(8pts)Problem 1
Find the area of the region bounded by the curves

$$f(x) = x^2 \text{ and } \quad g(x) = 2 - x^2, \quad 0 \leq x \leq 2.$$

Solution
<u>Method 1 without graphing</u>

$$\begin{aligned} A &= \int_0^2 |f(x) - g(x)|\, dx \\ &= \int_0^2 \left|x^2 \ - \left(2 - x^2\right)\right| dx \\ &= \int_0^2 \left|2x^2 - 2\right| dx \\ &= 2\int_0^2 \left|x^2 - 1\right| dx. \qquad \textbf{(4pts)} \end{aligned}$$

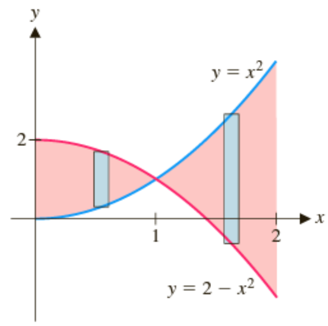
To remove the absolute value, we need to check the sign of $x^2 - 1$ between 0 and 2.

x	$-\infty$ -1	0	1	2	$+\infty$
x^2-1	$+$	$-$	$-$ 0	$+$	$+$
$\lvert x^2-1\rvert$	x^2-1	$-(x^2-1)$	$-(x^2-1)$	x^2-1	x^2-1

$0 \leq x \leq 2$

$$\begin{aligned} A &= 2\left[\int_0^1 -\left(x^2-1\right) dx + \int_1^2 \left(x^2-1\right) dx\right] \\ &= 2\left(\frac{2}{3} + \frac{4}{3}\right) = 4 \qquad \textbf{(4pts)} \end{aligned}$$

Method 2 by graphing



(4pts)

$$A = \int_0^1 [(2 - x^2) - x^2]\,dx + \int_1^2 [x^2 - (2 - x^2)]\,dx$$

$$= \int_0^1 (2 - 2x^2)\,dx + \int_1^2 (2x^2 - 2)\,dx = \left[2x - \frac{2x^3}{3}\right]_0^1 + \left[\frac{2x^3}{3} - 2x\right]_1^2$$

$$= \left(2 - \frac{2}{3}\right) - (0 - 0) + \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4.$$ **(4pts)**

(8pts)Problem 2

Find the arc length of the graph of

$$f(x) = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$$

on the interval $[-2,\ 2]$.

Solution

$$L = \int_{-2}^{2} \sqrt{1 + [f'(x)]^2} dx$$

$$f'(x) = \frac{1}{2}\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)$$

$$\begin{aligned} [f'(x)]^2 &= \frac{1}{4}\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2 \\ &= \frac{1}{4}e^x - \frac{1}{2} + \frac{1}{4}e^{-x} \end{aligned}$$

$$\begin{aligned} 1 + [f'(x)]^2 &= \frac{1}{4}e^x + 1 - \frac{1}{2} + \frac{1}{4}e^{-x} \\ &= \frac{1}{4}e^x + \frac{1}{2} + \frac{1}{4}e^{-x} \\ &= \left[\frac{1}{2}\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)\right]^2 \end{aligned}$$

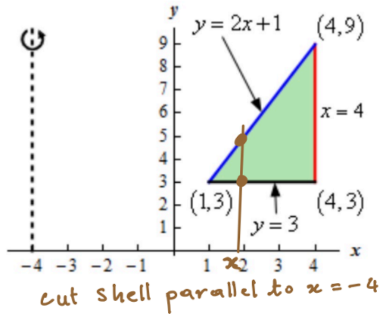
$$\begin{aligned} L &= \int_{-2}^{2} \sqrt{\left[\frac{1}{2}\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)\right]^2} dx \qquad (\mathbf{4pts}) \\ &= \int_{-2}^{2} \frac{1}{2}\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right) dx \\ &= \frac{1}{\cancel{2}}\left(\cancel{2}e^{\frac{x}{2}} - \cancel{2}e^{-\frac{x}{2}}\right)\Big|_{-2}^{2} \\ &= \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)\Big|_{-2}^{2} = \left(e - e^{-1}\right) - \left(e^{-1} - e\right) \\ &= 2e - 2e^{-1} = 4.7008 \qquad (\mathbf{4pts}) \end{aligned}$$

(8pts) Problem 3
Find the volume of the solid formed by revolving the region bounded by $y = 2x + 1,\ x = 4$ and $y = 3$ about the line $x = -4$.

