SPRING 2020 MATH 142-EXAM 1-UOWD

Part 1 MCQ (30%)

Directions: Circle the letter that corresponds to the correct answer. There is only one correct answer for each question. You do not need to show your work.

$(6pts)$**Problem 1**

$$\int x^3 \ln x dx$$

is equal to:

(a) $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

(b) $\ln x - \frac{1}{16}x^4 + C$

(c) $\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$

(d) $x^3 \ln x - \frac{1}{9}x^4 + C$

(e) $\frac{1}{4}x^4 \ln x - x^3 + C$

Solution

Using integration by parts,

$$u = \ln x \ , \qquad u' = \frac{1}{x}$$
$$v' = x^3, \qquad v = \frac{x^4}{4}$$

you get

$$\int x^3 \ln x dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C. \quad \textbf{Answer is (c)}$$

$(6pts)$**Problem 2** Let

$$F(x) = \int e^x \cos x dx \qquad \text{(without the arbitrary constant C).}$$

$F(0)$ is equal to

$$(a) \quad \sqrt{e}$$

$$(b) \quad 0$$

$$(c) \quad e$$

$$(d) \quad \tfrac{1}{2}$$

$$(e) \quad \tfrac{1}{4}$$

Solution
Using integration by parts once,

$$\begin{array}{cc} u = e^x \ , & u' = e^x \\ v' = \cos x, & v = \sin x \end{array} ,$$

you get

$$F(x) = e^x \sin x - \int e^x \sin x dx.$$

Repeating the integration by parts on $\int e^x \sin x dx$,

$$\begin{array}{cc} u = e^x \ , & u' = e^x \\ v' = \sin x, & v = -\cos x \end{array}$$

you obtain

$$F(x) = e^x \sin x - [-e^x \cos x + F(x)]$$

equivalently

$$F(x) = e^x \sin x + e^x \cos x - F(x).$$

Now solving for $F(x)$, you get

$$F(x) = \frac{1}{2} [e^x \sin x + e^x \cos x]$$

$$F(0) = \frac{1}{2}. \qquad \textbf{Answer is (d)}$$

(6*pts*)**Problem 3**

If the region enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $x = -1$, then the volume of the obtained solid is

(a) $\frac{\pi}{2}$

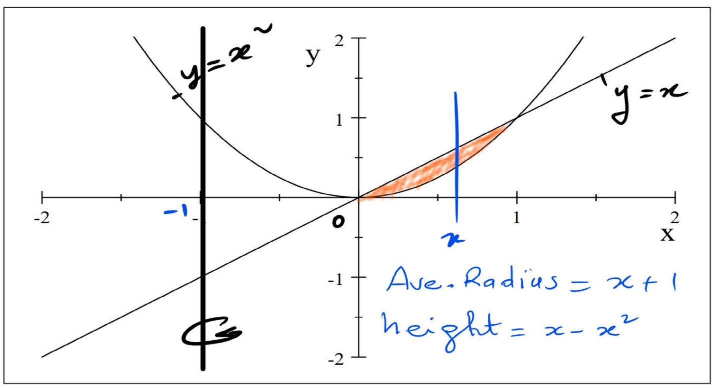
(b) $\frac{\pi}{4}$

(c) π

(d) $\frac{3\pi}{2}$

(e) $\frac{5\pi}{2}$

Solution



Using te shell method,

$$\begin{aligned} V &= \int_0^1 2\pi (x+1) \left(x - x^2\right) dx \\ &= \frac{\pi}{2} \qquad\qquad \textbf{Answer is (a)} \end{aligned}$$

(6*pts*)**Problem 4**

The area of the region enclosed by the curves $y = \dfrac{1}{x}, \quad y = 0, \ x = -3, \ x = -2$

