Math 142 *Power Series Practice Problems*

Power Series Mac Laurin Series

Assane Lo

- **Problem 1** For each of the following power series, find the interval of convergence and the radius of convergence:

- **Problem 1** For each of the following power series, find the interval of convergence and the radius of convergence:

1. $$\sum_{n=1}^{\infty}(-1)^n n^2 x^n$$

- **Problem 1** For each of the following power series, find the interval of convergence and the radius of convergence:

1. $$\sum_{n=1}^{\infty} (-1)^n n^2 x^n$$

2. $$\sum_{n=1}^{\infty} \frac{2^n}{n^2} (x-3)^n$$

- **Problem 1** For each of the following power series, find the interval of convergence and the radius of convergence:

1. $$\sum_{n=1}^{\infty}(-1)^n n^2 x^n$$

2. $$\sum_{n=1}^{\infty}\frac{2^n}{n^2}(x-3)^n$$

3. $$\sum_{n=1}^{\infty}\frac{n^3}{3^n}(x+1)^n$$

- **Problem 1** For each of the following power series, find the interval of convergence and the radius of convergence:

1. $$\sum_{n=1}^{\infty}(-1)^n n^2 x^n$$

2. $$\sum_{n=1}^{\infty}\frac{2^n}{n^2}(x-3)^n$$

3. $$\sum_{n=1}^{\infty}\frac{n^3}{3^n}(x+1)^n$$

4. $$\sum_{n=1}^{\infty}(-1)^n\frac{10^n}{n!}(x-10)^n$$

- **Problem 1** For each of the following power series, find the interval of convergence and the radius of convergence:

1. $$\sum_{n=1}^{\infty}(-1)^n n^2 x^n$$

2. $$\sum_{n=1}^{\infty}\frac{2^n}{n^2}(x-3)^n$$

3. $$\sum_{n=1}^{\infty}\frac{n^3}{3^n}(x+1)^n$$

4. $$\sum_{n=1}^{\infty}(-1)^n\frac{10^n}{n!}(x-10)^n$$

5. $$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n10^n}(x-2)^n$$

- **Solution**

- **Solution**

1. Notice that $a_{n+1} = (-1)^{n+1}(n+1)^2 x^{n+1}$. Then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)^2|x|^{n+1}}{n^2|x|^n} = \lim\limits_{n\to\infty}|x|\dfrac{n^2+2n+1}{n^2}$
$= |x|\lim\limits_{n\to\infty}\dfrac{2n+2}{2n} = |x|\lim\limits_{n\to\infty}\dfrac{2}{2} = |x|$, so this series converges absolutely for $-1 < x < 1$.
Notice when $x = 1$, we have $\sum\limits_{n=1}^{\infty}(-1)^n n^2 1^n = \sum\limits_{n=1}^{\infty}(-1)^n n^2$ which diverges by the nth term test.
Similarly, when $x = -1$, we have $\sum\limits_{n=1}^{\infty}(-1)^n n^2(-1)^n = \sum\limits_{n=1}^{\infty}(-1)^2 nn^2 = \sum\limits_{n=1}^{\infty}1$ which diverges by the nth term test.
Hence, the interval of convergence is: $(-1, 1)$ and the radius convergence is: $R = 1$.

2

Notice that $a_{n+1} = \dfrac{2^{n+1}}{(n+1)^2}(x-3)^{n+1}$. Then $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{2^{n+1}|x-3|^{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{2^n|x-3|^n}$ $= \lim_{n\to\infty}|x-3|\cdot 2\cdot\dfrac{n^2+2n+1}{n^2} = 2|x-3|\lim_{n\to\infty}\dfrac{2n+2}{2n} = 2|x-3|\lim_{n\to\infty}\dfrac{2}{2} = 2|x-3|$, so this series converges absolutely when $|x-3| < \frac{1}{2}$, or for $\dfrac{5}{2} < x < \dfrac{7}{2}$.

Notice when $x = \dfrac{5}{2}$, we have $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^2}(-\frac{1}{2})^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ Thus, since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series, the original series converges absolutely.

