Slope of polar curves

Consider the polar curve

$$r = f(\theta)$$

$$r = \cos\theta$$

$$r^2 = r\cos\theta$$

$$x^2 + y^2 = x$$

$$x^2 - x + y^2 = 0$$

$$\left(x - \frac{1}{2}\right) - \frac{1}{4} + y^2 = 0$$

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

$x = r\cos\theta \qquad y = r\sin\theta$

$x = f\theta\cos\theta \qquad y = f\theta\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example

Find the slope of the polar curve $r = e^{\cos\theta}$ at $\theta = 0$ and $\theta = \pi/2$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{-\sin\theta\, e^{\cos\theta}\sin\theta + \cos\theta\, e^{\cos\theta}}{-\sin\theta\, e^{\cos\theta}\cos\theta - \sin\theta\, e^{\cos\theta}}$$

$$= \frac{e^{\cos\theta}(\cos\theta - \sin^2\theta)}{-e^{\cos\theta}\sin\theta(1 + \cos\theta)}$$

@ $\theta = 0$ @ $\theta = \pi/2$

$$\frac{dy}{dx} = \frac{e(1-0)}{0} = \infty \qquad \frac{dy}{dx} = \frac{-1}{-1} = 1$$

Vertical tangent

Arc Length of Polar Curve

If $f(\theta)$ has continuous derivative on the interval $[\alpha, \beta]$ $(\alpha \leq \theta \leq \beta)$ then the length of the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

Example

Find the arc length of the polar curve $r = 2 - 2\cos\theta$ from $\theta = 0$ to $\theta = 2\pi$

$$[f(\theta)]^2 = (2 - 2\cos\theta)^2$$
$$= 4 + 4\cos^2\theta - 8\cos\theta$$

$$[f'(\theta)]^2 = (2\sin\theta)^2$$
$$= 4\sin^2\theta$$

$$[f(\theta)]^2 + [f'(\theta)]^2 = 4\sin^2\theta + 4 + 4\cos^2\theta - 8\cos\theta$$
$$= 8 - 8\cos\theta$$
$$= 8(1 - \cos\theta)$$
$$= 8\left(1 - (1 - 2\sin^2\theta/2)\right)$$
$$= 8(2\sin^2\theta/2)$$
$$= 16\sin^2\frac{\theta}{2}$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$
$$\cos\alpha = 1 - 2\sin^2\alpha/2$$

$$= \left(4 \sin \frac{\theta}{2}\right)^2$$

$$L = \int_0^{2\pi} \sqrt{(4 \sin \theta/2)^2} \, d\theta$$

$$= \int_0^{2\pi} |4 \sin \theta/2| \, d\theta$$

$$= -4 \times 2 \left[\cos \frac{\theta}{2}\right]_0^{2\pi}$$

$$= -8(\cos \pi - \cos 0)$$

$$= -8(-1-1)$$

$$= \underline{16}$$

Differential Equations

A differential equation is an equation that involves a function and its derivative.

$$x \frac{dy}{dx} + \cos y = e^x$$ → First Order Differential Equation

$$e^x \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 2$$ → Second Order Differential Equation

Types of Differential Equations

- Separable
- Linear
- Exact
- Non-Exact

- Non Exact
- Homogeneous
- Bernoulli

Separable Differential Equations

A first order differential equation is said to be separable if it is of the form

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

Example

Which of the following equations are separable?

1. $\frac{dy}{dx} - y^2 x e^{3x+4y} = 0$

2. $\frac{dy}{dx} - y = \sin x$

1. $$\frac{dy}{dx} = xy^2 e^{3x+4y}$$
$$= xe^{3x} \cdot e^{4y} \cdot y^2$$
$$= \underbrace{(xe^{3x})}_{g(x)} \underbrace{(y^2 e^{4y})}_{h(y)}$$

$\therefore$ Equation is separable

2. $$\frac{dy}{dx} = y + \sin x \neq (g(x))(h(y))$$

Not separable

Example

Show that the equation is separable and solve it.

$$(1+x)\,dy - y\,dx = 0$$

$$(1+x)\,dy = y\,dx$$
$$\frac{dy}{dx} = \frac{y}{(1+x)}$$

$$= \left(\frac{1}{1+x}\right)(y)$$

$\therefore$ Separable

$$\frac{dy}{dx} = \frac{y}{1+x}$$

$$\frac{1}{y}\,dy = \frac{1}{1+x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

$$e^{\ln|y|} = e^{\ln|1+x|+c}$$

$$|y| = e^{c}\cdot e^{\ln|1+x|}$$

$$|y| = |1+x|\,e^{c}$$

$$|y| = |e^{c}(1+x)|$$

$$y = \pm e^{c}(1+x)$$

$$= k(1+x)$$

$$k = \pm e^{c}$$

$$\ln|y| = \ln|1+x| + C \rightarrow \ln|a|,\ a>0$$

$$\ln|y| = \ln|1+x| + \ln|a|$$

$$|y| = |(1+x)\,a|$$

$$y = \pm a(1+x)$$

$$y = k(1+x)$$

$\ln|a| + \ln|b| = \ln|ab|$

$|A| = |B| \Rightarrow A = \pm B$

k arbitrary

A differential equation with an initial condition is called an initial value problem.

Example

Show that the equation is separable and solve the initial value problem.

$$\frac{dy}{dx} = -\frac{x}{y} \qquad y(4) = -3$$

$$\frac{dy}{dx} = (-x)\left(\frac{1}{y}\right)$$

$\therefore$ Separable

$$y\,dy = -x\,dx$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

when $x = 4$, $y = (-3)$

$$(-3)^2 = -(4)^2 + 2c$$

$$9 = -16 + R$$

$$\underline{R = 25}$$

Solution is

$$y^2 = -x^2 + 25$$

$$x^2 + y^2 = 25$$