EXAMINATION COVERSHEET

Winter 2024 Quiz 2

Solution KEY

UNIVERSITY
OF WOLLONGONG
IN DUBAI

THIS EXAMINATION CONTENT IS STRICTLY CONFIDENTIAL Students must comply with requirements stated in the Examination Policy & Procedures	
Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	**03/14/2024**
Subject Code:	**Math 142**
Subject Title:	**Essentials of Engineering Mathematics**
Time Permitted to Write Exam:	1 Hour
Total Number of Questions:	**5 written questions**
Total Number of Pages *(including this page):*	**6**

INSTRUCTIONS TO STUDENTS FOR THE EXAM

1. Please note that subject lecturer/tutor will be unavailable during exams. *If there is a doubt in any of the exam questions i.e. problem solving etc. students should proceed by assuming values etc. Students should mention their assumption on the question paper.*
2. Answers must be written (and drawn) in black or blue ink
3. Any mistakes must be crossed out. Whitener and ink erasers must not be used.
4. Answer ALL/ 6 questions. The marks for each question are shown next to each question.
5. Total marks: 40.

EXAMINATION MATERIALS/AIDS ALLOWED
Approved Calculators and Formula Sheet

Exam Unauthorised Items - Students bringing these items to the examination room must follow the instructions of the invigilators with regards to these items.

6. Bags, including carrier bags, backpacks, shoulder bags and briefcases
7. Any form of electronic device including but not limited to mobile phones, smart watches, MP3 players, handheld computers and unauthorised calculators;
8. Calculator cases and covers, opaque pencil cases
9. Blank paper
10. Any written material

NOTE: The University does not guarantee the safe-keeping of students' personal items during examinations. Students concerned about the safety of their valuable items should make alternative arrangements for their care.

(8pts) **Problem 1**

Show that the equation is separable and solve the initial-value problem

$$\left(x^2-2\right)dx+ydy=0, \qquad y(0)=1.$$

Solution

$$\left(x^2-2\right)dx+ydy=0, \qquad y(0)=1.$$

We have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-\left(x^2-2\right)}{y}\\
&= -\left(x^2-2\right)\left(\frac{1}{y}\right)\\
&= g(x)h(y) \Rightarrow \text{The equation is separable.} \qquad [\,2 \text{ points}\,]
\end{aligned}$$

$$ydy=-\left(x^2-2\right)dx \Rightarrow \int ydy=-\int\left(x^2-2\right)dx$$

$$\frac{1}{2}y^2=-\frac{1}{3}\left(x^3-6x\right)+C$$

$$y^2=-\frac{2}{3}\left(x^3-6x\right)+K. \qquad [\,4 \text{ points}\,]$$

Now using $y(0)=1$, we get

$$K=1. \qquad [\,2 \text{ points}\,]$$

The solution is

$$y^2=-\frac{2}{3}\left(x^3-6x\right)+1.$$

$$y^2=-\frac{2}{3}x^3+4x+1.$$

(8pts) **Problem 2**
Show that the equation is linear and solve the initial-value problems

$$x^2 \frac{dy}{dx} + xy = 1, \qquad x > 0 \text{ and } \quad y(1) = 2.$$

Solution

$$\begin{aligned} x^2 \frac{dy}{dx} + xy = 1, &\qquad x > 0 \text{ and } \quad y(1) = 2. \\ \frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x^2}, &\qquad x > 0 \text{ and } \quad y(1) = 2. \end{aligned}$$

This shows that the equation is linear with $P(x) = \dfrac{1}{x}$. [2 points]

$$IF = e^{\int P(x)dx} = e^{\displaystyle\int \frac{1}{x} dx} = e^{\ln x} = x. \qquad [\text{ 2 points }]$$

$$y = \frac{1}{IF} \int IF \cdot \frac{1}{x^2} dx \quad \text{gives}$$

$$\begin{aligned} y &= \frac{1}{x} \int \frac{1}{x} dx \\ &= \frac{1}{x} (\ln x + C) . \qquad [\text{ 2 points }] \end{aligned}$$

Now using $y(1) = 2$ we get

$$C = 2. \qquad [\text{ 2 points }]$$

The solution is

$$y = \frac{1}{x} (\ln x + 2) .$$

(8pts) **Problem 3**
Show that the differential equation is not exact. make it exact but DO NOT Solve.

