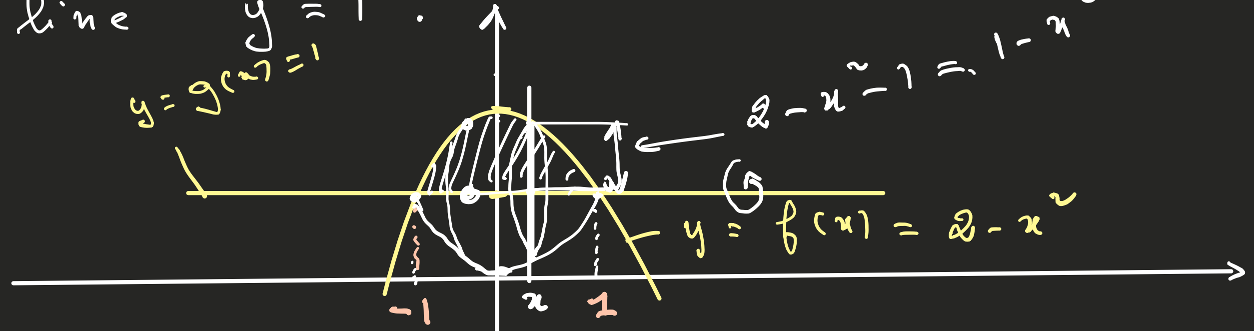
§ Volume by the disc Method.

Recall Volume $= \int$ Area of slice dx or dy

→ the slice should be chosen perpendicular to the axis of revolution.

Exple Use the disc method to find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$.



$$2 - x^2 = 1$$
$$x^2 - 1 = 0$$
$$x = \pm 1$$

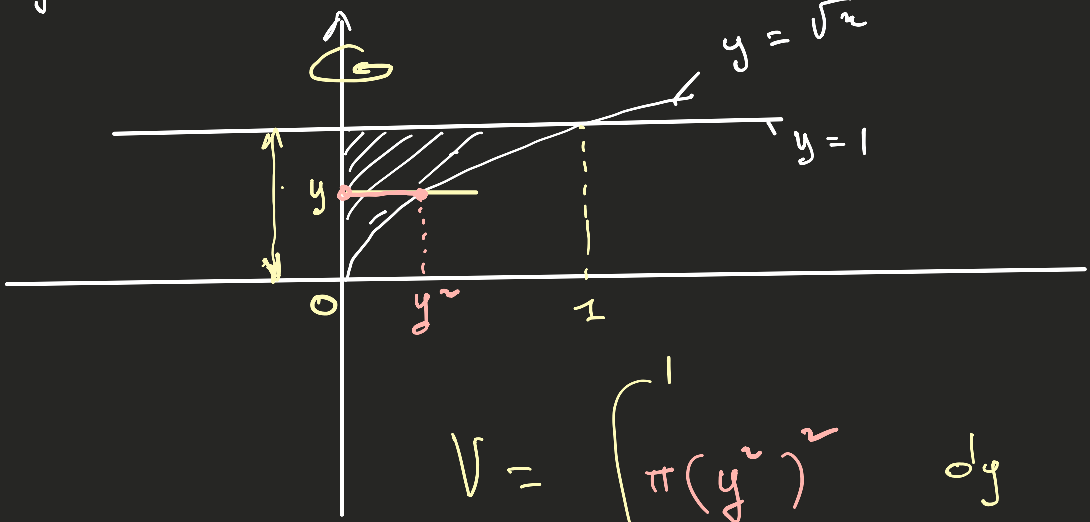
$$V = \int \text{Area of slice } dx \text{ or } dy$$

$$V = \int_{-1}^{1} \pi (1 - x^2)^2 \, dx = \pi \int_{-1}^{1} (1 - 2x^2 + x^4) \, dx$$

$$= \pi \left(x - \frac{2x^3}{3} + \frac{x^5}{5} \Big|_{-1}^{1} \right)$$

$$= \frac{16\pi}{15}.$$

Exple Use the disc method to find the Volume of the Solid obtained by rotating the region bounded by $y = \sqrt{x}$, $x = 0$ and $y = 1$ about the y-axis.

