Ex 1 In each part, find a formula for the general term of the sequence starting with $n=1$.

a) $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \ldots \rightarrow \frac{1}{5^0}, \frac{1}{5^1}, \frac{1}{5^2}, \frac{1}{5^3}, \ldots$

→ term

→ term #

$n=1 \quad n=2 \quad n=3 \quad n=4$

$$a_n = \frac{1}{5^{n-1}}$$

Def: A sequence is a function whose domain is a set of nonnegative integers (or positive integers)

$n = 0, 1, 2, 3, \ldots$ or

$n = 1, 2, 3, \ldots$

b) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$

$\frac{1}{3^0}, -\frac{1}{3^1}, \frac{1}{3^2}, -\frac{1}{3^3}, \ldots$

$$a_n = \frac{(-1)^{n-1}}{3^{n-1}} = \left(\frac{-1}{3}\right)^{n-1}$$

c) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$

$n=1 \quad n=2 \quad n=3 \quad n=4$

$$a_n = \frac{2n-1}{2n}$$

d) $\frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \frac{16}{\sqrt[5]{\pi}}, \ldots$

$$a_n = \frac{n^2}{\sqrt[n+1]{\pi}}$$

Ex. 2 Write out the first five terms of the sequence whose nth term is given below.
Determine whether the sequence converges, and if so find its limits.

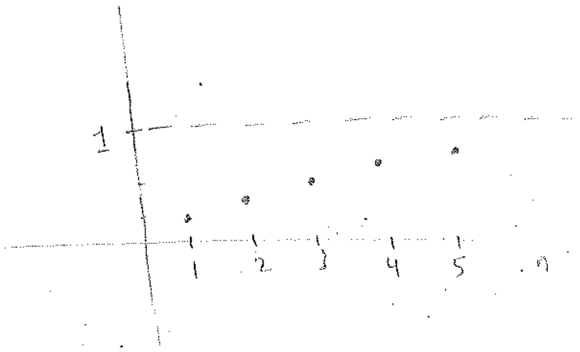
$$\boxed{\text{thm. } \lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} f(n) = L}$$

1). $a_n = \frac{n}{n+3}$

nth term of the sequence $a_n = \frac{n}{n+3} \to f(n)$

$a_1 = \frac{1}{4} \to 0.25$ $\quad a_2 = \frac{2}{5} \to 0.4$ $\quad a_3 = \frac{3}{6} = \frac{1}{2} \to 0.5$

$a_4 = \frac{4}{7} \to 0.57$ $\quad a_5 = \frac{5}{8} \to 0.63$



Converges!

$$\lim_{n\to\infty} \frac{n}{n+3} = \lim_{n\to\infty} \frac{n}{n} = \lim_{n\to\infty} 1 = 1$$

$f(x) = \frac{x}{x+3}$

$x \geq 1$

$\boxed{\text{Converges to } 1}$

$\therefore \left\{\frac{n}{n+3}\right\}_{n=1}^{\infty}$ converges

2) $a_n = \frac{\ln n}{n}$

$a_1 = \frac{\ln 1}{1} = 0$ $\quad a_2 = \frac{\ln 2}{2}$ $\quad a_3 = \frac{\ln 3}{3}$ $\quad a_4 = \frac{\ln 4}{4}$

$a_5 = \frac{\ln 5}{5}$ $\quad \lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$\therefore \left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ converges

3) $a_n = n^2 e^{-n}$

$a_1 = e^{-1}$ $a_2 = 4e^{-2}$ $a_3 = 9e^{-3}$ $a_4 = 4e^{-4}$ $a_5 = 5e^{-5}$

$$\lim_{n\to\infty} n^2 e^{-n} = \lim_{n\to\infty} \frac{n^2}{e^n}$$

$$= \lim_{x\to\infty} \frac{x^2}{e^x}$$

$$= \lim_{x\to\infty} \frac{2x}{e^x}$$

$$= \lim_{x\to\infty} \frac{2}{e^x}$$

$$= 0 \quad \text{converges}$$

Note $\boxed{\text{If } \lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0}$ } If the absolute value sequence converges to 0, the original sequence converges to 0.

4) $a_n = \frac{(-1)^{n+1}}{n^2}$

$a_1 = \frac{1}{1}$ $a_2 = -\frac{1}{4}$ $a_3 = \frac{1}{9}$ $a_4 = -\frac{1}{16}$ $a_5 = \frac{1}{25}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \lim_{n\to\infty} \frac{1}{n^2} = 0 \Rightarrow$$

$$\lim_{n\to\infty} \frac{(-1)^{n+1}}{n^2} = 0$$ & the sequence converges to 0.

5) $a_n = (-1)^n \frac{2n^3}{n^3+1}$

$$\lim_{n\to\infty} (-1)^n \frac{2n^3}{n^3+1} = \lim_{n\to\infty} (-1)^n (2) = \pm 2$$

Diverges.

6) $a_n = \frac{2+\cos n}{\sqrt{n}}$

$$-1 \le \cos n \le 1 \Rightarrow 1 \le 2+\cos n \le 3$$

$$\frac{1}{\sqrt{n}} \le \frac{2+\cos n}{\sqrt{n}} \le \frac{3}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{3}{\sqrt{n}} = 0 . \Rightarrow \lim_{n\to\infty} \frac{2+\cos n}{\sqrt{n}} = 0$$

Arithmetic Sequence

Ex. 3 show that the sequence $5, -6, -17, -28, \ldots$ is an arithmetic sequence and find its nth term a_n.

we need to show that $a_{n+1} - a_n = d$ (common difference of the sequence)

$-6 - 5 = -11$

$-17 - (-6) = -11$

$-28 - (-17) = -11$.

the nth term of an arithmetic sequence is given by

$a_n = a_1 + (n-1)d$ $\quad a_1 = 5 \quad d = -11$

$a_n = 5 + (n-1)(-11) \Rightarrow a_n = 5 - 11n + 11$

$$\boxed{a_n = -11n + 16}$$

Geometric Sequence

Ex.4 Show that the sequence $-2, 6, -18, 54, \ldots$ is a geometric sequence and find its nth term.

