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Students must comply with requirements stated in the Examination Policy & Procedures

Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	**05/18/2023**
Subject Code:	**Math 142**
Subject Title:	**Essentials of Engineering Mathematics**
Time Permitted to Write Exam:	1 Hour
Total Number of Questions:	**5 written questions**
Total Number of Pages *(including this page)*:	**6**

INSTRUCTIONS TO STUDENTS FOR THE EXAM

1. Please note that subject lecturer/tutor will be unavailable during exams. *If there is a doubt in any of the exam questions i.e. problem solving etc. students should proceed by assuming values etc. Students should mention their assumption on the question paper.*
2. Answers must be written (and drawn) in black or blue ink
3. Any mistakes must be crossed out. Whitener and ink erasers must not be used.
4. Answer ALL/ 5 questions. The marks for each question are shown next to each question.
5. Total marks: 40.

(8pts) Problem 1.
Determine convergence or divergence of the following improper integrals

$$1. \int_e^\infty \frac{dx}{x(\ln x)^2}, \qquad\qquad 2. \quad \int_0^2 \frac{dx}{x-1}$$

1. **Solution**

Solution: Use the definition of improper integral and make the substitution $u = \ln x$ with $dx = xdu$. Then

$$\begin{aligned}\int_e^\infty \frac{1}{x(\ln x)^2}dx &= \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^2}dx = \lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u^2}du \\ &= \lim_{t\to\infty}\left[-\frac{1}{u}\right]_1^{\ln t} = \lim_{t\to\infty}(-\frac{1}{\ln t}+1) = 1.\end{aligned}$$

(4pts)

2.**Solution**

Note that this integral is an improper integral as $\frac{1}{x-1}$ is not defined at $x = 1$. Now,

$$\int_0^2 \frac{1}{x-1}dx = \int_0^1 \frac{1}{x-1}dx + \int_1^2 \frac{1}{x-1}dx.$$

By definition,

$$\int_0^1 \frac{1}{x-1}dx = \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}dx.$$

$$\begin{aligned}\lim_{t\to 1^-}\int_0^t \frac{1}{x-1}dx &= \lim_{t\to 1^-}[\ln|x-1|]_0^t \\ &= \lim_{t\to 1^-}\ln|t-1| = +\infty.\end{aligned}$$

This means that $\int_0^2 \frac{1}{x-1}dx$ diverges.

Note: You could equally consider $\int_1^2 \frac{1}{x-1}dx$ and get the same conclusion. The point is that one of these integrals is divergent is enough to conclude $\int_0^2 \frac{1}{x-1}dx$ is divergent. **(4pts)**

(8pts)Problem 2.
Show that the equation is separable and find the general solution.

$$2\frac{dy}{dx} = (y-1)\,e^x.$$

Solution
This is a separable equation.

$$2\frac{dy}{y-1} = e^x dx \qquad (\textbf{4pts})$$

$$\int \frac{1}{y-1}dy = \frac{1}{2}\int e^x dx$$

$$\ln|y-1| = \frac{1}{2}e^x + \ln C \qquad (\textbf{2pts})$$

$$y-1 = Ce^{\frac{e^x}{2}}.$$

Expanding and solving for y, we get

$$y = Ce^{\frac{e^x}{2}} + 1 \qquad (\textbf{2pts})$$

(8pts)Problem 3.
Solve the initial value problem for the linear equation below

$$\frac{dy}{dx} = -\frac{1}{x}y + \sin x, \qquad y(\pi) = 1, \quad x > 0.$$

Solution
This is a linear first order equation.

$$\frac{dy}{dx} + \frac{1}{x}y = \sin x. \qquad (\mathbf{2pts})$$

The integrating factor is

$$e^{\int \frac{1}{x} dx} = x \qquad\qquad x > 0.$$

$$\frac{d}{dx}[xy] = x \sin x.$$

$$xy = \int x \sin x dx.$$

Integrating by parts, we get

$$xy = -x\cos x + \sin x + C$$

So

$$\begin{aligned} y &= \frac{-\cancel{x}\cos x}{\cancel{x}} + \frac{\sin x}{x} + \frac{C}{x} \\ &= -\cos x + \frac{\sin x}{x} + \frac{C}{x}. \qquad (\mathbf{4pts}) \end{aligned}$$

Now using the initial condition, we get

$$1 = 1 + 0 + \frac{C}{\pi}.$$

$$C = 0. \qquad (\mathbf{2pts})$$

(8pts)Problem 4.
Show that the differential equation is exact and find the general solution.

$$\left(xe^{2y} - x^2\right) dx + \left(x^2 e^{2y} + e^y\right) dy = 0.$$

Solution
Put

$$M = xe^{2y} - x^2 \qquad \text{and} \qquad N = x^2 e^{2y} + e^y$$

$$M_y = 2xe^{2y} = N_x \quad \text{shows that the equation is exact.} \qquad (\textbf{2pts})$$

There exist a function f such that

$$\begin{cases} \dfrac{\partial f}{\partial x} = xe^{2y} - x^2 \\ \dfrac{\partial f}{\partial y} = x^2 e^{2y} + e^y \end{cases} \qquad (\textbf{2pts})$$

$$\frac{\partial f}{\partial y} = x^2 e^{2y} + e^y \Rightarrow f(x, y) = \frac{1}{2} x^2 e^{2y} + e^y + g(x).$$

Now using the expression for $\dfrac{\partial f}{\partial x}$, we have

$$xe^{2y} + g'(x) = xe^{2y} - x^2$$

$$g'(x) = -x^2$$

and

$$g(x) = -\frac{x^3}{3} + C.$$

The general solution is

$$\frac{1}{2} x^2 e^{2y} + e^y - \frac{x^3}{3} = C \qquad (\textbf{4pts})$$

(8pts)Problem 5.
(a) Solve the homogeneous equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}.$$

(b) Find an explicit solution of the initial value problem

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}, \qquad y(e) = 2e.$$

Solution
(a)

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy} = \left(\frac{y}{x}\right)^{-1} + \frac{y}{x}. \qquad \textbf{(2pts)}$$

Put

$$u = \frac{y}{x} \Rightarrow y = xu$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

The equation becomes

$$u + x\frac{du}{dx} = \frac{1}{u} + u$$

$$x\frac{du}{dx} = \frac{1}{u}$$

$$udu = \frac{1}{x}dx$$

Integrating, we get

$$\frac{u^2}{2} = \ln|x| + C$$

$$\frac{1}{2}\frac{y^2}{x^2} = \ln|x| + C. \qquad \textbf{(3pts)}$$

(b)

$$y^2 = 2x^2\left(\ln|x| + C\right).$$

$$y(e) = 2e \Rightarrow C = 1 \qquad \textbf{(3pts)}$$

$$y^2 = 2x^2\left(\ln|x| + 1\right)$$