(12pts)Problem 1.
Evaluate the following integrals

1. $\int_0^1 (x-1)\,e^{-x}dx$ 2. $\int x^{11}\ln x dx$

Solution

1. $\int_0^1 (x-1)\,e^{-x}dx$? by parts, put

$$\begin{aligned} u &= x-1, \quad u' = 1 \\ v' &= e^{-x}, \qquad v = -e^{-x} \end{aligned}$$

$$\begin{aligned} \int_0^1 (x-1)\,e^{-x}dx &= -(x-1)\,e^{-x}\Big|_0^1 + \int_0^1 e^{-x}dx \qquad \textbf{(3pts)} \\ &= -(x-1)\,e^{-x}\Big|_0^1 + -e^{-x}\Big|_0^1 \\ &= -e^{-1} = \frac{-1}{e} = -0.367\,88 \qquad \textbf{(3pts)} \end{aligned}$$

2. $\int x^{11}\ln x dx$? Again by parts

$$\begin{aligned} u &= \ln x \quad u' = \frac{1}{x} \\ v' &= x^{11}, \qquad v = \frac{x^{12}}{12} \end{aligned}$$

$$\begin{aligned} \int x^{11}\ln x dx &= \frac{x^{12}}{12}\ln x - \int \frac{1}{x}\cdot\frac{x^{12}}{12}dx \qquad \textbf{(3pts)} \\ &= \frac{x^{12}}{12}\ln x - \frac{1}{12}\int x^{11}dx \\ &= \frac{x^{12}}{12}\ln x - \frac{x^{12}}{(12)^2} + C \\ &= \frac{1}{12}x^{12}\ln x - \frac{1}{144}x^{12} + C \qquad \textbf{(3pts)} \end{aligned}$$

(11pts)Problem 2.

Find the area of the region bounded by the graphs of $y = \dfrac{2}{x}$ and $y = 3 - x$.

Solution

Method 1 (without graphing)

$$A = \int_a^b \left| \frac{2}{x} - (3-x) \right| dx.$$

To find a and b, we solve the equation

$$\begin{aligned} \frac{2}{x} &= (3-x) \Leftrightarrow x^2 - 3x + 2 = 0 \quad (x \neq 0) \\ x &= 1 \text{ or } x = 2 \end{aligned}$$

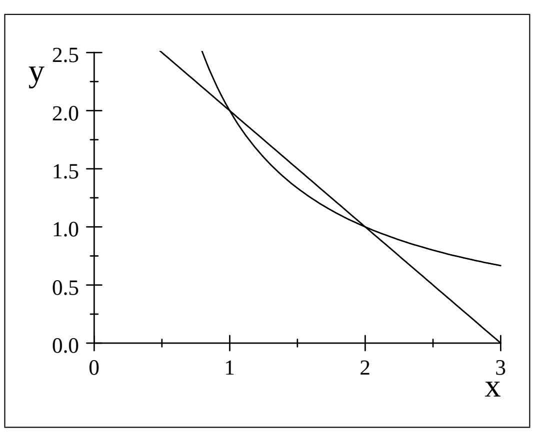
$$\begin{aligned} A &= \int_1^2 \left| \frac{2}{x} - (3-x) \right| dx \\ &= \int_1^2 \left| \frac{x^2 - 3x + 2}{x} \right| dx \qquad (\mathbf{5pts}) \end{aligned}$$

Since x is positive between 1 and 2, the sign of $\dfrac{x^2 - 3x + 2}{x}$ depends on the sign of $x^2 - 3x + 2$ between 1 and 2.

It is clear that $x^2 - 3x + 2$ has the opposite sign of $a = 1$ (negative) between the roots 1 and 2.

$$\begin{aligned} A &= \int_1^2 \frac{-(x^2 - 3x + 2)}{x} dx \\ &= \int_1^2 \left(-x + 3 - \frac{2}{x} \right) dx \\ &= \left. \frac{-x^2}{2} + 3x - 2\ln x \right|_1^2 \\ &= \frac{3}{2} - 2\ln 2 = 0.113\,71 \qquad (\mathbf{6pts}) \end{aligned}$$

Method 2 (with graphing)



(**5pts**)

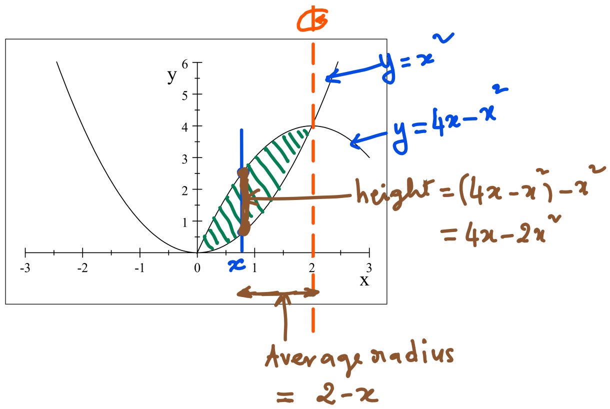
From the graph, you can see that the line is on top of the parabola between 1 and 2.

$$\begin{aligned}
A &= \int_1^2 \left[(3-x) - \frac{2}{x}\right] dx \\
&= \int_1^2 \left(-x + 3 - \frac{2}{x}\right) dx \\
&= \left.\frac{-x^2}{2} + 3x - 2\ln x\right|_1^2 \\
&= \frac{3}{2} - 2\ln 2 = 0.113\,71 \qquad (\textbf{6pts})
\end{aligned}$$

(10pts)Problem 3

Sketch the region bounded by the curves $y = x^2$ and $y = 4x - x^2$, and use the method of **cylindrical shells** to find the volume obtained by rotating the region about the line $x = 2$.

