(12pts)Problem 1.

Evaluate the following integrals

$$1.\ \int_0^{\frac{\pi}{4}} x\sin(2x)dx \qquad\qquad 2.\ \int \cos(\ln x)\,dx$$

Solution

1.

$$\int_0^{\frac{\pi}{4}} x\sin(2x)dx$$

Using integration by parts, put

$$\begin{array}{cc} u = x, & u' = 1 \\ v' = \sin(2x), & v = \frac{-1}{2}\cos(2x). \end{array} \qquad \textbf{(2pts)}$$

Applying the integration by parts formula, we get

$$\begin{aligned} \int_0^{\frac{\pi}{4}} x\sin(2x)dx &= \frac{-x}{2}\cos(2x)\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}}\cos(2x)\,dx \\ &= \frac{-x}{2}\cos(2x)\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\,\frac{1}{2}\sin(2x)\Big|_0^{\frac{\pi}{4}} \\ &= 0 + \frac{1}{4} \\ &= \frac{1}{4} = 0.25 \qquad \textbf{(3pts)} \end{aligned}$$

2.

$$\int \cos(\ln x)\,dx$$

We first perform a substitution. Put

$$X = \ln x, \qquad dX = \frac{1}{x}dx$$

$$x = e^X \ \text{ and } dx = e^X dX$$

$$\begin{aligned} \int \cos(\ln x)\,dx &= \int (\cos X)\left(e^X dX\right) \\ &= \int e^X\cos X dX. \qquad \textbf{(3pts)} \end{aligned}$$

Now put

$$A = \int e^X \cos X dX.$$

Using integration by parts, we get

$$\begin{array}{cc} u = e^X, & u' = e^X \\ v' = \cos X, & v = \sin X. \end{array}$$

$$A = \int e^X \cos X dX = e^X \sin X - \int e^X \sin X dX$$

Again by parts

$$\begin{array}{cc} u = e^X, & u' = e^X \\ v' = \sin X, & v = -\cos X. \end{array}$$

Hence,

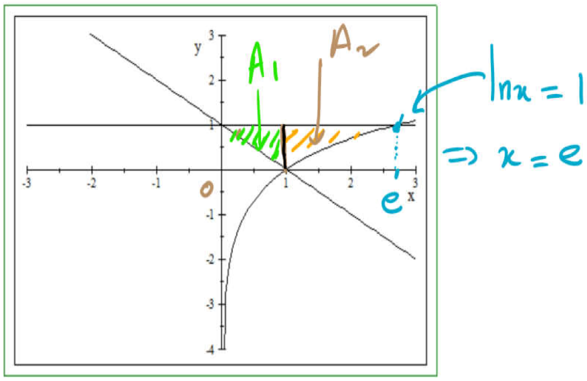
$$\begin{aligned} A = e^X \sin X - \left[-e^X \cos X + A\right] \Leftrightarrow & \qquad (\textbf{2pts}) \\ A = e^X \sin X + e^X \cos X - A & \\ 2A = e^X \sin X + e^X \cos X & \\ A = \frac{1}{2}\left(e^X \sin X + e^X \cos X\right) + C & \end{aligned}$$

Now using $X = \ln x$,

$$\begin{aligned} \int \cos(\ln x)\, dx &= A = \frac{1}{2}\left(e^{\ln x} \sin(\ln x) + e^{\ln x} \cos(\ln x)\right) + C \\ &= \frac{x}{2}\left(\sin(\ln x) + \cos(\ln x)\right) + C \qquad (\textbf{2pts}) \end{aligned}$$

(10pts)Problem 2.
Sketch the region bounded by the graphs of $y = \ln x$, $x + y = 1$, and the line $y = 1$, and find the area of that region.

Solution



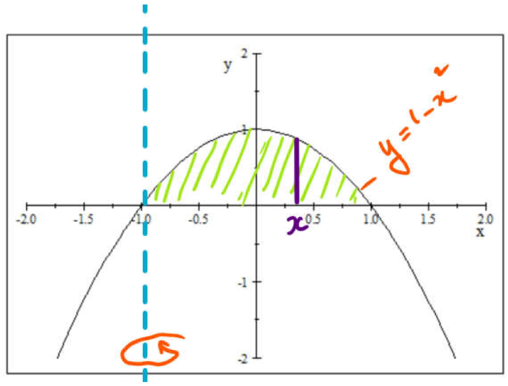
(3pts)

$$
\begin{aligned}
A &= A_1 + A_2 \qquad \textbf{(2pts)} \\
&= \int_0^1 [1-(1-x)]\,dx + \int_1^e (1-\ln x)\,dx \\
&= \int_0^1 x dx + \int_1^e (1-\ln x)\,dx \qquad \textbf{(3pts)} \\
&= \frac{1}{2} + e - 2 \\
&= e - \frac{3}{2} = 1.2183 \qquad \textbf{(2pts)}
\end{aligned}
$$

(10pts)Problem 3

Sketch the region bounded by the curves $y = 1 - x^2$ and $y = 0$, and use the method of **cylindrical shells** to find the volume obtained by rotating the region about the line $x = -1$.

