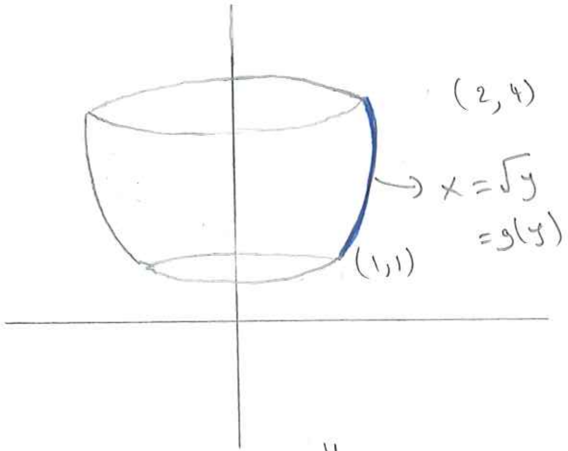
Ex. 8 Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the $y$-axis.




$x^2 = y$

$x = \pm\sqrt{y}$

$\boxed{x = \sqrt{y}}$

$$S = 2\pi \int_1^4 g(y) \sqrt{1 + [g'(y)]^2}\, dy$$

$$1 + [g'(y)]^2 = 1 + \left(\frac{1}{2\sqrt{y}}\right)^2$$

$$= \frac{4y+1}{4y}$$

$$S = 2\pi \int_1^4 \sqrt{y} \sqrt{\frac{4y+1}{4y}}\, dy$$

$$= 2\pi \int_1^4 \cancel{\sqrt{y}} \frac{\sqrt{4y+1}}{2\cancel{\sqrt{y}}}\, dy$$

$$= \frac{\pi}{4} \int_1^4 4(4y+1)^{\frac{1}{2}}\, dy$$

$$= \frac{\pi}{4} \left[ \frac{2}{3}(4y+1)^{\frac{3}{2}} \right]_1^4$$

$$= \frac{\pi}{6} \left[ (17)^{\frac{3}{2}} - (5)^{\frac{3}{2}} \right]$$

Ex. 9 Find the area of the surface obtained by rotating the curve $y = \sqrt{x+1}$ $\quad 0 \le x \le 4$ about the x-axis.

$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

$$= 2\pi \int_0^4 \sqrt{x+1}\ \sqrt{1 + \left(\frac{1}{2\sqrt{x+1}}\right)^2}\ dx$$

$$= 2\pi \int_0^4 \cancel{\sqrt{x+1}}\ \frac{\sqrt{4x+5}}{2\,\cancel{\sqrt{x+1}}}\ dx$$

$$= \pi \int_0^4 \sqrt{4x+5}\ dx$$

$$= \frac{\pi}{4} \int_0^4 4(4x+5)^{\frac{1}{2}} dx$$

$$= \frac{\pi}{4}\left[\frac{2}{3}(4x+5)^{\frac{3}{2}}\right]_0^4$$

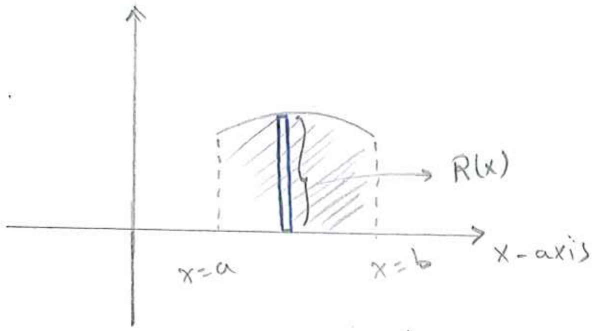
$$= \frac{\pi}{6}(4x+5)^{\frac{3}{2}}\Big|_0^4$$

