UNIVERSITY OF WOLLONGONG IN DUBAI

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Tutorial 7

Q1. Find $\lim_{n\to\infty} a_n$.

(a). $1,\ \frac{-1}{2},\ \frac{1}{3},\ \frac{-1}{4},\ \frac{1}{5},\ldots$

(b) $a_n = 3n\sin\frac{\pi}{2n}$

(c) $\ln\left(\frac{2}{1}\right),\ \ln\left(\frac{3}{2}\right),\ \ln\left(\frac{4}{3}\right),\ldots$

Solution

(a). $1,\ \frac{-1}{2},\ \frac{1}{3},\ \frac{-1}{4},\ \frac{1}{5},\ldots$

$$a_n = \frac{(-1)^{n+1}}{n}$$

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$$

(b)

$$\begin{aligned} a_n &= 3n\sin\frac{\pi}{2n} \\ &= 3\frac{\sin\frac{\pi}{2n}}{\frac{1}{n}} \\ &= 3\pi\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{n}} \\ &= \frac{3\pi}{2}\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} 3n\sin\frac{\pi}{2n} &= \lim_{n\to\infty}\frac{3\pi}{2}\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} \\ &= \frac{3\pi}{2}. \end{aligned}$$

(c) a_n converges to 0

Q2. Find $\lim_{n\to\infty} a_n$.

(i) $a_n = n\sin\left(\pi + \frac{3}{n}\right)$

(ii) $\frac{4}{1},\ \frac{7}{3},\ \frac{10}{5},\ \frac{13}{7},\ldots$

(Show your work)

Solution

(i) $a_n = n\sin\left(\pi + \frac{3}{n}\right)$

$$\begin{aligned}\lim_{n\to\infty} n\sin\left(\pi+\frac{3}{n}\right) &= \lim_{n\to\infty}\frac{\sin\left(\pi+\frac{3}{n}\right)}{\frac{1}{n}}\\ &= \lim_{x\to\infty}\frac{\sin\left(\pi+\frac{3}{x}\right)}{\frac{1}{x}}\end{aligned}$$

Using L'Hopital's rule,

$$\begin{aligned}\lim_{x\to\infty}\frac{\sin\left(\pi+\frac{3}{x}\right)}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{-\frac{3}{x^2}\cos\left(\pi+\frac{3}{x}\right)}{\frac{-1}{x^2}}\\ &= \lim_{x\to\infty} 3\cos\left(\pi+\frac{3}{x}\right) = 3\cos\pi = -3.\end{aligned}$$

(ii) $\frac{4}{1}, \frac{7}{3}, \frac{10}{5}, \frac{13}{7}, \ldots$

The numerator is an arithmetic sequence with first term 4 and common difference 3. Thus, a formula for the numerator is

$$4 + 3(n-1) = 3n + 1$$

The denominator is an arithmetic sequence with first term 1 and common difference 2. Thus, a formula for the numerator is

$$1 + 2(n-1) = 2n - 1.$$

Hence,

$$a_n = \frac{3n+1}{2n-1} \quad \text{and} \quad \lim_{n\to\infty} a_n = \frac{3}{2}.$$

Q3. Determine whether the series converges or diverges. If it converges, find the sum

1. $\sum_{n=1}^{\infty} 4^n \cdot 5^{1-n}$

2. $\sum_{n=1}^{\infty} \frac{1}{(n+1)!n!}\left((n+1)! - n!\right)$

Solution

1.

$$\sum_{n=1}^{\infty} 4^n \cdot 5^{1-n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{5}\right)^{n-1}.$$

This is a geometric series with $a_1 = 4$ and $r = \frac{4}{5}$.

$$\left|\frac{4}{5}\right| < 1.$$

The series converges and its sum is

$$S = \frac{a}{1-r} = \frac{4}{1-\frac{4}{5}} = 20$$

2.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(n+1)!n!}\left((n+1)! - n!\right) &= \sum_{n=1}^{\infty}\left[\frac{1}{n!} - \frac{1}{(n+1)!}\right] \\
&= \lim_{N\to\infty} \sum_{n=1}^{N}\left[\frac{1}{n!} - \frac{1}{(n+1)!}\right] \\
&= \lim_{N\to\infty}\left[\left(1-\frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \ldots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)\right] \\
&= \lim_{N\to\infty}\left(1 - \frac{1}{(n+1)!}\right) = 1.
\end{aligned}$$

The series converges and its value is $S = 1$.

Q4. Find the sum of the following series

(i) $\sum_{n=1}^{\infty}\left[(-0.2)^n + (0.6)^{n-1}\right]$ $\qquad$ (ii) $\sum_{n=0}^{\infty}\left[6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\right]$.

(Show your work)

Solution

(i)

$$\begin{aligned}
\sum_{n=1}^{\infty}\left[(-0.2)^n + (0.6)^{n-1}\right] &= \sum_{n=1}^{\infty}(-0.2)^n + \sum_{n=1}^{\infty}(0.6)^{n-1} \\
&= \sum_{n=1}^{\infty}(-0.2)(-0.2)^{n-1} + \sum_{n=1}^{\infty}(0.6)^{n-1} \\
&= \frac{-0.2}{1-(-0.2)} + \frac{1}{1-0.6} \\
&= 2.3333
\end{aligned}$$

(ii)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left[6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\right] \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\left[6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\right] \\
&= \lim_{N\to\infty}\left[\begin{array}{c} 6\tan^{-1}\left(\sqrt{3}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) + 6\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) \\ + \ldots + 6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\end{array}\right] \\
&= \lim_{N\to\infty}\left[6\tan^{-1}\left(\sqrt{3}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\right] = 6\tan^{-1}\left(\sqrt{3}\right) - 0 \\
&= 2\pi
\end{aligned}$$

Q5. Determine whether the series converges or diverges. (Justify you answer)

1. $\sum_{n=1}^{\infty} \sqrt[3]{\frac{-8n}{n+4}}$ 2. $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

Solution

1. $\sum_{n=1}^{\infty} \sqrt[3]{\frac{-8n}{n+4}}$

$$\lim_{n\to\infty} \sqrt[3]{\frac{-8n}{n+4}} = -2 \neq 0. \text{ The series cannot converge. Diverges.}$$

2. $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$. Here we can apply the ratio test or you can simply observe that

$$\begin{aligned} \lim_{n\to\infty} \frac{e^n}{n^3} &= \lim_{x\to\infty} \frac{e^x}{x^3} \\ &= +\infty \quad \text{(L'Hospital's rule)} \\ &\neq 0 \quad \text{Diverges} \end{aligned}$$

Q6. Determine whether the following series converges or diverges. (Justify your answer and show your work)

(i) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ (Show your work) (ii) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

Solution

(i)

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n} \text{ is an alternating series with } b_n = \frac{1}{n \ln n}.$$

Since for $n \geq 2, b_n$ is positive, decreasing and $\lim_{n\to\infty} b_n = 0$, the series converges by the alternating series test.

(ii)

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$\frac{1}{n \ln n} = f(n)$ where $f(x) = \frac{1}{x \ln x}$ is continuous, positive and decreasing when $x \geq 2$.

We will use the integral test.

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x \ln x} &= \lim_{t\to\infty} \int_2^t \frac{dx}{x \ln x} \\ &= \lim_{t\to\infty} \ln |\ln x| \big|_2^t \\ &= \lim_{t\to\infty} [\ln |\ln t| - \ln \ln 2] \\ &= +\infty. \quad \text{The series diverges.} \end{aligned}$$