We need to show that $\frac{a_{n+1}}{a_n} = r$ (common ratio of the sequence)

$\frac{6}{-2} = -3 \qquad \frac{-18}{6} = -3 \qquad \frac{54}{-18} = -3$

$a_n = a_1 r^{n-1} \Rightarrow a_n = -2(-3)^{n-1}$

Monotone Sequences

Note: A monotone sequence is a sequence that is either increasing or decreasing. A strictly monotone sequence is a sequence that is either strictly increasing or strictly decreasing.

Ex.5

a) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2}, \ldots$ strictly decreasing

b) $1, 1, \frac{1}{4}, \frac{1}{4}, \frac{1}{9}, \frac{1}{9}, \ldots$ decreasing

c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ strictly increasing

d) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$ Neither increasing or decreasing

Note: • To show that $\{a_n\}$ is strictly increasing we verify that $a_{n+1} - a_n > 0$ or $\frac{a_{n+1}}{a_n} > 1$

• To show that $\{a_n\}$ is strictly decreasing we verify that $a_{n+1} - a_n < 0$ or $\frac{a_{n+1}}{a_n} < 1$

Ex. Show that the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$

is strictly increasing sequence.

1st way $a_{n+1} - a_n > 0$

$$\frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}$$

$$= \frac{n^2 + 2n + 1 - \cancel{n^2} - \cancel{2n}}{(n+2)(n+1)}$$

$$= \frac{1}{(n+2)(n+1)} > 0 \quad n \geq 1$$

2nd way let $f(x) = \frac{x}{x+1}$ $x \geq 1$ and $a_n = f(n)$

$$f'(x) = \frac{(1)(x+1) - (1)(x)}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2} > 0 \quad x \geq 1 \Rightarrow a_n \text{ strictly increasing.}$$

Series

Geometric Series.

A geometric series $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots + ar^k + \dots$ $(a \neq 0)$

converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series

converges, then the sum is $\sum_{k=0} ar^k = \frac{a}{1-r}$

Ex. Determine whether each series converges, and if so find its sum.

$$\sum_{k=0}^{\infty} ar^k$$

a) $\displaystyle\sum_{k=0}^{\infty} \frac{5}{4^k}$

$\displaystyle\sum_{k=0}^{\infty} 5\left(\frac{1}{4}\right)^k \qquad r = \frac{1}{4} \qquad \left|\frac{1}{4}\right| < 1 \Rightarrow$ converge.

$$\sum_{k=0}^{\infty} 5\left(\frac{1}{4}\right)^k = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$$

b) $\displaystyle\sum_{k=0}^{\infty} \frac{2^{k+3}}{3^k} = \sum_{k=0}^{\infty} 2^3 \cdot \frac{2^k}{3^k}$

$$= \sum_{k=0}^{\infty} 8\left(\frac{2}{3}\right)^k$$

$r = \frac{2}{3} \qquad \left|\frac{2}{3}\right| < r \Rightarrow$ converge.

$$\sum_{k=0}^{\infty} 8\left(\frac{2}{3}\right)^k = \frac{8}{1-2/3} = 24$$

c) $\sum_{n=1}^{\infty} 5 \cdot 7^{1-n}$

$\sum_{n=1}^{\infty} 5 \cdot 7^{-(n-1)} = \sum_{n=1}^{\infty} 5 \cdot \left(\frac{1}{7}\right)^{n-1}$

$\left|\frac{1}{7}\right| < 1 \Rightarrow$ converge.

$\sum_{n=1}^{\infty} 5\left(\frac{1}{7}\right)^{n-1} = \frac{5}{1-\frac{1}{7}}$

$= \frac{5}{6/7}$

$= 35/6$

OR

$\sum_{n=1}^{\infty} 5 \cdot 7^1 \cdot 7^{-n}$

$\sum_{n=1}^{\infty} 35\left(\frac{1}{7}\right)^n$

converges

$S = \frac{35/7}{1-1/7} \quad \frac{35}{6}$

d) $\sum_{n=1}^{\infty} 3\left(\frac{11}{8}\right)^{n+1}$

$$\sum_{n=1}^{\infty} 3\left(\frac{11}{8}\right)^{\boxed{n-1}+1+1} = \sum_{n=1}^{\infty} 3\left(\frac{11}{8}\right)^{n-1} \cdot \left(\frac{11}{8}\right)^{2}$$

OR $\boxed{\sum_{n=1}^{\infty} 3\left(\frac{11}{8}\right)\left(\frac{11}{8}\right)^{n}}$ $= \sum_{n=1}^{\infty} 3\left(\frac{11}{8}\right)^{2}\left(\frac{11}{8}\right)^{n-1}$

e) $\sum_{k=1}^{\infty} 3^{2k} 5^{1-k}$ $\qquad a(r)^{k-1}$

$$\frac{3^{2k}}{5^{k-1}} = \frac{9^k}{5^{k-1}}$$

$$= 9 \cdot \frac{9^k \cdot 9^{-1}}{5^{k-1}}$$

$$= 9\left(\frac{9}{5}\right)^{k-1}$$

$$\sum_{k=1}^{\infty} 9\left(\frac{9}{5}\right)^{k-1}$$

$\left|\frac{9}{5}\right| > 1$ Diverges

and No

sum.

OR

$$\sum_{k=1}^{\infty} 9^k \cdot 5^{-k} \cdot 5$$

$$= \sum_{k=1}^{\infty} 5\left(\frac{9}{5}\right)^{k}$$

c) $\sum_{k=0}^{\infty} 5 \cdot 7^{1-k} = \sum_{k=0}^{\infty} 5 \cdot 7^{1} \cdot 7^{-k}$

$= \sum_{k=0}^{\infty} 35 \left(\frac{1}{7}\right)^{k} \qquad |r| = \left|\frac{1}{7}\right| < 1$

$= \dfrac{35}{1-\frac{1}{7}} = \dfrac{35 \times 7}{6}$

$= \dfrac{245}{6}$

d) $\sum_{k=0}^{\infty} 3^{2k} \cdot 5^{1-k} = \sum_{k=0}^{\infty} 9^{k} \cdot 5^{1} \cdot 5^{-k}$

$= \sum_{k=0}^{\infty} 5 \left(\frac{9}{5}\right)^{k} \qquad \left|\frac{9}{5}\right| \geq 1$

Diverges.