Solution

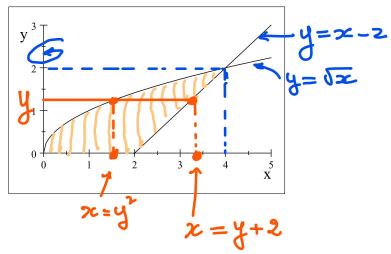


(**4pts**)

$$
\begin{aligned}
V &= \int_0^2 2\pi\,(2-x)\left(4x - 2x^2\right) dx \quad (\mathbf{4pts}) \\
&= 2\pi \int_0^2 \left(2x^3 - 8x^2 + 8x\right) dx \\
&= \frac{16}{3}\pi = 16.755 \qquad (\mathbf{2pts})
\end{aligned}
$$

(10pts)Problem 4.
Sketch the region above the x-axis bounded by the curve $y = \sqrt{x}$, $y = x - 2$ use the **disk method** to find the volume obtained by rotating the region about the y-axis.

Solution



(**4pts**)

$$\begin{aligned} V &= \int_0^2 \left[\pi (y+2)^2 - \pi \left(y^2\right)^2\right] dy \qquad (\mathbf{4pts}) \\ &= \pi \int_0^2 \left(-y^4 + y^2 + 4y + 4\right) dy \\ &= \frac{184}{15}\pi = 38.537. \quad (\mathbf{2pts}) \end{aligned}$$

(10pts)Problem 5.

Find the arc length of the curve $y = \dfrac{3}{2}x^{2/3} + 4$ from $x = 1$ to $x = 27$.

Solution

$L = \int\limits_1^{27} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int\limits_1^{27} \sqrt{1 + x^{\frac{-2}{3}}} dx$? cannot evaluate $\Rightarrow$ we consider x as function of y. (**5pts**)

$$x = \frac{2\sqrt{2}}{3\sqrt{3}}(y-4)^{3/2}$$

When $x = 1$, $y = \dfrac{3}{2} + 4 = \dfrac{11}{2}$.

When $x = 27$, $y = \dfrac{3}{2}(27)^{2/3} + 4 = \dfrac{35}{2}$

$$L = \int\limits_{\frac{11}{2}}^{\frac{35}{2}} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$\frac{dx}{dy} = \frac{3}{2}\frac{2\sqrt{2}}{3\sqrt{3}}(y-4)^{1/2}$$

$$\left(\frac{dx}{dy}\right)^2 = \left(\frac{3}{2}\frac{2\sqrt{2}}{3\sqrt{3}}\right)^2 (y-4) = \frac{2}{3}y - \frac{8}{3}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{2}{3}y - \frac{5}{3}$$

$$\begin{aligned} L &= \int\limits_{\frac{11}{2}}^{\frac{35}{2}} \sqrt{\frac{2}{3}y - \frac{5}{3}} dy \\ &= \frac{1}{\sqrt{3}} \int\limits_{\frac{11}{2}}^{\frac{35}{2}} \sqrt{2y-5} dy \\ &= 10\sqrt{10} - 2\sqrt{2} = 28.794 \qquad (\mathbf{5pts}) \end{aligned}$$

(10pts)Problem 6.
Use trigonometric substitution to evaluate the integral

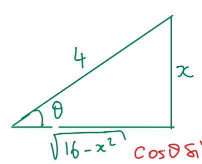
$$\int \frac{x^2}{\sqrt{16 - x^2}} dx.$$

Solution
We do the substitution

$$x = 4\sin\theta, \qquad dx = 4\cos\theta d\theta \qquad (\mathbf{2pts})$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^2}} dx &= \int \frac{16\sin^2\theta}{4\cos\theta} 4\cos\theta d\theta \\
&= \int 16\sin^2\theta d\theta \\
&= 16 \int \frac{1-\cos 2\theta}{2} d\theta \\
&= 8\left(\theta - \frac{1}{2}\sin 2\theta\right) + C \\
&= 8\left(\theta - \cos\theta\sin\theta\right) + C \qquad (\mathbf{4pts})
\end{aligned}$$

$$\sin\theta = \frac{x}{4}, \qquad \theta = \sin^{-1}\left(\frac{x}{4}\right)$$



$\sin\theta = \frac{x}{4}$

$\cos\theta = \frac{\sqrt{16-x^2}}{4}$

$\cos\theta\sin\theta = \frac{x\sqrt{16-x^2}}{16}$

$(\mathbf{4pts})$

$$\int \frac{x^2}{\sqrt{16-x^2}} dx = 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16-x^2} + C$$