Solution

By Shell Method

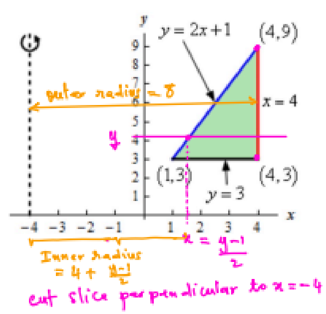
The shell method is more convenient for this problem.



(2pts)

$$
\begin{aligned}
V &= \int_a^b 2\pi \cdot \text{average radius} \ \cdot \ \text{height } dx \\
&= \int_1^4 2\pi\,(x+4)\,[(2x+1)-3]\,dx \quad (\mathbf{4pts}) \\
&= \int_1^4 2\pi\,(x+4)\,(2x-2)\,dx \\
&= 126\pi = 395.84 \qquad (\mathbf{2pts})
\end{aligned}
$$

By the Disc Method



(2pts)

$$\text{Inner Radius} = 4 + \frac{1}{2}(y-1) = \frac{1}{2}y + \frac{7}{2} \qquad \text{Outer Radius} = 4 + 4 = 8$$

$$
\begin{aligned}
A(x) &= \pi\left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] \\
&= \pi\left[(8)^2 - \left(\frac{1}{2}y + \frac{7}{2}\right)^2\right] = \pi\left(\frac{207}{4} - \frac{7}{2}y - \frac{1}{4}y^2\right)
\end{aligned}
$$

(3pts)

$$V = \int_3^9 \pi\left(\frac{207}{4} - \frac{7}{2}y - \frac{1}{4}y^2\right) dy = \pi\left(\frac{207}{4}y - \frac{7}{4}y^2 - \frac{1}{12}y^3\right)\Bigg|_3^9 = \boxed{126\pi}$$

(3pts)

(**8pts**)**Problem 4.**
Find the area of the surface generated by revolving the parametric curve

$$x = e^t - t \quad \text{and} \quad y = 4e^{t/2}, \qquad 0 \le t \le 1.$$

about the x-axis.

Solution

$$S = 2\pi \int_0^1 y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \left(e^t - 1\right)^2 + \left(2e^{t/2}\right)^2 \\
&= e^{2t} - 2e^t + 4e^t + 1 \\
&= e^{2t} + 2e^t + 1 \\
&= \left(e^t + 1\right)^2 \qquad (\mathbf{3pts})
\end{aligned}$$

$$\begin{aligned}
S &= 2\pi \int_0^1 4e^{t/2} \sqrt{\left(e^t + 1\right)^2} dt \\
&= 2\pi \int_0^1 4e^{t/2} \left(e^t + 1\right) dt \\
&= 8\pi \int_0^1 \left(e^{3t/2} + e^{t/2}\right) dt \\
&= 8\pi \left(2e^{\frac{1}{2}} + \frac{2}{3}e^{\frac{3}{2}} - \frac{8}{3}\right) = 90.945 \qquad (\mathbf{5pts})
\end{aligned}$$

(8pts)Problem 5.
Find the slope and the equation of the tangent line to the graph of the polar curve

$$r = e^{2\theta}$$

at $\theta = 0$.

Solution

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$x = e^{2\theta}\cos\theta \quad \text{and} \quad y = e^{2\theta}\sin\theta$$

$$\text{Slope } = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{2e^{2\theta}\sin\theta + e^{2\theta}\cos\theta}{2e^{2\theta}\cos\theta - e^{2\theta}\sin\theta} \qquad (\mathbf{4pts})$$

At $\theta = 0$, the slope is

$$m = \frac{0+1}{2-0} = \frac{1}{2}. \qquad (\mathbf{2pts})$$

When $\theta = 0,\ x = 1$ and $y = 0$.
The equation of the tangent line is

$$y = \left(\frac{1}{2}\right)(x-1) + 0$$

$$y = \frac{1}{2}x - \frac{1}{2} \qquad (\mathbf{2pts})$$