UNIVERSITY
OF WOLLONGONG
IN DUBAI

Exam 1 Review Problems

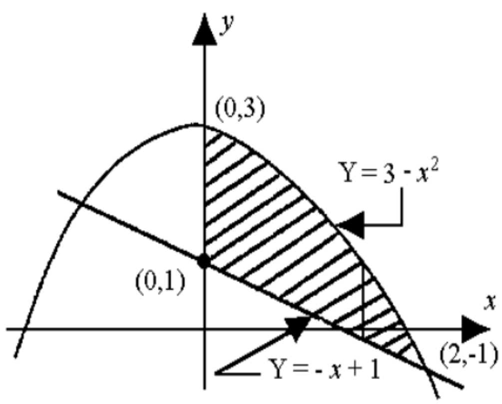
Problem 1.

Sketch the region bounded by $y = 3 - x^2$, $y = -x + 1$, $x = 0$ and $x = 2$, and find its area.

Sketch the region bounded by $x = 3y - y^2$ and $x + y = 3$, and find its area.

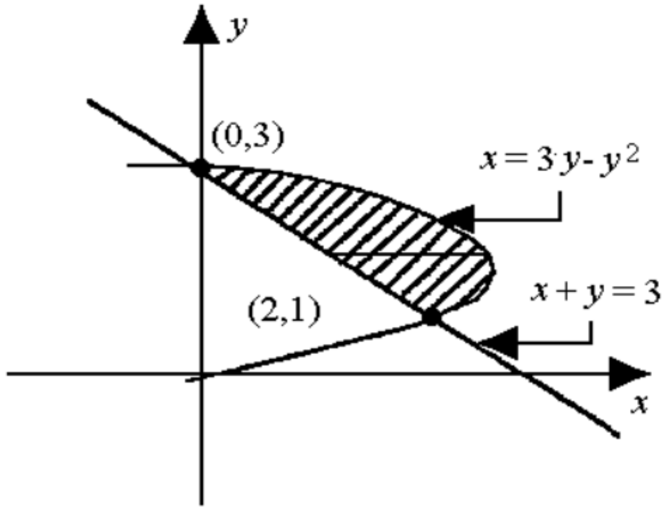
Solution

For the first one, we have



$$
\begin{aligned}
A &= \int_0^2 \left[\left(3 - x^2\right) - (-x + 1)\right] dx \\
&= \int_0^2 \left(-x^2 + x + 2\right) dx \\
&= \frac{10}{3}.
\end{aligned}
$$

For the second one



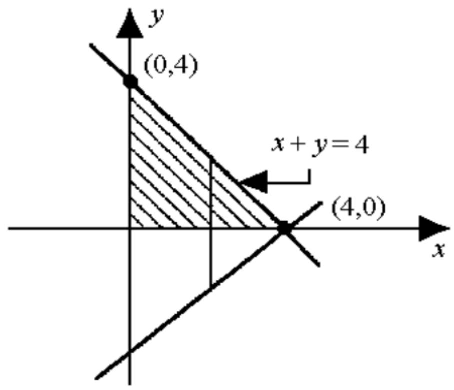
$$
\begin{aligned}
A &= \int_1^3 \left[\left(3y - y^2\right) - (3 - y)\right] dy \\
&= \int_1^3 \left(-y^2 + 4y - 3\right) dy \\
&= \frac{4}{3}.
\end{aligned}
$$

Problem 2.
Use the disc method to
A)

Sketch the region Ω bounded by $x + y = 4$, $y = 0$, and $x = 0$, and find the volume of the solid generated by revolving the region about the x-axis.

Solution

$$V = \int_a^b \text{Area of slice dx or dy}$$

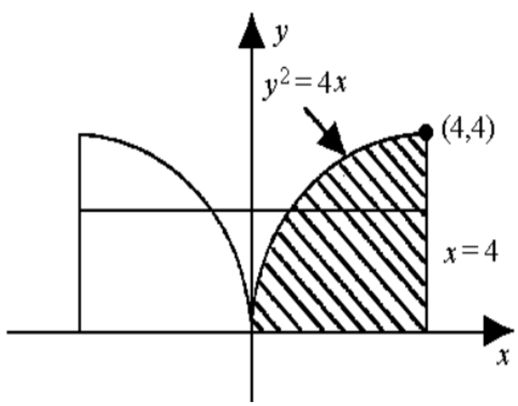


$$V = \int_0^4 \pi (4 - x)^2 \, dx = \frac{64}{3}\pi$$

B)

Sketch the region Ω bounded by $y^2 = 4x$, $x = 4$, and $y = 0$, and find the volume of the solid generated by revolving the region about the y-axis.

Solution



$$\begin{aligned} V &= \int_0^4 \pi \left[4^2 - \left(\frac{y^2}{4} \right)^2 \right] dy \\ &= \int_0^4 \pi \left(16 - \frac{1}{16} y^4 \right) dy = \frac{256}{5}\pi \end{aligned}$$

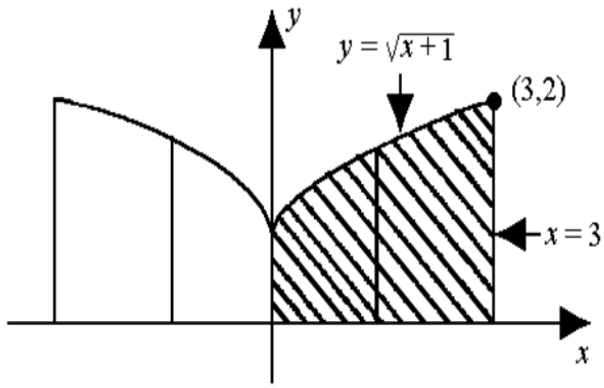
Problem 3.

Sketch the region Ω bounded by $y = \sqrt{x+1}$, $x = 0$, $y = 0$, and $x = 3$, and use the shell method to find the volume of the solid generated by revolving Ω about the y-axis.

Sketch the region Ω bounded by $y = x^2$, $y = 4$, and $x = 0$, and use the shell method to find the volume of the solid generated by revolving Ω about the x-axis.

Solution

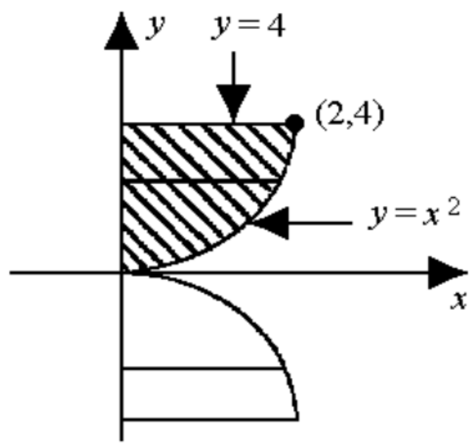
First one:



$$V = \int_a^b 2\pi \cdot average\ radius \cdot height \cdot dx$$

$$V = \int_0^3 2\pi x\sqrt{x+1}dx = \frac{232}{15}\pi$$

For the second one



$$V = \int_0^4 2\pi y\sqrt{y}dy = \frac{128}{5}\pi.$$

Problem 4.

Find the area of the surface generated by revolving $f(x) = 2\sqrt{x+1}$, $x \in [-1, 1]$ about the x-axis.

Solution

$$\begin{aligned} \mathrm{S} &= 2\pi \int_{-1}^{1} 2\sqrt{x+1}\sqrt{1+[f\,'(x)]^2}dx \\ &= 2\pi \int_{-1}^{1} 2\sqrt{x+1}\sqrt{1+\left(\frac{1}{\sqrt{x+1}}\right)^2}dx \\ &= 2\pi\left(4\sqrt{3}-\frac{4}{3}\right). \end{aligned}$$

Problem 5.

A)

Find the arc length of the curve $f(x) = x^{2/3}, x \in [0, 8]$ and compare it to the straight-line distance between the endpoints.

Solution

$$\begin{aligned} L &= \int_a^b \sqrt{1 + [f\,'(x)]^2}dx \\ &= \int_0^8 \sqrt{1 + \left(\frac{2}{3\sqrt[3]{x}}\right)^2}\,dx \\ &= \int_0^8 \sqrt{1 + \frac{4}{9}x^{\frac{-2}{3}}}\,dx. \quad \text{Do not evaluate.} \end{aligned}$$

B)

Find the arc length of the curve $f(x) = \frac{x^3}{6} + \frac{1}{2x}, x \in [1, 3]$ and compare it to the straight-line distance between the endpoints.

