Autumn 2022

Math 141 Midterm

Part 1 MCQ 30% (circle your choice)

(6pts)Problem 1

Find the value of c for which the function

$$f(x) = \begin{cases} c^2 + cx^2, & x < 2 \\ 6c, & x = 2 \\ cx + x^3, & x > 2 \end{cases}$$

is continuous at $x = 2$.

$(A)\ \ c = -4 \qquad (B)\ c = 4 \qquad (C)\ \ c = -2 \qquad (D)\ \ c = 2 \qquad (E)\ \ c = 6$

Solution

The function f is continuous at $x = 2$ if and only if

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2).$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (c^2 + cx^2) = c^2 + 4c$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (cx + x^3) = 2c + 8$$
$$f(2) = 6c.$$

The condition of continuity gives

$$c^2 + 4c = 2c + 8 = 6c \Rightarrow c = 2$$

$$Ans:\ (D)$$

(6pts)Problem 2

If $f(x) = \dfrac{xe^x}{x^2 + e^x}$, then $f'(0) =$

$(A)\quad 1 \qquad (B)\ \ e \qquad (C)\quad 0 \qquad (D)\ \ e^{-1} \qquad (E)\quad e^2$

Solution

Here you apply the product and quotient rule to get

$$f'(x) = \frac{1}{(e^x + x^2)^2}\left(x^3e^x - x^2e^x + e^{2x}\right)$$
$$f'(0) = 1.$$

$$Ans:\ (A)$$

(6pts)Problem 3

$$\lim_{x\to-\infty}\frac{\sqrt{4x^6-x}}{x^3+2}$$

$(A)\ \dfrac{1}{2}\qquad (B)\quad 2\qquad (C)\quad 1\qquad (D)\quad -\infty\qquad (E)\quad -2$

Solution

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{4x^6-x}}{x^3+2} &= \lim_{x\to-\infty}\frac{\sqrt{4x^6}}{x^3}\\ &= \lim_{x\to-\infty}\frac{2\left|x^3\right|}{x^3}\\ &= \lim_{x\to-\infty}\frac{-2\cancel{x^3}}{\cancel{x^3}} = -2.\end{aligned}$$

$$Ans:(E)$$

(6pts)Problem 4

If $F(x)=\displaystyle\int_{\sqrt{x}}^{x}(1+t^2)\,dt,$ then $F'(1)$ is equal to

$(A)\ \ \frac{1}{2}\qquad (B)\quad 2\qquad (C)\quad 1\qquad (D)\ \ \frac{3}{2}\qquad (E)\ \ \frac{-1}{2}$

Solution

$$F'(x)=\left(1+x^2\right)-\frac{1}{2\sqrt{x}}(1+x)$$

$$F'(1)=1.$$

$$Ans:(C)$$

(6pts)Problem 5
The length of a rectangle of constant area 800 square millimeters is increasing at the rate of 4 millmeters per second. What is the width of the rectangle at the moment the width is decreasing at the rate of 0.5 millimeter per second ?

$$(A)\ \ 15 \qquad (B)\ 50 \qquad (C)\ \ 80 \qquad (D)\ \ 10 \qquad (E)\ \ 20$$

Solution
The area

$$800 = \ell w$$

Differentiating,

$$0 = \ell \frac{dw}{dt} + w \frac{d\ell}{dt}.$$

We are given $\frac{d\ell}{dt} = 4$, So

$$0 = \ell \frac{dw}{dt} + 4w.$$

When $\frac{dw}{dt} = -0.5$,

$$0 = -0.5\ell + 4w.$$

Combining this with

$$\ell = \frac{800}{w},$$

we obtain

$$w = 10$$

$$Ans:\ (D)$$

Part 2 Written 70%

$(12pts)$**Problem 1**
Find the equations for the tangent and normal at the point $P(1,2)$ for

$$x^2 - xy + y^2 = 3.$$

Solution
An implicit differentiation gives

$$2x - y - xy' + 2yy' = 0$$

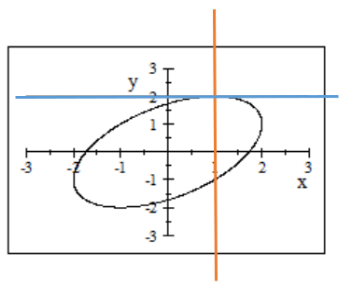
$$y'(2y - x) = y - 2x$$

$$y' = \frac{y - 2x}{2y - x} \qquad [\mathbf{6\ points}]$$

The slope at $x = 1$ and $y = 2$ is

$$m = \frac{2 - 2}{4 - 1} = 0. \qquad [2\ \mathbf{points}]$$

Thus the equation of the tangent line is $y = 2$ and the equation of the normal line is $x = 1$. [4 **points**]



$(10pts)$**Problem 2**
Find the absolute extrema of

$$f(x) = \frac{x^2}{16} + \frac{1}{x} \quad \text{on the interval } [1,\ 4].$$

Solution

$$\begin{aligned} f'(x) &= \frac{1}{8}x - \frac{1}{x^2} \qquad && [2 \textbf{ points}] \\ &= \frac{1}{8x^2}\left(x^3 - 8\right) \end{aligned}$$

$$f'(x) = 0 \Leftrightarrow x^3 - 8 = 0 \Leftrightarrow x = 2.$$

$x = 2$ is the only critical number. [2 **points**]

$$f(1) = \frac{1}{16} + \frac{1}{1} = \frac{17}{16} = 1.0625$$

$$f(4) = \frac{16}{16} + \frac{1}{4} = \frac{5}{4} = 1.25$$

$$f(2) = \frac{4}{16} + \frac{1}{2} = \frac{3}{4} = 0.75$$

Absolute Maximum $= \frac{5}{4} = 1.25$ [3 **points**]

