Part A MCQ (30%)

(5*pts*) **Problem 1**
If a_n is the sequence given by

$$\ln\left(\frac{2}{1}\right),\ \ln\left(\frac{3}{2}\right),\ \ln\left(\frac{4}{3}\right),\ldots$$

Evaluate $\lim_{n\to\infty} a_n$.

(a) a_n converges to 1

(b) a_n converges to $\ln 2$

(c) a_n converges to 0

(d) a_n converges to $\ln 3$

(e) a_n diverges

Answer is (c)

(5*pts*) **Problem 2**
The sum of the geometric series

$$4 - 1 + \frac{1}{4} - \frac{1}{16} + \ldots$$

is

(a) $\frac{17}{16}$

(b) $\frac{-19}{4}$

(c) $\frac{145}{16}$

(d) $\frac{-14}{3}$

(e) $\frac{16}{5}$

Answer is (e)

$(5pts)$ **Problem 3**
The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$$

(a) converges absolutely

(b) converges conditionally

(c) diverges

(d) is a convergent geometric series

(e) is a divergent telescoping series

$\underline{\textit{Answer is } (a)}$

$(5pts)$ **Problem 4**

The radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n+1}$ is

(a) 2

(b) $\frac{1}{2}$

(c) 1

(d) ∞

(e) 0

$\underline{\textit{Answer is } (b)}$

($5pts$) **Problem 5**

The power series representation of the function $\dfrac{1}{4-x^2}$ is equal to

$$(a) \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{4^n}, \qquad |x| < 2$$

$$(b) \quad \sum_{n=0}^{\infty} \frac{x^{2n}}{4^{n+1}}, \qquad |x| < 2$$

$$(c) \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{4^{n+1}}, \qquad |x| < 2$$

$$(d) \quad \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^{n+1}}, \qquad |x| < 2$$

$$(e) \quad \sum_{n=0}^{\infty} \frac{x^{4n}}{2^{n+1}}, \qquad |x| < 2$$

$\underline{Answer\ is\ (b)}$

($5pts$) **Problem 6**

The coefficient of x^3 in Maclaurin series of the function $f(x) = \ln(1-x)$ equal to

$$(a) \quad \frac{-1}{3}$$

$$(b) \quad \frac{-1}{6}$$

$$(c) \quad \frac{5}{6}$$

$$(d) \quad \frac{1}{2}$$

$$(e) \quad -1$$

$\underline{Answer\ is\ (a)}$

Part B Written Questions (70%)

$(15pts)$**Problem 1**
Find the interval of convergence of the following power series

$$1.\ \sum_{n=1}^{\infty} \frac{x^n}{n2^n} \qquad\qquad 2.\ \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}.$$

Solution

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n2^n}$ is a power series centered at $a = 0$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\,2^{n+1}}\cdot\frac{n2^n}{x^n}\right| &= \lim_{n\to\infty}\left|\frac{\cancel{x^n}x}{(n+1)\,\cancel{2^n}2}\cdot\frac{n\cancel{2^n}}{\cancel{x^n}}\right| \\ &= \lim_{n\to\infty}\frac{n\,|x|}{(n+1)\,2} = \frac{|x|}{2}.\end{aligned}$$

$$R = \frac{1}{\frac{1}{2}} = 2 \qquad [\ \textbf{2 points}\ ]$$

$$a - R = -2 \ \text{ and } a + R = 2.$$

- When $x = -2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges by A.S.T} \qquad [\ \textbf{2 points}\ ]$$

- When $x = 2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges by p-test.} \qquad [\ \textbf{2 points}\ ]$$

$$\text{IC} = [-2,\ 2) \qquad [\ \textbf{2 points}\ ]$$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}$ is a power series centered at $a = -2$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)!}\cdot\frac{n!}{(x+2)^n}\right| &= \lim_{n\to\infty}\left|\frac{(x+2)}{(n+1)\,\cancel{n!}}\cdot\frac{\cancel{n!}}{1}\right| \\ &= \lim_{n\to\infty}\frac{|x+2|}{n+1} = 0 \qquad [\ \textbf{3 points}\ ]\end{aligned}$$

$$R = \infty \ \text{ and } \ \text{IC} = (-\infty,\ \infty)\,. \qquad [\ \textbf{4 points}\ ]$$