Telescoping Sum

consider $\sum_{k=1}^{\infty} a_k$ and let $S_n = \sum_{k=1}^{n} a_k$ (series of partial sums)

We have $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k$.

If $\lim_{n\to\infty} S_n = L$ (real number), then $\sum_{k=1}^{\infty} a_k = L$ (converges)

If $\lim_{n\to\infty} S_n$ does not exist, then $\sum_{k=1}^{\infty} a_k$ diverges.

Ex. Find the sum of the series.

$$\sum_{k=1}^{\infty} \frac{1}{k^2+7k+12}$$

$$\frac{1}{k^2+7k+12} = \frac{1}{(k+3)(k+4)}$$

$$= \frac{A}{k+3} + \frac{B}{k+4}$$

$$A(k+4) + B(k+3) = 1$$

$$(A+B)k + 4A + 3B = 1$$

$A + B = 0$ and $4A + 3B = 1$

$B = -A$ $\qquad 4A - 3A = 1$

$$A = 1 \Rightarrow B = -1$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{k+3} - \frac{1}{k+4} \right)$$

Let $S_n = \sum_{k=1}^{n} \left(\frac{1}{k+3} - \frac{1}{k+4} \right)$

$$= \left(\frac{1}{4} - \cancel{\frac{1}{5}} \right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} \right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{7}} \right)$$

$$\cancel{+ \cdots +} \left(\frac{1}{n+2} - \cancel{\frac{1}{n+3}} \right) + \left(\cancel{\frac{1}{n+3}} - \frac{1}{n+4} \right)$$

$$= \frac{1}{4} - \frac{1}{n+4}$$

$$\lim_{n \to \infty} S_n = \sum_{k=1}^{\infty} \left(\frac{1}{k+3} - \frac{1}{k+4} \right) = \frac{1}{4}$$

Hence, $\sum_{k=1}^{\infty} \frac{1}{k^2+7k+12} = \frac{1}{4}$ converges

$\underline{Ex.}$ $\sum_{k=1}^{\infty}\left(\frac{1}{2^k}-\frac{1}{2^{k+1}}\right)$

$$S_n=\sum_{k=1}^{n}\left(\frac{1}{2^k}-\frac{1}{2^{k+1}}\right)$$

$$=\left(\frac{1}{2^1}-\cancel{\frac{1}{2^2}}\right)+\left(\cancel{\frac{1}{2^2}}-\cancel{\frac{1}{2^3}}\right)+\left(\cancel{\frac{1}{2^3}}-\cancel{\frac{1}{2^4}}\right)$$

$$+\cdots+\left(\cancel{\frac{1}{2^{n-1}}}-\cancel{\frac{1}{2^n}}\right)+\left(\cancel{\frac{1}{2^n}}-\frac{1}{2^{n+1}}\right)$$

$$=\frac{1}{2}-\frac{1}{2^{n+1}}$$

$$\lim_{n\to\infty}S_n=\frac{1}{2}\implies\sum_{k=1}^{\infty}\left(\frac{1}{2^k}-\frac{1}{2^{k+1}}\right)=\frac{1}{2}.$$

3

3 $\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)$

$$S_n = \sum_{k=1}^{n} \ln\left(\frac{k}{k+1}\right)$$

$$= \sum_{k=1}^{n} \left[\ln k - \ln(k+1)\right]$$

$$= \left[\ln 1 - \cancel{\ln 2}\right] + \left[\cancel{\ln 2} - \cancel{\ln 3}\right] + \left[\cancel{\ln 3} - \cancel{\ln 4}\right] +$$

$$\left[\cancel{\ln 4} - \cancel{\ln 5}\right] + \left[\cancel{\ln 5} - \cancel{\ln 6}\right] + \left[\cancel{\ln 6} - \cancel{\ln 7}\right] +$$

$$\cancel{\ldots} \cancel{+} \left[\cancel{\ln n} - \ln(n+1)\right]$$

$$= \ln 1 - \ln(n+1)$$

$$= -\ln(n+1)$$

$\lim_{n\to\infty} S_n = -\infty$ DNE $\Rightarrow \sum_{k=1}^{\infty} \ln\left(\frac{k}{k+2}\right)$ Diverge.

More Practice

Ex. Give an explicit formula for the n^{th} term a_n of the sequence

a) $1, \frac{4}{5}, \frac{6}{8}, \frac{8}{11}, \frac{10}{14}, \frac{12}{17}, \ldots$

b) $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$

Ans. a) $a_n = \frac{2n}{3n-1}$

b) $a_n = \frac{(-1)^n}{3^n}$

Ex. Consider the sequence given by

$$a_1 = 1, \quad a_2 = 4, \quad a_{n+1} = 2a_n - a_{n-1}$$

a) Write the first six terms of the sequence

b) Find a formula for the general term a_n.

Ans. a) $a_1 = 1$, $a_2 = 4$, $a_3 = 7$, $a_4 = 10$, $a_5 = 13$ and $a_6 = 16$

$1, 4, 7, 10, 13, 16, \ldots$

b) $a_n = a_1 + (n-1)d$ $\qquad d = 4 - 1 = 3 \qquad a_1 = 1$

$= 1 + (n-1)(3)$

$= 3n - 3 + 1$

$$\boxed{a_n = 3n - 2}$$

Ex. State whether or not the sequence converges as $n \to \infty$, if it does, find the limit.

a) $a_n = \dfrac{\ln n}{n^2}$

$$\lim_{n \to \infty} \frac{\ln n}{n^2} = \frac{\infty}{\infty}$$

$$\text{L'Hop.} \Rightarrow \lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{1/x}{2x}$$

$$= \lim_{x \to \infty} \frac{1}{2x^2}$$

$= 0 \Rightarrow$ the sequence converges to 0.

b) $b_n = \dfrac{1 - n^2}{3n^2 + 9n + 7}$

$$\lim_{n \to \infty} \frac{1 - n^2}{3n^2 + 9n + 7} = \lim_{n \to \infty} \frac{-n^2}{3n^2}$$

$$= \lim_{n \to \infty} \left(-\frac{1}{3}\right)$$

$= -\frac{1}{3} \Rightarrow$ the sequence converge.

