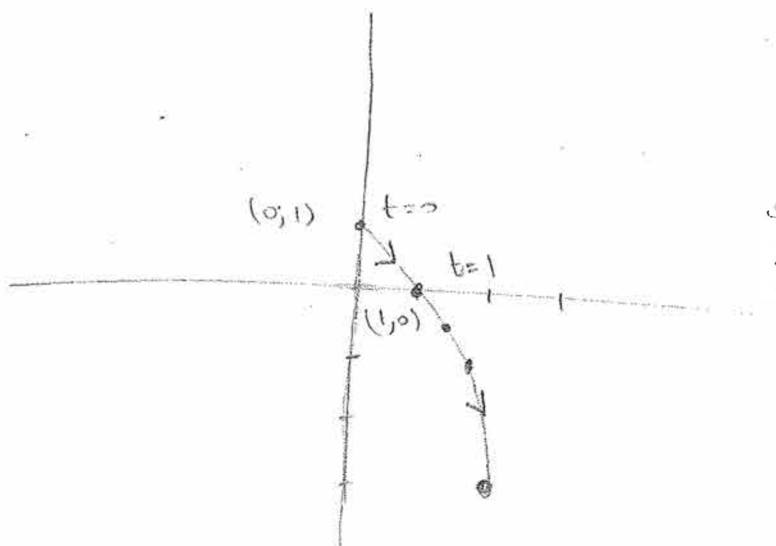
Parametric Curves and Parametric Equations.

Ex. 1 Sketch the curve described by the parametric equations.

a) $x = \sqrt{t}$ $\quad y = 1 - t$ $\quad 0 \le t \le 4$

t	0	1	2	3	4
x	0	1	1.4	1.7	2
y	1	0	-0.4	-0.7	-3



$t = 0 \quad (0,1)$

$t = 1 \quad (1,0)$

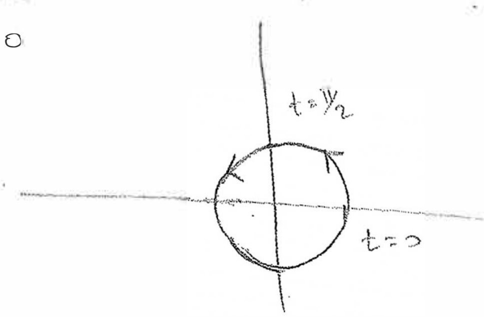
orientation: In the direction of increasing parameter.

b) $x = \cos t$ $\quad y = \sin t$ $\quad 0 \le t \le 2\pi$

t	0	$\pi/2$	π	$3\pi/2$	2π
x	1	0	-1	0	1
y	0	1	0	-1	0

$t = 0 \quad (1,0)$

$t = \pi/2 \quad (0,1)$



Eliminating the Parameter

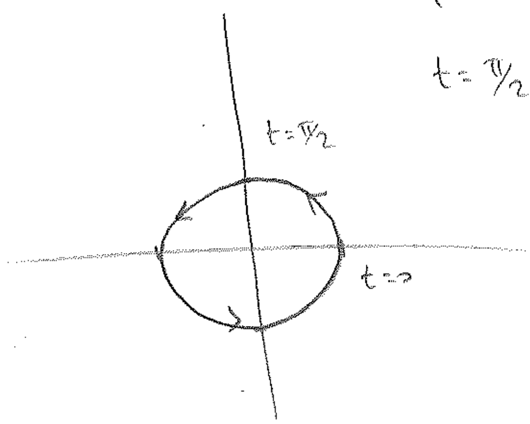
Ex.2 Sketch the curve represented by the following parametric equations by eliminating the parameter.

a) $x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$

$$\cos^2 t + \sin^2 t = 1 \Rightarrow \boxed{x^2 + y^2 = 1} \quad \text{(rectangular equation)}$$

