$\boxed{1}$ Give explicit solutions to the initial value problem $\frac{dy}{dx} = xy^3$ with $y(0) = 1$, $y(0) = 1/2$, and $y(0) = -2$. Then determine the domains of each of these solutions.

Solution: First, we separate the differential equation and solve it:

$$\begin{aligned} \int y^{-3} \frac{dy}{dx}\, dx &= \int x\, dx, \\ \int y^{-3}\, dy &= \frac{x^2}{2} + C, \\ \frac{y^{-2}}{-2} &= \frac{x^2}{2} + C. \end{aligned}$$

By our standard abuse of constants, we then get

$$y^2 = \frac{-1}{x^2 + C},$$

so

$$y = \pm\sqrt{\frac{-1}{x^2 + C}}.$$

Now we want to match the various initial conditions. For each, we need to decide whether to take the $+$ or the $-$ of the $\pm$, and then we need to determine C:

initial condition	$\pm$	C	solution
$y(0) = 1$	$+$	$C = -1$	$y(x) = \sqrt{\frac{-1}{x^2-1}}$
$y(0) = 1/2$	$+$	$C = -4$	$y(x) = \sqrt{\frac{-1}{x^2-4}}$
$y(0) = -2$	$-$	$C = -1/2$	$y(x) = -\sqrt{\frac{-1}{x^2-1/4}}$

We now need to determine the domains of these solutions. For all of them, we must be careful not to get 0 in the denominator or to take the square root of a negative number. For example, for the first solution, we have a problem when $x = 1$ (division by 0) or when $|x| > 1$ (square root of a negative number), so the domain is $|x| < 1$.

solution	domain
$y(x) = \sqrt{\frac{-1}{x^2-1}}$	$\lvert x\rvert < 1$
$y(x) = \sqrt{\frac{-1}{x^2-4}}$	$\lvert x\rvert < 2$
$y(x) = -\sqrt{\frac{-1}{x^2-1/4}}$	$\lvert x\rvert < 1/2$

$\boxed{2}$ Show that every separable first-order differential equation can easily be converted into an exact equation.

Solution: A first-order differential equation is *separable* if it can be put in the form

$$\frac{dy}{dx} = g(x)p(y).$$

To test for exactness, we put this equation into the form $M(x, y) + N(x, y)\frac{dy}{dx} = 0$. In this case we see that

$$M(x, y) = -g(x) \text{ and } N(x, y) = \frac{1}{p(y)}.$$

Now we only need that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, which is true because both partials are 0.

$\boxed{3}$ For each of the following differential equations, indicate whether they are separable, linear, or can easily be converted into an exact equation. *Note that some equations may be more than one type, while others may not be any of these types.* Then, solve the equations which are separable, linear, or exact.

a. $\frac{dy}{dx} = \frac{-2xy}{x^2+y^2}$.

Solution: This equation is not separable, because there is no way[1] to write it in the form $\frac{dy}{dx} = g(x)p(y)$. The equation is also not linear, because the y^2 term prevents us from putting it in the form $\frac{dy}{dx} + P(x)y = Q(x)$. To test for exactness we put the equation in the form $M(x, y) + N(x, y)\frac{dy}{dx} = 0$:

$$2xy + (x^2 + y^2)\frac{dy}{dx} = 0.$$

Now we see that the relevant second partials are in fact equal:

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x},$$

so the equation is exact.

[1] If we wanted to, we could verify this by considering the four points $(x, y) = (1, 1), (1, 2), (2, 1), (2, 2)$, but that won't be required on the midterm.

