(10pts)Problem 1
Solve the first order differential equation

$$\left(x^2+4\right)\frac{dy}{dx}-2xy=x\left(x^2+4\right)^3. \qquad \text{(Show your work)}$$

Solution

$$\left(x^2+4\right)\frac{dy}{dx}-2xy=x\left(x^2+4\right)^3$$

$\Leftrightarrow$

$$\frac{dy}{dx}-\frac{2x}{x^2+4}y=x\left(x^2+4\right)^2. \qquad \textbf{(2pts)}$$

This equation is linear with integrating factor

$$IF=e^{\int -\frac{2x}{x^2+4}dx}=e^{-\ln\left(x^2+4\right)}=\frac{1}{x^2+4}. \qquad \textbf{(3pts)}$$

We have

$$\frac{d}{dx}\left[\frac{y}{x^2+4}\right]=x\left(x^2+4\right)=x^3+4x$$

$\Rightarrow$

$$\frac{y}{x^2+4}=\frac{x^4}{4}+2x^2+C$$

$\Leftrightarrow$

$$y=\left(x^2+4\right)\left(\frac{x^4}{4}+2x^2+C\right). \qquad \textbf{(5pts)}$$

(**10pts**)**Problem 2**

Show that the differential equation is exact and solve the initial value problem

$$\left(-y\cos x+\sec^2 x\right)dx+(2y-\sin x)\,dy=0,\qquad y(0)=2\qquad (\text{ Show your work})$$

Solution

$$M=-y\cos x+\sec^2 x \quad \text{and} \quad N=2y-\sin x$$

$$M_y=-\cos x,\quad N_x=-\cos x$$

$$M_y=N_x \Rightarrow \text{ the equation is exact.} \qquad (\mathbf{3pts})$$

We have

$$\begin{cases} f_x=-y\cos x+\sec^2 x \\ f_y=2y-\sin x \end{cases}.$$

$$f_x=-y\cos x+\sec^2 x \Rightarrow f(x,y)=-y\sin x+\tan x+g(y). \qquad (\mathbf{3pts})$$

$f_y=2y-\sin x$ gives

$$-\sin x+g'(y)=2y-\sin x$$

$\Leftrightarrow$

$$g'(y)=2y \Rightarrow g(y)=y^2+K.$$

$f(x,y)$ is finally given by

$$f(x,y)=-y\sin x+\tan x+y^2$$

and the solution of the equation is

$$-y\sin x+\tan x+y^2=C.$$

Now using the initial condition $y(0)=2$, we obtain

$$C=4.$$

Solution is

$$-y\sin x+\tan x+y^2=4. \qquad (\mathbf{4pts})$$

(12pts)Problem 3
A) Show that the differential equation is not exact, find a special integrating factor and make it exact.

$$\left(y + \frac{1}{3}y^3 + \frac{1}{3}x^2\right) dx + \frac{1}{4}\left(x + xy^2\right) dy = 0. \qquad \text{(Show your work)}$$

Solution

$$M = y + \frac{1}{3}y^3 + \frac{1}{3}x^2 \text{ and } N = \frac{1}{4}\left(x + xy^2\right)$$

$$\frac{M_y - N_x}{N} = \frac{1 + y^2 - \frac{1}{4}\left(1 + y^2\right)}{\frac{1}{4}\left(x + xy^2\right)} = \frac{3}{x}, \text{ a function of } x \text{ alone.} \qquad \textbf{(2pts)}$$

$$SIF = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3. \qquad \textbf{(2pts)}$$

To make the equation exact, we multiply both sides of the equation by x^3.

$$\left(yx^3 + \frac{x^3}{3}y^3 + \frac{1}{3}x^5\right) dx + \frac{1}{4}\left(x^4 + x^4y^2\right) dy = 0. \qquad \textbf{(2pts)}$$

$$\frac{\partial}{\partial y}\left(yx^3 + \frac{x^3}{3}y^3 + \frac{1}{3}x^5\right) = x^3\left(y^2 + 1\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{4}\left(x^4 + x^4y^2\right)\right) = x^3\left(y^2 + 1\right) = \frac{\partial}{\partial y}\left(yx^3 + \frac{x^3}{3}y^3 + \frac{1}{3}x^5\right)$$

This shows that the equation is exact.