$t = 0 \quad (1, 0)$

$t = \pi/2 \quad (0, 1)$



b) $x = 2t - 3 \qquad y = 6t - 7$

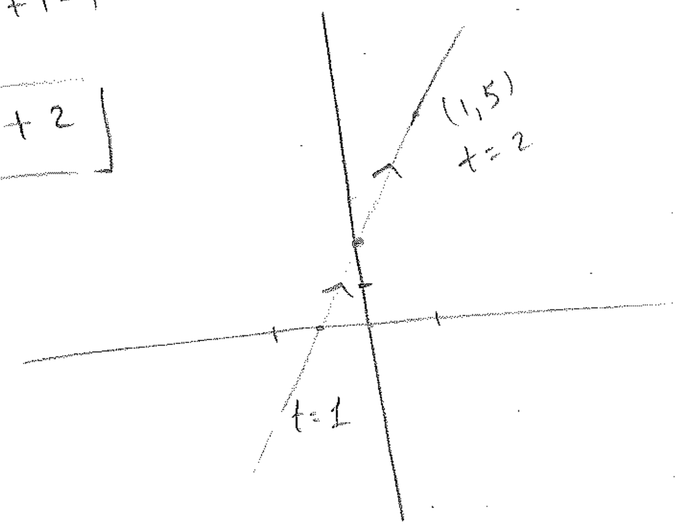
$x = 2t - 3 \Rightarrow 2t = x + 3$

$t = \dfrac{x+3}{2}$

$y = 6\left(\dfrac{x+3}{2}\right) - 7 \Rightarrow y = 3(x+3) - 7$

$y = 3x + 9 - 7$

$$\boxed{y = 3x + 2}$$



$t = 1 \quad (-1, -1)$

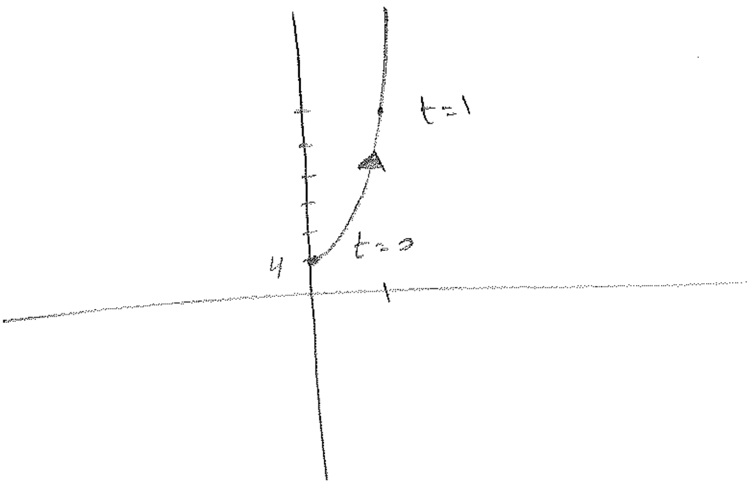
$t = 2 \quad (1, 5)$

(c) $x = \sqrt{t}$ $\quad y = 2t + 4$

$x = \sqrt{t} \Rightarrow x^2 = t \Rightarrow \boxed{y = 2x^2 + 4}$ $\quad x \geq 0$

