Faculty of Engineering and Information Sciences

UNIVERSITY OF WOLLONGONG IN DUBAI

MATH 142, Quiz 2, Winter 2023, Duration: 60 minutes

Name: ____________________, ID Number: ____________________

Time Allowed: 1 Hour
Total Number of Questions: 6
Total Number of Pages (incl. this page): 7
EXAM UNAUTHORISED ITEMS
Students bringing these items to the examination room shall be required to leave the items at the front of the room or outside the examination room. The University does not guarantee the safe-keeping of students' personal items during examinations. Students concerned about the safety of their valuable items should make alternative arrangements for their care.

1. Bags, including carry bags, backpacks, shoulder bags and briefcases
2. Any form of electronic device including but not limited to mobile phones, smart watches, laptops, iPads, MP3 players, handheld computers and electronic dictionaries,
3. Calculator cases and covers
4. blank paper
5. Any written material

DIRECTIONS TO CANDIDATES

1. Total marks: 40
2. All questions are compulsory.
3. Answer all questions on the given exam paper sheets.
4. Write your name and Id number on the papers provided for rough work.

(6pts)**Problem 1.**
Determine convergence or divergence of the following improper integrals,

1. $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$ 2. $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^3}$

Solution

1.

Find $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$, if it converges.

Solution: We might think just to do

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \left[3(x-1)^{1/3}\right]_0^3,$$

but this is not okay: The function $f(x) = \frac{1}{(x-1)^{2/3}}$ is undefined when $x = 1$, so we need to split the problem into two integrals.

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx.$$

The two integrals on the right hand side both converge and add up to $3[1+2^{1/3}]$, so $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3[1+2^{1/3}]$. $(3pts)$

2.

$$\begin{aligned}
\int_2^\infty \frac{dx}{x(\ln x)^3} &= \lim_{t\to\infty}\int_2^t \frac{1}{x}(\ln x)^{-3}\,dx \\
&= \lim_{t\to\infty}\left[\frac{(\ln x)^{-2}}{-2}\right]_2^t \\
&= \lim_{t\to\infty}\left[\frac{-1}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2}\right] \\
&= \frac{1}{2(\ln 2)^2} = 1.0407 \qquad (3pts)
\end{aligned}$$

(7pts)**Problem 2.**
Solve the following differential equation

$$\frac{dy}{dx} = \frac{(y+3)(x-1)}{(y-2)(x+4)}$$

Solution
The equation is separable

$$\frac{y-2}{y+3}dy = \frac{x-1}{x+4}dx$$

$\Rightarrow$

$$\int \frac{y-2}{y+3}dy = \int \frac{x-1}{x+4}dx$$

$\Rightarrow$

$$\int \left(1 - \frac{5}{y+3}\right) dy = \int \left(1 - \frac{5}{x+4}\right) dx \qquad (4pts)$$

$\Rightarrow$

$$y - 5\ln|y+3| = x - 5\ln|x+4| + C$$

or

$$(y-x) + 5\ln\left|\frac{x+4}{y+3}\right| = C \qquad (3pts)$$

(7pts)**Problem 3.**
Show that the equation is linear and solve the initial value problem

$$x\frac{dy}{dx} - 2y = x^2, \qquad y(1) = 3$$

SOLUTION Begin by writing the equation in standard form.

$$y' + \left(-\frac{2}{x}\right)y = x \qquad \text{Standard form, } y' + P(x)y = Q(x)$$

In this form, you can see that $P(x) = -2/x$ and $Q(x) = x$. So,

$$\begin{aligned}\int P(x)\,dx &= -\int \frac{2}{x}\,dx \\ &= -2\ln x \\ &= -\ln x^2\end{aligned}$$

which implies that the integrating factor is

$$\begin{aligned}u(x) &= e^{\int P(x)\,dx} \\ &= e^{-\ln x^2} \\ &= \frac{1}{e^{\ln x^2}} \\ &= \frac{1}{x^2}. \qquad \text{Integrating factor}\end{aligned}$$

