EXAMINATION COVERSHEET

Winter 2023 Midterm Exam

UNIVERSITY
OF WOLLONGONG
IN DUBAI

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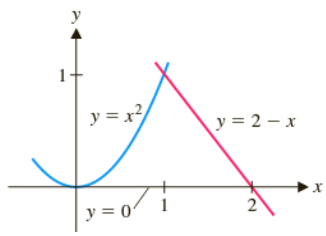
Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	**02/08/2023**
Subject Code:	**Math 142**
Subject Title:	**Essentials of Engineering Mathematics**
Time Permitted to Write Exam:	2 Hour
Total Number of Questions:	**6 MCQ's and 6 Written Questions**

Part 1 MCQ's 30% (Circle Your Choice)

(5pts) Problem 1

Find the area bounded by the graphs of $y = x^2, y = 2 - x$ and $y = 0$



$A)\ \frac{3}{2}$

$B)\ \frac{7}{3}$

$C)\ \frac{5}{6}$

$D)\ \frac{10}{3}$

$E)\ \frac{9}{2}$

Solution

$$A = \int_0^1 \left(x^2 - 0\right) dx + \int_1^2 (2 - x)\, dx = \frac{5}{6}$$

Answer is $C)$

(5pts) Problem 2

$$\int 9\sqrt{x} \ln x dx =$$

$A)\quad 6x^{3/2} \ln x - 4x^{3/2} + C$

$B)\quad 9x^{3/2} \ln x + 2x^{3/2} + C$

$C)\quad 2x^{3/2} \ln x - 6x^{3/2} + C$

$D)\quad -6x^{3/2} \ln x + 4x^{3/2} + C$

$E)\quad 6x^{3/2} \ln x - 6x^{3/2} + C$

Solution

Integrate by parts

$$\begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = x^{1/2} & v = \frac{2}{3}x^{3/2} \end{array}$$

$$\begin{aligned} \int 9\sqrt{x} \ln x dx &= 9\left[\frac{2}{3}x^{3/2} \ln x - \int \left(\frac{2}{3}x^{3/2}\right)\left(\frac{1}{x}\right) dx\right] \\ &= 9\left[\frac{2}{3}x^{3/2} \ln x - \int \left(\frac{2}{3}x^{1/2}\right) dx\right] \\ &= 6x^{3/2} \ln x - 4x^{3/2} + C \end{aligned}$$

Answer is $A)$

(5pts) Problem 3
The equation of the tangent line to the curve $x = 3e^t, \quad y = 5e^{-t}$ at $t = 0$ is

$$A) \quad y - 5x = 3$$

$$B) \quad 15y + x = 3$$

$$C) \quad x + y = 15$$

$$D) \quad 3x - 5y = 30$$

$$E) \quad 5x + 3y = 30$$

Solution

$$\text{Slope } m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-5}{3}e^{-2t}. \text{ At } t = 0, \quad m = \frac{-5}{3}, \ x = 3 \text{ and } y = 5.$$

The equation is $y = \frac{-5}{3}(x-3) + 5 \Leftrightarrow \ y = 10 - \frac{5}{3}x \Leftrightarrow 5x + 3y = 30$ Answer is $E)$

(5pts) Problem 4
The graph of the polar $r = 4\cos\theta - 5\sin\theta$ is

$A)$ A circle centered at $(4, \ -5)$ with radius 9

$B)$ A parabola with vertex $(4, \ -5)$

$C)$ A line with slope $\frac{-5}{4}$

$D)$ A circle centered at $\left(2, \ -\frac{5}{2}\right)$ with radius $\frac{\sqrt{41}}{2}$

$E)$ A circle centered at $(4, \ -5)$ with radius $\frac{33}{4}$

Solution

$$\begin{aligned} r &= 4\cos\theta - 5\sin\theta \Leftrightarrow r^2 = 4r\cos\theta - 5r\sin\theta \\ &\Leftrightarrow x^2 + y^2 = 4x - 5y \Leftrightarrow x^2 - 4x + y^2 + 5y = 0 \\ &\Leftrightarrow (x-2)^2 - 4 + \left(y + \frac{5}{2}\right)^2 - \frac{25}{4} = 0 \\ &\Leftrightarrow (x-2)^2 + \left(y + \frac{5}{2}\right)^2 = 4 + \frac{25}{4} = \frac{41}{4} \end{aligned}$$