B) Identify the equation and use the appropriate substitution to transform it into a linear equation. (DO NOT SOLVE)

$$\left(\sin x - y^2 \cos x\right) dx + \frac{1}{y} dy = 0. \qquad \text{(Show your work)}$$

Solution
The equation is equivalent to

$$\frac{dy}{dx} + y \sin x = y^3 \cos x.$$

This is a Bernoulli equation with $n = 3$. **(3pts)**
We have

$$y^{-3}\frac{dy}{dx} + y^{-2} \sin x = \cos x.$$

We do the substitution

$$u = y^{-2}$$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow y^{-3}\frac{dy}{dx} = \frac{-1}{2}\frac{du}{dx}.$$

The equation becomes the linear equation

$$\frac{-1}{2}\frac{du}{dx} + u \sin x = \cos x$$

$\Leftrightarrow$

$$\frac{du}{dx} - 2u \sin x = -2 \cos x. \qquad (\mathbf{3pts})$$

(10pts)Problem 4

Show that the equation is separable and solve the equation

$$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}. \qquad \text{(Show your work)}$$

Solution

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} \\ &= \frac{(y+3)(x-1)}{(y-2)(x+4)} \\ &= \left[\frac{(y+3)}{(y-2)}\right]\left[\frac{(x-1)}{(x+4)}\right] = f(x)g(y) \Rightarrow \text{ equation is separable.} \quad (\mathbf{4pts}) \end{aligned}$$

$$\frac{y-2}{y+3}dy = \frac{x-1}{x+4}dx$$

$\Rightarrow$

$$\int \frac{y-2}{y+3}dy = \int \frac{x-1}{x+4}dx$$

$$\begin{aligned} \int \frac{y-2}{y+3}dy &= \int \frac{y+3-3-2}{y+3}dy \\ &= \int \left(1 - \frac{5}{y+3}\right) dy \\ &= y - 5\ln|y+3| + C_1 \end{aligned}$$

Similarly,

$$\int \frac{x-1}{x+4}dx = x - 5\ln|x+4| + C_2.$$

We have

$$y - 5\ln|y+3| + C_1 = x - 5\ln|x+4| + C_2$$

$\Leftrightarrow$

$$y - x = 5\ln\left|\frac{y+3}{x+4}\right| + C \quad \text{where} \quad C = C_2 - C_1 \quad (\mathbf{6pts})$$

(10pts)Problem 5

Find $\lim_{n\to\infty} a_n$.

$(i)\quad a_n = n\sin\left(\pi + \frac{3}{n}\right)$ $\qquad$ $(ii)\quad \frac{4}{1}, \frac{7}{3}, \frac{10}{5}, \frac{13}{7}, \ldots$ $\qquad$ (Show your work)

Solution

$(i)\quad a_n = n\sin\left(\pi + \frac{3}{n}\right)$

$$\begin{aligned}\lim_{n\to\infty} n\sin\left(\pi+\frac{3}{n}\right) &= \lim_{n\to\infty}\frac{\sin\left(\pi+\frac{3}{n}\right)}{\frac{1}{n}}\\ &= \lim_{x\to\infty}\frac{\sin\left(\pi+\frac{3}{x}\right)}{\frac{1}{x}}\end{aligned}$$

Using L'Hopital's rule,

$$\begin{aligned}\lim_{x\to\infty}\frac{\sin\left(\pi+\frac{3}{x}\right)}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{-\frac{3}{x^2}\cos\left(\pi+\frac{3}{x}\right)}{\frac{-1}{x^2}}\\ &= \lim_{x\to\infty} 3\cos\left(\pi+\frac{3}{x}\right) = 3\cos\pi = -3. \qquad (\mathbf{5pts})\end{aligned}$$

$(ii)\quad \frac{4}{1}, \frac{7}{3}, \frac{10}{5}, \frac{13}{7}, \ldots$

The numerator is an arithmetic sequence with first term 4 and common difference 3. Thus, a formula for the numerator is

$$4 + 3(n-1) = 3n+1$$

The denominator is an arithmetic sequence with first term 1 and common difference 2. Thus, a formula for the numerator is

$$1 + 2(n-1) = 2n-1.$$

Hence,

$$a_n = \frac{3n+1}{2n-1} \quad \text{and} \quad \lim_{n\to\infty} a_n = \frac{3}{2}. \qquad (\mathbf{5pts})$$

(12pt$s$)Problem 6

Find the sum of the following series

$$(i)\ \sum_{n=1}^{\infty}\left[(-0.2)^n+(0.6)^{n-1}\right] \qquad (ii)\ \sum_{n=0}^{\infty}\left[6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right)-6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\right].$$

(Show your work)

Solution

(i)

$$\begin{aligned}\sum_{n=1}^{\infty}\left[(-0.2)^n+(0.6)^{n-1}\right] &= \sum_{n=1}^{\infty}(-0.2)^n+\sum_{n=1}^{\infty}(0.6)^{n-1}\\ &= \sum_{n=1}^{\infty}(-0.2)\,(-0.2)^{n-1}+\sum_{n=1}^{\infty}(0.6)^{n-1}\\ &= \frac{-0.2}{1-(-0.2)}+\frac{1}{1-0.6}\\ &= 2.333\,3 \qquad (\ (\mathbf{3pts})+\ (\mathbf{3pts}))\end{aligned}$$