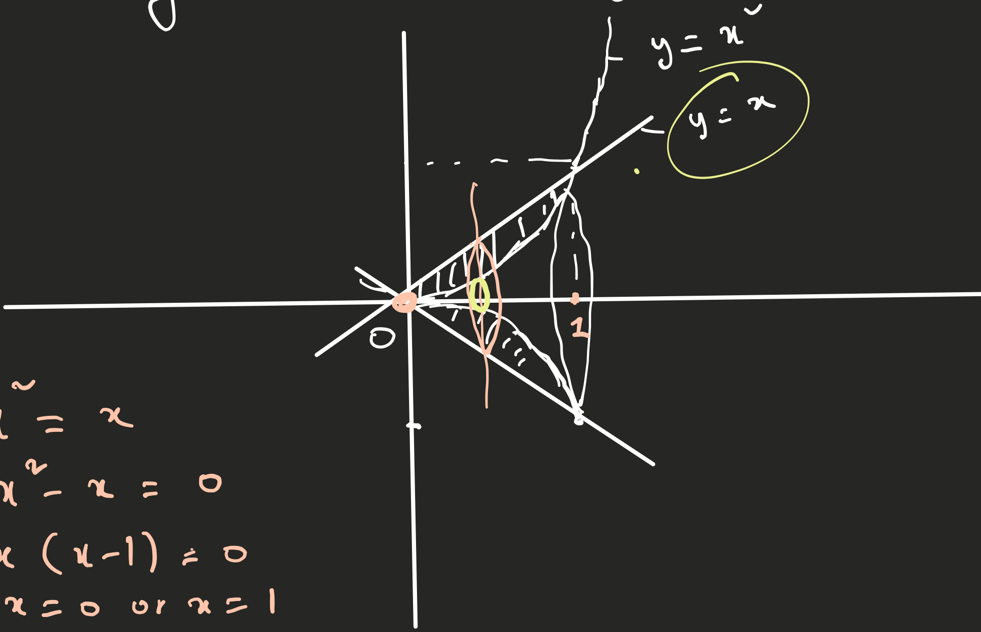


$$V = \int_0^1 \pi (y^2)^2 \, dy$$

$$= \pi \int_0^1 y^4 \, dy =$$

$$= \pi \left(\frac{y^5}{5} \Big|_0^1 \right) = \frac{\pi}{5}$$

Exple Find the Volume of the solid obtained by rotating the region enclosed by the curve $y = x$ and $y = x^2$ about the x-axis.



$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ or $x = 1$

$$V = \int_0^1 \pi x^2 - \pi (x^2)^2 \, dx$$

$$= \pi \int_0^1 (x^2 - x^4) \, dx$$

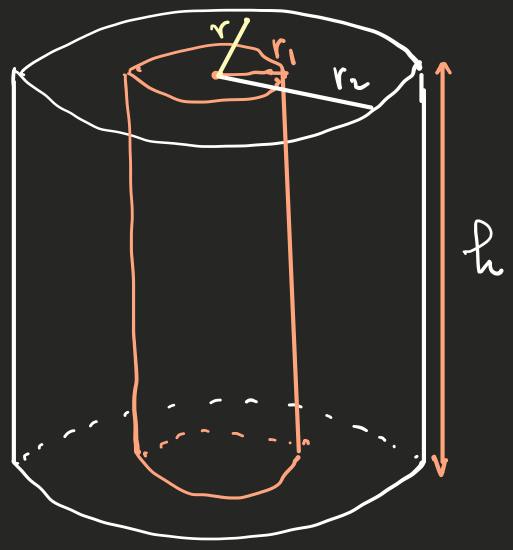
$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \Big|_0^1 \right)$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}.$$

