$$\frac{d^{999}}{dx^{999}}(\cos x)$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d^2}{dx^2}(\cos x) = -\cos x$$

$$\frac{d^3}{dx^3}(\cos x) = \sin x$$

$$\frac{d^4}{dx^4}(\cos x) = \cos x$$

$$\frac{d^{996}}{dx^{996}}(\cos x) = \cos x$$

$$999 = 996 + 3$$
$$= 4(249) + 3$$

$$\frac{d^{999}}{d^{999}} = \frac{d^3}{dx^3}\frac{d^{996}}{dx^{996}}(\cos x)$$

$$= \frac{d^3}{dx^3}\cos x$$

$$= \underline{\sin x}$$

$$\frac{d}{dx}e^{-x} = -e^{-x}$$

$$\frac{d^2}{dx^2}e^{-x} = e^{-x}$$

Implicit Differentiation

Consider function given by $yx\cos x + 3 = x^2$

Find $\frac{dy}{dx}$

$$y = \frac{x^2 - 3}{x\cos x}$$

y is given explicitly as a function of x

$$xy^3 + y^2 = e^x$$

y is given implicitly as a function of x

$$\frac{d}{dx}(y^n) = ny'y^{n-1}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

To perform an implicit differentiation to find $y' = \frac{dy}{dx}$ do the following

1. Differentiate both sides with respect to x, keeping in mind y is a function of x
2. Keep all terms with y' on the left side and move all others terms to the right
3. Factor and solve for y'

$$xy^3 + y^2 = e^x$$

$$y^3 + x\,3y^2 y' + 2yy' = e^x$$

$$x\,3y^2 y' + 2yy' = e^x - y^3$$

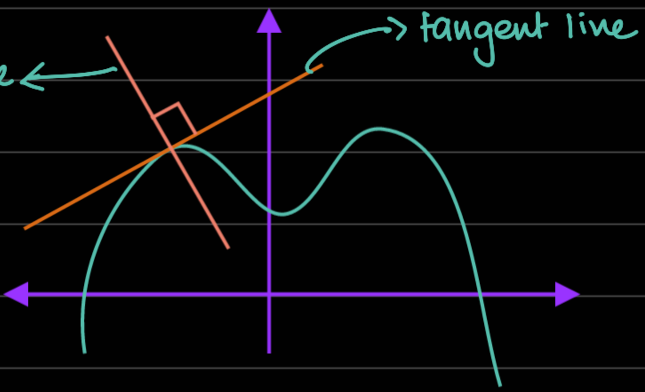
$$y'(3xy^2 + 2y) = e^x - y^3$$

$$\boxed{y' = \frac{e^x - y^3}{3xy^2 + 2y}}$$

Example

Find the slope of the normal line to the curve given by $y^3 + y^2 + 5\cos y - x^2 = 4$ at the point $(1, 0)$

line perpendicular to tangent line



$$y^3 + y^2 + 5\cos y - x^2 = 4$$

$$3y^2 y' + 2y y' - 5 y' \sin y - 2x = 0$$

$$y'(3y^2 + 2y - 5\sin y) = 2x$$

$$y' = \frac{2x}{3y^2 + 2y - 5\sin y}$$

@ $(1, 0)$

$$m = \frac{2(1)}{3(0)^2 + 2(0) - 5\sin 0)}$$

$$m = \frac{2}{0} = \infty \Rightarrow \text{Vertical line}$$

Slope $= \infty \Rightarrow$ Vertical

Slope $= 0 \Rightarrow$ Horizontal

Equation of tangent line $x = 1$

$\therefore$ Equation of normal line $y = 0$

Example

Find the slope of the curve given by $3(x^2 + y^2)^2 = 100xy$ at the point $(3, 1)$

