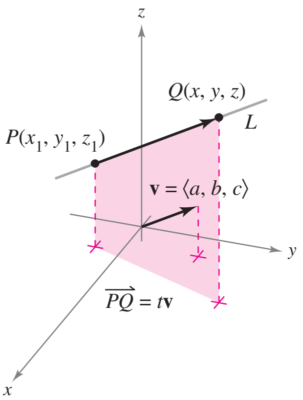
Section 11.5 Lines and Planes in Space

- **Write a set of parametric equations for a line in space.**
- **Write a linear equation to represent a plane in space.**
- **Sketch the plane given by a linear equation.**
- **Find the distances between points, planes, and lines in space.**

Lines in Space



Line L and its direction vector $\mathbf{v}$
Figure 11.43

In the plane, *slope* is used to determine an equation of a line. In space, it is more convenient to use *vectors* to determine the equation of a line.

In Figure 11.43, consider the line L through the point $P(x_1, y_1, z_1)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$. The vector $\mathbf{v}$ is a **direction vector** for the line L, and a, b, and c are **direction numbers.** One way of describing the line L is to say that it consists of all points $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$. This means that $\overrightarrow{PQ}$ is a scalar multiple of $\mathbf{v}$, and you can write $\overrightarrow{PQ} = t\mathbf{v}$, where t is a scalar (a real number).

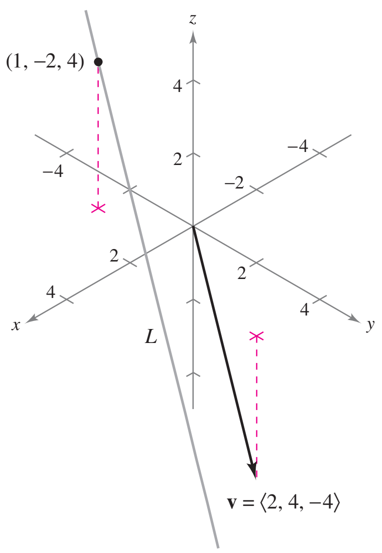
$$\overrightarrow{PQ} = \langle x - x_1, y - y_1, z - z_1\rangle = \langle at, bt, ct\rangle = t\mathbf{v}$$

By equating corresponding components, you can obtain **parametric equations** of a line in space.

THEOREM 11.11 Parametric Equations of a Line in Space

A line L parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the **parametric equations**

$$x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.$$



The vector $\mathbf{v}$ is parallel to the line L.
Figure 11.44

If the direction numbers a, b, and c are all nonzero, you can eliminate the parameter t to obtain **symmetric equations** of the line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \text{Symmetric equations}$$

EXAMPLE 1 Finding Parametric and Symmetric Equations

Find parametric and symmetric equations of the line L that passes through the point $(1, -2, 4)$ and is parallel to $\mathbf{v} = \langle 2, 4, -4\rangle$.

Solution To find a set of parametric equations of the line, use the coordinates $x_1 = 1$, $y_1 = -2$, and $z_1 = 4$ and direction numbers $a = 2$, $b = 4$, and $c = -4$ (see Figure 11.44).

$$x = 1 + 2t, \quad y = -2 + 4t, \quad z = 4 - 4t \qquad \text{Parametric equations}$$

Because a, b, and c are all nonzero, a set of symmetric equations is

$$\frac{x - 1}{2} = \frac{y + 2}{4} = \frac{z - 4}{-4}. \qquad \text{Symmetric equations}$$

Neither parametric equations nor symmetric equations of a given line are unique. For instance, in Example 1, by letting $t = 1$ in the parametric equations you would obtain the point $(3, 2, 0)$. Using this point with the direction numbers $a = 2$, $b = 4$, and $c = -4$ would produce a different set of parametric equations

$$x = 3 + 2t, \quad y = 2 + 4t, \quad \text{and} \quad z = -4t.$$

EXAMPLE 2 Parametric Equations of a Line Through Two Points

Find a set of parametric equations of the line that passes through the points $(-2, 1, 0)$ and $(1, 3, 5)$.

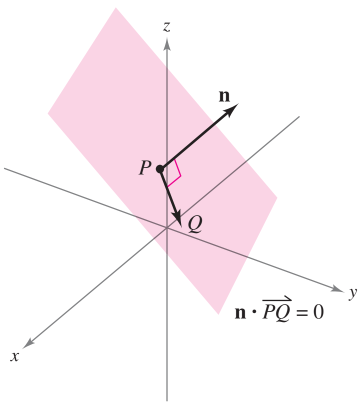
Solution Begin by using the points $P(-2, 1, 0)$ and $Q(1, 3, 5)$ to find a direction vector for the line passing through P and Q, given by

$$\mathbf{v} = \overrightarrow{PQ} = \langle 1 - (-2), 3 - 1, 5 - 0 \rangle = \langle 3, 2, 5 \rangle = \langle a, b, c \rangle.$$

Using the direction numbers $a = 3$, $b = 2$, and $c = 5$ with the point $P(-2, 1, 0)$, you can obtain the parametric equations

$$x = -2 + 3t, \quad y = 1 + 2t, \quad \text{and} \quad z = 5t.$$

NOTE As t varies over all real numbers, the parametric equations in Example 2 determine the points (x, y, z) on the line. In particular, note that $t = 0$ and $t = 1$ give the original points $(-2, 1, 0)$ and $(1, 3, 5)$.



The normal vector $\mathbf{n}$ is orthogonal to each vector $\overrightarrow{PQ}$ in the plane.
Figure 11.45

Planes in Space

You have seen how an equation of a line in space can be obtained from a point on the line and a vector *parallel* to it. You will now see that an equation of a plane in space can be obtained from a point in the plane and a vector *normal* (perpendicular) to the plane.

