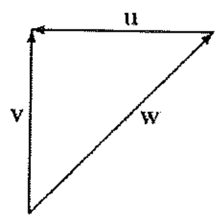
Solution Key

Part 1 MCQ (30%)

(6pts)**Problem 1.**
Consider the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ below.



Which of the following is TRUE

(a) $\mathbf{w}+\mathbf{u}+\mathbf{v}=\mathbf{0}$

(b) $\mathbf{w}+\mathbf{u}-\mathbf{v}=\mathbf{0}$

(c) $\mathbf{w}-\mathbf{u}-\mathbf{v}=\mathbf{0}$

(d) $-\mathbf{w}+\mathbf{u}+\mathbf{v}=\mathbf{0}$

(e) None of the above

Solution

$$\mathbf{v}=\mathbf{w}+\mathbf{u} \quad \text{or} \quad \mathbf{w}+\mathbf{u}-\mathbf{v}=\mathbf{0}$$

Answer is (b).
(6pts)**Problem 2.**
If $P(1,-2), Q(-3,1), R(2,4)$ and $S(-1,6)$ are points in the Cartesian plane, then $\overrightarrow{PQ}+2\overrightarrow{RS}$ is equal to:

(a) $\langle 1,\ 6\rangle$

(b) $\langle 0,\ 11\rangle$

(c) $\langle -10,\ 12\rangle$

(d) $\langle -1,\ 17\rangle$

(e) $\langle -10,\ 7\rangle$

Solution

$$\overrightarrow{PQ}=\langle -4,\ 3\rangle$$
$$\overrightarrow{RS}.=\langle -3,\ 2\rangle$$

$$
\begin{aligned}
\overrightarrow{PQ} + 2\overrightarrow{RS} &= \langle -4,\ 3\rangle + 2\langle -3,\ 2\rangle \\
&= \langle -10,\ 7\rangle
\end{aligned}
$$

Answer is (e)

(6pts)**Problem 3.**
If $\overrightarrow{a} = \langle 5, \ -2\rangle$ and $\overrightarrow{b} = \langle 15, c\rangle$ are parallel, c is equal to

(a) $c = 20$

(b) $c = -4$

(c) $c = -6$

(d) $c = -2$

(e) $c = 3$

Solution
One vector should be a constant multiple of the other. i.e.

$$\overrightarrow{b} = t\overrightarrow{a}$$

$$\langle 15, c\rangle = \langle 5t, \ -2t\rangle$$

$$\begin{cases} 5t = 15 \\ c = -2t \end{cases} \Rightarrow t = 3 \ \text{ and } c = -6$$

Answer is (c)

(6pts)**Problem 4.**
Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be two unit vectors. If the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$ is $\theta = \frac{\pi}{3}$, then $\|\overrightarrow{u} + \overrightarrow{v}\|$ is equal to

(a) $\sqrt{3}\pi$

(b) π

(c) $\sqrt{2}$

(d) 2

(e) $\sqrt{3}$

Solution

$$\begin{aligned} \|\overrightarrow{u} + \overrightarrow{v}\|^2 &= (\overrightarrow{u} + \overrightarrow{v}) \cdot (\overrightarrow{u} + \overrightarrow{v}) \\ &= \|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2 + 2\overrightarrow{u} \cdot \overrightarrow{v} \\ &= 1 + 1 + 2(1)(1)\left(\frac{1}{2}\right) = 3 \end{aligned}$$

Answer is (e)
(6pts)**Problem 5**

If $\overrightarrow{a} = \langle 1,\ 6,\ -2\rangle$ and $\overrightarrow{b} = \langle 1,\ 2,\ -2\rangle$, then $\Pr oj_{\overrightarrow{b}.}\overrightarrow{a} =$

(a) $\left\langle \frac{17}{9},\ \frac{34}{9},\ -\frac{-34}{9}\right\rangle$

(b) $\langle 2,\ 12,\ -4\rangle$

(c) $\left\langle \frac{1}{9},\ 6,\ \frac{-2}{9}\right\rangle$

(d) $\langle 2,\ 8,\ -4\rangle$

(e) $\langle 2,\ 4,\ -4\rangle$

Solution

$$\Pr oj_{\overrightarrow{b}.}\overrightarrow{a} = \left(\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{\left\|\overrightarrow{b}\right\|^2}\right)\overrightarrow{b} = \frac{17}{9}\langle 1,\ 2,\ -2\rangle.$$

Answer is (a)

Part 2 Written questions (70%)

(10pts)**Problem 1**
Consider the points $P(1,-2,3), Q(2,4,0), R(2,0,1)$.
(a) Find the area of the triangle PQR.
(b) Find the equation of the plane containing the points P, Q and R.