Solution

$$f\,'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$\begin{aligned} [f\,'(x)]^2 &= \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 \\ &= \frac{1}{4x^4} + \frac{1}{4}x^4 - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 1 + [f\,'(x)]^2 &= \frac{1}{4x^4} + \frac{1}{4}x^4 + \frac{1}{2} \\ &= \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2 \end{aligned}$$

$$\begin{aligned} L &= \int_a^b \sqrt{1 + [f\,'(x)]^2}dx \\ &= \int_1^3 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}dx = \\ &= \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx \\ &= \frac{14}{3}. \end{aligned}$$

Problem 6
Evaluate the following integrals
(a) $\int_0^2 (x^2+3x-1)\,dx$ (b) $\int_0^1 2\sqrt{1-x^2}dx$ (c) $\int_0^1 2x\sqrt{1-x^2}dx$

(d) $\int x^4 \ln x dx$ (e) $\int \sqrt{\sin x}\cos^3 x dx$

__Solution__
(a)

$$\begin{aligned}\int_0^2 \left(x^2+3x-1\right)dx &= \left.\frac{x^3}{3}+3\frac{x^2}{2}-x\right|_0^2 \\ &= \frac{20}{3}\end{aligned}$$

(b)

$$\begin{aligned}\int_0^1 2\sqrt{1-x^2}dx &= 2\int_0^1 \sqrt{1-x^2}dx \\ &= 2\left(\frac{\pi 1^2}{4}\right) \\ &= \frac{1}{2}\pi\end{aligned}$$

(c)

$$\begin{aligned}\int_0^1 2x\sqrt{1-x^2}dx &= -\int_0^1 -2x\left(1-x^2\right)^{1/2}dx \\ &= \left.-\frac{2}{3}\left(1-x^2\right)^{3/2}\right|_0^1 \\ &= \frac{2}{3}\end{aligned}$$

(d) By parts:

$$\begin{aligned}\int x^4 \ln x dx &= \frac{1}{5}x^5\ln x-\frac{1}{5}\int x^4 dx \\ &= \frac{1}{5}x^5\ln x-\frac{1}{25}x^5+C\end{aligned}$$

(e) The power of $\cos x$ is odd and positive. save one $\cos x$ next to the dx, convert the rest into $\sin x$ and do the substitution $u=\sin x$.

$$\int \sqrt{\sin x}\cos^3 x dx = \int \sqrt{\sin x}\left(1-\sin^2 x\right)\cos x dx.$$

Put

$$u=\sin x, \qquad du=\cos x dx$$

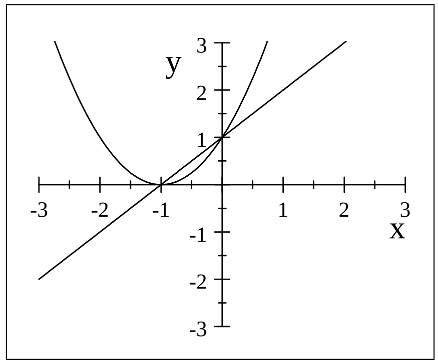
and the integral becomes

$$
\begin{aligned}
\int \sqrt{u}\left(1-u^2\right) du &= \int \left(u^{1/2}-u^{5/2}\right) du \\
&= \frac{2}{3}u^{3/2}-\frac{2}{7}u^{7/2}+C \\
&= \frac{2}{3}\sin^{3/2} x-\frac{2}{7}\sin^{7/2} x+C.
\end{aligned}
$$

Problem 7.
Find the area of the region bounded by the curves

$$f(x) = x^2 + 2x + 1 \text{ and } \quad g(x) = x + 1$$

Solution



$$A = \int_a^b \left[(x+1) - \left(x^2 + 2x + 1\right)\right] dx.$$

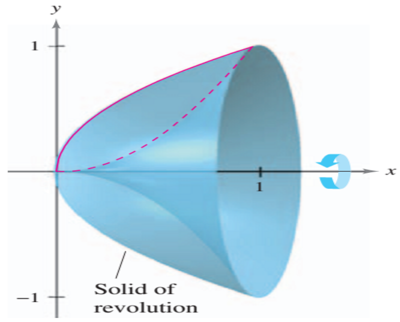
a and b are obtained fron the intersecting points.

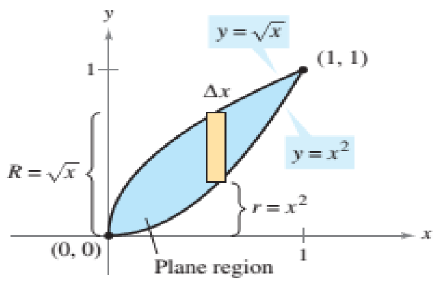
$$x^2 + 2x + 1 = x + 1 \Leftrightarrow x^2 + x = 0 \Rightarrow x = 0 \text{ or } x = -1$$

$$\begin{aligned} A &= \int_{-1}^{0} \left(-x^2 - x\right) dx \\ &= \frac{1}{6}. \end{aligned}$$

Problem 8

Use the disc (washer) method to find the volume of the solid formed by revolving the region bounded by the graphs of
$y = \sqrt{x}$ and $y = x^2$ about the x-axis.





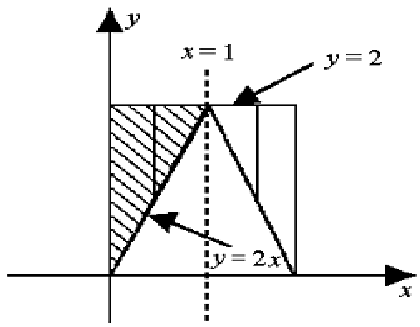
$$V = \int_a^b \pi \left[\left(\sqrt{x}\right)^2 - \left(x^2\right)^2 \right] dx$$

$\sqrt{x} = x^2 \Rightarrow a = 0$ and $b = 1$.

$$\begin{aligned} V &= \int_0^1 \pi \left(x - x^4\right) dx \\ &= \frac{3}{10}\pi \ . \end{aligned}$$

Problem 9
Sketch the region Ω bounded by $y = 2x$, $x = 0$, and $y = 2$, use the shell method to find the volume of the solid generated by revolving Ω about $x = 1$.

Solution



$$
\begin{aligned}
V &= \int_a^b 2\pi\,(\text{average radius})\,(\text{height})\,dx \\
&= \int_0^1 2\pi\,(1-x)\,(2-2x)\,dx \\
&= \frac{4}{3}\pi.
\end{aligned}
$$

Problem 10
Use trigonometric substitution to evaluate

$$\int \frac{dx}{x^2\sqrt{x^2+25}}$$

Solution
Put

$$x = 5\tan\theta, \qquad dx = 5\sec^2\theta d\theta$$

The integral becomes

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2+25}} &= \int \frac{5\sec^2\theta d\theta}{25\tan^2\theta\sqrt{25\tan^2\theta+25}} \\
&= \int \frac{5\sec^2\theta d\theta}{25\tan^2\theta\,(5\sec\theta)} \\
&= \int \frac{\cos\theta d\theta}{25\sin^2\theta}.
\end{aligned}$$

Put

$$u = \sin\theta, \qquad du = \cos\theta d\theta$$

$$\begin{aligned}
\int \frac{\cos\theta d\theta}{25\sin^2\theta} &= \frac{1}{25}\int \frac{du}{u^2} \\
&= \frac{-1}{25u} + C \\
&= \frac{-1}{25\sin\theta} + C
\end{aligned}$$

From the triangle

$$\sin\theta = \frac{x}{\sqrt{x^2+25}}.$$

Hence,

$$\int \frac{dx}{x^2\sqrt{x^2+25}} = \frac{-\sqrt{x^2+25}}{25x} + C$$

Problem 11

$$\text{Evaluate the integral } I = \int \frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-8)(x-9)}\,dx$$

Solution

Using the method of partial fractions the integrand can be expressed in the simpler form

$$\frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-8)(x-9)} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} + \frac{B_1}{x-8} + \frac{C_1}{x-9}$$

where A_1, A_2, A_3, B_1, C_1 are constants to be determined. The constants B_1 and C_1 are found as in the previous example. One can verify that

$$C_1 = \frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-8)}\Big|_{x=9} = 2$$

and

$$B_1 = \frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-9)}\Big|_{x=8} = 3$$

One can then write

$$\frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-8)(x-9)} - \frac{2}{x-9} - \frac{3}{x-8} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3}$$

which simplifies to

$$\frac{3x^2 - 8x + 3}{(x-2)^3} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} \tag{3.44}$$