c) $c_n = \ln\left(\dfrac{2n}{4n+1}\right)$

$$\lim_{n \to \infty} \ln\left(\frac{2n}{4n+1}\right) = \ln\left(\lim_{n \to \infty} \frac{2n}{4n+1}\right)$$

$$= \ln\left(\frac{2}{4}\right)$$

$= \ln\left(\frac{1}{2}\right) = -\ln 2 \Rightarrow$ converges

d) $d_n = n\ln\left(1+\frac{1}{n}\right)$

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \infty \cdot \ln(1)$$
$$= \infty \cdot 0 \quad \text{Indeterminate Form}$$

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \ln e^1$$
$$= 1 \Rightarrow \text{converges.}$$

Recall $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$ where k = constant.

e) $e_n = \left(1+\frac{5}{n}\right)^{-3n}$

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^{-3n} = \lim_{n\to\infty} \frac{1}{\left[\left(1+\frac{5}{n}\right)^n\right]^3}$$

$$= \frac{1}{\left[\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n\right]^3}$$

$$= \frac{1}{(e^5)^3}$$

$$= \frac{1}{e^{15}} \Rightarrow \text{converges.}$$

Divergence Test

Consider the series $\sum a_n$.

a) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges

b) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ may either converge or diverge.

Ex. Determine the convergence and divergence of the following series

a) $\sum_{k=1}^{\infty} \frac{k}{k+1}$ $\qquad \lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0 \Rightarrow \sum_{k=1}^{\infty} \frac{k}{k+1}$ diverges.

b) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^k$ $\qquad \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = 1^{\infty}$ Ind. form

$$\boxed{\text{Note } \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k \quad k = \text{constant}}$$

$(\div k) \Rightarrow \lim_{k\to\infty}\left(\frac{1}{1+\frac{1}{k}}\right)^k = \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^k} = \frac{1}{e}$

$\neq 0 \Rightarrow$ Diverges.

c) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}$ $\qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{1}{2}}$

$= e^{\frac{1}{2}}$

$\neq 0 \Rightarrow$ Diverges.

d) $\sum_{k=1}^{\infty} \frac{k}{e^k}$ $\lim_{k \to \infty} \frac{k}{e^k} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x}$

$= 0 \Rightarrow$ The series may either converge or diverge.

P- Series Test

Consider $\sum_{k=1}^{\infty} \frac{1}{k^p}$

a) $p > 1 \Rightarrow$ The series converges

b) $0 \leq p \leq 1 \Rightarrow$ The series diverges

Ex. Determine whether the following series converge or diverge.

1) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ $p = \frac{1}{2} < 1$

diverges

2) $\sum_{k=1}^{\infty} 2k^{-\frac{5}{3}} = 2\sum \frac{1}{k^{5/3}}$ $p = \frac{5}{3} > 1$ converges

3) $\sum_{k=1}^{\infty} \frac{1}{k}$ $p = 1 \leq 1$ diverges

gral Test

If f is positive, continuous, and decreasing for $x \geq a$ and $a_n = f(n)$ $n \geq a$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_a^{\infty} f(x)\,dx \quad \text{either both converge or}$$

diverge.

Ex. Show that the integral test applies, and use the integral test to determine whether the following series converge or diverge.

a) $$\sum_{n=1}^{\infty} n e^{-n^2}$$

let $f(x) = x e^{-x^2}$ $\quad x \geq 1$

1) $f(x) = x e^{-x^2} > 0$

2) $f'(x) = (1) e^{-x^2} + (-2x) x e^{-x^2}$

$= (1 - 2x^2) e^{-x^2} \quad (e^{-x^2} > 0)$

$1 - 2x^2 = 0 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$

$- \quad -\frac{\sqrt{2}}{2} \quad + \quad \frac{\sqrt{2}}{2} \quad -$

for $x \geq 1$ $\quad f'(x) < 0 \Rightarrow f$ is decreasing

3) $f(x) = x e^{-x^2}$ is cont. $\quad x \geq 1$

$$\int_1^\infty x e^{-x^2} dx = -\frac{1}{2}\int_1^\infty -2x e^{-x^2} dx$$

OR $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

$$= -\frac{1}{2}\left[e^{-x^2} \right]_1^\infty$$

$$= -\frac{1}{2}\left(0 - e^{-1}\right)$$

$$= +\frac{1}{2e}$$ converges $\Rightarrow$ The series converges.

b) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2-4n+5} = \underset{n=1}{\frac{1}{2}} + \underset{n=2}{\frac{1}{1}} + \underset{n=3}{\frac{1}{2}} + \underset{n=4}{\frac{1}{5}} + \dots$

$f(x) = \dfrac{1}{x^2-4x+5}$ $\quad x \geq 2$

1) $f(x) > 0$

2) $f'(x) = \dfrac{0(\quad) - (2x-4)}{(x^2-4x+5)^2} = \dfrac{-2x+4}{(x^2-4x+5)^2}$

$-2x+4 = 0 \Rightarrow x = \frac{-4}{-2} = 2$

$$\underset{+}{\quad\quad}\overset{2}{|}\underset{-}{\quad\quad}$$

for $x \geq 2 \Rightarrow f'(x) < 0 \Rightarrow f$ is decreasing

3) $x^2 - 4x + 5 \neq 0 \Rightarrow f(x)$ is cont $x \geq 2$

$\boxed{3}$

$$\int_2^{\infty} \frac{1}{x^2-4x+5}\,dx = \int_2^{\infty} \frac{1}{x^2-4x+4-4+5}\,dx$$

$$= \int_2^{\infty} \frac{1}{(x-2)^2+1}\,dx$$

let $u = x-2$
$du = dx$

$$\int \frac{1}{u^2+1}\,du = \tan^{-1} u$$
$$= \tan^{-1}(x-2)$$

$$= \left[\tan^{-1}(x-2)\right]_2^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1} 0$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2} \Rightarrow \text{converges}$$

c) $$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} = \underset{n=1}{0} + \underset{n=2}{\frac{\ln 2}{4}} + \underset{n=3}{\frac{\ln 3}{9}} + \underset{n=4}{\frac{\ln 4}{16}} + \cdots$$