$$= \frac{\pi}{6}\left[(21)^{\frac{3}{2}} - (5)^{\frac{3}{2}}\right]$$

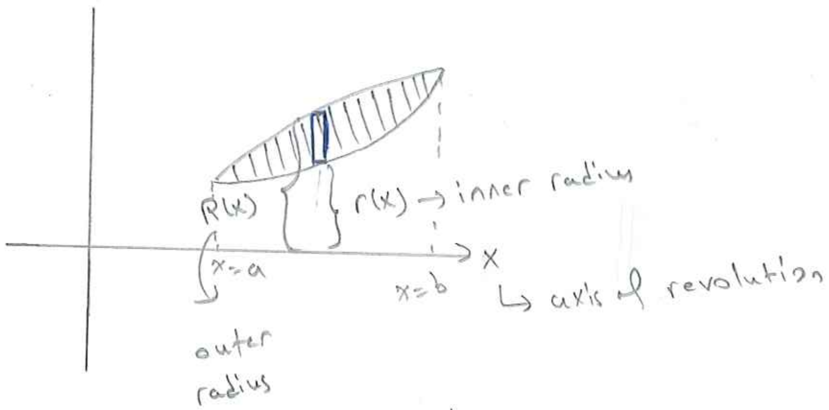
## Volumes

**Disk / Washer Method** (Take a representative rectangle perpendicular to the axis of revolution)

- Horizontal Axis of Revolution

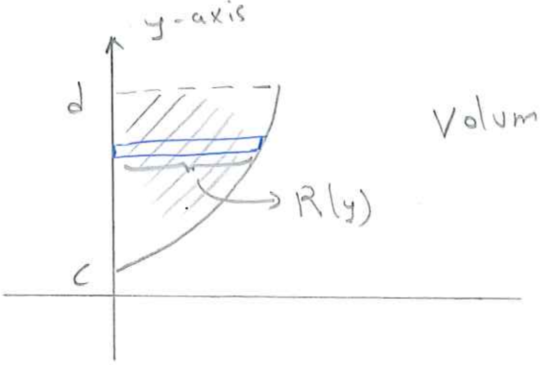



$$\text{Volume} = \pi \int_a^b [R(x)]^2 \, dx$$

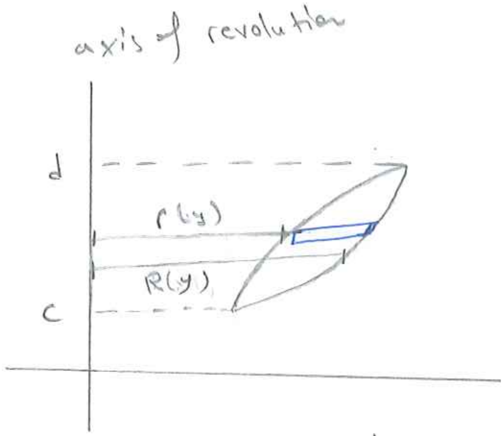



$$\text{Volume} = \pi \int_a^b \left( [R(x)]^2 - [r(x)]^2 \right) dx$$

- Vertical Axis of Revolution

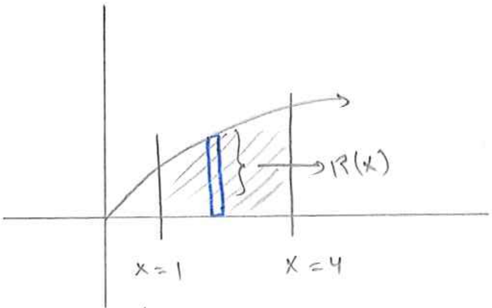



$$\text{Volume} = \pi \int_c^d [R(y)]^2 \, dy$$



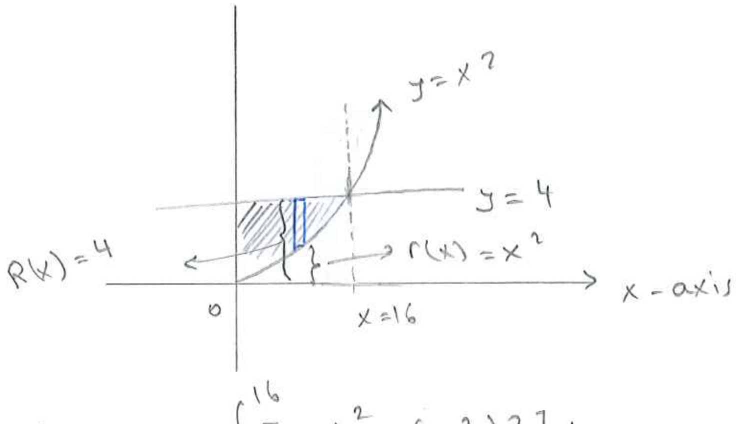

$$\text{Volume} = \pi \int_c^d \left( [R(y)]^2 - [r(y)]^2 \right) dy$$

Ex. 10 Find the volume of the solid that is obtained when the region under $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis.