$$y = 2x^2 - x^3$$

§ The cylindrical shell Method

* Cylindrical Shell



$r = \frac{r_1 + r_2}{2}$

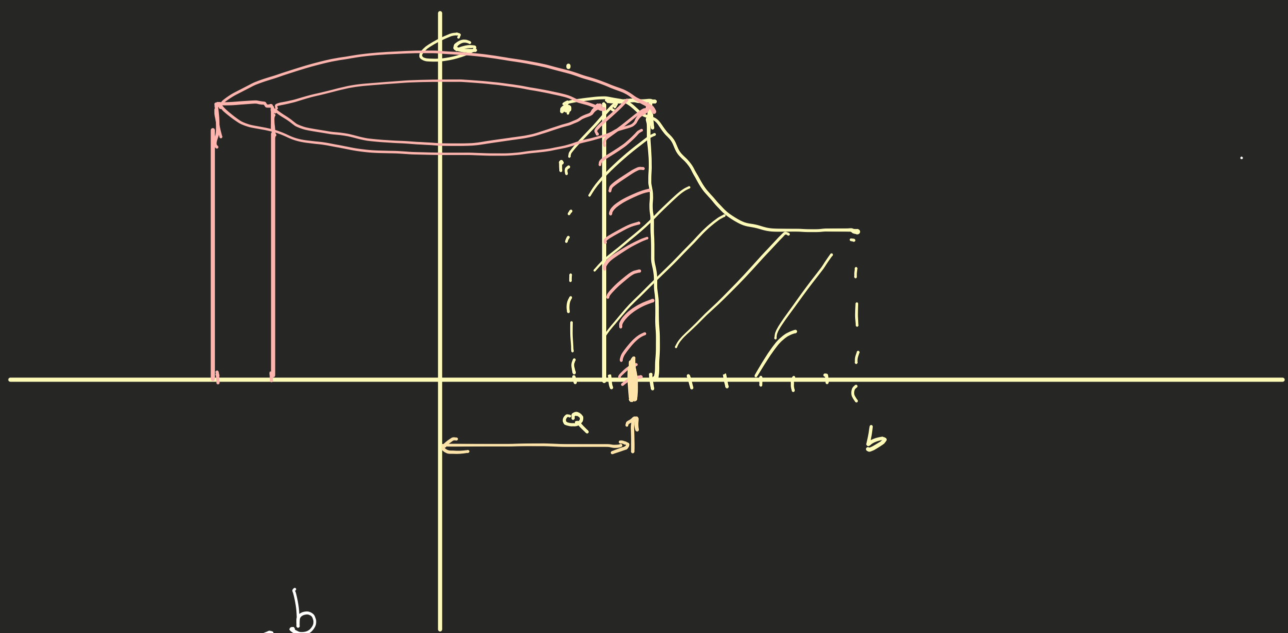
$$V = \pi r_2^2 h - \pi r_1^2 h$$

$$= \pi (r_2^2 - r_1^2) h$$

$$= \pi (r_2 + r_1)(r_2 - r_1) h$$

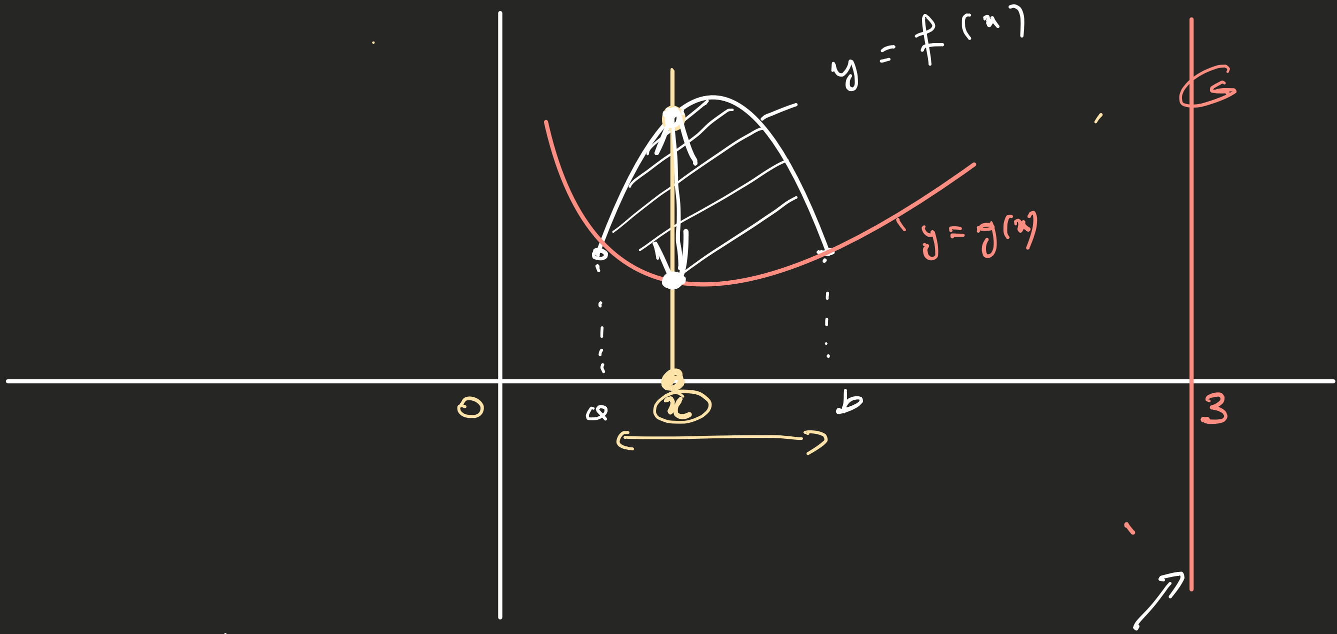
$$= 2\pi \underbrace{\frac{(r_2 + r_1)}{2}} \cdot h \underbrace{(r_2 - r_1)}_{\Delta r}$$

$$= 2\pi r \cdot h \, \Delta r$$



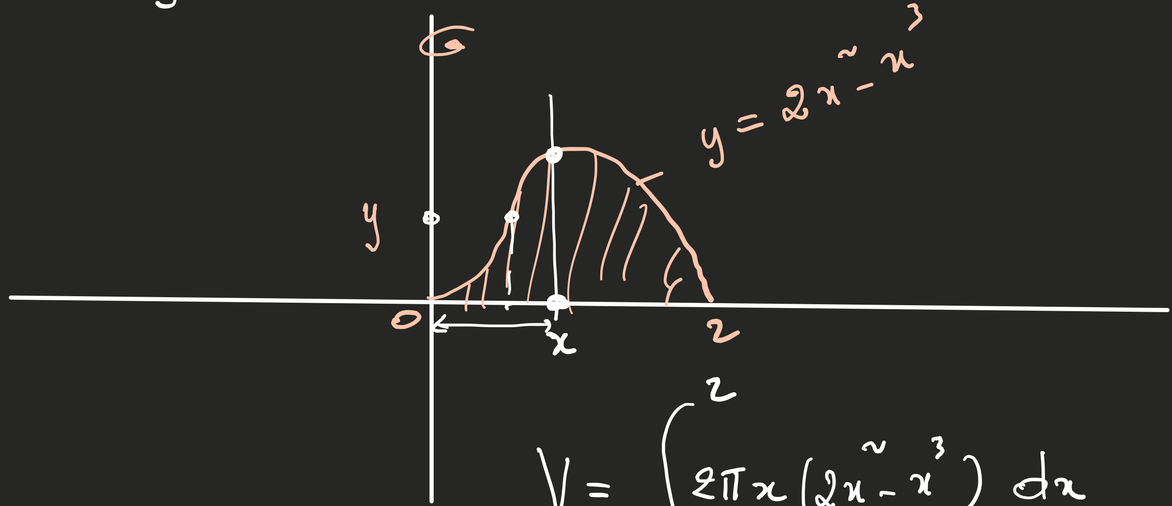
$$V = \int_a^b 2\pi \cdot \text{Average radius} \cdot \text{height} \cdot \begin{matrix} dx \\ \text{or} \\ dy \end{matrix}$$