Multiply both sides of equation (3.44) by $(x-2)^3$ to obtain

$$3x^2 - 8x + 3 = A_1(x-2)^2 + A_2(x-2) + A_3 \tag{3.45}$$

Differentiate equation (3.45) and show

$$6x - 8 = 2A_1(x-2) + A_2 \tag{3.46}$$

Differentiate equation (3.46) and show

$$6 = 2A_1 \tag{3.47}$$

giving $A_1 = 3$. Evaluate equations (3.45) and (3.46) at $x = 2$ to show $A_3 = -1$ and $A_2 = 4$. The given integral can now be represented in the form

$$I = 3\int \frac{dx}{x-2} + 4\int \frac{dx}{(x-2)^2} - \int \frac{dx}{(x-2)^3} + 3\int \frac{dx}{x-8} + 2\int \frac{dx}{x-9}$$

where each term can be integrated to obtain

$$I = 3\ln|x-2| - \frac{4}{x-2} + \frac{1}{2}\frac{1}{(x-2)^2} + 3\ln|x-8| + 2\ln|x-9| + C$$

or

$$I = \ln\left|(x-2)^3(x-8)^3(x-9)^2\right| + \frac{1}{2}\frac{1}{(x-2)^2} - \frac{4}{x-2} + C$$

Problem 12

Compute each of the following integrals.

1. $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$

2. $\displaystyle\int \sqrt{1-x^2}\,dx$

3. $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$

4. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

5. $\displaystyle\int \sqrt{x^2+4}\,dx$

6. $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$

Answer

1.) $\ln\left|x+\sqrt{x^2+4}\right|+C$ 2.) $\dfrac{1}{2}\sin^{-1}x+\dfrac{1}{2}x\sqrt{1-x^2}+C$ 3.) $\ln\left|x+\sqrt{x^2-9}\right|+C$

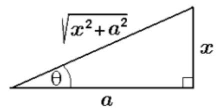
4.) $8\sin^{-1}\left(\dfrac{x}{4}\right)-\dfrac{1}{2}x\sqrt{16-x^2}+C$ 5.) $\dfrac{1}{2}x\sqrt{x^2+4}+2\ln\left|x+\sqrt{x^2+4}\right|+C$

6.) $\dfrac{1}{2}x\sqrt{x^2+9}-\dfrac{9}{2}\ln\left|x+\sqrt{x^2+9}\right|+C$

Solution

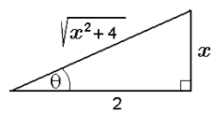
Trigonometric substitution is a technique of integration. It is especially useful in handling expressions under a square root sign.

Case 1. The substitution $x = a\tan\theta$. This is useful in handling an integral involving $\sqrt{x^2+a^2}$.
Let $x = a\tan\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. (That is the same thing as stating that $\theta = \tan^{-1}\frac{x}{a}$. The interval between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is the domain of the inverse function $\tan^{-1} x$.)
The picture below shows the reference triangle we use for this substitution.



Using this triangle, we do not have to do heavy duty algebra because we can read (up to sign) the trigonometric functions of θ in terms of x and a.

Example 1: Compute the integral $\int \frac{1}{\sqrt{x^2+4}}\,dx$.
Solution: We will use a trigonometric substitution. We start with a reference triangle where the hypotenuse is the denominator. Using the substitution $x = 2\tan\theta$, (where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$) we will transform the integral into one in θ.



From the triangle, $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$. The expression $\sqrt{x^2+4}$ becomes $2\sec\theta$ - using the picture, or using algebra. Recall the identity $\tan^2 x + 1 = \sec^2 x$

$$\sqrt{x^2+4} = \sqrt{(2\tan\theta)^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4}\sqrt{\tan^2\theta+1} = 2\sqrt{\sec^2\theta} = 2\,|\sec\theta|$$

Because θ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\sec x$ is positive and so $|\sec x| = \sec x$.

$$\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{1}{2\sec\theta}\,\left(2\sec^2\theta\,d\theta\right) = \int \sec\theta\,d\theta$$

This is an integral we have already seen: we can either use substitution (see in that handout) or partial fraction(see in that handout). Either way,

$$\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy.

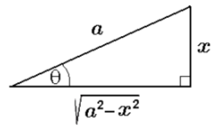
$$\sec\theta = \frac{\sqrt{x^2+4}}{2} \qquad \text{and} \qquad \tan\theta = \frac{x}{2}$$

Thus the answer is $\int \frac{1}{\sqrt{x^2+4}}\,dx = \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$. This expression can be further simplified:

$$\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + C = \ln\left|\sqrt{x^2+4}+x\right| - \ln 2 + C = \ln\left|\sqrt{x^2+4}+x\right| + C$$

and so the final answer is $\boxed{\ln\left|\sqrt{x^2+4}+x\right|+C}$.

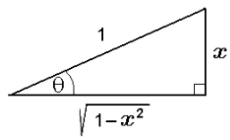
Case 2. The substitution $x = a\sin\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. This is useful in handling an integral involving $\sqrt{a^2-x^2}$. The picture below shows the reference triangle we use for this substitution.



Using this triangle, we can read (up to sign) the trigonometric functions of θ in terms of x and a.

Example 2: Compute the integral $\int \sqrt{1-x^2}\, dx$.

Solution: This is a very famous integral because it leads to the area formula of the unit circle. We will use a trigonometric substitution. We start with a reference triangle where the $\sqrt{1-x^2}$ is one of the legs. Using the substitution $x = \sin\theta$, $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$ we will transform the integral into one in θ.



From the triangle, $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$. The expression $\sqrt{1-x^2}$ becomes

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$$

Because θ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos x$ is positive and so $|\cos x| = \cos x$.

$$\int \sqrt{1-x^2}\, dx = \int \cos\theta\ (\cos\theta\, d\theta) = \int \cos^2\theta\, d\theta$$

This is an integral we have already seen; we can simplify it using the double angle formula for cosine.

$$\cos 2\theta = 2\cos^2\theta - 1 \quad \Longrightarrow \quad \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$\begin{aligned}\int \cos^2\theta\, d\theta &= \int \frac{1}{2}(\cos 2\theta + 1)\ d\theta = \frac{1}{2}\int \cos 2\theta + 1\ d\theta = \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + C \\ &= \frac{1}{2}\left(\frac{1}{2}(2\sin\theta\cos\theta) + \theta\right) + C = \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C\end{aligned}$$

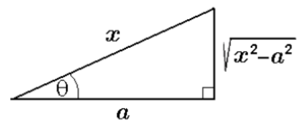
Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy.

$$\sin\theta = x, \quad \cos\theta = \sqrt{1-x^2} \quad \text{and} \quad \theta = \sin^{-1} x$$

Thus the answer is $\int \sqrt{1-x^2}\, dx = \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C = \boxed{\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x + C}$

Note that if we now compute $\int_{-1}^{1} \sqrt{1-x^2}\, dx$ the result is the area of the unit semi-circle, $\frac{\pi}{2}$.

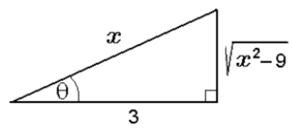
Case 3. The substitution $x = a\sec\theta$ where $0 < \theta < \frac{\pi}{2}$. This is useful in handling an integral involving $\sqrt{x^2 - a^2}$. The picture below shows the reference triangle we use for this substitution.



Using this triangle, we can read (up to sign) the trigonometric functions of θ in terms of x and a.

Example 3: Compute the integral $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$.

Solution: We will use a trigonometric substitution. We start with a reference triangle where the hypotenuse is x and one shorter side is 3. Using the substitution $x = 3\sec\theta$, we will transform the integral into one in θ.



From the triangle, $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\ d\theta$. The expression $\sqrt{x^2-9}$ becomes $3\tan\theta$ - either from the picture or using algebra. Recall the identity $\sec^2 x = \tan^2 x + 1$

$$\sqrt{x^2-9} = \sqrt{(3\sec\theta)^2 - 9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9}\sqrt{\sec^2\theta - 1} = 3\sqrt{\tan^2\theta} = 3\,|\tan\theta|$$

Because $0 < \theta < \frac{\pi}{2}$, $\tan\theta$ is positive and so $|\tan\theta| = \tan\theta$.

