Differential Equations

Definition

A differential equation (DE) is an equation involving a function and its derivative(s).

Ordinary Differential Equations (ODEs)

An ODE is an equation that relates a function to its ordinary derivates. (Leibniz)

Example ① a) $\frac{dy}{dx} + xy = 0$ or $y' + xy = 0$

b) $\frac{d^2y}{dx^2} + 4y^3 = 0$ or $y'' + 4y^3 = 0$

c) $x\frac{d^4y}{dx^4} + \frac{dy}{dx} + y = xe^x$ or $xy^{(4)} + y' + y = xe^x$

Note: $y = f(x)$; y is the dependent variable and x is the independent variable.

Order of ODEs

The order of an ODE is the order of the highest derivative appearing in the ODE.

Linearity of ODEs

An ODE is linear if it has the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

Where the coefficients $a_n(x), a_{n-1}(x), \dots, a_0(x)$ and the function $F(x)$ depend only on the independent variable x.

Example (2) Determine the orders of the following equations and state whether the equation is linear or nonlinear.

a) $\frac{dy}{dx} + xy = 0$

Ans. 1^{st} order linear

b) $\frac{d^2y}{dx^2} + y^3 = 0$

Ans. 2^{nd} order nonlinear

c) $\left(\frac{d^4y}{dx^4}\right)^2 + xy = 2$

Ans. 4^{th} order nonlinear

d) $5\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = 2\cos 3t$

Ans. 2^{nd} order linear

e) $\sqrt{1-y}\,\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

Ans. 2^{nd} order nonlinear (because of $\sqrt{1-y}\,\frac{d^2y}{dx^2}$ term)

f) $\frac{d^2y}{dx^2} + \sin y = 0$

Ans. 2^{nd} order nonlinear

Solution of an ODE

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an n^{th}-order ODE reduce the equation to an identity.

Example 3: Verify that the indicated function is a solution of the given DE on the interval $(-\infty, \infty)$

a) $\frac{dy}{dx} = xy^{\frac{1}{2}}$; $y = \frac{1}{16}x^4$

$$y = \frac{1}{16}x^4 \Rightarrow \frac{dy}{dx} = 4\left(\frac{1}{16}\right)x^3 = \frac{1}{4}x^3$$

substitute y and $\frac{dy}{dx}$ in the given DE

$$\frac{1}{4}x^3 = x\left[\frac{1}{16}x^4\right]^{\frac{1}{2}}$$

$$\frac{1}{4}x^3 = x\left[\frac{1}{4}x^2\right]$$

$$\frac{1}{4}x^3 = \frac{1}{4}x^3 \quad \text{(identity)}$$

b) $y'' - 2y' + y = 0$; $y = xe^x$

$$y = xe^x \Rightarrow y' = e^x + xe^x \text{ and } y'' = e^x + e^x + xe^x$$

$$y'' = 2e^x + xe^x$$

substitute y' and y'' in $y'' - 2y' + y = 0$

$$2e^x + xe^x - 2(e^x + xe^x) + xe^x = 0$$

$$0 = 0 \quad \text{(identity)}$$

Initial Value Problems (IVPs)

An initial value problem (IVP) is a differential equation (DE) along with an appropriate number of initial conditions.

Example 4 : show that $y(x) = \sin x - \cos x$ is a solution to the IVP

$$\frac{d^2y}{dx^2} + y = 0 \qquad y(0) = -1 , \quad y'(0) = 1$$

$$y = \sin x - \cos x \Rightarrow y' = \cos x + \sin x \text{ and } y'' = -\sin x + \cos x$$

$$-\sin x + \cos x + \sin x - \cos x = 0$$

$$0 = 0 \quad (\text{identity})$$

$$y(0) = \sin 0 - \cos 0 = 1$$

$$y'(0) = \cos 0 + \sin 0 = 1$$

Motivation

In sciences and engineering, DEs model real world problems and hence help us better understand and interpret such problems.

Example : The rate of loss of mass of an isotope is proportional to its mass.