Consider the plane containing the point $P(x_1, y_1, z_1)$ having a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$, as shown in Figure 11.45. This plane consists of all points $Q(x, y, z)$ for which vector $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$. Using the dot product, you can write the following.

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$
$$\langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

The third equation of the plane is said to be in **standard form.**

THEOREM 11.12 Standard Equation of a Plane in Space

The plane containing the point (x_1, y_1, z_1) and having a normal vector $\mathbf{n} = \langle a, b, c \rangle$ can be represented, in **standard form,** by the equation

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

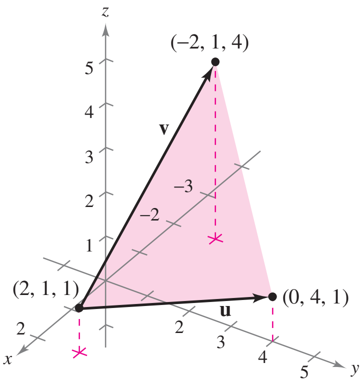
By regrouping terms, you obtain the **general form** of the equation of a plane in space.

$$ax + by + cz + d = 0 \qquad \text{General form of equation of plane}$$

Given the general form of the equation of a plane, it is easy to find a normal vector to the plane. Simply use the coefficients of x, y, and z and write $\mathbf{n} = \langle a, b, c \rangle$.

EXAMPLE 3 Finding an Equation of a Plane in Three-Space

Find the general equation of the plane containing the points $(2, 1, 1)$, $(0, 4, 1)$, and $(-2, 1, 4)$.



A plane determined by **u** and **v**
Figure 11.46

Solution To apply Theorem 11.12 you need a point in the plane and a vector that is normal to the plane. There are three choices for the point, but no normal vector is given. To obtain a normal vector, use the cross product of vectors **u** and **v** extending from the point $(2, 1, 1)$ to the points $(0, 4, 1)$ and $(-2, 1, 4)$, as shown in Figure 11.46. The component forms of **u** and **v** are

$$\mathbf{u} = \langle 0 - 2, 4 - 1, 1 - 1 \rangle = \langle -2, 3, 0 \rangle$$
$$\mathbf{v} = \langle -2 - 2, 1 - 1, 4 - 1 \rangle = \langle -4, 0, 3 \rangle$$

and it follows that

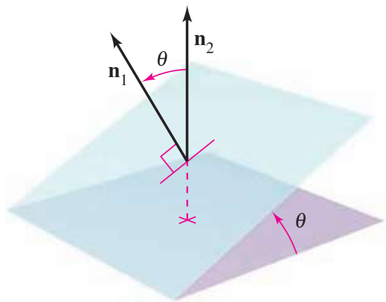
$$\begin{aligned} \mathbf{n} &= \mathbf{u} \times \mathbf{v} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix} \\ &= 9\mathbf{i} + 6\mathbf{j} + 12\mathbf{k} \\ &= \langle a, b, c \rangle \end{aligned}$$

is normal to the given plane. Using the direction numbers for **n** and the point $(x_1, y_1, z_1) = (2, 1, 1)$, you can determine an equation of the plane to be

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$
$$9(x - 2) + 6(y - 1) + 12(z - 1) = 0 \qquad \text{Standard form}$$
$$9x + 6y + 12z - 36 = 0 \qquad \text{General form}$$
$$3x + 2y + 4z - 12 = 0. \qquad \text{Simplified general form}$$

NOTE In Example 3, check to see that each of the three original points satisfies the equation $3x + 2y + 4z - 12 = 0$.

Two distinct planes in three-space either are parallel or intersect in a line. If they intersect, you can determine the angle $(0 \le \theta \le \pi/2)$ between them from the angle between their normal vectors, as shown in Figure 11.47. Specifically, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, the angle θ between the normal vectors is equal to the angle between the two planes and is given by



The angle θ between two planes
Figure 11.47

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|}. \qquad \text{Angle between two planes}$$

Consequently, two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are

1. *perpendicular* if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.
2. *parallel* if $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

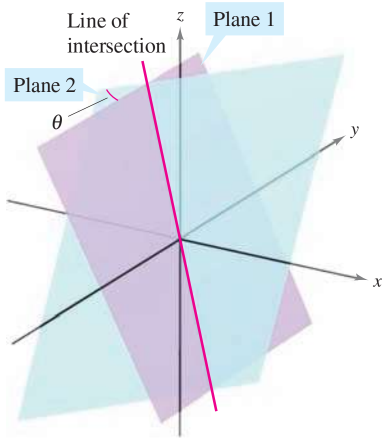


Figure 11.48

EXAMPLE 4 Finding the Line of Intersection of Two Planes

Find the angle between the two planes given by

$$x - 2y + z = 0 \qquad \text{Equation of plane 1}$$
$$2x + 3y - 2z = 0 \qquad \text{Equation of plane 2}$$

and find parametric equations of their line of intersection (see Figure 11.48).

