Math141

Tutorial 8

Prob.1 $P(1,-1,0)$ $Q(2,1,-1)$ $R(-1,1,3)$

a) $\overrightarrow{PQ} = \langle 2-1, 1-(-1), -1-0 \rangle$

$= \langle 1, 2, -1 \rangle$

$\overrightarrow{PR} = \langle -1-1, 1-(-1), 3-0 \rangle$

$= \langle -2, 2, 3 \rangle$

b) $\text{proj}_{\overrightarrow{PR}} \overrightarrow{PQ} = \left(\dfrac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PR}\|^2} \right) \overrightarrow{PR}$

$= \left(\dfrac{(1)(-2) + (2)(2) + (-1)(3)}{(-2)^2 + (2)^2 + (3)^2} \right) \langle -2, 0, 3 \rangle$

$= -\dfrac{1}{17} \langle -2, 2, 3 \rangle$

$= \left\langle \dfrac{2}{17}, \dfrac{-2}{17}, \dfrac{-3}{17} \right\rangle$

c) Area of ΔPQR is $\frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\|$

$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ -2 & 2 & 3 \end{vmatrix}$

$= \langle 6+2, -(3-2), 2+4 \rangle$

$= \langle 8, -1, 6 \rangle$

$$\text{Area} = \frac{1}{2}\sqrt{8^2+(-1)^2+6^2}$$

$$= \frac{1}{2}\sqrt{101}$$

Prob 2. $\vec{u} = \langle 2,-3,1\rangle$ $\quad \vec{v} = \langle 3,1,-1\rangle$

a)

$$\operatorname{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\right)\vec{u}$$

$$= \left(\frac{(2)(3)+(-3)(1)+(1)(-1)}{(2)^2+(-3)^2+(1)^2}\right)\langle 2,-3,1\rangle$$

$$= \frac{1}{7}\langle 2,-3,1\rangle$$

$$= \left\langle \frac{2}{7}, -\frac{3}{7}, \frac{1}{7}\right\rangle$$

b)

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= \langle -4+2, -(6-1), -6+2\rangle$$

$$= \langle -2, -5, -4\rangle$$

$$\|\vec{u}\times\vec{v}\| = \sqrt{(-2)^2+(-5)^2+(-4)^2}$$

$$= \sqrt{4+25+16}$$

$$= \sqrt{45}$$

The required vector is $7 \cdot \dfrac{\vec{u}\times\vec{v}}{\|\vec{u}\times\vec{v}\|} = \dfrac{7}{\sqrt{45}}\langle -2,-5,-4\rangle$

c) $\vec{u} = \langle 4, -2, 1 \rangle$ $\quad \vec{v} = \langle 5, 0, -3 \rangle$

$$\vec{w} = \text{proj}_{\vec{v}}\,\vec{u}$$

$$\vec{w} \cdot \left[(2\vec{u} \times \vec{v}) - 3\vec{v} \right] = ?$$

$\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$

$\vec{w}$ is parallel to $\vec{v}$ $\Rightarrow$ $\vec{u} \times \vec{v}$ is perpendicular to $\vec{w}$

$$\therefore \quad \vec{w} \cdot (2\vec{u} \times \vec{v}) = 0$$

$$\vec{w} \cdot \left[(2\vec{u} \times \vec{v}) - 3\vec{v} \right] = -3\,\vec{w} \cdot \vec{v}$$

$$= -3 \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} \cdot \vec{v}$$

$$= -3 \left(\frac{\vec{u} \cdot \vec{v}}{\cancel{\|\vec{v}\|^2}} \right) \cancel{\|\vec{v}\|^2}$$

$$= -3\,\vec{u} \cdot \vec{v}$$

$$= -3 \big((4)(5) + (-2)(0) + (1)(-3) \big)$$

$$= -3\,(17)$$

$$= -51$$

d) $\|\vec{u}\| = 3$ $\|\vec{v}\| = 2$ $\theta = \frac{\pi}{3}$

$\|\vec{u}-\vec{v}\| = ?$

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 - 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \qquad \vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$= (3)^2 - 2\left((3)(2)\cos\frac{\pi}{3}\right) + (2)^2$$

$$= 9 - 6 + 4$$

$$= 7$$

$$\|\vec{u}-\vec{v}\| = \sqrt{7}$$

Prob 3.

a) $\vec{u} = \overrightarrow{AB}$ $\qquad A = (7, 6, 4)$ $B = (4, 6, 5)$

$\qquad\qquad\qquad\qquad\qquad x_0\ y_0\ z_0$

$= \langle 4-7,\ 6-6,\ 5-4 \rangle$

$= \langle -3, 0, 1 \rangle$ $\quad \vec{u}$ is a vector parallel to the given line.

