Faculty of Engineering and Information Sciences

UNIVERSITY
OF WOLLONGONG
IN DUBAI

MATH 142, Quiz 1, Spring 2021, Duration: 90 minutes

Name:____________________, ID Number:____________________

Time Allowed: 1 hour 30 minutes
Total Number of Questions: 6
Total Number of Pages (incl. this page): 7

EXAM UNAUTHORISED ITEMS

Students bringing these items to the examination room shall be required to leave the items at the front of the room or outside the examination room. The University does not guarantee the safe-keeping of students' personal items during examinations. Students concerned about the safety of their valuable items should make alternative arrangements for their care.

1. Bags, including carry bags, backpacks, shoulder bags and briefcases
2. Any form of electronic device including but not limited to mobile phones, smart watches, laptops, iPads, MP3 players, handheld computers and electronic dictionaries,
3. Calculator cases and covers
4. blank paper
5. Any written material

DIRECTIONS TO CANDIDATES

1. Total marks: 40
2. All questions are compulsory.
3. Answer all questions on the given exam paper sheets.
4. Write your name and Id number on the papers provided for rough work.

(6pts) Problem 1.
Find the area enclosed by the graphs of

$$f(x) = 2x^2 \quad \text{and} \quad g(x) = x^4 - 2x^2.$$

Solution

Method 1 without graphing:

$$\begin{aligned} A &= \int_a^b |f(x) - g(x)|\, dx \\ &= \int_a^b \left|4x^2 - x^4\right| dx. \qquad \textbf{(2pts)} \end{aligned}$$

To find a and b, we do $f(x) = g(x)$ and solve for x.

$$4x^2 - x^4 = 0 \Leftrightarrow x^2\left(4 - x^2\right) = 0 \Rightarrow x = 0, \text{ or } x = -2, \text{ or } x = 2.$$

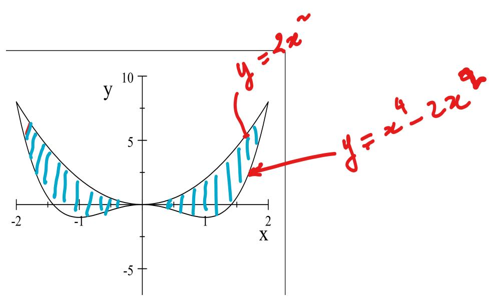
$$a = -2 \quad \text{and } b = 2.$$

$$A = \int_{-2}^{2} \left|x^2\left(4 - x^2\right)\right| dx$$

Since $4 - x^2$ has the sign of $a = -1$ outside of the root i.e. negative on $(-\infty,\ -2] \cup [2,\ \infty)$ and the opposite sign of a inside of the root i.e. positive on $[-2,\ 2]$.
Since the integration is between -2 and 2,

$$\begin{aligned} A &= \int_{-2}^{2} x^2\left(4 - x^2\right) dx \\ &= \frac{128}{15} = 8.5333 \qquad \textbf{(4pts)} \end{aligned}$$

Method 2 with graphing:
Graph both functions on the same window to determine which one is the top and which one is the bottom function.



(2pts)

You can see that between -2 and 2, $f(x) = 2x^2$ is on top of $g(x) = x^4 - 2x^2$.

$$\begin{aligned} A &= \int_{-2}^{2} [f(x) - g(x)]\, dx \\ &= \int_{-2}^{2} \left[2x^2 - \left(x^4 - 2x^2\right)\right] dx \\ &= \frac{128}{15} = 8.5333 \qquad \textbf{(4pts)} \end{aligned}$$