Solution Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, -2, 1\rangle$ and $\mathbf{n}_2 = \langle 2, 3, -2\rangle$. Consequently, the angle between the two planes is determined as follows.

$$\begin{aligned}\cos\theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \qquad \text{Cosine of angle between } \mathbf{n}_1 \text{ and } \mathbf{n}_2\\ &= \frac{|-6|}{\sqrt{6}\sqrt{17}}\\ &= \frac{6}{\sqrt{102}}\\ &\approx 0.59409\end{aligned}$$

This implies that the angle between the two planes is $\theta \approx 53.55°$. You can find the line of intersection of the two planes by simultaneously solving the two linear equations representing the planes. One way to do this is to multiply the first equation by -2 and add the result to the second equation.

$$\begin{array}{lcl} x - 2y + z = 0 & \Rightarrow & -2x + 4y - 2z = 0\\ 2x + 3y - 2z = 0 & & \underline{2x + 3y - 2z = 0}\\ & & 7y - 4z = 0 \quad \Rightarrow \quad y = \dfrac{4z}{7}\end{array}$$

Substituting $y = 4z/7$ back into one of the original equations, you can determine that $x = z/7$. Finally, by letting $t = z/7$, you obtain the parametric equations

$$x = t, \quad y = 4t, \quad \text{and} \quad z = 7t \qquad \text{Line of intersection}$$

which indicate that 1, 4, and 7 are direction numbers for the line of intersection.

Note that the direction numbers in Example 4 can be obtained from the cross product of the two normal vectors as follows.

$$\begin{aligned}\mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 1\\ 2 & 3 & -2\end{vmatrix}\\ &= \begin{vmatrix} -2 & 1\\ 3 & -2\end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1\\ 2 & -2\end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2\\ 2 & 3\end{vmatrix}\mathbf{k}\\ &= \mathbf{i} + 4\mathbf{j} + 7\mathbf{k}\end{aligned}$$

This means that the line of intersection of the two planes is parallel to the cross product of their normal vectors.

NOTE The three-dimensional rotatable graphs that are available in the *HM mathSpace®* CD-ROM and the online *Eduspace®* system for this text can help you visualize surfaces such as those shown in Figure 11.48. If you have access to these graphs, you should use them to help your spatial intuition when studying this section and other sections in the text that deal with vectors, curves, or surfaces in space.

Sketching Planes in Space

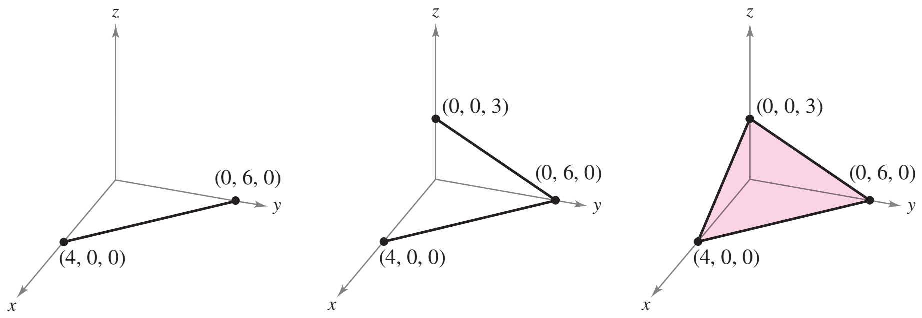
If a plane in space intersects one of the coordinate planes, the line of intersection is called the **trace** of the given plane in the coordinate plane. To sketch a plane in space, it is helpful to find its points of intersection with the coordinate axes and its traces in the coordinate planes. For example, consider the plane given by

$$3x + 2y + 4z = 12. \qquad \text{Equation of plane}$$

You can find the xy-trace by letting $z = 0$ and sketching the line

$$3x + 2y = 12 \qquad xy\text{-trace}$$

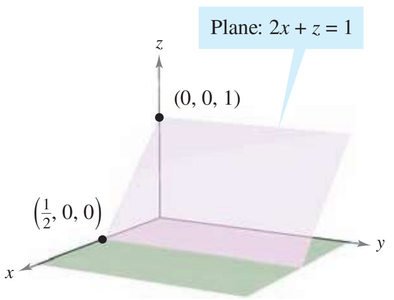
in the xy-plane. This line intersects the x-axis at $(4, 0, 0)$ and the y-axis at $(0, 6, 0)$. In Figure 11.49, this process is continued by finding the yz-trace and the xz-trace, and then shading the triangular region lying in the first octant.



xy-trace ($z = 0$): $3x + 2y = 12$ yz-trace ($x = 0$): $2y + 4z = 12$ xz-trace ($y = 0$): $3x + 4z = 12$

Traces of the plane $3x + 2y + 4z = 12$

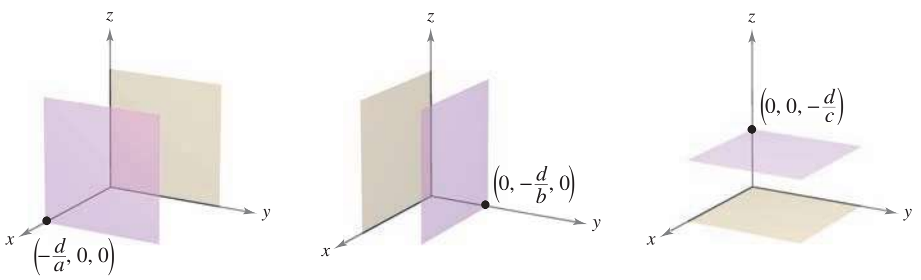
Figure 11.49



Plane $2x + z = 1$ is parallel to the y-axis.

Figure 11.50

If an equation of a plane has a missing variable, such as $2x + z = 1$, the plane must be *parallel to the axis* represented by the missing variable, as shown in Figure 11.50. If two variables are missing from an equation of a plane, it is *parallel to the coordinate plane* represented by the missing variables, as shown in Figure 11.51.



Plane $ax + d = 0$ is parallel to the yz-plane. Plane $by + d = 0$ is parallel to the xz-plane. Plane $cz + d = 0$ is parallel to the xy-plane.

