EXAMINATION COVERSHEET
Winter 2023 Final Examination

UNIVERSITY OF WOLLONGONG IN DUBAI

THIS EXAMINATION CONTENT IS STRICTLY CONFIDENTIAL Students must comply with requirements stated in the Examination Policy & Procedures	
Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	**01/04/2023**
Subject Code:	**Math 142**
Subject Title:	**Essentials of Engineering Mathematics**
Time Permitted to Write Exam:	2 Hours
Total Number of Questions:	**11 (6 MCQ's + 5 written questions)**
Total Number of Pages *(including this page):*	**9**

INSTRUCTIONS TO STUDENTS FOR THE EXAM

1. Please note that subject lecturer/tutor will be unavailable during exams. *If there is a doubt in any of the exam questions i.e. problem solving etc. students should proceed by assuming values etc. Students should mention their assumption on the question paper.*
2. Answers must be written (and drawn) in black or blue ink
3. Any mistakes must be crossed out. Whitener and ink erasers must not be used.
4. Part A (MCQ): Answer ALL/ 6 questions. The marks for each question are shown next to each question. The total for Part A is 30 marks.
5. Part B (Written): Answer ALL/ 5 questions. The marks for each question are shown next to each question. The total for Part B is 70 marks.)
6. Total marks: 100. This Exam is worth 40% of your final marks for MATH 142.

Part 1 MCQ's 30% (Circle Your Choice)

(5pts) Problem 1

Evaluate the improper integral

$$I = \int_6^8 \frac{4}{\sqrt{x-6}}dx$$

$A)\qquad I = 8$

$B)\qquad I = 8\sqrt{2}$

$C)\qquad I = 6$

$D)\qquad I = \infty$

$E)\qquad I = 2\sqrt{2}$

Solution

$$\begin{aligned} I &= \int_6^8 \frac{4}{\sqrt{x-6}}dx = \lim_{t\to 6^+}\int_t^8 4\,(x-6)^{-\frac{1}{2}}\,dx \\ &= \lim_{t\to 6^+}\left[4\,\frac{(x-6)^{-\frac{1}{2}+1}}{\frac{-1}{2}+1}\Bigg|_t^8\right] \\ &= \lim_{t\to 6^+}\left[8\,\sqrt{x-6}\Big|_t^8\right] \\ &= \lim_{t\to 6^+} 8\left(\sqrt{2}-\sqrt{t-6}\right) = 8\sqrt{2} \end{aligned}$$

Answer is B)

(5pts) Problem 2

Evaluate the improper integral

$$I = \int_{10}^{\infty} \frac{1}{x\ln x}dx$$

$A)\qquad I = 10\ln 10$

$B)\qquad I = 100$

$C)\qquad I = \sqrt{10}$

$D)\qquad I = 10\sqrt{10}$

$E)\qquad I = \infty$

Solution

$$\begin{aligned} I &= \int_{10}^{\infty} \frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_{10}^{t} \frac{\frac{1}{x}}{\ln x}dx \\ &= \lim_{t\to\infty} \ln|\ln x|\big|_{10}^{t} \\ &= \lim_{t\to\infty} (\ln|\ln t| - \ln\ln 10) = \infty \end{aligned}$$

Answer is E)

(5pts) Problem 3

Consider the differential equation

$$\frac{dy}{dx} = \frac{-2xy}{x^2+y^2}$$

Which of the following is TRUE.

$A)$ The differential equation is linear

$B)$ The differential equation is separable

$C)$ The differential equation is homogeneous

$D)$ The differential equation is exact

$E)$ None of the above is true

Solution

This equation is not separable, because there is no way to write it in the form

$$\frac{dy}{dx} = f(x)g(y)$$

The equation is also not linear, because the y^2 term prevents us from putting it in the form $\frac{dy}{dx} + P(x)y = Q(x)$. The equation is not homogeneous because the right-hand side cannot be written as a function of $\frac{y}{x}$ alone.

To test for exactness we put the equation in the form

$$M(x,y)dx + N(x,y)dy = 0$$

$$2xydx + \left(x^2+y^2\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

So the equation is exact. **D**) is correct.

Moreover,

$$\frac{dy}{dx} = \frac{-2xy}{x^2+y^2} = \frac{-2\cancel{x}y}{x^{\cancel{2}}\left(1+\left(\frac{y}{x}\right)^2\right)} = \frac{-2\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}$$

The equation is homogeneous. **C**) is correct.