Absolute Minimum $= \frac{3}{4} = 0.75$ [3 **points**]

(12pts)**Problem 3**

Find the open interval(s) where the function $f(x)$ is concave upward and the open interval(s) where the function $f(x)$ is concave downward.

$$f(x) = x^4 - 4x^3.$$

Solution

$$f'(x) = 4x^3 - 12x^2 \qquad [2 \textbf{ points}]$$

$$\begin{aligned} f''(x) &= 12x^2 - 24x \\ &= 12x(x-2). \qquad [4 \textbf{ points}] \end{aligned}$$

x	$-\infty$	0		2	$+\infty$
$f'' = 12x^2 - 24x$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$		$\cap$		$\cup$

[4 **points**]

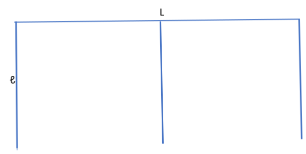
Concave up on $(-\infty,\ 0) \cup (2,\ \infty)$ [2 **points**]
Concave down on $(0,\ 2)$. [2 **points**]

(12pts)**Problem 4**

A rancher is going to build a 3-sided cattle enclosure with a divider down the middle as shown below.

The cost per foot of the three side walls will be \$6/foot, while the back wall, being taller, will be \$10/foot. If the rancher wishes to enclose an area of 180 ft^2, what dimensions of the enclosure will minimize his cost?

Solution



$$\text{Area } = \ell L \Leftrightarrow \ell L = 180$$

$$L = \frac{180}{\ell}, \quad \ell > 0. \qquad [2 \textbf{ points}]$$

$$\text{Cost} = \; C = (6)(3\ell) + 10L \qquad [2 \textbf{ points}]$$

$$C = 18\ell + \frac{1800}{\ell}$$

$$C' = 18 - \frac{1800}{\ell^2} \qquad [2 \textbf{ points}]$$

$$18 - \frac{1800}{\ell^2} = 0 \Rightarrow \ell = 10$$

$$C'' = \frac{3600}{\ell^3} > 0 \Rightarrow C \text{ is minimized at } \ell = 10. \qquad [2 \textbf{ points}]$$

The dimensions of the enclosure will minimize his cost are

$$\ell = 10 \text{ and } L = \frac{180}{10} = 18. \qquad [4 \textbf{ points}]$$

(12pts)**Problem 5**
Use definite integrals to evaluate the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\cos\left(\frac{\pi}{2}+i\frac{\pi}{2n}\right).$$

Solution
We will use the formula

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[f\left(a+i\frac{b-a}{n}\right)\right]\frac{b-a}{n}=\int_a^b f(x)dx.$$

$$a=\frac{\pi}{2} \text{ and } \frac{b-\frac{\pi}{2}}{\cancel{n}}=\frac{\pi}{2\cancel{n}}\Rightarrow b=\frac{\pi}{2}+\frac{\pi}{2}=\pi \qquad [4 \textbf{ points}]$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\cos\left(\frac{\pi}{2}+i\frac{\pi}{2n}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\left[2\cos\left(\frac{\pi}{2}+i\frac{\pi}{2n}\right)\right]\frac{\pi}{2n}$$

$$f(x)=2\cos x. \qquad [4 \textbf{ points}]$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\cos\left(\frac{\pi}{2}+i\frac{\pi}{2n}\right) &= \int_{\frac{\pi}{2}}^{\pi}2\cos x dx\\ &= -2. \qquad [4 \textbf{ points}]\end{aligned}$$

$(12pts)$**Problem 6**
Evaluate

$$(a)\ \int x^2\sqrt[3]{x^3+5}dx, \qquad (b)\ \int \frac{\sin^3 x}{\cos x}dx$$

Solution
(a)

$$\int x^2\sqrt[3]{x^3+5}dx = \int \sqrt[3]{x^3+5}x^2dx$$

Put

$$u = x^3+5, \quad du = 3x^2dx \Rightarrow x^2dx = \frac{du}{3}. \qquad [3\ \textbf{points}]$$

The integral becomes

$$\begin{aligned}\frac{1}{3}\int u^{\frac{1}{3}}du &= \frac{1}{4}u^{\frac{4}{3}}+C\\ &= \frac{1}{4}\left(x^3+5\right)^{\frac{4}{3}}+C \qquad [3\ \textbf{points}]\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{\sin^3 x}{\cos x}dx &= \int \frac{\sin^2 x}{\cos x}\sin x dx\\ &= \int \frac{1-\cos^2 x}{\cos x}\sin x dx. \qquad [2\ \textbf{points}]\end{aligned}$$

Now put

$$u=\cos x, \qquad du = -\sin x dx \Leftrightarrow \sin x dx = -du. \qquad [2\ \textbf{points}]$$

The integral becomes

$$\begin{aligned}\int \frac{\sin^3 x}{\cos x}dx &= \int \frac{u^2-1}{u}du\\ &= \int \left(u-\frac{1}{u}\right)du\\ &= \frac{1}{2}u^2 - \ln|u| + C\\ &= \frac{1}{2}\cos^2 x - \ln|\cos x| + C \qquad [2\ \textbf{points}]\end{aligned}$$