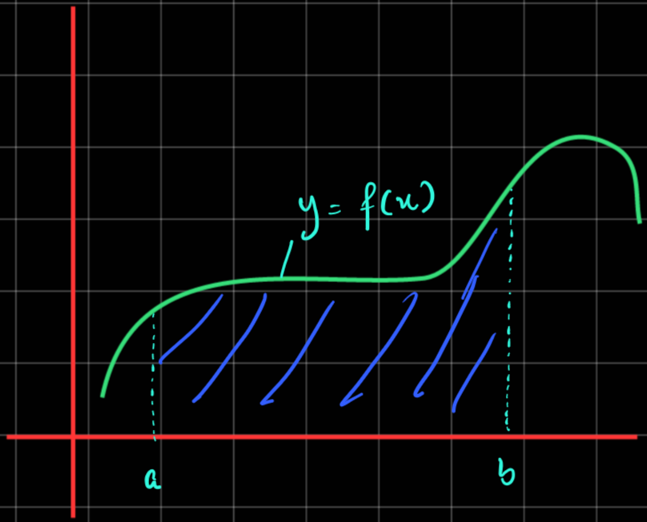
Application of Definite Integral

1. Area between two curves

If f is continuous on $[a, b]$

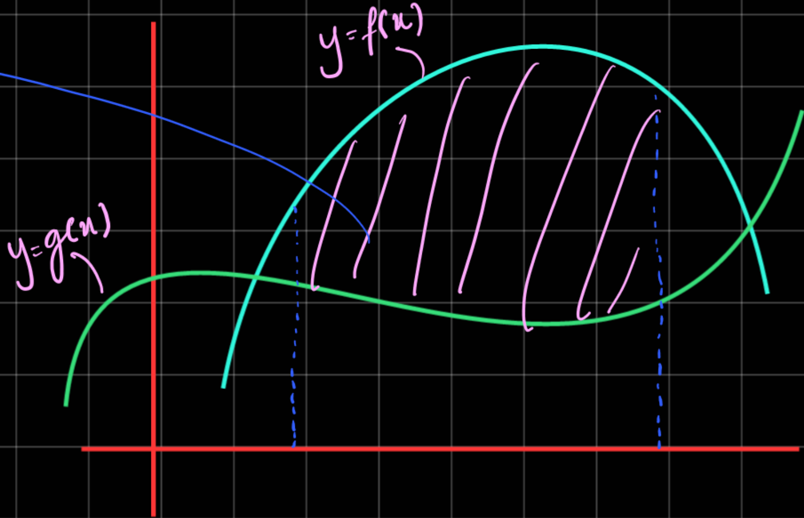
$$\int_a^b f(x)\,dx = \text{Area under the curve}$$



Area bounded by two curves

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$A = \int_a^b [f(x) - g(x)]\,dx$$



Example

Find the area of the region bounded by the graphs of

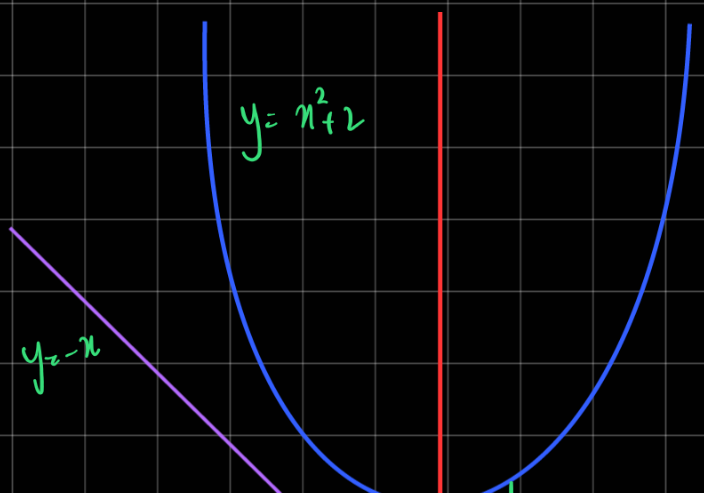
$y = x^2 + 2$, $y = -x$

$x = 0$ and $x = 1$

$$A = \int_0^1 (x^2+2) - (-x)\,dx$$

$$= \int_0^1 x^2 + x + 2\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1 dx$$