Remark: *You should circle both* ***C)*** *and* ***D).*** *You loose 2 points if you miss one choice.*

(5pts) Problem 4
Let a_n be the sequence given by

$$\ln\frac{2}{1},\ \ln\frac{3}{2},\ \ln\frac{4}{3}, \ldots$$

$\lim\limits_{n\to\infty} a_n$ is equal to

$$A)\quad 0$$

$$B)\quad 1$$

$$C)\quad 2$$

$$D)\quad 3$$

$$E)\quad 4$$

Solution

$$a_n = \ln\left(\frac{n+1}{n}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \ln 1 = 0$$

Answer is A)

(5pts) Problem 5

If

$$S = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + ...,$$

then

$$A) \qquad S = \infty$$

$$B) \qquad S = 1$$

$$C) \qquad S = 10000$$

$$D) \qquad S = 0.0001$$

$$E) \qquad S = 90909$$

Solution

$$S = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + ..., = \sum_{n=1}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^{n-1}$$

S is a geometric series with $a_1 = \dfrac{9}{10}$ and $r = \dfrac{1}{10}$.

$$|r| = \frac{1}{10} < 1 \Rightarrow S = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = 1$$

Answer is B)

(5pts) Problem 6
If

$$\mathcal{L} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1},$$

then

$$A) \quad \mathcal{L} = \frac{1}{4}$$

$$B) \quad \mathcal{L} = \frac{1}{2}$$

$$C) \quad \mathcal{L} = \infty$$

$$D) \quad \mathcal{L} = 4$$

$$E) \quad \mathcal{L} = 7$$

Solution

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \\ \frac{1}{(2n-1)(2n+1)} &= \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)} \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} &= \lim_{N\to\infty} \frac{1}{2} \sum_{n=1}^{N} \left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)} \right) \\ &= \lim_{N\to\infty} \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + ... + \left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)} \right) \right] \\ &= \lim_{N\to\infty} \frac{1}{2} \left(1 - \frac{1}{(2n+1)} \right) = \frac{1}{2} \end{aligned}$$

Answer is B)

Part 2 Written Questions (70%)

(16pts) Problem 1

Determine convergence or divergence of the following series.

$$\text{(A)} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{n!} \qquad \text{(B)} \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n+1}$$
$$\text{(C)} \sum_{n=0}^{\infty} \frac{n+1}{\sqrt{n^2+n+1}} \qquad \text{(D)} \sum_{n=0}^{\infty} \frac{7}{n\sqrt{n}}$$

Solution

(A) $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$. Here we apply the ratio test with $a_n = (-1)^{n+1} \frac{2^n}{n!}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+2} \frac{2^{n+1}}{(n+1)!}}{(-1)^{n+1} \frac{2^n}{n!}} \right| \\
&= \lim_{n\to\infty} \left| (-1)^{n+2} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^{n+1} 2^n} \right| \\
&= \lim_{n\to\infty} \left| -(\cancel{-1})^{n+1} \frac{\cancel{2^n} \cdot 2}{(n+1)\, \cancel{n!}} \cdot \frac{\cancel{n!}}{(\cancel{-1})^{n+1} \cancel{2^n}} \right| \\
&= \lim_{n\to\infty} \frac{2}{(n+1)} \cdot \frac{1}{1} = 0 \qquad (\textbf{4 points})
\end{aligned}$$

(B) $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n+1}$. This is an alternating series with $b_n = \frac{1}{2n+1}$. Since

$$\begin{cases} b_n = \dfrac{1}{2n+1} > 0 \\ b_n = \dfrac{1}{2n+1} \text{ is decreasing} \\ \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{1}{2n+1} = 0 \end{cases},$$

the series converges by the alternating series test. (**4 points**)

(C) $\sum_{n=0}^{\infty} \frac{n+1}{\sqrt{n^2+n+1}}$. Since $\lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = 1 \neq 0$, the series diverges by the divergence test. (**4 points**)

(D) $\sum_{n=0}^{\infty} \frac{7}{n\sqrt{n}} = \sum_{n=0}^{\infty} \frac{7}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$. It converges by the p-series test. (**4 points**)

(14pts) Problem 2
Find the interval of convergence of the following power series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}.$$

Solution

$$\text{Center } a = 2.$$

Radius of convergence:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)^2+1}}{\frac{(x-2)^n}{n^2+1}} \right| &= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{(x-2)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(x-2)}{(n+1)^2+1} \times \frac{n^2+1}{1} \right| = |x-2|
\end{aligned}$$

$$R = 1 \qquad (\textbf{4 points})$$

$$\begin{aligned}
a - R &= 2 - 1 = 1 \qquad (\textbf{2 points}) \\
a + R &= 2 + 1 = 3 \qquad (\textbf{2 points})
\end{aligned}$$

- When $x = 1$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} \text{ which is a convergent alternating series.} \qquad (\textbf{2 points})$$

- When $x = 3$, the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}.$$

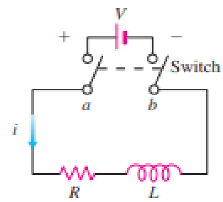
Using the limit comparison test to compare the series with $\sum_{n=0}^{\infty} \frac{1}{n^2}$, we see that it converges by the $p-$ series test. (**2 points**) Hence,

the interval of convergence is

$$IC = [1, 3]. \qquad (\textbf{2 points})$$

(12pts) Problem 3

The diagram in the Figure below represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts.