A circle centered at $\left(2, \ -\frac{5}{2}\right)$ with radius $\frac{\sqrt{41}}{2}$ Answer is $D)$

(5pts) Problem 5
The area of the region bounded by $x = 3 - y^2$ and $x = y + 1$ is equal to

$A)\quad \frac{9}{2}$

$B)\quad \frac{3}{2}$

$C)\quad 3$

$D)\quad \frac{7}{3}$

$E)\quad 4$

Solution

$$\begin{aligned} A &= \int_{-2}^{1} [(3 - y^2) - (y + 1)]\, dy \\ &= \int_{-2}^{1} (-y^2 - y + 2)\, dy \\ &= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1} \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) \\ &= \frac{9}{2}. \end{aligned}$$

Answer is $A)$

(5pts) Problem 6
If

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1},$$

then $A + B + C$ is equal to

$A)\quad -5$

$B)\quad 4$

$C)\quad 3$

$D)\quad -3$

$E)\quad -1$

Solution

$$\begin{aligned} \frac{x+1}{x^2(x-1)} &= -\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1} \\ A &= -2,\ B = -1 \text{ and } C = 2 \\ A + B + C &= -1 \end{aligned}$$

Answer is $E)$

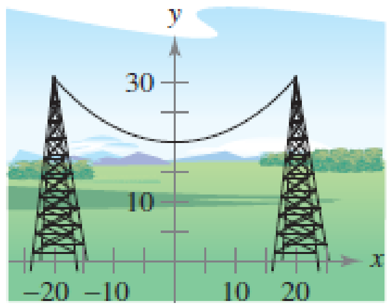
Part 2 Written Questions (70%)

(12pts) Problem 1

Electrical wires suspended between two towers form a catenary (see figure) modeled by the equation

$$y = 10\left(e^{x/20} + e^{-x/20}\right), \qquad -20 \le x \le 20$$

where x and $y$are measured in meters. The towers are 40 meters apart. Find the length of the suspended cable.



Solution

We need to find the arclength of the curve.

$$L = \int_{-20}^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \textbf{(2pts)}$$

$$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{20}x} - \frac{1}{2}e^{-\frac{1}{20}x}$$

$$\begin{aligned}\left(\frac{dy}{dx}\right)^2 &= \left(\frac{1}{2}e^{\frac{1}{20}x} - \frac{1}{2}e^{-\frac{1}{20}x}\right)^2 \\ &= \frac{1}{4}e^{\frac{1}{10}x} - \frac{1}{2} + \frac{1}{4}e^{-\frac{1}{10}x}\end{aligned}$$

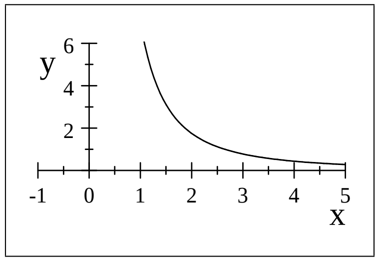
$$\begin{aligned}1 + \left(\frac{dy}{dx}\right)^2 &= \frac{1}{4}e^{\frac{1}{10}x} + \frac{1}{2} + \frac{1}{4}e^{-\frac{1}{10}x} \\ &= \left(\frac{1}{2}e^{\frac{1}{20}x} + \frac{1}{2}e^{-\frac{1}{20}x}\right)^2 \qquad \textbf{(5pts)}\end{aligned}$$

$$\begin{aligned}L &= \int_{-20}^{20} \sqrt{\left(\frac{1}{2}e^{\frac{1}{20}x} + \frac{1}{2}e^{-\frac{1}{20}x}\right)^2}\, dx \\ &= \int_{-20}^{20} \left(\frac{1}{2}e^{\frac{1}{20}x} + \frac{1}{2}e^{-\frac{1}{20}x}\right) dx \\ &= 10e^{\frac{1}{20}x} - 10e^{\frac{-1}{20}x}\Big|_{-20}^{20} \\ &= 20\left(e - \frac{1}{e}\right) = 47.008 \qquad \textbf{(5pts)}\end{aligned}$$

