Integrating Improper Rational Functions

↳ (Degree of the numerator > Degree of the denomenator)

$$I = \int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\, dx$$

$$\begin{array}{r l}
 & 3x^2 + 1 \rightarrow \text{quotient} \\
\text{divisor} \leftarrow x^2 + x - 2 \,\big) & \cancel{3x^4} + \cancel{3x^3} - 5x^2 + x - 1 \rightarrow \text{dividend} \\
- & \cancel{3x^4} + \cancel{3x^3} - 6x^2 \\
\hline
 & \cancel{x^2} + \cancel{x} - 1 \\
- & \cancel{x^2} + \cancel{x} - 2 \\
\hline
 & 1 \rightarrow \text{remainder}
\end{array}$$

$$\frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$$= 3x^2 + 1 + \frac{1}{x^2 + x - 2}$$

$$\therefore \quad I = \int \left(3x^2 + 1 + \frac{1}{x^2 + x - 2}\right) dx$$

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$x = -2 \qquad x = 1$$

Cover-up Method → $A = \frac{1}{x-1} = \frac{1}{-2-1} = -\frac{1}{3}$

$$B = \frac{1}{x+2} = \frac{1}{1+2} = \frac{1}{3}$$

$$I = \int \left(3x^2 + 1 + \frac{-\frac{1}{3}}{x+2} + \frac{\frac{1}{3}}{x-1}\right) dx = x^3 + x - \frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$$

$$= x^3 + x + \frac{1}{3}\left(\ln|x-1| - \ln|x+2|\right) + C$$

$$= x^3 + x + \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$