(6pts) Problem 2
Find the arclength of the function

$$f(x) = \frac{x^3}{12} + \frac{1}{x}$$

from $x = 1$ to $x = 4$.
Solution

$$L = \int_1^4 \sqrt{1 + [f\ '(x)]^2}dx.$$

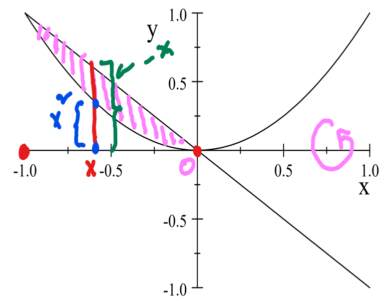
$$f\ '(x) = \frac{1}{4}x^2 - \frac{1}{x^2} \qquad \textbf{(2pts)}$$

$$\begin{aligned} [f\ '(x)]^2 &= \left(\frac{1}{4}x^2 - \frac{1}{x^2}\right)^2 = \frac{1}{x^4} + \frac{1}{16}x^4 - \frac{1}{2} \\ 1 + [f\ '(x)]^2 &= \frac{1}{x^4} + \frac{1}{16}x^4 + \frac{1}{2} = \left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)^2 \end{aligned}$$

$$\begin{aligned} L &= \int_1^4 \sqrt{\left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)^2} dx \qquad \textbf{(2pts)} \\ &= \int_1^4 \left(\frac{1}{4}x^2 + \frac{1}{x^2}\right) dx \\ &= 6 \qquad \textbf{(2pts)} \end{aligned}$$

(6pts) Problem 3
Sketch the region bounded by $y = -x, \ y = x^2$, and use the disc method to find the volume of the solid generated by revolving the region about the x-axis.

Solution



(**2pts**)

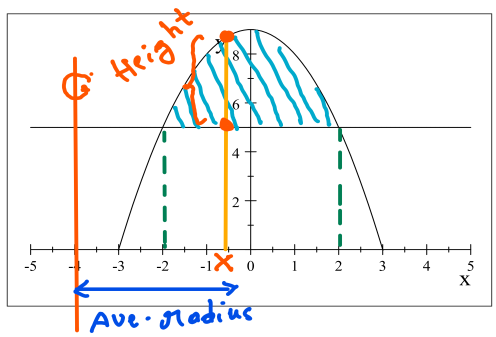
$$\text{Radius of big disc} = -x$$
$$\text{Radius of big disc} = x^2$$

$$\begin{aligned} V &= \int_{-1}^{0} \pi \left[(-x)^2 - x^4\right] dx \\ &= \int_{-1}^{0} \pi \left(x^2 - x^4\right) dx \\ &= \frac{2}{15}\pi = 0.418\,88 \qquad (\textbf{4pts}) \end{aligned}$$

(6pts) Problem 4

Sketch the region bounded by $y = 9 - x^2,\ y = 5$, and use the shell method to find the volume of the solid generated by revolving the region about the line $x = -4$.

Solution



(2pts)

$$\begin{aligned} \text{Average Radius} &= x - (-4) = x + 4 \\ \text{Height of the shell} &= \left(9 - x^2\right) - (5) \end{aligned}$$

$$\begin{aligned} V &= \int_{-2}^{2} 2\pi\,(x+4)\left(4 - x^2\right) dx \\ &= \int_{-2}^{2} 2\pi\left(-x^3 - 4x^2 + 4x + 16\right) dx \\ &= \frac{256}{3}\pi = 268.08. \qquad \textbf{(4pts)} \end{aligned}$$

(8pts) Problem 5
Use trigonometric substitution to evaluate the integral

$$\int \frac{\sqrt{x^2-1}dx}{x^2}.$$

Solution
We do the substitution

$$x = \sec\theta, \qquad dx = \sec\theta\tan\theta d\theta \qquad\qquad (\mathbf{2pts})$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}dx}{x^2} &= \int \frac{\sqrt{\sec^2\theta-1}\sec\theta\tan\theta d\theta}{\sec^2\theta} \\
&= \int \frac{\tan^2\theta}{\sec\theta}d\theta \qquad\qquad (\mathbf{2pts}) \\
&= \int \frac{\sin^2\theta}{\cos^2\theta}\cos\theta d\theta = \int \frac{\sin^2\theta}{1-\sin^2\theta}\cos\theta d\theta
\end{aligned}$$