This implies that the general solution is

$$\begin{aligned}y &= \frac{1}{u(x)}\int Q(x)u(x)\,dx && \text{Form of general solution} \\ &= \frac{1}{1/x^2}\int x\left(\frac{1}{x^2}\right)dx && \text{Substitute.} \\ &= x^2\int \frac{1}{x}\,dx && \text{Simplify.} \\ &= x^2(\ln x + C). && \text{General solution}\end{aligned}$$

$(5pts)$

$$y(1) = 3 \text{ gives } C = 3$$

The solution is

$$y = x^2(\ln x - 3) \qquad (2pts)$$

(7pts)**Problem 4.**
Show that the differential equation is exact and solve the equation.

$$\left(1 + 2x - y^3\right) dx + \left(2y - 3xy^2\right) dy = 0$$

Solution

$$M(x, y) = 1 + 2x - y^3 \quad \text{and} \quad N(x, y) = 2y - 3xy^2 \qquad (2pts)$$

$$M_y = -3y^2 = N_x \qquad \therefore \text{ The differential equation is exact.}$$

So, there is a function f such that

$$\frac{\partial f}{\partial x} = 1 + 2x - y^3 \ \text{ and } \frac{\partial f}{\partial y} = 2y - 3xy^2$$

$$\frac{\partial f}{\partial x} = 1 + 2x - y^3 \Rightarrow f(x, y) = x + x^2 - xy^3 + g(y). \qquad (3pts)$$

Plugging this into $\dfrac{\partial f}{\partial y} = 2y - 3xy^2$, we obtain

$$-3xy^2 + g'(y) = 2y - 3xy^2 \Rightarrow g'(y) = 2y.$$

Thus

$$g(y) = y^2 + C_1$$

and

$$f(x, y) = x + x^2 - xy^3 + y^2 + C_1$$

The solution of the differential equation is

$$x + x^2 - xy^3 + y^2 + C_1 = C_2 \quad \text{or} \quad x + x^2 - xy^3 + y^2 = C \qquad (2pts)$$

(6pts)**Problem 5.**
Determine convergence or divergence of the series. Justify your answer by applying the appropriate test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$ 2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+1}{\pi n+3}\right)^n$

Solution
1.

$$\sum_{n=1}^{\infty} \frac{n!}{3^n}$$

We will use the ratio test to test the convergence of the series.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)n!}{3\cdot 3^n} \cdot \frac{3^n}{n!}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)}{3}\right| = +\infty > 1 \\
&\text{diverges} \qquad (3pts)
\end{aligned}$$

2.

$$\sum_{n=1}^{\infty} \left(\frac{3n+1}{\pi n+3}\right)^n$$

Here, we shall use the root test.

$$\begin{aligned}
\lim_{n\to\infty} \left|\left(\frac{3n+1}{\pi n+3}\right)^n\right|^{1/n} &= \lim_{n\to\infty} \left|\frac{3n+1}{\pi n+3}\right| = \frac{3}{\pi} < 1 \\
&\text{The series converges absolutely} \qquad (3pts)
\end{aligned}$$

(7pts)**Problem 6.**

Using the Integral Test Remainder Estimate for the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ to find that the smallest number of terms needed to ensure that the sum is accurate to within 0.009.

Solution

$$\frac{1}{n^4} = f(n), \text{ where } f(x) = \frac{1}{x^4}.$$

Using the formula

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx,$$

we have

$$\begin{aligned} R_n &\leq \int_n^{\infty} \frac{1}{x^4}dx \qquad (3pts) \\ &= \lim_{t\to\infty} \int_n^t x^{-4}dx \\ &= \lim_{t\to\infty} \left. \frac{-1}{3x^3} \right|_n^t \\ &= \lim_{t\to\infty} \left[\frac{-1}{3t^3} + \frac{1}{3n^3} \right] = \frac{1}{3n^3} \end{aligned}$$

$$\frac{1}{3n^3} < 0.009 \Rightarrow n > 3.3333$$

The smallest number of terms needed to ensure that the sum is accurate to within 0.009 is $\underline{\mathbf{n = 4}}$. $(4pts)$