UNIVERSITY OF WOLLONGONG IN DUBAI

Math 142 - Exam 2 Review Problems.

Problem 1.

Find the area of the surface generated by revolving the parametric curve

$$x = \cos^3\theta \quad \text{and} \quad y = \sin^3\theta, \qquad 0 \le t \le \frac{\pi}{2}.$$

about the x-axis.

Solution

We will rotate the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

about the x or y-axis. We are going to assume that the curve is traced out exactly once as t increases from α to β. At this point there actually isn't all that much to do. We know that the surface area can be found by using one of the following two formulas depending on the axis of rotation (recall the Surface Area section of the Applications of Integrals chapter).

$$S = \int 2\pi y\, ds \qquad \text{rotation about } x-\text{axis}$$

$$S = \int 2\pi x\, ds \qquad \text{rotation about } y-\text{axis}$$

All that we need is a formula for ds to use and from the previous section we have,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{if } x = f(t), y = g(t), \alpha \le t \le \beta$$

which is exactly what we need.

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = -3\cos^2\theta\sin\theta \qquad \frac{dy}{dt} = 3\sin^2\theta\cos\theta$$

Before plugging into the surface area formula let's get the ds out of the way.

$$
\begin{aligned}
ds &= \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}\; dt \\
&= 3\left|\cos\theta\sin\theta\right|\sqrt{\cos^2\theta + \sin^2\theta} \\
&= 3\cos\theta\sin\theta
\end{aligned}
$$

Notice that we could drop the absolute value bars since both sine and cosine are positive in this range of θ given.

Now let's get the surface area and don't forget to also plug in for the y.

$$
\begin{aligned}
S &= \int 2\pi y\, ds \\
&= 2\pi\int_0^{\frac{\pi}{2}} \sin^3\theta\left(3\cos\theta\sin\theta\right)\, d\theta \\
&= 6\pi\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\; d\theta && u = \sin\theta \\
&= 6\pi\int_0^1 u^4\, du \\
&= \frac{6\pi}{5}
\end{aligned}
$$

Problem 2.
Find the equation of the tangent line to the polar curve

$$r = 3 + 8\sin\theta$$

at $\theta = \dfrac{\pi}{6}$.

Solution
We'll first need the following derivative.

$$\frac{dr}{d\theta} = 8\cos\theta$$

The formula for the derivative $\frac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3+8\sin\theta)\cos\theta}{8\cos^2\theta - (3+8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

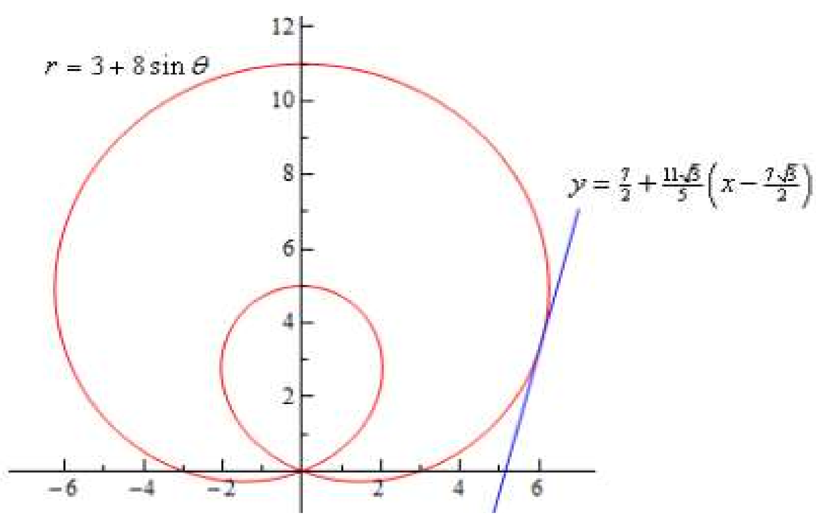
Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$. We'll need to get the corresponding x-y coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad\qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

For the sake of completeness here is a graph of the curve and the tangent line.



Problem 3.
(a) Replace the polar equation $r = 2\tan\theta\sec\theta$ with an equivalent Cartesian equation.
Solution

$$r = 2\tan\theta\sec\theta \Leftrightarrow r = \frac{2\sin\theta}{\cos^2\theta}$$

$\Leftrightarrow$

$$r\cos^2\theta = 2\sin\theta.$$

Multiplying by r on both sides, we get

$$\begin{aligned} r^2\cos^2\theta &= 2r\sin\theta && \Leftrightarrow \\ 2y &= x^2 && \Leftrightarrow \\ y &= \frac{x^2}{2}. \end{aligned}$$

(b) Sketch the polar curve

$$r = 2\sin\theta + 3\cos\theta$$

Solution
Multiplying by r on both sides of the equation, we get

$$\begin{aligned} r^2 &= 2r\sin\theta + 3r\cos\theta && \Leftrightarrow \\ x^2 + y^2 &= 2y + 3x && \Leftrightarrow \\ x^2 - 3x + y^2 - 2y &= 0. \end{aligned}$$

Completing the squares, we get

$$\left(x - \frac{3}{2}\right)^2 + (y-1)^2 = \frac{9}{4} + 1 = \frac{13}{4}.$$

This is a circle centered at $\left(\frac{3}{2}, 1\right)$ with radius $\frac{\sqrt{13}}{2} = 1.8028$.