Integration by Parts

$$\int u \cdot dv = u \cdot v - \int v\,du$$

LIATE

Logarithms, Inverse Trigonometric, Algebraic, Trigonometric, Exponential

We let $u =$ to the function whose category occurs earlier in the list.

$dv =$ rest of the integrand.

Ex. Evaluate the integral

a) $I = \int (2x-3)e^x\,dx$

let $u = 2x-3 \Rightarrow du = 2\,dx$

$dv = e^x dx \Rightarrow v = e^x$

$$\int u\,dv = u.v - \int v\,du$$

$$= (2x-3)e^x - \int 2e^x dx$$

$$= (2x-3)e^x - 2e^x + C$$

$$= 2xe^x - 5e^x + C$$

2nd way (Tabular Integration)

D		I
$2x-3$		e^x
	$+$	
2		e^x
	$-$	
0		e^x

$$I = (2x-3)e^x - 2e^x + C$$

$$= 2xe^x - 5e^x + C$$

b) $\int x^2 e^{2x}\,dx$

$u = x^2 \Rightarrow du = 2x\,dx$

$dv = e^{2x}\,dx \Rightarrow v = \frac{1}{2}e^{2x}$

$$\int \underbrace{x^2}_{u}\,\underbrace{e^{2x}\,dx}_{dv} = \frac{1}{2}x^2e^{2x} - \int \frac{1}{\cancel{2}}e^{2x}\cdot\cancel{2}x\,dx$$

$$= \frac{1}{2}x^2e^{2x} - \boxed{\int x e^{2x}\,dx} \longrightarrow \begin{matrix} u = x & dv = e^{2x}dx \\ du = dx & v = \frac{1}{2}e^{2x} \end{matrix}$$

$$= \frac{1}{2}x^2e^{2x} - \left[\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx\right]$$

$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$

<u>2nd way</u>

D		I
x^2		e^{2x}
	$+$	
$2x$		$\frac{1}{2}e^{2x}$
	$-$	
2		$\frac{1}{4}e^{2x}$
	$+$	
0		$\frac{1}{8}e^{2x}$

$$I = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$

c) $\displaystyle\int \frac{\ln x}{x^2}\,dx$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$dv = \frac{dx}{x^2} \Rightarrow v = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$= -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x} + C$$

2nd <u>way</u>

$$\begin{array}{ccc} D & & I \\ \ln x & + & \frac{1}{x^2} \\ \frac{1}{x} & -\int & -\frac{1}{x} \end{array}$$

$$-\frac{1}{x}\ln x + \int \frac{1}{x}\cdot\frac{1}{x}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x} + C$$

d) $\int x \sec^2 x \, dx$

$u = x \qquad dv = \sec^2 x \, dx$

$du = dx \qquad v = \tan x$

$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx$ Recall $\int \tan x \, dx = -\ln|\cos x| +$

$= x \tan x + \ln|\cos x| + C$

2nd way

D		I
x	$+$	$\sec^2 x$
1	$-$	$\tan x$
0		$-\ln|\cos x|$

$\int x \sec^2 x \, dx = x \tan x + \ln|\cos x| + C$

e) $\int x \ln x \, dx$

$u = \ln x \qquad dv = x \, dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^2}{2}$

$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C$

D		I
$\ln x$	$+$	x
$\frac{1}{x}$	$-\int$	$\frac{x^2}{2}$

$\frac{x^2}{2} \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C$

f) $\int \tan^{-1} x \, dx$

let $u = \tan^{-1} x \qquad dv = dx$

$du = \dfrac{1}{1+x^2}\,dx \qquad v = x$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

$$= x \tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$$

D		I
$\tan^{-1} x$	$+$	1
$\dfrac{1}{1+x^2}$	$-\int$	x

$$\longrightarrow \; x \tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

$$= x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$$

g) $\int x \tan^{-1} x \, dx$

h) $\int \sin^{-1} x \, dx \qquad u = \sin^{-1} x \Rightarrow du = \dfrac{1}{\sqrt{1-x^2}}\,dx$