$$\left(x^2+xy\right)\frac{dy}{dx}+\left(3xy+y^2\right)=0$$

Solution

$$\left(x^2+xy\right)\frac{dy}{dx}+\left(3xy+y^2\right)=0$$

$\Leftrightarrow$

$$\left(3xy+y^2\right)dx+\left(x^2+xy\right)dy=0$$

$$M=3xy+y^2, \qquad N=x^2+xy$$

$$M_y=3x+2y, \qquad N_x=2x+y$$

$$M_y\neq N_x \Rightarrow \text{The equation is not exact.} \qquad [\,2 \text{ points}\,]$$

$$\frac{M_y-N_x}{N}=\frac{3x+2y-2x-y}{x^2+xy}=\frac{1}{x}.$$

$$SIF=e^{\int \frac{1}{x}dx}=e^{\ln x}=x, \qquad x>0. \qquad [\ 2 \text{ points}\ ]$$

To turn it into an exact equation, we multiply both sides of the equation by x to get

$$\left(3x^2y+y^2x\right)dx+\left(x^3+x^2y\right)dy=0. \qquad [\,2 \text{ points}\,]$$

To check that this equation is exact, Put

$$M=3x^2y+y^2x, \qquad N=x^3+x^2y.$$

$$M_y=3x^2+2xy, \qquad N_x=3x^2+2xy$$

$$M_y=N_x \Rightarrow \text{The new equation is exact.} \qquad [\,2 \text{ points}\,]$$

(8pts) **Problem 4**
Evaluate

1. $\lim\limits_{n\to\infty} n\sin\frac{3}{n}$ 2. $\lim\limits_{n\to\infty} \frac{n^2+1}{e^n}$

Solution

1.

$$\begin{aligned}\lim_{n\to\infty} n\sin\frac{3}{n} &= \lim_{n\to\infty}\frac{\sin\frac{3}{n}}{\frac{1}{n}}\\ &= \lim_{n\to\infty}\frac{\sin\frac{3}{n}}{\frac{1}{3}\cdot\frac{3}{n}}\\ &= \lim_{n\to\infty} 3\frac{\sin\frac{3}{n}}{\frac{3}{n}}.\end{aligned}$$

Now using

$$\lim_{\boxdot\to 0}\frac{\sin\boxdot}{\boxdot}=1,\ \text{with } \boxdot\ =\frac{3}{n},\ \text{we have}$$

$$\begin{aligned}\lim_{n\to\infty} n\sin\frac{3}{n} &= 3\lim_{n\to\infty}\frac{\sin\frac{3}{n}}{\frac{3}{n}}\\ &= (3)\,(1)=3. \qquad [\,4\text{ points}\,]\end{aligned}$$

2.

$$\begin{aligned}\lim_{n\to\infty}\frac{n^2+1}{e^n} &= \lim_{x\to\infty}\frac{x^2+1}{e^x}\\ &= \lim_{x\to\infty}\frac{2x}{e^x}\\ &= \lim_{x\to\infty}\frac{2}{e^x}=0. \qquad [\,4\text{ points}\,]\end{aligned}$$

(8pts) **Problem 5**

Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} 2^{3-n}$$

Solution

$$\begin{aligned}
\sum_{n=1}^{\infty} 2^{3-n} &= \sum_{n=1}^{\infty} 2^3 \left(\frac{1}{2}\right)^n \\
&= \sum_{n=1}^{\infty} 2^3 \left(\frac{1}{2}\right)^{n-1+1} \\
&= \sum_{n=1}^{\infty} 2^2 \left(\frac{1}{2}\right)^{n-1} \\
&= \sum_{n=1}^{\infty} 4 \left(\frac{1}{2}\right)^{n-1}. \qquad \text{[4 points]}
\end{aligned}$$

This is a geometric series with $a_1 = 4$ and $r = \frac{1}{2}$.

$$\begin{aligned}
|r| &= \frac{1}{2} < 1 \Rightarrow \text{ the series converges and its sum is} \\
S &= \frac{a_1}{1-r} = \frac{4}{1-\frac{1}{2}} = 8. \qquad \text{[4 points]}
\end{aligned}$$