$t = 0 \quad (0, 4)$

$t = 1 \quad (1, 6)$



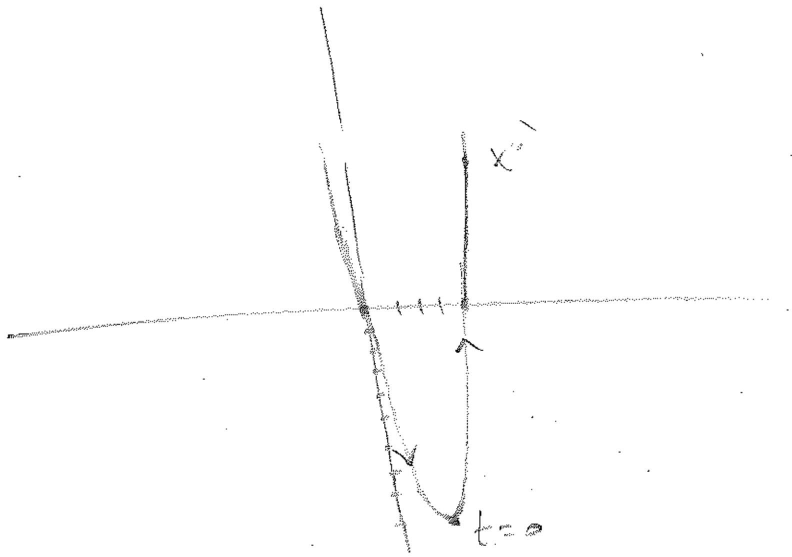
(d) $x = 4t + 3$ $\quad y = 16t^2 - 9$

$x = 4t + 3 \Rightarrow t = \frac{x-3}{4} \Rightarrow y = 16\left(\frac{x-3}{4}\right)^2 - 9$

$y = (x-3)^2 - 9$

$t = 0 \quad (3, -9)$

$t = 1 \quad (7, 7)$



e) $x = \sec^2 t \qquad y = \tan^2 t \qquad 0 \le t < \frac{\pi}{2}$

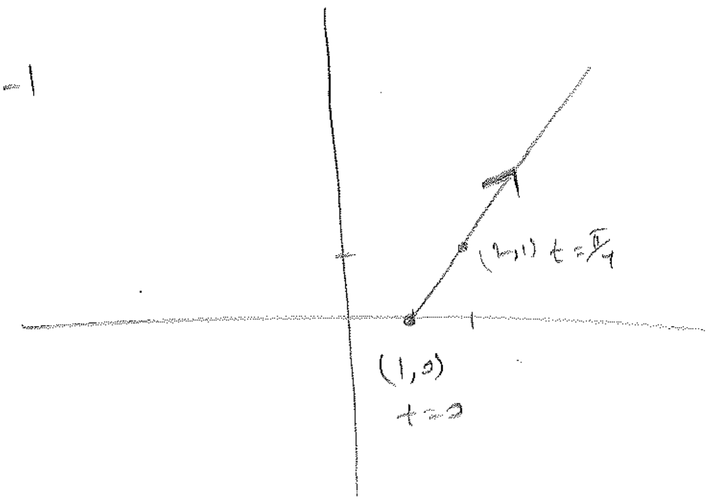
$1 + \tan^2 t = \sec^2 t$

$1 + y = x \Rightarrow y = x - 1$

$y \geq 0 \Rightarrow x - 1 \geq 0$

$x \geq 1$

$$\boxed{y = x - 1 \qquad x \geq 1}$$



$t = 0 \quad (1, 0)$

$t = \frac{\pi}{4} \quad (2, 1)$

f) $x = 2\cos t \qquad y = 5 \sin t \qquad 0 \le t \le 2\pi$

$\cos t = \frac{x}{2} \qquad \sin t = \frac{y}{5}$

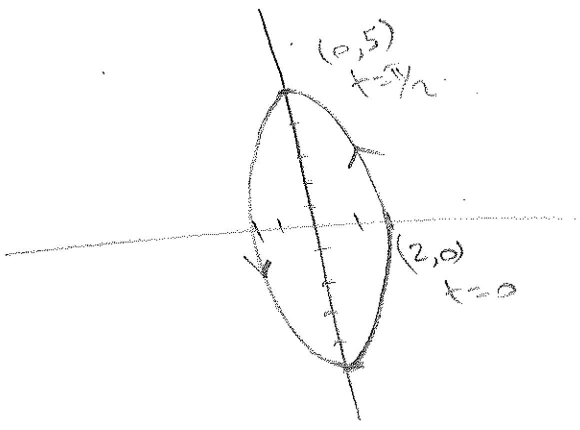
$\sin^2 t + \cos^2 t = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$

$$\boxed{\frac{x^2}{4} + \frac{y^2}{25} = 1}$$

Ellipse with vertical axis.

