Math 141

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Eigenvalues and EigenVectors

Definition

Suppose that $A = (a_{ij})_{ij}$ is an $n \times n$ matrix with entries in $\mathbb{R}$. Suppose further that there exist a number $\lambda \in \mathbb{R}$ and a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Then we say that λ is an eigenvalue of the matrix A, and that $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue λ.

- Suppose that λ is an eigenvalue of the n $\times$ n matrix A, and that **v** is an eigenvector corresponding to the eigenvalue λ. Then

$$A\mathbf{v} = \lambda\mathbf{v} = \lambda I\mathbf{v},$$

where I is the n $\times$ n identity matrix, so that

$$(A - \lambda I)\mathbf{v} = 0.$$

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where I is the n $\times$ n identity matrix, so that

$$(A - \lambda I)\mathbf{v} = 0.$$

- Since $\mathbf{v} \in \mathbb{R}^n$ is non-zero, it follows that we must have

$$\det(A - \lambda I) = 0$$

- In other words, we must have

$$\det\begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{pmatrix} = 0.$$

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- Note that $\det(A - \lambda I) = 0$ is a polynomial equation in λ. Solving this equation gives the eigenvalues of the matrix A.

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- Note that $\det(A - \lambda I) = 0$ is a polynomial equation in λ. Solving this equation gives the eigenvalues of the matrix A.
- On the other hand, for any eigenvalue λ of the matrix A, the set

$$\{\mathbf{v} \in \mathbb{R}^n : (A - \lambda I)\mathbf{v} = \mathbf{0}\}$$

is the nullspace of the matrix $A - \lambda I$, a subspace of $\mathbb{R}^n$.

Definition

The polynomial $P(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of the matrix A. For any root λ of $P(\lambda)$, the space

$$\{\mathbf{v} \in \mathbb{R}^n : (A - \lambda I)\mathbf{v} = \mathbf{0}\}$$

is called the eigenspace corresponding to the eigenvalue λ.

- **Example 7** The matrix

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- in other words,

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Hence the eigenvalues are

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- has characteristic polynomial
- $$(3 - \lambda)(5 - \lambda) - 3 = 0;$$
- in other words,

$$\lambda^2 - 8\lambda + 12 = 0.$$

Hence the eigenvalues are
- $\lambda_1 = 2$ and $\lambda_2 = 6$, with corresponding eigenvectors

- Substituting $\lambda = 2$ into

$$(A - \lambda I)\mathbf{v} = 0,$$

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- we obtain

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with root} \qquad \mathbf{v}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

- Substituting $\lambda = 6$ into

$$(A - \lambda I)\mathbf{v} = 0,$$

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with root} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- Substituting $\lambda = 6$ into

$$(A - \lambda I)\mathbf{v} = 0,$$

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with root} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

respectively.

- The eigenspace corresponding to the eigenvalue 2 is

$$\left\{\mathbf{v} \in \mathbb{R}^2 : \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \mathbf{v} = \mathbf{0}\right\} = \left\{c \begin{pmatrix} 3 \\ -1 \end{pmatrix} : c \in \mathbb{R}\right\}.$$

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- The eigenspace corresponding to the eigenvalue 6 is

$$\left\{ \mathbf{v} \in \mathbb{R}^2 : \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0} \right\} = \left\{ c \begin{pmatrix} 1 \\ 1 \end{pmatrix} : c \in \mathbb{R} \right\}.$$

- Consider the matrix

$$A = \begin{pmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{pmatrix}.$$

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$$A = \begin{pmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{pmatrix}.$$

- To find the eigenvalues of A, we need to find the roots of

$$\det \begin{pmatrix} -1-\lambda & 6 & -12 \\ 0 & -13-\lambda & 30 \\ 0 & -9 & 20-\lambda \end{pmatrix} = 0;$$

- in other words,
$$(\lambda+1)(\lambda-2)(\lambda-5)=0.$$

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$$(\lambda + 1)(\lambda - 2)(\lambda - 5) = 0.$$
- The eigenvalues are therefore $\lambda_1 = -1, \lambda_2 = 2$ and $\lambda_3 = 5$.

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$$(A + I)\mathbf{v} = \begin{pmatrix} 0 & 6 & -12 \\ 0 & -12 & 30 \\ 0 & -9 & 21 \end{pmatrix} \mathbf{v} = \mathbf{0}, \quad \text{with root} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

- An eigenvector corresponding to the eigenvalue 2 is a solution of the system

$$(A-2I)\mathbf{v} = \begin{pmatrix} -3 & 6 & -12 \\ 0 & -15 & 30 \\ 0 & -9 & 18 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with root} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

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$$(A-2I)\mathbf{v} = \begin{pmatrix} -3 & 6 & -12 \\ 0 & -15 & 30 \\ 0 & -9 & 18 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with root} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