* For the Shell method, the slice should be chosen parallel to the axis of rotation.



$$V = \int_a^b 2\pi(3-x)\left(f(x) - g(x)\right)\,dx$$

Exple Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$, $y = 0$, between $x = 0$ and $x = 2$.



$$V = \int_0^2 2\pi x\,(2x^2 - x^3)\,dx$$

$$V = 2\pi \int_0^2 (2x^3 - x^4)\, dx = 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5} \Big|_0^2 \right)$$

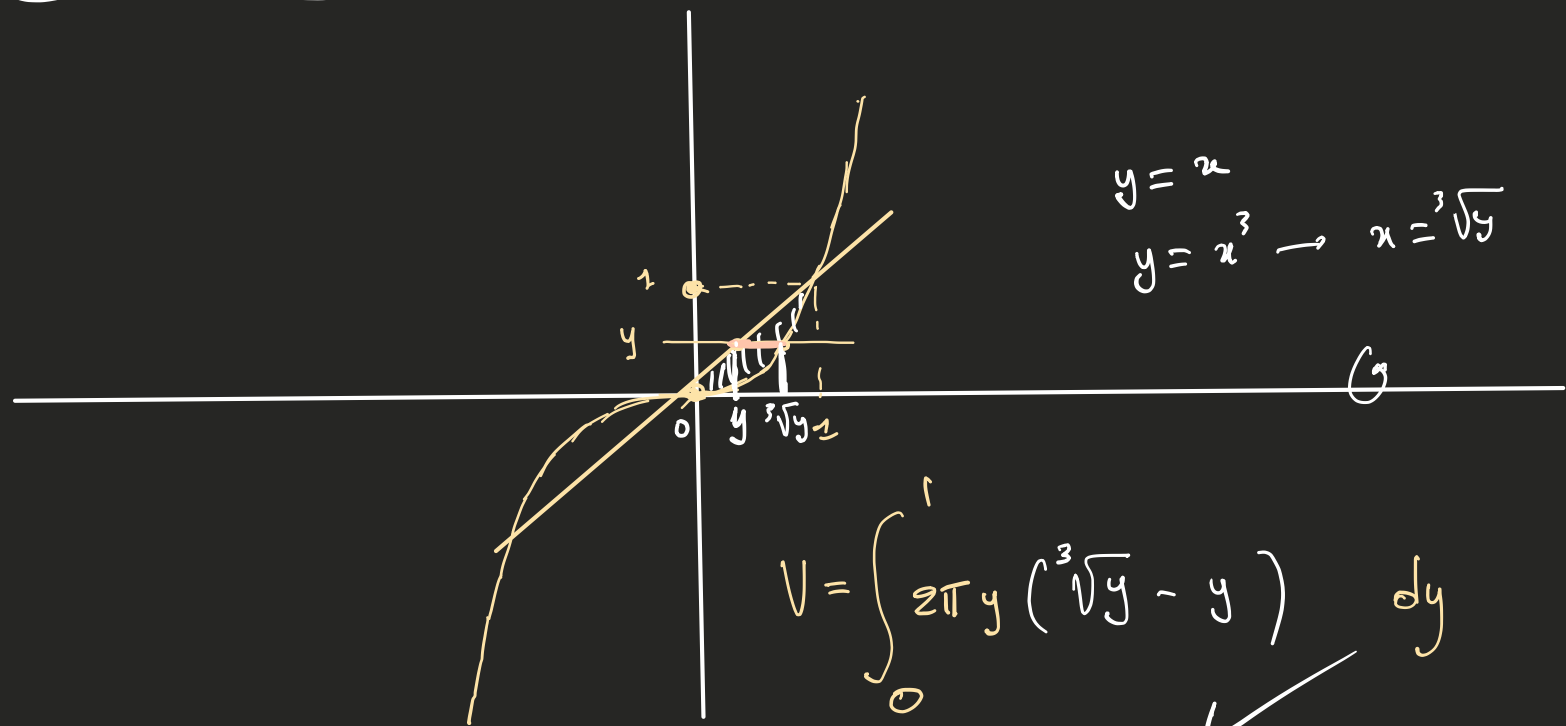
$$= 2\pi \left(8 - \frac{32}{5} \right)$$

$$= 2\pi \left(\frac{8}{5} \right) = \frac{16\pi}{5}.$$

Exple Use the shell method to set up (DO Not evaluate) the integral for finding the volume of the solid obtained by rotating the region in the first quadrant bounded by $y = x$ and $y = x^3$