$a^2 = 25 \qquad b^2 = 4$

$a = 5 \qquad b = 2$



$t = 0 \quad (2, 0)$

$t = \pi/2 \quad (0, 5)$

8) $x = 3\sin^2 t \qquad y = 5\sec t$

$$-\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\cos t = \sqrt{1-\sin^2 t}$$

$$= \sqrt{1-\frac{x}{3}}$$

$$y = 5\sec t = \frac{5}{\cos t}$$

$$y = \frac{5}{\sqrt{1-\frac{x}{3}}}$$

$0 \le \sin^2 t < 1$

$0 \le \frac{x}{3} < 1$

$\boxed{0 < x < 3}$

Finding Parametric Equations from Rectangular Equations.

Ex. 3 Find a set of parametric equations for the following rectangular equation.

$y = 2x^2 + 1$

let $x = t \Rightarrow y = 2t^2 + 1$

$$\boxed{x = t \;,\; y = 2t^2 + 1}$$

OR $x = 2t \Rightarrow y = 2(2t)^2 + 1$

$y = 8t^2 + 1$

$$\boxed{x = 2t \;,\; y = 8t^2 + 1}$$

OR $x = -t \Rightarrow y = 2(-t)^2 + 1$

$y = 2t^2 + 1$

$$\boxed{x = -t \;,\; y = 2t^2 + 1}$$

Slope, Concavity and Tangent lines.

$x = f(t) \qquad y = g(t)$. let's find $\left|\frac{dy}{dx}\right|$

If we eliminate the parameter we get $y = F(x)$

$g(t) = F(f(t))$

$g'(t) = F'(f(t)) \cdot f'(t)$

$F'(f(t)) = \frac{g'(t)}{f'(t)}$

$F'(x) = \frac{g'(t)}{f'(t)} \Rightarrow \boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$

$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$

$\boxed{\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}}$

Ex. 4 find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and evaluate each at the indicated value of the parameter.

a) $x = 2\cos t \qquad y = 2\sin t \qquad t = \frac{\pi}{4}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t}$

$= -\cot t$

$\left.\frac{dy}{dx}\right|_{t=\pi/4} = -\cot \pi/4$

$= -1$

$\frac{d^2y}{dx^2} = \frac{+\csc^2 t}{-2\sin t} = -\frac{\csc^3 t}{2}$

$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/4} = -\sqrt{2}$

Note If we eliminate the parameter.

$\sin^2 t + \cos^2 t = \frac{y^2}{4} + \frac{x^2}{4} = 1$

$x^2 + y^2 = 4$

$2x + 2yy' = 0$

$2yy' = -2x$

$y' = -\frac{2x}{2y}$

$y' = -\frac{x}{y}$

$\frac{dy}{dx} = \frac{-x}{y}$

$t = \pi/4 \quad (\sqrt{2}, \sqrt{2})$

$\left.\frac{dy}{dx}\right|_{(\sqrt{2},\sqrt{2})} = -1$

b) $x = t+1 \qquad y = t^2+3t \qquad t=-1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+3}{1} = 2t+3$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = 2(-1)+3 = 1$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{2}{1}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=-1} = 2$$

c) $x = \frac{1}{2}t^2+1 \qquad y = \frac{1}{3}t^3-t \qquad t=4$

$$\frac{dy}{dx} = \frac{t^2-1}{t} = t-\frac{1}{t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=4} = \frac{15}{4}$$

$$\frac{d^2y}{dx^2} = \frac{1+1/t^2}{t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=4} = \left.\frac{t^2+1}{t^3}\right|_{t=4} = \frac{15}{64}$$

Ex. 5 for the curve given by

$$x = \sqrt{t} \qquad y = \frac{1}{4}(t^2-4) \qquad t \geq 0$$

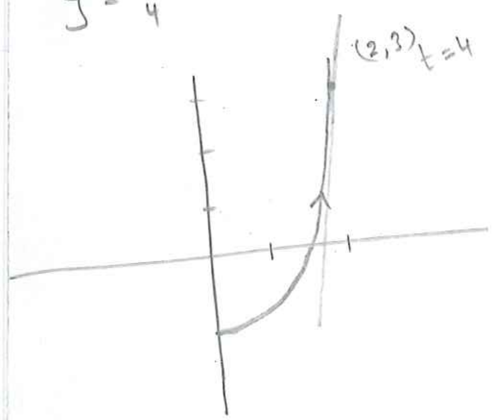
find the slope and concavity at the point $(2,3)$

