Tutorial 2

Question 1

Evaluate the following integrals

1. $\displaystyle\int_0^1 (x-1)\, e^{-x} dx$ 2. $\displaystyle\int x^{11} \ln x dx$

Solution

1. $\displaystyle\int_0^1 (x-1)\, e^{-x} dx$? by parts, put

$$\begin{aligned} u &= x-1, \quad u' = 1 \\ v' &= e^{-x}, \quad v = -e^{-x} \end{aligned}$$

$$\begin{aligned} \int_0^1 (x-1)\, e^{-x} dx &= -(x-1)\, e^{-x}\Big|_0^1 + \int_0^1 e^{-x} dx \\ &= -(x-1)\, e^{-x}\Big|_0^1 + -e^{-x}\Big|_0^1 \\ &= -e^{-1} = \frac{-1}{e} = -0.367\,88 \end{aligned}$$

2. $\int x^{11} \ln x dx$? Again by parts

$$\begin{aligned} u &= \ln x \quad u' = \frac{1}{x} \\ v' &= x^{11}, \quad v = \frac{x^{12}}{12} \end{aligned}$$

$$\begin{aligned} \int x^{11} \ln x dx &= \frac{x^{12}}{12} \ln x - \int \frac{1}{x} \cdot \frac{x^{12}}{12} dx \\ &= \frac{x^{12}}{12} \ln x - \frac{1}{12} \int x^{11} dx \\ &= \frac{x^{12}}{12} \ln x - \frac{x^{12}}{(12)^2} + C \\ &= \frac{1}{12} x^{12} \ln x - \frac{1}{144} x^{12} + C \end{aligned}$$

Question 2

Use trigonometric substitution to evaluate the integral

$$\int \frac{x^2}{\sqrt{16-x^2}}dx.$$

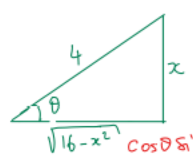
Solution

We do the substitution

$$x = 4\sin\theta, \qquad dx = 4\cos\theta d\theta$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^2}}dx &= \int \frac{16\sin^2\theta}{4\cos\theta}4\cos\theta d\theta \\
&= \int 16\sin^2\theta d\theta \\
&= 16\int \frac{1-\cos 2\theta}{2}d\theta \\
&= 8\left(\theta - \frac{1}{2}\sin 2\theta\right) + C \\
&= 8\left(\theta - \cos\theta\sin\theta\right) + C
\end{aligned}$$

$$\sin\theta = \frac{x}{4}, \qquad \theta = \sin^{-1}\left(\frac{x}{4}\right)$$



$\sin\theta = \frac{x}{4}$

$\cos\theta = \frac{\sqrt{16-x^2}}{4}$

$\cos\theta\sin\theta = \frac{x\sqrt{16-x^2}}{16}$

$$\int \frac{x^2}{\sqrt{16-x^2}}dx = 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16-x^2} + C$$

Question 3

Evaluate the following integrals

1. $\int_0^{\frac{\pi}{4}} x\sin(2x)dx$ 2. $\int \cos(\ln x)\,dx$

Solution

1.

$$\int_0^{\frac{\pi}{4}} x\sin(2x)dx$$

Using integration by parts, put

$$u = x, \qquad u' = 1$$
$$v' = \sin(2x), \qquad v = \tfrac{-1}{2}\cos(2x).$$

Applying the integration by parts formula, we get

$$\begin{aligned}\int_0^{\frac{\pi}{4}} x\sin(2x)dx &= \frac{-x}{2}\cos(2x)\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos(2x)\,dx \\ &= \frac{-x}{2}\cos(2x)\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\,\frac{1}{2}\sin(2x)\Big|_0^{\frac{\pi}{4}} \\ &= 0 + \frac{1}{4} \\ &= \frac{1}{4} = 0.25 \qquad (\mathbf{3pts})\end{aligned}$$

2.

