EXAMINATION COVERSHEET

Winter 2024 Quiz 1

UNIVERSITY OF WOLLONGONG IN DUBAI

THIS EXAMINATION CONTENT IS STRICTLY CONFIDENTIAL Students must comply with requirements stated in the Examination Policy & Procedures	
Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	**02/08/2024**
Subject Code:	**Math 142**
Subject Title:	**Essentials of Engineering Mathematics**
Time Permitted to Write Exam:	1 Hour
Total Number of Questions:	**5 written questions**
Total Number of Pages *(including this page)*:	**6**

INSTRUCTIONS TO STUDENTS FOR THE EXAM

1. Please note that subject lecturer/tutor will be unavailable during exams. *If there is a doubt in any of the exam questions i.e. problem solving etc. students should proceed by assuming values etc. Students should mention their assumption on the question paper.*
2. Answers must be written (and drawn) in black or blue ink
3. Any mistakes must be crossed out. Whitener and ink erasers must not be used.
4. Answer ALL/ 6 questions. The marks for each question are shown next to each question.
5. Total marks: 40.

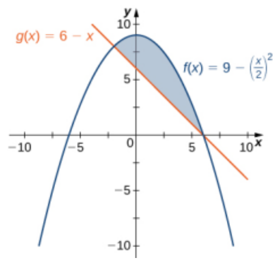
The Solution Key

(8pts)Problem 1
Find the area enclosed by the curves

$$f(x) = 9 - \frac{x^2}{4} \text{ and } g(x) = 6 - x.$$

Solution
Method 1 by graphing

$$A = \int_a^b (Topfunction - Bottom\ function)\ dx$$



[3 points]

To find a and b we need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$9 - \frac{x^2}{4} = 6 - x \iff$$
$$x^2 - 4x - 12 = 0 \iff$$
$$(x-6)(x+2) = 0.$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. [2 points]
Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$\begin{aligned} A &= \int_{-2}^{6} \left[\left(9 - \frac{x^2}{4}\right) - (\ 6 - x)\right] dx \\ &= \int_{-2}^{6} \left(-\frac{1}{4}x^2 + x + 3\right) dx \\ &= \left[\frac{-x^3}{12} + \frac{x^2}{2} + 3x\right]_{-2}^{6} \\ &= \frac{64}{3} = 21.333. \qquad [3 \text{ points}] \end{aligned}$$

Method 2 without graphing

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

To find a and b we need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$9 - \frac{x^2}{4} = 6 - x \iff$$

$$x^2 - 4x - 12 = 0 \Leftrightarrow$$
$$(x-6)(x+2) = 0.$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. Thus,

$$\begin{aligned} A &= \int_{-2}^{6} \left| -\frac{1}{4}x^2 + x + 3 \right| dx \qquad \text{[4 points]} \\ &= \int_{-2}^{6} \left| -\frac{1}{4}(x+2)(x-6) \right| dx \end{aligned}$$

Since we are integrating inside the root, the quadratic function will have the opposite sign of $a = -\frac{1}{4}$ which is positive inside roots -2 and 6. Hence,

$$\begin{aligned} A &= \int_{-2}^{6} \left(-\frac{1}{4}x^2 + x + 3 \right) dx \\ &= \frac{64}{3} = 21.333. \quad \text{[4 points]} \end{aligned}$$

(8pts)Problem 2

You are designing a roller coaster track, and you need to calculate the length of a specific curve on the track to ensure the safety and smoothness of the ride. The curve is described by the function $f(x) = \frac{2}{3}\left(x^2+1\right)^{3/2}, \quad 1 \le x \le 4$, where $f(x)$ represents the height of the track at each point along the x-axis. Find the arclength L of the curve $f(x)$.