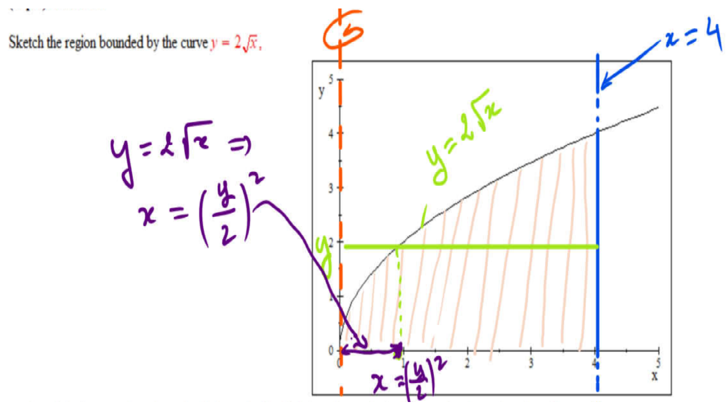
Solution



(**3pts**)

$$
\begin{aligned}
V &= \int_{-1}^{1} 2\pi \cdot (x+1) \cdot \left(1 - x^2\right) dx \qquad (\mathbf{6pts}) \\
&= \int_{-1}^{1} 2\pi \, (x+1) \left(1 - x^2\right) dx \\
&= \frac{8\pi}{3} = 8.3776. \qquad (\mathbf{1pt})
\end{aligned}
$$

(10pts)Problem 4.
Sketch the region bounded by the curve $y = 2\sqrt{x}$, the x-axis and the line $x = 4$, and use the **disk method** to find the volume obtained by rotating the region about the y-axis.
Solution



(**3pts**)

$$
\begin{aligned}
V &= \int_0^4 \pi \left[4^2 - \left(\frac{y^2}{4}\right)^2\right] dy \qquad (\mathbf{6pts}) \\
&= \int_0^4 \pi \left(16 - \frac{1}{16}y^4\right) dy \\
&= \frac{256}{5}\pi = 160.85. \qquad (\mathbf{1pt})
\end{aligned}
$$

(10pts)Problem 5.

Find the arc length of the curve $y = 3 - \ln\cos x$ from $x = 0$ to $x = \dfrac{\pi}{3}$.

Solution

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (\mathbf{2pts})$$

$$\frac{dy}{dx} = 0 - \frac{-\sin x}{\cos x} = \tan x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 = \tan^2 x = \sec^2 x \quad (\mathbf{2pts})$$

$$\begin{aligned} L &= \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x} dx \\ &= \int_0^{\frac{\pi}{3}} \sec x dx \qquad (\mathbf{3pts}) \\ &= \ln\left(\sec x + \tan x\right)\Big|_0^{\frac{\pi}{3}} \\ &= \ln\left(\sqrt{3} + 2\right) = 1.\,317\,0 \qquad (\mathbf{3pts}) \end{aligned}$$

(10pts)Problem 6.
Use trigonometric substitution to evaluate the integral

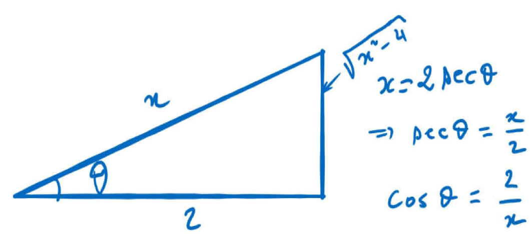
$$\int \frac{\sqrt{x^2-4}}{x}dx.$$

Solution
Here we do the trigonometric substitution

$$x = 2\sec\theta$$

$$dx = 2\sec\theta\tan\theta d\theta \qquad (\textbf{3pts})$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-4}}{x}dx &= \int \frac{\sqrt{4\sec^2\theta-4}}{2\sec\theta}2\sec\theta\tan\theta d\theta \\
&= \int \frac{2\tan\theta}{1}\tan\theta d\theta \\
&= 2\int \tan^2\theta d\theta \\
&= 2\int \left(\sec^2\theta - 1\right) d\theta \\
&= 2\left(\tan\theta - \theta\right) + C \qquad (\textbf{3pts})
\end{aligned}$$



$$(\textbf{2pts})$$

$$x = 2\sec\theta \Rightarrow \theta = \cos^{-1}\left(\frac{2}{x}\right).$$

From the triangle, we obtain

$$\tan\theta = \frac{\sqrt{x^2-4}}{2}$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-4}}{x}dx &= 2\left(\tan\theta - \theta\right) + C \\
&= \sqrt{x^2-4} - 2\cos^{-1}\left(\frac{2}{x}\right) + C. \qquad (\textbf{2pts})
\end{aligned}$$