Ohm's Law, $V = RI$, has to be modified for such a circuit. The modified form is a linear differential equation given by

$$L\frac{di}{dt} + Ri = V$$

where i is the intensity of the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed. If the switch is closed at time $t = 0$ ($i = 0$), How will the current flow as a function of time if $\frac{R}{L} = 3$ and $\frac{V}{L} = 5$?

Solution

We first start by writing the equation in standard form.

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L} \Leftrightarrow \frac{di}{dt} + 3i = 5$$

$$IF = e^{\int 3dt} = e^{3t} \qquad (\textbf{4 points})$$

$$\frac{d}{dt}\left[ie^{3t}\right] = 5e^{3t}$$

$$i = e^{-3t}\int 5e^{3t}dt + Ce^{-3t}$$

$$i = e^{-3t}\frac{5}{3}e^{3t} + Ce^{-3t}$$

$$i = \frac{5}{3} + Ce^{-3t} \qquad (\textbf{4 points})$$

When $t = 0$, $i = 0 \Rightarrow$

$$0 = \frac{5}{3} + C$$

$$C = \frac{-5}{3} \qquad (\textbf{4 points})$$

$$i = \frac{5}{3} - \frac{5}{3}e^{-3t}$$

(14pts) Problem 4

Show that the differential equation is exact and solve the initial value problem

$$\left(\cos x - x\sin x + y^2\right) dx + 2xydy = 0, \qquad y(\pi) = 1.$$

Solution

$$M = \cos x - x\sin x + y^2, \qquad N = 2xy.$$

Exactness condition:

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x} \Rightarrow \text{ The equation is exact.} \qquad (\mathbf{3\ points})$$

$$\begin{cases} f_x = \cos x - x\sin x + y^2 \\ f_y = 2xy \end{cases}.$$

Using the second equation, we have

$$f(x, y) = \int 2xydy = xy^2 + C(x). \qquad (\mathbf{3\ points})$$

Plugging this into the first equation we have

$$y^2 + C'(x) = \cos x - x\sin x + y^2$$

$$C'(x) = \cos x - x\sin x$$

$$\begin{aligned} C(x) &= \int (\cos x - x\sin x)\, dx \\ &= x\cos x + C \qquad (\mathbf{4\ points}) \end{aligned}$$

$$f(x, y) = xy^2 + x\cos x.$$

The solution is

$$xy^2 + x\cos x = C$$

$y(\pi) = 1 \Rightarrow$

$$C = \pi - \pi = 0$$

The particular solution is

$$xy^2 + x\cos x = 0 \qquad (\mathbf{4\ points})$$

(14pts) Problem 5
Show that the equation is Bernoulli and solve it.

$$\frac{dy}{dx} = y\left(xy^3 - 1\right)$$

Hint: $\int -3xe^{-3x}dx = \frac{1}{3}e^{-3x}\left(3x+1\right) + C$

Solution

We have

$$\frac{dy}{dx} + y = xy^4.$$

The equation is Bernoulli with $n = 4$. (**2 points**)

$$y^{-4}\frac{dy}{dx} + y^{-3} = x.$$

Put

$$u = y^{-3} \qquad (\textbf{2 points})$$

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$y^{-4}\frac{dy}{dx} = -\frac{1}{3}\frac{du}{dx}.$$

The equation becomes

$$-\frac{1}{3}\frac{du}{dx} + u = x$$

$$\frac{du}{dx} - 3u = -3x \qquad (\textbf{6 points})$$

$$IF = e^{-3x}$$

$$\frac{d}{dx}\left[e^{-3x}u\right] = -3xe^{-3x}$$

$$e^{-3x}u = \frac{1}{3}e^{-3x}\left(3x+1\right) + C$$

$$u = \frac{1}{3}\left(3x+1\right) + Ce^{3x}$$

$$y^{-3} = \frac{1}{3}\left(3x+1\right) + Ce^{3x}$$

$$y = \frac{1}{\sqrt[3]{x + \frac{1}{3} + Ce^{3x}}} \qquad (\textbf{4 points})$$