$$V = \pi \int_1^4 (\sqrt{x})^2 \, dx = \frac{15\pi}{2}$$

Ex. 11 Sketch the region bounded by $y = x^2$, $y = 4$ and $x = 0$ and use the disk method to find the volume of the solid generated by revolving the region about the x-axis.




$$\text{Volume} = \pi \int_0^{16} \left[ (4)^2 - (x^2)^2 \right] dx$$

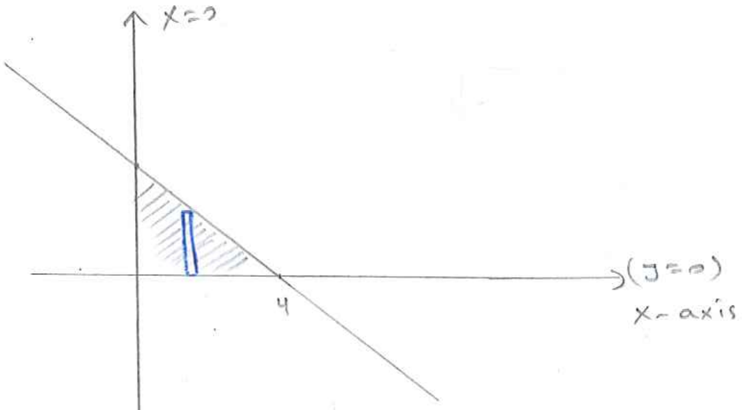
$$\text{Volume} = \pi \int_0^{16} (16 - x^4)\,dx$$

$$= \pi \left[ 16x - \frac{x^5}{5} \right]_0^{16}$$

$$= \frac{128}{5}\pi .$$

Ex. 12 Use the disk method to find:

a) the volume of the solid generated by revolving the region bounded by $x + y = 4$, $y = 0$, and $x = 0$ about the x-axis.




$x + y = 4 \rightarrow y = 4 - x$

$$V = \pi \int_0^4 (4-x)^2\,dx$$

$$= \frac{64}{3}\pi$$

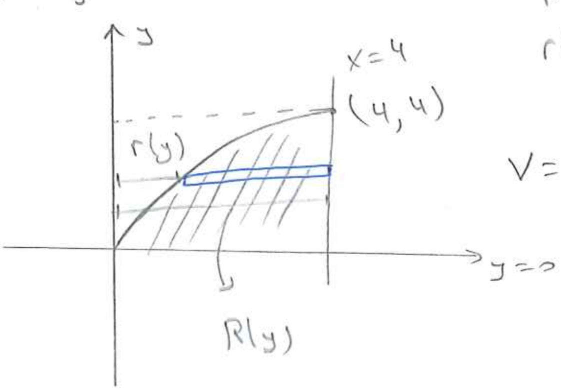
b) the volume of the solid generated by revolving the region bounded by $y^2 = 4x$, $x = 4$ and $y = 0$ about the y-axis.

$R(y) = 4$

$r(y) = \frac{1}{4}y^2$

$y^2 = 4x$

$x = \frac{1}{4}y^2$

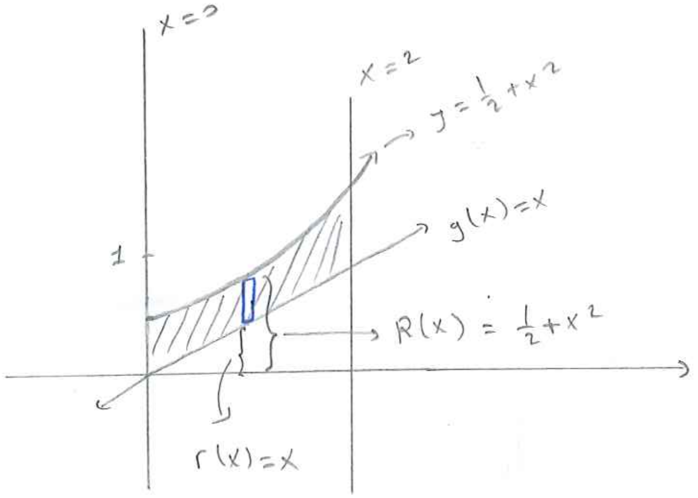



$$V = \pi \int_0^4 \left[ (4)^2 - \left(\frac{1}{4}y^2\right)^2 \right] dy$$

$$V = \pi \int_0^4 \left(16 - \frac{1}{16} y^4\right) dy$$

$$= \frac{256}{5} \pi$$

Ex. 13 Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.




