Name: ______________________________

Instructor: __________________________

Math 10560, Worksheet 15, Improper Integrals
February 22, 2016

- Please show all of your work for both MC and PC questions
- work without using a calculator.
- Multiple choice questions should take about 4 minutes to complete.
- Partial credit questions should take about 8 minutes to complete.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
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9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)

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Multiple Choice

1.(6 pts) What can be said about the integrals

$$(i)\int_0^1 \frac{e^x}{x^2}\,dx;$$

$$(ii)\int_1^\infty \frac{\cos^2 x}{x^2}\,dx?$$

Integral (i) diverges by the Comparison Theorem since the integrand is greater than $\frac{1}{x^2}$.

Integral (ii) converges by the Comparison Theorem since the integrand is less than $\frac{1}{x^2}$.

(a) (i) diverges and (ii) converges

(b) both (i) and (ii) converge

(c) neither integral (i) nor (ii) is improper

(d) both (i) and (ii) diverge

(e) (i) converges and (ii) diverges

2.(6 pts) Evaluate the improper integral

$$\int_4^\infty \frac{1}{(x-2)(x-3)}\,dx.$$

Using a partial fraction expansion $\int \frac{1}{(x-2)(x-3)}dx = \ln|\frac{x-3}{x-2}| + C$.

Therefore $\int_4^\infty \frac{1}{(x-2)(x-3)}dx = \lim_{t\to\infty} \ln|\frac{t-3}{t-2}| - \ln|\frac{1}{2}| = 0 + \ln 2$.

(a) $\ln 3$ (b) $\ln \frac{1}{2}$ (c) the integral diverges

(d) $\ln 2$ (e) $3\ln 2$

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3.(6 pts) Evaluate the following improper integral:

$$\int_e^\infty \frac{1}{x(\ln x)^2}dx$$

Solution: Use the definition of improper integral and make the substitution $u = \ln x$ with $dx = xdu$. Then

$$\int_e^\infty \frac{1}{x(\ln x)^2}dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^2}dx = \lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u^2}du$$

$$= \lim_{t\to\infty}\left[-\frac{1}{u}\right]_1^{\ln t} = \lim_{t\to\infty}(-\frac{1}{\ln t}+1) = 1.$$

(a) 0 (b) -1 (c) 1 (d) $\frac{1}{e}$ (e) divergent

4.(6 pts) Find $\int_{-2}^{2}\frac{1}{x+1}dx$.

Solution: Function $\frac{1}{x+1}$ has an infinite discontinuity at the point $x = -1$. Therefore

$$\int_{-2}^{2}\frac{1}{x+1}dx = \int_{-2}^{-1}\frac{1}{x+1}dx + \int_{-1}^{2}\frac{1}{x+1}dx,$$

where each of the integrals is improper. Compute the first integral as follows

$$\int_{-2}^{-1}\frac{1}{x+1}dx = \lim_{t\to\ -1}\int_{-2}^{t}\frac{1}{x+1}dx = \lim_{t\to\ -1}[\ln|x+1|]_{-2}^{t} = \lim_{t\to\ -1}\ln|t+1| - \ln 1 = -\infty.$$

Since $\int_{-2}^{-1}\frac{1}{x+1}dx$ diverges, then the initial integral diverges as well.

(a) diverges (b) 0 (c) $\frac{1}{2}\ln 3$ (d) $\frac{8}{9}$ (e) $\ln 3$

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5.(6 pts) Evaluate the integral $\int_2^\infty xe^{-x}\,dx$.

Solution: First we find the definite integral using integration by parts: Let $u = x$ and $dv = e^{-x}$ so that $du = dx$ and $v = -e^{-x}$. So we have that

$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C$$

Then we see that

$$\int_2^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_2^b xe^{-x}\,dx = \lim_{b\to\infty}\left(-xe^{-x} - e^{-x}\right)\Big|_2^b$$

$$= \lim_{b\to\infty}\left((-be^{-b} - e^{-b}) - (-2e^{-2} - e^{-2})\right) = 0 - (-3e^{-2}) = \frac{3}{e^2}$$

(a) $-\frac{2}{e^2}$ (b) $\frac{1}{e^2}$ (c) divergent (d) $\frac{3}{e^2}$ (e) 1

6.(6 pts) Compute the integral

$$\int_{-3}^3 \frac{1}{(x+2)^3}\,dx.$$

Solution: We have to be careful at the point where the function does not exist, namely $x = -2$. So we see that

$$\int_{-3}^3 \frac{1}{(x+2)^3}\,dx = \int_{-3}^{-2} \frac{1}{(x+2)^3}\,dx + \int_{-2}^3 \frac{1}{(x+2)^3}\,dx$$

We work first on the part $\int_{-2}^3 \frac{1}{(x+2)^3}\,dx$. We will solve this using u-substitution. If we let $u = x+2$ (so $du = dx$), then the bounds change from $x = -2$ to $u = 0$ and $x = 3$ to $u = 5$. Making the substitution we see that

$$\int_{-2}^3 \frac{1}{(x+2)^3}\,dx = \int_0^5 \frac{1}{u^3}\,du = \lim_{b\to 0}\left(\int_b^5 u^{-3}\,du\right)$$

$$= \lim_{b\to 0}\left(-\frac{u^{-2}}{2}\right)\Big|_b^5 = \lim_{b\to 0}\left(-\frac{5^{-2}}{2} + \frac{b^{-2}}{2}\right) = \lim_{b\to 0}\left(-\frac{1}{50} + \frac{1}{2b^2}\right) = \infty$$

So the integral is **divergent**.