A) About the x-axis

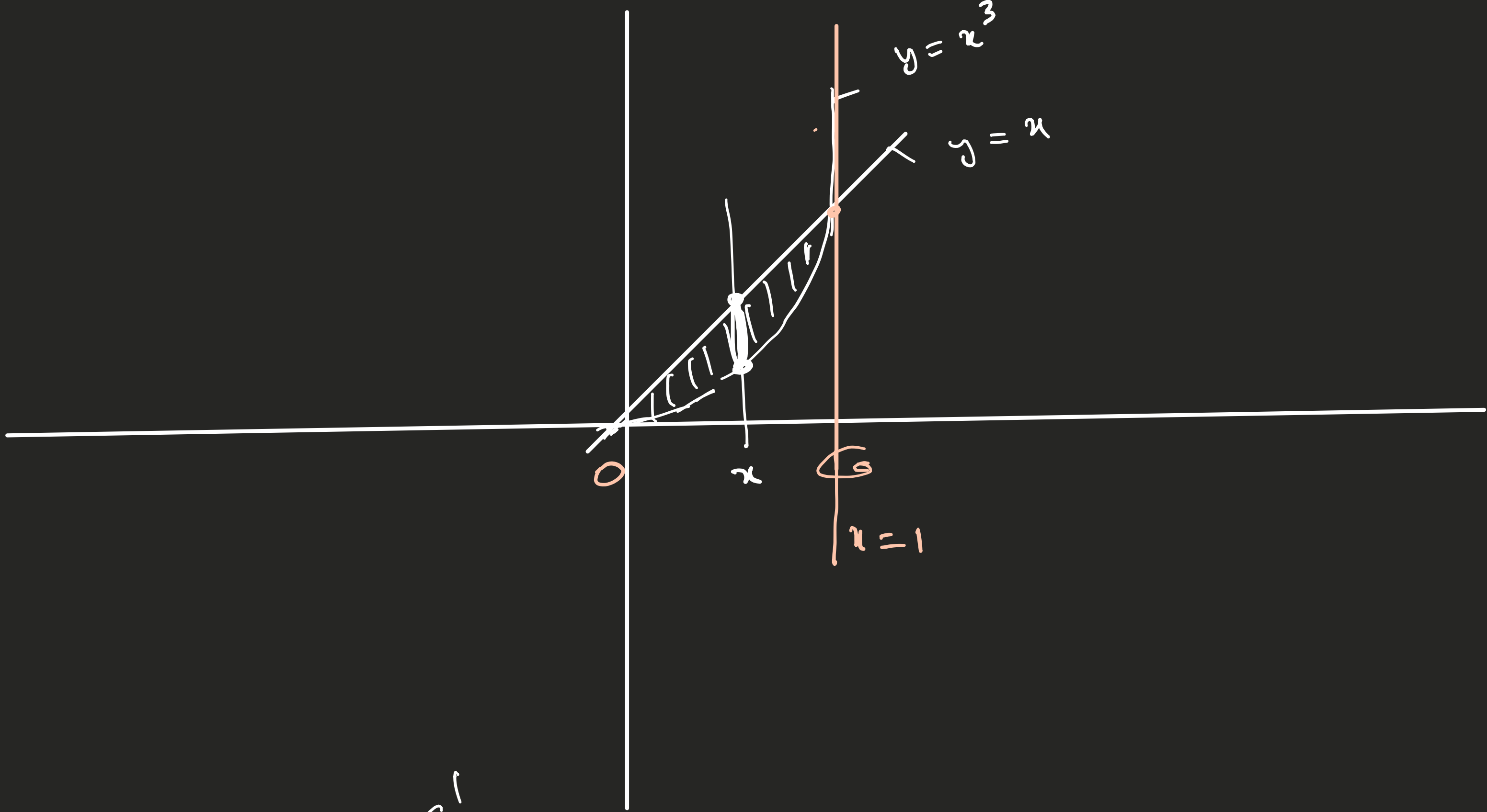
B) About the line $x = 1$.



$y = x$

$y = x^3 \to x = \sqrt[3]{y}$

$$V = \int_0^1 2\pi y \left(\sqrt[3]{y} - y \right) dy$$



$$V = \int_0^1 2\pi(1-x)(x-x^3)\,dx$$

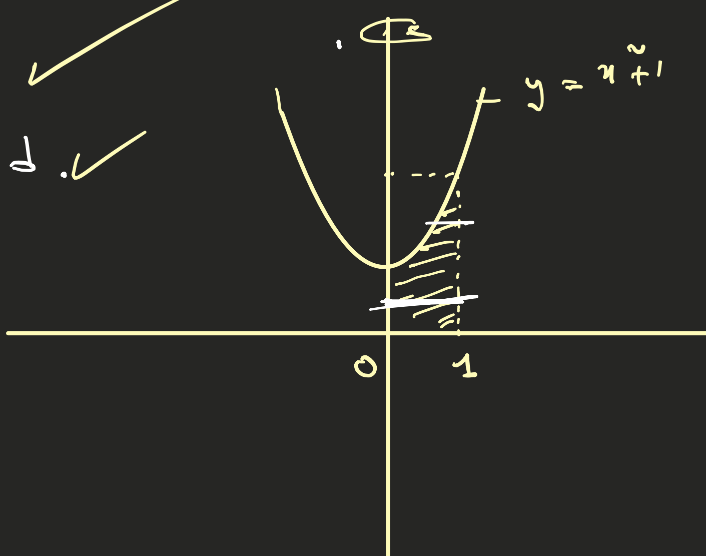
<u>Exple</u> Find the Volume of the solid formed by rotating the region bounded by the graphs of