$(15pts)$**Problem 2**

Solve the initial value problem for the separable equation below

$$\frac{dy}{dx} = 3x^2y^2, \qquad\qquad y(0) = \frac{1}{2}.$$

Solution

$$\int \frac{dy}{y^2} = \int 3x^2 dx, \qquad\qquad [\ \textbf{3 points}\ ]$$

$$\frac{-1}{y} = x^3 + C,$$

$$y = \frac{-1}{x^3 + C}. \qquad\qquad [\ \textbf{5 points}\ ]$$

$$y(0) = \frac{1}{2} \Rightarrow C = -2. \qquad\qquad [\ \textbf{5 points}\ ]$$

Hence,

$$y = \frac{-1}{x^3 - 2}. \qquad\qquad [\ \textbf{2 points}\ ]$$

(20*pts*)**Problem 3**
Show that the differential equation is exact and solve the equation.

$$(\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy = 0.$$

Solution

$$M = \cos y + y\cos x \text{ and } N = \sin x - x\sin y.$$

$$M_y = -\sin y + \cos x, \qquad N_x = \cos x - \sin y$$

$$M_y = N_x \Rightarrow \text{Equation is exact.} \qquad [\ \mathbf{5\ points}\ ]$$

$$\begin{cases} \frac{\partial f}{\partial x} = \cos y + y\cos x \\ \frac{\partial f}{\partial y} = \sin x - x\sin y \end{cases},$$

$$\frac{\partial f}{\partial x} = \cos y + y\cos x \Rightarrow f(x,y) = x\cos y + y\sin x + C(y). \qquad [\ \mathbf{5\ points}\ ]$$

Now using this temporary expression of f,

$$\frac{\partial f}{\partial y} = \sin x - x\sin y \Leftrightarrow -x\sin y + \sin x + C'(y) = \sin x - x\sin y,$$

$$C'(y) = 0 \Rightarrow C(y) = C, \text{ a constant.} \qquad [\ \mathbf{5\ points}\ ]$$

Thus,

$$f(x,y) = x\cos y + y\sin x$$

and the solution is given by

$$x\cos y + y\sin x = C. \qquad [\ \mathbf{5\ points}\ ]$$

(20*pts*)**Problem 4**
Solve the initial value problem for the Bernoulli equation below

$$x\frac{dy}{dx} - 2y = 4x^3y^{1/2}, \qquad y(1) = 0.$$

Solution
The given equation is equivalent to equivalent to

$$y^{-1/2}\frac{dy}{dx} - \frac{2}{x}y^{1/2} = 4x^2.$$

Put

$$u = y^{1/2}. \qquad [\ \textbf{3 points}\ ]$$

$$\frac{du}{dx} = \frac{1}{2}\frac{dy}{dx}y^{-1/2},$$

$$\frac{dy}{dx}y^{-1/2} = 2\frac{du}{dx}.$$

The equation becomes

$$2\frac{du}{dx} - \frac{2}{x}u = 4x^2,$$

$$\frac{du}{dx} - \frac{1}{x}u = 2x^2. \qquad [\ \textbf{5 points}\ ]$$

This is a linear equation with

$$P(x) = \frac{-1}{x}.$$

$$\text{IF} = e^{\int P(x)dx} = e^{-\ln x} = \frac{1}{x} \qquad (x > 0). \qquad [\ \textbf{2 points}\ ]$$

$$\frac{u}{x} = \int \left(2x^2\right)\left(\frac{1}{x}\right)dx,$$

$$\frac{u}{x} = \int 2x dx = x^2 + C.$$

$$u = x\left(x^2 + C\right) \Leftrightarrow y^{1/2} = x\left(x^2 + C\right),$$

$$y = x^2\left(x^2 + C\right)^2. \qquad [\ \textbf{5 points}\ ]$$

$$y(1) = 0 \Leftrightarrow (1 + C)^2 = 0,$$

$$C = -1.$$

Hence,

$$\begin{aligned} y &= x^2\left(x^2 - 1\right)^2 \\ &= x^6 - 2x^4 + x^2. \qquad [\ \textbf{5 points}\ ] \end{aligned}$$