$$\int \cos(\ln x)\,dx$$

We first perform a substitution. Put

$$X = \ln x, \qquad dX = \frac{1}{x}dx$$
$$x = e^X \quad \text{and } dx = e^X dX$$

$$\begin{aligned}\int \cos(\ln x)\,dx &= \int (\cos X)\left(e^X dX\right) \\ &= \int e^X \cos X dX.\end{aligned}$$

Now put

$$A = \int e^X \cos X dX.$$

Using integration by parts, we get

$$\begin{aligned} u &= e^X, & u' &= e^X \\ v' &= \cos X, & v &= \sin X. \end{aligned}$$

$$A = \int e^X \cos X dX = e^X \sin X - \int e^X \sin X dX$$

Again by parts

$$\begin{aligned} u &= e^X, & u' &= e^X \\ v' &= \sin X, & v &= -\cos X. \end{aligned}$$

Hence,

$$\begin{aligned} A = e^X \sin X - \left[-e^X \cos X + A\right] &\Leftrightarrow \\ A &= e^X \sin X + e^X \cos X - A \\ 2A &= e^X \sin X + e^X \cos X \\ A &= \frac{1}{2}\left(e^X \sin X + e^X \cos X\right) + C \end{aligned}$$

Now using $X = \ln x$,

$$\begin{aligned} \int \cos(\ln x)\, dx &= A = \frac{1}{2}\left(e^{\ln x} \sin(\ln x) + e^{\ln x} \cos(\ln x)\right) + C \\ &= \frac{x}{2}\left(\sin(\ln x) + \cos(\ln x)\right) + C \end{aligned}$$

Question 4

Use trigonometric substitution to evaluate the integral

$$\int \frac{\sqrt{x^2 - 4}}{x} dx.$$

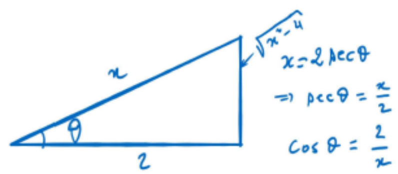
Solution

Here we do the trigonometric substitution

$$x = 2\sec\theta$$

$$dx = 2\sec\theta\tan\theta d\theta$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-4}}{x}dx &= \int \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta} 2\sec\theta\tan\theta d\theta \\
&= \int \frac{2\tan\theta}{1}\tan\theta d\theta \\
&= 2\int \tan^2\theta d\theta \\
&= 2\int \left(\sec^2\theta - 1\right) d\theta \\
&= 2(\tan\theta - \theta) + C
\end{aligned}$$



$$x = 2\sec\theta \Rightarrow \theta = \cos^{-1}\left(\frac{2}{x}\right).$$

From the triangle, we obtain

$$\tan\theta = \frac{\sqrt{x^2-4}}{2}$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-4}}{x}dx &= 2(\tan\theta - \theta) + C \\
&= \sqrt{x^2-4} - 2\cos^{-1}\left(\frac{2}{x}\right) + C.
\end{aligned}$$

Question 5

Use Trigonometric Substitution to evaluate

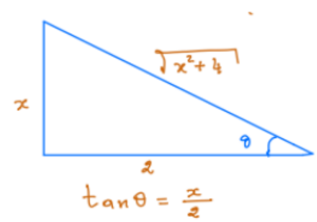
$$\int \frac{4}{x^2\left(\sqrt{x^2+4}\right)}dx$$

Solution
Put
$$x = 2\tan\theta, \quad \text{then } dx = 2\sec^2\theta d\theta.$$

The integral becomes

$$\begin{aligned}
\int \frac{4}{x^2\left(\sqrt{x^2+4}\right)}dx &= 4\int \frac{1}{4\tan^2\theta\,(2\sec\theta)}2\sec^2\theta d\theta \\
&= \int \frac{\sec\theta}{\tan^2\theta}d\theta \\
&= \int \cos\theta\left(\sin^{-2}\theta\right)d\theta \\
&= \frac{\sin^{-2+1}\theta}{-2+1}+C \\
&= -\csc\theta + C
\end{aligned}$$