Solution
(a)

$$\overrightarrow{PQ} = \langle 1,\ 6,\ -3\rangle \qquad \text{and} \qquad \overrightarrow{PR} = \langle 1,\ 2,\ -2\rangle \qquad \textbf{(2pts)}$$

$$\text{Area} = \frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|$$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 6 & -3 \\ 1 & 2 & -2 \end{vmatrix} = \langle -6,\ -1,\ -4\rangle \quad \textbf{(2pts)}$$

$$\begin{aligned} \text{Area} &= \frac{1}{2}\sqrt{36+1+16} \\ &= \frac{1}{2}\sqrt{53} = 3.6401 \qquad \textbf{(2pts)} \end{aligned}$$

(b) The equation of the plane is

$$-6(x-1) - (y+2) - 4(z-3) = 0$$
$$6x + y + 4z = 16 \qquad \textbf{(4pts)}$$

(10pts)**Problem 2.**

(a) Find the parametric equations of the line L passing through the point $(2,4,1)$ that is perpendicular to the plane

$$3x - y + 5z = 77.$$

(b) Find the intersection point of the line L in part (a) and the plane $3x - y + 5z = 77$.

Solution

(a) The direction vector of the line is

$$\overrightarrow{u} = \langle 3,\ -1,\ 5\rangle \qquad \textbf{(2pts)}$$

The parametric equations of the line L are

$$\begin{cases} x = 2 + 3t \\ y = 4 - t \\ z = 1 + 5t \end{cases} \qquad \textbf{(3pts)}$$

(b) We have

$$3\,(2+3t) - (4-t) + 5\,(1+5t) = 77 \qquad \textbf{(2pts)}$$

$$35t + 7 = 77$$

Solving for t, we get

$$t = 2$$

The intersection point with the line L is

$$(8,\ 2,\ 11) \qquad \textbf{(3pts)}$$

(10pts)**Problem 3.**
Find the parametric equations for the line L of intersection of the planes

$$x - 2y + z = 5 \quad \text{and} \quad 2x + y - z = 0.$$

Solution
A vector $\mathbf{v}$ parallel to the line is the cross product of the normal vectors of the planes:

$$\mathbf{v} = \langle 1, -2, 1 \rangle \times \langle 2, 1, -1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$
$$= \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \mathbf{k} = \langle 1, 3, 5 \rangle.$$ **(3pts)**

A point on L is any (x_0, y_0, z_0) that satisfies both of the plane equations. Setting $z = 0$, we obtain the equations $x - 2y = 5$ and $2x + y = 0$ and find such a point $(1, -2, 0)$. **(3pts)**

Therefore parametric equations for **L** are:

$$x = 1 + t$$
$$y = -2 + 3t$$
$$z = 5t.$$ **(4pts)**

(10pts)**Problem 4.**
Use the Gauss elimination method to solve the linear system

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 5 \\ x_1 - x_2 \qquad &= -1 \\ -x_1 + x_2 + x_3 &= 5 \end{aligned}$$

Solution

The augmented matrix is

$$A = \begin{bmatrix} 1 & -2 & 2 & 5 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 5 \end{bmatrix}. \quad \textbf{(3pts)}$$

After reducing in echellon form, you get , row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}. \qquad \textbf{(4pts)}$$

Now doing the back substitution you obtain

$$x_1 = 1, \quad x_2 = 2 \text{ and } x_3 = 4. \quad \textbf{(3pts)}$$

(15pts)**Problem 5.**

Let

$$A = \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & -1 \\ -3 & -5 & 6 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ -3 & 4 \end{bmatrix}.$$

Find

1. $A + B$ 2. $C + B^t$ 3. AC

Solution

1.

$$A + B = \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -1 \\ -3 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -3 \\ -2 & 0 & 8 \end{bmatrix} \qquad \textbf{(5pts)}$$

2.

$$C + B^t = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & -5 \\ -4 & 10 \end{bmatrix} \qquad \textbf{(5pts)}$$

3.

$$AC = \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ -10 & 11 \end{bmatrix} \qquad \textbf{(5pts)}$$

(15pts)**Problem 6.**
Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & -1 \\ -1 & 0 & -3 \end{bmatrix}$$

(a) Find the determinant of A.
(b) Use the Gauss Jordan method to find the inverse of A.

Solution

(a)

$$\det A = 1 \neq 0 \qquad \textbf{(7pts)}$$

(b) The Augmented matrix is

$$\begin{bmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -3 & 0 & 0 & 1 \end{bmatrix}$$

After using the Gauss Jordan method, we find that

$$A^{-1} = \begin{bmatrix} -3 & 0 & -4 \\ -2 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix} \qquad \textbf{(8pts)}$$