$$\text{Volume} = \pi \int_0^2 \left[\left(\frac{1}{2} + x^2\right)^2 - (x)^2\right] dx$$

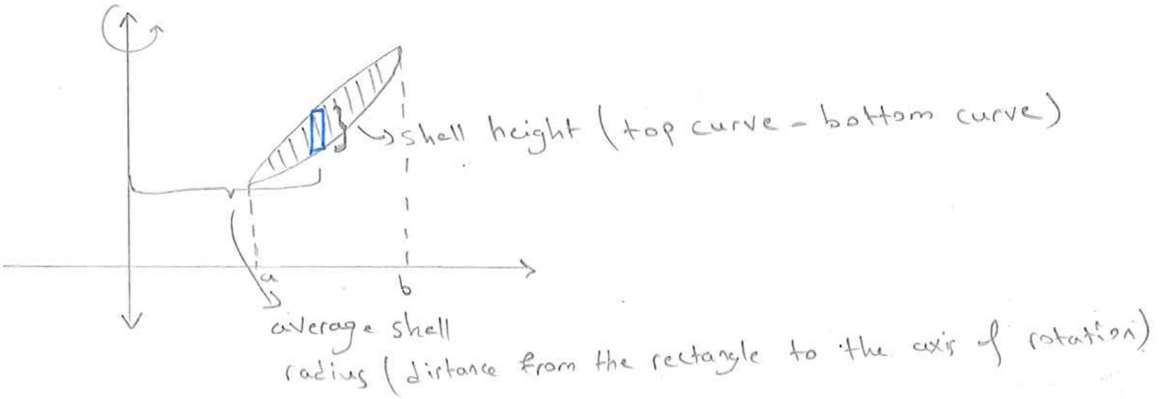
$$= \pi \int_0^2 \left(\frac{1}{4} + \cancel{x^2} + x^4 - \cancel{x^2}\right) dx$$

$$= \pi \left[\frac{1}{4} x + \frac{x^5}{5}\right]_0^2$$

$$= \frac{69\pi}{10}.$$

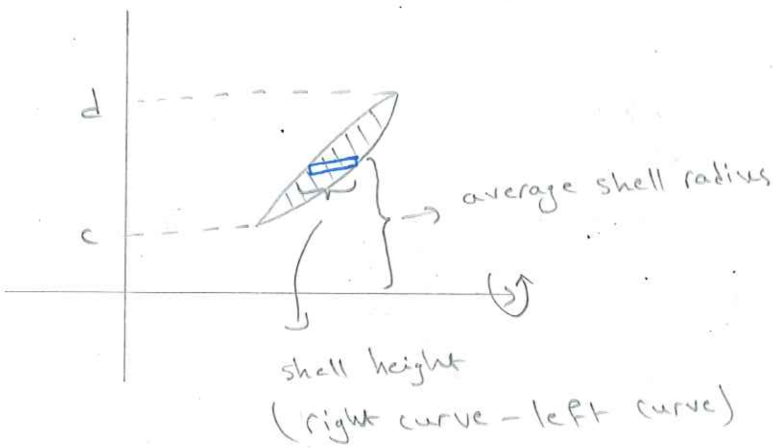
# Volume - The Shell Method (Take a representative rectangle that is parallel to the axis of revolution)

## Vertical Axis of Revolution



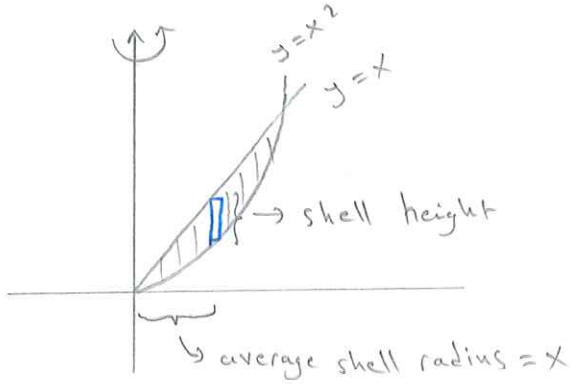

$$\text{Volume} = 2\pi \int_a^b (\text{average shell radius})(\text{shell height})\, dx$$

## Horizontal Axis of Revolution




$$\text{Volume} = 2\pi \int_c^d (\text{average shell radius})(\text{shell height})\, dy$$

Ex. 1 Use the shell method to find the volume of the solid generated when the region in the first quadrant between $y = x$ and $y = x^2$ is revolved about the $y$-axis.