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{1}{3\tan\theta}\,(3\sec\theta\tan\theta\ d\theta) = \int \sec\theta\ d\theta$$

Again,

$$\int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + C$$

Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy.

$$\sec\theta = \frac{x}{3} \quad \text{and} \quad \tan\theta = \frac{\sqrt{x^2-9}}{3}$$

Thus the answer is $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$. We can still simplify this result a bit:

$$\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C = \ln\left|x + \sqrt{x^2-9}\right| - \ln 3 + C = \ln\left|x + \sqrt{x^2-9}\right| + C_2$$

Thus the final answer is $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx = \boxed{\ln\left|x+\sqrt{x^2-9}\right| + C}$.

Example 4: Compute the integral $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\, dx$

Solution: Let $x = 4\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 4\cos\theta\ d\theta$ and

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16}\sqrt{1-\sin^2\theta} = 4\sqrt{\cos^2\theta} = 4\,|\cos\theta|$$

Because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos\theta$ is non-negative, and $|\cos\theta| = \cos\theta$. So the integral is

$$\int \frac{x^2}{\sqrt{16-x^2}}\, dx = \int \frac{16\sin^2\theta}{4\cos\theta}\ (4\cos\theta\ d\theta) = \int 16\sin^2\theta\ d\theta = 16\int \sin^2\theta\ d\theta$$

By the double angle formula for cosine, $\cos 2\theta = 1 - 2\sin^2\theta \quad\Longrightarrow\quad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\begin{aligned}\int \frac{x^2}{\sqrt{16-x^2}}\, dx &= 16\int \sin^2\theta\ d\theta = 16\int \frac{1}{2}(1-\cos 2\theta)\ d\theta = 8\int 1-\cos 2\theta\ d\theta = 8\left(\theta - \frac{1}{2}\sin 2\theta + C\right)\\ &= 8\theta - 4\sin 2\theta + C\end{aligned}$$

Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy. Recall that $x = 4\sin\theta$ and so

$$\begin{aligned}\theta &= \sin^{-1}\left(\frac{x}{4}\right) \quad \text{and}\\ \sin 2\theta &= 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta} = 2\left(\frac{x}{4}\right)\sqrt{1-\left(\frac{x}{4}\right)^2} = \frac{x}{2}\sqrt{\frac{1}{16}(16-x^2)}\\ &= \frac{x}{2}\left(\frac{1}{4}\right)\sqrt{16-x^2} = \frac{1}{8}x\sqrt{16-x^2}\end{aligned}$$

And so the final answer is $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\, dx = 8\theta - 4\sin 2\theta + C = \boxed{8\sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16-x^2} + C}$

Example 5: Compute the integral $\displaystyle\int \sqrt{x^2+4}\ dx$

Solution: Let $x = 2\tan\beta$ where $-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$. Then $dx = 2\sec^2\beta\ d\beta$ and

$$\begin{aligned}\int \sqrt{x^2+4}\ dx &= \int \sqrt{4\tan^2\beta+4}\ \left(2\sec^2\beta\ d\beta\right) = \int 2\sqrt{\tan^2\beta+1}\ \left(2\sec^2\beta\ d\beta\right) = 4\int |\sec\beta|\ \left(\sec^2\beta\ d\beta\right) =\\ &= 4\int \sec\beta\ \left(\sec^2\beta\ d\beta\right) = 4\int \sec^3\beta\ d\beta\end{aligned}$$

We will compute $\displaystyle\int \sec^3\beta\ d\beta$ by parts.
Let $u = \sec\beta$ and $dv = \sec^2\beta\ d\beta$. Then

$$du = \sec\beta\tan\beta d\beta \quad\text{and}\quad v = \int dv = \int \sec^2\beta\ d\beta$$

In short,

$u = \sec\beta$	$v = \tan\beta$
$du = \sec\beta\tan\beta\ d\beta$	$dv = \sec^2\beta\ d\beta$

$$\begin{aligned}
\int u\ dv &= uv - \int v\ du \text{ becomes} \\
\int \sec\beta\ \sec^2\beta\ d\beta &= \sec\beta\tan\beta - \int \tan\beta\ \sec\beta\tan\beta\ d\beta \\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \tan^2\beta\ \sec\beta\ d\beta \qquad\qquad \text{recall } \tan^2\beta + 1 = \sec^2\beta \\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \left(\sec^2\beta - 1\right)\sec\beta\ d\beta \\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \sec^3\beta - \sec\beta\ d\beta \\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \sec^3\beta\ d\beta + \int \sec\beta\ d\beta \\
2\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta + \int \sec\beta\ d\beta \\
2\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta + \ln|\sec\beta + \tan\beta| + C \\
\int \sec^3\beta\ d\beta &= \frac{1}{2}\sec\beta\tan\beta + \frac{1}{2}\ln|\sec\beta + \tan\beta| + C
\end{aligned}$$

Now the original integral is

$$\begin{aligned}
\int \sqrt{x^2+4}\ dx &= 4\int \sec^3\beta\ d\beta = 4\left(\frac{1}{2}\sec\beta\tan\beta + \frac{1}{2}\ln|\sec\beta + \tan\beta|\right) + C \\
&= 2\sec\beta\tan\beta + 2\ln|\sec\beta + \tan\beta| + C
\end{aligned}$$

Now we need to reverse the substitution and write the result as an expression of x. Recall that $x = 2\tan\beta$. Then $\tan\beta = \dfrac{x}{2}$ and

$$\sec\beta = \sqrt{\tan^2\beta + 1} = \sqrt{\left(\frac{x}{2}\right)^2 + 1} = \sqrt{\frac{1}{4}x^2 + 1} = \sqrt{\frac{1}{4}\left(x^2+4\right)} = \frac{1}{2}\sqrt{x^2+4}$$

and so

$$\begin{aligned}
\int \sqrt{x^2+4}\ dx &= 2\sec\beta\tan\beta + 2\ln|\sec\beta + \tan\beta| + C = 2\left(\frac{1}{2}\sqrt{x^2+4}\right)\left(\frac{x}{2}\right) + 2\ln\left|\frac{1}{2}\sqrt{x^2+4} + \frac{x}{2}\right| + C \\
&= \frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| + C = \frac{1}{2}x\sqrt{x^2+4} + 2\left(\ln\left|x+\sqrt{x^2+4}\right| - \ln 2\right) + C \\
&= \frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|x+\sqrt{x^2+4}\right| - 2\ln 2 + C = \boxed{\frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|x+\sqrt{x^2+4}\right| + C}
\end{aligned}$$

Example 6: Compute the integral $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$

Let $x = 3\tan\theta$ where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Then $dx = 3\sec^2\theta d\theta$ and

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2+9}}\,dx &= \int \frac{9\tan^2\theta}{\sqrt{9\tan^2\theta+9}}\left(3\sec^2\theta\,d\theta\right) = \int \frac{9\tan^2\theta}{3\sqrt{\tan^2\theta+1}}\left(3\sec^2\theta\,d\theta\right) = \int \frac{9\tan^2\theta}{3\left|\sec\theta\right|}\left(3\sec^2\theta\,d\theta\right) \\
&= \int \frac{9\tan^2\theta}{3\sec\theta}\left(3\sec^2\theta\,d\theta\right) = 9\int \tan^2\theta\sec\theta\,d\theta = 9\int\left(\sec^2\theta-1\right)\sec\theta\,d\theta \\
&= 9\int \sec^3\theta - \sec\theta\,d\theta = 9\int\sec^3\theta\,d\theta - 9\int\sec\theta\,d\theta
\end{aligned}$$

We know that $\displaystyle\int \sec\theta\,d\theta = \ln\left|\sec\theta+\tan\theta\right| + C$ and from the previous computation we have that $\displaystyle\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C$. So that the integral is

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2+9}}\,dx &= 9\int\sec^3\theta\,d\theta - 9\int\sec\theta\,d\theta = 9\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|\right) - 9\ln\left|\sec\theta+\tan\theta\right| + C \\
&= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln\left|\sec\theta+\tan\theta\right| - 9\ln\left|\sec\theta+\tan\theta\right| + C = \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln\left|\sec\theta+\tan\theta\right| + C
\end{aligned}$$

Now we need to reverse the substitution and write the result as an expression of x. Recall that $x = 3\tan\theta$. Then $\tan\theta = \dfrac{x}{3}$ and

$$\sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{\left(\frac{x}{3}\right)^2+1} = \sqrt{\frac{1}{9}x^2+1} = \sqrt{\frac{1}{9}\left(x^2+9\right)} = \frac{1}{3}\sqrt{x^2+9}$$

and so

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2+9}}\,dx &= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln\left|\sec\theta+\tan\theta\right| + C = \frac{9}{2}\left(\frac{1}{3}\sqrt{x^2+9}\right)\left(\frac{x}{3}\right) - \frac{9}{2}\ln\left|\left(\frac{1}{3}\sqrt{x^2+9}\right)+\frac{x}{3}\right| + C \\
&= \frac{1}{2}x\sqrt{x^2+9} - \frac{9}{2}\ln\left|\frac{x+\sqrt{x^2+9}}{3}\right| + C = \frac{1}{2}x\sqrt{x^2+9} - \frac{9}{2}\left(\ln\left|x+\sqrt{x^2+9}\right| - \ln 3\right) + C \\
&= \frac{1}{2}x\sqrt{x^2+9} - \frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right| + \frac{9}{2}\ln 3 + C = \boxed{\frac{1}{2}x\sqrt{x^2+9} - \frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right| + C}
\end{aligned}$$

Problem 13

Compute each of the following integrals.