We now proceed to solve the equation; remember that the solution will be an implicit solution of the form $F(x,y) = C$ where $\frac{d}{dx}F(x,y) = M(x,y) + N(x,y)\frac{dy}{dx}$. First we integrate $M(x,y)$ with respect to x:

$$F(x,y) = \int 2xy\,dx = x^2y + g(y).$$

Note that, as usual, our constant of integration *is a function of* y. Now we want to have

$$\frac{\partial F}{\partial y} = N(x,y),$$

so we get

$$x^2 + \frac{dg}{dy} = x^2 + y^2.$$

Therefore $dg/dy = y^2$, so $g(y) = y^3/3 + C$. Thus our solution can be written as

$$x^2y + \frac{y^3}{3} + C = D,$$

where D is another arbitrary constant. Since we don't need both of these constants, we can also write this solution as

$$x^2y + \frac{y^3}{3} = C.$$

In this particular case, we *could* solve to get y as an explicit function of x, but there is no need, so we might as well leave it in the implicit form above.

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b. $\frac{dy}{dx} = xy\sin x$.

Solution: This equation is separable, because we can write it as

$$\frac{1}{y}\frac{dy}{dx} = x\sin x.$$

Therefore, by our previous problem, this equation is also exact. Furthermore, the equation is also linear, because we can express it as

$$\frac{dy}{dx} - (x\sin x)y = 0.$$

Therefore we have three methods available to solve the equation. Viewing it as a separable equation, we separate to get

$$\frac{1}{y}\frac{dy}{dx} = x\sin x.$$

We then integrate both sides with respect to x,

$$\begin{aligned}
\int \frac{1}{y}\frac{dy}{dx}\,dx &= \int x\sin x\,dx,\\
\int \frac{1}{y}\,dy &= \sin x - x\cos x + C \text{ (obtained by integration by parts)},\\
\ln|y| &= \sin x - x\cos x + C,\\
y(x) &= \pm e^C e^{\sin x - x\cos x},\\
y(x) &= Ce^{\sin x - x\cos x}.
\end{aligned}$$

Note that in going from the second-to-last line to the last line above, we have done our standard abuse of constants, replacing $\pm e^C$ by C. It is worth noting that C cannot be 0.

Important note: because we divided by y to put the equation in the desired (separable) form, we must also consider the possible solution $y \equiv 0$ (y identically equal to 0). This equation does happen to satisfy our differential equation, so it is also a solution.

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c. $\frac{dy}{dx} = \sin(x + y^2)$.

Solution: The equation is not separable[2]. The equation is also not linear, because we cannot express it in the form $\frac{dy}{dx} + P(x)y = Q(x)$. Finally, the equation is not exact; to see this, we express it in the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$, where we see that

$$M(x,y) = -\sin(x + y^2) \text{ and } N(x,y) = 1,$$

and the partials do not match:

$$\frac{\partial M}{\partial y} = 2y\cos(x + y^2) \neq 0 = \frac{\partial N}{\partial x}.$$

Since none of our methods apply to this equation, we cannot solve it (yet).

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d. $\frac{dy}{dx} = \frac{x-y}{2x}$.

[2]Again, if we wanted to, we could verify this by considering the three points $(x,y) = (0,0), (\pi/2, 0), (0, \sqrt{\pi/2})$, but again, that won't be required on the midterm.

Solution: The equation is not separable (again this could be verified with a chart). The equation is linear, however, because we can express it as

$$\frac{dy}{dx} + \frac{1}{2x}y = \frac{1}{2}.$$

The equation is not exact, because if we express it as $M(x,y) + N(x,y)\frac{dy}{dx} = 0$, we see that

$$M(x,y) = \frac{y}{2x} - \frac{1}{2} \text{ and } N(x,y) = 1,$$

and the partials do not match:

$$\frac{\partial M}{\partial y} = \frac{1}{2x} \neq 0 = \frac{\partial N}{\partial x}.$$

Therefore we can only solve this equation by use of an integrating factor. Recall that this integrating factor is $e^{\int P(x)\,dx}$ when the equation is put in the form $\frac{dy}{dx} + P(x)y = Q(x)$:

$$e^{\int P(x)\,dx} = e^{\int \frac{1}{2x}\,dx} = e^{\frac{1}{2}\int \frac{1}{x}\,dx} = e^{\ln x^{1/2}} = x^{1/2}.$$

