Tutorial (6)

Homogeneous Equations

Def.

A DE $\frac{dy}{dx} = f(x,y)$ is homogeneous if $f(x,y)$ can be expressed as $f(x,y) = F\left(\frac{y}{x}\right)$ [$f(x,y)$ can be expressed as a function of $\frac{y}{x}$ only].

Example : 1) $\frac{dy}{dx} = \sin\left(\frac{y}{x}\right) - 2$ Homogeneous

2) $\frac{dy}{dx} = \frac{x^2 - 3y^2}{2xy}$ Homogeneous

why? $\frac{dy}{dx} = \frac{x^2}{2xy} - \frac{3}{2}\frac{y^2}{xy}$

$= \frac{1}{2}\left(\frac{x}{y}\right) - \frac{3}{2}\left(\frac{y}{x}\right)$

$= \frac{1}{2}\left(\frac{y}{x}\right)^{-1} - \frac{3}{2}\left(\frac{y}{x}\right)$

$= F\left(\frac{y}{x}\right)$ Homogeneous.

Note : Solving a homogeneous DE.

1) We use the substitution $v = \frac{y}{x}$

2) we chang the homogeneous DE into separable DE and solve it.

Ex. 1 a) Solve

$$\frac{dy}{dx} = \frac{x-2y}{x}$$

$$\frac{dy}{dx} = 1-2\left(\frac{y}{x}\right) \qquad (I)$$

let $\boxed{v = \frac{y}{x}} \Rightarrow y = xv$

$$\frac{dy}{dx} = (1)v + x\frac{dv}{dx} \quad (v \text{ is a function of } x)$$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

substitute $v = \frac{y}{x}$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (I)

we get $v + x\frac{dv}{dx} = 1-2v$

simplify $x\frac{dv}{dx} = 1-3v$ (separable DE)

$$\frac{dv}{1-3v} = \frac{dx}{x}$$

$$-\frac{1}{3}\int\frac{-3\,dv}{1-3v} = \int\frac{dx}{x}$$

$$-\frac{1}{3}\ln|1-3v| = \ln|x| + C_1$$

$$\ln|1-3v| = -3\ln|x| - 3C_1$$

$$\ln|1-3v| + 3\ln|x| = -3C_1 \qquad \text{Note: } \ln|x|^3 = \ln|x^3|$$

$$\ln|1-3v| + \ln|x^3| = -3C_1$$

$$\ln|(1-3v)x^3| = -3C_1$$

$$|(1-3v)x^3| = e^{-3c_1}$$

$$(1-3v)x^3 = \pm e^{-3c_1}$$

$$(1-3v)x^3 = C_2 \qquad \text{where } C_2 = \pm e^{-3c_1}$$

$$1-3v = \frac{C_2}{x^3}$$

$$v = \frac{C_2}{-3x^3} + \frac{1}{3}$$

$$v = \frac{C}{x^3} + \frac{1}{3} \qquad C = \frac{C_2}{-3}$$

$$\frac{y}{x} = \frac{C}{x^3} + \frac{1}{3} \Rightarrow \boxed{y = \frac{C}{x^2} + \frac{1}{3}x}$$

b) Easier!

$$xy' = \frac{y^2}{x} + y$$

$$x\frac{dy}{dx} = \frac{y^2}{x} + y$$

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x}$$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) \qquad (I)$$

let $v = \frac{y}{x} \Rightarrow y = xv$ and $\frac{dy}{dx} = v + \frac{dv}{dx}x$

substitute in (I) $\Rightarrow$ $v + \frac{dv}{dx}x = v^2 + v$

$$x\frac{dv}{dx} = v^2 \qquad \text{separable}$$

$$\frac{dv}{v^2} = \frac{dx}{x} \Rightarrow \int \frac{dv}{v^2} = \int \frac{dx}{x}$$

$$-\frac{1}{v} = \ln|x| + C$$

$$-v = \frac{1}{\ln|x| + C} \Rightarrow v = \frac{-1}{\ln|x| + C}$$

$$\frac{y}{x} = \frac{-1}{\ln|x| + C}$$

$$\boxed{y = \frac{-x}{\ln|x| + C}}$$

c) $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

$$2xy\,dy = (3y^2 - x^2)\,dx$$

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$$

$$\frac{dy}{dx} = \frac{3y^2}{2xy} - \frac{x^2}{2xy}$$

$$\frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{x}{y}\right)$$

$$\frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{y}{x}\right)^{-1}$$

Let $\frac{y}{x} = v \Rightarrow y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{3}{2}v - \frac{1}{2}v^{-1}$$

$$x\frac{dv}{dx} = \frac{1}{2}v - \frac{1}{2}v^{-1}$$ separable

$$x\frac{dv}{dx} = \frac{1}{2}\left(\frac{v^2-1}{v}\right)$$

$$\frac{dv}{\frac{v^2-1}{v}} = \frac{1}{2}\frac{dx}{x}$$

$$\frac{v}{v^2-1}dv = \frac{1}{2}\frac{dx}{x} \Rightarrow \frac{1}{2}\int\frac{2v}{v^2-1}dv = \frac{1}{2}\int\frac{dx}{x}$$

$$\ln|v^2-1| = \ln|x| + C_1$$

$$\ln\left|\frac{v^2-1}{x}\right| = C_1$$

$$\left|\frac{v^2-1}{x}\right| = e^{C_1}$$

$$\frac{(v^2-1)}{x} = \pm e^{C_1}$$

$$v^2-1 = Cx \qquad C = \pm e^{C_1}$$

$$v^2 = Cx + 1$$

$$\frac{y^2}{x^2} = Cx + 1$$

$$\boxed{y^2 = Cx^3 + x^2}$$

Exact Differential Equations

A first order DE of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be an exact equation if the expression on the left-hand side is an exact differential $\left(df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)$.

