

**(12pts) Problem 1.**

Evaluate the following integrals

1.  $\int x \sin x dx$

2.  $\int 3 \sec^3 x dx.$

**Solution**

1. By parts

$$\begin{aligned} u &= x, & u' &= 1 \\ v' &= \sin x & v &= -\cos x \end{aligned}$$

$$\int x \sin x dx = \sin x - x \cos x + C \quad (6pts)$$

2.

$$\int 3 \sec^3 x dx = \int \sec x \sec^2 x dx$$

Use integration by parts. Let  $u = \sec x$ ,  $dv = \sec^2 x dx$ . Then  $du = \sec x \tan x dx$  and  $v = \tan x$ :

$$\begin{aligned} \int \sec^3 x dx &= \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx = \sec x \tan x + \ln |\sec x + \tan x| - \int \sec^3 x dx \end{aligned}$$

Notice on the right side we have the same integral as what we started with, so move it over to the left side:

$$2 \int \sec^3 x dx = \sec x \tan x + \ln |\sec x + \tan x|$$

Divide by 2 and add C:

$$\int \sec^3 x dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C$$

$$\int 3 \sec^3 x dx = \frac{3}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C \quad (6pts)$$

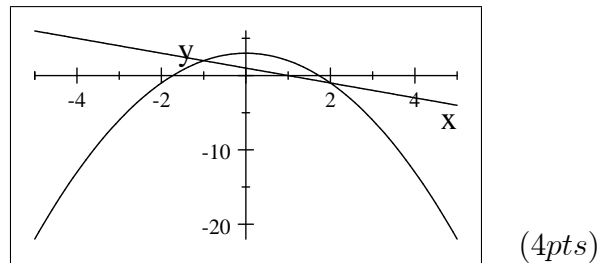
**(12pts) Problem 2.**

Find the area of the region bounded by the graphs of

$$f(x) = 3 - x^2 \quad \text{and} \quad g(x) = -x + 1$$

from  $x = 0$  to  $x = 2$ .

**Solution**



$$\begin{aligned} A &= \int_0^2 [(3 - x^2) - (-x + 1)] dx \\ &= \int_0^2 (-x^2 + x + 2) dx \\ &= \frac{10}{3} = 3.3333. \quad (8pts) \end{aligned}$$

**(12pts) Problem 3**

Find the arclength of the function

$$f(x) = \frac{x^3}{6} + \frac{1}{2x}$$

on the interval  $[1, 3]$ .

**Solution**

$$L = \int_1^3 \sqrt{1 + [f'(x)]^2} dx. \quad (4pts)$$

$$f'(x) = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

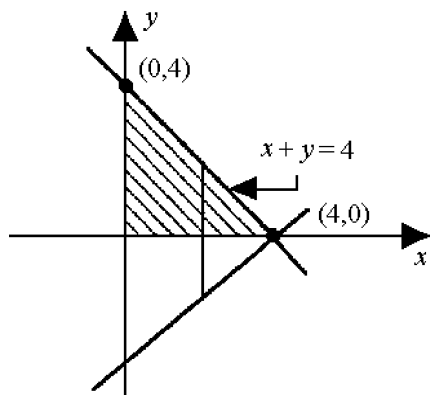
$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left( \frac{1}{2}x^2 - \frac{1}{2x^2} \right)^2 \\ &= \frac{1}{4x^4} + \frac{1}{4}x^4 + \frac{1}{2} \\ &= \left( \frac{1}{2}x^2 + \frac{1}{2x^2} \right)^2 \end{aligned}$$

$$\begin{aligned} L &= \int_1^3 \sqrt{\left( \frac{1}{2}x^2 + \frac{1}{2x^2} \right)^2} dx \\ &= \int_1^3 \left( \frac{1}{2}x^2 + \frac{1}{2x^2} \right) dx \\ &= \frac{14}{3} = 4.666\overline{7} \quad (8pts) \end{aligned}$$

**(10pts) Problem 4**

Sketch the region bounded by  $x + y = 4$ ,  $y = 0$  and  $x = 0$ , and use the disc method to find the volume of the solid generated by revolving the region about the  $x$ -axis.