1. $\displaystyle\int \frac{1}{x^2-4}\,dx$

2. $\displaystyle\int \frac{2x}{(x+3)(3x+1)}\,dx$

3. $\displaystyle\int \frac{x+5}{x^2-2x-3}\,dx$

4. $\displaystyle\int \frac{x^4+x^3-5x^2+26x-21}{x^2+3x-4}\,dx$

5. $\displaystyle\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)}\,dx$

6. $\displaystyle\int \frac{2x-1}{(x-5)^2}\,dx$

7. $\displaystyle\int \frac{x+3}{(x-1)^3}\,dx$

Answer

1.) $\frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+C$ 2.) $\frac{3}{4}\ln|x+3|-\frac{1}{12}\ln|3x+1|+C$ 3.) $2\ln|x-3|-\ln|x+1|+C$

4.) $\frac{x^3}{3}-x^2+5x+\frac{13}{5}\ln|x+4|+\frac{2}{5}\ln|x-1|+C$ 5.) $-\frac{1}{6}\ln|x+1|-\frac{1}{3}\ln|x-2|+\frac{2}{3}\ln|x-5|+C$

6.) $2\ln|x-5|-\frac{9}{x-5}+C$ 7.) $-\frac{1}{x-1}-\frac{2}{(x-1)^2}+C=\frac{-x-1}{(x-1)^2}+C$

Solution

1. $\displaystyle\int \frac{1}{x^2-4}\,dx$

Solution: We factor the denominator: $x^2-4=(x+2)(x-2)$. Next, we re-write the fraction $\frac{1}{x^2-4}$ as a sum (or difference) of fractions with denominators $x+2$ and $x-2$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+2}+\frac{B}{x-2}=\frac{1}{x^2-4}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A(x-2)}{(x+2)(x-2)}+\frac{B(x+2)}{(x-2)(x+2)}=\frac{Ax-2A+Bx+2B}{x^2-4}=\frac{(A+B)x-2A+2B}{x^2-4}$$

Thus we have

$$\frac{(A+B)x-2A+2B}{x^2-4}=\frac{1}{x^2-4}$$

We clear the denominators by multiplication

$$(A+B)x-2A+2B=1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. In other words,

$$(A+B)x-2A+2B=0x+1$$

This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} A+B &= 0\\ -2A+2B &= 1 \end{aligned}$$

We solve this system and obtain $A=-\frac{1}{4}$ and $B=\frac{1}{4}$.

So our fraction, $\frac{1}{x^2-4}$ can be re-written as $\dfrac{-\frac{1}{4}}{x+2}+\dfrac{\frac{1}{4}}{x-2}$. We check:

$$\begin{aligned}\frac{-\frac{1}{4}}{x+2}+\frac{\frac{1}{4}}{x-2} &= \frac{-\frac{1}{4}(x-2)}{(x+2)(x-2)}+\frac{\frac{1}{4}(x+2)}{(x-2)(x+2)}=\frac{-\frac{1}{4}(x-2)+\frac{1}{4}(x+2)}{(x+2)(x-2)}=\frac{-\frac{1}{4}x+\frac{1}{2}+\frac{1}{4}x+\frac{1}{2}}{(x+2)(x-2)}\\ &= \frac{1}{x^2-4}\end{aligned}$$

Now we can easily integrate:

$$\int \frac{1}{x^2-4}\,dx=\int \frac{-\frac{1}{4}}{x+2}+\frac{\frac{1}{4}}{x-2}\,dx=-\frac{1}{4}\int\frac{1}{x+2}\,dx+\frac{1}{4}\int\frac{1}{x-2}\,dx=\boxed{-\frac{1}{4}\ln|x+2|+\frac{1}{4}\ln|x-2|+C}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+2} + \frac{B}{x-2} = \frac{1}{x^2-4}$$

We bring the fractions to the common denominator:

$$\frac{A(x-2)}{(x+2)(x-2)} + \frac{B(x+2)}{(x-2)(x+2)} = \frac{1}{x^2-4}$$

and then multiply both sides by the denominator:

$$A(x-2) + B(x+2) = 1$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=2$ into both sides:

$$\begin{aligned} A(0) + B(4) &= 1 \\ B &= \frac{1}{4} \end{aligned}$$

Let us substitute $x=-2$ into both sides:

$$\begin{aligned} A(-4) + B(0) &= 1 \\ A &= -\frac{1}{4} \end{aligned}$$

and so $A = -\dfrac{1}{4}$ and $B = \dfrac{1}{4}$.

2. $\displaystyle\int \frac{2x}{(x+3)(3x+1)}\,dx$

Solution: We re-write the fraction $\dfrac{2x}{(x+3)(3x+1)}$ as a sum (or difference) of fractions with denominators $x+3$ and $3x+1$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+3} + \frac{B}{3x+1} = \frac{2x}{(x+3)(3x+1)}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{A}{x+3} + \frac{B}{3x+1} &= \frac{A(3x+1)}{(x+3)(3x+1)} + \frac{B(x+3)}{(x+3)(3x+1)} = \frac{A(3x+1)+B(x+3)}{(x+3)(3x+1)} = \frac{3Ax+A+Bx+3B}{(x+3)(3x+1)} \\ &= \frac{(3A+B)x+A+3B}{(x+3)(3x+1)} \end{aligned}$$

Thus we have

$$\frac{(3A+B)x+A+3B}{(x+3)(3x+1)} = \frac{2x}{(x+3)(3x+1)}$$

We clear the denominators by multiplication

$$(3A+B)x + A + 3B = 2x$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. In other words,

$$(3A+B)x + A + 3B = 2x + 0$$

This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} 3A+B &= 2 \\ A+3B &= 0 \end{aligned}$$

We solve this system and obtain $A=\frac{3}{4}$ and $B=-\frac{1}{4}$.

So our fraction, $\frac{2x}{(x+3)(3x+1)}$ can be re-written as $\frac{\frac{3}{4}}{x+3}+\frac{-\frac{1}{4}}{3x+1}$. We check:

$$\begin{aligned} \frac{\frac{3}{4}}{x+3}+\frac{-\frac{1}{4}}{3x+1} &= \frac{\frac{3}{4}(3x+1)}{(x+3)(3x+1)}+\frac{-\frac{1}{4}(x+3)}{(x+3)(3x+1)}=\frac{\frac{3}{4}(3x+1)-\frac{1}{4}(x+3)}{(x+3)(3x+1)}=\frac{\frac{9}{4}x+\frac{3}{4}-\frac{1}{4}x-\frac{3}{4}}{(x+3)(3x+1)} \\ &= \frac{2x}{(x+3)(3x+1)} \end{aligned}$$

Now we can easily integrate:

$$\int \frac{2x}{(x+3)(3x+1)}\,dx=\int \frac{\frac{3}{4}}{x+3}+\frac{-\frac{1}{4}}{3x+1}\,dx=\frac{3}{4}\int \frac{1}{x+3}\,dx-\frac{1}{4}\int \frac{1}{3x+1}\,dx=\boxed{\frac{3}{4}\ln|x+3|-\frac{1}{12}\ln|3x+1|+C}$$

The second integral can be computed using the substitution $u=3x+1$.