Multiplying through by the integrating factor, we obtain

$$x^{1/2}\frac{dy}{dx} + \frac{1}{2x^{1/2}}y = \frac{1}{2}x^{1/2}.$$

Now we integrate both sides with respect to x. We know that the left-hand side of this equation is the derivative (with respect to x) of the integrating factor times y: $x^{1/2}y$. On the right-hand side, we have

$$\int \frac{1}{x^{1/2}}\,dx = \frac{1}{2}\left(\frac{2}{3}x^{3/2}\right) + C = \frac{1}{3}x^{3/2} + C,$$

so we obtain

$$x^{1/2}y = \frac{1}{3}x^{3/2} + C,$$

and thus

$$y(x) = \frac{1}{3}x + \frac{C}{x^{1/2}}.$$

It is important to note that the term on the right of this solution is a constant *times a function of* x, and therefore we cannot replace this term by a simple constant.

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e. $\frac{dy}{dx} = \frac{5x^4}{\cos y + e^y}$.

Solution: This equation is separable, because we can express it as

$$(\cos y + e^y)\frac{dy}{dx} = 5x^4.$$

Therefore the equation is also exact. The equation is not linear, however, because e^y term prevents us from expressing it in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

To solve the equation, we integrate both sides of its separated form above (with respect to x):

$$\begin{aligned}
\int \cos y + e^y \frac{dy}{dx}\, dx &= \int 5x^4\, dx, \\
\int \cos y + e^y\, dy &= x^5 + C, \\
\sin y + e^y &= x^5 + C.
\end{aligned}$$

We have no way to solve this equation for $y(x)$, so we have to be satisfied with this implicit solution.

$\boxed{4}$ For each of the following initial value problems, determine if they have zero, one, or more than one solution(s). *You do not need to solve these equations.*

a. $y\frac{dy}{dx} + x = 0$; $y(1) = 0$.

Solution: This equation has *no* solutions. Plugging in the initial conditions $(x_0, y_0) = (1, 0)$ gives

$$(0)\frac{dy}{dx}(1) + 1 = 0,$$

i.e., $1 = 0$, which has no solution.

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b. $\frac{dy}{dx} = 3y^{2/3}$; $y(0) = 0$.

Solution: This solution has *more than one solution.* In particular, $y \equiv 0$ (y identically equal to 0 solves the differential equation, and $y(x) = x^3$ also solves the equation (this solution can be found by separating the equation). In fact, this equation has *infinitely many solutions*: for every number $a \geq 0$, the function

$$y(x) = \begin{cases} 0 & \text{for } x \leq a, \\ (x-a)^3 & \text{for } x > a \end{cases}$$

solves the equation.

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c. $y\frac{dy}{dx} = \arctan(x+y)$; $y(1) = 1$.

Solution: To use the Existence and Uniqueness Theorem for first-order differential equations, we must first put this equation in the form

$$\frac{dy}{dx} = f(x, y).$$

In this case we get

$$\frac{dy}{dx} = \frac{\arctan(x+y)}{y}.$$

Now we test whether f and $\partial f/\partial y$ are continuous in some neighborhood of $(x_0, y_0) = (1, 1)$. The only discontinuity of f is along the line $y = 0$, so f is certainly continuous in a neighborhood of $(1, 1)$. Using the quotient rule, we see that $\partial f/\partial y$ is

$$\frac{\partial f}{\partial y} = \frac{\frac{y}{(x+y)^2+1} - \arctan(x+y)}{y^2} = \frac{1}{y((x+y)^2+1)} - \frac{\arctan(x+y)}{y^2}.$$

This function is also continuous except along the line $y = 0$, and so it is continuous in a neighborhood of $(1, 1)$. We can then conclude, via the Existence and Uniqueness Theorem, that the given IVP has a unique solution.

$\boxed{5}$ Make an appropriate substitution in order to solve the following differential equations.

a. $\frac{dy}{dx} = \frac{2y}{x} - x^2y^2$.

