

Tutorial 3

Question 1

Find the equation of the tangent line to the curve $x = 3e^t, y = 5e^{-t}$ at $t = 0$.

Solution

$$\text{Slope } m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-5}{3}e^{-2t}. \text{ At } t = 0, \quad m = \frac{-5}{3}, \quad x = 3 \text{ and } y = 5.$$

$$\text{The equation is } y = \frac{-5}{3}(x - 3) + 5 \Leftrightarrow y = 10 - \frac{5}{3}x \Leftrightarrow 5x + 3y = 30$$

Question 2

Find the graph of the polar equation $r = 4\cos\theta - 5\sin\theta$.

Solution

$$\begin{aligned} r &= 4\cos\theta - 5\sin\theta \Leftrightarrow r^2 = 4r\cos\theta - 5r\sin\theta \\ \Leftrightarrow x^2 + y^2 &= 4x - 5y \Leftrightarrow x^2 - 4x + y^2 + 5y = 0 \\ \Leftrightarrow (x - 2)^2 - 4 + \left(y + \frac{5}{2}\right)^2 - \frac{25}{4} &= 0 \\ \Leftrightarrow (x - 2)^2 + \left(y + \frac{5}{2}\right)^2 &= 4 + \frac{25}{4} = \frac{41}{4} \end{aligned}$$

A circle centered at $\left(2, -\frac{5}{2}\right)$ with radius $\frac{\sqrt{41}}{2}$

Question 3

Find the area of the surface generated by revolving the parametric curve

$$x = \frac{1}{2}t^2 \quad \text{and} \quad y = \frac{1}{3}(2t + 1)^{3/2}, \quad 0 \leq t \leq 1.$$

about the y-axis.



Solution

$$\begin{aligned}\text{Area} &= 2\pi \int_0^1 \frac{1}{2}t^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ \left(\frac{dx}{dt}\right)^2 &= t^2, \quad \left(\frac{dy}{dt}\right)^2 = 2t + 1\end{aligned}$$

$$\begin{aligned}\text{Area} &= 2\pi \int_0^1 \frac{1}{2}t^2 \sqrt{t^2 + 2t + 1} dt \\ &= 2\pi \int_0^1 \frac{1}{2}t^2 \sqrt{(t+1)^2} dt \\ &= 2\pi \int_0^1 \frac{1}{2}t^2 (t+1) dt \\ &= \frac{7}{12}\pi = 1.8326\end{aligned}$$

Question 4

Find the slope of the line that is tangent to the polar curve

$$r = 3 \sin \theta$$

$$\text{at } \theta = \frac{\pi}{2}.$$

Solution

We have

$$x = 3 \sin \theta \cos \theta = \frac{3}{2} \sin 2\theta \quad \text{and} \quad y = 3 \sin^2 \theta$$

$$\begin{aligned}\text{Slope} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{6 \cos \theta \sin \theta}{3 \cos 2\theta}\end{aligned}$$

The slope at $\theta = \frac{\pi}{2}$ is

$$\text{Slope} = \frac{0}{-1} = 0.$$



Question 5

Find the area of the surface generated by revolving the parametric curve

$$x = e^t - t \quad \text{and} \quad y = 4e^{t/2}, \quad 0 \leq t \leq 1.$$

about the x-axis.

Solution

$$S = 2\pi \int_0^1 y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (e^t - 1)^2 + (2e^{t/2})^2 \\ &= e^{2t} - 2e^t + 4e^t + 1 \\ &= e^{2t} + 2e^t + 1 \\ &= (e^t + 1)^2 \end{aligned}$$

$$\begin{aligned} S &= 2\pi \int_0^1 4e^{t/2} \sqrt{(e^t + 1)^2} dt \\ &= 2\pi \int_0^1 4e^{t/2} (e^t + 1) dt \\ &= 8\pi \int_0^1 (e^{3t/2} + e^{t/2}) dt \\ &= 8\pi \left(2e^{\frac{3}{2}} + \frac{2}{3}e^{\frac{3}{2}} - \frac{8}{3} \right) = 90.945 \end{aligned}$$

Question 6

Find the slope and the equation of the tangent line to the graph of the polar curve

$$r = e^{2\theta}$$

at $\theta = 0$.



