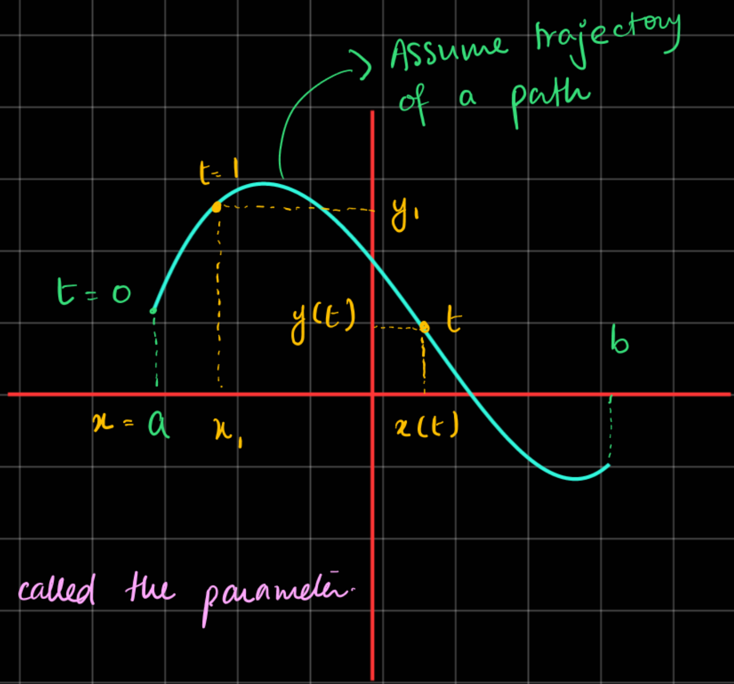
Parametric Equations

$y = f(x) \Rightarrow$ Rectangular equations



$x = f(t)$, $y = g(t)$

$x = f(t)$ and $y = g(t)$ $\quad a \leq t \leq b$
are called parametric equations and t is called the parameter.

$x = x_0 + at = f(t)$

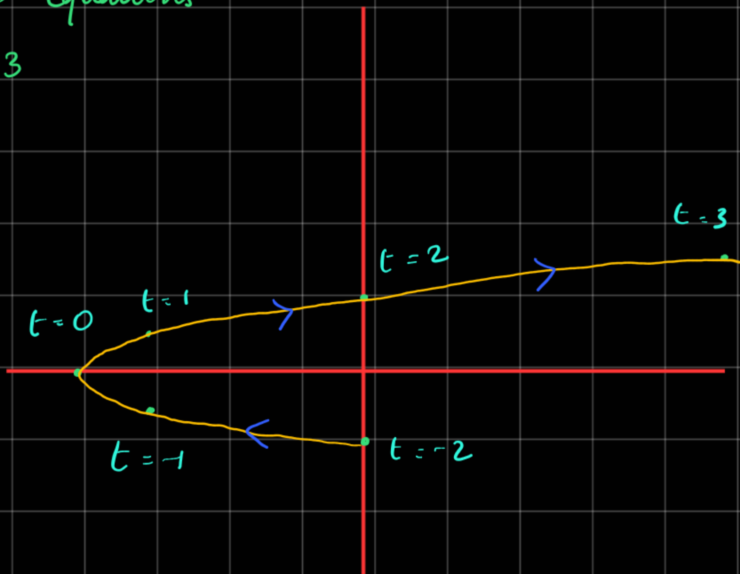
$y = y_0 + bt = g(t)$

$z = z_0 + ct = h(t)$

Example

Sketch the curve described by the parametric equations
$x(t) = t^2 - 4$ and $y = \frac{t}{2}$, $-2 \leq t = 3$