$f(x) = \dfrac{\ln x}{x^2} \quad x \geq 2$

1) $f(x) > 0$

2) $$f'(x) = \frac{1/x \cdot x^2 - 2x\ln x}{x^4} = \frac{x - 2x\ln x}{x^4}$$
$$= \frac{x(1-2\ln x)}{x^3}$$
$$= \frac{1-2\ln x}{x^3}$$

$1 - 2\ln x = 0 \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{\frac{1}{2}}$ $\quad$ $+ \quad e^{\frac{1}{2}} \approx 1.65$

$f'(x) < 0 \Rightarrow f(x)$ is decreasing for $x \geq 2$

3) $f(x)$ cont.

$$\int_2^\infty \frac{\ln x}{x^2}\,dx = \int_2^\infty x^{-2}\ln x\,dx$$

D		I
$\ln x$	$+$	x^{-2}
$\frac{1}{x}$	$-$	$\frac{x^{-1}}{-1}$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int x^{-2}\,dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x}$$

$$= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^\infty$$

$\frac{\frac{1}{x}}{1}$

$$= (0-0) - \left(-\frac{\ln 2}{2} - \frac{1}{2}\right)$$

$$= \frac{\ln 2}{2} + \frac{1}{2} \qquad \text{conv.}$$

(Direct) Comparison Test

Consider $\sum a_n$ and $\sum b_n$

Suppose $a_n \le b_n$.

1) $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

2) $\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

Informal principle
p.632
- Discard constants in the denominator
- Consider leading terms in polynomials

Ex. Use the comparison test to determine whether the following series converge or diverge

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-\frac{1}{2}}$

$$\sqrt{k} - \tfrac{1}{2} \le \sqrt{k} \qquad k \ge 1$$

$$\frac{1}{\sqrt{k}-\frac{1}{2}} \ge \frac{1}{\sqrt{k}}$$

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ Diverges by the p-series test $\Rightarrow$ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-\frac{1}{2}}$ Diverges

OR $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-\frac{1}{2}} \sim \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ Diverges

$$\sum_{n=1}^{\infty} \frac{1}{2^n+n}$$

$2^n+n \geq 2^n \quad n \geq 1$

$$\frac{1}{2^n+n} \leq \frac{1}{2^n}$$

$\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ converges G.S

$\Rightarrow \sum \frac{1}{2^n+n}$ converges.

b) $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$

$$2k^2+k \geq 2k^2$$

$$\frac{1}{2k^2+k} \leq \frac{1}{2k^2}$$

$\sum \frac{1}{2k^2} = \frac{1}{2}\sum \frac{1}{k^2}$ converges by the p-series test

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2k^2+k}$ converges.

---

c) $\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{4n^2+5} \sim \sum_{n=1}^{\infty} \frac{3\sqrt{n}}{4n^2} = \frac{3}{4}\sum_{n=1}^{\infty} n^{-\frac{3}{2}}$

$$= \frac{3}{4}\sum \frac{1}{n^{3/2}}$$

converges

d) $\sum_{k=1}^{\infty} \frac{k}{k^3+1} \sim \sum_{k=1}^{\infty} \frac{k}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

e) $\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \sim \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

f) $\sum_{k=1}^{\infty} \frac{\ln k}{k^3}$

$\boxed{\text{Note } \quad x > \ln x.}$

$k > \ln k \Rightarrow \frac{k}{k^3} \geq \frac{\ln k}{k^3}$

$\frac{1}{k^2} \geq \frac{\ln k}{k^3}$ or $\frac{\ln k}{k^3} \leq \frac{1}{k^2}$

$\sum \frac{1}{k^2}$ converges $\Rightarrow \sum \frac{\ln k}{k^3}$ converges

g) $\sum_{k=1}^{\infty} \frac{1}{1+2\ln k} \sim \sum_{k=1}^{\infty} \frac{1}{2\ln k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{\ln k}$

$k > \ln k \Rightarrow \frac{1}{k} < \frac{1}{\ln k}$ (or $\frac{1}{\ln k} > \frac{1}{k}$)

$\sum \frac{1}{k}$ diverges $\Rightarrow \sum_{k=1}^{\infty} \frac{1}{\ln k}$ diverges

h) $\sum \frac{2+\sin k}{k^2}$

$-1 \leq \sin k \leq 1$

$1 \leq 2+\sin k \leq 3$

$\frac{1}{k^2} \leq \frac{2+\sin k}{k^2} \leq \frac{3}{k^2}$

$\sum \frac{3}{k^2}$ converges $\Rightarrow \sum \frac{2+\sin k}{k^2}$ converges

i) $\sum \frac{2+\cos k}{\sqrt{k+1}}$

$$\sum \frac{2+\cos k}{\sqrt{k+1}} \sim \sum \frac{2+\cos k}{\sqrt{k}}$$

$-1 \le \cos k \le 1 \Rightarrow 1 \le 2+\cos k \le 3$

$$\frac{1}{\sqrt{k}} \le \frac{2+\cos k}{\sqrt{k}} \le \frac{3}{\sqrt{k}} \Rightarrow \frac{2+\cos k}{\sqrt{k}} \geq \frac{1}{\sqrt{k}}$$

$$\sum \frac{1}{\sqrt{k}} = \sum \frac{1}{k^{\frac{1}{2}}} \text{ diverges} \Rightarrow$$

$$\sum \frac{2+\cos k}{\sqrt{k}} \text{ diverges.}$$

Limit Comparison Test

let $\sum a_n$ and $\sum b_n$ be two series with positive terms. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ = finite and positive, then the series both converge or both diverge.