Put

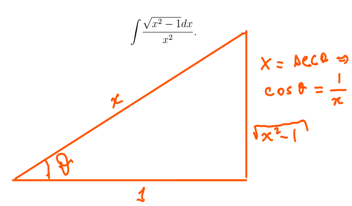
$$u = \sin\theta, \qquad du = \cos\theta d\theta.$$

$$\begin{aligned}
\int \frac{\sin^2\theta}{1-\sin^2\theta}\cos\theta d\theta &= \int \frac{u^2}{1-u^2}du \\
&= \int \frac{1-1+u^2}{1-u^2}du = \int \frac{1-(1-u^2)}{1-u^2}du \\
&= \int \left(\frac{1}{1-u^2}-1\right)du = \int \frac{du}{1-u^2} - \int 1du
\end{aligned}$$

Using partial fraction decomposition, we have

$$\int \frac{du}{1-u^2}du = \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1| + C$$

$$\begin{aligned}
\int \frac{du}{1-u^2} - \int 1du &= \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1| - u + C \\
&= \frac{1}{2}\ln|\sin\theta+1| - \frac{1}{2}\ln|\sin\theta-1| - \sin\theta + C
\end{aligned}$$



From the triangle, we get

$$\sin\theta = \frac{\sqrt{x^2-1}}{x}.$$

Thus

$$\int \frac{\sqrt{x^2-1}dx}{x^2} = \frac{1}{2}\ln\left|\frac{\sqrt{x^2-1}}{x}+1\right| - \frac{1}{2}\ln\left|\frac{\sqrt{x^2-1}}{x}-1\right| - \frac{\sqrt{x^2-1}}{x} + C \qquad (\mathbf{4pts})$$

(8pts) Problem 6
Use partial fraction decomposition to evaluate the integrals

$$\int \frac{x^3 - 4x - 1}{(x^2-1)(x-1)} dx$$

Solution

$$\int \frac{x^3 - 4x - 1}{(x^2-1)(x-1)} dx = \int \frac{x^3 - 4x - 1}{x^3 - x^2 - x + 1} dx$$

Since the numerator and denominator have the same degree, we first do the long division to obtain

$$\begin{aligned} \frac{x^3 - 4x - 1}{x^3 - x^2 - x + 1} &= 1 + \frac{-x^2 + 3x + 2}{-x^3 + x^2 + x - 1} \\ &= 1 - \frac{-x^2 + 3x + 2}{(x+1)(x-1)^2} \qquad \textbf{(2pts)} \end{aligned}$$

$$\frac{-x^2 + 3x + 2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2}$$

$$A = -\frac{1}{2}, \quad B_1 = -\frac{1}{2} \text{ and } B_2 = 2 \qquad \textbf{(3pts)}$$

$$\frac{-x^2 + 3x + 2}{(x+1)(x-1)^2} = \frac{2}{(x-1)^2} - \frac{1}{2(x+1)} - \frac{1}{2(x-1)}$$

$$\begin{aligned} \int \frac{-x^2 + 3x + 2}{(x+1)(x-1)^2} dx &= \int \left[\frac{2}{(x-1)^2} - \frac{1}{2(x+1)} - \frac{1}{2(x-1)} \right] dx \\ &= \frac{-2}{x-1} - \frac{1}{2} \ln|x+1| - \frac{1}{2} \ln|x-1| + C. \end{aligned}$$

Thus,

$$\begin{aligned} \int \frac{x^3 - 4x - 1}{(x^2-1)(x-1)} dx &= x - \left(\frac{-2}{x-1} - \frac{1}{2} \ln|x+1| - \frac{1}{2} \ln|x-1| \right) + C \\ &= x + \frac{2}{x-1} + \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C\ . \qquad \textbf{(3pts)} \end{aligned}$$