Problem 1.
Use Gauss elimination method to solve the linear system

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0. \end{aligned}$$

<u>Solution</u>
The augmented matrix is

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 2 & 1 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 2 & 1 & -17 \\ 0 & 3 & 1 & -27 \end{bmatrix}$$

Multiply the second row by $\dfrac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 1 & \frac{1}{2} & \frac{-17}{2} \\ 0 & 3 & 1 & -27 \end{bmatrix}$$

Add -3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 1 & \frac{1}{2} & \frac{-17}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{-3}{2} \end{bmatrix}$$

Multiply the third row by -2 to obtain

$$\begin{bmatrix} 1 & 1 & -2 & 9 \\ 0 & 1 & \frac{1}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \textbf{(4pts)}$$

Now doing the back substitution, we get

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 9 \\ x_2 + \tfrac{1}{2}x_3 &= \tfrac{-17}{2} \\ x_3 &= 3. \end{aligned} \qquad \textbf{(1pt)}$$

Thus,

$$x_2 = \frac{-17}{2} - \frac{1}{2}(3) = -10 \text{ and } x_1 = 9 - (-10) + 2(3) = 25$$

$$x_1 = 25, \qquad x_2 = -10 \text{ and } \qquad x_3 = 3. \qquad \textbf{(3pts)}$$

Problem 2.
(a) Let

$$A = \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix},\ B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -6 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix}.$$

Perform the following calculations

1. $A + B$ 2. $C + B^t$ 3. $-2AC$

<u>**Solution**</u>
1.
$$A + B = \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ -2 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 6 & -2 \\ 4 & -3 & 5 \end{bmatrix}$$

2.
$$C + B^t = \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & -6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 6 & -6 \\ -1 & 7 \end{bmatrix}$$

3.
$$-2AC = (-2) \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -48 & 4 \\ -48 & -48 \end{bmatrix}$$

(b) Solve the given vector equation for x, or explain why no solution exists:

$$2\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}.$$

<u>**Solution**</u>

$$(2)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix} - (3)\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}$$

$\Leftrightarrow$
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & 4 - 3x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}$$

This equation is valid only if
$$4 - 3x = -2 \Longrightarrow x = 2.$$

Problem 3.
Use the Gauss Jordan method to find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}.$$

<u>**Solution**</u>
To find A^{-1} we consider the array

$$(A \mid I_3) = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We now perform elementary row operations on this array and try to reduce the left hand half to the matrix I_3.
Note that if we succeed, then the final array is clearly in reduced row echelon form.
We therefore follow the same procedure as reducing an array to reduced row echelon form.
Adding -3 times row 1 to row 2, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Adding 2 times row 1 to row 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 5 & 4 & 2 & 0 & 1 \end{pmatrix}.$$

Multiplying row 3 by 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 15 & 12 & 6 & 0 & 3 \end{pmatrix}.$$

Adding 5 times row 2 to row 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Multiplying row 1 by 3, we obtain

$$\begin{pmatrix} 3 & 3 & 6 & 3 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Adding 2 times row 3 to row 1, we obtain

$$\begin{pmatrix} 3 & 3 & 0 & -15 & 10 & 6 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Adding -1 times row 3 to row 2, we obtain

$$\begin{pmatrix} 3 & 3 & 0 & -15 & 10 & 6 \\ 0 & -3 & 0 & 6 & -4 & -3 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Adding 1 times row 2 to row 1, we obtain

$$\begin{pmatrix} 3 & 0 & 0 & -9 & 6 & 3 \\ 0 & -3 & 0 & 6 & -4 & -3 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Multiplying row 1 by 1/3, row 2 by -1/3 and row 3 by -1/3, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & 1 & 0 & -2 & 4/3 & 1 \\ 0 & 0 & 1 & 3 & -5/3 & -1 \end{pmatrix}.$$

Note now that the array is in reduced row echelon form, and that the left hand half is the identity matrix I_3.
It follows that the right hand half of the array represents the inverse A^{-1}. Hence

$$A^{-1} = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 4/3 & 1 \\ 3 & -5/3 & -1 \end{pmatrix}.$$

