Faculty of Engineering and Information Sciences

UNIVERSITY OF WOLLONGONG IN DUBAI

MATH 142, Quiz 2, Winter 2021, Duration: 1hour 45 minutes

Name:____________________, ID Number:____________________

Time Allowed: 1 hour 30 minutes
Total Number of Questions: 6
Total Number of Pages (incl. this page): 7

EXAM UNAUTHORISED ITEMS
Students bringing these items to the examination room shall be required to leave the items at the front of the room or outside the examination room. The University does not guarantee the safe-keeping of students' personal items during examinations. Students concerned about the safety of their valuable items should make alternative arrangements for their care.

1. Bags, including carry bags, backpacks, shoulder bags and briefcases
2. Any form of electronic device including but not limited to mobile phones, smart watches, laptops, iPads, MP3 players, handheld computers and electronic dictionaries,
3. Calculator cases and covers
4. blank paper
5. Any written material

DIRECTIONS TO CANDIDATES

1. Total marks: 40
2. All questions are compulsory and you must show all your work.
3. Answer all questions on the given exam paper sheets.
4. Write your name and Id number on the papers provided for rough work.

(6pts) Problem 1.
Show that the equation is separable and solve the equation.

$$\frac{dy}{dx} = \frac{2xy^2 + 2x}{x^2y + y}.$$

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2xy^2 + 2x}{x^2y + y} \\
&= \frac{2x\left(y^2+1\right)}{y\left(x^2+1\right)} \\
&= \left(\frac{2x}{x^2+1}\right)\left(\frac{y^2+1}{y}\right) = f(x)g(y). \qquad (\mathbf{2pts})
\end{aligned}$$

The equation is separable.

$$\frac{y}{y^2+1}dy = \frac{2x}{x^2+1}dx$$

$$\int \frac{y}{y^2+1}dy = \int \frac{2x}{x^2+1}dx$$

$$\frac{1}{2}\ln\left(y^2+1\right) = \ln\left(x^2+1\right) + \ln K, \qquad K > 0$$

$$\ln\left(y^2+1\right) = \ln\left(x^2+1\right)^2 + \ln K^2, \qquad K > 0$$

$$\ln\left(y^2+1\right) = \ln\left[C\left(x^2+1\right)^2\right]$$

$$y^2 = C\left(x^2+1\right)^2 - 1. \qquad (\mathbf{4pts})$$

(6pts) Problem 2
Solve the linear differential equation

$$x^2\left(x-2\right)\frac{dy}{dx}+x\left(x-2\right)y=2.$$

Solution

$$x^2\left(x-2\right)\frac{dy}{dx}+x\left(x-2\right)y=2.$$
$$\frac{dy}{dx}+\frac{x\left(x-2\right)}{x^2\left(x-2\right)}y=\frac{2}{x^2\left(x-2\right)}$$
$$\frac{dy}{dx}+\frac{1}{x}y=\frac{2}{x^2\left(x-2\right)}$$
$$IF=e^{\int\frac{1}{x}dx}=x,\qquad x>0.\qquad (\mathbf{2pts})$$

We have

$$\frac{d}{dx}\left[yx\right]=\frac{2\cancel{x}}{x^{\cancel{2}}\left(x-2\right)}$$
$$y=\frac{1}{x}\int\frac{2}{x\left(x-2\right)}dx$$

$$\begin{aligned}\frac{2}{x\left(x-2\right)}&=\frac{x-x+2}{x\left(x-2\right)}\\&=\frac{x-\left(x-2\right)}{x\left(x-2\right)}\\&=\frac{1}{x-2}-\frac{1}{x}\end{aligned}$$

$$y=\frac{1}{x}\left(\ln\left|\frac{x-2}{x}\right|+C\right)$$
$$y=\frac{1}{x}\ln\left|\frac{x-2}{x}\right|+\frac{C}{x}.\qquad (\mathbf{4pts})$$

(8pts) Problem 3

Solve the differential equation

$$\frac{dy}{dx} = y\left(xy^3 - 1\right)$$

Solution

$$\frac{dy}{dx} + y = xy^4.$$

This is a Bernoulli equation.

$$y^{-4}\frac{dy}{dx} + y^{-3} = x. \qquad (\mathbf{2pts})$$

Put

$$u = y^{-3}$$

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx} \Rightarrow y^{-4}\frac{dy}{dx} = \frac{-1}{3}\frac{du}{dx}.$$