$y = y$

$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad x = 1$

$$\text{Volume} = 2\pi \int_0^1 x(x - x^2)\,dx$$

$$= 2\pi \int_0^1 (x^2 - x^3)\,dx$$

$$= 2\pi \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

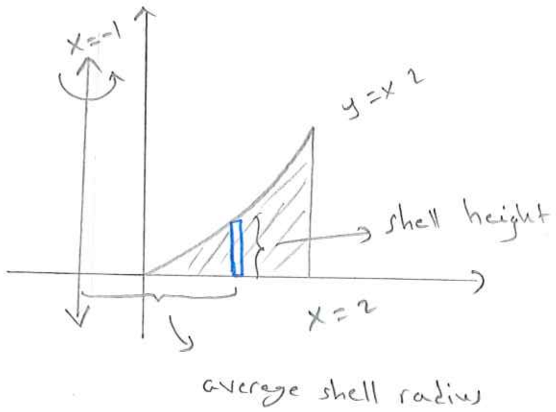
$$= \frac{\pi}{6}$$

Ex. 2 Use the shell method to find the volume of the solid generated when the region under $y = x^2$ over $[0, 2]$ is revolved about:

a) $x = -1$

b) $y = -1$

5



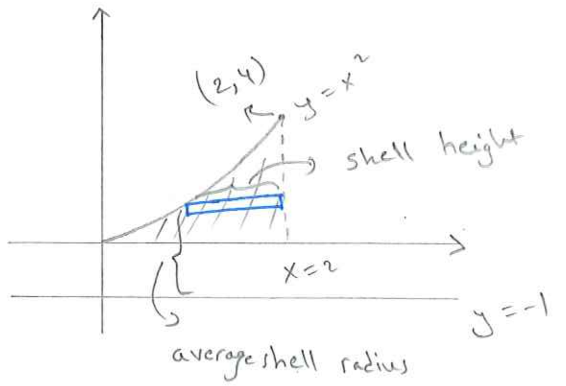

$$\text{Volume} = 2\pi \int_0^2 (x+1)(x^2-0)\,dx$$

$$= 2\pi \int_0^2 (x^3+x^2)\,dx$$

$$= 2\pi \left[ \frac{x^4}{4} + \frac{x^3}{3} \right]_0^2$$

$$= \frac{40\pi}{3}$$

b)




$y = x^2 \rightarrow x = \sqrt{y}$

$$\text{Volume} = 2\pi \int_0^4 (y+1)(2-\sqrt{y})\,dy$$

$$= 2\pi \int_0^4 (y+1)(2-y^{\frac{1}{2}})\,dy$$

$$= \frac{176\pi}{15}$$

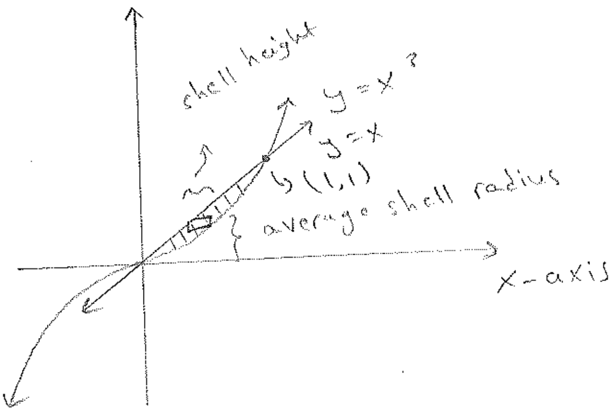
Ex. 3 Sketch the region $\Omega$ in the 1st quadrant by $y = x^3$ and $y = x$, and use the shell method to find the volume of the solid generated by revolving $\Omega$ about:

a) x-axis

b) $x = -1$ (Don't evaluate the integral)

c) $x = 1$ (Don't evaluate the integral)

a)




$y = y$

$x^3 = x$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x = 0$ or $x = \pm 1$