$(2,3) \quad x=2 \Rightarrow 2=\sqrt{t} \Rightarrow t=4$

$x=\sqrt{t} \Rightarrow x^2=t$

$y = \frac{1}{4}(x^4-4)$

$y = \frac{1}{4}x^4-1 \qquad x \geq 0$



$$\frac{dy}{dx} = \frac{\frac{1}{4}(2t)}{\frac{1}{2\sqrt{t}}} = t.\sqrt{t} = t^{3/2}$$

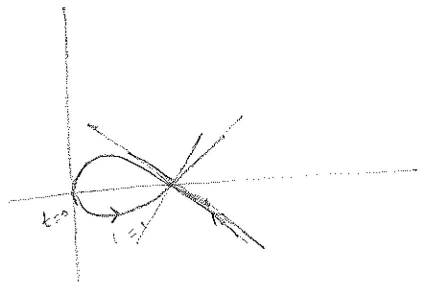
$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2}t^{1/2}}{\frac{1}{2\sqrt{t}}} = 3t^{1/2}.\sqrt{t} = 3t$$

$$\text{slope} = \left.\frac{dy}{dx}\right|_{t=4} = (4)^{3/2} = 8$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=4} = 12$$

Ex. 6 Show that the curve $x = t^2$ and $y = t^3 - 9t$ intersects itself at the point $(9, 0)$, and find equations for the two tangent lines to the curve at the point of intersection.

$(9,0) \Rightarrow x = 9$ and $y = 0$



$t^2 = 9$ \
$t = \pm 3$

$t^3 - 9t = 0$ \
$t(t^2 - 9) = 0$ \
$t = 0$ or $t^2 - 9 = 0$ \
$t^2 = 9$ \
$t = \pm 3$

$\therefore$ $(9,0)$ is reached when $t = \pm 3$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3t^2 - 9}{2t}$$

$$\left.\frac{dy}{dx}\right|_{t=3} = \frac{18}{6} = 3$$

$$\left.\frac{dy}{dx}\right|_{t=-3} = -3$$

$y - y_0 = m(x - x_0)$ $\quad (9, 0)$ — x_0 y_0

$m = \pm 3$

$y - 0 = \pm 3(x - 9)$

$$\boxed{y = \pm 3(x - 9)}$$

Horizontal and Vertical Tangents

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Horizontal tangent $\Rightarrow$ $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

Vertical tangent $\Rightarrow$ $\frac{dy}{dt} \neq 0$ and $\frac{dx}{dt} = 0$

Ex. Find all points of vertical and horizontal tangency to the curve.

$x = 3\cos\theta \quad y = 3\sin\theta \qquad 0 \le \theta \le 2\pi$

Horizontal $\quad \frac{dy}{d\theta} = \boxed{3\cos\theta = 0} \quad \frac{dx}{d\theta} = \boxed{-3\sin\theta \neq 0}$

$\theta = \pi/2 \,, \quad \theta = \frac{3\pi}{2} \Rightarrow$ corresponding points.

$(0, 3) \quad (0, -3)$

Vertical $\quad \frac{dy}{d\theta} = 3\cos\theta \qquad \frac{dx}{d\theta} = -3\sin\theta$

$3\cos\theta \neq 0$ and $-3\sin\theta = 0 \Rightarrow \theta = 0, \pi$

$\Rightarrow$ corresponding points.