Figure 11.51

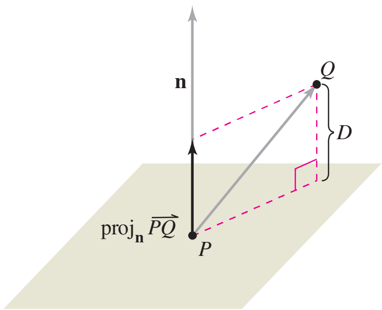
Distances Between Points, Planes, and Lines

This section is concluded with the following discussion of two basic types of problems involving distance in space.

1. Finding the distance between a point and a plane
2. Finding the distance between a point and a line

The solutions of these problems illustrate the versatility and usefulness of vectors in coordinate geometry: the first problem uses the *dot product* of two vectors, and the second problem uses the *cross product*.

The distance D between a point Q and a plane is the length of the shortest line segment connecting Q to the plane, as shown in Figure 11.52. If P is *any* point in the plane, you can find this distance by projecting the vector $\overrightarrow{PQ}$ onto the normal vector $\mathbf{n}$. The length of this projection is the desired distance.



$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\|$

The distance between a point and a plane
Figure 11.52

THEOREM 11.13 Distance Between a Point and a Plane

The distance between a plane and a point Q (not in the plane) is

$$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ}\cdot\mathbf{n}|}{\|\mathbf{n}\|}$$

where P is a point in the plane and $\mathbf{n}$ is normal to the plane.

To find a point in the plane given by $ax + by + cz + d = 0$ $(a \neq 0)$, let $y = 0$ and $z = 0$. Then, from the equation $ax + d = 0$, you can conclude that the point $(-d/a, 0, 0)$ lies in the plane.

EXAMPLE 5 Finding the Distance Between a Point and a Plane

Find the distance between the point $Q(1, 5, -4)$ and the plane given by

$$3x - y + 2z = 6.$$

Solution You know that $\mathbf{n} = \langle 3, -1, 2\rangle$ is normal to the given plane. To find a point in the plane, let $y = 0$ and $z = 0$, and obtain the point $P(2, 0, 0)$. The vector from P to Q is given by

$$\begin{aligned}\overrightarrow{PQ} &= \langle 1-2, 5-0, -4-0\rangle\\ &= \langle -1, 5, -4\rangle.\end{aligned}$$

Using the Distance Formula given in Theorem 11.13 produces

$$\begin{aligned}D = \frac{|\overrightarrow{PQ}\cdot\mathbf{n}|}{\|\mathbf{n}\|} &= \frac{|\langle -1, 5, -4\rangle \cdot \langle 3, -1, 2\rangle|}{\sqrt{9+1+4}} && \text{Distance between a point and a plane}\\ &= \frac{|-3-5-8|}{\sqrt{14}}\\ &= \frac{16}{\sqrt{14}}.\end{aligned}$$

NOTE The choice of the point P in Example 5 is arbitrary. Try choosing a different point in the plane to verify that you obtain the same distance.

From Theorem 11.13, you can determine that the distance between the point $Q(x_0, y_0, z_0)$ and the plane given by $ax + by + cz + d = 0$ is

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

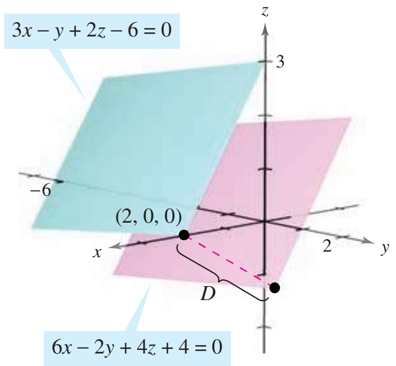
or

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance between a point and a plane

where $P(x_1, y_1, z_1)$ is a point in the plane and $d = -(ax_1 + by_1 + cz_1)$.

EXAMPLE 6 Finding the Distance Between Two Parallel Planes



The distance between the parallel planes is approximately 2.14.
Figure 11.53

Find the distance between the two parallel planes given by

$$3x - y + 2z - 6 = 0 \quad \text{and} \quad 6x - 2y + 4z + 4 = 0.$$

Solution The two planes are shown in Figure 11.53. To find the distance between the planes, choose a point in the first plane, say $(x_0, y_0, z_0) = (2, 0, 0)$. Then, from the second plane, you can determine that $a = 6$, $b = -2$, $c = 4$, and $d = 4$, and conclude that the distance is

$$\begin{aligned} D &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|6(2) + (-2)(0) + (4)(0) + 4|}{\sqrt{6^2 + (-2)^2 + 4^2}} \\ &= \frac{16}{\sqrt{56}} = \frac{8}{\sqrt{14}} \approx 2.14. \end{aligned}$$

Distance between a point and a plane

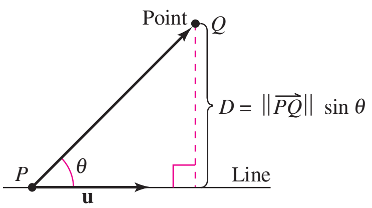
The formula for the distance between a point and a line in space resembles that for the distance between a point and a plane—except that you replace the dot product with the length of the cross product and the normal vector **n** with a direction vector for the line.

THEOREM 11.14 Distance Between a Point and a Line in Space

The distance between a point Q and a line in space is given by

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

where **u** is a direction vector for the line and P is a point on the line.