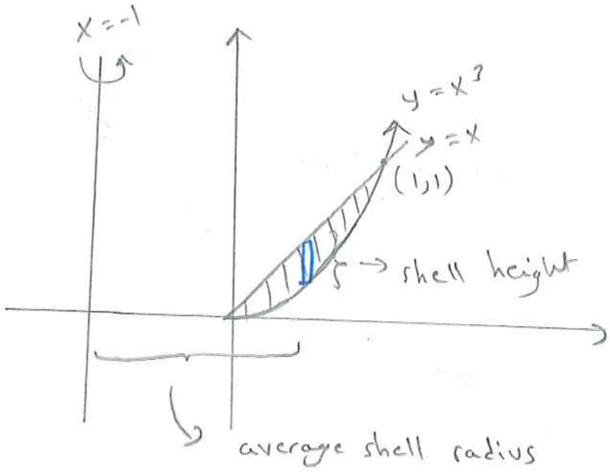
$$\text{Volume} = 2\pi \int_0^1 (y)(\sqrt[3]{y} - y)\,dy$$

$$= 2\pi \int_0^1 y(y^{1/3} - y)\,dy$$

$$= 2\pi \int_0^1 (y^{4/3} - y^2)\,dy$$

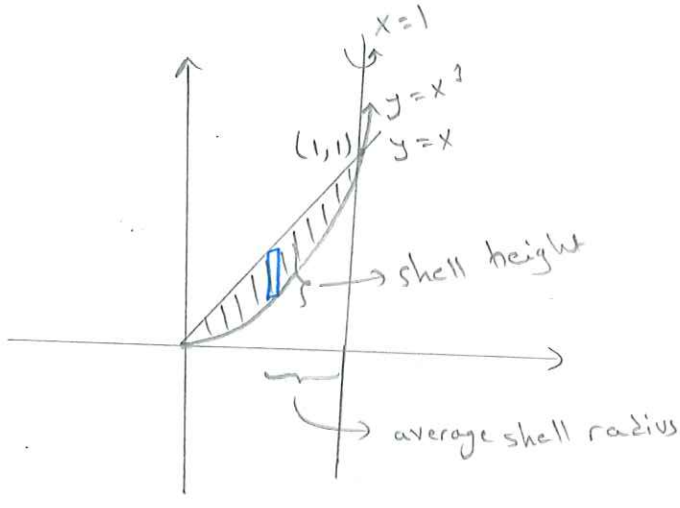
$$= \frac{4\pi}{21}$$

b)




$$\text{Volume} = 2\pi \int_0^1 (x+1)(x-x^3)\,dx$$

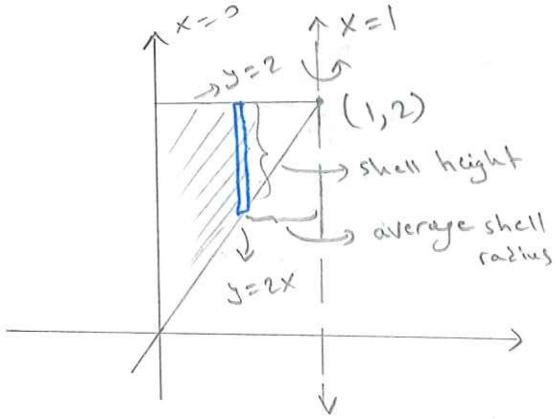
c)




$$\text{Volume} = 2\pi \int_0^1 (1-x)(x-x^3)\,dx$$

Ex. 4

Sketch the region bounded by $y = 2x$, $x = 0$ and $y = 2$ and use the shell method to find the volume of the solid generated by revolving about $x = 1$.




$y = y$

$2x = 2$

$x = 1$

$$\text{Volume} = 2\pi \int_0^1 (1-x)(2-2x)\,dx$$

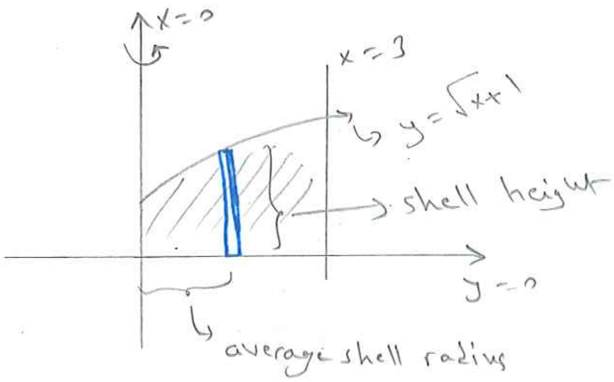
$$= 2\pi \int_0^1 2(1-x)(1-x)\,dx$$

$$= 4\pi \int_0^1 (1-x)^2\,dx$$

$$= -4\pi \left[\frac{(1-x)^3}{3}\right]_0^1$$

$$= -4\pi\left[0 - \frac{1}{3}\right] = \frac{4\pi}{3}$$

Ex. 5 Sketch the region $\Omega$ bounded by $y = \sqrt{x+1}$, $x = 0$, $y = 0$ and $x = 3$, and use the shell method to find the volume of the solid generated by revolving $\Omega$ about the y-axis