$dv = dx \Rightarrow v = x$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x \sin^{-1} x - \int x(1-x^2)^{-\frac{1}{2}}\,dx$$

$$= x \sin^{-1} x + \frac{1}{2}\int -2x(1-x^2)^{-\frac{1}{2}}\,dx$$

$$= x \sin^{-1} x + \frac{1}{2}\left[\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right] = x \sin^{-1} x + (1-x^2)^{\frac{1}{2}} + c$$

i) $\int x^3 e^{x^2} dx = \int x^2 \cdot x\, e^{x^2} dx$

1st by <u>substitution</u> let $u = x^2 \Rightarrow du = 2x\,dx$

$$x\,dx = \frac{du}{2}$$

$$\Rightarrow \frac{1}{2}\int u e^u du$$

Then by parts $= \frac{1}{2}\left(ue^u - e^u\right) + c$

$$= \frac{1}{2} x^2 e^{x^2} - e^{x^2} + c.$$

D		I
u		e^u
	$+$	
1		e^u
	$-$	
0		e^u

j) $\int (\ln x)^2 dx$ $\quad u = (\ln x)^2 \quad dv = dx$

$du = 2(\ln x)\cdot \frac{1}{x} dx \quad v = x$

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2(\ln x)\cdot \frac{1}{\cancel{x}}\cdot \cancel{x}\, dx$$

$$= x(\ln x)^2 - 2\int \ln x\, dx$$

$$= x(\ln x)^2 - 2x\ln x + 2x + c$$

D		I
$\ln x$		1
	$+$	
$\frac{1}{x}$	$-\int$	x

$x\ln x - \int \frac{1}{x}\cdot x\, dx$

$x\ln x - \int dx$

$x\ln x - x$

k) $\int e^{ax}\cos bx\,dx$ or $\int e^{ax}\sin bx\,dx$

$\int e^{x}\cos 2x\,dx$ Let $u=e^{x}$ or $u=\cos 2x$

$u=e^{x} \Rightarrow du=e^{x}dx$

$dv=\cos 2x\,dx \Rightarrow v=\frac{1}{2}\sin 2x$

$$\underline{\int e^{x}\cos 2x\,dx} = \frac{1}{2}e^{x}\sin 2x-\frac{1}{2}\underline{\int e^{x}\sin 2x\,dx} \quad (1)$$

Again let $u=e^{x} \Rightarrow du=e^{x}$

$dv=\sin 2x\,dx \Rightarrow v=-\frac{1}{2}\cos 2x$

$$(1) \Rightarrow \int e^{x}\cos 2x\,dx = \frac{1}{2}e^{x}\sin 2x-\frac{1}{2}\left[-\frac{1}{2}e^{x}\cos 2x+\frac{1}{2}\int e^{x}\cos 2x\,dx\right].$$

$$\int e^{x}\cos 2x\,dx = \frac{1}{2}e^{x}\sin 2x+\frac{1}{4}e^{x}\cos 2x-\frac{1}{4}\int e^{x}\cos 2x\,dx$$

$$\frac{5}{4}\int e^{x}\cos 2x\,dx = \frac{1}{2}e^{x}\sin 2x+\frac{1}{4}e^{x}\cos 2x$$

$$\therefore \int e^{x}\cos 2x\,dx = \frac{4}{5}\left(\frac{1}{2}e^{x}\sin 2x+\frac{1}{4}e^{x}\cos 2x\right)+C$$

$$=\frac{2}{5}e^{x}\sin 2x+\frac{1}{5}e^{x}\cos 2x+C$$

2nd way Tabular

$$\int e^x \cos 2x \, dx$$

D	I	or	D	I
e^x	$\cos 2x$		$\cos 2x$	e^x
e^x	$\frac{1}{2}\sin 2x$			
e^x	$-\frac{1}{4}\cos 2x$			

