EXAMINATION COVERSHEET

Winter 2024 Midterm Examination

UNIVERSITY OF WOLLONGONG IN DUBAI

Solution Key

THIS EXAMINATION CONTENT IS STRICTLY CONFIDENTIAL Students must comply with requirements stated in the Examination Policy & Procedures	
Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	
Subject Code:	**Math 142**
Subject Title:	**Essentials of Engineering Mathematics**
Time Permitted to Write Exam:	2 Hours
Total Number of Questions:	**6 MCQs + 5 written questions = 11**
Total Number of Pages *(including this page):*	**12**

INSTRUCTIONS TO STUDENTS FOR THE EXAM

1. Please note that subject lecturer/tutor will be unavailable during exams. *If there is a doubt in any of the exam questions i.e. problem solving etc. students should proceed by assuming values etc. Students should mention their assumption on the question paper.*
2. Answers must be written (and drawn) in black or blue ink
3. Any mistakes must be crossed out. Whitener and ink erasers must not be used.
4. Part A (MCQ): Answer ALL/ 6 questions. The marks for each question are shown next to each question. The total for Part A is 30 marks.
5. Part B (Written): Answer ALL/ 5 questions. The marks for each question are shown next to each question. The total for Part B is 70 marks.)
6. Total marks: 100. This Exam is worth 35% of your final marks for MATH 142.

EXAMINATION MATERIALS/AIDS ALLOWED
Approved Calculators and Formula Sheet

Exam Unauthorised Items - Students bringing these items to the examination room must follow the instructions of the invigilators with regards to these items.

7. Bags, including carrier bags, backpacks, shoulder bags and briefcases
8. Any form of electronic device including but not limited to mobile phones, smart watches, MP3 players, handheld computers and unauthorised calculators;
9. Calculator cases and covers, opaque pencil cases
10. Blank paper
11. Any written material

NOTE: The University does not guarantee the safe-keeping of students' personal items during examinations. Students concerned about the safety of their valuable items should make alternative arrangements for their care.

Part 1 MCQ 30% (circle your choice)

(5pts)Problem 1
Evalute the definite integral

$$\int_1^2 x^3 \ln x dx$$

(A) $2\ln 3 - \frac{6}{13}$

(B) $3\ln 3$

(C) $4\ln 3 - 3$

(D) $4\ln 2 - \frac{15}{16}$

(E) None of These

Solution
Answer is (D).
Using integration by parts

$$\begin{aligned} \int_1^2 x^3 \ln x dx &= \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right]_1^2 \\ &= 4\ln 2 - \frac{15}{16}. \end{aligned}$$

(5pts)Problem 2
Consider a cooling system for a nuclear reactor where the temperature of the coolant (water) as it flows through the reactor is modeled by the function

$$T(t) = \frac{100}{t^2},$$

where t represents the time in minutes since the start of the cooling process. What is the total amount of heat $\int_1^\infty T(t)dt$ absorbed by the coolant as it flows through the reactor from $t = 1$ minute to $t = \infty$?

(A) 50

(B) 100

(C) ∞

(D) 75

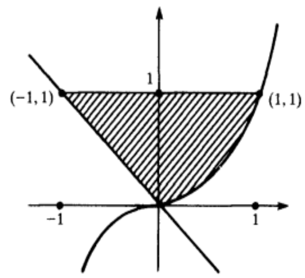
(E) 125

Solution
Answer is (B).

$$\begin{aligned} \int_1^\infty \frac{100}{t^2} dt &= \lim_{b\to\infty} \int_1^b \frac{100}{t^2} dt = \lim_{b\to\infty} \left[\frac{-100}{t}\right]_1^b \\ &= \lim_{b\to\infty} \left(\frac{-100}{b} + 100\right) = 100. \end{aligned}$$

(5pts)Problem 3

Find the area of the region between the curve $y = x^3$ and the lines $y = -x$ and $y = 1$ shown in the figure below



(A) $\frac{5}{4}$

(B) $\frac{3}{2}$

(C) 1

(D) $\frac{1}{2}$

(E) $\frac{4}{3}$

Solution

Answer is (A).

