# Integrals of Rational Functions by Partial Fraction Decomposition

$$\int \frac{P(x)}{Q(x)}\,dx \qquad P \,\&\, Q \text{ are polynomials}$$

$$\deg P(x) < Q(x)$$

**Rule 2** Non-distinct linear expansion

If the denominator $Q(x)$ can be decomposed as a product of linear factors, some of which are repeated

$$Q(x) = a(x - c_1) \cdots\cdots (x-b)^n \cdots\cdots$$

The partial fraction decomposition is given by

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-c_1} + \cdots\cdots + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \cdots + \frac{B_n}{(x-b)^n}$$

**Example**

$$\frac{x^2+x+1}{(x-2)(x+3)^2(x-1)^3} = \frac{A}{x-2} + \frac{B_1}{x+3} + \frac{B_2}{(x+3)^2} + \frac{C_1}{x-1} + \frac{C_2}{(x-1)^2} + \frac{C_2}{(x-1)^3}$$

**Example**

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$$

$$x^3+2x^2+x = x(x^2+2x+1)$$
$$= x(x+1)^2$$

$$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$5x^2 + 20x + 6 = A(x+1)^2 + Bx(x+1) + Cx$$
$$= A(x^2+2x+1) + Bx^2 + Bx + Cx$$
$$= Ax^2 + Bx^2 + 2Ax + Bx + Cx + A$$
$$= x^2(A+B) + x(2A+B+C) + A$$

Comparing LHS and RHS

$\underline{A = 6}$

$A + B = 5$

$6 + B = 5$

$\underline{B = -1}$

$2A + B + C = 20$

$12 - 1 + C = 20$

$\underline{C = 9}$

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \int \frac{6}{x}\,dx - \int \frac{dx}{x+1} + \int \frac{9}{(x+1)^2}\,dx$$
$$= 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + C$$

## Irreducible quadratic function

If $b^2 - 4ac < 0$, then the quadratic function $ax^2 + bx + c$ is called irreducible

## Rule 3 Distinct irreducible quadratic factors

If the decomposition of $Q(x)$ contains distinct irreducible quadratic factors

$Q(x) = m(x - c_1) \cdots (x - c_2) \cdots (ax^2 + bx + c) \Rightarrow$ with $b^2 - 4ac < 0$

Then the partial fraction decomposition is given by

$$\frac{P(x)}{Q(x)} = \frac{A}{x - c_1} + \cdots + \frac{B_1}{x - c_2} + \frac{B_2}{(x-c_2)^2} + \cdots \frac{B_n}{(x-c_2)^n} + \frac{Cx + D}{ax^2 + bx + c}$$

Example

$$\frac{2x^3+x-9}{x^3(x+1)(x^2+x+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{B}{x+1} + \frac{Cx+D}{x^2+x+1}$$

Example

Evaluate $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}$

$$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{2x^3-4x-8}{x(x-1)(x^2+4)}$$

$$\frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

$$2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)(x-1)(x)$$

$$= A(x^3+4x-x^2-4) + Bx^3 + 4Bx + (Cx+D)(x^2-x)$$

$$= Ax^3 + 4Ax - Ax^2 - 4A + Bx^3 + 4Bx + Cx^3 - Cx^2 + Dx^2 - Dx$$

$$= x^3(A+B+C) + x^2(-A-C+D) + x(4A+4B-D) - 4A$$

Comparing LHS & RHS

$$\begin{cases} A+B+C = 2 \\ -A-C+D = 0 \\ 4A+4B-D = -4 \\ -4A = -8 \end{cases} \Rightarrow \underline{A=2}$$

$$D = C+2$$

$$4A+4B-C-2 = -4$$

$$4(A+B) = C-2$$

$$A+B = \frac{C-2}{4}$$

$$\frac{C-2}{4} + \frac{4C}{4} = 2$$

$$5C - 2 = 8$$
$$5C = 10$$
$$C = 2$$

$$4(A+B) = C-2$$
$$8 + 4B = 0$$
$$B = -2$$

$$D = C+2$$
$$D = 4$$

$$\int \frac{2x^3 - 4x - 8}{(x^2-x)(x^2+4)}\,dx = \int \frac{2}{x}\,dx - \int \frac{2}{x-1}\,dx + \int \frac{2x+4}{x^2+4}\,dx$$
$$= 2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\tan^{-1}\left(\frac{x}{2}\right) + c$$