The distance between a point and a line
Figure 11.54

Proof In Figure 11.54, let D be the distance between the point Q and the given line. Then $D = \|\overrightarrow{PQ}\| \sin\theta$, where θ is the angle between **u** and $\overrightarrow{PQ}$. By Theorem 11.8, you have

$$\|\mathbf{u}\|\,\|\overrightarrow{PQ}\| \sin\theta = \|\mathbf{u} \times \overrightarrow{PQ}\| = \|\overrightarrow{PQ} \times \mathbf{u}\|.$$

Consequently,

$$D = \|\overrightarrow{PQ}\| \sin\theta = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}.$$

EXAMPLE 7 Finding the Distance Between a Point and a Line

Find the distance between the point $Q(3, -1, 4)$ and the line given by

$$x = -2 + 3t, \quad y = -2t, \quad \text{and} \quad z = 1 + 4t.$$

Solution Using the direction numbers 3, -2, and 4, you know that a direction vector for the line is

$$\mathbf{u} = \langle 3, -2, 4 \rangle. \qquad \text{Direction vector for line}$$

To find a point on the line, let $t = 0$ and obtain

$$P = (-2, 0, 1). \qquad \text{Point on the line}$$

So,

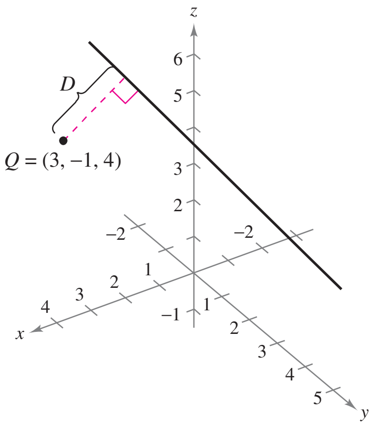
$$\overrightarrow{PQ} = \langle 3 - (-2), -1 - 0, 4 - 1 \rangle = \langle 5, -1, 3 \rangle$$

and you can form the cross product

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix} = 2\mathbf{i} - 11\mathbf{j} - 7\mathbf{k} = \langle 2, -11, -7 \rangle.$$

Finally, using Theorem 11.14, you can find the distance to be

$$\begin{aligned} D &= \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} \\ &= \frac{\sqrt{174}}{\sqrt{29}} \\ &= \sqrt{6} \approx 2.45. \end{aligned} \qquad \text{See Figure 11.55.}$$



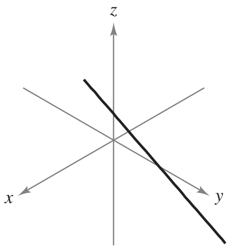
The distance between the point Q and the line is $\sqrt{6} \approx 2.45$.
Figure 11.55

Exercises for Section 11.5

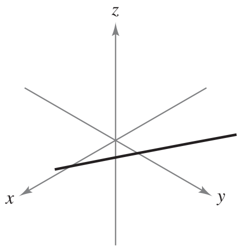
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, the figure shows the graph of a line given by the parametric equations. (a) Draw an arrow on the line to indicate its orientation. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*. (b) Find the coordinates of two points, P and Q, on the line. Determine the vector $\overrightarrow{PQ}$. What is the relationship between the components of the vector and the coefficients of t in the parametric equations? Why is this true? (c) Determine the coordinates of any points of intersection with the coordinate planes. If the line does not intersect a coordinate plane, explain why.

1. $x = 1 + 3t$
$y = 2 - t$
$z = 2 + 5t$



2. $x = 2 - 3t$
$y = 2$
$z = 1 - t$



In Exercises 3–8, find sets of (a) parametric equations and (b) symmetric equations of the line through the point parallel to the given vector or line. (For each line, write the direction numbers as integers.)

	Point	*Parallel to*
3.	$(0, 0, 0)$	$\mathbf{v} = \langle 1, 2, 3 \rangle$
4.	$(0, 0, 0)$	$\mathbf{v} = \left\langle -2, \frac{5}{2}, 1 \right\rangle$
5.	$(-2, 0, 3)$	$\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$
6.	$(-3, 0, 2)$	$\mathbf{v} = 6\mathbf{j} + 3\mathbf{k}$
7.	$(1, 0, 1)$	$x = 3 + 3t,\ y = 5 - 2t,\ z = -7 + t$
8.	$(-3, 5, 4)$	$\frac{x - 1}{3} = \frac{y + 1}{-2} = z - 3$

In Exercises 9–12, find sets of (a) parametric equations and (b) symmetric equations of the line through the two points. (For each line, write the direction numbers as integers.)

9. $(5, -3, -2), \left(-\frac{2}{3}, \frac{2}{3}, 1\right)$

10. $(2, 0, 2), (1, 4, -3)$

11. $(2, 3, 0), (10, 8, 12)$

12. $(0, 0, 25), (10, 10, 0)$

In Exercises 13–20, find a set of parametric equations of the line.

13. The line passes through the point $(2, 3, 4)$ and is parallel to the xz-plane and the yz-plane.

14. The line passes through the point $(-4, 5, 2)$ and is parallel to the xy-plane and the yz-plane.

15. The line passes through the point $(2, 3, 4)$ and is perpendicular to the plane given by $3x + 2y - z = 6$.

16. The line passes through the point $(-4, 5, 2)$ and is perpendicular to the plane given by $-x + 2y + z = 5$.

17. The line passes through the point $(5, -3, -4)$ and is parallel to $\mathbf{v} = \langle 2, -1, 3 \rangle$.

18. The line passes through the point $(-1, 4, -3)$ and is parallel to $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$.