Ex. $x^2y^3dx + x^3y^2dy = 0$ is an exact equation, because its left-hand side is an exact differential:

$$d\left(\tfrac{1}{3}x^3y^3\right) = x^2y^3dx + x^3y^2dy \quad , \text{ where}$$

$$f(x,y) = \tfrac{1}{3}x^3y^3.$$

Now $x^2y^3dx + x^3y^2dy = 0 \Rightarrow d\left(\tfrac{1}{3}x^3y^3\right) = 0$

$\therefore \ \tfrac{1}{3}x^3y^3 = C$ is a solution to the exact DE.

Test for Exactness

$M(x,y)\,dx + N(x,y)\,dy = 0$ is an exact differential if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Note Suppose $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact $\Rightarrow$

$$M(x,y)\,dx + N(x,y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$\therefore \ \frac{\partial f}{\partial x} = M(x,y) \quad , \quad \frac{\partial f}{\partial y} = N(x,y)$ and

the general solution is $\boxed{f(x,y) = C}$

Ex. Solve $\cos x - 2xy + (e^y - x^2)y' = 0 \qquad y(1) = 4$.

Write in differential form:

$$\cos x - 2xy + (e^y - x^2)\frac{dy}{dx} = 0$$

$$(\cos x - 2xy)\,dx + (e^y - x^2)\,dy = 0$$

$$M(x,y) = \cos x - 2xy \quad \text{and} \quad N(x,y) = e^y - x^2$$

$$\frac{\partial M}{\partial y} = -2x \qquad \frac{\partial N}{\partial x} = -2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{The equation is exact.}$$

we have $\frac{\partial f}{\partial x} = M(x,y) = \cos x - 2xy \Rightarrow$

$$f(x,y) = \int (\cos x - 2xy)\,dx + h(y)$$

$$f(x,y) = \sin x - x^2 y + h(y)$$

$$\downarrow$$

$$\frac{\partial f}{\partial y} = -x^2 + h'(y) = N(x,y)$$

$$-x^2 + h'(y) = e^y - x^2$$

$$h'(y) = e^y \Rightarrow h(y) = e^y + c_1$$

so $f(x,y) = \sin x - x^2 y + e^y + c_1$

and the general solution to the DE is $\sin x - x^2 y + e^y + c_1 = c_2$

or $\sin x - x^2 y + e^y = c$ where $c = c_2 - c_1$

$y(1) = 4 \Rightarrow \sin(1) - (1)^2(4) + e^4 = C$

$C = \sin(1) - 4 + e$

$\therefore$ the solution of the IVP is implicitly defined by $\sin x - x^2 y + e^y = e^4 - 4 + \sin 1$

Ex. $(x + \sin y)\,dx + (x\cos y - 2y)\,dy = 0$

$M(x,y) = x + \sin y$ and $N(x,y) = x\cos y - 2y$

$\frac{\partial M}{\partial y} = \cos y \qquad \frac{\partial N}{\partial x} = \cos y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ D.E. is exact

$\frac{\partial f}{\partial x} = x + \sin y \Rightarrow f(x,y) = \int (x + \sin y)\,dx + h(y)$

$f(x,y) = \frac{x^2}{2} + x\sin y + h(y)$

$\frac{\partial f}{\partial y} = x\cos y + h'(y) = N(x,y)$

$x\cos y + h'(y) = x\cos y - 2y$

$h'(y) = -2y \Rightarrow h(y) = -y^2$ (We can drop the constant)

$\therefore f(x,y) = \frac{x^2}{2} + x\sin y - y^2$ and the solution to the given DE is $\boxed{\frac{x^2}{2} + x\sin y - y^2 = C}$

Ex.

$$y' = \frac{2 + y e^{xy}}{2y - x e^{xy}}$$

$$\frac{dy}{dx} = \frac{2 + y e^{xy}}{2y - x e^{xy}}$$

$$(2 + y e^{xy})\,dx = (2y - x e^{xy})\,dy$$

$$(2 + y e^{xy})\,dx + (x e^{xy} - 2y)\,dy = 0$$

$$M(x,y) = 2 + y e^{xy} \qquad N(x,y) = x e^{xy} - 2y$$

$$\frac{\partial M}{\partial y} = e^{xy} + xy e^{xy} \qquad \frac{\partial N}{\partial x} = e^{xy} + xy e^{xy}. \qquad \text{Exact}$$

$$\frac{\partial f}{\partial x} = 2 + y e^{xy} \Rightarrow f(x,y) = \int (2 + y e^{xy})\,dx + h(y)$$

$$f(x,y) = 2x + e^{xy} + h(y)$$

$$\downarrow$$

$$\frac{\partial f}{\partial y} = x e^{xy} + h'(y) = N(x,y)$$

$$x e^{xy} + h'(y) = x e^{xy} - 2y$$

$$h'(y) = -2y \Rightarrow h(y) = -y^2$$

$$f(x,y) = 2x + e^{xy} - y^2$$

and the general solution is $\boxed{2x + e^{xy} - y^2 = c}$