Next,we need to plug back in x. Originally we had the substitution $x = 2\tan\theta$,so $\tan\theta = \frac{x}{2}$. This means our opposite side is x,our adjacent side is 2,and the hypotenuse is $\sqrt{x^2+4}$.



$$-\csc\theta = \frac{-1}{\sin\theta} = \frac{-1}{\frac{x}{\sqrt{x^2+4}}} = \frac{-\sqrt{x^2+4}}{x}$$

Then we have
$$\int \frac{4}{x^2\left(\sqrt{x^2+4}\right)}dx = \frac{-\sqrt{x^2+4}}{x}+C$$

Question 6

Use trigonometric substitution to evaluate the integral
$$\int \frac{\sqrt{x^2-1}dx}{x^2}.$$

Solution

We do the substitution

$$x = \sec\theta, \qquad dx = \sec\theta\tan\theta d\theta$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}dx}{x^2} &= \int \frac{\sqrt{\sec^2\theta - 1}\sec\theta\tan\theta d\theta}{\sec^2\theta} \\
&= \int \frac{\tan^2\theta}{\sec\theta} d\theta \\
&= \int \frac{\sin^2\theta}{\cos^2\theta}\cos\theta d\theta = \int \frac{\sin^2\theta}{1-\sin^2\theta}\cos\theta d\theta
\end{aligned}$$

Put

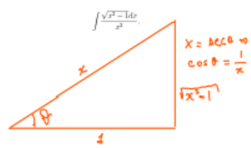
$$u = \sin\theta, \qquad du = \cos\theta d\theta.$$

$$\begin{aligned}
\int \frac{\sin^2\theta}{1-\sin^2\theta}\cos\theta d\theta &= \int \frac{u^2}{1-u^2} du \\
&= \int \frac{1-1+u^2}{1-u^2} du = \int \frac{1-(1-u^2)}{1-u^2} du \\
&= \int \left(\frac{1}{1-u^2} - 1\right) du = \int \frac{du}{1-u^2} - \int 1du
\end{aligned}$$

Using partial fraction decomposition, we have

$$\int \frac{du}{1-u^2} du = \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1| + C$$

$$\begin{aligned}
\int \frac{du}{1-u^2} - \int 1du &= \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1| - u + C \\
&= \frac{1}{2}\ln|\sin\theta+1| - \frac{1}{2}\ln|\sin\theta-1| - \sin\theta + C
\end{aligned}$$



From the triangle, we get

$$\sin\theta = \frac{\sqrt{x^2-1}}{x}.$$

Thus

$$\int \frac{\sqrt{x^2-1}dx}{x^2} = \frac{1}{2}\ln\left|\frac{\sqrt{x^2-1}}{x}+1\right| - \frac{1}{2}\ln\left|\frac{\sqrt{x^2-1}}{x}-1\right| - \frac{\sqrt{x^2-1}}{x} + C$$

Question 7

Evaluate the following integrals

$$(1)\quad \int_0^\pi x\sin\left(x-\frac{\pi}{2}\right)dx, \qquad (2)\quad \int x^2\ln x dx$$

Solution

(1)

$$\int_0^\pi x\sin\left(x-\frac{\pi}{2}\right)dx$$

By parts:

$$\begin{aligned} u &= x, \qquad u' = 1 \\ v' &= \sin\left(x-\frac{\pi}{2}\right), \qquad v = -\cos\left(x-\frac{\pi}{2}\right) \end{aligned}$$

$$\begin{aligned} \int_0^\pi x\sin\left(x-\frac{\pi}{2}\right)dx &= -x\cos\left(x-\frac{\pi}{2}\right)\Big|_0^\pi + \int_0^\pi \cos\left(x-\frac{\pi}{2}\right)dx \\ &= -x\cos\left(x-\frac{\pi}{2}\right)\Big|_0^\pi + \sin\left(x-\frac{\pi}{2}\right)\Big|_0^\pi \\ &= 0+2=2 \end{aligned}$$