$$\begin{aligned} A &= \int_{-1}^{0} [1 - (-x)]\, dx + \int_{0}^{1} \left(1 - x^3\right) dx \\ &= \int_{-1}^{0} (1 + x)\, dx + \int_{0}^{1} \left(1 - x^3\right) dx \\ &= \frac{1}{2} + \frac{3}{4} = \frac{5}{4}. \end{aligned}$$

(5pts)Problem 4

Imagine you're tasked with designing a curved road for a highway interchange. The curve follows the function $f(x) = \frac{x^3}{6} + \frac{1}{2x}$, where $f(x)$ represents the height of the road at a given horizontal distance x. Calculate the length of this curved road between $x = 1$ and $x = 3$, to estimate the amount of material needed for construction.

(A) $\frac{16}{7}$

(B) $\frac{14}{3}$

(C) $\frac{21}{5}$

(D) 6

(E) 12

Solution

Answer is (B).

$$f(x) = \frac{x^3}{6} + \frac{1}{2x}, \qquad f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 \\ &= \frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4x^4} \\ &= \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2 \end{aligned}$$

$$L = \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \frac{14}{3}$$

(5pts)**Problem 5**

Calculate the second derivative $\dfrac{d^2y}{dx^2}$ at $t = 2$ for the plane curve defined by the parametric equation

$$x = t^2 - 3 \quad \text{and} \quad y = 2t - 1.$$

(A) $\frac{-1}{16}$

(B) $\frac{\sqrt{3}}{4}$

(C) -8

(D) $\frac{7}{4}$

(E) None of these

Solution

Answer is (A).

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}. \\
\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} \\
&= \frac{\frac{-1}{t^2}}{2t} \\
&= -\frac{1}{2t^3}. \\
t &= 2 \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{2\,(2)^3} \\
&= -\frac{1}{16}.
\end{aligned}$$

(5pts)Problem 6

Identify the graph of the polar curve

$$r = -8\cos\theta.$$

(A) Circle centered at $(0,0)$ with radius 8.

(B) A line passing through the origin with slope 8

(C) Circle centered at $(-4,0)$ with radius 4.

(D) The vertical line $x = -8$

(E) Circle centered at $(-4,4)$ with radius 8.

Solution

Answer is (C).

$$\begin{aligned} r &= -8\cos\theta. \\ r^2 &= -8r\cos\theta \\ x^2 + y^2 &= -8x \\ x^2 + 8x + y^2 &= 0 \\ (x+4)^2 - 16 + y^2 &= 0 \\ (x+4)^2 + (y-0)^2 &= 16 = 4^2. \end{aligned}$$

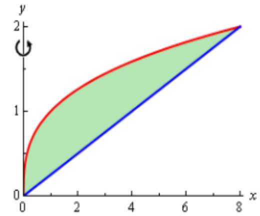
Part 2 Written 70%

(14*pts*)**Problem 1**

Determine the volume of the solid generated by rotating the region bounded by $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$ that lies in the first quadrant about the y-axis.

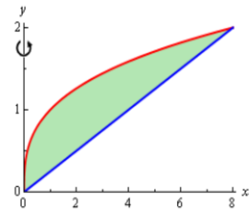
(A) By the disc (washer) method.

(B) By the cylindrical shell method.



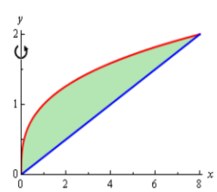
Solution

(A) By the disc (washer) method.



$$
\begin{aligned}
V &= \int_0^2 \left[\pi (4y)^2 - \pi \left(y^3\right)^2\right] dy && \text{[4 points]} \\
&= \pi \int_0^2 \left(16y^2 - y^6\right) dy. \\
&= \frac{512}{21}\pi = 76.595. && \text{[3 points]}
\end{aligned}
$$

(B) By the cylindrical shell method.



$$
\begin{aligned}
V &= \int_0^8 2\pi x \left(\sqrt[3]{x} - \frac{x}{4}\right) dx && \text{[4 points]} \\
&= \frac{512}{21}\pi = 76.595. && \text{[3 points]}
\end{aligned}
$$

