Math 142 Final Exam

Formula Sheet

Common Integrals

Polynomials

$$\int dx = x + c \qquad \int k\,dx = k\,x + c \qquad \int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$$

$$\int \frac{1}{x}dx = \ln|x| + c \qquad \int x^{-1}\,dx = \ln|x| + c \qquad \int x^{-n}dx = \frac{1}{-n+1}x^{-n+1} + c,\ n \neq 1$$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c \qquad \int x^{\frac{p}{q}}dx = \frac{1}{\frac{p}{q}+1}x^{\frac{p}{q}+1} + c = \frac{q}{p+q}x^{\frac{p+q}{q}} + c$$

Trig Functions

$$\int \cos u\,du = \sin u + c \qquad \int \sin u\,du = -\cos u + c \qquad \int \sec^2 u\,du = \tan u + c$$

$$\int \sec u \tan u\,du = \sec u + c \qquad \int \csc u \cot u\,du = -\csc u + c \qquad \int \csc^2 u\,du = -\cot u + c$$

$$\int \tan u\,du = \ln|\sec u| + c \qquad \int \cot u\,du = \ln|\sin u| + c$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + c \qquad \int \sec^3 u\,du = \frac{1}{2}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) + c$$

$$\int \csc u\,du = \ln|\csc u - \cot u| + c \qquad \int \csc^3 u\,du = \frac{1}{2}\left(-\csc u \cot u + \ln|\csc u - \cot u|\right) + c$$

Exponential/Logarithm Functions

$$\int \mathbf{e}^u\,du = \mathbf{e}^u + c \qquad \int a^u\,du = \frac{a^u}{\ln a} + c \qquad \int \ln u\,du = u\ln(u) - u + c$$

$$\int \mathbf{e}^{au}\sin(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\sin(bu) - b\cos(bu)\right) + c \qquad \int u\mathbf{e}^u du = (u-1)\mathbf{e}^u + c$$

$$\int \mathbf{e}^{au}\cos(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\cos(bu) + b\sin(bu)\right) + c \qquad \int \frac{1}{u\ln u}du = \ln|\ln u| + c$$

Inverse Trig Functions

$$\int \frac{1}{\sqrt{a^2-u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) + c \qquad \int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + c$$

$$\int \frac{1}{a^2+u^2}du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c \qquad \int \tan^{-1}u\,du = u\tan^{-1}u - \frac{1}{2}\ln\left(1+u^2\right) + c$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + c \qquad \int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2} + c$$

Miscellaneous

$$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + c \qquad \int \frac{1}{u^2-a^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + c$$

$$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right| + c$$

$$\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + c$$

$$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + c$$

Standard Integration Techniques

Note that all but the first one of these tend to be taught in a Calculus II class.

u Substitution

Given $\int_a^b f\left(g(x)\right)g'(x)\,dx$ then the substitution $u = g(x)$ will convert this into the integral, $\int_a^b f\left(g(x)\right)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$.

Integration by Parts

The standard formulas for integration by parts are,

$$\int u\,dv = uv - \int v\,du \qquad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Choose *u* and *dv* and then compute *du* by differentiating *u* and compute *v* by using the fact that $v = \int dv$.

Trig Substitutions

If the integral contains the following root use the given substitution and formula.

$$\sqrt{a^2-b^2x^2} \quad \Rightarrow \quad x = \frac{a}{b}\sin\theta \quad \text{and} \quad \cos^2\theta = 1-\sin^2\theta$$

$$\sqrt{b^2x^2-a^2} \quad \Rightarrow \quad x = \frac{a}{b}\sec\theta \quad \text{and} \quad \tan^2\theta = \sec^2\theta - 1$$

$$\sqrt{a^2+b^2x^2} \quad \Rightarrow \quad x = \frac{a}{b}\tan\theta \quad \text{and} \quad \sec^2\theta = 1+\tan^2\theta$$