(a) 1

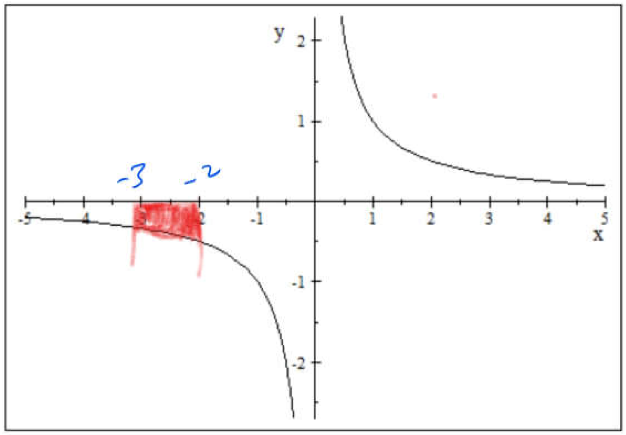
(b) $\ln\left(\frac{3}{2}\right)$

(c) 3

(d) $\ln 3$

(e) $\ln 2$

Solution



$$
\begin{aligned}
A &= \int_{-3}^{-2}\left(0-\frac{1}{x}\right)dx \\
&= \int_{-3}^{-2}-\frac{1}{x}dx = \ln\frac{3}{2}
\end{aligned}
$$

Answer is (b)

$(6pts)$**Problem 5**
The length of the curve

$$F(x) = \int_{-2}^{x} \sqrt{3t^4 - 1}dt, \qquad -2 \leq x \leq -1$$

is equal to

$$(a) \quad \tfrac{\sqrt{3}}{2}$$

$$(b) \quad \tfrac{\sqrt{3}}{3}$$

$$(c) \quad \sqrt{3}$$

$$(d) \quad \tfrac{3\sqrt{3}}{4}$$

$$(e) \quad \tfrac{7\sqrt{3}}{3}$$

$$Hint: \quad \frac{d}{dx}\int_{a}^{x} f(t)dt = f(x)$$

Solution

$$\begin{aligned} L &= \int_{-2}^{-1} \sqrt{1 + (F'(x))^2}dx \\ &= \int_{-2}^{-1} \sqrt{1 + \left(\sqrt{3x^4 - 1}\right)^2}dx \\ &= \int_{-2}^{-1} \sqrt{1 + 3x^4 - 1}dx \\ &= \int_{-2}^{-1} \sqrt{3x^4}dx \\ &= \int_{-2}^{-1} \sqrt{3}x^2dx = \frac{7}{3}\sqrt{3} \qquad \textbf{Answer is (e)} \end{aligned}$$

Part 2 Written Questions (70%)

$(10pts)$**Problem 1**
Find the area of the region enclosed by the curves $y = x^2 - 4x + 5$ and $y = 2x - 3$.
Solution
Method 1(without sketching)

$$\begin{aligned} A &= \int_a^b \left|\left(x^2 - 4x + 5\right) - (2x - 3)\right| dx \qquad (\mathbf{4pts}) \\ &= \int_a^b \left|x^2 - 6x + 8\right| dx. \end{aligned}$$

To find a and b, you do

$$x^2 - 4x + 5 = 2x - 3.$$

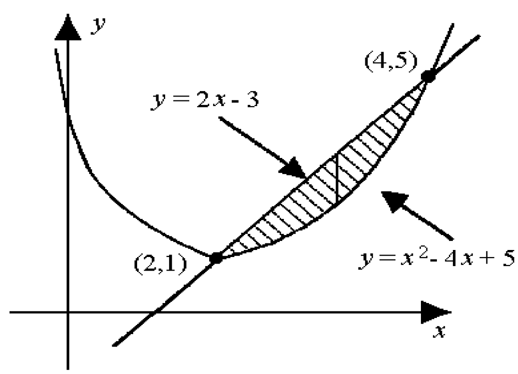
Solving, you get

$$\textbf{Solution is} : 4, 2 \qquad (\mathbf{3pts})$$

So $a = 2$ and $b = 4$.

$$\begin{aligned} A &= -\int_2^4 \left(x^2 - 6x + 8\right) dx \\ &= \frac{4}{3} \qquad (\mathbf{3pts}) \end{aligned}$$

Method 2 (with sketching)



(**4pts**)

$$\begin{aligned} A &= \int_2^4 \left[(2x - 3) - \left(x^2 - 4x + 5\right)\right] dx \\ &= \int_2^4 \left(-x^2 + 6x - 8\right) dx = \frac{4}{3} \qquad (\mathbf{3pts + 3pts}) \end{aligned}$$

$(10pts)$**Problem 2**

Find the area of the surface obtained by rotating the graph of

$$f(x) = 2\sqrt{x+1}\ , \qquad 0 \le x \le 1$$

about the x-axis.