Problem 4.
(a) Let $\mathbf{u}$ and $\mathbf{v}$ be two unit vectors. If $\theta = \dfrac{\pi}{4}$ is the angle between $\mathbf{u}$ and $\mathbf{v}$, find

$$\left\| \mathbf{u} - \sqrt{2}\mathbf{v} \right\|.$$

__**Solution**__

$$\begin{aligned}
\left\| \mathbf{u} - \sqrt{2}\mathbf{v} \right\|^2 &= \left(\mathbf{u} - \sqrt{2}\mathbf{v}\right) \cdot \left(\mathbf{u} - \sqrt{2}\mathbf{v}\right) \\
&= (\mathbf{u} \cdot \mathbf{u}) - \left(\mathbf{u} \cdot \sqrt{2}\mathbf{v}\right) - \left(\sqrt{2}\mathbf{v} \cdot \mathbf{u}\right) + \left(\sqrt{2}\mathbf{v} \cdot \sqrt{2}\mathbf{v}\right) \\
&= \|\mathbf{u}\|^2 - 2\sqrt{2}\,(\mathbf{u} \cdot \mathbf{v}) + \left\|\sqrt{2}\mathbf{v}\right\|^2 \\
&= \|\mathbf{u}\|^2 - 2\sqrt{2}\,\|\mathbf{u}\|\,\|\mathbf{v}\| \cos\frac{\pi}{4} + 2\,\|\mathbf{v}\|^2.
\end{aligned}$$

Now using

$$\|\mathbf{u}\| = \|\mathbf{v}\| = 1 \text{ and } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

we get

$$\begin{aligned}
\left\| \mathbf{u} - \sqrt{2}\mathbf{v} \right\|^2 &= 1 - \left(2\sqrt{2}\right)\left(\frac{1}{\sqrt{2}}\right) + 2 \\
&= 1.
\end{aligned}$$

Hence

$$\left\| \mathbf{u} - \sqrt{2}\mathbf{v} \right\| = 1.$$

(b) Find a set of scalar parametric equations for the line of intersection of the two planes

$$\mathcal{P}_1 : \quad x + y + 3z = 1 \qquad \text{and} \quad \mathcal{P}_2 : \; x - y + 2z = 0.$$

__**Solution**__

$$\mathbf{n}_1 = (1, 1, 3) \quad \text{and} \qquad \mathbf{n}_2 = (1, -1, 2).$$

A direction vector of the line of intersection is given by

$$\mathbf{u} = \mathbf{n}_1 \times \mathbf{n}_2 = (5, 1, -2).$$

To find a point on the line of intersection, we put $z = 0$ and solve for x and y to get

$$\begin{cases} x + y = 1 \\ x - y = 0 \end{cases} \Longrightarrow x = y = \frac{1}{2}.$$

The parametric equations of the line of intersection are

$$x = \frac{1}{2} + 5t, \qquad y = \frac{1}{2} + t, \qquad z = 0 - 2t.$$

Problem 5.
Use cofactor expansion to find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

<u>**Solution**</u>
It will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$\det A = (1)\det\begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

For this 3×3 determinant, it will be easiest to use cofactor expansion along its second column, since it has the most zeros:

$$\begin{aligned} \det A &= (1)(-2)\det\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= (1)(-2)(1+2) \\ &= -6 \end{aligned}$$

Problem 6.
Let A be the matrix given by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Suppose that $\det A = -6$. Find

1. $\det(2A)$
2. $\det(A^3)$
3. $\det\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$
4. $\det\begin{bmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{bmatrix}$

Solution
1. Since A is a 3×3 matrix

$$\begin{aligned} \det(2A) &= 2^3 \det(A) \\ &= (8)(-6) \\ &= -48 \end{aligned}$$

2.

$$\begin{aligned} \det\left(A^3\right) &= \det(AAA) = \det(A)\det(A)\det(A) \\ &= [\det(A)]^3 = (-6)^3 \\ &= -216 \end{aligned}$$

3. The matrix

$$\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$$

is obtained from A by interchanging row 1 and row, then row 2 and row 3.

$$\begin{aligned} \det\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} &= (-1)(-1)\det A \\ &= \det A = -6 \end{aligned}$$

4. The matrix

$$\begin{bmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{bmatrix}$$

is obtained from the matrix A by multiplying the first row of A by (-3) and by adding (-4) times row 2 to row 3.
As a result of these elementary operations, we get

$$\begin{aligned} \det \begin{bmatrix} -3a & -3b & -3c \\ d & e & f \\ g - 4d & h - 4e & i - 4f \end{bmatrix} &= (-3) \det A \\ &= 18. \end{aligned}$$