19. The line passes through the point $(2, 1, 2)$ and is parallel to the line $x = -t,\ y = 1 + t,\ z = -2 + t$.

20. The line passes through the point $(-6, 0, 8)$ and is parallel to the line $x = 5 - 2t,\ y = -4 + 2t,\ z = 0$.

In Exercises 21–24, find the coordinates of a point P on the line and a vector v parallel to the line.

21. $x = 3 - t, \quad y = -1 + 2t, \quad z = -2$

22. $x = 4t, \quad y = 5 - t, \quad z = 4 + 3t$

23. $\dfrac{x-7}{4} = \dfrac{y+6}{2} = z + 2$

24. $\dfrac{x+3}{5} = \dfrac{y}{8} = \dfrac{z-3}{6}$

In Exercises 25 and 26, determine if any of the lines are parallel or identical.

25. L_1: $x = 6 - 3t, \quad y = -2 + 2t, \quad z = 5 + 4t$

L_2: $x = 6t, \quad y = 2 - 4t, \quad z = 13 - 8t$

L_3: $x = 10 - 6t, \quad y = 3 + 4t, \quad z = 7 + 8t$

L_4: $x = -4 + 6t, \quad y = 3 + 4t, \quad z = 5 - 6t$

26. L_1: $\dfrac{x-8}{4} = \dfrac{y+5}{-2} = \dfrac{z+9}{3}$

L_2: $\dfrac{x+7}{2} = \dfrac{y-4}{1} = \dfrac{z+6}{5}$

L_3: $\dfrac{x+4}{-8} = \dfrac{y-1}{4} = \dfrac{z+18}{-6}$

L_4: $\dfrac{x-2}{-2} = \dfrac{y+3}{1} = \dfrac{z-4}{1.5}$

In Exercises 27–30, determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.

27. $x = 4t + 2, \quad y = 3, \quad z = -t + 1$

$x = 2s + 2, \quad y = 2s + 3, \quad z = s + 1$

28. $x = -3t + 1,\ y = 4t + 1,\ z = 2t + 4$

$x = 3s + 1,\ y = 2s + 4,\ z = -s + 1$

29. $\dfrac{x}{3} = \dfrac{y-2}{-1} = z + 1, \quad \dfrac{x-1}{4} = y + 2 = \dfrac{z+3}{-3}$

30. $\dfrac{x-2}{-3} = \dfrac{y-2}{6} = z - 3, \quad \dfrac{x-3}{2} = y + 5 = \dfrac{z+2}{4}$

In Exercises 31 and 32, use a computer algebra system to graph the pair of intersecting lines and find the point of intersection.

31. $x = 2t + 3,\ y = 5t - 2,\ z = -t + 1$

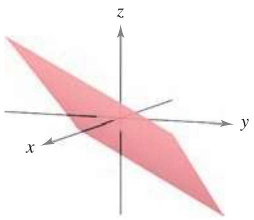
$x = -2s + 7,\ y = s + 8,\ z = 2s - 1$

32. $x = 2t - 1,\ y = -4t + 10,\ z = t$

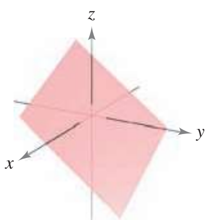
$x = -5s - 12,\ y = 3s + 11,\ z = -2s - 4$

Cross Product **In Exercises 33 and 34, (a) find the coordinates of three points P, Q, and R in the plane, and determine the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. (b) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$. What is the relationship between the components of the cross product and the coefficients of the equation of the plane? Why is this true?**

33. $4x - 3y - 6z = 6$



34. $2x + 3y + 4z = 4$



In Exercises 35–40, find an equation of the plane passing through the point perpendicular to the given vector or line.

	Point	*Perpendicular to*
35.	$(2, 1, 2)$	$\mathbf{n} = \mathbf{i}$
36.	$(1, 0, -3)$	$\mathbf{n} = \mathbf{k}$
37.	$(3, 2, 2)$	$\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
38.	$(0, 0, 0)$	$\mathbf{n} = -3\mathbf{i} + 2\mathbf{k}$
39.	$(0, 0, 6)$	$x = 1 - t,\ y = 2 + t,\ z = 4 - 2t$
40.	$(3, 2, 2)$	$\dfrac{x-1}{4} = y + 2 = \dfrac{z+3}{-3}$

In Exercises 41–52, find an equation of the plane.

41. The plane passes through $(0, 0, 0)$, $(1, 2, 3)$, and $(-2, 3, 3)$.

42. The plane passes through $(2, 3, -2)$, $(3, 4, 2)$, and $(1, -1, 0)$.

43. The plane passes through $(1, 2, 3)$, $(3, 2, 1)$, and $(-1, -2, 2)$.

44. The plane passes through the point $(1, 2, 3)$ and is parallel to the yz-plane.

45. The plane passes through the point $(1, 2, 3)$ and is parallel to the xy-plane.

46. The plane contains the y-axis and makes an angle of $\pi/6$ with the positive x-axis.

47. The plane contains the lines given by

$$\frac{x-1}{-2} = y - 4 = z \quad \text{and} \quad \frac{x-2}{-3} = \frac{y-1}{4} = \frac{z-2}{-1}.$$

48. The plane passes through the point $(2, 2, 1)$ and contains the line given by

$$\frac{x}{2} = \frac{y-4}{-1} = z.$$

49. The plane passes through the points $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to the plane $2x - 3y + z = 3$.

50. The plane passes through the points $(3, 2, 1)$ and $(3, 1, -5)$ and is perpendicular to the plane $6x + 7y + 2z = 10$.