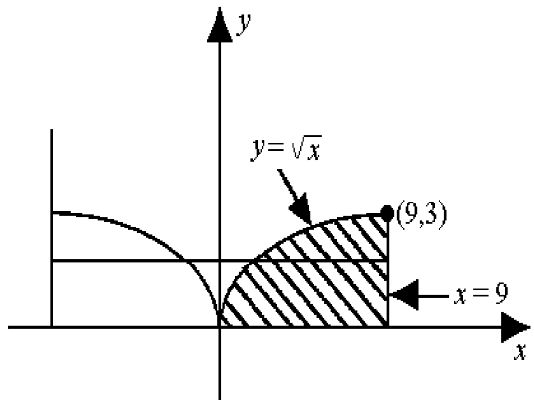
Solution

$$\begin{aligned}
S &= 2\pi \int_0^1 2\sqrt{x+1}\sqrt{1+[f\,'(x)]^2}dx \\
&= 2\pi \int_0^1 2\sqrt{x+1}\sqrt{1+\left(\frac{1}{\sqrt{x+1}}\right)^2}dx \qquad (\mathbf{4pts}) \\
&= 2\pi \int_0^1 2\frac{\sqrt{x+2}}{\sqrt{x+1}}\sqrt{x+1}dx \\
&= 4\pi \int_0^1 \sqrt{x+2}dx \qquad (\mathbf{4pts}) \\
&= 4\pi\left(2\sqrt{3}-\frac{4}{3}\sqrt{2}\right) = 19.836. \qquad (\mathbf{2pts})
\end{aligned}$$

$(10pts)$**Problem 3**

Sketch the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$, and use the disc method to find the volume of the solid generated by revolving the region about the y-axis.

Solution

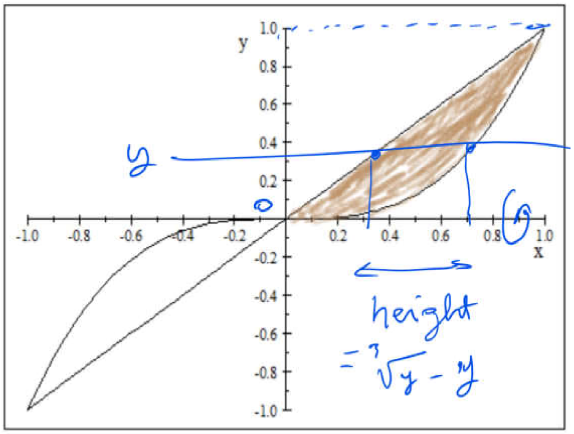


(**2pts**)

$$
\begin{aligned}
V &= \int_0^3 \left(81\pi - \pi y^4\right) dy \quad (\mathbf{2pts} + \mathbf{2pts} + \mathbf{2pts}) \\
&= \frac{972}{5}\pi = 610.73 \qquad (\mathbf{2pts})
\end{aligned}
$$

(10pts)**Problem 4**
Sketch the region bounded by $y = x^3$ and $y = x$, and use the shell method to find the volume of the solid generated by revolving the region about the x-axis.