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	-1/2	0	1/2	1	3/2

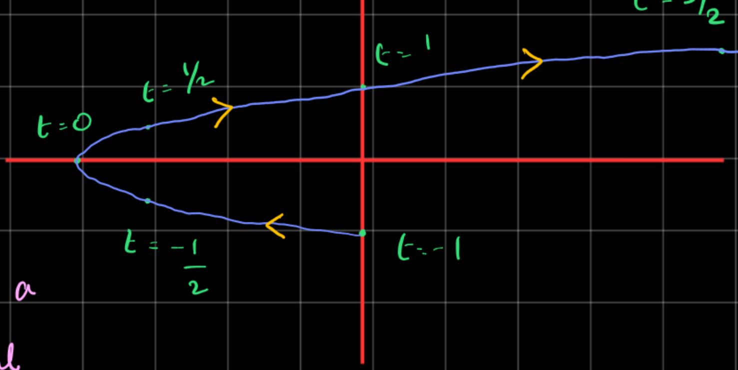


The graph of a set of parametric equations is called a parametrized curve.

Example

Sketch the parametrized curve given by $x = 4t^2 - 4$ and $y = t$, $-1 \leq t \leq 3/2$

t	-1	-1/2	0	1/2	1	3/2
x	0	-3	-4	-3	0	5
y	-1	-1/2	0	1/2	1	3/2



From the examples above, we observe that a given parametrized curve may have several different parametric equations.

Eliminating the parameter 't' to obtain the rectangular equation $(y = f(x))$
or Cartesian

Consider the parametric equations

$x = f(t)$ and $y = g(t)$

To eliminate the parameter 't' do the following

1. Out of the two equations choose the one that is easier to solve for t and solve for t.
2. Plug the expression of t into the other equation to obtain the corresponding rectangular equation.