(14pts)**Problem 2**

Evaluate the integral

$$\int \frac{3x}{2x^2 - x - 1} dx.$$

Solution

Factor the denominator:

$$2x^2 - x - 1 = (2x+1)(x-1) \qquad [2 \text{ points}]$$

$$\frac{3x}{2x^2 - x - 1} = \frac{3x}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1} \qquad [2 \text{ points}]$$

$$A = 1, \quad B = 1 \qquad [6 \text{ points}]$$

$$\begin{aligned} \int \frac{3x}{2x^2 - x - 1} dx &= \int \left(\frac{1}{2x+1} + \frac{1}{x-1} \right) dx \\ &= \frac{1}{2} \ln|2x+1| + \ln|x-1| + C \qquad [4 \text{ points}] \end{aligned}$$

(14pts)**Problem 3**
Evaluate the intergals

$$\int \frac{\sqrt{x}}{x+1}dx$$

Solution
Put

$$\begin{aligned} u &= \sqrt{x}. \\ \text{We have } x &= u^2 \text{ and } dx = 2udu \qquad\qquad [4 \text{ points}] \end{aligned}$$

$$\begin{aligned} \int \frac{\sqrt{x}}{x+1}dx &= \int \frac{2u^2du}{u^2+1} \\ &= 2\int \frac{u^2+1-1du}{u^2+1} \qquad\qquad [3 \text{ points}] \\ &= 2\int \left(\frac{u^2+1}{u^2+1} - \frac{1}{u^2+1}du\right) \\ &= 2\int \left(1 - \frac{1}{u^2+1}\right) du \qquad\qquad [4 \text{ points}] \\ &= 2u - 2\tan^{-1} u + C \\ &= 2\sqrt{x} - 2\tan^{-1} \sqrt{x} + C. \qquad\qquad [3 \text{ points}] \end{aligned}$$

(14*pts*)**Problem 4**

Use trigonometric substitution to evaluate the integral

$$\int \frac{dx}{\sqrt{x^2-9}}$$

Solution

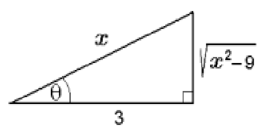
Put

$$x = 3\sec\theta, \qquad 0 < \theta < \frac{\pi}{2}. \qquad [3 \text{ points}]$$

$$dx = 3\sec\theta\tan\theta d\theta \ \text{ and } \sqrt{x^2-9} = 3\tan\theta. \qquad [3 \text{ points}]$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-9}} &= \int \frac{3\sec\theta\tan\theta d\theta}{3\tan\theta} \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \qquad [4 \text{ points}]\end{aligned}$$

Now using the triangle



,

we have

$$\sec\theta = \frac{x}{3} \ \text{ and } \ \tan\theta = \frac{\sqrt{x^2-9}}{3}. \qquad [2 \text{ points}]$$

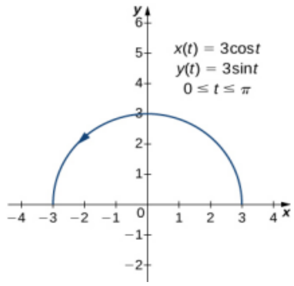
Hence,

$$\int \frac{dx}{\sqrt{x^2-9}} = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C. \qquad [2 \text{ points}]$$

$(14pts)$**Problem 5**
Imagine you're designing a roller coaster track, and one section of the track consists of a semicircular loop whose parametric equations are given by

$$x = 3\cos t \quad \text{and} \quad y = 3\sin t, \qquad 0 \leq t \leq \pi.$$

as shown below.



To ensure the safety of the riders and proper construction of the track, calculate the length of this semicircular loop accurately.

Solution

$$\begin{aligned}
L &= \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt && \text{[2 points]} \\
&= \int_0^\pi \sqrt{(-3\sin t)^2 + (3\cos t)^2} dt && \text{[6 points]} \\
&= \int_0^\pi \sqrt{9\sin^2 t + 9\cos^2 t} dt \\
&= \int_0^\pi \sqrt{9} dt && \text{[4 points]} \\
&= 3\pi = 9.4248. && \text{[2 points]}
\end{aligned}$$