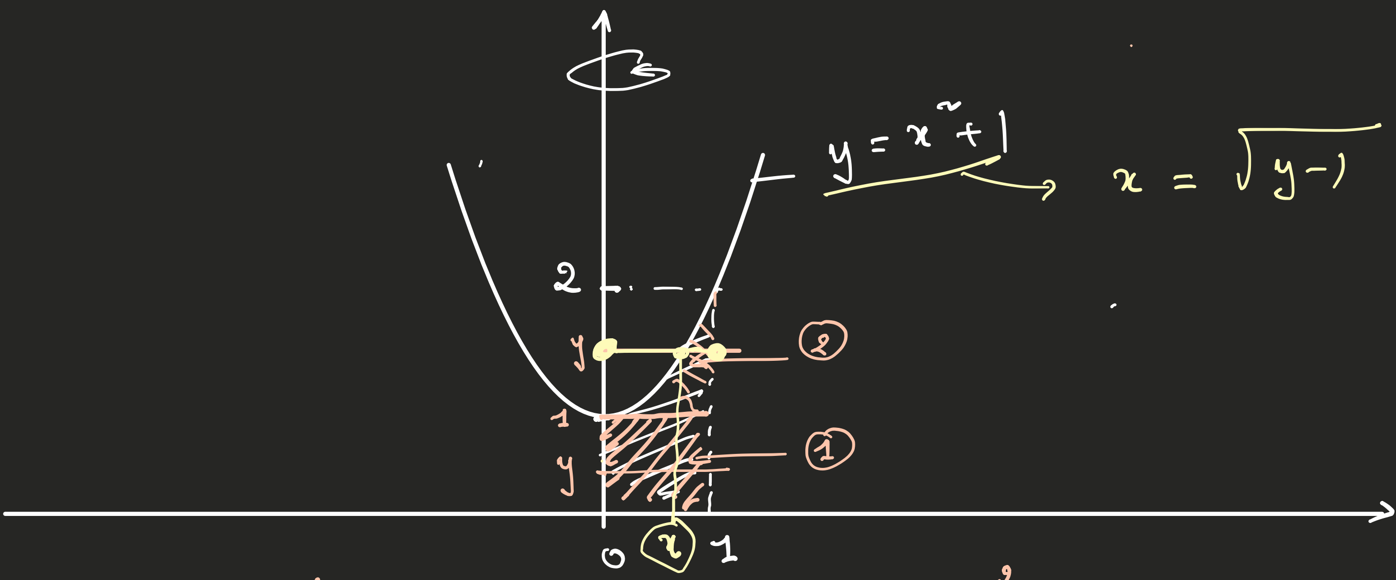
$y = x^2+1$, $y = 0$, $x = 0$, and $x = 1$

about the y-axis.

A) By the disc Method

B) By the Shell Method.





$$V_1 = \int_0^1 \pi 1^2 \, dy \quad , \quad V_2 = \int_1^2 \left(\pi 1^2 - \pi\left(\sqrt{y-1}\right)^2\right) dy$$

$$V_1 = \pi$$

$$V_2 = \int_1^2 [\pi - \pi(y-1)] \, dy$$

$$= \pi \int_1^2 (1 - y + 1) \, dy$$

$$= \pi \int_1^2 (2 - y) \, dy$$

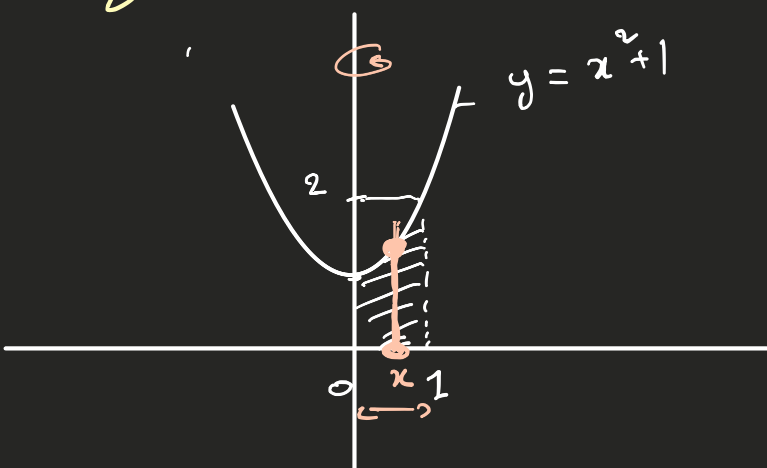
$$= \pi \left[2y - \frac{y^2}{2} \right]_1^2$$

$$= \pi \left(4 - 2 - \left(2 - \frac{1}{2} \right) \right)$$

$$= \pi \left(2 - \frac{3}{2} \right) = \frac{\pi}{2}$$

$$V = V_1 + V_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

B) The Shell Method



$$V = \int_0^1 2\pi x (x^2 + 1) \, dx$$

$$= 2\pi \int_0^1 (x^3 + x) \, dx$$

$$= 2\pi \left(\frac{x^4}{4} + \frac{x^2}{2} \Big|_0^1 \right) = 2\pi \left(\frac{1}{4} + \frac{1}{2} \right) = 2\pi \cdot \frac{3}{4}$$

