Linear Equations

Def. A first-order DE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (I)$$

is said to be a linear equation in the variable y.

Equation (I) can be expressed in standard form as:

$$\boxed{\frac{dy}{dx} + P(x)\,y = \varphi(x)}$$

Solving a linear First-order DE

① Write the DE in standard form.

② Find the integrating factor $\mu(x) = e^{\int P(x)\,dx}$

③ Multiply by $\mu(x)$ both sides of the DE in standard form.

④ From step (3) we will get $\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)\varphi(x)$

⑤ Integrate both sides of the last equation and solve for y

to get $y = \frac{1}{\mu(x)}\left[\int \mu(x)\varphi(x)\,dx + C\right]$.

Ex. Solve

a) $\frac{1}{x}\frac{dy}{dx} - 4y = 1$

$$\frac{dy}{dx} - 4xy = x \quad \text{(I) standard form}$$

$P(x) = -4x$ and $\varphi(x) = x$

Integrating factor $\mu(x) = e^{\int p(x)\,dx}$

$$= e^{\int -4x\,dx}$$

$$= e^{-4\frac{x^2}{2}} \Rightarrow \boxed{\mu(x) = e^{-2x^2}}$$

Multiply (I) by e^{-2x^2}

$$e^{-2x^2}\frac{dy}{dx} - 4xe^{-2x^2}y = xe^{-2x^2}$$

we have $\frac{d}{dx}\left[e^{-2x^2}y\right] = xe^{-2x^2}$

$$d\left[e^{-2x^2}y\right] = xe^{-2x^2}dx$$

$$\int d\left[e^{-2x^2}y\right] = -\frac{1}{4}\int -4xe^{-2x^2}dx$$

$$e^{-2x^2}y = -\frac{1}{4}e^{-2x^2} + C$$

$$\boxed{y = -\frac{1}{4} + \frac{C}{e^{-2x^2}}}$$

OR $y = -\frac{1}{4} + Ce^{2x^2}$

b) $x \dfrac{dy}{dx} - 4y = x^6 e^x$

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x \qquad (\mathrm{I})$$

$$p(x) = -\frac{4}{x}$$

$$\mu(x) = e^{\int p(x)\,dx}$$

$$= e^{\int -\frac{4}{x}\,dx}$$

$$= e^{-4\ln|x|}$$

$$= \cancel{e^{\ln}}|x|^{-4}$$

$$= |x|^{-4} \Rightarrow \mu(x) = x^{-4}$$

Multiply by the integrating factor both sides of (I)

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = x e^x$$

$$\frac{d}{dx}\left[x^{-4}y\right] = x e^x$$

$$\int d\left[x^{-4}y\right] = \int x e^x\,dx$$

D		I
x	$+$	e^x
1	$-$	e^x
0		e^x

$$x^{-4}y = xe^x - e^x + C$$

$$\boxed{y = x^5 e^x - x^4 e^x + Cx^4}$$

c) Solve the IVP

$$y' + y = x \qquad y(0) = 4$$

$$\frac{dy}{dx} + y = x \qquad (I)$$

$$p(x) = 1 \qquad \Rightarrow \mu(x) = e^{\int 1\,dx}$$

$$= e^{x}$$

$$(I) \Rightarrow e^{x}\frac{dy}{dx} + e^{x}y = e^{x}x$$

$$\frac{d}{dx}\left[e^{x}y\right] = xe^{x}$$

D		I
x	$+$	e^x
1	$-$	e^x
0		e^x

$$\int d\left[e^{x}y\right] = \int xe^{x}\,dx$$

$$e^{x}y = xe^{x} - e^{x} + C$$

$$y = x - 1 + \frac{C}{e^{x}}$$

$$y(0) = 4 \Rightarrow 4 = 0 - 1 + \frac{C}{e^{0}}$$

$$4 = -1 + C \Rightarrow \underline{C = 5}$$

$$\therefore \boxed{y = x - 1 + \frac{5}{e^{x}}}$$