Faculty of Engineering and Information Sciences

MATH 142,   Quiz 2,   Spring 2019,   Duration: 60 minutes

Name:_____,   ID Number:_____

Time Allowed: 1 Hour
Total Number of Questions: 6
Total Number of Pages (incl. this page): 7

EXAM UNAUTHORISED ITEMS
Students bringing these items to the examination room shall be required to leave the items at the front of the room or outside the examination room. The University does not guarantee the safe-keeping of students' personal items during examinations. Students concerned about the safety of their valuable items should make alternative arrangements for their care.
1.   Bags, including carry bags, backpacks, shoulder bags and briefcases
2.   Any form of electronic device including but not limited to mobile phones, smart watches, laptops, iPads, MP3 players, handheld computers and electronic dictionaries,
3.   Calculator cases and covers
4.   blank paper
5.   Any written material

DIRECTIONS TO CANDIDATES
1.   Total marks: 40
2.   All questions are compulsory.
3.   Answer all questions on the given exam paper sheets.
4.   Write your name and Id number on the papers provided for rough work.

(6pts)**Problem 1.**
Determine convergence or divergence of the following improper integrals,

$$1. \quad \int_0^3 \frac{dx}{(x-1)^{2/3}} \qquad\qquad 2. \quad \int_2^\infty \frac{dx}{x(\ln x)^3}$$

Solution

1.

$$\text{Find} \quad \int_0^3 \frac{1}{(x-1)^{2/3}}\, dx, \quad \text{if it converges.}$$

Solution: We might think just to do

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = \left[3(x-1)^{1/3}\right]_0^3,$$

but this is not okay: The function $f(x) = \frac{1}{(x-1)^{2/3}}$ is undefined when $x = 1$, so we need to split the problem into two integrals.

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\, dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\, dx.$$

The two integrals on the right hand side both converge and add up to $3[1 + 2^{1/3}]$, so $\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = 3[1 + 2^{1/3}]$. $(3pts)$

2.

$$
\begin{aligned}
\int_2^\infty \frac{dx}{x(\ln x)^3} &= \lim_{t\to\infty} \int_2^t \frac{1}{x}(\ln x)^{-3}\, dx \\
&= \lim_{t\to\infty} \left[\frac{(\ln x)^{-2}}{-2}\right]_2^t \\
&= \lim_{t\to\infty} \left[\frac{-1}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2}\right] \\
&= \frac{1}{2(\ln 2)^2} = 1.040\,7 \qquad\qquad (3pts)
\end{aligned}
$$

(6pts)**Problem 2.**

Solve the following differential equation

$$\frac{dy}{dx} = \frac{(y+3)(x-1)}{(y-2)(x+4)}$$

Solution

The equation is separable

$$\frac{y-2}{y+3}dy = \frac{x-1}{x+4}dx$$

$\Rightarrow$

$$\int \frac{y-2}{y+3}dy = \int \frac{x-1}{x+4}dx$$

$\Rightarrow$

$$\int \left(1 - \frac{5}{y+3}\right) dy = \int \left(1 - \frac{5}{x+4}\right) dx \qquad (3pts)$$

$\Rightarrow$

$$y - 5\ln|y+3| = x - 5\ln|x+4| + C$$

or

$$(y-x) + 5\ln\left|\frac{x+4}{y+3}\right| = C \qquad (3pts)$$

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(7pts)**Problem 3.**
Show that the equation is linear and solve the initial value problem

$$x\frac{dy}{dx} - 2y = x^2, \qquad y(1) = 3$$

SOLUTION Begin by writing the equation in standard form.

$$y' + \left(-\frac{2}{x}\right)y = x \qquad\qquad \text{Standard form, } y' + P(x)y = Q(x)$$