Partial Fractions

If integrating $\int \frac{P(x)}{Q(x)} dx$ where the degree (largest exponent) of $P(x)$ is smaller than the degree of $Q(x)$ then factor the denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Formula: If $f'(x)$ is continuous on $[a,b]$, then the *arc length of the curve* $y=f(x)$ on the interval $[a,b]$ is given by

$$s=\int_a^b \sqrt{1+(f'(x))^2}\,dx.$$

Formula: If $f'(x)$ is continuous on $[a,b]$, then the *surface area of a solid of revolution* obtained by rotating the curve $y=f(x)$

1. Around the y-axis on the interval $[a,b]$ is given by (provided that $x \geq 0$)

$$S=\int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx.$$

2. Around the x-axis on the interval $[a,b]$ is given by (provided that $y=f(x)\geq 0$)

$$S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx.$$

Notation:

Number of terms in the series: n
First term: a_1
N^{th} **term:** a_n
Sum of the first n **terms:** S_n
Difference between successive terms: d
Common ratio: q
Sum to infinity: S

Arithmetic Series Formulas:

1. $$a_n = a_1 + (n-1)d$$

2. $$a_i = \frac{a_{i-1} + a_{i+1}}{2}$$

3. $$S_n = \frac{a_1 + a_n}{2} \cdot n$$

4. $$S_n = \frac{2 \cdot a_1 + (n-1) \cdot d}{2} \cdot n$$

Geometric Series Formulas:

5. $$a_n = a_1 \cdot q^{n-1}$$

6. $$a_i = \sqrt{a_{i-1} \cdot a_{i+1}}$$

7. $$S_n = \frac{a_n q - a_1}{q - 1}$$

8. $$S_n = \frac{a_1 \cdot (q^n - 1)}{q - 1}$$

9. $$S = \frac{a_1}{1-q}, \quad (\text{for } -1 < q < 1)$$

Function	Maclaurin Series	
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
e^x	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$R = \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$R = \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} x^{2n} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$R = \infty$
$\arctan(x)$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)} x^{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} x^n = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$R = 1$

Alternating Series Estimation Theorem If $s = \sum(-1)^{n-1}b_n$, $b_n > 0$ is the sum of an alternating series that satisfies

$$(i)\quad b_{n+1} < b_n \quad \text{for all } n$$

$$(ii)\quad \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Bound for the Remainder in the Integral Test: Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive terms with $a_k = f(k)$, where $f(x)$ is a continuous positive decreasing function of x for all $x \geq n$ and that $\sum_{k=1}^{\infty} a_k$ converges to S. Then the remainder $R_n = R - S_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Differential Equations

General form of first order equation:

$$\frac{dy}{dx} = f(x, y)$$

General form of homogeneous equation:

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right).$$

p

General exact equation

Equation considered:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

Hypothesis: there exists $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial \psi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x, y).$$

Conclusion: general solution to (20) is given by:

$$\psi(x, y) = c, \quad \text{with} \quad c \in \mathbb{R},$$

provided this relation defines $y = y(x)$ implicitely.

Special Integrating Factors

Consider the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$.

1. If

$$\frac{1}{N(x, y)}[M_y(x, y) - N_x(x, y)] = h(x)$$

is a function of x alone, then $e^{\int h(x)\,dx}$ is an integrating factor.

2. If

$$\frac{1}{M(x, y)}[N_x(x, y) - M_y(x, y)] = k(y)$$

is a function of y alone, then $e^{\int k(y)\,dy}$ is an integrating factor.

Bernoulli's Equation

The differential equation

$$y' + P(x)y = Q(x)y^n$$

is known as **Bernoulli's equation.** If $n = 0$, Bernoulli's equation reduces immediately to the standard form first-order linear equation:

$$y' + P(x)y = Q(x)$$

If $n = 1$, the equation can also be written as a linear equation:

$$y' + P(x)y = Q(x)y \Rightarrow y' + [P(x) - Q(x)]y = 0$$

However, if n is not 0 or 1, then Bernoulli's equation is not linear. Nevertheless, it can be *transformed* into a linear equation by first multiplying through by y^{-n},