(11pts)Problem 7.
Use partial fraction decomposition to evaluate

$$\int \frac{3x-4}{x^2-2x+1}dx$$

Solution

$$\frac{3x-4}{x^2-2x+1} = \frac{3x-4}{(x-1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} \qquad (\mathbf{4pts})$$

$$A_1 = 3 \text{ and } A_2 = -1 \qquad (\mathbf{4pts})$$

$$\begin{aligned}\int \frac{3x-4}{x^2-2x+1}dx &= \int \left(\frac{3}{x-1} - \frac{1}{(x-1)^2}\right) dx \\ &= 3\ln|x-1| + \frac{1}{x-1} + C \qquad (\mathbf{3pts})\end{aligned}$$

(13pts)Problem 8.

Evaluate the following integrals

$$1.\ \int \frac{\sqrt{x}dx}{x-4}, \qquad\qquad 2.\ \int \frac{x^3+2x^2-4}{x^2-x}dx$$

Solution

1. $\displaystyle\int \frac{\sqrt{x}dx}{x-4}?$

$$\text{Put } u=\sqrt{x}, \quad u^2 = x \Rightarrow dx = 2udu \ \text{ and } x = u^2. \qquad (\mathbf{2pts})$$

The integral becomes

$$\begin{aligned}
\int \frac{\sqrt{x}dx}{x-4} &= \int \frac{2u^2du}{u^2-4} \\
&= 2\int \frac{u^2du}{u^2-4} = 2\int \frac{(u^2-4+4)\,du}{u^2-4} \\
&= 2\int \left(1+\frac{4}{u^2-4}\right)du \qquad (\mathbf{2pts})
\end{aligned}$$

$$\frac{4}{u^2-4} = \frac{4}{(u-2)(u+2)} = \frac{2}{u-2} - \frac{2}{u+2}$$

$$\begin{aligned}
\int \frac{\sqrt{x}dx}{x-4} &= 2\int \left(1+\frac{4}{u^2-4}\right)du \\
&= 2\int \left(1+\frac{2}{u-2}-\frac{2}{u+2}\right)du \\
&= 2u + 4\ln|u-2| - 4\ln|u+2| + C \\
&= 2\sqrt{x} + 4\ln\left|\sqrt{x}-2\right| - 4\ln\left|\sqrt{x}+2\right| + C \qquad (\mathbf{2pts})
\end{aligned}$$

2. $\displaystyle\int \frac{x^3+2x^2-4}{x^2-x}dx?$

$$\frac{x^3+2x^2-4}{x^2-x} = x+3+\frac{3x-4}{x^2-x} \qquad (\mathbf{2pts})$$

$$\frac{3x-4}{x^2-x} = \frac{3x-4}{x(x-1)} = \frac{4}{x} - \frac{1}{x-1}.$$

Thus,

$$\begin{aligned}
\int \frac{x^3+2x^2-4}{x^2-x}dx &= \int \left(x+3+\frac{4}{x}-\frac{1}{x-1}\right)dx \qquad (\mathbf{3pts}) \\
&= \frac{x^2}{2} + 3x + 4\ln|x| - \ln|x-1| + C \qquad (\mathbf{2pts})
\end{aligned}$$

(13pts)Problem 9.
Determine the convergence or divergence the following improper integrals. If the integral is convergent, then find its value.

1. $\displaystyle\int_0^{12} \frac{9}{\sqrt{12-x}}dx,$ 2. $\displaystyle\int_0^{\infty} \frac{e^x}{1+e^x}dx$

Solution
1. $\int_0^{12} \frac{9}{\sqrt{12-x}}dx$?

$$\begin{aligned}\int_0^{12} \frac{9}{\sqrt{12-x}}dx &= \lim_{t\to 12^-}\int_0^{t} \frac{9}{\sqrt{12-x}}dx \qquad (\textbf{3pts})\\ &= \lim_{t\to 12}\left(36\sqrt{3}-18\sqrt{12-t}\right)\\ &= 36\sqrt{3} = 62.354. \qquad (\textbf{4pts})\end{aligned}$$

The improper integral converges to $36\sqrt{3} = 62.354$

2. $\int_0^{\infty} \frac{e^x}{1+e^x}dx$?

$$\begin{aligned}\int_0^{\infty} \frac{e^x}{1+e^x}dx &= \lim_{t\to\infty}\int_0^{t} \frac{e^x}{1+e^x}dx \qquad (\textbf{3pts})\\ &= \lim_{t\to\infty} \ln\left(1+e^x\right)\Big|_0^t\\ &= \lim_{t\to\infty}\left[1+e^t-\ln 2\right] = \infty. \qquad (\textbf{3pts})\end{aligned}$$

The improper integral diverges.