$$e^x \xrightarrow{+} \tfrac{1}{2}\sin 2x, \quad e^x \xrightarrow{-} -\tfrac{1}{4}\cos 2x, \quad e^x \xrightarrow{+\int} -\tfrac{1}{4}\cos 2x$$

$$\int e^x \cos 2x \, dx = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x \, dx$$

$$\frac{5}{4}\int e^x \cos 2x \, dx = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x$$

$$\therefore \int e^x \cos 2x \, dx = \frac{4}{5}\left(\frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x\right) + C$$

$$= \frac{2}{5} e^x \sin 2x + \frac{1}{5} e^x \cos 2x + C.$$

(l) $\int x^4 \log_2 x \, dx$

$u = \log_2 x \rightarrow du = \frac{1}{(\ln 2) x} dx$

$dv = x^4 dx \rightarrow v = \frac{x^5}{5}$

$$\int x^4 \log_2 x \, dx = \frac{x^5}{5} \log_2 x - \int \frac{x^5}{5} \frac{1}{(\ln 2) x} dx$$

$$= \frac{x^5}{5} \log_2 x - \frac{1}{5 \ln 2} \int x^4 dx$$

$$= \frac{x^5}{5} \log_2 x - \frac{1}{25 \ln 2} x^5 + C$$

Practice

Evaluate the following integrals:

1) $\int \cos(\ln x) dx$

2) $\int x \tan^2 x \, dx$

3) $\int \cos^{-1}(2x) dx$

4) $\int_0^2 \ln(x^2+1) dx$

5) $\int \sqrt{x} \ln x \, dx$

torial (7) Trigonometric Substitutions

Expression in the Integrand	Substitution ($a>0$)	Restriction on θ
$\sqrt{a^2-u^2}$	$u = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2+u^2}$	$u = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{u^2-a^2}$	$u = a\sec\theta$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } u \ge a \\ \frac{\pi}{2} < \theta \le \pi & \text{if } u \le -a \end{cases}$

Ex. Evaluate:

$$I = \int \frac{1}{x^2\sqrt{16-x^2}}\, dx$$

We use the substitution $u = a\sin\theta$

$a = 4 \qquad u = x \;\rightarrow\; x = 4\sin\theta$

$\qquad\qquad\qquad\qquad dx = 4\cos\theta\, d\theta$

$$\therefore\; I = \int \frac{4\cos\theta}{(4\sin\theta)^2\sqrt{16-16\sin^2\theta}}\, d\theta$$

$$I = \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16(1-\sin^2\theta)}}\, d\theta$$

$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta\sqrt{\cos^2\theta}}\, d\theta$$

$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta \cdot \cancel{\cos\theta}}\, d\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \rightarrow \cos\theta > 0$$

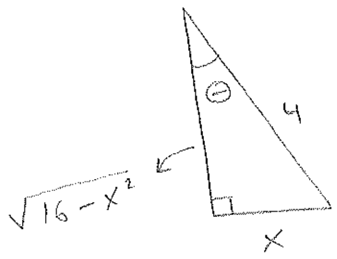
$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{16}\int \csc^2\theta \, d\theta$$

$$= -\frac{1}{16}\cot\theta + C$$

$x = 4\sin\theta$

$\sin\theta = \frac{x}{4}$



$$= -\frac{1}{16}\cdot\frac{\sqrt{16-x^2}}{x} + C$$

Ex. Evaluate $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\, dx$.

we use the substitution $u = a\tan\theta$

$a = 3 \quad u = x \longrightarrow \quad x = 3\tan\theta$

$\qquad\qquad\qquad\qquad dx = 3\sec^2\theta \, d\theta$

$$\int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+9\tan^2\theta}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9(1+\tan^2\theta)}}$$

$$= \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\, d\theta$$

$$= \int \sec\theta \, d\theta$$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2} \longrightarrow$

