UNIVERSITY OF WOLLONGONG IN DUBAI

MATH 142 -Exam 2 Review

Problem 1

Find

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

(if it even converges)

Solution: By definition,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \int_{-\infty}^{c} \frac{1}{1+x^2}\, dx + \int_{c}^{\infty} \frac{1}{1+x^2}\, dx,$$

where we get to pick whatever c we want. Let's pick $c = 0$.

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\, dx = \lim_{b\to-\infty} \Big[\arctan(x)\Big]_b^0 = \lim_{b\to-\infty} [\arctan(0) - \arctan(b)]$$
$$= 0 - \lim_{b\to-\infty} \arctan(b) = \frac{\pi}{2}$$

Similarly,

$$\int_{0}^{\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{2}.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Problem 2

Find

$$\int_{0}^{2} \frac{2x}{x^2-4}\, dx.$$

(if it converges)

Solution: The denominator of $\frac{2x}{x^2-4}$ is 0 when $x = 2$, so the function is not even defined when $x = 2$. So

$$\int_{0}^{2} \frac{2x}{x^2-4}\, dx = \lim_{c\to2-} \int_{0}^{c} \frac{2x}{x^2-4}\, dx = \lim_{c\to2-} \Big[\ln|x^2-4|\Big]_0^c$$
$$= \lim_{c\to2-} \ln|x^2-4| - \ln(4) = -\infty,$$

so the integral diverges.

Problem 3

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} \qquad \leftarrow \text{laws of exponents}$$

$$= |1 + x|\,e^{c_1} \qquad \leftarrow \begin{cases} |1 + x| = 1 + x, & x \geq -1 \\ |1 + x| = -(1 + x), & x < -1 \end{cases}$$

$$= \pm e^{c_1}(1 + x).$$

Relabeling $\pm e^{c_1}$ as c then gives $y = c(1 + x)$.

Problem 4

Solve $\dfrac{dy}{dx} + y = x, \quad y(0) = 4.$

SOLUTION The equation is in standard form, and $P(x) = 1$ and $f(x) = x$ are continuous on $(-\infty, \infty)$. The integrating factor is $e^{\int dx} = e^x$, so integrating

$$\frac{d}{dx}[e^x y] = xe^x$$

gives $e^x y = xe^x - e^x + c$. Solving this last equation for y yields the general solution $y = x - 1 + ce^{-x}$. But from the initial condition we know that $y = 4$ when $x = 0$. Substituting these values into the general solution implies that $c = 5$. Hence the solution of the problem is

$$y = x - 1 + 5e^{-x}, \quad -\infty < x < \infty. \tag{12}$$

Problem 5

Solve $2xy\,dx + (x^2 - 1)\,dy = 0$.

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and so by Theorem 2.4.1 there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2y + g(y).$$

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y$. Hence $f(x, y) = x^2y - y$, so the solution of the equation in implicit form is $x^2y - y = c$. The explicit form of the solution is easily seen to be $y = c/(1 - x^2)$ and is defined on any interval not containing either $x = 1$ or $x = -1$. ■

Problem 6

Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$.

SOLUTION The equation is exact because

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Hence a function $f(x, y)$ exists for which

$$M(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y}.$$

Now for variety we shall start with the assumption that $\partial f/\partial y = N(x, y)$; that is,

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$f(x, y) = 2x\int e^{2y}\,dy - x\int \cos xy\,dy + 2\int y\,dy.$$

Remember, the reason x can come out in front of the symbol $\int$ is that in the integration with respect to y, x is treated as an ordinary constant. It follows that

$$f(x, y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy, \quad \leftarrow M(x, y)$$

and so $h'(x) = 0$ or $h(x) = c$. Hence a family of solutions is

$$xe^{2y} - \sin xy + y^2 + c = 0.$$

Problem 7

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2.$

SOLUTION By writing the differential equation in the form

$$(\cos x \sin x - xy^2)\, dx + y(1 - x^2)\, dy = 0,$$

we recognize that the equation is exact because

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}.$$

Now
$$\frac{\partial f}{\partial y} = y(1 - x^2)$$

$$f(x, y) = \frac{y^2}{2}(1 - x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2.$$

The last equation implies that $h'(x) = \cos x \sin x$. Integrating gives

$$h(x) = -\int (\cos x)(-\sin x\, dx) = -\frac{1}{2}\cos^2 x.$$

Thus
$$\frac{y^2}{2}(1 - x^2) - \frac{1}{2}\cos^2 x = c_1 \qquad \text{or} \qquad y^2(1 - x^2) - \cos^2 x = c, \tag{7}$$

where $2c_1$ has been replaced by c. The initial condition $y = 2$ when $x = 0$ demands that $4(1) - \cos^2(0) = c$, and so $c = 3$. An implicit solution of the problem is then $y^2(1 - x^2) - \cos^2 x = 3$.

