Properties of Definite Integrals

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. If c is in the interval (a,b) then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Anti Derivatives

$f(x) = 2x$ $\qquad$ $F(x) = x^2$

$= x^2+1$

Anti Derivative

General form x^2+c → c is an arbitrary constant

$$\int 2x\,dx = x^2+c$$

This is an indefinite integral

If $F'(x) = f(x)$

$$\int f(x)\,dx = F(x)+c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int e^x\,dx = e^x + c$$

Properties of Indefinite Integrals

1. $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

2. $\int k\, f(x)\, dx = k \int f(x)\, dx$

3. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$ if $n \neq 1$ POWER RULE

4. $\int x^{-1}\, dx = \ln|x| + c$

5. $\int k\, dx = kx + c$

Example

$$\int (2x^3 - x^2 + 1)\, dx$$

$$\int 2x^3\, dx - \int x^2\, dx + \int 1\, dx$$

$$= \frac{\cancel{2}x^4}{\cancel{4}\,2} - \frac{x^3}{3} + x + c$$

$$= \frac{x^4}{2} - \frac{x^3}{3} + x + c$$

$$\int \left(\frac{x^2 + x + 1}{\sqrt{x}}\right) dx$$

$$\int \frac{x^2}{\sqrt{x}}\, dx + \int \frac{x}{\sqrt{x}}\, dx + \int \frac{1}{\sqrt{x}}\, dx$$

$$= \frac{2x^{5/2}}{5} + \frac{2x^{3/2}}{3} + 2x^{1/2} + c$$

First Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ then the function

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable and

$$F'(x) = f(x)$$

Example

Find $\dfrac{d}{dx} \displaystyle\int_1^{x^2} \cos t \, dt$

Put $F(x) = \displaystyle\int_1^x \cos t \, dt \Rightarrow F'(x) = \cos x$

$$F(x^2) = \int_1^{x^2} \cos t \, dt$$

$$\frac{d}{dx} F(x^2) = 2x \cdot F'(x^2)$$

$$= 2x \cos(x^2)$$

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = v'(x)\, f[v(x)] - u'(x)\, f[u(x)]$$

Example

$$\frac{d}{dx}\int_1^{x^2} \cos t \, dt = 2x \cos x^2 - 0 = 2x \cos x^2$$

$u = 1 \quad v = x^2 \quad f(t)$

Second Fundamental Theorem of Calculus

If f is continuous on $[a,b]$ and F is an antiderivative of f

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example

Evaluate $\int_0^1 (x^2+2)\,dx$

$$\int_0^1 (x^2+2)\,dx$$

$$= \frac{x^3}{3} + 2x\Big|_0^1$$

$$= \frac{1}{3} + 2 - 0$$

$$= 7/3$$

Example

Evaluate $\int_1^2 \frac{x^2}{3}\,dx$

$$= \frac{1}{3}\int_1^2 x^2\,dx$$

$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3}\left[\frac{8}{3} - \frac{1}{3}\right]$$

$$= \frac{1}{3} \times \frac{7}{3}$$

$$= 7/9$$

Example

Use definite integrals to evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{i}{n}\right)\frac{1}{3n}$$

NEVER USE THIS

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a+i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)$$

Quantity without $i \Rightarrow$ Always a

next to $i \Rightarrow \frac{b-a}{n}$

$$\frac{b-a}{n} = \frac{1}{n}$$

$$b-1=1$$

$$b=2$$

$$x = a+i\left(\frac{b-a}{n}\right)$$

$$\therefore f(x) = \frac{1}{3}x^2 = \frac{x^2}{3}$$

$$\therefore \int_1^2 \frac{x^2}{3}\,dx$$

Now use second fundamental theorem of calculus