51. The plane passes through the points $(1, -2, -1)$ and $(2, 5, 6)$ and is parallel to the x-axis.

52. The plane passes through the points $(4, 2, 1)$ and $(-3, 5, 7)$ and is parallel to the z-axis.

In Exercises 53 and 54, sketch a graph of the line and find the points (if any) where the line intersects the xy-, xz-, and yz-planes.

53. $x = 1 - 2t, \quad y = -2 + 3t, \quad z = -4 + t$

54. $\dfrac{x-2}{3} = y + 1 = \dfrac{z-3}{2}$

In Exercises 55 and 56, find an equation of the plane that contains all the points that are equidistant from the given points.

55. $(2, 2, 0), \quad (0, 2, 2)$

56. $(-3, 1, 2), \quad (6, -2, 4)$

In Exercises 57–62, determine whether the planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.

57. $5x - 3y + z = 4$
$x + 4y + 7z = 1$

58. $3x + y - 4z = 3$
$-9x - 3y + 12z = 4$

59. $x - 3y + 6z = 4$
$5x + y - z = 4$

60. $3x + 2y - z = 7$
$x - 4y + 2z = 0$

61. $x - 5y - z = 1$
$5x - 25y - 5z = -3$

62. $2x - z = 1$
$4x + y + 8z = 10$

In Exercises 63–70, label any intercepts and sketch a graph of the plane.

63. $4x + 2y + 6z = 12$

64. $3x + 6y + 2z = 6$

65. $2x - y + 3z = 4$

66. $2x - y + z = 4$

67. $y + z = 5$

68. $x + 2y = 4$

69. $x = 5$

70. $z = 8$

In Exercises 71–74, use a computer algebra system to graph the plane.

71. $2x + y - z = 6$

72. $x - 3z = 3$

73. $-5x + 4y - 6z = -8$

74. $2.1x - 4.7y - z = -3$

In Exercises 75 and 76, determine if any of the planes are parallel or identical.

75. P_1: $3x - 2y + 5z = 10$
P_2: $-6x + 4y - 10z = 5$
P_3: $-3x + 2y + 5z = 8$
P_4: $75x - 50y + 125z = 250$

76. P_1: $-60x + 90y + 30z = 27$
P_2: $6x - 9y - 3z = 2$
P_3: $-20x + 30y + 10z = 9$
P_4: $12x - 18y + 6z = 5$

In Exercises 77–80, describe the family of planes represented by the equation, where c is any real number.

77. $x + y + z = c$

78. $x + y = c$

79. $cy + z = 0$

80. $x + cz = 0$

In Exercises 81 and 82, find a set of parametric equations for the line of intersection of the planes.

81. $3x + 2y - z = 7$
$x - 4y + 2z = 0$

82. $6x - 3y + z = 5$
$-x + y + 5z = 5$

In Exercises 83–86, find the point(s) of intersection (if any) of the plane and the line. Also determine whether the line lies in the plane.

83. $2x - 2y + z = 12, \quad x - \dfrac{1}{2} = \dfrac{y + (3/2)}{-1} = \dfrac{z+1}{2}$

84. $2x + 3y = -5, \quad \dfrac{x-1}{4} = \dfrac{y}{2} = \dfrac{z-3}{6}$

85. $2x + 3y = 10, \quad \dfrac{x-1}{3} = \dfrac{y+1}{-2} = z - 3$

86. $5x + 3y = 17, \quad \dfrac{x-4}{2} = \dfrac{y+1}{-3} = \dfrac{z+2}{5}$

In Exercises 87–90, find the distance between the point and the plane.

87. $(0, 0, 0)$
$2x + 3y + z = 12$

88. $(0, 0, 0)$
$8x - 4y + z = 8$

89. $(2, 8, 4)$
$2x + y + z = 5$

90. $(3, 2, 1)$
$x - y + 2z = 4$

In Exercises 91–94, verify that the two planes are parallel, and find the distance between the planes.

91. $x - 3y + 4z = 10$
$x - 3y + 4z = 6$

92. $4x - 4y + 9z = 7$
$4x - 4y + 9z = 18$

93. $-3x + 6y + 7z = 1$
$6x - 12y - 14z = 25$

94. $2x - 4z = 4$
$2x - 4z = 10$

In Exercises 95–98, find the distance between the point and the line given by the set of parametric equations.

95. $(1, 5, -2)$; $x = 4t - 2$, $y = 3$, $z = -t + 1$

96. $(1, -2, 4)$; $x = 2t$, $y = t - 3$, $z = 2t + 2$

97. $(-2, 1, 3)$; $x = 1 - t$, $y = 2 + t$, $z = -2t$

98. $(4, -1, 5)$; $x = 3$, $y = 1 + 3t$, $z = 1 + t$

In Exercises 99 and 100, verify that the lines are parallel, and find the distance between them.

99. L_1: $x = 2 - t$, $y = 3 + 2t$, $z = 4 + t$
L_2: $x = 3t$, $y = 1 - 6t$, $z = 4 - 3t$

100. L_1: $x = 3 + 6t$, $y = -2 + 9t$, $z = 1 - 12t$
L_2: $x = -1 + 4t$, $y = 3 + 6t$, $z = -8t$

Writing About Concepts

101. Give the parametric equations and the symmetric equations of a line in space. Describe what is required to find these equations.

102. Give the standard equation of a plane in space. Describe what is required to find this equation.

103. Describe a method of finding the line of intersection of two planes.

104. Describe each surface given by the equations $x = a$, $y = b$, and $z = c$.

105. Describe a method for determining when two planes

$$a_1x + b_1y + c_1z + d_1 = 0$$

and

$$a_2x + b_2y + c_2z + d_2 = 0$$

are (a) parallel and (b) perpendicular. Explain your reasoning.