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+3}+\frac{B}{3x+1}=\frac{2x}{(x+3)(3x+1)}$$

We bring the fractions to the common denominator:

$$\frac{A(3x+1)}{(x+3)(3x+1)}+\frac{B(x+3)}{(x+3)(3x+1)}=\frac{2x}{(x+3)(3x+1)}$$

and then multiply both sides by the denominator:

$$A(3x+1)+B(x+3)=2x$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=-\frac{1}{3}$ into both sides:

$$\begin{aligned} A(0)+B\left(-\frac{1}{3}+3\right) &= 2\left(-\frac{1}{3}\right) \\ \frac{8}{3}B &= -\frac{2}{3} \\ B &= -\frac{1}{4} \end{aligned}$$

Let us substitute $x=-3$ into both sides:

$$\begin{aligned} A(3(-3)+1)+B(-3+3) &= 2(-3) \\ -8A+B(0) &= -6 \\ -8A &= -6 \\ A &= \frac{3}{4} \end{aligned}$$

and so $A=\frac{3}{4}$ and $B=-\frac{1}{4}$.

3. $\displaystyle\int \frac{x+5}{x^2-2x-3}\,dx$

Solution: We factor the denominator: $x^2-2x-3=(x+1)(x-3)$. Next, we re-write the fraction $\dfrac{x+5}{x^2-2x-3}$ as a sum (or difference) of fractions with denominators $x+1$ and $x-3$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+1}+\frac{B}{x-3}=\frac{x+5}{x^2-2x-3}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A(x-3)}{(x+1)(x-3)}+\frac{B(x+1)}{(x-3)(x+1)}=\frac{Ax-3A+Bx+B}{x^2-2x-3}=\frac{(A+B)x-3A+B}{x^2-2x-3}$$

Thus

$$\frac{(A+B)x-3A+B}{x^2-2x-3}=\frac{x+5}{x^2-2x-3}$$

We clear the denominators by multiplication

$$(A+B)x-3A+B=x+5$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} A+B &= 1\\ -3A+B &= 5 \end{aligned}$$

We solve the system and obtain $A=-1$ and $B=2$.

So we have that our fraction, $\dfrac{x+5}{x^2-2x-3}$ can be re-written as $\dfrac{-1}{x+1}+\dfrac{2}{x-3}$. We check:

$$\frac{-1}{x+1}+\frac{2}{x-3}=\frac{-1(x-3)}{(x+1)(x-3)}+\frac{2(x+1)}{(x-3)(x+1)}=\frac{-(x-3)+2(x+1)}{(x+1)(x-3)}=\frac{-x+3+2x+2}{x^2-2x-3}=\frac{x+5}{x^2-2x-3}$$

Now we can easily integrate:

$$\int \frac{x+5}{x^2-2x-3}\,dx=\int \frac{-1}{x+1}+\frac{2}{x-3}\,dx=-\int \frac{1}{x+1}\,dx+2\int \frac{1}{x-3}\,dx=\boxed{-\ln|x+1|+2\ln|x-3|+C}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+1}+\frac{B}{x-3}=\frac{x+5}{x^2-2x-3}$$

We bring the fractions to the common denominator:

$$\frac{A(x-3)}{(x-3)(x+1)}+\frac{B(x+1)}{(x-3)(x+1)}=\frac{x+5}{x^2-2x-3}$$

and then multiply both sides by the denominator:

$$A(x-3)+B(x+1)=x+5$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=3$ into both sides:

$$\begin{aligned} A(0)+B(4) &= 3+5\\ 4B &= 8\\ B &= 2 \end{aligned}$$

Let us substitute $x=-1$ into both sides:

$$\begin{aligned} A(-4)+B(0) &= -1+5 \\ -4A &= 4 \\ A &= -1 \end{aligned}$$

and so $A=-1$ and $B=2$.

4. $\displaystyle\int \frac{x^4+x^3-5x^2+26x-21}{x^2+3x-4}\,dx$

Solution: This rational function is an improper fraction since the numerator has a higher degree than the denominator. We first perform long division. This process is similar to long division among numbers. For example, to simplify $\frac{38}{7}$, we perform the long division $38 \div 7 = 5$ R 3 which is the same thing as to say that $\frac{38}{7} = 5\frac{3}{7}$. The division:

$$\begin{array}{rrrrrrrrrrrrrrrr}
 & & & & & & & x^2 & - & 2x & + & 5 & & & & \\ \cline{7-16}
x^2 & + & 3x & - & 4 &) & & x^4 & + & x^3 & - & 5x^2 & + & 26x & - & 21 \\
 & & & & & & & -x^4 & - & 3x^3 & + & 4x^2 & & & & \\ \cline{8-16}
 & & & & & & & & - & 2x^3 & - & x^2 & + & 26x & - & 21 \\
 & & & & & & & & & 2x^3 & + & 6x^2 & - & 8x & & \\ \cline{10-16}
 & & & & & & & & & & & 5x^2 & + & 18x & - & 21 \\
 & & & & & & & & & & - & 5x^2 & - & 15x & + & 20 \\ \cline{11-16}
 & & & & & & & & & & & & & 3x & - & 1
\end{array}$$

Step 1: $\dfrac{x^4}{x^2} = x^2$

$x^2\left(x^2+3x-4\right) = x^4+3x^3-4x^2$

$-\left(x^4+3x^3-4x^2\right) = -x^4-3x^3+4x^2$

We add that to the original polynomial shown above.

Step 2: $\dfrac{-2x^3}{x^2} = -2x$

$-2x\left(x^2+3x-4\right) = -2x^3-6x^2+8x$

$-1\left(-2x^3-6x^2+8x\right) = 2x^3+6x^2-8x$

We add that to the original polynomial shown above.

Step 3: $\dfrac{5x^2}{x^2} = 5$

$5\left(x^2+3x-4\right) = 5x^2+15x-20$

$-1\left(5x^2+15x-20\right) = -5x^2-15x+20$

We add that to the original polynomial shown above.

The result of this computation is that

$$\frac{x^4+x^3-5x^2+26x-21}{x^2+3x-4} = x^2-2x+5+\frac{3x-1}{x^2+3x-4}$$

very much like $\frac{38}{7} = 5+\frac{3}{7}$. Thus

$$\begin{aligned} \int \frac{x^4+x^3-5x^2+26x-21}{x^2+3x-4}\,dx &= \int x^2-2x+5+\frac{3x-1}{x^2+3x-4}\,dx = \int x^2-2x+5\,dx + \int \frac{3x-1}{x^2+3x-4}\,dx \\ &= \frac{x^3}{3}-x^2+5x+C_1+\int \frac{3x-1}{x^2+3x-4}\,dx \end{aligned}$$

We apply the method of partial fractions to compute $\displaystyle\int \frac{3x-1}{x^2+3x-4}\,dx$.

We factor the denominator: $x^2 + 3x - 4 = (x+4)(x-1)$. Next, we re-write the fraction $\dfrac{3x-1}{x^2+3x-4}$ as a sum (or difference) of fractions with denominators $x+4$ and $x-1$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+4} + \frac{B}{x-1} = \frac{3x-1}{x^2+3x-4}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A(x-1)}{(x+4)(x-1)} + \frac{B(x+4)}{(x+4)(x-1)} = \frac{Ax - A + Bx + 4B}{x^2+3x-4} = \frac{(A+B)x - A + 4B}{x^2+3x-4}$$

Thus

$$\frac{(A+B)x - A + 4B}{x^2+3x-4} = \frac{3x-1}{x^2+3x-4}$$

We clear the denominators by multiplication

$$(A+B)x - A + 4B = 3x - 1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. This gives us an equation for each coefficient that forms a system of linear equations:

$$\begin{aligned} A + B &= 3 \\ -A + 4B &= -1 \end{aligned}$$

We solve the system and obtain $A = \dfrac{13}{5}$ and $B = \dfrac{2}{5}$.