(a) $\frac{12}{25}$ (b) 0 (c) $\frac{13}{25}$ (d) divergent (e) $-\frac{13}{25}$

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7.(6 pts) Consider the three integrals

$$\text{I.} \int_{-1}^{1} \frac{dx}{x^2} \,. \qquad \text{II.} \int_{0}^{1} \frac{dx}{\sqrt{x}} \,. \qquad \text{III.} \int_{-1}^{1} \frac{dx}{1+x} \,.$$

One of the following statements is true. Which one?

Solution. We know that $\int_0^1 \frac{1}{x^p}$ diverges if $p \geq 1$, and converges if $p < 1$.

Integral I: The integrand is discontinuous at $x = 0$, and the integral is therefore given as the sum of two improper integrals:

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}.$$

The the second integral on the right hand side is $\int_0^1 \frac{1}{x^p}$ for $p = 2 \geq 1$, and so is divergent (the first one is too). Therefore integral I is divergent.

Integral II: The integral is $\int_0^1 \frac{1}{x^p}$ for $p = \frac{1}{2} < 1$ and thus diverges.

Integral III: Substitute $u = 1 + x$ with $du = dx$.

$$\int_{-1}^{1} \frac{dx}{1+x} = \int_{0}^{2} \frac{du}{u} = \int_{0}^{1} \frac{du}{u} + \int_{1}^{2} \frac{du}{u}.$$

The the first integral on the right hand side is $\int_0^1 \frac{1}{x^p}$ for $p = 1 \geq 1$, and so is divergent. Therefore integral III diverges.

(a) They are all convergent.

(b) They are all divergent.

(c) I is convergent; II and III are divergent.

(d) II and III are convergent; I is divergent.

(e) II is convergent; I and III are divergent.

8.(6 pts) The improper integral

$$\int_{0}^{\infty} \frac{dx}{x^2 + 4}$$

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$$\int_0^\infty \frac{dx}{x^2+4} = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+4}dx = \lim_{t\to\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^t = \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

(a) diverges to ∞ (b) converges to $\frac{\pi}{2}$ (c) converges to $\frac{1}{4}$

(d) diverges to $-\infty$ (e) converges to $\frac{\pi}{4}$

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9.(6 pts) The improper integral

$$\int_0^1 x \ln x dx$$

$$\int_0^1 x \ln x dx \quad = \lim_{t \to 0} \int_t^1 x \ln x dx$$

Using integration by parts with $u = \ln x$, $dv = x dx$, $du = \frac{1}{x} dx$, $v = \frac{x^2}{2}$, we get

$$\lim_{t \to 0} \int_t^1 x \ln x dx = \lim_{t \to 0} \left[\frac{x^2}{2} \ln x \right]_t^1 - \lim_{t \to 0} \int_t^1 \frac{x^2}{2} \cdot \frac{1}{x} dx = \lim_{t \to 0} \left[\frac{-t^2}{2} \ln t \right] - \lim_{t \to 0} \frac{x^2}{4} \Bigg]_t^1$$

Using L'Hospital's Rule, we get

$$= \lim_{t \to 0} \left[\frac{-t^2}{2} \ln t \right] = \lim_{t \to 0} \left[\frac{-\ln t}{2/t^2} \right] = -\frac{1}{2} \lim_{t \to 0} \frac{1/t}{-2/t^3} = -\frac{1}{2} \lim_{t \to 0} \frac{-t^3}{2t} = 0$$

Hence

$$\lim_{t \to 0} \int_t^1 x \ln x dx = -\lim_{t \to 0} \frac{x^2}{4} \Bigg]_t^1 = -\lim_{t \to 0} \left[\frac{1}{4} - \frac{t^2}{4} \right] = -\frac{1}{4}.$$

(a) diverges to ∞ (b) converges to $-\frac{1}{4}$ (c) converges to $-e^2$

(d) converges to $\frac{1}{4}$ (e) diverges to $-\infty$

10.(6 pts) Evaluate the following integral $\int_0^{+\infty} x e^{-x^2} \, dx$.

Put $u = x^2$ then $dx = \frac{1}{2} du$. Also $x = 0 \Rightarrow u = 0$ and $x = +\infty \Rightarrow u = +\infty$. So,

$$\int_0^{+\infty} x e^{-x^2} \, dx = \frac{1}{2} \int_0^{+\infty} e^{-u} \, du.$$

By definition

$$\frac{1}{2} \int_0^{+\infty} e^{-u} \, du = \lim_{a \to +\infty} \frac{1}{2} \int_0^a e^{-u} \, du = \lim_{a \to +\infty} \frac{1}{2} \left[-e^{-u} \right]_0^a = \frac{1}{2}$$

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(a) 1

(b) $2e$

(c) $\dfrac{1}{2}$

(d) Diverges and the limit is not ∞

(e) Diverges and the limit is ∞

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11.(6 pts) Compute
$$\int_0^2 \frac{1}{x-1}\,dx.$$

Note that this integral is an improper integral as $\dfrac{1}{x-1}$ is not defined at $x=1$. Now,
$$\int_0^2 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x-1}\,dx.$$

By definition,
$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx.$$
$$\begin{aligned}\lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx &= \lim_{t\to 1^-} \left[\ln|x-1|\right]_0^t \\ &= \lim_{t\to 1^-} \ln|t-1| = +\infty.\end{aligned}$$

This means that $\int_0^2 \dfrac{1}{x-1}\,dx$ diverges.

Note: You could equally consider $\int_1^2 \dfrac{1}{x-1}\,dx$ and get the same conclusion. The point is that one of these integrals is divergent is enough to conclude $\int_0^2 \dfrac{1}{x-1}\,dx$ is divergent.

(a) Diverges (b) 0 (c) 2

(d) -2 (e) 4

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