$$= \frac{3\pi}{2}$$

$\int$ Integration by Parts

$$\frac{d}{dx}(u \cdot v) = u'v + uv'$$

$$\int \frac{d}{dx}(u \cdot v)\, dx = \int u'v\, dx + \int uv'\, dx$$

$$u \cdot v = \int u'v\, dx + \int uv'\, dx$$

* $\int u'v\, dx = uv - \int uv'\, dx$
* $\int uv'\, dx = uv - \int u'v\, dx$

The integration by parts formulas

Exple Evaluate

$u = x$ $\quad u' = 1$

$v' = \sin x$ $\quad v = -\cos x$

$$\int x \sin x\, dx$$

$$\int x \sin x\, dx = -x\cos x - \int (1)(-\cos x)\, dx$$

$$= -x\cos x + \int \cos x\, dx$$

$$= -x\cos x + \sin x + C$$

Exple Evaluate

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

1) $\int x^2 e^x\, dx$

2) $\int x^3 \ln x\, dx$

3) $\int \ln x\, dx$

4) $\int e^x \sin x\, dx$

5) $\int \sec^3 x\, dx$

1.) $\int x^2 e^x \, dx$

$u = x^2 \qquad u' = 2x$

$v' = e^x \qquad v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

$$= x^2 e^x - 2 \int x e^x \, dx$$

$\int x e^x \, dx = \, ?$

$u = x \qquad u' = 1$

$v' = e^x \qquad v = e^x$

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$$

$$\int x^2 e^x \, dx = x^2 e^x - 2 \left[x e^x - e^x + c \right]$$

$$= x^2 e^x - 2x e^x + 2 e^x + C$$

2.) $\int x^3 \ln x \, dx$.

$u = \ln x \, , \quad u' = \frac{1}{x}$

$v' = x^3 \qquad v = \frac{x^4}{4}$

$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \left(\frac{1}{\cancel{x}} \right) \left(\frac{x^{\cancel{4}\,3}}{4} \right) dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \, \frac{x^4}{4} + C$$

$$= \frac{x^4}{4} \left(\ln x - \frac{1}{4} \right) + C$$

3.) $\int 1 . \ln x \, dx$

$u = \ln x \quad , \quad u' = \frac{1}{x}$

$v' = 1 \quad , \quad v = x$

$$\int \ln x \, dx = x \ln x - \int \left(\frac{1}{\cancel{x}}\right)(\cancel{x}) \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

4) $\int e^x \sin x \, dx$

$u = e^x \quad , \quad u' = e^x$

$v' = \sin x \quad , \quad v = -\cos x$

Let $A = \int e^x \sin x \, dx.$

$$A = -e^x \cos x - \int -e^x \cos x \, dx$$

$$A = -e^x \cos x + \int e^x \cos x \, dx$$

$u = e^x \quad , \quad u' = e^x$

$v' = \cos x \quad , \quad v = \sin x$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

A

$$A = -e^x \cos x + e^x \sin x - A$$

$$2A = -e^x \cos x + e^x \sin x$$

$$A = \frac{1}{2}(e^x \sin x - e^x \cos x) + C$$

5.) $A = \int \sec^3 x \, dx = \int \sec^2 x \cdot \sec x \, dx$

$u = \sec x \qquad u' = \sec x \tan x$

$v' = \sec^2 x \qquad v = \tan x$

$$A = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \underbrace{\int \sec^3 x \, dx}_{A} + \int \sec x \, dx$$

$$A = \sec x \tan x - A + \ln|\sec x + \tan x| + C$$

$$2A = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$A = \int \sec^3 x \, dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

* Integration by parts and definite integrals.

$$\int_a^b u' v \, dx = uv\Big|_a^b - \int_a^b u v' \, dx$$

Exple Evaluate

1) $\int_0^1 x \cos x \, dx$ 2) $\int_1^2 x \ln x \, dx$.

1) $\int_0^1 x\cos x\,dx$.

$u = x \quad u' = 1$

$v' = \cos x \quad v = \sin x$

$$\int_0^1 x\cos x\,dx = \underbrace{x\sin x\Big|_0^1}_{} - \int_0^1 \sin x\,dx$$

$$= \overbrace{(1)\sin(1) - (0)\sin(0)} - \left(-\cos x\Big|_0^1\right)$$

$$= \sin 1 - (-\cos 1 + \cos 0)$$

$$= \sin 1 + \cos 1 - 1$$

2) $\int_1^2 x\ln x\,dx$.

$u = \ln x, \quad u' = \frac{1}{x}$

$v' = x, \quad v = \frac{x^2}{2}$

$$\int_1^2 x\ln x\,dx = \frac{x^2}{2}\ln x\Big|_1^2 - \int_1^2 \left(\frac{1}{\cancel{x}}\right)\left(\frac{x^{\cancel{2}}}{2}\right)dx$$

$$= 2\ln 2 - \frac{1}{2}\ln 1 - \frac{1}{2}\left(\frac{x^2}{2}\Big|_1^2\right)$$

$$= 2\ln 2 - \frac{1}{4}(4-1)$$

$$= 2\ln 2 - \frac{3}{4}.$$

§ Integration of Rational functions by Partial Fraction Decomposition

$\frac{\sqrt{2}}{3}$

$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$$\frac{x^2+x+1}{3x^3-4x+7} \checkmark \qquad \frac{2\sqrt{x}+1}{x^3+x-5} \times$$

$$\int \frac{P(x)}{Q(x)}\,dx$$ where both $P(x)$ and $Q(x)$ are polynomial functions.