So our fraction, $\dfrac{3x-1}{x^2+3x-4}$ can be re-written as $\dfrac{\frac{13}{5}}{x+4} + \dfrac{\frac{2}{5}}{x-1}$. We check:

$$\begin{aligned} \frac{\frac{13}{5}}{x+4} + \frac{\frac{2}{5}}{x-1} &= \frac{\frac{13}{5}(x-1)}{(x+1)(x-4)} + \frac{\frac{2}{5}(x+4)}{(x-4)(x+1)} = \frac{\frac{13}{5}(x-1) + \frac{2}{5}(x+4)}{(x+1)(x-4)} \\ &= \frac{\frac{13}{5}x - \frac{13}{5} + \frac{2}{5}x + \frac{8}{5}}{(x+1)(x-4)} = \frac{\frac{15}{5}x - \frac{5}{5}}{x^2+3x-4} = \frac{3x-1}{x^2+3x-4} \end{aligned}$$

Now we can easily integrate:

$$\begin{aligned} \int \frac{3x-1}{x^2+3x-4}\, dx &= \int \frac{\frac{13}{5}}{x+4} + \frac{\frac{2}{5}}{x-1}\, dx = \int \frac{\frac{13}{5}}{x+4}\, dx + \int \frac{\frac{2}{5}}{x-1}\, dx \\ &= \frac{13}{5}\int \frac{1}{x+4}\, dx + \frac{2}{5}\int \frac{1}{x-1}\, dx = \frac{13}{5}\ln|x+4| + \frac{2}{5}\ln|x-1| + C \end{aligned}$$

Thus the final answer is

$$\begin{aligned} &\int \frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2+3x-4}\, dx = \\ &= \frac{x^3}{3} - x^2 + 5x + C_1 + \int \frac{3x-1}{x^2+3x-4}\, dx = \frac{x^3}{3} - x^2 + 5x + C_1 + \frac{13}{5}\ln|x+4| + \frac{2}{5}\ln|x-1| + C_2 \\ &= \boxed{\frac{x^3}{3} - x^2 + 5x + \frac{13}{5}\ln|x+4| + \frac{2}{5}\ln|x-1| + C} \end{aligned}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+4}+\frac{B}{x-1}=\frac{3x-1}{x^2+3x-4}$$

We bring the fractions to the common denominator:

$$\frac{A\,(x-1)}{(x+4)\,(x-1)}+\frac{B\,(x+4)}{(x+4)\,(x-1)}=\frac{3x-1}{(x+1)\,(x-4)}$$

and then multiply both sides by the denominator:

$$A\,(x-1)+B\,(x+4)=3x-1$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=1$ into both sides:

$$\begin{aligned} A\,(1-1)+B\,(1+4) &= 3x-1 \\ A\cdot 0+B\cdot 5 &= 3\cdot 1-1 \\ 5B &= 2 \\ B &= \frac{2}{5} \end{aligned}$$

Let us substitute $x=-4$ into both sides:

$$\begin{aligned} A\,(-4-1)+B\,(-4+4) &= 3\,(-4)-1 \\ -5A &= -13 \\ A &= \frac{13}{5} \end{aligned}$$

and so $A=\frac{4}{5}$ and $B=\frac{11}{5}$.

5. $\displaystyle\int \frac{x^2+x-3}{(x+1)\,(x-2)\,(x-5)}\,dx$

Solution: We re-write the fraction $\dfrac{x^2+x-3}{(x+1)\,(x-2)\,(x-5)}$ as a sum (or difference) of fractions with denominators $x+1$, $x-2$ and $x-5$. This means that we need to solve for A, B, and C in the equation

$$\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x-5}=\frac{x^2+x-3}{(x+1)\,(x-2)\,(x-5)}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x-5} &= \frac{A\,(x-2)\,(x-5)}{(x+1)\,(x-2)\,(x-5)}+\frac{B\,(x+1)\,(x-5)}{(x+1)\,(x-2)\,(x-5)}+\frac{C\,(x+1)\,(x-2)}{(x+1)\,(x-2)\,(x-5)} \\ &= \frac{A\left(x^2-7x+10\right)}{(x+1)\,(x-2)\,(x-5)}+\frac{B\left(x^2-4x-5\right)}{(x+1)\,(x-2)\,(x-5)}+\frac{C\left(x^2-x-2\right)}{(x+1)\,(x-2)\,(x-5)} \\ &= \frac{A\left(x^2-7x+10\right)+B\left(x^2-4x-5\right)+C\left(x^2-x-2\right)}{(x+1)\,(x-2)\,(x-5)} \\ &= \frac{Ax^2-7Ax+10A+Bx^2-4Bx-5B+Cx^2-Cx-2C}{(x+1)\,(x-2)\,(x-5)} \\ &= \frac{(A+B+C)\,x^2+(-7A-4B-C)\,x+10A-5B-2C}{(x+1)\,(x-2)\,(x-5)} \end{aligned}$$

Thus
$$\frac{(A+B+C)\,x^2+(-7A-4B-C)\,x+10A-5B-2C}{(x+1)\,(x-2)\,(x-5)}=\frac{x^2+x-3}{(x+1)\,(x-2)\,(x-5)}$$
We clear the denominators by multiplication
$$(A+B+C)\,x^2+(-7A-4B-C)\,x+10A-5B-2C=x^2+x-3$$
The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. We have an equation for each coefficient that gives us a system of linear equations:
$$\begin{array}{rcrcrcr} A & + & B & + & C & = & 1 \\ -7A & - & 4B & - & C & = & 1 \\ 10A & - & 5B & - & 2C & = & -3 \end{array}$$
We solve the system by elimination: first we will eliminate C from the second and third equations. To eliminate C from the second equation, we simply add the first and second equations.
$$\begin{array}{rcrcrcr} A & + & B & + & C & = & 1 \\ -7A & - & 4B & - & C & = & 1 \\ \hline -6A & - & 3B & & & = & 2 \end{array}$$
To eliminate C from the third equation, we multiply the first equation by 2 and add that to the third equation.
$$\begin{array}{rcrcrcr} 2A & + & 2B & + & 2C & = & 2 \\ 10A & - & 5B & - & 2C & = & -3 \\ \hline 12A & - & 3B & & & = & -1 \end{array}$$
We now have a system of linear equations in two variables:
$$\begin{aligned} -6A-3B &= 2 \\ 12A-3B &= -1 \end{aligned}$$
We will eliminate B by adding the opposite of the first equation to the second equation.
$$\begin{array}{rcrcr} 6A & + & 3B & = & -2 \\ 12A & - & 3B & = & -1 \\ \hline & & 18A & = & -3 \\ & & A & = & -\dfrac{1}{6} \end{array}$$
Using the equation $6A+3B=-2$ we can now solve for B.
$$\begin{aligned} 6\left(-\frac{1}{6}\right)+3B &= -2 \\ -1+3B &= -2 \\ 3B &= -1 \\ B &= -\frac{1}{3} \end{aligned}$$
Using the first equation, we can now solve for C.
$$\begin{aligned} A+B+C &= 1 \\ -\frac{1}{6}+\left(-\frac{1}{3}\right)+C &= 1 \\ -\frac{1}{2}+C &= 1 \\ C &= \frac{3}{2} \end{aligned}$$

Thus $A = -\frac{1}{6}$, $B = -\frac{1}{3}$, and $C = \frac{3}{2}$

So we have that our fraction, $\frac{x^2+x-3}{(x+1)(x-2)(x-5)}$ can be re-written as $\frac{-\frac{1}{6}}{x+1} + \frac{-\frac{1}{3}}{x-2} + \frac{\frac{3}{2}}{x-5}$. We check:

$$\begin{aligned}
\frac{-\frac{1}{6}}{x+1} + \frac{-\frac{1}{3}}{x-2} + \frac{\frac{3}{2}}{x-5} &= \frac{-\frac{1}{6}(x-2)(x-5)}{(x+1)(x-2)(x-5)} + \frac{-\frac{1}{3}(x+1)(x-5)}{(x+1)(x-2)(x-5)} + \frac{\frac{3}{2}(x+1)(x-2)}{(x+1)(x-2)(x-5)} \\
&= \frac{-\frac{1}{6}(x-2)(x-5) - \frac{1}{3}(x+1)(x-5) + \frac{3}{2}(x+1)(x-2)}{(x+1)(x-2)(x-5)} \\
&= \frac{-\frac{1}{6}\left(x^2-7x+10\right) - \frac{1}{3}\left(x^2-4x-5\right) + \frac{3}{2}\left(x^2-x-2\right)}{(x+1)(x-2)(x-5)} \\
&= \frac{-\frac{1}{6}x^2 + \frac{7}{6}x - \frac{5}{3} - \frac{1}{3}x^2 + \frac{4}{3}x + \frac{5}{3} + \frac{3}{2}x^2 - \frac{3}{2}x - 3}{(x+1)(x-2)(x-5)} \\
&= \frac{\left(-\frac{1}{6} - \frac{1}{3} + \frac{3}{2}\right)x^2 + \left(\frac{7}{6} + \frac{4}{3} - \frac{3}{2}\right)x - \frac{5}{3} + \frac{5}{3} - 3}{(x+1)(x-2)(x-5)} \\
&= \frac{x^2+x-3}{(x+1)(x-2)(x-5)}
\end{aligned}$$

Now we can easily integrate:

$$\begin{aligned}
\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)}\,dx &= \int \frac{-\frac{1}{6}}{x+1} + \frac{-\frac{1}{3}}{x-2} + \frac{\frac{3}{2}}{x-5}\,dx \\
&= -\frac{1}{6}\int \frac{1}{x+1}\,dx - \frac{1}{3}\int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{x-5}\,dx \\
&= \boxed{-\frac{1}{6}\ln|x+1| - \frac{1}{3}\ln|x-2| + \frac{2}{3}\ln|x-5| + C}
\end{aligned}$$

