$ $b=0$

$|z| = \sqrt{(-2)^2+0^2}$

$= \sqrt{4}$

$= 2$

ii) $z = 3 + 2i$

$a = 3 \quad b = 2$

$$|z| = \sqrt{(3)^2 + (2)^2}$$
$$= \sqrt{9 + 4}$$
$$= \sqrt{13}$$

iii) $z = 5i \qquad a = 0 \quad b = 5$

$$|z| = \sqrt{(0)^2 + (5)^2}$$
$$= 5$$

iv) $z = (2, 0) = 2 + 0.i$

$$|z| = \sqrt{(2)^2}$$
$$= 2$$

v) $z = (-2, 1) \qquad |z| = \sqrt{(-2)^2 + (1)^2}$
$$= \sqrt{4 + 1}$$
$$= \sqrt{5}$$

vi) $(-2, -1) = z \qquad |z| = \sqrt{(-2)^2 + (-1)^2}$
$$= \sqrt{4 + 1}$$
$$= \sqrt{5}$$

vii) $\frac{1+2i}{2-i} = \frac{1+2i}{2-i} \cdot \frac{2+i}{2+i}$

$= \frac{2+i+4i-2}{(2)^2-(i)^2}$

$= \frac{5i}{4+1}$

$= \frac{5i}{5}$

$= i$

$z = i \Rightarrow |z| = |i| = \sqrt{(0)^2+(1)^2}$

$= 1$

viii) $\frac{(3-5i)(1+i)}{4+2i} = \frac{3+3i-5i+5}{4+2i}$

$= \frac{8-2i}{4+2i} \cdot \frac{4-2i}{4-2i}$

$= \frac{32-16i-8i-4}{(4)^2-(2i)^2}$

$= \frac{28-24i}{20}$

$= \frac{28}{20} - \frac{24}{20}i$

$= \frac{7}{5} - \frac{6}{5}i$

$\left|\frac{7}{5} - \frac{6}{5}i\right| = \sqrt{\left(\frac{7}{5}\right)^2 + \left(-\frac{6}{5}\right)^2} = \frac{\sqrt{85}}{5}$

Recall $z = a + i \cdot b$ in polar form is $z = r(\cos\theta + i \cdot \sin\theta)$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$

7. i) $z = 2 + 2\sqrt{3}\,i$

$r = |z| = \sqrt{(2)^2 + (2\sqrt{3})^2}$

$= \sqrt{4 + 12}$

$= \sqrt{16}$

$= 4$

$\tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3}$ $\quad \theta = \tan^{-1}(\sqrt{3})$

$= 60^\circ$

$= \frac{\pi}{3}$

$\therefore z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

ii) $z = 1 - i = (1, -1)$

$r = |z| = \sqrt{(1)^2 + (-1)^2}$

$= \sqrt{2}$

$\tan\theta = -\frac{1}{1} = -1$ $\quad \theta = \tan^{-1}(\overset{+ve}{1}) = \frac{\pi}{4}$ but $(1,-1)$ is in

↳ reference angle in QI

QIV $\Rightarrow$ $\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$r = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

iii) $z=-1-i$

$r=|z|=\sqrt{(-1)^2+(-1)^2}$

$=\sqrt{2}$

$\tan\theta=\frac{b}{a}=\frac{-1}{-1}=1$ $\theta=\tan^{-1}(1)=\frac{\pi}{4}$ but since

$(-1,-1)$ belongs to Q III we have $\theta=\pi+\frac{\pi}{4}$

$=\frac{5\pi}{4}$

$\therefore\ z=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$