Solution: This equation is of the form

$$a(x)\frac{dy}{dx} = b(x)y + c(x)y^n$$

(here $n = 2$), so it is Bernoulli. To solve a Bernoulli equation we divide by the greatest power of y and then let our substitution v equal the second smallest power of y in the remaining equation. So in this case we have

$$y^{-2}\frac{dy}{dx} = \frac{2}{x}y^{-1} - x^2.$$

We set $v = y^{-1}$, and then have

$$v = -y^{-2}\frac{dy}{dx}$$

by the Chain Rule. Therefore we have

$$-\frac{dv}{dx} = \frac{2}{x}v - x^2.$$

This is a linear equation (all Bernoulli equations transform into linear equations), so we put it in standard form,

$$\frac{dv}{dx} + \frac{2}{x}v = x^2.$$

Our integrating factor is then

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = e^{|x|^2} = |x|^2 = x^2.$$

After multiplying through by x^2, we have

$$x^2\frac{dv}{dx} + 2xv = x^4.$$

We now integrate both sides by x. We know that the left-hand side will become $\mu(x)v$, while the right-hand side is just $x^5/5 + C$, so

$$\begin{aligned}
x^2 v &= \frac{x^5}{5} + C,\\
v &= \frac{x^3}{5} + \frac{C}{x^2},\\
\frac{1}{y} &= \frac{x^3}{5} + \frac{C}{x^2},\\
y(x) &= \frac{1}{\frac{x^3}{5} + \frac{C}{x^2}}.
\end{aligned}$$

Important note: $y \equiv 0$ is also a solution. (We didn't find that solution above because in the beginning we divided by y^2.)

b. $x^2\frac{dy}{dx} = xy - y^2$.

Solution: This is a homogeneous equation[3], that is, an equation of the form $\frac{dy}{dx} = G(y/x)$, so we make the substitution $v = y/x$. First we need to translate the $\frac{dy}{dx}$ into these terms:

$$\begin{aligned} y &= xv, \text{so} \\ \frac{dy}{dx} &= x\frac{dv}{dx} + v \text{ by the product rule.} \end{aligned}$$

Maying this substitution, we transform the equation into

$$x\frac{dv}{dx} + v = v - v^2,$$

which separates (after canceling a v on each side) as

$$-v^{-2}\frac{dv}{dx} = x^{-1}.$$

Integrating both sides, we obtain

$$\begin{aligned} \int -v^{-2}\frac{dv}{dx}\,dx &= \int x^{-1}\,dx, \\ v^{-1} &= \ln|x| + C, \\ v &= \frac{1}{\ln|x| + C}. \end{aligned}$$

Finally we put this back in terms of $y(x)$:

$$\frac{y}{x} = \frac{1}{\ln|x| + C},$$

so

$$y(x) = \frac{x}{\ln|x| + C}.$$

Important note: when we separated this equation we divided by v^2, so we should check the solution $v \equiv 0$. This is the same as $y \equiv 0$, which you can see is a solution to the original differential equation. Therefore $y \equiv 0$ is also a solution.

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[3]This equation is also a Bernoulli equation, so you could solve it by dividing by y^2 and then setting $v = y^{-1}$.

c. $\frac{dy}{dx} = \frac{1}{(2x+y)e^{2x+y}} - 2.$

Solution: This is an equation of the form $\frac{dy}{dx} = G(ax + by)$, so we make the substitution $v = 2x + y$. We then see that

$$\frac{dv}{dx} = 2 + \frac{dy}{dx},$$

so the equation transforms into

$$\frac{dv}{dx} - 2 = \frac{1}{ve^v} - 2.$$

After canceling the -2s and separating, we just need to integrate both sides:

$$\begin{aligned}
\int ve^v \frac{dv}{dx}\,dx &= \int 1\,dx, \\
\int ve^v\,dv &= x + C, \\
e^v(v-1) &= x + C \text{ (integration by parts).}
\end{aligned}$$

Finally, we replace v by $2x + y$ to obtain the solution

$$e^{2x+y}(2x + y - 1) = x + C.$$

Because we can't solve this equation for $y(x)$, we have to be satisfied with it in implicit form.