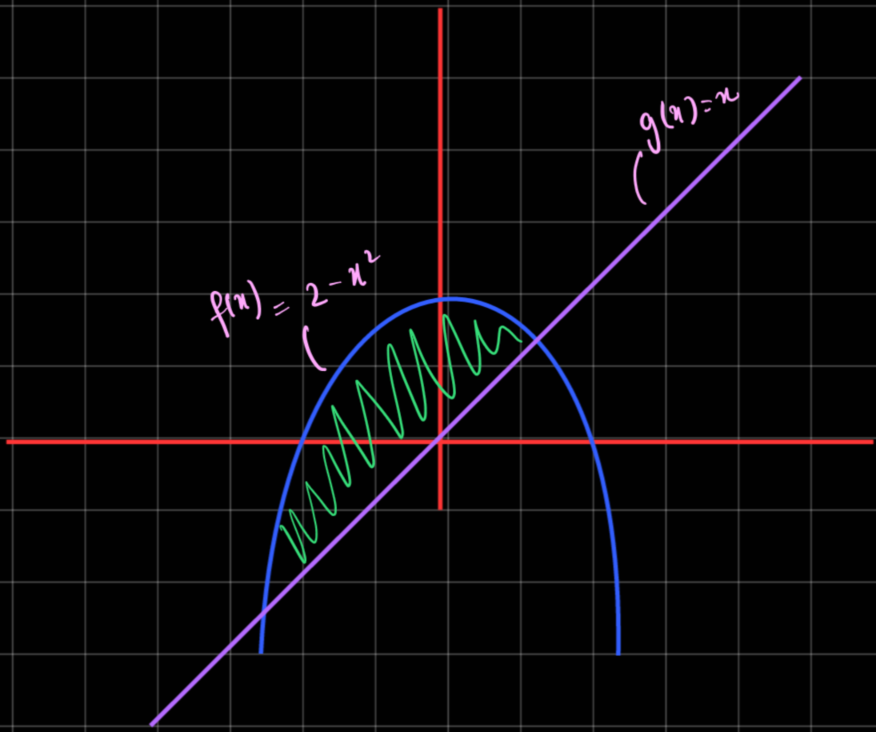
$= \frac{1}{3} + \frac{1}{2} + 2(1)$

$= \frac{17}{6}$ units

Example

Find the area bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$

$A = \int_a^b [(2 - x^2) - (x)]\, dx$



Finding a and b

$2 - x^2 = x$

$x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

$x(x+2) - 1(x+2) = 0$

$x = -2$ OR $x = 1$

$\therefore\ a = -2 \quad b = 1$

$A = \int_{-2}^{1} [2 - x^2 - x]\, dx$

$= \left[2x - \frac{x^3}{3} - \frac{x^2}{2} \right]_{-2}^{1}$

$= \left[2 - \frac{1}{3} - \frac{1}{2} \right] - \left[-4 + \frac{8}{3} - 2 \right]$

$= \frac{9}{2}$ units

Let f and g be continuous on $[a, b]$.

The area bounded by the graph of f and g between $x=a$ and $x=b$ is given by the formula

$$A = \int_a^b |f(x) - g(x)|\, dx$$

$$A = \int_a^b |g(x) - f(x)|\, dx$$

Example

Find the area of the region bounded by the graphs of

$y = x^2+2$, $y = -x$

$x=0$ and $x=1$

$$A = \int_0^1 |-x - (x^2+2)|\, dx$$

$$= \int_0^1 |-x - x^2 - 2|\, dx$$

$a = -1,\ b = -1,\ c = -2$

$$b^2 - 4ac = 1 - 4(-1)(-2)$$

$$= 1 - 8$$

$$= -7 < 0$$

x	$-\infty$		∞
$-x^2-x-2$		$-$	

$$A = -\int_0^1 (-x^2 - x - 2)\, dx$$

$$= \int_0^1 x^2 + x + 2\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^1 dx$$