$$6(x^2 + y^2)(2x + 2y y') = 100(y + xy')$$

$$6(x^2 + y^2) 2y y' - 100 xy' = 100y - 6(x^2 + y^2)(2x)$$

$$y'(6(x^2 + y^2)2y - 100x) = 100y - 6(x^2 + y^2)(2x)$$

$$y' = \frac{100y - 6(x^2 + y^2)(2x)}{6(x^2 + y^2)2y - 100x}$$

@ $(3, 1)$

$$m = \frac{100 - 6(9 + 1)(6)}{6(9 + 1)(2) - 300} = \frac{13}{9}$$

Example

Find $\frac{d^2}{dx^2}$ at the point $(0,2)$ $\quad 3x^3 - 2y^2 = 8$

$$9x^2 - 4y'y = 0$$

$$y' = \frac{9x^2}{4y}$$

@ $(0,2)$ $\quad y' = \frac{0}{8} = \underline{0}$

$\frac{d}{dx}$ again

$$18x - 4y''y - 4y'y' = 0$$

$$18x - 4y''y - 4(y')^2 = 0$$

$$18x - 4(y')^2 = 4y\,y''$$

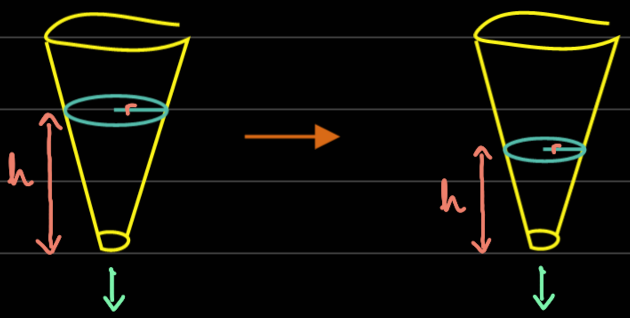
$$y'' = \frac{18x - 4(y')^2}{4y}$$

$$\frac{18x - 4(y')^2}{4y} = y''$$

@ $(0,2)$

$$y'' = \frac{0 - 4(0)^2}{8} = \underline{0}$$

Related Rates



$h = h(t)$

$r = r(t)$

$V = V(t)$

$$V = \frac{1}{3}\pi r^2 h$$

Given $\frac{dV}{dt}$, $\frac{dh}{dt}$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right] \Rightarrow \text{Related Rate Equation}$$

Example

Suppose x and y are both differentiable functions of t and are related by the equation $xy + x^2 = 3$

Find $\frac{dy}{dt}$ when $x = 1$ given that $\frac{dx}{dt} = 2$ when $x = 1$.

$$x\frac{dy}{dt} + \frac{dx}{dt}y + 2x\frac{dx}{dt} = 0$$

$$xy + x^2 = 3$$
$$y + 1 = 3$$
$$y = 2$$

@ $x = 1$

$$\frac{dy}{dt} + 4 + 2(1)(2) = 0$$

$$\frac{dy}{dt} = \underline{-8}$$

Example

Air is being pumped into a spherical balloon at a rate of 4.5 cubic feet/min

Find the rate of change of the radius when $r = 2ft$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4.5 \text{ cubic ft/min}$$

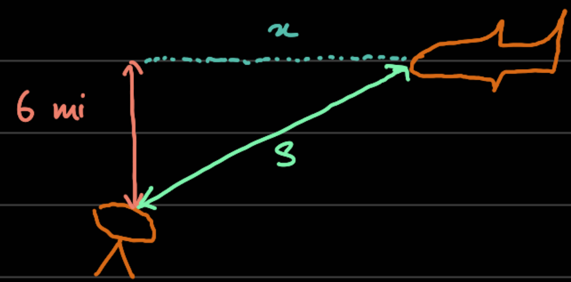
$$\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt}$$

$$\frac{9}{2} = \frac{4}{\cancel{3}}\pi \times \cancel{3}(2)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{9}{32\pi} \text{ ft/min}$$

Example

An airplane is flying on a flight path that will take it directly over a radar tracking station as shown below.



If s is decreasing at a rate of 400 mi/hr when s is 10 miles what is the speed of the plane?

$$s^2 = x^2 + 6^2 \qquad \frac{ds}{dt} = -400 \text{ mi/hr} \qquad s = 10 \text{ mi} \qquad h = 6 \text{ mi}$$

$$\cancel{2}s \frac{ds}{dt} = \cancel{2}x \frac{dx}{dt} + 0$$

$$s \frac{ds}{dt} = x \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$$

$$x = \sqrt{10^2 - 6^2}$$
$$= 8$$

$$= \frac{10}{8} (-400)$$

$$= \frac{-4000}{8}$$

$$= -500 \text{ mi/hr}$$

The L'Hospital Rule

If $\lim\limits_{\substack{x \to c \\ x \to \pm\infty}} \frac{f(x)}{g(x)}$ produces indeterminate of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$ or $\frac{\infty}{-\infty}$ or $\frac{-\infty}{-\infty}$ then

$$\lim_{\substack{x \to c \\ x \to \pm\infty}} \frac{f(x)}{g(x)} = \lim_{\substack{x \to c \\ x \to \pm\infty}} \frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

Example

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \frac{0}{0}$$

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{2\sqrt{x}}$$