Ex. Use the limit comparison test to determine whether the series converge or diverge. a) $\sum \frac{3k+4}{2k^3}$

b) $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$

Consider $\sum \frac{1}{k^2}$ converges

$$\lim_{k\to\infty} \frac{\frac{3k+4}{2k^3}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{3k^3}{2k^3} = \frac{3}{2}$$

$\therefore \sum \frac{3k+4}{2k^3}$ converges

Consider the series $\sum \frac{1}{n}$ diverges.

$$\lim_{n \to \infty} \frac{\sqrt[n]{2} - 1}{\frac{1}{n}} = \lim_{n \to \infty} \frac{2^{\frac{1}{n}} - 1}{\frac{1}{n}}$$

$$= \frac{0}{0}$$

$$\lim_{x \to \infty} \frac{2^{\frac{1}{x}} - 1}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\cancel{-\frac{1}{x^2}} 2^{\frac{1}{x}}(\ln 2)}{\cancel{-\frac{1}{x^2}}} = \ln 2$$

positive and finite

$$\Rightarrow \sum \sqrt[n]{2} - 1 \quad \text{diverges}$$

<u>Alternating Series Test</u>

Let $a_n > 0$. The alternating series

$\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if:

1. $a_{n+1} \leq a_n$ $\left[\text{or } \frac{a_{n+1}}{a_n} \leq 1\right]$

2. $\lim_{n \to \infty} a_n = 0$

<u>Ex.</u> Use the alternating series test to show that the following series converge.

a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+2}{n^2+n}$

$$a_n = \frac{n+2}{n^2+n} = \frac{n+2}{n(n+1)}$$

$$\frac{a_{n+1}}{a_n} = \frac{n+3}{(n+1)(n+2)} \cdot \frac{n(n+1)}{n+2} = \frac{n^2+3n}{(n+2)^2}$$

$$= \frac{n^2+3n}{n^2+4n+4}$$

$$= \frac{n^2+3n}{(n^2+3n)+(n+4)}$$

$$< 1$$

$$\lim_{n\to\infty} \frac{n+2}{n^2+n} = \lim_{n\to\infty} \frac{n}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$= 0$$

$\therefore$ The series converges

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+4}$

let $f(x) = \dfrac{x}{x^3+4}$ $x \geq 1$ let $a_n = f(n)$ $n \geq 1$

$$f'(x) = \frac{(1)(x^3+4) - (3x^2)(x)}{(x^3+4)^2} = \frac{x^3+4-3x^3}{(x^3+4)^2}$$

$$= \frac{-2x^3+4}{(x^3+4)^2} < 0 \quad \forall x \geq 1$$

$\{a_n\}$ is decreasing $\Rightarrow$ $a_{n+1} < a_n$ $\forall n \geq 1$

$$\lim_{x\to\infty} \frac{x}{x^3+4} = \lim_{x\to\infty} \frac{1}{x^2} = 0$$

$\therefore$ converges.

Alternating Series Remainder (error)

If $\sum_{n=1}^{\infty} (-1)^n a_n$ is a convergent alternating series then

$\boxed{|R_n| = |S - S_n| \leq a_{n+1}}$ → The absolute value of the remainder R_n involved in approximating the sum S by S_n is less than or equal to the first neglected term.

Ex. Approximate the sum of the following series by its first 5 terms.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n}$$

1) $$\frac{a_{n+1}}{a_n} = \frac{1}{3^{n+1}} \cdot 3^n$$

$$= \frac{1}{\cancel{3^n} \cdot 3} \cdot \cancel{3^n}$$

$$= \frac{1}{3} < 1 \quad \text{decreasing}$$

2) $$\lim_{n \to \infty} \frac{1}{3^n} = 0$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n} \quad \text{Convergent alternating series.}$$

$$S_5 = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} = \frac{61}{243} \approx 0.25103$$

$$a_6 = \frac{1}{3^6} = \frac{1}{729} \approx 0.00137$$

$$|S - 0.25103| \leq 0.00137$$

$$-0.00137 \leq S - 0.25103 \leq 0.00137$$

$$0.24966 \leq S \leq 0.2524$$

Remainder Estimate for the Integral Test

Suppose $a_n = f(n)$ satisfies the conditions of the Integral Test and $\sum_{n=1}^{\infty} a_n$ converges.

If $R_n = S - S_n$, then $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$

Ex. Use the sum of the first ten terms to approximate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$f(x) = \frac{1}{x^2}$

1) $f(x) > 0$

2) $f(x)$ cont.

3) $f'(x) = \frac{-2}{x^3} < 0$

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{10^2} \approx 1.549768$$

$$\int_{11}^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{11}^{\infty} = +\frac{1}{11}$$

$$\int_{10}^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{10}^{\infty} = \frac{1}{10}$$

$$\frac{1}{11} \le S - S_{10} \le \frac{1}{10}$$

$$\frac{1}{11} + S_{10} \le S \le \frac{1}{10} + S_{10}$$

$$1.640677 \le S \le 1.649768$$

Ex. Determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001.

a) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{8n^2+1}$

this is a convergent alternating series.

$a_n = \frac{1}{8n^2+1}$

Error $|S - S_n| \leq a_{n+1}$

Take $|S - S_n| = a_{n+1}$

We have $a_{n+1} \leq 0.001$

$$\frac{1}{8(n+1)^2+1} \leq 0.001$$

$$8(n+1)^2+1 \geq 1000$$

$$8(n+1)^2 \geq 999$$

$$(n+1)^2 \geq 124.875 \longrightarrow n+1 \geq 11.1747$$

$$n \geq 10.1747$$

Hence $\boxed{n \geq 11}$

b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$\sum \frac{1}{n^3}$ converges by the integral test

The remainder estimate $\Rightarrow R_n \leq \int_n^{\infty} f(x)\,dx$

Take $R_n = \int_n^{\infty} f(x)\,dx = \int_n^{\infty} \frac{1}{x^3}\,dx \leq 0.001$

$R_n = \left[\frac{-1}{2x^2} \right]_n^{\infty} \leq 0.001 \Rightarrow 0 + \frac{1}{2n^2} \leq 0.001$

$2n^2 \geq 1000$

$n^2 \geq 500 \rightarrow n \geq 22.36$

Hence $n \geq 23$

/

Absolute and Conditional Convergence

1) $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

2) $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Ex. Determine whether the series converges absolutely or converges conditionally.

a) $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$$

$$\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{k+3}{k(k+1)} \right| = \sum_{k=1}^{\infty} \frac{k+3}{k(k+1)}$$

$$\sim \sum \frac{1}{k} \quad \text{diverges}$$

and $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$ converges by the alternating series test. [why? $a_k = \frac{k+3}{k(k+1)} > 0$ $\lim_{k\to\infty} \frac{k+3}{k(k+1)} = 0$ and
$\frac{4}{2} + \frac{5}{6} + \frac{6}{12} + \cdots$ decreasing]

Hence $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$ converges conditionally.

b) $\sum \frac{(-1)^{\frac{n(n+1)}{2}}}{2^n} = -\frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - $ [not an alternating series] $\boxed{21}$

$$\sum \left| \frac{(-1)^{\frac{n(n+1)}{2}}}{2^n} \right| = \sum \frac{1}{2^n}$$

$$= \sum \left(\frac{1}{2}\right)^n \quad \left[\text{geometric series } r = \tfrac{1}{2} < 1\right]$$

converges.