$\quad a\ \ b\ \ c$

parametric equations of a line

$x = x_0 + at \rightarrow x = 7 - 3t$

$y = y_0 + bt \rightarrow y = 6 + 0.t$

$z = z_0 + ct \rightarrow z = 4 + t$

b) $\frac{x+1}{5} = \frac{y-2}{3} = \frac{z-7}{2}$ $\quad p(1,-3,2)$

The direction vector is $\vec{u} = \langle 5, 3, 2 \rangle$

$\therefore$ $x = 1 + 5t$

$y = -3 + 3t$

$z = 2 + 2t$

prob 4 $\quad (1,1,1) \quad (2,2,2)$
$\quad\quad a\ b\ c$

a)

The radius of the sphere is $r = \sqrt{(2-1)^2 + (2-1)^2 + (2-1)^2}$
$= \sqrt{3}$

Equation of a sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

where (a,b,c) is the center and r is the radius.

$(x-1)^2 + (y-1)^2 + (z-1)^2 = (\sqrt{3})^2$

$x^2 - 2x + y^2 - 2y + z^2 - 2z = 0$

b) $x^2-4x+y^2+4y+z^2=8$

$x^2-4x+4+y^2+4y+4+z^2=8+4+4$

$(x-2)^2+(y+2)^2+z^2=16$

center $(2,-2,0)$ radius 4

$x^2+y^2+z^2-4x-6z-3=0$

$x^2-4x+y^2+z^2-6z=3$

$x^2-4x+4+y^2+z^2-6z+9=3+4+9$

$(x-2)^2+y^2+(z-3)^2=16$

center $(2,0,3)$ radius 4

$x^2+y^2+2y+z^2+4z=20$

$x^2+y^2+2y+1+z^2+4z+4=20+1+4$

$x^2+(y+1)^2+(z+2)^2=25$

center $(0,-1,-2)$ radius 5

prob 5. $x - 2y + z = 10$ $\qquad$ $2x + y - z = 0$

$\vec{n}_1 = \langle 1, -2, 1 \rangle$ $\qquad$ $\vec{n}_2 = \langle 2, 1, -1 \rangle$

The direction vector to the line of intersection is $\vec{u} = \vec{n}_1 \times \vec{n}_2$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \langle 1, 3, 5 \rangle$$

put $z = 0$ we get $x - 2y = 10$

$2x + y = 0$

solve the system $\to$ $x = 2$ $\quad$ $y = -4$

$(2, -4, 0)$ is a point on the line of intersection

$\vec{u} = \langle \underset{a}{1}, \underset{b}{3}, \underset{c}{5} \rangle$ $\qquad$ $(\underset{x_0}{2}, \underset{y_0}{-4}, \underset{z_0}{0})$

The parametric equations are:

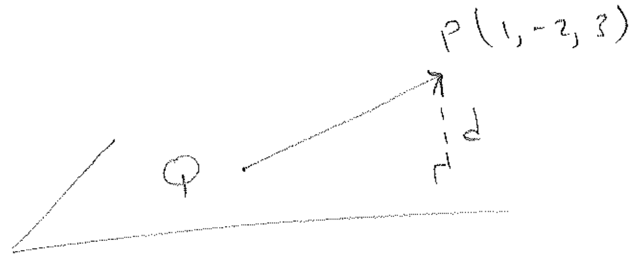
$x = 2 + t$

$y = -4 + 3t$

$z = 0 + 5t$

$= 5t$

Prob 6.



$2x - 2y + z = 5$

$\vec{n} = \langle 2, -2, 1 \rangle$

put $x = y = 0 \Rightarrow z = 5$

$Q(0, 0, 5)$ is a point in the plane.

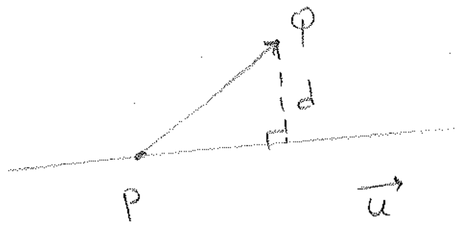
$$d = \frac{|\vec{QP} \cdot \vec{n}|}{\|\vec{n}\|}$$

$\vec{QP} = \langle 1, -2, 3 \rangle$

$$d = \frac{|(1)(2) + (-2)(-2) + (3)(1)|}{\sqrt{(2)^2 + (-2)^2 + (1)^2}}$$

$$= \frac{4}{3}$$

prob 7. $Q(3, -1, 4)$ $x = -2 + 3t$ $y = -2t$ $z = 1 + 4t$



$$d = \frac{\|\vec{PQ} \times \vec{u}\|}{\|\vec{u}\|}$$

let $t = 0 \Rightarrow P(-2, 0, 1)$ is a point on the line

$\overrightarrow{PQ} = \langle 5, -1, 3 \rangle \qquad \vec{u} = \langle 3, -2, 4 \rangle$

$$\overrightarrow{PQ} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix}$$

$$= \langle 2, -11, -7 \rangle$$

$$d = \frac{\sqrt{174}}{\sqrt{29}} \approx 2.45$$