* $\underline{\text{CASE I}}$: $\deg P(x) < \deg Q(x)$
  * Rule 1
  * Rule 2
  * Rule 3
  * Rule 4 .

* $\underline{\text{CASE II}}$ $\deg P(x) \geq \deg Q(x)$.

$$\frac{P(x)}{Q(x)} = f(x) + \frac{r(x)}{Q(x)}$$ with $\deg r(x) < \deg Q(x)$

$$\int \frac{P(x)}{Q(x)}\,dx = \int f(x)\,dx + \underbrace{\int \frac{r(x)}{Q(x)}\,dx}$$

* $\longrightarrow$ $$\int \frac{P(x)}{Q(x)}\,dx$$

CASE I : $\deg P(x) < \deg Q(x)$

Rule 1 If $Q(x)$ can be decomposed as a product of distinct linear factors (none of which is repeated)

$$Q(x) = a(x-c_1)(x-c_2)\cdots(x-c_n),$$

then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ is given by

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-c_1} + \frac{A_2}{x-c_2} + \cdots + \frac{A_n}{x-c_n}$$

Exple Evaluate $\int \frac{dx}{x^2-5x+6}$

$$\underbrace{x^2-5x+6} = (x-3)(x-2)$$

$$\frac{1}{(x-3)(x-2)} = \frac{A_1}{x-3} + \frac{A_2}{x-2}$$

for A_1

$$* \quad \frac{1}{x-2} = A_1 + \frac{(x-3)A_2}{x-2} \leftarrow$$

$x = 3$

$$\frac{1}{3-2} = A_1 + 0 \qquad A_1 = \frac{1}{1} = 1$$

for A_2

$$* \quad \frac{1}{x-3} = \frac{(x-2)A_1}{x-3} + A_2$$

$x = 2$

$$\frac{1}{2-3} = 0 + A_2 \rightarrow A_2 = \frac{1}{-1} = -1$$

$$\frac{1}{x^2-5x+6} = \frac{1}{x-3} - \frac{1}{x-2}$$

$$\int \frac{dx}{x^2-5x+6} = \int \frac{dx}{x-3} - \int \frac{dx}{x-2}$$

$$= \ln|x-3| - \ln|x-2| + C$$

$$= \ln\left|\frac{x-3}{x-2}\right| + C$$

General Method

$$\frac{1}{(x-3)(x-2)} = \frac{A_1}{x-3} + \frac{A_2}{x-2}$$

* Multiply both sides by $(x-3)(x-2)$ to get

$$1 = A_1(x-2) + A_2(x-3)$$

$$1 = A_1x + A_2x - 2A_1 - 3A_2$$

$$0x + 1 = (A_1 + A_2)x - 2A_1 - 3A_2$$

$$2\begin{cases} A_1 + A_2 = 0 \longrightarrow A_1 = -A_2 \\ -2A_1 - 3A_2 = 1 \end{cases}$$

$$-A_2 = 1 \Rightarrow \underline{A_2 = -1}$$

$$A_1 = -(-1) = 1$$

$$A_1 = 1$$

Ex ple   Evaluate

$$\int \frac{x+2}{x^3 - x^2 - 2x} \, dx$$

$$x^3 - x^2 - 2x = x\,(\underbrace{x^2 - x - 2})$$

$$= x\,(x+1)(x-2)$$

$$\frac{x+2}{x(x+1)(x-2)} = \frac{A_1}{x} + \frac{A_2}{x+1} + \frac{A_3}{x-2}$$

$A_1 = ?$

$$\frac{x+2}{(x+1)(x-2)} = A_1 + x\left(\frac{A_2}{x+1} + \frac{A_3}{x-2} \right)$$

$x = 0$

$$\frac{2}{-2} = A_1 + 0 \Rightarrow A_1 = -1$$

$A_2 = ?$

$$\frac{x+2}{x(x-2)} = A_2 + (x+1)\left(\frac{A_1}{x} + \frac{A_3}{x-2} \right)$$

$x = -1$

$$\frac{-1+2}{(-1)(-1-2)} = A_2 + 0 \qquad A_2 = \frac{1}{3}$$

$A_3 = ?$

$$\frac{x+2}{x(x+1)} = A_3 + (x-2)\left(\frac{A_1}{x} + \frac{A_2}{x+1} \right)$$

$x = 2$

$$\frac{4}{2(3)} = A_3 \qquad A_3 = \frac{4}{6} = \frac{2}{3}$$

$$\frac{x+2}{x(x+1)(x-2)} = -\frac{1}{x} + \frac{1}{3}\cdot\frac{1}{x+1} + \frac{2}{3}\frac{1}{x-2}$$

$$\int \frac{x+2}{x^3 - x^2 - 2x}\,dx = \int -\frac{1}{x}\,dx + \frac{1}{3}\int \frac{dx}{x+1} + \frac{2}{3}\int \frac{dx}{x-2}$$

$$= -\ln|x| + \frac{1}{3}\ln|x+1| + \frac{2}{3}\ln|x-2| + C$$