Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

is called a power series centred at 0.

$\sum_{n=0}^{\infty} a_n (x-a)^n$ is a power series centred at a.

Example

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} \frac{(x-2)^n}{n+1}$$

$$a_n = \frac{1}{\sqrt{n}} \qquad a_n = \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \qquad x = -1$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad b_n = \frac{1}{\sqrt{n}}$$ converges by AST

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \qquad x = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$p = 1/2 < 1$

$\therefore$ Diverges

Radius of Convergence & Interval of Convergance

Consider the power series

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$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

To find the radius and interval of convergance of the power series do the following

1. Compute $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n}\right| =$

Ans	Radius of convergence	Interval of convergence
0	∞	$(-\infty, \infty)$
$+\infty$ or diverges, except at center $\{a\}$	0	$[a, a]$
$L\lvert x-a\rvert$	$1/L$	interval includes $\left(a-\frac{1}{L}, a+\frac{1}{L}\right)$

$L|x-a| < 1$ converges

$L|x-a| = 1$ inconclusive

$L|x-a| > 1$ diverges

Since the ratio test is inconclusive when lim = 1, manually plug $x = a \pm 1/L$ and determine convergance. If converges, the bracket closes and $\boxed{a \pm 1/L}$ is part of the interval of convergance

either +ve, -ve or both. Check both +ve & -ve

for convergance

$$L|x-a| < 1$$

$$|x-a| < \frac{1}{L}$$

$$-\frac{1}{L} < x-a < \frac{1}{L}$$

$$a - \frac{1}{L} < x < a + \frac{1}{L}$$

Example

Find the radius and interval of convergance of the following power series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{n+1}}{(n+1)!} \times \frac{n!}{(-1)^n x^n}\right|$$

$$\lim \left| \frac{\cancel{(-1)^n}(-1)\cancel{x^n}x}{\quad} \times \frac{\cancel{n!}}{\quad} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cdots}{(n+1)\,\cancel{n!}} \cdot \frac{\cdots}{\cancel{(-1)^n x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$$

$$= 0$$

Radius of convergence $= +\infty$

Interval of convergence $= (-\infty, \infty)$

2. $\sum_{n=0}^{\infty} n 2^n (x+1)^n$

$a = -1$

$$\lim_{n\to\infty} \left| \frac{(n+1)\, 2^{n+1} (x+1)^{n+1}}{n\, 2^n (x+1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2(n+1)(x+1)}{n} \right|$$

$$= \lim_{n\to\infty} \frac{2(n+1)}{n} |x+1|$$

$$= 2|x+1|$$

Radius of convergence $= \frac{1}{2}$

Interval of convergance

Endpoints $-\frac{3}{2}$, $-\frac{1}{2}$

@ $x = -3/2$, the series becomes

$$\sum_{n=0}^{\infty} n 2^n \left(-3/2 + 1\right)^n$$

$$= \sum_{n=0}^{\infty} n \cancel{2^n} \frac{(-1)^n}{\cancel{2^n}}$$

$$= \sum_{n=0}^{\infty} n(-1)^n$$

$$\lim_{n\to\infty} n(-1)^n \neq 0 \quad \therefore \text{Diverges}$$

@ $x = -1/2$, the series becomes

$$\sum_{n=0}^{\infty} n 2^n \left(-1/2 + 1\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{n \cancel{2^n}}{\cancel{2^n}}$$

$$\lim_{n\to\infty} n = \infty \neq 0$$

$\therefore$ Diverges

$IC = \left(-3/2, -1/2\right)$

3. $\sum_{n=0}^{\infty} n!(x-3)^n$

$a = 3$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x-3)^{n+1}}{n!\,(x-3)^n} \right|$$

$$= \lim_{n\to\infty} |(n+1)(x-3)|$$

$$= \lim_{n\to\infty} (n+1)|x-3|$$

$$= \infty$$

Radius of convergence $= 0$

Interval of convergance $= \{3\} = [3,3]$

4. $\sum_{n=0}^{\infty} \frac{3^n (x-2)^n}{n+1}$

$$\lim_{n\to\infty} \left| \frac{3^{n+1}(x-2)^{n+1}}{n+2} \times \frac{n+1}{3^n (x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3(x-2)(n+1)}{(n+2)} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n+2} \times 3|x-2|$$

$$= 3|x-2|$$

Radius of convergance $= \frac{1}{3}$ $\qquad$ $IC = \left[\frac{5}{3}, \frac{7}{3}\right)$

Endpoints are $2 - 1/3 = 5/3$

$2 + 1/3 = 7/3$

@ $x = 5/3$ $\qquad$ @ $x = 7/3$

$$\sum_{n=0}^{\infty} \frac{3^n (5/3 - 2)^n}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n+1}\left(\frac{-1}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

$\therefore$ converges by AST

$$\sum_{n=0}^{\infty} \frac{3^n (7/3 - 2)^n}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n+1}\left(\frac{1}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1}$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \sim \sum_{n=1}^{\infty} \frac{1}{n}$$

$\therefore$ diverges by limit comparison test

Representation of functions as power series

$$\sum_{n=0}^{\infty} a_1 z^n = \frac{a_1}{1-z} \quad \text{if } |z| < 1$$

$$\frac{a_1}{1-z} = \sum_{n=0}^{\infty} a_1 z^n \quad \text{if } |z| < 1$$

let $a_1 = 1$ and $z = x$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x| < 1$$

$$\frac{1}{1-\square} = \sum_{n=0}^{\infty} \square^n \quad \text{if } |\square| < 1$$

Example

Find the power series of

1. $f(x) = \dfrac{4}{x+2}$ with center 0

$$\frac{4}{x+2} = \frac{4}{2+x} = \frac{4}{2}\left(\frac{1}{1+x/2}\right)$$

$$= 2\left(\frac{1}{1-\left(-\frac{x}{2}\right)}\right)$$

$$2\sum_{n=0}\left(\frac{-x}{2}\right)^n \quad \text{if } \left|\frac{-x}{2}\right|<1$$

$$=\sum_{n=0}^{\infty} 2\left(\frac{-x}{2}\right)^n \quad \text{if } \left|\frac{-x}{2}\right|<1$$

$$=\sum_{n=0}^{\infty}\frac{2(-1)^n x^n}{2^n} \quad \text{if } |x|<2$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n-1}}x^n \quad \text{if } |x|<2$$

2. $f(x)=\frac{1}{x}$ with center 1