**Solution**



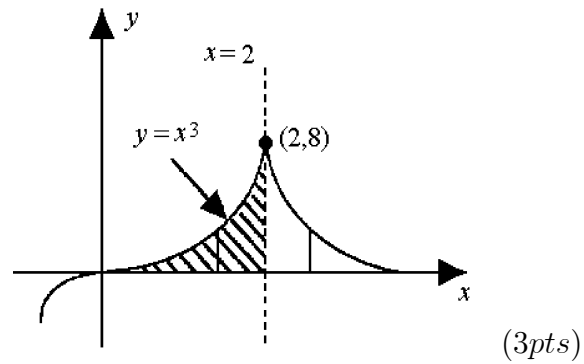
(3pts)

$$V = \int_0^4 \pi (4 - x)^2 dx$$
$$\frac{64}{3}\pi = 67.021 \quad (7pts)$$

**(10pts) Problem 5**

Sketch the region bounded by  $y = x^3$ ,  $x = 2$ , and  $y = 0$ , and use the shell method to find the volume of the solid generated by revolving the region about the line  $x = 2$ .

### Solution



$$\begin{aligned} V &= \int_0^2 2\pi(2-x)(x^3) dx \\ &= \frac{16}{5}\pi = 10.053 \quad (7pts) \end{aligned}$$

**(12pts) Problem 6**

Use trigonometric substitution to evaluate the integral

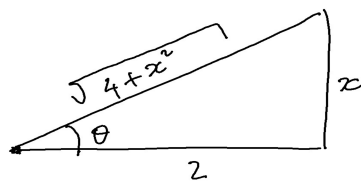
$$\int \frac{dx}{(x^2 + 4) \sqrt{x^2 + 4}}.$$

**Solution**

Put

$$\begin{aligned} x &= 2 \tan \theta \\ dx &= 2 \sec^2 \theta d\theta \end{aligned} \quad (2pts)$$

$$\begin{aligned} \int \frac{dx}{(x^2 + 4) \sqrt{x^2 + 4}} &= \int \frac{2 \sec^2 \theta d\theta}{(4 \tan^2 \theta + 4) \sqrt{4 \tan^2 \theta + 4}} \\ &= \int \frac{2 \sec^2 \theta d\theta}{4 \sec^2 \theta (2 \sec \theta)} \\ &= \int \frac{d\theta}{2 \sec \theta} \\ &= \frac{1}{4} \sin \theta + C \end{aligned} \quad (4pts)$$



$$\sin \theta = \frac{x}{\sqrt{4+x^2}} \quad (2pts)$$

$$\int \frac{dx}{(x^2 + 4) \sqrt{x^2 + 4}} = \frac{x}{4\sqrt{x^2 + 4}} + C. \quad (2pts)$$

**(12pts) Problem 7**

Use partial fraction decomposition to evaluate the integrals

1.  $\int \frac{(x+1) dx}{x^2(x-1)}$

**Solution**

$$\frac{x+1}{x^2(x-1)} = -\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1} \quad (6pts)$$

$$\int \frac{(x+1) dx}{x^2(x-1)} = -2 \ln |x| + \frac{1}{x} + 2 \ln |x-1| + C \quad (6pts)$$

**(10pts) Problem 8**

Find the area of the surface obtained by rotating the graph of

$$f(x) = \sqrt{x+1} \ , \quad -1 \leq x \leq 1$$

about the x-axis.

**Solution**

$$S = 2\pi \int_{-1}^1 \sqrt{x+1} \sqrt{1 + [f'(x)]^2} dx \quad (3pts)$$

$$= 2\pi \int_{-1}^1 \sqrt{x+1} \sqrt{1 + \left( \frac{1}{\sqrt{x+1}} \right)^2} dx \quad (4pts)$$

$$= \pi \left( 4\sqrt{3} - \frac{4}{3} \right)$$

$$= 17.577. \quad (3pts)$$



**(10pts) Problem 9**

Evaluate the integral

$$\int \frac{1}{x + x\sqrt{x}} dx$$

**Solution**

Put

$$u = \sqrt{x}$$

$$x = u^2 \quad \text{and} \quad dx = 2u du \quad (3pts)$$

$$\begin{aligned} \int \frac{1}{x + x\sqrt{x}} dx &= \int \frac{2u du}{u^2 + u^3} \\ &= \int \frac{2 du}{u(1 + u)} \end{aligned}$$

$$\frac{2}{u(1 + u)} = \frac{2}{u} - \frac{2}{u + 1}$$

$$\int \frac{2 du}{u(1 + u)} = 2 \ln |u| - 2 \ln |u + 1| + C \quad (5pts)$$

$$\int \frac{1}{x + x\sqrt{x}} dx = 2 \ln \sqrt{x} - 2 \ln |\sqrt{x} + 1| + C \quad (2pts)$$