Bernoulli Equation

A differential equation of the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$

is called a Bernoulli Equation.

If $n=1$ or $n=0$, then equation is linear.

Solving

To solve the Bernoulli equation,

$$\frac{dx}{dy} + P(x)y = Q(x)\, y^n$$

Do the following

1. Divide both sides by y^n

$$y^{-n} \frac{dy}{dx} + P(x)\, y^{1-n} = Q(x)$$

2. Do the substitution $u = y^{1-n}$

$$u = y^{1-n}$$

$$du = (1-n)\, y^{1-n-1} \frac{dy}{dx}$$

$$du = (1-n)\, y^{-n} \frac{dy}{dx}$$

$$\frac{1}{1-n} \frac{du}{dx} = y^{-n} \frac{dy}{dx}$$

$$\frac{1}{1-n} \frac{du}{dx} + P(x)\, u = Q(x)$$

$$\frac{du}{dx} + (1-n)\, P(x)\, u = Q(x)$$

This is a simple linear equation in u.

Example

Show that the equation is Bernoulli and solve the initial value problem.

$$x\frac{dy}{dx} + 5y = 2x^2y^4 \qquad y(1) = 3$$

$$\frac{dy}{dx} + \frac{5}{x}y = 2xy^4$$

$$y^{-4}\frac{dy}{dx} + \frac{5}{x}y^{-3} = 2x$$

$$u = y^{-3}$$

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$y^{-4}\frac{dy}{dx} = \frac{-1}{3}\frac{du}{dx}$$

$$\frac{du}{dx} - \frac{15}{x}u = -6x$$

$$IF = e^{-\int\frac{15}{x}dx} = e^{-15\ln|x|} = x^{-15}$$

$$u = \frac{1}{x^{-15}}\int x^{-15}(-6x)\,dx$$

$$= -6x^{15}\int x^{-14}\,dx$$

$$= \frac{\cancel{-}6x^{15}\times x^{-13}}{\cancel{-}13} + x^{15}c$$

$$= \frac{6x^2}{13} + x^{15}c$$

$$u = y^{-3}$$

$$\frac{1}{y^3} = \frac{6x^2}{13} + x^{15}c$$

$$y = \sqrt[3]{\frac{13}{6x^2} + \frac{1}{cx^{15}}}$$

@ $x = 1$, $y = 3$

$$3 = \sqrt[3]{\frac{13}{6} + \frac{1}{c}}$$

$$27 = \frac{13}{6} + \frac{1}{c}$$

$$\frac{149}{6} = \frac{1}{c}$$

$$c = \frac{6}{149}$$

Sequences & Series

Just a function, a special type of function

A sequence is a function defined on a set of natural numbers

$a_n = 2n - 1$ with $n \in \{1, 2, 3, 4, 5\}$

$a(n) = \{1, 3, 5, 7, 9\} \Rightarrow$ Finite sequence

Given a sequence a_n, $n \in \mathbb{N}$,

a_1 is called the first term of the sequence a_n

a_2 " " second " " " " "

$\vdots$

a_n " " n^{th} " " " " " "

Exa

Example

Find the first 4 terms of the sequence

$$a_n = \frac{\log(n)}{n+1}$$

$$a_1 = \frac{\log(1)}{1+1} = 0$$

$$a_2 = \frac{\log 2}{3}$$

$$a_3 = \frac{\log 3}{4}$$

$$a_4 = \frac{\log 4}{5}$$

Example

Find the n^{th} term (a_n) of the sequence $\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \frac{11}{10}$

$$\frac{1+2n}{2n}$$

Arithmetic Sequences

A sequence a_n is said to be arithmetic if $a_{n+1} - a_n = d$ is constant.

d is called the common difference.

If a_n is arithmetic with first term a_1 and common difference d then

$$a_n = a_1 + (n-1)d$$

Geometric Sequences

A sequence a_n is called geometric if

$$\frac{a_{n+1}}{a_n} = r \text{, is constant}$$

r is called the common ratio of the geometric sequence.

If a_n is a geometric sequence with first term a_1 and common ratio r, then its n-th term is given by $a_n = a_1 r^{n-1}$

Example

Find the n^{th} term of the sequence $1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}$

$$\frac{a_2}{a_1} = \frac{3/5}{1} \qquad \frac{a_3}{a_2} = \frac{9/125}{3/5} = \frac{3}{5} \qquad \therefore r = \frac{3}{5}$$

$$a_n = 1\left(\frac{3}{5}\right)^{n-1}$$

Recursive Representation of Sequences

A recursive representation is a representation that gives the first term(s) of the sequence along with a relationship between the remaining terms.

Example

$a_1 = 2$

$a_{n+1} = 3a_n - a_n^2 + 1$

Find a_6

$$a_2 = 3a_1 - a_1^2 + 1$$
$$= 3(2) - 2^2 + 1$$
$$= 3$$

$$a_3 = 3a_2 - a_2^2 + 1$$
$$= 3(3) - 3^2 + 1$$
$$= 1$$

$$a_4 = 3a_3 - a_3^2 + 1$$

$= 3 - 1 + 1$

$= \underline{3}$

$a_5 = 3a_4 - a_4^3 + 1$

$= 3(3) - 3^2 + 1$

$= \underline{1}$

$a_6 = 3a_5 - a_5^2 + 1$

$= 3 - 1 + 1$

$= \underline{3}$

Limit of Sequences

Because sequences are functions defined on set of natural numbers 1, 2, 3, 4, - - -
we will be interested in finding limit of a_n when $n \to +\infty$

$$\lim_{n \to +\infty} a_n$$

All properties for limit of functions are valid for limit of sequences.

Example

Evaluate the following limits.

1. $\displaystyle\lim_{n \to \infty} \frac{5n^3 + 4n - 6}{7 - 3n^3}$

$\displaystyle\lim_{n \to \infty} \frac{5\cancel{n^3}}{-3\cancel{n^3}}$

$= \underline{\dfrac{-5}{3}}$

2. $\displaystyle\lim_{n \to \infty} n \sin\left(\frac{4}{n}\right)$

$\displaystyle\lim_{n \to \infty} \frac{\sin 4/n}{1/n}$

$= \lim 4 \sin 4/n$

$$\lim_{n\to\infty} \frac{}{4/n}$$

$$= \lim_{n\to\infty} 4 \times 1$$

$$= 4$$

3. $$\lim_{n\to\infty} \frac{n^2}{2^n - 1}$$

$$\boxed{\text{If } a_n = f(n) \text{ and } \lim_{x\to\infty} f(x) = L, \text{ then } \lim_{n\to\infty} a_n = L}$$

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1}$$

$$= \lim_{x\to\infty} \frac{2x}{2^x \ln 2}$$

$$= \lim_{x\to\infty} \frac{2}{(\ln 2)^2\, 2^x}$$

$$= 0$$

4. $$\lim_{n\to\infty} \frac{n!}{n^n}$$