(12pts)Problem 7.
Use partial fraction decomposition to evaluate

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$$

Solution

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x\,(x+2)\,(2x-1)}$$

Next, you do the partial fraction decomposition

$$\frac{x^2+2x-1}{x\,(x+2)\,(2x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1} \qquad (\mathbf{3pts})$$

After solving, you get

$$A = \frac{1}{2}, \qquad B = \frac{-1}{10} \text{ and } C = \frac{1}{5} \qquad (\mathbf{3pts})$$

$$\frac{x^2+2x-1}{x\,(x+2)\,(2x-1)} = \frac{1}{2x} - \frac{1}{10\,(x+2)} + \frac{1}{5\,(2x-1)}$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \frac{1}{2}\ln|x| - \frac{1}{10}\ln|x+2| + \frac{1}{10}\ln|2x+1| + C \qquad (\mathbf{6pts})$$

(13pts)Problem 8.
Evaluate the following integrals

$$1. \int \frac{dx}{2\sqrt{x}+2x}, \qquad\qquad 2. \int \frac{3x^3-3x^2+4}{x^2-x}dx$$

Solution

1.

$$\int \frac{dx}{2\sqrt{x}+2x}.$$

We rationalize by putting

$$u=\sqrt{x}$$
$$x=u^2 \text{ and } dx=2udu \qquad (\mathbf{3pts})$$

The integral becomes

$$\begin{aligned}\int \frac{dx}{2\sqrt{x}+2x} &= \int \frac{2udu}{2u+2u^2}\\ &= \int \frac{\cancel{2u}du}{\cancel{2u}(1+u)}\\ &= \int \frac{du}{1+u} = \ln|1+u|+C\\ &= \ln\left|1+\sqrt{x}\right|+C \qquad (\mathbf{3pts})\end{aligned}$$

2.

$$\int \frac{3x^3-3x^2+4}{x^2-x}dx$$

Since the degree of the numerator is higher, we first perform a long division.

$$\frac{3x^3-3x^2+4}{x^2-x} = 3x - \frac{4}{x-x^2}. \qquad (\mathbf{3pts})$$

Next, we do a partial fraction decomposition on $\frac{4}{x-x^2}$.

$$\frac{4}{x-x^2} = \frac{4}{-x(x-1)} = \frac{4}{x} - \frac{4}{x-1}$$

$$\begin{aligned}\int \frac{3x^3-3x^2+4}{x^2-x}dx &= \int \left[3x - \left(\frac{4}{x} - \frac{4}{x-1}\right)\right]dx\\ &= \int \left(3x - \frac{4}{x} + \frac{4}{x-1}\right)dx\\ &= \frac{3}{2}x^2 - 4\ln|x| + 4\ln|x-1| + C \qquad (\mathbf{4pts})\end{aligned}$$

(13pts)Problem 9.
Determine the convergence or divergence the following improper integrals. If the integral is convergent, then find its value.

1. $\displaystyle\int_0^\infty \frac{x}{1+x^2}dx,$ 2. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

Solution
1.

$$\begin{aligned}\int_0^\infty \frac{x}{1+x^2}dx &= \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}dx \\ &= \lim_{t\to\infty}\frac{1}{2}\int_0^t \frac{2x}{1+x^2}dx \\ &= \lim_{t\to\infty}\frac{1}{2}\ln\left(1+x^2\right)\Big|_0^t \qquad (\mathbf{4pts}) \\ &= \lim_{t\to\infty}\frac{1}{2}\ln\left(1+t^2\right) \\ &= \infty \quad \text{diverges} \qquad (\mathbf{2pts})\end{aligned}$$

2.

$$\begin{aligned}\int_{-2}^{14}\frac{dx}{\sqrt[4]{x+2}} &= \lim_{t\to-2^-}\int_t^{14}\frac{dx}{\sqrt[4]{x+2}} \qquad (\mathbf{2pts}) \\ &= \lim_{t\to-2^-}\int_t^{14}(x+2)^{-1/4}\,dx \\ &= \lim_{t\to-2^-}\frac{4}{3}(x+2)^{\frac{3}{4}}\Big|_t^{14} \qquad (\mathbf{3pts}) \\ &= \lim_{t\to-2^-}\left[\frac{4}{3}(14+2)^{\frac{3}{4}}-\frac{4}{3}(t+2)^{\frac{3}{4}}\right] \\ &= \frac{32}{3} = 10.667.\end{aligned}$$

Converges and its value is 10.667 $(\mathbf{2pts})$.