The solution curve of the IVP is the curve drawn in dark blue in Figure 2.4.1; it is part of an interesting family of curves. The graphs of the members of the one-parameter family of solutions given in (7) can be obtained in several ways, two of which are using software to graph level curves (as discussed in Section 2.2) and using a graphing utility to carefully graph the explicit functions obtained for various values of c by solving $y^2 = (c + \cos^2 x)/(1 - x^2)$ for y. ■

Problem 8

Solve the differential equation $(y^2 - x)\,dx + 2y\,dy = 0$.

Solution The given equation is not exact because $M_y(x, y) = 2y$ and $N_x(x, y) = 0$. However, because

$$\frac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \frac{2y - 0}{2y} = 1 = h(x)$$

it follows that $e^{\int h(x)\,dx} = e^{\int dx} = e^x$ is an integrating factor. Multiplying the given differential equation by e^x produces the exact differential equation

$$(y^2e^x - xe^x)\,dx + 2ye^x\,dy = 0$$

whose solution is obtained as follows.

$$f(x, y) = \int N(x, y)\,dy = \int 2ye^x\,dy = y^2e^x + g(x)$$

$$f_x(x, y) = y^2e^x + g'(x) = \overbrace{y^2e^x - xe^x}^{M(x,y)}$$

$$\boxed{g'(x) = -xe^x}$$

Therefore, $g'(x) = -xe^x$ and $g(x) = -xe^x + e^x + C_1$, which implies that

$$f(x, y) = y^2e^x - xe^x + e^x + C_1.$$

The general solution is $y^2e^x - xe^x + e^x = C$, or $y^2 - x + 1 = Ce^{-x}$.

Problem 9

Show that the equation is homogeneous and solve.

$$(x^2y + 2xy^2 - y^3)\mathrm{d}x - (2y^3 - xy^2 + x^3)\mathrm{d}y = 0.$$

Solution. The differential equation is homogeneous. Denote $y = vx$. Then

$$(x^3v + 2x^3v^2 - x^3v^3)\mathrm{d}x - (2x^3v^3 - x^3v^2 + x^3)(v\mathrm{d}x + x\mathrm{d}v) = 0,$$
$$(x^3v + 2x^3v^2 - x^3v^3)\mathrm{d}x - (2x^3v^4 - x^3v^3 + x^3v)\mathrm{d}x - (2x^4v^3 - x^4v^2 + x^4)\mathrm{d}v = 0,$$
$$x^3(2v^2 - 2v^4)\mathrm{d}x - x^4(2v^3 - v^2 + 1)\mathrm{d}v = 0,$$
$$\frac{\mathrm{d}x}{x} = \frac{2v^3 - v^2 + 1}{2v^2 - 2v^4}\mathrm{d}v.$$
$$2\log|x| = c_1 - \frac{1}{v} - \log|1 - v^2|,$$
$$x^2\mathrm{e}^{-v}(1 - v^2) = c,$$
$$c = (x^2 - y^2)\mathrm{e}^{x/y}.$$

Problem 10

Solve $x\dfrac{dy}{dx} + y = x^2y^2$.

SOLUTION We first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

by dividing by x. With $n = 2$ we have $u = y^{-1}$ or $y = u^{-1}$. We then substitute

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx} \qquad \leftarrow \text{Chain Rule}$$

into the given equation and simplify. The result is

$$\frac{du}{dx} - \frac{1}{x}u = -x.$$

The integrating factor for this linear equation on, say, $(0, \infty)$ is

$$e^{-\int dx/x} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}.$$

Integrating $\qquad \dfrac{d}{dx}[x^{-1}u] = -1$

gives $x^{-1}u = -x + c$ or $u = -x^2 + cx$. Since $u = y^{-1}$, we have $y = 1/u$, so a solution of the given equation is $y = 1/(-x^2 + cx)$. ■

Problem 11

Find the general solution to

$$y' + ty = ty^3.$$

Solution

We put $v = y^{-2}$

We get

$$v' = (-2)y^{-3}y', \; y = y^3 v$$

So,

$$y' + ty = ty^3$$

$$(-1/2)y^3 v' + ty^3 v = ty^3$$

$$v' - 2tv = -2t$$

$$\mu = e^{-t^2}$$

$$\begin{aligned} v &= e^{t^2}\left(\int^t e^{-t^2}(-2t)dt + c\right) \\ &= e^{t^2}\left(e^{-t^2} + c\right) \\ &= 1 + ce^{t^2}, \end{aligned}$$

and,

$$y = v^{-\frac{1}{2}} = \left[1 + ce^{t^2}\right]^{-\frac{1}{2}}.$$