Solution

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

$$x = e^{2\theta} \cos \theta \quad \text{and} \quad y = e^{2\theta} \sin \theta$$

$$\text{Slope} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2e^{2\theta} \sin \theta + e^{2\theta} \cos \theta}{2e^{2\theta} \cos \theta - e^{2\theta} \sin \theta}$$

At $\theta = 0$, the slope is

$$m = \frac{0 + 1}{2 - 0} = \frac{1}{2}.$$

When $\theta = 0$, $x = 1$ and $y = 0$.

The equation of the tangent line is

$$y = \left(\frac{1}{2}\right)(x - 1) + 0$$

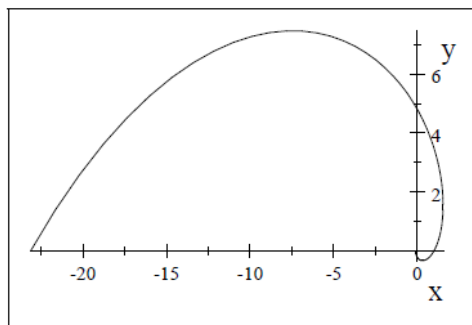
$$y = \frac{1}{2}x - \frac{1}{2}$$

Question 7

Find the arc length of the polar curve $r = e^\theta$ from $\theta = 0$ to $\theta = \ln 2$.



Solution



$$\begin{aligned}
 L &= \int_0^{\ln 2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
 &= \int_0^{\ln 2} \sqrt{e^{2\theta} + e^{2\theta}} d\theta \\
 &= \sqrt{2} \int_0^{\ln 2} e^{\theta} d\theta \\
 &= \sqrt{2} = 1.4142
 \end{aligned}$$

Question 8

Consider the curve given by

$$x^{2/3} + y^{2/3} = 4, \quad 1 \leq x \leq 8.$$

- Find the arclength of the curve.
- Find the area of the surface obtained by rotating the curve about the x-axis.

Hint: Use implicit differentiation to find $\frac{dy}{dx}$.



Solution

$$x^{2/3} + y^{2/3} = 4 \Leftrightarrow y^{2/3} = 4 - x^{2/3}$$

(a)

$$L = \int_1^8 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Next, we use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{y^{2/3}}{x^{2/3}} \\ &= 1 + \frac{4 - x^{2/3}}{x^{2/3}} \\ &= \frac{4}{x^{2/3}} = \left(\frac{2}{x^{1/3}}\right)^2 \end{aligned}$$

Thus,

$$\begin{aligned} L &= \int_1^8 \sqrt{\left(\frac{2}{x^{1/3}}\right)^2} dx \\ &= \int_1^8 \frac{2}{x^{1/3}} dx \\ &= 9. \end{aligned}$$

(b)

$$\begin{aligned} S &= 2\pi \int_1^8 (4 - x^{2/3})^{3/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= 2\pi \int_1^8 (4 - x^{2/3})^{3/2} \frac{2}{x^{1/3}} dx. \end{aligned}$$



Now put

$$u = 4 - x^{2/3}, \quad du = \frac{1}{3} \frac{-2}{x^{1/3}} dx \Rightarrow \frac{2}{x^{1/3}} dx = -3du.$$

when $x = 1$, $u = 3$ and when $x = 8$, $u = 0$.

Thus,

$$\begin{aligned} S &= 2\pi \int_3^0 u^{3/2} (-3du) \\ &= 6\pi \int_0^3 u^{3/2} du \\ &= \frac{108}{5} \sqrt{3}\pi = 37.412\pi = 117.53. \end{aligned}$$

Question 9

Find the area of the surface obtained by rotating the graph of

$$f(x) = 2\sqrt{x+1}, \quad 0 \leq x \leq 1$$

about the x-axis.

Solution

$$\begin{aligned} S &= 2\pi \int_0^1 2\sqrt{x+1} \sqrt{1 + [f'(x)]^2} dx \\ &= 2\pi \int_0^1 2\sqrt{x+1} \sqrt{1 + \left(\frac{1}{\sqrt{x+1}}\right)^2} dx \\ &= 2\pi \int_0^1 2 \frac{\sqrt{x+2}}{\sqrt{x+1}} \sqrt{x+1} dx \\ &= 4\pi \int_0^1 \sqrt{x+2} dx \\ &= 4\pi \left(2\sqrt{3} - \frac{4}{3}\sqrt{2} \right) = 19.836. \end{aligned}$$