- An eigenvector corresponding to the eigenvalue 5 is a solution of the system

$$(A-5I)\mathbf{v} = \begin{pmatrix} -6 & 6 & -12 \\ 0 & -18 & 30 \\ 0 & -9 & 15 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with root} \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}.$$

- **Example 8** Consider the matrix

$$A = \begin{pmatrix} 17 & -10 & -5 \\ 45 & -28 & -15 \\ -30 & 20 & 12 \end{pmatrix}.$$

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- To find the eigenvalues of A, we need to find the roots of

$$\det \begin{pmatrix} 17-\lambda & -10 & -5 \\ 45 & -28-\lambda & -15 \\ -30 & 20 & 12-\lambda \end{pmatrix} = 0;$$

- in other words,

$$(\lambda + 3)(\lambda - 2)^2 = 0.$$

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$$(\lambda+3)(\lambda-2)^2=0.$$
- The eigenvalues are therefore $\lambda_1=-3$ and $\lambda_2=2$. An eigenvector corresponding to the eigenvalue -3 is a solution of the system

$$(A+3I)\mathbf{v}=\begin{pmatrix} 20 & -10 & -5 \\ 45 & -25 & -15 \\ -30 & 20 & 15 \end{pmatrix}\mathbf{v}=\mathbf{0}, \quad \text{with root} \quad \mathbf{v}_1=\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}.$$

- An eigenvector corresponding to the eigenvalue 2 is a solution of the system

$$(A-2I)\mathbf{v} = \begin{pmatrix} 15 & -10 & -5 \\ 45 & -30 & -15 \\ -30 & 20 & 10 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with roots} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

- An eigenvector corresponding to the eigenvalue 2 is a solution of the system

$$(A-2I)\mathbf{v} = \begin{pmatrix} 15 & -10 & -5 \\ 45 & -30 & -15 \\ -30 & 20 & 10 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with roots} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

- Note that the eigenspace corresponding to the eigenvalue -3 is a line through the origin, while the eigenspace corresponding to the eigenvalue 2 is a plane through the origin.

- **Example 9** Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

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$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

- To find the eigenvalues of A, we need to find the roots of

$$\det \begin{pmatrix} 2-\lambda & -1 & 0 \\ 1 & 0-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0;$$

- in other words,

$$(\lambda - 3)(\lambda - 1)^2 = 0.$$

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- The eigenvalues are therefore $\lambda_1 = 3$ and $\lambda_2 = 1$.

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- The eigenvalues are therefore $\lambda_1 = 3$ and $\lambda_2 = 1$.
- An eigenvector corresponding to the eigenvalue 3 is a solution of the system

$$(A - 3I)\mathbf{v} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0}, \quad \text{with root} \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- An eigenvector corresponding to the eigenvalue 1 is a solution of the system

$$(A - I)\mathbf{v} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with root} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

- An eigenvector corresponding to the eigenvalue 1 is a solution of the system

$$(A - I)\mathbf{v} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0}, \quad \text{with root} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

- Note that the eigenspace corresponding to the eigenvalue 3 is a line through the origin.

- An eigenvector corresponding to the eigenvalue 1 is a solution of the system

$$(A - I)\mathbf{v} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0}, \qquad \text{with root} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

- Note that the eigenspace corresponding to the eigenvalue 3 is a line through the origin.
- On the other hand, the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

- **Example 9** Consider the matrix

$$A = \begin{pmatrix} 3 & -3 & 2 \\ 1 & -1 & 2 \\ 1 & -3 & 4 \end{pmatrix}.$$

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- To find the eigenvalues of A, we need to find the roots of

$$\det \begin{pmatrix} 3-\lambda & -3 & 2 \\ 1 & -1-\lambda & 2 \\ 1 & -3 & 4-\lambda \end{pmatrix} = 0;$$

- in other words,
$$(\lambda - 2)^3 = 0.$$

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- The eigenvalue is therefore $\lambda = 2$. An eigenvector corresponding to the eigenvalue 2 is a solution of the system

$$(A - 2I)\mathbf{v} = \begin{pmatrix} 1 & -3 & 2 \\ 1 & -3 & 2 \\ 1 & -3 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0}, \quad \text{with roots} \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Remark

- Suppose that λ is an eigenvalue of a matrix A, with corresponding eigenvector **v**. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

Remark

- Suppose that λ is an eigenvalue of a matrix A, with corresponding eigenvector $\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

- Hence λ^2 is an eigenvalue of the matrix A^2, with corresponding eigenvector $\mathbf{v}$.

Remark

- Suppose that λ is an eigenvalue of a matrix A, with corresponding eigenvector $\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

- Hence λ^2 is an eigenvalue of the matrix A^2, with corresponding eigenvector $\mathbf{v}$.
- In fact, it can be proved by induction that for every natural number k $\in \mathbb{N}$,λ^k is an eigenvalue of the matrix A^k, with corresponding eigenvector $\mathbf{v}$.