Tutorial 8

1.
a) $f(x) = \frac{1}{x^3}$

f is continuous, positive and decreasing on $[1, \infty)$

$n = 10$

$$\int_{n+1}^{\infty} f(x)\,dx \leq S - S_n \leq \int_{10}^{\infty} f(x)\,dx$$

$$S_{10} + \int_{11}^{\infty} \frac{dx}{x^3} \leq S \leq S_{10} + \int_{10}^{\infty} \frac{dx}{x^3}$$

$$S_{10} = \sum_{k=1}^{10} \frac{1}{k^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} = 1.1975$$

$$\int_{11}^{\infty} x^{-3}\,dx = \lim_{t\to\infty} \int_{11}^{t} x^{-3}\,dx$$
$$= \lim_{t\to\infty} \frac{-1}{2x^2}\Big|_{11}^{t}$$
$$= \frac{1}{242}$$
$$= 0.0041322$$

$$\int_{10}^{\infty} x^{-3}\,dx = \lim_{t\to\infty} \int_{10}^{t} x^{-3}\,dx$$
$$= \lim_{t\to\infty} \frac{-1}{2x^2}\Big|_{10}^{t}$$
$$= \frac{1}{200}$$
$$= 0.005$$

$$1.20163 \leq S \leq 1.2025$$

b) $$R_n = S - S_n \leq \int_{n}^{\infty} \frac{dx}{x^3}$$

$$S - S_n \leq \frac{1}{2n^2} \leq 0.0005$$

$$\frac{1}{2n^2} \leq 0.0005$$

$$\frac{1}{n^2} \leq 0.001$$

$$n^2 \geq 1000$$

$$n \geq \sqrt{1000}$$

$$n \geq 31.62$$

$$\underline{n \geq 32}$$

2. a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+3)}{n(n+1)}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}(n+3)}{n(n+1)} \right|$$

$$= \sum_{n=1}^{\infty} \left| \frac{n+3}{n^2+n} \right| \sim \sum_{n=1}^{\infty} \frac{n}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

$p = 1 \leq 1$

$\therefore$ Diverges by p-series test

$\therefore \displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}(n+3)}{n(n+1)} \right|$ diverges

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n+3)}{n(n+1)}$$

$b_n = \dfrac{n+3}{n^2+n} \Rightarrow$ positive, decreasing and $\lim_{n\to\infty} b_n = 0$

$\therefore$ Converges by AST

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+3)}{n(n+1)}$ converges conditionally

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b) $\displaystyle\sum_{n=2} \frac{(-1)^n}{n\ln n}$

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n\ln n} \right|$$

$$= \sum_{n=2}^{\infty} \left| \frac{1}{n\ln n} \right|$$

Using Integral Test

$$\int_2^{\infty} \frac{dx}{x\ln x}$$

$$= \lim_{t\to\infty} \int_2^{t} \frac{dx}{x\ln x}$$

$$= \lim_{t\to\infty} \ln(\ln x) \Big|_2^t$$

$$= \infty$$

$\therefore$ Diverges

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$$

$b_n = \dfrac{1}{n\ln n}$ $\Rightarrow$ positive, decreasing, and $\lim_{n\to\infty} b_n = 0$

$\therefore$ Converges by AST

$\therefore$ Series converges conditionally

3. a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{e^{n+1}}{(n+1)!} \times \frac{n!}{e^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{e}{n+1} \right|$$

$$= 0 < 1$$

$\therefore$ converges by ratio test

b) $$\sum_{n=1}^{\infty} \left(\frac{2n}{13n+1} \right)^n$$

$$\lim_{n\to\infty} \left| \left(\frac{2n}{13n+1} \right)^n \right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left| \frac{2n}{13n+1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2n}{13n} \right|$$

$$= \frac{2}{13} < 1$$

$\therefore$ converges by root test

4. a) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} (x+1)^n$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} (x+1)^{n+1}}{2^{n+1}} \times \frac{2^n}{(-1)^n (x+1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x+1)}{2} \right|$$

$$= \frac{1}{2} |x+1|$$

Radius of convergance $= \frac{1}{L} = 2$

Endpoints $a - R = -1 - 2 = -3$

$a - R = -1 + 2 = 1$

@ $x = -3$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} (-3+1)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{(-1)(-2)}{2} \right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{2} \right)^n$$

$$= \sum_{n=1}^{\infty} 1^n$$

Diverges by geometric series test

@ $x = 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} (1+1)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\cancel{2^n}} \times \cancel{2^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n$$

$b_n = 1 \Rightarrow$ positive, constant, $\lim_{n\to\infty} b_n \neq 0$

Diverges by alternating series test

$\therefore IC = (-3, 1)$

$\boxed{\text{Series of a non-zero number always diverges}}$