Ratio Test

$$\sum_{n=1}^{\infty} a_n \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \quad : \quad \begin{cases} L<1 & \text{converges absolutely} \\ L>1 & \text{diverges} \\ L=1 & \text{inconclusive} \end{cases}$$

Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$

If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L = 1$ then the test cannot decide.

Example

Determine convergence or divergence of the following series.

1. $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

$$= \sum_{n=1}^{\infty} \left(\frac{e^2}{n}\right)^n$$

$$\lim_{n\to\infty} \left|\left(\frac{e^2}{n}\right)^n\right|^{1/n}$$

$$= \lim_{n\to\infty} \frac{e^2}{n}$$

$$= 0 < 1$$

$\therefore$ By root test

$\sum^{\infty} e^{2n}$ converges absolutely

$\sum_{n=1} \frac{\quad}{n^n}$ converges absolutely

2. $$\sum_{n=1}^{\infty} \left(\frac{3n^2 + 4n + 1}{2n^2 + 6} \right)^n$$

$$\lim_{n \to \infty} \left| \left(\frac{3n^2 + 4n + 1}{2n^2 + 6} \right)^n \right|^{\frac{1}{n}}$$

$$= \lim_{n \to \infty} \left| \frac{3\cancel{n^2}}{2\cancel{n^2}} \right|$$

$$= \lim_{n \to \infty} \frac{3}{2}$$

$$= \frac{3}{2} > 1$$

$\therefore$ The series diverges by the root test.

Strategies for testing series

Consider the series $\sum_{n=1}^{\infty} a_n$

To test the convergence of the series you may do the following

1. Evaluate $\lim_{n \to \infty} a_n$. If $\lim_{n \to \infty} a_n \neq 0$, then series diverges

2. If $\lim_{n \to \infty} a_n = 0$ proceed with the appropiate test.

3. Check if the series is one of the special types
   * Telescoping
   * Geometric
   * p- series
   * Alternating series.

4. Try the limit comparision test

5. Try the ratio / root test.

6. See if you can apply the integral test.

Example

Determine convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$

$$\lim_{n\to\infty} \frac{n+1}{3n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{3\cancel{n}}$$

$$= \frac{1}{3} \neq 0$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ diverges

2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$

$$\lim_{n\to\infty} |a_n|^{1/n}$$

$$= \lim_{n\to\infty} \left|\left(\frac{\pi}{6}\right)^n\right|^{1/n}$$

$$= \frac{\pi}{6}$$

$$\approx \frac{3.14}{6} < 1$$

$\therefore$ Converges absolutely by root test

$\displaystyle\sum^{\infty} \left(\pi\right)^{n-1+1}$

$$\sum_{n=1} \left(\frac{\pi}{6}\right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)\left(\frac{\pi}{6}\right)^{n-1}$$

$$|r| = \frac{\pi}{6} < 1$$

$\therefore$ Converges by geometric series test.

3. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

$$a_n = n^2 e^{-n^3} = f(n) = \frac{n^2}{e^{n^3}}$$

f is positive, decreasing and continuous for $n \geq 1$

$$\int_1^{\infty} x^2 e^{-x^3} = \lim_{t\to\infty} \int_1^t x^2 e^{-x^3}\,dx$$

$$= \lim_{t\to\infty} \frac{-1}{3} \int 3x^2 e^{-x^3}\,dx$$

$$= \lim_{t\to\infty} \frac{-1}{3} e^{-x^3}\Big|_1^t$$

$$= \frac{-1}{3} \lim_{t\to\infty} e^{-t^3} - e^{-1}$$

$$= \frac{-1}{3} \lim \frac{1}{e^{t^3}} - \frac{1}{e}$$

$$= \frac{-1}{3}\left(\frac{-1}{e}\right)$$

$$= \frac{1}{3e}$$

$\because$ limit converges

$\therefore$ series converges

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1} \sim \sum_{n=1}^{\infty} \frac{1}{3n}$

$\displaystyle = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$

p- series $p = 1$

$\because p \leq 1$

$\therefore$ Series diverges

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3(-1)^n}{4n+1}$

$\displaystyle b_n = \frac{3}{4n+1} \Rightarrow$ positive

decreasing

$\displaystyle\lim_{n\to\infty} b_n = 0$

$\therefore$ Converges by AST

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

$\displaystyle a_n = \frac{n!}{10^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{10^{n+1}} \times \frac{10^n}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}}{10\cdot\cancel{10^n}} \times \frac{\cancel{10^n}}{\cancel{n!}}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{10}$$

$$= \infty > 1$$

$\therefore$ Series diverges by ratio test

7. $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

$$\lim_{n\to\infty} |a_n|^{1/n}$$

$$= \lim_{n\to\infty} \left|\left(\frac{n+1}{2n+1}\right)^n\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n+1}$$

$$= \lim_{n\to\infty} \frac{n}{2n}$$

$$= \frac{1}{2} < 1$$

$\therefore$ Converges absolutely by root test