(a) Special Integrating Factors

$M(x,y)\,dx + N(x,y)\,dy = 0$. Suppose $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Find:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \longrightarrow \mu(x) = e^{\int f(x)\,dx}$$

OR

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y) \longrightarrow \mu(y) = e^{\int g(y)\,dy}$$

and multiply the DE by one of the integrating factors to change it to an exact DE.

Ex. $\frac{y}{x^2} + 1 + \frac{1}{x}\frac{dy}{dx} = 0$

$$\left(\frac{y}{x^2}+1\right)dx + \frac{1}{x}\,dy = 0 \qquad \frac{\partial M}{\partial y} = \frac{1}{x^2} \qquad \frac{\partial N}{\partial x} = -\frac{1}{x^2}$$

Not exact!

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\frac{1}{x^2} + \frac{1}{x^2}}{\frac{1}{x}} = \frac{\frac{2}{x^2}}{\frac{1}{x}}$$

$$= \frac{2}{x} = f(x)$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-\frac{1}{x^2} - \frac{1}{x^2}}{\frac{y}{x^2}+1} = \frac{-\frac{2}{x^2}}{\frac{y}{x^2}+1} \neq g(y)$$

So we use $\mu(x) = e^{\int f(x)\,dx}$.

$$= e^{\int \frac{2}{x}\,dx}$$

$$= e^{2\ln|x|}$$

$$= e^{\ln|x|^2}$$

$$= e^{\ln x^2}$$

$$= x^2$$

Multiply by x^2 the DE we get:

$(y + x^2)\,dx + x\,dy = 0$ Exact !

$$\frac{\partial f}{\partial x} = y + x^2 \Rightarrow f(x,y) = \int (y + x^2)\,dx + h(y)$$

$$f(x,y) = yx + \frac{x^3}{3} + h(y)$$

$$\downarrow$$

$$\frac{\partial f}{\partial y} = x + h'(y) = x \Rightarrow$$

$$h'(y) = 0 \Rightarrow h(y) = c_1$$

$$f(x,y) = yx + \frac{x^3}{3} + c_1$$

The general solution $yx + \frac{x^3}{3} + c_1 = c_2$

$$\boxed{yx + \frac{x^3}{3} = C} \qquad C = c_2 - c_1$$

Ex. $(2xy)dx + (y^2 - 3x^2)dy = 0$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = -6x \qquad \text{Not exact!}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x+6x}{y^2-3x^2} = \frac{8x}{y^2-3x^2} \neq f(x)$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x-2x}{2xy}$$

$$= \frac{-8x}{2xy}$$

$$= \frac{-4}{y}$$

$$= g(y) \to \mu(y) = e^{\int -\frac{4}{y}dy}$$

$$\mu(y) = e^{-4\ln|y|} = \cancel{e}^{\ln|y|^{-4}}$$

$$= \frac{1}{|y|^4} \qquad |y|^4 = y^4$$

$$= y^{-4}$$

Multiply by y^{-4} both sides of the DE, we get:

$$(2xy^{-3})dx + (y^{-2} - 3x^2y^{-4})dy = 0 \qquad \text{Exact!}$$

$$\frac{\partial f}{\partial x} = 2xy^{-3} \Rightarrow f(x,y) = \int 2xy^{-3}dx + h(y)$$

$$= x^2y^{-3} + h(y)$$

$$\frac{\partial f}{\partial y} = -3y^{-4}x^2 + h'(y) \Rightarrow -3y^{-4}x^2 + h'(y) = y^{-2} - 3x^2y^{-4}$$

$$h'(y) = y^{-2} = \frac{1}{y^2} \Rightarrow h(y) = -\frac{1}{y}$$

$$f(x,y) = x^2 y^{-3} - \frac{1}{y}$$

$$\boxed{x^2 y^{-3} - \frac{1}{y} = c}$$

Bernoulli Equations

A Bernoulli equation is a DE of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Note when $n=0$ or $n=1$, the above equation becomes linear.

Divide by $y^n \Rightarrow y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$ (I)

and use the substitution $v = y^{1-n}$ to change (I) to a linear equation.

Ex. $\frac{dy}{dx} + \frac{1}{x}y = x^2 y^2$ $(n=2)$

Divide by $y^2 \Rightarrow y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x^2$. (I)

Let $v = y^{-1}$ $(v = y^{1-n})$

$$\frac{dv}{dx} = -1\,y^{-2}\cdot\frac{dy}{dx} \Rightarrow -\frac{dv}{dx} = y^{-2}\frac{dy}{dx}$$

substitute in (I) $\Rightarrow -\frac{dv}{dx} + \frac{1}{x}v = x^2$

$$\boxed{\frac{dv}{dx} - \frac{1}{x}v = -x^2}$$

(II) is a linear equation.

$$\mu(x) = e^{\int p(x)\,dx}$$

$$= e^{\int \frac{-1}{x}\,dx}$$

$$= e^{-\ln|x|}$$

$$= \cancel{e^{\ln|x|^{-1}}}$$

$$= |x|^{-1}$$

Consider $\mu(x) = x^{-1}$

Multiply (II) by x^{-1} $\Rightarrow$ $x^{-1}\frac{dv}{dx} - x^{-2}v = -x$

$$\frac{d}{dx}\left[x^{-1}v\right] = -x$$

$$d\left[x^{-1}v\right] = -x\,dx$$

$$x^{-1}v = \frac{-x^2}{2} + C_1$$

$$v = \frac{-x^3}{2} + C_1 x$$

$$y^{-1} = \frac{-x^3 + Cx}{2}$$

$$\boxed{y = \frac{2}{-x^3 + Cx}}$$

Ex. $\frac{dy}{dx} + y = e^x y^{-2}$

Bernoulli $n=-2$. Divide by y^{-2} $\Longrightarrow$

(I) $\quad y^2 \underbrace{\frac{dy}{dx}} + \underbrace{y^3}_{\hookrightarrow V} = e^X \qquad$ let $V = y^3$ or $[V = y^{1-n}]$

$$\frac{dv}{dx} = 3y^2 \frac{dy}{dx} \Longrightarrow \frac{1}{3}\frac{dv}{dx} = y^2 \frac{dy}{dx}$$

substitute in (I) $\Longrightarrow \frac{1}{3}\frac{dv}{dx} + V = e^X$

$$\boxed{\frac{dv}{dx} + 3V = 3e^X} \quad (\text{II}) \quad \text{linear equation.}$$

$$\mu(x) = e^{\int 3\,dx} = e^{3x}$$

Multiply by e^{3x} (II) $\Longrightarrow e^{3x}\frac{dv}{dx} + 3e^{3x}V = 3e^{4x}$

$$\frac{d}{dx}\left[e^{3x}V\right] = 3e^{4x}$$

$$d\left[e^{3x}V\right] = 3e^{4x}\,dx$$

$$e^{3x}V = 3.\frac{1}{4}e^{4x} + C_1$$

$$V = \frac{3}{4}e^x + C_1 e^{-3x}$$

$$\boxed{y^3 = \frac{3e^x + Ce^{-3x}}{4}} \quad \text{where} \quad C = 4C_1$$

OR $\quad y = \left(\frac{3e^x + Ce^{-3x}}{4}\right)^{1/3}$