In this form, you can see that $P(x) = -2/x$ and $Q(x) = x$. So,

$$\int P(x)\,dx = -\int \frac{2}{x}\,dx$$

$$= -2\ln x$$

$$= -\ln x^2$$

which implies that the integrating factor is

$$u(x) = e^{\int P(x)\,dx}$$

$$= e^{-\ln x^2}$$

$$= \frac{1}{e^{\ln x^2}}$$

$$= \frac{1}{x^2}. \qquad\qquad \text{Integrating factor}$$

This implies that the general solution is

$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx \qquad\qquad \text{Form of general solution}$$

$$= \frac{1}{1/x^2}\int x\left(\frac{1}{x^2}\right)dx \qquad\qquad \text{Substitute.}$$

$$= x^2\int \frac{1}{x}\,dx \qquad\qquad \text{Simplify.}$$

$$= x^2(\ln x + C). \qquad\qquad \text{General solution} \qquad\qquad (5pts)$$

$$y(1) = 3 \text{ gives } C = 3$$

The solution is

$$y = x^2\,(\ln x - 3) \qquad\qquad\qquad (2pts)$$

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(7pts)**Problem 4.**

Show that the differential equation is exact and solve the equation.

$$\left(1 + 2x - y^3\right) dx + \left(2y - 3xy^2\right) dy = 0$$

Solution

$$M(x, y) = 1 + 2x - y^3 \quad \text{and} \quad N(x, y) = 2y - 3xy^2 \qquad (2pts)$$

$$M_y = -3y^2 = N_x \qquad \therefore \text{ The differential equation is exact.}$$

So, there is a function f such that

$$\frac{\partial f}{\partial x} = 1 + 2x - y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y - 3xy^2$$

$$\frac{\partial f}{\partial x} = 1 + 2x - y^3 \Rightarrow f(x, y) = x + x^2 - xy^3 + g(y). \qquad (3pts)$$

Plugging this into $\dfrac{\partial f}{\partial y} = 2y - 3xy^2$, we obtain

$$-3xy^2 + g'(y) = 2y - 3xy^2 \Rightarrow g'(y) = 2y.$$

Thus

$$g(y) = y^2 + C_1$$

and

$$f(x, y) = x + x^2 - xy^3 + y^2 + C_1$$

The solution of the differential equation is

$$x + x^2 - xy^3 + y^2 + C_1 = C_2 \quad \text{or} \quad x + x^2 - xy^3 + y^2 = C \qquad (2pts)$$

(7pts)**Problem 5.**

Show that the differential equation is homogeneous and solve it.

$$(x - 2y)dx + xdy = 0.$$

Solution

Simplifying we get

$$\frac{dy}{dx} = \frac{2y - x}{x} = 2\frac{y}{x} - 1 = F(\frac{y}{x})$$

This shows that the equation is homogeneous.

Put

$$u = \frac{y}{x}$$

The equation becomes

$$\frac{dy}{u - 1} = \frac{dx}{x}$$

Integrating, we get

$$\ln|u - 1| = \ln|x| + \ln C \qquad \text{with } C > 0.$$

Thus

$$|u - 1| = C|x|$$

$$u - 1 = Ax \qquad \text{where } A = \pm C$$

$$\frac{y}{x} = Ax + 1$$

and

$$y = Ax^2 + x. \qquad \text{(7pts)}$$

(7pts)**Problem 6**

Solve the following Bernoulli differential equation

$$x\frac{dy}{dx} + y = x^2 y^2$$

The equation can be re-written in form (1) simply dividing by x:

$$y' + \frac{1}{x}y = xy^2$$

The substitution to be used in this case is $u = y^{1-2} = 1/y$, or $y = 1/u$. Given that,

$$y' = -\frac{1}{u^2}u'$$

the initial equation becomes,

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = x\frac{1}{u^2}$$

$$\Downarrow$$

$$u' - \frac{1}{x}u = -x$$

We have, thus, obtained a first order linear differential equation with $P(x) = -1/x$ and $Q(x) = -x$. The general solution of this equation is:

$$u = -x^2 + cx$$

To conclude, given that $y = 1/u$, we have the following general solution for the given Bernoulli's equation:

$$y = \frac{1}{-x^2 + cx}$$

(*7pts*)