Rule 4 Repeated irreducible quadratic factors

If the decomposition of $Q(x)$ contains repeated irreducible quadratic factors

$Q(x) = \cdots\cdots (ax^2+bx+c)^m \cdots\cdots \Rightarrow b^2 - 4ac < 0$

then the partial fraction decomposition is given by

$$\frac{P(x)}{Q(x)} = \cdots\cdots \frac{A_1x + B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots\cdots \frac{A_mx + B_m}{(ax^2+bx+c)^m}$$

Example

Give the partial fraction decomposition of

$$f(x) = \frac{5x^4 - 6x + 9}{x(x+3)^3(x^2+1)(x^2+x+1)^3}$$

Do NOT find the value of the coefficient

$$\frac{A}{x} + \frac{B_1}{x+3} + \frac{B_2}{(x+3)^2} + \frac{B_3}{(x+3)^3} + \frac{Cx+D}{x^2+1} + \frac{E_1x+F_1}{x^2+x+1} + \frac{E_2x+F_2}{(x^2+x+1)^2} + \frac{E_3x+F_3}{(x^2+x+1)^3}$$

Example

Evaluate $\int \frac{8x^3+13x}{(x^2+2)^2}\,dx$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

$$\begin{aligned} 8x^3+13x &= (Ax+B)(x^2+2) + Cx+D \\ &= Ax^3 + 2Ax + Bx^2 + 2B + Cx + D \\ &= x^3(A) + x^2(B) + x(2A+C) + 2B+D \end{aligned}$$

$$\begin{aligned} A &= 8 &\Rightarrow A &= 8 \\ B &= 0 &\Rightarrow B &= 0 \\ 2A+C &= 13 &\Rightarrow C &= -3 \\ 2B+D &= 0 &\Rightarrow D &= 0 \end{aligned}$$

$$16 + C = 13$$
$$\underline{C = -3}$$

$$\begin{aligned} \int \frac{8x^3+13x}{(x^2+2)^2}\,dx &= \int \frac{8x}{x^2+2}\,dx - \int \frac{3x}{(x^2+2)^2}\,dx \\ &= 4\ln|x^2+2| - \frac{3}{2}\int 2x(x^2+2)^{-2}\,dx \\ &= 4\ln|x^2+2| + \frac{3}{2(x^2+2)} + C \end{aligned}$$

**Case IV** $\quad \deg P(x) \geq Q(x)$

In this case, we can perform the long division to get

$$\frac{P(x)}{Q(x)} = f(x) + \frac{r(x)}{Q(x)} \quad \text{where } \deg r(x) < \deg Q(x)$$

$$\int \frac{P(x)}{Q(x)}\,dx = \int f(x)\,dx + \int \frac{r(x)}{Q(x)}\,dx$$

Standard integration

Case I

Example

Evaluate $\displaystyle\int \frac{5x^3 + 3x^2 - 2}{x^2 - x}\, dx$

$$
\begin{array}{r|l}
 & 5x + 8 \\
\hline
x^2 - x & 5x^3 + 3x^2 - 2 \\
 & \underline{\overset{(-)}{5x^3} \overset{(+)}{-} 5x^2} \\
 & 8x^2 + 0x - 2 \\
 & \underline{\overset{(-)}{8x^2} \overset{(+)}{-} 8x} \\
 & 8x - 2
\end{array}
$$

$$\int \frac{5x^3 + 3x^2 - 2}{x^2 - x}\, dx = \int (5x + 8)\, dx + \int \frac{8x - 2}{x^2 - x}\, dx$$

$$\frac{8x - 2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$8x - 2 = A(x-1) + Bx$$

@ $x = 1$

$B = 6$

@ $x = 0$

$-A = -2$

$A = 2$

$$\int \frac{8x - 2}{x^2 - x}\, dx = \int \frac{2}{x}\, dx + \int \frac{6}{x-1}\, dx$$

$$= 2\ln|x| + 6\ln|x-1| + C$$

$$\int \frac{5x^3 + 3x^2 - 2}{x^2 - x}\, dx = \int (5x + 8)\, dx + \int \frac{8x - 2}{x^2 - x}\, dx$$

$$= \frac{5x^2}{2} + 8x + 2\ln|x| + 6\ln|x-1| + C$$