$$= \frac{1}{3} + \frac{1}{2} + 2(1)$$

$$= \frac{17}{6} \text{ units}$$

Example

Find the area between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$

$$A = \int_a^b \left| f(x) - g(x) \right| dx$$

$$= \int_a^b \left| 3x^3 - x^2 - 10x - (-x^2 + 2x) \right| dx$$

Finding a and b

$$3x^3 - \cancel{x^2} - 10x = -\cancel{x^2} + 2x$$

$$3x^3 - 10x = 2x$$

$$3x^3 - 12x = 0$$

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$$x(x+2)(x-2) = 0$$

$$x = -2, \quad x = 0, \quad x = 2$$

$$a = -2 \qquad b = 2$$

$$A = \int_{-2}^2 \left| 3x^3 - \cancel{x^2} - 10x + \cancel{x^2} - 2x \right| dx$$

$$= \int_{-2}^2 \left| 3x^3 - 12x \right| dx$$

$$= \int_{-2}^{2} |3x(x^2-4)|\, dx$$

x	$-\infty$		-2		0		2		$+\infty$
$3x$		$-$		$-$	0	$+$		$+$	
x^2-4		$+$	0	$-$		$-$	0	$+$	
$3x^3-12x$		$-$		$+$		$-$		$+$	

$$A = \int_{-2}^{0} (3x^3 - 12x)\, dx - \int_{0}^{2} (3x^3-12x)\, dx$$

$$A = \left[\frac{3x^4}{4} - \frac{12x^2}{2}\right]_{-2}^{0} - \left[\frac{3x^4}{4} - \frac{12x^2}{2}\right]_{0}^{2}$$

$$= 0 - \left[\frac{\cancel{16}^{4}\times 3}{\cancel{4}} - \frac{12(4)}{2}\right] - [12 - 24]$$

$$= 0 - [12 - 24] - [12 - 24]$$

$$= 24 \text{ units}$$

Integration of Even and Odd Functions

Even Function

A function f is said to be even if for any x in the domain of f,　　　　o in

$-x$ is also in the domain of f

$$f(x) = f(-x)$$

E.g $f(x) = x^2$

$f(x) = \cos x$

If f is even, then

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$$

Example

Find the area enclosed by the curve $y^2 = 2x + 6$ and $y = x - 1$

$y^2 = 2x + 6$ $\qquad$ $y = x - 1$

$x = \frac{y^2 - 6}{2}$ $\qquad$ $x = y + 1$

$$A = \int_a^b \left| \frac{y^2-6}{2} - (y+1) \right| dy$$

$$\frac{y^2-6}{2} = y + 1$$

$$y^2 - 6 = 2y + 2$$
$$y^2 - 2y - 8 = 0$$
$$y^2 - 4y + 2y - 8 = 0$$
$$y(y-4) + 2(y-4) = 0$$
$$y = -2 \text{ OR } y = 4$$

$$A = \int_{-2}^{4} \left| \frac{y^2 - 6}{2} - y + 1 \right| dy$$

$$= \int_{-2}^{4} \left| \frac{y^2 - 6 - 2y - 2}{2} \right| dy$$

$$= \int_{-2}^{4} \left| \frac{1}{2} y^2 - 2y - 8 \right| dy$$

$$= \frac{-1}{2} \left[\frac{y^3}{3} - \frac{\cancel{2}y^2}{\cancel{2}} - 8y \right]_{-2}^{4}$$

$$= \frac{-1}{2}\left[\frac{64}{3} - 16 - 32 \right] + \frac{1}{2}\left[\frac{-8}{3} - 4 + 16 \right]$$

$$= \frac{40}{3} + \frac{14}{3}$$

$$= \frac{54}{3}$$

$$= \underline{18 \text{ units}}$$