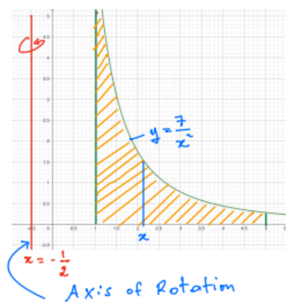
(13pts) Problem 2

A) Use the cylindrical shell method to find the volume of the solid generated by rotating about the $x = -\frac{1}{2}$ the region in the first quadrant bounded by the curves

$$y = \frac{7}{x^2},\ y = 0,\ x = 1,\ x = 5$$

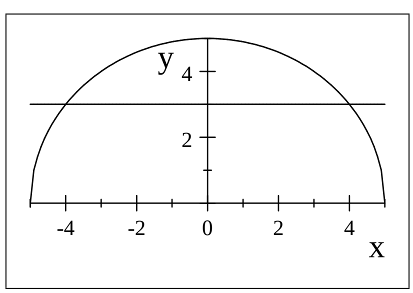


Solution

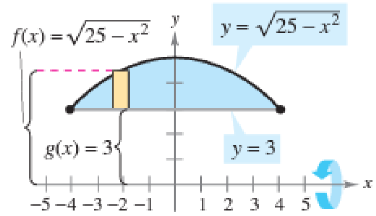


$$\begin{aligned}
V &= \int_1^5 2\pi \cdot \textit{average radius} \cdot \textit{height}\ dx \\
&= \int_1^5 2\pi\left(x+\frac{1}{2}\right)\left(\frac{7}{x^2}\right) dx \qquad (\ \textbf{2pts} + \textbf{2pts}+\ \textbf{2pts}) \\
&= 7\pi\int_1^5\left(\frac{2}{x}+\frac{1}{x^2}\right) dx \\
&= 7\pi\left(2\ln 5+\frac{4}{5}\right) = 88.380 \qquad (\textbf{1pt})
\end{aligned}$$

B) Use the washer (disk) method to find the volume of the solid generated by rotating about the x-axis the region bounded by the graphs of $f(x) = \sqrt{25 - x^2}$ and $g(x) = 3$.



Solution



$$
\begin{aligned}
V &= \int_{-4}^{4} Area\ of\ slice\ dx \\
&= \int_{-4}^{4} \left[\pi\left(\sqrt{25-x^2}\right)^2 - \pi\,(3)^2\right] dx \qquad (\ \mathbf{2pts} + \mathbf{2pts}) \\
&= \pi \int_{-4}^{4} \left(16 - x^2\right) dx \\
&= \frac{256}{3}\pi = 268.08 \qquad (\mathbf{2pts})
\end{aligned}
$$

(11pts) Problem 3
Evaluate the integral

$$\int \frac{x^2+1}{x\,(x-1)^3}\,dx$$

Solution

$$\frac{x^2+1}{x\,(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} \qquad (\mathbf{4pts})$$

Clear the denominators:

$$x^2+1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$$

Combine like terms:

$$x^2+1 = (A+B)x^3 + (-3A-2B+C)x^2 + (3A+B-C+$$

Then we get the system of equations: $\begin{cases} A+B=0 \\ -3A-2B+C=1 \\ 3A+B-C+D=0 \\ -A=1 \end{cases}$

Solving this system we see that $A=-1, B=1, C=0, D=2$. Then

$$\frac{x^2+1}{x(x-1)^3} = \frac{-1}{x} + \frac{1}{x-1} + \frac{2}{(x-1)^3}$$

$$\int \frac{x^2+1}{x\,(x-1)^3}\,dx = -\ln|x| + \ln|x-1| - \frac{1}{(x-1)^2} \qquad (\mathbf{3pts})$$

