
SOLUTION

($8pts$) **Problem 1**
Evaluate the following limits

$$(a)\quad \lim_{x\to 0}\frac{\tan x - x}{x^2} \qquad\qquad (b)\quad \lim_{x\to -\infty}\frac{-3x^2\,|x| + 5x + 1}{5x^3 + 2x^2 - 9}$$

Solution

(a)

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^2} &= \lim_{x\to 0}\frac{\sec^2 x - 1}{2x}\\ &= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{2}\\ &= 0. \qquad\qquad [\textbf{4 points}]\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to -\infty}\frac{-3x^2\,|x| + 5x + 1}{5x^3 + 2x^2 - 9} &= \lim_{x\to -\infty}\frac{-3x^2\,|x|}{5x^3}\\ &= \lim_{x\to -\infty}\frac{3\cancel{x^3}}{5\cancel{x^3}}\\ &= \frac{3}{5}. \qquad\qquad [\textbf{4 points}]\end{aligned}$$

$(8pts)$**Problem 2**

Find $\dfrac{dy}{dx}$ for

$$(a) \quad y = \ln\left[\frac{\sqrt[3]{2x+1}}{(2x-1)(x+3)}\right] \qquad\qquad (b) \;\; e^x y^2 + y^5 = 5$$

(a)

$$\begin{aligned} y &= \ln\left[\frac{\sqrt[3]{2x+1}}{(2x-1)(x+3)}\right] = \ln(2x+1)^{\frac{1}{3}} - \ln[(2x-1)(x+3)] \\ &= \frac{1}{3}\ln(2x+1) - \ln(2x-1) - \ln(x+3) \\ & \frac{dy}{dx} = \frac{2}{3(2x+1)} - \frac{2}{2x-1} - \frac{1}{x+3} \qquad [\textbf{4 points}] \end{aligned}$$

(b)

$$e^x y^2 + y^5 = 5$$

$$e^x (y)^2 + 2e^x y' y + 5y' (y)^4 = 0$$

$$\frac{dy}{dx} = y' = \frac{-y^2 e^x}{2ye^x + 5y^4} = \frac{-ye^x}{2e^x + 5y^3} \qquad [\textbf{4 points}]$$

$(8pts)$ **Problem 3**
Find the absolute extrema of the function $f(x) = 4x^3 - 6x^2 - 9x$, on the interval $[-1,\ 2]$.

Solution

$$\begin{aligned} f'(x) &= 12x^2 - 12x - 9 \\ &= 3(2x+1)(2x-3) \end{aligned}$$

The critical numbers are

$$\boxed{x = -\frac{1}{2}},\ \boxed{x = \frac{3}{2}} \qquad [\mathbf{4\ points}]$$

$$f(-1) = 4(-1)^3 - 6(-1)^2 - 9(-1) = -1$$

$$f(2) = 4(2)^3 - 6(2)^2 - 9(2) = -10$$

$$f(-\frac{1}{2}) = 4\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right)^2 - 9\left(-\frac{1}{2}\right) = 2.5$$

$$f(\frac{3}{2}) = 4\left(\frac{3}{2}\right)^3 - 6\left(\frac{3}{2}\right)^2 - 9\left(\frac{3}{2}\right) = -13.5$$

$$\begin{aligned} \text{Absolute Min} &= -13.5 \qquad [\mathbf{2\ points}] \\ \text{Absolute Max} &= 2.5 \qquad [\mathbf{2\ points}] \end{aligned}$$

($8pts$)**Problem 4**
Use definite integrals to evaluate

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\left(1+\frac{3i}{n}\right)^{\frac{3}{2}}$$

Solution
We have

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[f\left(a+i\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)=\int_a^b f(x)dx$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\left(1+\frac{3i}{n}\right)^{\frac{3}{2}}=?$$

$$a=1 \quad \text{and}\ \frac{b-(1)}{\cancel{n}}=\frac{3}{\cancel{n}}\Rightarrow b=3+1=4 \qquad [\textbf{2 points}]$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}2\left(1+\frac{3i}{n}\right)^{\frac{3}{2}}\frac{3}{n}\Rightarrow f(x)=2x^{\frac{3}{2}}. \qquad [\textbf{2 points}]$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\left(1+\frac{3i}{n}\right)^{\frac{3}{2}} &= \int_1^4 2x^{\frac{3}{2}}dx\\ &= 2\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\Bigg|_1^4\\ &= 24.8 \qquad [4\ \textbf{points}]\end{aligned}$$

$(8pts)$**Problem 5**

Find the critical numbers and the local extrema of the function

$$F(x) = \int_0^{x^2} \left(1 - t^2\right) dt$$

Solution

$$\begin{aligned} F'(x) &= 2x\left(1-x^4\right) \\ &= 2x\left(1-x^2\right)\left(1+x^2\right) \\ &= -2x\,(x-1)\,(x+1)\left(x^2+1\right) \end{aligned}$$

The critical numbers are

$$[x=0]\,,\quad [x=1]\ \text{ and }\ [x=-1] \qquad [3\ \textbf{points}]$$

x	$-\infty$		-1		0		1		$+\infty$
$2x$		$-$		$-$	0	$+$		$+$	
$1-x^2$		$-$	0	$+$		$+$	0	$-$	
$1+x^2$		$+$		$+$		$+$		$+$	
$F'(x)$		$+$		$-$		$+$		$-$	
$F(x)$		$\nearrow$	$F(-1)$	$\searrow$	$F(0)$	$\nearrow$	$F(1)$	$\searrow$	

[1 **point**]

$$\begin{aligned} F(-1) &= \int_0^1 \left(1-t^2\right) dt = 0.666\,67 \\ F(0) &= 0. \\ F(1) &= \int_0^1 \left(1-t^2\right) dt = 0.666\,67 \end{aligned}$$

Local Max $= 0.666\,67$ and Local Min $= 0.$ $\qquad$ [4 **points**]