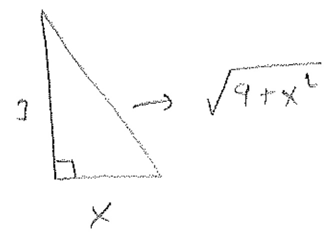
$\sec\theta > 0$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

$x = 3\tan\theta$

$\tan\theta = \frac{x}{3}$



Ex. Evaluate $\displaystyle\int \frac{dx}{(4x^2-9)^{3/2}}$, $x > \frac{3}{2}$

let $I = \displaystyle\int \frac{dx}{\left[(2x)^2-(3)^2\right]^{3/2}}$ we use the substitution $u = a\sec\theta$

$u = 2x \qquad a = 3$

$2x = 3\sec\theta \rightarrow x = \frac{3}{2}\sec\theta$

$\therefore\ dx = \frac{3}{2}\sec\theta\tan\theta\, d\theta$.

$$I = \frac{3}{2}\int \frac{\sec\theta\tan\theta\, d\theta}{\left[(3\sec\theta)^2 - 3^2\right]^{3/2}} = \frac{3}{2}\int \frac{\sec\theta\tan\theta\, d\theta}{\left[\sqrt{9(\sec^2\theta-1)}\right]^3}$$

$$= \frac{1}{18}\int \frac{\sec\theta\tan\theta}{\left(\sqrt{\tan^2\theta}\right)^3}\, d\theta$$

$$= \frac{1}{18}\int \frac{\sec\theta\tan\theta}{\tan^3\theta}\, d\theta$$

$$= \frac{1}{18}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{18}\int \csc\theta\cdot\cot\theta\, d\theta$$

$$= -\frac{1}{18}\csc\theta + C$$

$$\frac{\sec\theta}{\tan^2\theta} = \sec\theta\cdot\frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{\cos\theta}{\sin^2\theta}$$

$$= \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}$$

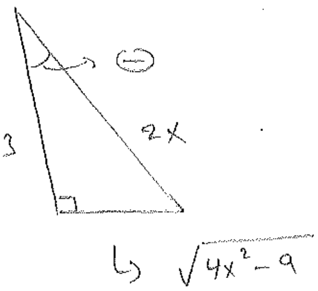
$$= \csc\theta\cdot\cot\theta$$

$$= -\frac{1}{18} \cdot \frac{2x}{\sqrt{4x^2-9}} + C$$

$x = \frac{3}{2}\sec\theta$

$\sec\theta = \frac{2x}{3}$

$$= -\frac{1}{9} \cdot \frac{x}{\sqrt{4x^2-9}} + C$$



Ex. Evaluate $\displaystyle\int_0^3 \frac{x^3}{(3+x^2)^{\frac{5}{2}}}\,dx$ (challenging)

We use the substitution $u = a\tan\theta$

$a = \sqrt{3}$ $\quad u = x \to x = \sqrt{3}\tan\theta$

$dx = \sqrt{3}\sec^2\theta\, d\theta$

Change the limits of integration

$x = 0 \to \tan\theta = 0 \Rightarrow \theta = 0$

$x = 3 \to \tan\theta = \frac{3}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{3}$

$$I = \int_0^{\pi/3} \frac{3\sqrt{3}\tan^3\theta\,(\sqrt{3}\sec^2\theta\, d\theta)}{\left(\sqrt{3+3\tan^2\theta}\right)^5}$$