$(3, 0) \quad (-3, 0)$

Practice

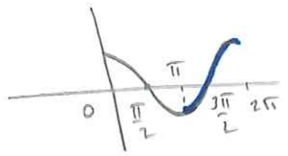
Ex.1 sketch the curve represented by the following parametric equations by eliminating the parameter.

a) $x = 2\cos t \ , \quad y = 2\sin t \qquad \pi \le t \le 2\pi$

$\sin^2 t + \cos^2 t = 1 \Rightarrow \left(\frac{y}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = 1$

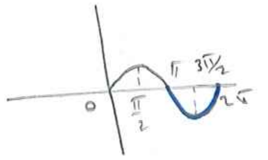
$\frac{x^2}{4} + \frac{y^2}{4} = 1 \Rightarrow \boxed{x^2 + y^2 = 4}$

$\pi \le t \le 2\pi \Rightarrow -1 \le \cos t \le 1$



$-2 \le 2\cos t \le 2$

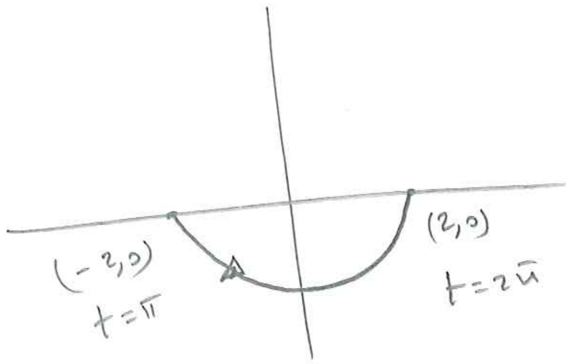
$\boxed{-2 \le x \le 2}$



$\pi \le t \le 2\pi \Rightarrow -1 \le \sin t \le 0$

$-2 \le 2\sin t \le 0$

$\boxed{-2 \le y \le 0}$



counterclockwise

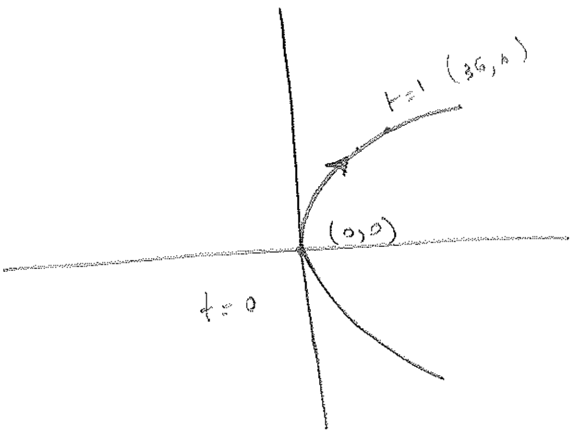
b) $x = 36t^2 \quad y = 6t \quad -\infty < t < \infty$

$y = 6t \Rightarrow t = \frac{y}{6}$

$x = 36\left(\frac{y}{6}\right)^2 = \cancel{36}\left(\frac{y^2}{\cancel{36}}\right)$

$x = y^2$

$0 \leq x < \infty$



Ex 2. Find an equation for the line tangent to the curve at the point defined by the given value of t.

$x = t + \cos t \quad y = 2 - \sin t \quad t = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\cos t}{1 - \sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = -\sqrt{3}$$

$t = \frac{\pi}{6} \Rightarrow x_0 = \frac{\pi}{6} + \frac{\sqrt{3}}{2} \qquad y = 2 - \frac{1}{2} = \frac{3}{2}$

$y - y_0 = m(x - x_0) \Rightarrow y - \frac{3}{2} = -\sqrt{3}\left(x - \frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)$

$$\boxed{y = -\sqrt{3}x + \frac{\sqrt{3}}{6}\pi + 3}$$

Ex. For the given curve find the concavity at the indicated value of t.

3

$$x = 8t^2 - 5 \quad , \quad y = t^3 \quad , \quad t = 1$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$$

$$\frac{dy'}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{16t} = \frac{3t}{16}$$

$$\frac{d^2y}{dx^2} = \frac{3/16}{\frac{16t}{1}} = \frac{3}{256t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{3}{256(1)} = \frac{3}{256} > 0 \Rightarrow \text{concave up at } (3,1).$$