(2)

$$\int x^2\ln x dx$$

By parts:

$$\begin{aligned} u &= \ln x, \qquad u' = \frac{1}{x} \\ v' &= x^2, \qquad v = \frac{x^3}{3} \end{aligned}$$

$$\begin{aligned} \int x^2\ln x dx &= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}dx \\ &= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C \end{aligned}$$

Question 8

Evaluate the following integrals

$$(1)\quad \int_0^\pi x\cos(3x-\pi)\,dx, \qquad (2)\quad \int \ln\left(x^2+1\right)dx$$

Solution

(1)

$$\int_0^{\pi} x\cos(3x-\pi)\,dx = ?$$

Here, we integrate by parts.

$$\begin{aligned} u &= x, \qquad u' = 1 \\ v' &= \cos(3x-\pi), \qquad v = \frac{1}{3}\sin(3x-\pi) \end{aligned}$$

$$\begin{aligned} \int_0^{\pi} x\cos(3x-\pi)\,dx &= \frac{x}{3}\sin(3x-\pi)\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{3}\sin(3x-\pi)\,dx \\ &= 0-\left(-\frac{2}{9}\right) \\ &= \frac{2}{9} = 0.222\,22 \end{aligned}$$

(2)

$$\int \ln\left(x^2+1\right)dx = ?$$

Again, we integrate by parts.

$$\begin{aligned} u &= \ln\left(x^2+1\right), \qquad u' = \frac{2x}{x^2+1} \\ v' &= 1, \qquad v = x \end{aligned}$$

$$\begin{aligned} \int \ln\left(x^2+1\right)dx &= x\ln\left(x^2+1\right) - 2\int \frac{x^2}{x^2+1}dx \\ &= x\ln\left(x^2+1\right) - 2\int \frac{x^2+1-1}{x^2+1}dx \\ &= x\ln\left(x^2+1\right) - 2\left(\int 1-\frac{1}{x^2+1}dx\right) \\ &= x\ln\left(x^2+1\right) - 2x + 2\tan^{-1}x + C \end{aligned}$$

Question 9

Evaluate the integral

$$\int \frac{dx}{(4-x^2)\sqrt{4-x^2}}$$

Solution

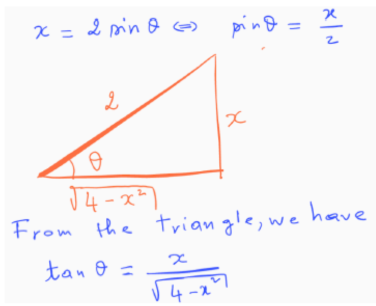
Since the integral involves $\sqrt{a^2 - x^2}$ with $a = 2$, we do the substitution

$$x = 2\sin\theta.$$

$$dx = 2\cos\theta d\theta \quad \text{and} \quad \sqrt{4 - x^2} = 2\cos\theta.$$

The integral becomes

$$\begin{aligned}\int \frac{dx}{(4 - x^2)\sqrt{4 - x^2}} &= \int \frac{2\cos\theta d\theta}{(4\cos^2\theta)(2\cos\theta)} \\ &= \frac{1}{4}\int \sec^2\theta d\theta \\ &= \frac{1}{4}\tan\theta + C\end{aligned}$$



$$\int \frac{dx}{(4 - x^2)\sqrt{4 - x^2}} = \frac{1}{4}\frac{x}{\sqrt{4 - x^2}} + C$$

Question 10

Evaluate the following integrals

1. $\int x^{10} \ln x dx$ 2. $\int \cos 2x \sin 5x dx$.

Solution

1. By parts: put

$$\begin{aligned} u &= \ln x, & u' &= \frac{1}{x} \\ v' &= x^{10}, & v &= \frac{x^{11}}{11} \end{aligned}$$

$$\begin{aligned}\int x^{10}\ln x dx &= \frac{x^{11}}{11}\ln x - \frac{1}{11}\int x^{10} dx \\ &= \frac{x^{11}}{11}\ln x - \frac{1}{121}x^{11} + C\end{aligned}$$

2.

$$\int \cos 2x \sin 5x dx = -\frac{1}{6}\cos 3x - \frac{1}{14}\cos 7x + C$$