(12pts) Problem 4
Evaluate the integral

$$\int \frac{x-3}{x^3+3x} dx$$

Solution

$$\frac{x-3}{x^3+3x} = \frac{A}{x} + \frac{Bx+C}{x^2+3} \qquad (\mathbf{3pts})$$

Clear the denominators:

$$x-3 = A(x^2+3) + (Bx+C)x$$

Combine like terms:

$$x-3 = (A+B)x^2 + Cx + 3A$$

Then we get the system of equations: $\begin{cases} A+B=0 \\ C=1 \\ 3A=-3 \end{cases}$

Solving this system we see that $A=-1, B=1, C=1$. Then

$$\frac{x-3}{x(x^2+3)} = -\frac{1}{x} + \frac{x+1}{x^2+3} \qquad (\mathbf{6pts})$$

$$\int \frac{x-3}{x^3+3x} dx = -\ln|x| + \frac{1}{2}\ln\left(x^2+3\right) + \frac{1}{3}\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + C \qquad (\mathbf{3pts})$$

(12pts) Problem 5
Use Trigonometric Substitution to evaluate

$$\int \frac{4}{x^2\left(\sqrt{x^2+4}\right)}dx$$

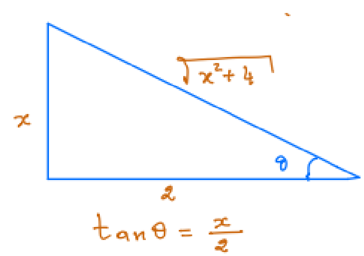
Solution
Put

$$x = 2\tan\theta, \quad \text{then } dx = 2\sec^2\theta d\theta. \qquad (\textbf{4pts})$$

The integral becomes

$$\begin{aligned}\int \frac{4}{x^2\left(\sqrt{x^2+4}\right)}dx &= 4\int \frac{1}{4\tan^2\theta\left(2\sec\theta\right)}2\sec^2\theta d\theta \\ &= \int \frac{\sec\theta}{\tan^2\theta}d\theta \\ &= \int \cos\theta\left(\sin^{-2}\theta\right)d\theta \\ &= \frac{\sin^{-2+1}\theta}{-2+1}+C \\ &= -\csc\theta + C \qquad (\textbf{4pts})\end{aligned}$$

Next,we need to plug back in x. Originally we had the substitution $x = 2\tan\theta$,so $\tan\theta = \frac{x}{2}$. This means our opposite side is x,our adjacent side is 2,and the hypotenuse is $\sqrt{x^2+4}$.



$$-\csc\theta = \frac{-1}{\sin\theta} = \frac{-1}{\frac{x}{\sqrt{x^2+4}}} = \frac{-\sqrt{x^2+4}}{x}$$

Then we have

$$\int \frac{4}{x^2\left(\sqrt{x^2+4}\right)}dx = \frac{-\sqrt{x^2+4}}{x} + C \qquad (\textbf{4pts})$$

(10pts) Problem 6
Evaluate

$$\int \frac{\sqrt{x}}{2\,(1+x)}dx$$

Solution
Put

$$u = \sqrt{x}. \text{ then } x = u^2 \text{ and } dx = 2udu. \quad (\mathbf{2pts})$$

The integral becomes

$$\begin{aligned}
\int \frac{\sqrt{x}}{2\,(1+x)}dx &= \int \frac{u}{2\,(1+u^2)}2udu \\
&= \int \frac{u^2}{1+u^2}du \\
&= \int \frac{u^2+1-1}{1+u^2}du \qquad (\mathbf{4pts}) \\
&= \int \left(1 - \frac{1}{1+u^2}\right) du \\
&= u - \tan^{-1} u + C \\
&= \sqrt{x} + \tan^{-1} \sqrt{x} + C \qquad (\mathbf{4pts})
\end{aligned}$$