$\hookrightarrow 3(1+\tan^2\theta)$

$$= \int_0^{\pi/3} \frac{\cancel{9}\tan^3\theta\sec^2\theta\, d\theta}{\cancel{9}\sqrt{3}\left(\sqrt{\sec^2\theta}\right)^5}$$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2} \Rightarrow \sec\theta > 0$

$$= \frac{1}{\sqrt{3}} \int_0^{\pi/3} \frac{\tan^3\theta\sec^2\theta}{\sec^5\theta}\, d\theta = \frac{1}{\sqrt{3}} \int_0^{\pi/3} \frac{\tan^3\theta}{\sec^3\theta}\, d\theta$$

$$\frac{1}{\sqrt{3}} \int_0^{\pi/3} \sin^3\theta \, d\theta$$

$\hookrightarrow \sin\theta \cdot \sin^2\theta$

$$= \frac{1}{\sqrt{3}} \int_0^{\pi/3} \sin\theta(1-\cos^2\theta)\, d\theta$$

By substitution again, let $t = \cos\theta$

$dt = -\sin\theta \, d\theta$

$\theta = 0 \to t = 1$

$\theta = \frac{\pi}{3} \to t = \frac{1}{2}$

$$= \frac{1}{\sqrt{3}} \int_1^{\frac{1}{2}} (1-t^2)(-dt)$$

$$= \frac{1}{\sqrt{3}} \int_1^{\frac{1}{2}} (t^2-1)\, dt = \frac{1}{\sqrt{3}} \left[\frac{t^3}{3} - t \right]_1^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{3}} \left[\frac{1}{24} - \frac{1}{2} - \frac{1}{3} + 1 \right]$$

$$= \frac{1}{\sqrt{3}} \left(\frac{5}{24} \right)$$

$$= \frac{5}{24\sqrt{3}} \text{ or } \frac{5\sqrt{3}}{72}$$

Ex. Evaluate $\displaystyle\int_{\sqrt{2}}^{2} \frac{\sqrt{2x^2-4}}{x}\,dx$.

We use the substitution $u = a\sec\theta$

$a = 2 \qquad u = \sqrt{2}x \rightarrow \sqrt{2}x = 2\sec\theta$

$$x = \frac{2}{\sqrt{2}}\sec\theta = \sqrt{2}\sec\theta .$$

$$dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$$

$x \geq \sqrt{2} \rightarrow$
$\sqrt{2}x \geq 2$
$u \geq 2 \rightarrow u \geq a$
$\downarrow$
$0 \leq \theta \leq \frac{\pi}{2}$

Change of limits.

$x = \sqrt{2} \rightarrow \sqrt{2} = \sqrt{2}\sec\theta$

$\sec\theta = 1 \Rightarrow \theta = 0$

$x = 2 \rightarrow 2 = \sqrt{2}\sec\theta$

$\sec\theta = \frac{2}{\sqrt{2}}$
$\downarrow$
$\cos\theta = \frac{\sqrt{2}}{2} \longrightarrow \theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{\sqrt{2(\sqrt{2}\sec\theta)^2-4}}{\sqrt{2}\sec\theta} \cdot \sqrt{2}\sec\theta\tan\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{\sqrt{4\sec^2\theta-4}}{\cancel{\sqrt{2}\sec\theta}}\,\cancel{\sqrt{2}\sec\theta}\tan\theta\,d\theta$$

$$= 2\int_0^{\pi/4} \sqrt{\sec^2\theta-1}\,\tan\theta\,d\theta$$

$$= 2\int_0^{\pi/4} \sqrt{\tan^2\theta}\,\tan\theta\,d\theta$$

$$= 2\int_0^{\pi/4} \tan\theta \cdot \tan\theta \, d\theta \qquad 0 \le \theta \le \frac{\pi}{4} \qquad \tan\theta \ge 0$$

$$= 2\int_0^{\pi/4} \tan^2\theta \, d\theta$$

$$= 2\int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta$$

$$= 2\Big[\tan\theta - \theta\Big]_0^{\pi/4}$$

$$= 2\left[1 - \frac{\pi}{4}\right] = 2 - \frac{\pi}{4}$$

Integration of Rational Functions Using Partial Fractions

(1) Partial Fractions – Distinct Linear Factors

$$\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

Ex. $\displaystyle\int \frac{dx}{x^2+x-2}$

$$\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)}$$

$$\frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$