$\Rightarrow \sum \frac{(-1)^{\frac{n(n+1)}{2}}}{2^n}$ converges absolutely [hence converges]

c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} = \frac{1}{2\ln 2} - \frac{1}{3\ln 3} + \frac{1}{4\ln 4} - \frac{1}{5\ln 5} + \cdots$

$= 0.72 - 0.30 + 0.18 - 0.12 + \cdots$

Note $\sum \left| \frac{(-1)^n}{n \ln n} \right| = \sum \frac{1}{n \ln n}$

$n > \ln n \Rightarrow n^2 > n \ln n \quad (n \geq 2)$

$\frac{1}{n^2} < \frac{1}{n \ln n}$

No conclusion!

$\sum \left| \frac{(-1)^n}{n \ln n} \right| = \sum \frac{1}{n \ln n}$ diverges By the Integral Test

$n \ln n \quad \frac{1}{n \ln 2}$

$\sum \frac{(-1)^n}{n \ln n}$ converges by the alternating series test. Why?

1) $a_n = \frac{1}{n \ln n} > 0$

2) decreasing

3) $\lim_{n \to \infty} \frac{1}{n \ln n} = 0$

$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges conditionally.

Ratio Test (Try this test when a_n involves factorials or n^{th} power)

Consider $\sum a_n$ and Let $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

1) $L < 1 \Rightarrow \sum a_n$ converges absolutely (converges)

2) $L > 1$ or $L = \infty \Rightarrow \sum a_n$ diverges

3) $L = 1$ test fails

Ex. Determine whether the following series converge or diverge.

a) $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{|a_{k+1}|}{|a_k|} = \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k}$$

$$= \frac{\cancel{2^k} \cdot 2}{\cancel{k!}\,(k+1)} \cdot \frac{\cancel{k!}}{\cancel{2^k}}$$

$$= \frac{2}{k+1}$$

$\lim_{k\to\infty} \frac{2}{k+1} = 0 < 1 \Rightarrow$ The series converges.

b) $$\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{3^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{[2(n+1)-1]!}{3^{n+1}} \cdot \frac{3^n}{(2n-1)!}$$

$$= \frac{(2n+1)!}{\cancel{3^n}\cdot 3} \cdot \frac{\cancel{3^n}}{(2n-1)!}$$

$$= \frac{\cancel{(2n-1)!}(2n)(2n+1)}{3\cancel{(2n-1)!}}$$

$$= \frac{4n^2+2n}{3}$$

$\lim\limits_{n\to\infty} \frac{4n^2+2n}{3} = \infty \Rightarrow$ the series diverges.

Note $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \cdot n$

$$= \frac{n}{n+1} \qquad \lim_{n\to\infty} \frac{n}{n+1} = 1 \quad \text{inconclusive}$$

yet we know that $\sum \frac{1}{n}$ diverges.

c) $$\sum_{k=0}^{\infty} \frac{(-1)^k k!}{e^k} \qquad \left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)!}{e^{k+1}} \cdot \frac{e^k}{k!}$$

$$= \frac{k!(k+1)}{e^k \cdot e} \cdot \frac{\cancel{e^k}}{\cancel{k!}}$$

$$= \frac{k+1}{e}$$

$\lim\limits_{k\to\infty} \frac{k+1}{e} = \infty$ diverges

Root Test (Try this test when a_n involves n^{th} power)

Consider $\sum_{n=1}^{\infty} a_n$ Let $\lim_{n \to \infty} |a_n|^{\frac{1}{n}} = L$

1) $L < 1 \Rightarrow \sum a_n$ converges

2) $L > 1$, $L = \infty \Rightarrow \sum a_n$ diverges

3) $L = 1$ the test fails

Ex. Determine the convergence and divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{12^n}{n^n}$

$$|a_n|^{\frac{1}{n}} = \left(\frac{12^n}{n^n}\right)^{\frac{1}{n}} = \frac{12}{n}$$

$$\lim_{n \to \infty} \frac{12}{n} = 0 < 1 \Rightarrow \text{The series converges.}$$

b) $\sum_{n=2}^{\infty} \left(\frac{4n-5}{2n+1}\right)^n$

$$|a_n|^{\frac{1}{n}} = \left|\left(\frac{4n-5}{2n+1}\right)^n\right|^{\frac{1}{n}}$$

$$= \left[\left(\frac{4n-5}{2n+1}\right)^n\right]^{\frac{1}{n}}$$

$$= \frac{4n-5}{2n+1}$$

$$\lim_{n \to \infty} \frac{4n-5}{2n+1} = 2 > 1 \Rightarrow \text{Diverges}$$

c) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

Representation of Functions by Power Series

Recall $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, $|r|<1$ $\left\{ \frac{1}{1-\square} = \sum_{n=0}^{\infty} \square^n \quad |\square|<1 \right.$

Ex. Find a power series for each of the following.