(ii)

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right)-6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\right]\\ =\ &\lim_{N\to\infty}\sum_{n=0}^{N}\left[6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right)-6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\right]\\ =\ &\lim_{N\to\infty}\left[\begin{array}{c}6\tan^{-1}\left(\sqrt{3}\right)-6\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)+6\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)-6\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\\ +\ldots+6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right)-6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\end{array}\right] (\mathbf{3pts})\\ =\ &\lim_{N\to\infty}\left[6\tan^{-1}\left(\sqrt{3}\right)-6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right)\right]=6\tan^{-1}\left(\sqrt{3}\right)-0\\ =\ &2\pi \qquad (\mathbf{3pts})\end{aligned}$$

(12pt$s$)Problem 7
Determine whether the following series converges or diverges. (Justify your answer and show your work)

$$(i) \quad \sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n} \qquad\qquad (ii) \quad \sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

(Show your work)

Solution
(i)

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n} \text{ is an alternating series with } b_n = \frac{1}{n \ln n}.$$

Since for $n \geq 2, b_n$ is positive, decreasing and $\lim\limits_{n\to\infty} b_n = 0$, the series converges by the alternating series test. (**6pts**)

(ii)

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$$\frac{1}{n \ln n} = f(n) \text{ where } f(x) = \frac{1}{x \ln x} \text{ is continuous, positive and decreasing when } x \geq 2.$$

We will use the integral test.

$$\begin{aligned}
\int_2^{\infty} \frac{dx}{x \ln x} &= \lim_{t\to\infty} \int_2^t \frac{dx}{x \ln x} \\
&= \lim_{t\to\infty} \ln |\ln x| \Big|_2^t \\
&= \lim_{t\to\infty} [\ln |\ln t| - \ln \ln 2] \\
&= +\infty. \text{ The series diverges.} \qquad (\textbf{6pts})
\end{aligned}$$

(12pts)Problem 8

Find the interval of convergence of the following series

$$(i)\ \sum_{n=1}^{\infty} \frac{(x+3)^n}{n5^n} \qquad (ii)\ \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+2)!}. \qquad \text{(Show your work)}$$

Solution

$$(i)\ \sum_{n=1}^{\infty} \frac{(x+3)^n}{n5^n}$$

Center c $= -3$

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{(n+1)\,5^{n+1}} \cdot \frac{n5^n}{(x+3)^n} \right| &= \lim_{n\to\infty} \left| \frac{(x+3)}{(n+1)\,5} \cdot \frac{n}{1} \right| \\ &= \frac{1}{5}\,|x+3| \end{aligned}$$

$$R = 5. \qquad \textbf{(3pts)}$$

$$c - R = -3 - 5 = -8 \quad \text{and} \quad c + R = -3 + 5 = 2.$$

When $x = -8$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This series converges by the alternating series test.

When $x = 2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{5^n}{n5^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This series diverges by the p-series test.

$$\text{Interval of convergence } = [-8,\ 2). \qquad \textbf{(3pts)}$$

$$(ii)\ \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+2)!}.$$

$$\begin{aligned} &\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{(-1)^n x^n} \right| \\ = &\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^n x}{(n+3)\,(n+2)!} \cdot \frac{(n+2)!}{(-1)^n x^n} \right| \\ = &\lim_{n\to\infty} \left| \frac{x}{(n+3)} \cdot \frac{1}{1} \right| = 0 \end{aligned}$$

$$R = \infty \qquad \textbf{(3pts)}$$

$$IC = (-\infty,\ \infty) \qquad \textbf{(3pts)}$$

(12pts)Problem 9

A) Find the Maclaurin series of

$$f(x) = \frac{x^2}{3-x}. \qquad \text{(Show your work)}$$

Solution

$$\begin{aligned}
\frac{1}{3-x} &= \frac{1}{3\left(1-\frac{x}{3}\right)} \\
&= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \quad \text{if} \quad \left|\frac{x}{3}\right| < 1. \qquad (\mathbf{3pts}) \\
&= \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}} \text{ if } \quad |x| < 3.
\end{aligned}$$

$$f(x) = \frac{x^2}{3-x} = \sum_{n=0}^{\infty}\frac{x^{n+2}}{3^{n+1}} \text{ if } \quad |x| < 3 \qquad (\mathbf{3pts})$$

B) Find the sum of the series

$$\frac{\pi}{2}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n+1)!}. \qquad \text{(Show your work)}$$

Solution

We know that

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \text{ for all } x. \qquad (\mathbf{2pts})$$

$$\begin{aligned}
\frac{\pi}{2}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n+1)!} &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!} \\
&= \frac{1}{2}\sin\pi = 0 \qquad (\mathbf{4pts})
\end{aligned}$$