Solution

$$L = \int_1^4 \sqrt{1 + [f'(x)]^2}\,dx$$

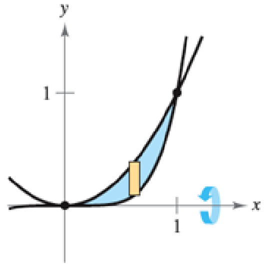
$$f'(x) = \frac{\cancel{2}}{\cancel{3}}\frac{\cancel{3}}{\cancel{2}}(2x)\left(x^2+1\right)^{1/2} \qquad \text{[3 points]}$$

$$\begin{aligned} [f'(x)]^2 &= \left[(2x)\left(x^2+1\right)^{1/2}\right]^2 \\ &= 4x^2\left(x^2+1\right) \\ &= 4x^4 + 4x^2 \end{aligned}$$

$$\begin{aligned} 1 + [f'(x)]^2 &= 4x^4 + 4x^2 + 1 \\ &= \left(2x^2+1\right)^2 \qquad \text{[3 points]} \end{aligned}$$

$$\begin{aligned} L &= \int_1^4 \left(2x^2+1\right)dx \\ &= \left[\frac{2x^3}{3} + x\right]_1^4 = 45. \qquad \text{[2 points]} \end{aligned}$$

(8pts) Problem 3
Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = x^2$ and $y = x^5$.

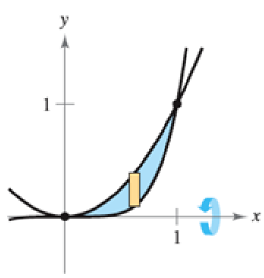


Solution
Method 1 (using the disc method).

$$R = x^2 \text{ (radius of big disc) and } r = x^5 \text{ (radius of hole).}$$

$$\begin{aligned} V &= \int_0^1 Area\ of\ slice\ dx \\ &= \int_0^1 \left[\pi \left(x^2\right)^2 - \pi \left(x^5\right)^2\right] dx \qquad \text{[6 points]} \\ &= \pi \int_0^1 \left(x^4 - x^{10}\right) dx \\ &= \frac{6}{55}\pi = 0.342\,72. \qquad \text{[2 points]} \end{aligned}$$

Method 2 (using the shell method).



$$\begin{aligned} V &= \int_0^1 (2\pi \cdot average\ radius \cdot height)\ dy \\ &= 2\pi \int_0^1 y\left(\sqrt[5]{y} - \sqrt{y}\right) dy \qquad \text{[5 points]} \\ &= 2\pi \int_0^1 \left(y^{6/5} - y^{3/2}\right) dy \\ &= \frac{6}{55}\pi = 0.342\,72. \quad \text{[3 points]} \end{aligned}$$

(8pts)Problem 4.
Evaluate the integral

$$\int \frac{3x dx}{(2x+1)(x-1)}$$

Solution

$$\frac{3x}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1} \qquad \text{[2 points]}$$
$$A = 1 \quad \text{and} \quad B = 1. \qquad \text{[2 points]}$$

$$\begin{aligned} \int \frac{3x dx}{(2x+1)(x-1)} &= \int \left(\frac{1}{2x+1} + \frac{1}{x-1} \right) dx \\ &= \ln|x-1| + \frac{1}{2}\ln\left|x+\frac{1}{2}\right| + C \qquad \text{[4 points]} \end{aligned}$$

(8pts)Problem 5.
Use trigonometric substitution to evaluate the integral

$$\int \frac{dx}{\sqrt{1+x^2}}.$$

Solution

We have an integral involving an expression of the form $\sqrt{a^2+x^2}$, with $a=1$. Put

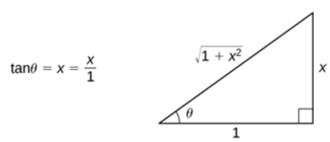
$$x = \tan\theta, \qquad \frac{-\pi}{2} < \theta < \frac{\pi}{2}.$$

$$dx = \sec^2\theta d\theta. \qquad \text{[2 points]}$$

The integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\sec^{\cancel{2}}\theta d\theta}{\sec\theta} \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C. \qquad \text{[2 points]}\end{aligned}$$

Draw the reference triangle in the following figure.



[2 points]

$$\begin{aligned}\cos\theta &= \frac{1}{\sqrt{1+x^2}} \Rightarrow \sec\theta = \sqrt{1+x^2} \\ \tan\theta &= \frac{x}{1}\end{aligned}$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|\sqrt{1+x^2}+x\right| + C \qquad \text{[2 points]}$$