a) $f(x) = \frac{9}{x+3}$

$$\frac{9}{x+3} = \frac{3}{\frac{x}{3}+1} = \frac{3}{1-\left(-\frac{x}{3}\right)}$$

$$= \sum 3\left(-\frac{x}{3}\right)^n \qquad \left|-\frac{x}{3}\right|<1$$

$$|x|<3$$

b) $f(x) = \frac{x}{3+x^2}$

$$\frac{x}{3+x^2} = \frac{\frac{x}{3}}{1+\frac{x^2}{3}} = \frac{\frac{x}{3}}{1-\left(-\frac{x^2}{3}\right)}$$

$$= \sum \frac{x}{3}\left(-\frac{x^2}{3}\right)^n \qquad \left|-\frac{x^2}{3}\right|<1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{n+1}} \qquad |x^2|<3$$

$$x^2<3$$

Interval and Radius of Convergence of a Power Series

* Use the Ratio test $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ and check the endpoints of the interval

Ex. Find the radius and interval of convergence of the following series.

a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)}$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{3^{n+1} (n+2)} \cdot \frac{3^n (n+1)}{x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{\cancel{x^n} \cdot x}{\cancel{3^n} \cdot 3 (n+2)} \cdot \frac{3^n (n+1)}{\cancel{x^n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x (n+1)}{3 (n+2)} \right|$$

$$= |x| \lim_{n \to \infty} \left| \frac{n+1}{3(n+2)} \right| = |x| \cdot \frac{1}{3}$$

$$\frac{|x|}{3} < 1 \Rightarrow |x| < 3 \Rightarrow -3 < x < 3.$$

$\underline{x = -3}$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n (-3)^n}{3^n (n+1)} = \sum \frac{(-1)^n (-1)^n 3^n}{3^n (n+1)} = \sum \frac{3^n}{3^n (n+1)}$

$$= \sum \frac{1}{n+1}$$

Diverges

$\underline{x = 3}$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{3^n}}{\cancel{3^n} (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Convergent alternating series.

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Interval of convergence $(-3, 3]$

Radius of convergence $R = \frac{3-(-3)}{2} = 3$.

b) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| = \lim_{k\to\infty} \left| \frac{x^k \cdot x}{k!(k+1)} \cdot \frac{k!}{x^k} \right|$$

$$= |x| \lim_{k\to\infty} \left| \frac{1}{k+1} \right|$$

$$= |x| \cdot 0$$

$|x| \cdot 0 < 1$ true $\forall x$

Interval of convergence $(-\infty, \infty)$

$R = \infty$

c) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\ln(n+4)}$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{\ln(n+5)} \cdot \frac{\ln(n+4)}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(x-2)^n}(x-2)}{\ln(n+5)} \cdot \frac{\ln(n+4)}{\cancel{(x-2)^n}} \right|$$

$$= |x-2| \lim_{n\to\infty} \left| \frac{\ln(n+4)}{\ln(n+5} \right|$$

$$= |x-2| \cdot 1$$

$|x-2| \cdot 1 < 1 \quad \Rightarrow \quad -1 < x-2 < 1$

$$1 < x < 3$$

$x = 1 \quad \Rightarrow \quad \sum \frac{(-1)^n}{\ln(n+4)} = \frac{1}{\ln 5} + \frac{1}{\ln 6} - \frac{1}{\ln 7} + \cdots$

- decreasing
- $\lim_{n\to\infty} \frac{1}{\ln(n+4)} = 0$

converges.

$x = 3 \quad \Rightarrow \quad \sum \frac{(1)^n}{\ln(n+4)} = \sum \frac{1}{\ln(n+4)}$

$n+4 > \ln(n+4)$

$\frac{1}{n+4} < \frac{1}{\ln(n+4)}$ or $\frac{1}{\ln(n+4)} > \frac{1}{n+4}$

$\sum \frac{1}{\ln(n+4)}$ diverges

Interval of convergence $\quad [1, 3)$

$R = \frac{3-1}{2} = \frac{2}{2} = 1$

Ex. $\sum_{k=0}^{\infty} k!\, x^k \qquad \lim_{k\to\infty} \left| \frac{(k+1)!\, x^{k+1}}{k!\, x^k} \right| = \lim_{k\to\infty} \left| \frac{\cancel{k!}(k+1)\, \cancel{x^k} \cdot x}{\cancel{k!} \cdot \cancel{x^k}} \right|$

$= |x| \lim_{k\to\infty} (k+1)$

$= |x| \cdot \infty$

$= \infty \quad \forall\, x \neq 0$

The series diverges $\forall\, x$ except $x = 0$.
The interval of convergence is $x = 0$ and $R = 0$.

Tutorial (3) cont.

Basic Taylor Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (-\infty, \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad [-1, 1]$$

Ex. Find the Maclaurin Series for the given function.

$f(x) = 2\sin(2x)\cos(2x)$

$2\sin(2x)\cos(2x) = \sin(4x)$ why? recall $\sin(2x) = 2\sin x\cos x$

We know that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

so $\sin(4x) = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n+1}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{2n+1}}{(2n+1)!}$$

Ex. Use a known series to find a power series in x that has the given function as its sum.

a) $x\sin(x^3)$

Recall $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$$x\sin(x^3) = x \cdot \sum_{n=0}^{\infty}(-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!}$$

$$= x \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+4}}{(2n+1)!}$$

b) $\frac{\ln(1+x)}{x}$

Recall

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

$$\frac{\ln(1+x)}{x} = \frac{1}{x} \cdot \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n+1}$$

(d)

c) $\dfrac{x-\tan^{-1}x}{x^3}$

Recall

$$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

$$x-\tan^{-1}x = x-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots\right)$$

$$= \frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7}-\dots$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}$$

$$\frac{x-\tan^{-1}x}{x^3} = \frac{1}{x^3}\cdot\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-2}}{2n+1}$$

Ex. Find the sum of the following series.

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)}$

Recall

$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$x = \frac{1}{2}$

$$\tan^{-1}\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)}$$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{100^{n+1}(n+1)}$

Recall

$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

$x = \frac{1}{100}$

$= 0.01$

$$\ln\left(1 + \frac{1}{100}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(1/100)^{n+1}}{n+1}$$

$$\ln(1.01) = \sum_{n=0}^{\infty} \frac{(-1)^n}{100^{n+1}(n+1)}$$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{10^{2n+1}(2n+1)!}$

Recall

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$x = \frac{\pi}{10}$

$$\sin(\pi/10) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{10^{2n+1}(2n+1)!}$$