Method 2: The values of A, B, and C can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-5} = \frac{x^2+x-3}{(x+1)(x-2)(x-5)}$$

We bring the fractions to the common denominator:

$$\frac{A(x-2)(x-5)}{(x+1)(x-2)(x-5)} + \frac{B(x+1)(x-5)}{(x+1)(x-2)(x-5)} + \frac{C(x+1)(x-2)}{(x+1)(x-2)(x-5)} = \frac{x^2+x-3}{(x+1)(x-2)(x-5)}$$

and then multiply both sides by the denominator:

$$A(x-2)(x-5) + B(x+1)(x-5) + C(x+1)(x-2) = x^2+x-3$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x = 2$ into both sides:

$$\begin{aligned}
A(2-2)(2-5) + B(2+1)(2-5) + C(2+1)(2-2) &= 2^2+2-3 \\
0A - 9B + 0C &= 3 \\
-9B &= 3 \\
B &= -\frac{1}{3}
\end{aligned}$$

Let us substitute $x = -1$ into both sides:

$$\begin{aligned}
A(-1-2)(-1-5) + B(-1+1)(-1-5) + C(-1+1)(-1-2) &= (-1)^2 + (-1) - 3 \\
A(-3)(-6) + 0B + 0C &= -3 \\
18A &= -3 \\
A &= -\frac{1}{6}
\end{aligned}$$

Let us substitute $x = 5$ into both sides:

$$\begin{aligned}
A(5-2)(5-5) + B(5+1)(5-5) + C(5+1)(5-2) &= 5^2 + 5 - 3 \\
A(0) + B(0) + C(6)(3) &= 27 \\
18C &= 27 \\
C &= \frac{3}{2}
\end{aligned}$$

and so $A = -\frac{1}{6}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$.

6. $\displaystyle\int \frac{2x-1}{(x-5)^2}\, dx$

Solution: We will re-write the fraction $\dfrac{2x-1}{(x-5)^2}$ as a sum (or difference) of fractions with denominators $x-5$ and $(x-5)^2$. This means that we need to solve for A and B in the equation

$$\frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{A(x-5)}{(x-5)^2} + \frac{B}{(x-5)^2} = \frac{A(x-5)+B}{(x-5)^2} = \frac{Ax-5A+B}{(x-5)^2}$$

Thus we have

$$\frac{Ax-5A+B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

We clear the denominators by multiplication

$$Ax - 5A + B = 2x - 1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned}
A &= 2 \\
-5A + B &= -1
\end{aligned}$$

We solve this system and obtain $A = 2$ and $B = 9$.

So our fraction, $\dfrac{2x-1}{(x-5)^2}$ can be re-written as $\dfrac{2}{x-5} + \dfrac{9}{(x-5)^2}$. We check:

$$\frac{2}{x-5} + \frac{9}{(x-5)^2} = \frac{2(x-5)}{(x-5)^2} + \frac{9}{(x-5)^2} = \frac{2x-10+9}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

Now we can easily integrate:

$$\int \frac{2x-1}{(x-5)^2}\,dx = \int \frac{2}{x-5} + \frac{9}{(x-5)^2}\,dx = 2\int \frac{1}{x-5}\,dx + 9\int \frac{1}{(x-5)^2}\,dx = \boxed{2\ln|x-5| - \frac{9}{x-5} + C}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

We bring the fractions to the common denominator:

$$\frac{A(x-5)}{(x-5)^2} + \frac{B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

and then multiply both sides by the denominator:

$$A(x-5) + B = 2x-1$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=5$ into both sides:

$$\begin{aligned} A(0) + B &= 9 \\ B &= 9 \end{aligned}$$

The other value of x can be arbitrarily chosen. (There is no value that would eliminate B from the equation.) For easy substitution, let us substitute $x=0$ into both sides and also substitute $B=9$:

$$\begin{aligned} A(-5) + 9 &= -1 \\ -5A &= -10 \\ A &= 2 \end{aligned}$$

and so $A=2$ and $B=9$.

7. $\displaystyle\int \frac{x+3}{(x-1)^3}\,dx$

Solution: We re-write the fraction $\dfrac{x+3}{(x-1)^3}$ as a sum (or difference) of fractions with denominators $x-1$, $(x-1)^2$ and $(x-1)^3$. This means that we need to solve for A, B, and C in the equation

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{x+3}{(x-1)^3}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} &= \frac{A(x-1)^2}{(x-1)^3} + \frac{B(x-1)}{(x-1)^3} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3} \\ &= \frac{A\left(x^2-2x+1\right) + B(x-1) + C}{(x-1)^3} = \frac{Ax^2 - 2Ax + A + Bx - B + C}{(x-1)^3} \\ &= \frac{Ax^2 + (-2A+B)x + A - B + C}{(x-1)^3} \end{aligned}$$

Thus

$$\frac{Ax^2+(-2A+B)\,x+A-B+C}{(x-1)^3}=\frac{x+3}{(x-1)^3}$$

We clear the denominators by multiplication

$$Ax^2+(-2A+B)\,x+A-B+C=x+3$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. We have an equation for each coefficient that gives us a system of linear equations:

$$\begin{array}{rcrcrcl} A & & & & & = & 0 \\ -2A & + & B & & & = & 1 \\ A & - & B & + & C & = & 3 \end{array}$$

Since $A=0$, this is really a system in two variables:

$$\begin{aligned} B &= 1 \\ -B+C &= 3 \end{aligned}$$

We solve this system and obtain $B=1$ and $C=4$.

So our fraction, $\dfrac{x+3}{(x-1)^3}$ can be re-written as $\dfrac{1}{(x-1)^2}+\dfrac{4}{(x-1)^3}$. We check:

$$\frac{1}{(x-1)^2}+\frac{4}{(x-1)^3}=\frac{1\,(x-1)}{(x-1)^3}+\frac{4}{(x-1)^3}=\frac{x-1+4}{(x-1)^3}=\frac{x+3}{(x-1)^3}$$

Now we can easily integrate:

$$\begin{aligned} \int\frac{x+3}{(x-1)^3}\,dx &= \int\frac{1}{(x-1)^2}+\frac{4}{(x-1)^3}\,dx=\int\frac{1}{(x-1)^2}\,dx+4\int\frac{1}{(x-1)^3}\,dx \\ &= -\frac{1}{x-1}-\frac{4}{2}\cdot\frac{1}{(x-1)^2}+C=\boxed{-\frac{1}{x-1}-\frac{2}{(x-1)^2}+C} \\ &= \frac{-1\,(x-1)}{(x-1)^2}-\frac{2}{(x-1)^2}+C=\frac{-x+1-2}{(x-1)^2}+C=\boxed{\frac{-x-1}{(x-1)^2}+C} \end{aligned}$$

Both final answers are acceptable.

Method 2: The values of A, B, and C can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}=\frac{x+3}{(x-1)^3}$$

We bring the fractions to the common denominator:

$$\frac{A\,(x-1)^2}{(x-1)^3}+\frac{B\,(x-1)}{(x-1)^3}+\frac{C}{(x-1)^3}=\frac{x+3}{(x-1)^3}$$

and then multiply both sides by the denominator:

$$A\,(x-1)^2+B\,(x-1)+C=x+3$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=1$ into both sides:

$$\begin{aligned} A\,(0)+B\,(0)+C &= 1+3 \\ C &= 4 \end{aligned}$$

There is no value other than 1 that would eliminate A or B from the equation. Our method will still work. For easy substitution, let us substitute $x = 0$ into both sides and also substitute $C = 4$:

$$\begin{aligned} A(x-1)^2 + B(x-1) + C &= x+3 \\ A(0-1)^2 + B(0-1) + 4 &= 0+3 \\ A - B + 4 &= 3 \\ A - B &= -1 \end{aligned}$$

Let us substitute $x = 2$ into both sides:

$$\begin{aligned} A(x-1)^2 + B(x-1) + C &= x+3 \\ A(2-1)^2 + B(2-1) + 4 &= 2+3 \\ A + B + 4 &= 5 \\ A + B &= 1 \end{aligned}$$

We now solve the system of equations

$$\begin{aligned} A - B &= -1 \\ A + B &= 1 \end{aligned}$$

and obtain $A = 0$ and $B = 1$. Recall that we already have $C = 4$.