Example

Eliminate the parameter and sketch the corresponding parametrized curve

1. $x = t^2 - 4 \qquad y = t/2 \qquad -2 \leq t \leq 3$

$y = \frac{t}{2}$

$t = 2y$

$\boxed{x = 4y^2 - 4} \qquad -1 \leq y \leq \frac{3}{2}$

2. $x = 4t^2 - 4 \qquad y = t \qquad -1 \leq t \leq 3/2$

$y = t$

$x = 4y^2 - 4 \qquad -1 \leq y \leq 3/2$

3. $x = \dfrac{1}{\sqrt{t+1}} \quad y = \dfrac{t}{t+1}, \quad t > -1$

$$x^2 = \frac{1}{t+1}$$

$$x^2(t+1) = 1$$

$$t + 1 = \frac{1}{x^2}$$

$$t = \frac{1}{x^2} - 1$$

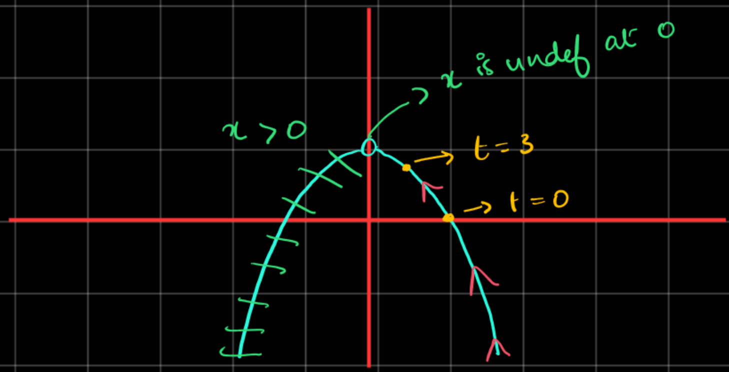
$$y = \frac{t}{t+1}$$

$$= \left(\frac{1}{x^2} - 1\right) x^2$$

$$= \frac{1 - x^2}{\cancel{x^2}} \times \cancel{x^2}$$

$$= 1 - x^2 \qquad x > 0$$

t	0	3
x	1	1/2
y	0	3/4



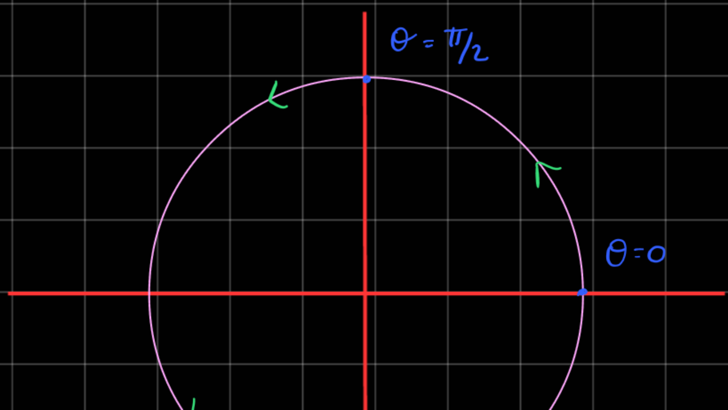
4. $x = 3\cos\theta \quad y = 3\sin\theta, \quad 0 \le \theta \le 2\pi$

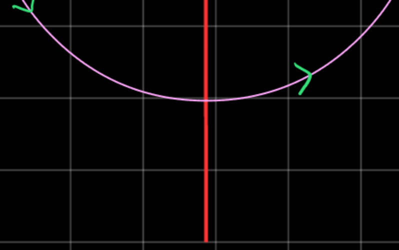
$$\cos\theta = \frac{x}{3} \qquad \sin\theta = \frac{y}{3}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{x^2 + y^2}{9} = 1$$

$$x^2 + y^2 = 9$$





Calculus with parametric equations

$x = f(t) \qquad y = g(t) \longrightarrow$

$\dfrac{dy}{dx} = ?$

$$y = F(x)$$
$$g(t) = F(f(t))$$
$$g'(t) = F'(f(t)) \cdot f'(t)$$
$$g'(t) = F'x \cdot f'(t)$$
$$F'(x) = \frac{g'(t)}{f'(t)} = \frac{dy/dt}{dx/dt}$$

Let c be the parametized curve given by $x = f(t)$ and $y = g(t)$. Then the slope of the curve at (x, y) is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \qquad dx/dt \neq 0$$

Second Derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

Example

Consider the curve given by $x = \sqrt{t}$ and $y = \frac{1}{4}(t^2 - 4)$, $t \geq 0$

a) Find the equation of the tangent line to the curve at the point $(2, 3)$

b) Find the second derivative at the point $(2, 3)$

a) $y = m(x - 2) + 3$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dt} = \frac{2t}{4} = \frac{t}{2}$$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dx} = \frac{t/2}{1/2\sqrt{t}}$$

$$= \frac{\cancel{2}t\sqrt{t}}{\cancel{2}}$$

$$= t\sqrt{t}$$

$$= t^{3/2}$$

$\frac{dy}{dx}$ at $(2,3)$

$x = \sqrt{t}$ @ $x = 2$ $2 = \sqrt{t}$ $\Rightarrow t = 4$

$y = \frac{1}{4}(t^2 - 4)$ @ $y = 3$ $\frac{1}{4}(t^2 - 4) = 3$

$$\frac{1}{4}(t^2 - 4) = 3$$

$$t^2 - 4 = 12$$

$$t^2 = 16$$

$$t = \pm 4$$

$\because t = -4$ doesn't solve x

$\therefore t = 4$

$$\frac{dy}{dx} = 4^{3/2}$$

$$= \sqrt{64}$$

$$= 8$$

$$y = 8(x - 2) + 3$$

$y = 8x - 16 + 3$

$\underline{y = 8x - 13}$

b) $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$

$= \dfrac{\frac{3}{2} t^{1/2}}{1/2\sqrt{t}}$

$= \dfrac{3}{\cancel{2}} \sqrt{t} \times \cancel{2}\sqrt{t}$

$= 3t$

$\dfrac{d^2y}{dx^2}$ at $(2,3) = 3(4)$

$= \underline{12}$