(8).



$$\text{Volume} = 2\pi \int_0^3 x\left(\sqrt{x+1}\right) dx$$

$$= 2\pi \int_0^3 x\sqrt{x+1}\, dx$$

$u = x+1 \Rightarrow x = u-1$

$du = dx$

$x = 0 \rightarrow u = 1$

$x = 3 \rightarrow u = 4$

$$= 2\pi \int_1^4 (u-1)\sqrt{u}\, du$$

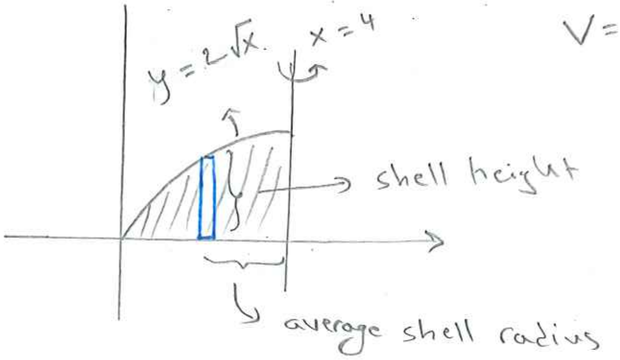
$$= 2\pi \int_1^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$$

$$= 2\pi \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$

$$= \frac{232}{15}\pi$$

Ex. 6

Use the shell method to find the volume of the solid generated by revolving the region bounded by $y^2 = 4x$, $x = 4$ and $y = 0$ about $x = 4$.




$$V = 2\pi \int_0^4 (4-x)(2\sqrt{x})\, dx$$

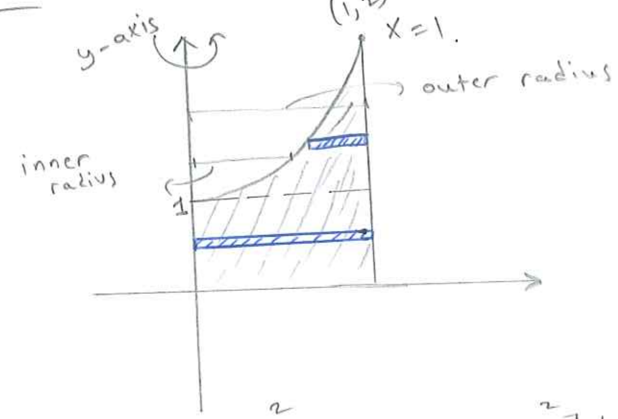
$$V = 4\pi \int_0^4 (4-x)x^{\frac{1}{2}} dx$$

$$= 4\pi \int_0^4 \left(4x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) dx$$

$$= \frac{512}{15}\pi$$

Ex. 7 Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 1$ about the $y$-axis.

Method 1. Applying the disk method.




$y = x^2 + 1$

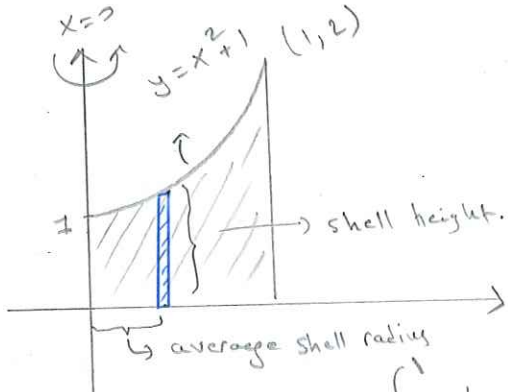
$x^2 = y - 1$

$x = \sqrt{y-1}$

$$\text{Volume} = \pi \int_1^2 \left[ (1)^2 - \left(\sqrt{y-1}\right)^2 \right] dy + \pi \int_0^1 (1)^2 \, dy$$

$$= \frac{3\pi}{2}$$

Method 2 Applying the shell method (easier)




$$\text{Volume} = 2\pi \int_0^1 x(x^2+1)\, dx$$

$$= \frac{3\pi}{2}$$