Let $m(t)$ be the mass of an isotope at time t $\Rightarrow \frac{dm}{dt} = -k\,m(t)$
(Decay)

First - Order Differential Equations

* Separable Equations $\frac{dy}{dx} = f(x,y)$

Def. A first-order DE $\frac{dy}{dx} = f(x,y)$ is separable if it can be written in the form $\frac{dy}{dx} = g(x) \cdot h(y)$

Example 4 : a) $\frac{dy}{dx} = y^2 x e^{3x+4y}$ is separable

why? $\frac{dy}{dx} = \underbrace{xe^{3x}}_{g(x)} \cdot \underbrace{y^2 e^{4y}}_{h(y)}$

b) $\frac{dy}{dx} = y + \sin x$ is not separable.

why? $\frac{dy}{dx} \neq g(x) \cdot h(y)$

Note : To solve a separable DE we separate the variables and integrate both sides.

Ex. 5 a) Solve $(1+x)\,dy - y\,dx = 0$

$$(1+x)\,dy = y\,dx$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

$$\ln|y| = \ln|1+x| + C_1$$

$$\ln|y| - \ln|1+x| = C_1$$

$$\ln\left|\frac{y}{1+x}\right| = C_1$$

$$\left|\frac{y}{1+x}\right| = e^{C_1}$$

$$\frac{y}{1+x} = \pm e^{C_1}$$

$$\frac{y}{1+x} = C \qquad \text{where } C = \pm e^{C_1}$$
(arbitrary constant)

$$y = C(1+x)$$

b) Show that the differential equation

$$2xy + 6x + (x^2-4)y' = 0$$

is separable and solve the equation.

$$2xy + 6x + (x^2-4)\frac{dy}{dx} = 0$$

$$(x^2-4)\frac{dy}{dx} = -2xy - 6x$$

$$(x^2-4)\frac{dy}{dx} = -2x(y+3)$$

$$\frac{dy}{dx} = \underbrace{\frac{-2x}{x^2-4}}_{g(x)} \cdot \underbrace{y+3}_{h(y)}$$

Hence separable.

<u>Solving</u>

$$\frac{dy}{y+3} = \frac{-2x}{x^2-4}\,dx$$

$$\int \frac{dy}{y+3} = \int \frac{-2x}{x^2-4}\,dx$$

$$\ln|y+3| = -\ln|x^2-4| + c_1$$

$$\ln|y+3| + \ln|x^2-4| = c_1$$

$$\ln|(y+3)(x^2-4)| = c_1$$

$$|(y+3)(x^2-4)| = e^{c_1}$$

$$(y+3)(x^2-4) = \pm e^{c_1}$$

$$(y+3)(x^2-4) = C \qquad (C = \pm e^{c_1})$$

$$y+3 = \frac{C}{x^2-4}$$

$$\boxed{y = \frac{C}{x^2-4} - 3}$$

c) Solve the IVP

$$(e^{2y}-y)\cos x \frac{dy}{dx} = e^y \sin 2x \qquad y(0)=0$$

$$\frac{dy}{dx} = \underbrace{\frac{\sin 2x}{\cos x}}_{g(x)} \cdot \underbrace{\frac{e^y}{e^{2y}-y}}_{h(y)} \qquad \text{separable}$$

$$\frac{dy}{\frac{e^y}{e^{2y}-y}} = \frac{\sin 2x}{\cos x}\, dx$$

$$\frac{e^{2y}-y}{e^y}\, dy = \frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}}\, dx$$

$$(e^y - ye^{-y})\, dy = 2\sin x\, dx$$

$$\int (e^y - ye^{-y})\, dy = \int 2\sin x\, dx$$

Note $\int y e^{-y}\,dy$

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y	$+$	e^{-y}
1	$-$	$-e^{-y}$
0		e^{-y}

$= -ye^{-y} - e^{-y}$

So

$$e^{y} - \left(-ye^{-y} - e^{-y}\right) = -2\cos x + C$$

$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + C$$

$$y(0) = 0 \Rightarrow e^{0} + \cancel{0 \cdot e^{0}} + e^{0} = -2\cos 0 + C$$

$$2 = -2 + C$$

$$\underline{C = 4}$$

$\therefore$ The solution of the IVP is given implicitly as

$$\boxed{e^{y} + ye^{-y} + e^{-y} = -2\cos x + 4}$$