106. Let L_1 and L_2 be nonparallel lines that do not intersect. Is it possible to find a nonzero vector $\mathbf{v}$ such that $\mathbf{v}$ is perpendicular to both L_1 and L_2? Explain your reasoning.

107. Find an equation of the plane with x-intercept $(a, 0, 0)$, y-intercept $(0, b, 0)$, and z-intercept $(0, 0, c)$. (Assume a, b, and c are nonzero.)

108. (a) Describe and find an equation for the surface generated by all points (x, y, z) that are four units from the point $(3, -2, 5)$.

(b) Describe and find an equation for the surface generated by all points (x, y, z) that are four units from the plane

$$4x - 3y + z = 10.$$

109. ***Modeling Data*** Per capita consumptions (in gallons) of different types of plain milk in the United States from 1994 to 2000 are shown in the table. Consumption of light and skim milks, reduced-fat milk, and whole milk are represented by the variables x, y, and z, respectively. *(Source: U.S. Department of Agriculture)*

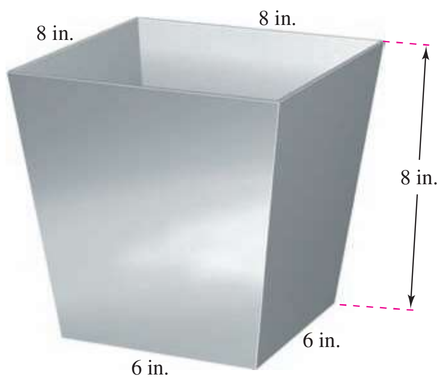
Year	1994	1995	1996	1997	1998	1999	2000
x	5.8	6.2	6.4	6.6	6.5	6.3	6.1
y	8.7	8.2	8.0	7.7	7.4	7.3	7.1
z	8.8	8.4	8.4	8.2	7.8	7.9	7.8

A model for the data is given by $0.04x - 0.64y + z = 3.4$.

(a) Complete a fourth row in the table using the model to approximate z for the given values of x and y. Compare the approximations with the actual values of z.

(b) According to this model, any increases in consumption of two types of milk will have what effect on the consumption of the third type?

110. ***Mechanical Design*** A chute at the top of a grain elevator of a combine funnels the grain into a bin (see figure). Find the angle between two adjacent sides.



111. ***Distance*** Two insects are crawling along different lines in three-space. At time t (in minutes), the first insect is at the point (x, y, z) on the line

$$x = 6 + t, \quad y = 8 - t, \quad z = 3 + t.$$

Also, at time t, the second insect is at the point (x, y, z) on the line

$$x = 1 + t, \quad y = 2 + t, \quad z = 2t.$$

Assume distances are given in inches.

(a) Find the distance between the two insects at time $t = 0$.

(b) Use a graphing utility to graph the distance between the insects from $t = 0$ to $t = 10$.

(c) Using the graph from part (b), what can you conclude about the distance between the insects?

(d) How close do the insects get?

112. Find the standard equation of the sphere with center $(-3, 2, 4)$ that is tangent to the plane given by $2x + 4y - 3z = 8$.

113. Find the point of intersection of the plane $3x - y + 4z = 7$ and the line through $(5, 4, -3)$ that is perpendicular to this plane.

114. Show that the plane $2x - y - 3z = 4$ is parallel to the line $x = -2 + 2t$, $y = -1 + 4t$, $z = 4$, and find the distance between them.

115. Find the point of intersection of the line through $(1, -3, 1)$ and $(3, -4, 2)$, and the plane given by $x - y + z = 2$.

116. Find a set of parametric equations for the line passing through the point $(1, 0, 2)$ that is parallel to the plane given by $x + y + z = 5$, and perpendicular to the line $x = t$, $y = 1 + t$, $z = 1 + t$.

True or False? **In Exercises 117–120, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

117. If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ is any vector in the plane given by $a_2x + b_2y + c_2z + d_2 = 0$, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

118. Every pair of lines in space are either intersecting or parallel.

119. Two planes in space are either intersecting or parallel.

120. If two lines L_1 and L_2 are parallel to a plane P, then L_1 and L_2 are parallel.

Section Project: Distances in Space

You have learned two distance formulas in this section—the distance between a point and a plane, and the distance between a point and a line. In this project you will study a third distance problem—the distance between two skew lines. Two lines in space are *skew* if they are neither parallel nor intersecting (see figure).

(a) Consider the following two lines in space.

L_1: $x = 4 + 5t$, $y = 5 + 5t$, $z = 1 - 4t$

L_2: $x = 4 + s$, $y = -6 + 8s$, $z = 7 - 3s$

(i) Show that these lines are not parallel.

(ii) Show that these lines do not intersect, and therefore are skew lines.

(iii) Show that the two lines lie in parallel planes.

(iv) Find the distance between the parallel planes from part (iii). This is the distance between the original skew lines.

(b) Use the procedure in part (a) to find the distance between the lines.

L_1: $x = 2t$, $y = 4t$, $z = 6t$

L_2: $x = 1 - s$, $y = 4 + s$, $z = -1 + s$

(c) Use the procedure in part (a) to find the distance between the lines.

L_1: $x = 3t$, $y = 2 - t$, $z = -1 + t$

L_2: $x = 1 + 4s$, $y = -2 + s$, $z = -3 - 3s$

(d) Develop a formula for finding the distance between the skew lines.

L_1: $x = x_1 + a_1t$, $y = y_1 + b_1t$, $z = z_1 + c_1t$

L_2: $x = x_2 + a_2s$, $y = y_2 + b_2s$, $z = z_2 + c_2s$