Solution



(**2pts**)

$$
\begin{aligned}
V &= \int_0^1 2\pi y \left(\sqrt[3]{y} - y\right) dy \qquad (\mathbf{2pts} + \mathbf{2pts} + \mathbf{2pts}) \\
&= 2\pi \int_0^1 \left(y^{\frac{4}{3}} - y^2\right) dy \\
&= \frac{4}{21}\pi = 0.598\,40 \qquad (\mathbf{2pts})
\end{aligned}
$$

$(10pts)$**Problem 5**

Evaluate the integral

$$\int \frac{(x+1)(x+3)}{(2+x)^3 - (2+x)^2} dx$$

Solution

You first factor the denominator

$$\begin{aligned} \frac{(x+1)(x+3)}{(2+x)^3 - (2+x)^2} &= \frac{(x+1)(x+3)}{(2+x)^2 [(2+x-1)]} \\ &= \frac{(x+3)}{(2+x)^2} \qquad (\mathbf{2pts}) \end{aligned}$$

The partial fraction decomposition is

$$\frac{(x+3)}{(2+x)^2} = \frac{A_1}{(2+x)} + \frac{A_2}{(2+x)^2}. \qquad (\mathbf{2pts})$$

Solving, you get

$$A_1 = A_2 = 1 \qquad (\mathbf{4pts})$$

$$\begin{aligned} \int \frac{(x+1)(x+3)}{(2+x)^3 - (2+x)^2} dx &= \int \left[\frac{1}{(2+x)} + \frac{1}{(2+x)^2} \right] dx \\ &= \ln(x+2) + \frac{-1}{x+2} + C \qquad (\mathbf{2pts}) \end{aligned}$$

(10pts)**Problem 6**
Use trigonometric substitution to evaluate

$$\int \frac{4dx}{x^2\sqrt{x^2+4}}$$

Solution
$a = 2$
Put

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta d\theta \qquad (\textbf{3pts})$$

The integral becomes

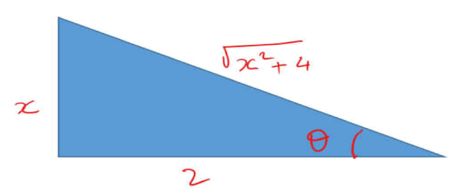
$$\begin{aligned}
\int \frac{4dx}{x^2\sqrt{x^2+4}} &= \int \frac{(4)\,2\sec^2\theta d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} \\
&= \int \frac{2\sec^2\theta d\theta}{\tan^2\theta\,(2\sec\theta)} \\
&= \int \frac{\cos\theta d\theta}{\sin^2\theta}. \qquad (\textbf{2pts})
\end{aligned}$$

Put

$$u = \sin\theta, \qquad du = \cos\theta d\theta$$

$$\begin{aligned}
\int \frac{\cos\theta d\theta}{\sin^2\theta} &= \int \frac{du}{u^2} \\
&= \frac{-1}{u} + C \\
&= \frac{-1}{\sin\theta} + C \qquad (\textbf{2pts})
\end{aligned}$$

From the triangle



$$\sin\theta = \frac{x}{\sqrt{x^2+4}}.$$

Hence,

$$\int \frac{4dx}{x^2\sqrt{x^2+4}} = \frac{-\sqrt{x^2+4}}{x} + C \qquad (\textbf{3pts})$$

$(10pts)$**Problem 7**
Evaluate the integral

$$\int \frac{\sqrt{x}}{2\left(1+\sqrt{x}\right)} dx$$

Solution
You rationalize the integral the integral by doing the substitution

$$\begin{aligned} u &= \sqrt{x} \rightarrow u^2 = x \\ 2udu &= dx \qquad \textbf{(3pts)} \end{aligned}$$

The integral becomes

$$\begin{aligned}
\int \frac{\sqrt{x}}{2\left(1+\sqrt{x}\right)} dx &= \int \frac{u}{2\left(1+u\right)} 2udu \\
&= \int \frac{u^2}{1+u} du \qquad \textbf{(2pts)} \\
&= \int \frac{u^2-1+1}{1+u} du \\
&= \int \frac{u^2-1}{1+u} du - \int \frac{1}{1+u} du \\
&= \int \left(u-1\right) du - \int \frac{1}{1+u} du \qquad \textbf{(3pts)} \\
&= \frac{u^2}{2} - u - \ln\left|1+u\right| + C \\
&= \frac{x}{2} - \sqrt{x} - \ln\left|1+\sqrt{x}\right| + C \qquad \textbf{(2pts)}
\end{aligned}$$