Faculty of Engineering and Information Sciences

UNIVERSITY OF WOLLONGONG IN DUBAI

MATH 142, Quiz 2, Summer 2019, Duration: 60 minutes

Name:____________________, ID Number:____________________

Time Allowed: 1 Hour
Total Number of Questions: 5
Total Number of Pages (incl. this page): 6

EXAM UNAUTHORISED ITEMS
Students bringing these items to the examination room shall be required to leave the items at the front of the room or outside the examination room. The University does not guarantee the safe-keeping of students' personal items during examinations. Students concerned about the safety of their valuable items should make alternative arrangements for their care.

1. Bags, including carry bags, backpacks, shoulder bags and briefcases
2. Any form of electronic device including but not limited to mobile phones, smart watches, laptops, iPads, MP3 players, handheld computers and electronic dictionaries,
3. Calculator cases and covers
4. blank paper
5. Any written material

DIRECTIONS TO CANDIDATES

1. Total marks: 40
2. All questions are compulsory.
3. Answer all questions on the given exam paper sheets.
4. Write your name and Id number on the papers provided for rough work.

(8pts) Problem 1.
Determine convergence or divergence of the following improper integrals

$$1. \int_e^{\infty} \frac{dx}{x\,(\ln x)^2}, \qquad 2. \int_0^2 \frac{dx}{x-1}$$

1. **Solution**

Solution: Use the definition of improper integral and make the substitution $u = \ln x$ with $dx = xdu$. Then

$$\begin{aligned}\int_e^{\infty} \frac{1}{x(\ln x)^2}dx &= \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^2}dx = \lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u^2}du \\ &= \lim_{t\to\infty}\left[-\frac{1}{u}\right]_1^{\ln t} = \lim_{t\to\infty}(-\frac{1}{\ln t}+1) = 1.\end{aligned}$$

(4pts)

2.**Solution**

Note that this integral is an improper integral as $\frac{1}{x-1}$ is not defined at $x = 1$. Now,

$$\int_0^2 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x-1}\,dx.$$

By definition,

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,dx.$$

$$\begin{aligned}\lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,dx &= \lim_{t\to 1^-}\left[\ln|x-1|\right]_0^t \\ &= \lim_{t\to 1^-}\ln|t-1| = +\infty.\end{aligned}$$

This means that $\int_0^2 \frac{1}{x-1}\,dx$ diverges.

Note: You could equally consider $\int_1^2 \frac{1}{x-1}\,dx$ and get the same conclusion. The point is that one of these integrals is divergent is enough to conclude $\int_0^2 \frac{1}{x-1}\,dx$ is divergent. **(4pts)**

(8pts)Problem 2.
Show that the equation is separable and find the general solution.

$$2\frac{dy}{dx} = (y-1)\,e^x.$$

Solution
This is a separable equation.

$$\begin{aligned} 2\frac{dy}{y-1} &= e^x dx \qquad\qquad (\mathbf{4pts}) \\ \int \frac{1}{y-1} dy &= \frac{1}{2}\int e^x dx \\ \ln|y-1| &= e^x + \ln C \qquad\qquad (\mathbf{2pts}) \\ y-1 &= Ce^{e^x}. \end{aligned}$$

Expanding and solving for y, we get

$$y = Ce^{e^x} + 1 \qquad\qquad (\mathbf{2pts})$$

(8pts)Problem 3.
Solve the initial value problem for the linear equation below

$$\frac{dy}{dx} = -\frac{1}{x}y + \sin x, \qquad y(\pi) = 1, \quad x > 0.$$

Solution
This is a linear first order equation.

$$\frac{dy}{dx} + \frac{1}{x}y = \sin x. \qquad \textbf{(2pts)}$$

The integrating factor is

$$e^{\int \frac{1}{x} dx} = x \qquad x > 0.$$

$$\frac{d}{dx}[xy] = x \sin x.$$

$$xy = \int x \sin x dx.$$

Integrating by parts, we get

$$xy = -x\cos x + \sin x + C$$

So

$$\begin{aligned} y &= \frac{-\cancel{x}\cos x}{\cancel{x}} + \frac{\sin x}{x} + \frac{C}{x} \\ &= -\cos x + \frac{\sin x}{x} + \frac{C}{x}. \qquad \textbf{(4pts)} \end{aligned}$$

Now using the initial condition, we get

$$1 = 1 + 0 + \frac{C}{\pi}.$$

$$C = 0. \qquad \textbf{(2pts)}$$

(8pts)Problem 4.
Show that the differential equation is exact and find the general solution.

$$\left(xe^{2y} - x^2\right) dx + \left(x^2 e^{2y} + e^y\right) dy = 0.$$

Solution
Put

$$M = xe^{2y} - x^2 \qquad \text{and} \qquad N = x^2 e^{2y} + e^y$$

$$M_y = 2xe^{2y} = N_x \quad \text{shows that the equation is exact.} \qquad (\mathbf{2pts})$$

There exist a function f such that

$$\begin{cases} \dfrac{\partial f}{\partial x} = xe^{2y} - x^2 \\ \dfrac{\partial f}{\partial y} = x^2 e^{2y} + e^y \end{cases} \qquad (\mathbf{2pts})$$

$$\frac{\partial f}{\partial y} = x^2 e^{2y} + e^y \Rightarrow f(x, y) = \frac{1}{2} x^2 e^{2y} + e^y + g(x).$$

Now using the expression for $\dfrac{\partial f}{\partial x}$, we have

$$xe^{2y} + g'(x) = xe^{2y} - x^2$$

$$g'(x) = -x^2$$

and

$$g(x) = -\frac{x^3}{3} + C.$$

The general solution is

$$\frac{1}{2} x^2 e^{2y} + e^y - \frac{x^3}{3} = C \qquad (\mathbf{4pts})$$

(8pts)Problem 5.
(a) Solve the homogeneous equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}.$$

(b) Find an explicit solution of the initial value problem

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}, \qquad y(e) = 2e.$$

Solution
(a)

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy} = \left(\frac{y}{x}\right)^{-1} + \frac{y}{x}. \qquad \textbf{(2pts)}$$

Put

$$u = \frac{y}{x} \Rightarrow y = xu$$
$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

The equation becomes

$$u + x\frac{du}{dx} = \frac{1}{u} + u$$
$$x\frac{du}{dx} = \frac{1}{u}$$
$$udu = \frac{1}{x}dx$$

Integrating, we get

$$\frac{u^2}{2} = \ln|x| + C$$
$$\frac{1}{2}\frac{y^2}{x^2} = \ln|x| + C. \qquad \textbf{(3pts)}$$

(b)

$$y^2 = 2x^2\left(\ln|x| + C\right).$$
$$y(e) = 2e \Rightarrow C = 1 \qquad \textbf{(3pts)}$$
$$y^2 = 2x^2\left(\ln|x| + 1\right)$$