The equation becomes

$$\frac{-1}{3}\frac{du}{dx} + u = x \Leftrightarrow \frac{du}{dx} - 3u = -3x$$

$$IF = e^{-3x}. \qquad (\mathbf{2pts})$$

$$\frac{d}{dx}\left[ue^{-3x}\right] = -3xe^{-3x}$$

$$\begin{aligned} ue^{-3x} &= \int -3xe^{-3x}dx \\ &= \frac{1}{3}e^{-3x}(3x+1) + C \end{aligned}$$

$$u = \frac{1}{3}(3x+1) + Ce^{3x}$$

$\Leftrightarrow$

$$y^{-3} = \frac{1}{3}(3x+1) + Ce^{3x}$$

$$y = \frac{1}{\sqrt[3]{\frac{1}{3}(3x+1) + Ce^{3x}}}. \qquad (\mathbf{4pts})$$

(6pts) Problem 4

Find $\lim_{n\to\infty} a_n$.

$$(a).\ \ 1,\ \frac{-1}{2},\ \frac{1}{3},\ \frac{-1}{4},\ \frac{1}{5},\ldots \qquad\qquad (b)\ a_n = 3n\sin\frac{\pi}{2n}$$

Solution

$(a).\ \ 1,\ \frac{-1}{2},\ \frac{1}{3},\ \frac{-1}{4},\ \frac{1}{5},\ldots$

$$a_n = \frac{(-1)^{n+1}}{n}$$

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0 \qquad (\mathbf{3pts})$$

(b)

$$\begin{aligned} a_n &= 3n\sin\frac{\pi}{2n} \\ &= 3\frac{\sin\frac{\pi}{2n}}{\frac{1}{n}} \\ &= 3\pi\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{n}} \\ &= \frac{3\pi}{2}\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} 3n\sin\frac{\pi}{2n} &= \lim_{n\to\infty}\frac{3\pi}{2}\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} \\ &= \frac{3\pi}{2}. \qquad (\mathbf{3pts}) \end{aligned}$$

(6pts) Problem 5

Determine whether the series converges or diverges. (Justify you answer)

1. $\displaystyle\sum_{n=1}^{\infty} \sqrt[3]{\frac{-8n}{n+4}}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

Solution

1. $\displaystyle\sum_{n=1}^{\infty} \sqrt[3]{\frac{-8n}{n+4}}$

$$\lim_{n\to\infty} \sqrt[3]{\frac{-8n}{n+4}} = -2 \neq 0. \text{ The series cannot converge. Diverges.} \qquad (\mathbf{3pts})$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3}$. Here we can apply the ratio test or you can simply observe that

$$\begin{aligned} \lim_{n\to\infty} \frac{e^n}{n^3} &= \lim_{x\to\infty} \frac{e^x}{x^3} \\ &= +\infty \quad \text{(L'Hospital's rule)} \\ &\neq 0 \quad \text{Diverges} \quad (\mathbf{3pts}) \end{aligned}$$

(8pts) Problem 6
Determine whether the series converges or diverges. If it converges, find the sum

1. $\sum_{n=1}^{\infty} 4^n \cdot 5^{1-n}$

2. $\sum_{n=1}^{\infty} \frac{1}{(n+1)!n!}((n+1)!-n!)$

Solution
1.

$$\sum_{n=1}^{\infty} 4^n \cdot 5^{1-n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{5}\right)^{n-1}.$$

This is a geometric series with $a_1 = 4$ and $r = \frac{4}{5}$.

$$\left|\frac{4}{5}\right| < 1.$$

The series converges and its sum is

$$S = \frac{a}{1-r} = \frac{4}{1-\frac{4}{5}} = 20 \qquad (\mathbf{4pts})$$

2.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(n+1)!n!}((n+1)!-n!) &= \sum_{n=1}^{\infty}\left[\frac{1}{n!}-\frac{1}{(n+1)!}\right] \\
&= \lim_{N\to\infty}\sum_{n=1}^{N}\left[\frac{1}{n!}-\frac{1}{(n+1)!}\right] \\
&= \lim_{N\to\infty}\left[\left(1-\frac{1}{2!}\right)+\left(\frac{1}{2!}-\frac{1}{3!}\right)+...+\left(\frac{1}{n!}-\frac{1}{(n+1)!}\right)\right] \\
&= \lim_{N\to\infty}\left(1-\frac{1}{(n+1)!}\right) = 1.
\end{aligned}$$

The series converges and its value is $S = 1$. (**4pts**).