Similarly, when $x = \dfrac{7}{2}$, we have $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^2}(\frac{1}{2})^n = \sum_{n=1}^{\infty}\frac{(1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which is a convergent p-series.

Hence, the interval of convergence is: $\left[\dfrac{5}{2}, \dfrac{7}{2}\right]$ and the radius convergence is: $R = \dfrac{1}{2}$.

3

Notice that $a_{n+1} = \dfrac{(n+1)^3}{3^{n+1}}(x+1)^{n+1}$. Then $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{(n+1)^3|x+1|^{n+1}}{3^{n+1}}\cdot\dfrac{3^n}{n^3|x+1|^n}$ $= \dfrac{1}{3}|x+1|\lim_{n\to\infty}\dfrac{(n+1)^3}{n^3}$, which, after a few applications of L'Hôpital's Rule, is $\dfrac{|x+1|}{3}$, so this series converges absolutely when $|x+1| < 3$ or for $-4 < x < 2$.

Notice when $x = -4$, we have $\sum_{n=1}^{\infty}\dfrac{n^3}{3^n}(-3)^n = \sum_{n=1}^{\infty}(-1)^n n^3$, which diverges by the nth term test.

Similarly, when $x = 2$, we have $\sum_{n=1}^{\infty}\dfrac{n^3}{3^n}3^n = \sum_{n=1}^{\infty}n^3$ which diverges by the nth term test.

Hence, the interval of convergence is: $(-4, 2)$ and the radius convergence is: $R = 3$.

4

Notice that $a_{n+1} = (-1)^{n+1} \frac{10^{n+1}}{(n+1)!}(x-10)^{n+1}$. Then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{10^{n+1}|x-10|^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n |x-10|^n}$
$= |x-10| \lim_{n\to\infty} \frac{10}{n+1} = 0$
Hence the interval of convergence is $(-\infty, \infty)$ and $R = \infty$.

5

Notice that $a_{n+1} = (-1)^{n+1}\dfrac{1}{(n+1)10^{n+1}}(x-2)^{n+1}$. Then $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{|x-2|^{n+1}}{(n+1)10^{n+1}}\cdot\dfrac{n10^n}{|x-2|^n}$ $= \dfrac{1}{10}|x-2|\lim_{n\to\infty}\dfrac{n}{n+1} = \dfrac{1}{10}|x-2|$, so this series converges absolutely when $|x-2| < 10$ or for $-8 < x < 12$.

Notice when $x = -8$, we have $\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{n10^n}(-10)^n = \sum_{n=1}^{\infty}(-1)^n\dfrac{1}{n}(-1)^n = \sum_{n=1}^{\infty}\dfrac{1}{n}$, which diverges since it is the harmonic series.

Similarly, when $x = 10$, we have $\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{n10^n}10^n = \sum_{n=1}^{\infty}(-1)^n\dfrac{1}{n}$ which converges by the Alternating Series Test.

Hence, the interval of convergence is: $(-8, 10]$ and the radius convergence is: $R = 10$.

- **Problem 2** Use a known series to find a power series in x that has the given function as its sum:

- **Problem 2** Use a known series to find a power series in x that has the given function as its sum:

1.
$$x \sin\left(x^3\right)$$

- **Problem 2** Use a known series to find a power series in x that has the given function as its sum:

1. $$x \sin\left(x^3\right)$$

2. $$\frac{\ln(1+x)}{x}$$

- **Problem 2** Use a known series to find a power series in x that has the given function as its sum:

1. $$x \sin\left(x^3\right)$$
2. $$\frac{\ln(1+x)}{x}$$
3. $$\frac{x - \arctan x}{x^3}$$

- **Solution**

- **Solution**

1. Recall the Maclaurin series for $\sin u = \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)!}$

   Therefore, $\sin(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{(x)^{6n+3}}{(2n+1)!}$.

   Hence $x\sin(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x)^{6n+4}}{(2n+1)!}$.

- **Solution**

1. Recall the Maclaurin series for $\sin u = \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)!}$

   Therefore, $\sin(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{(x)^{6n+3}}{(2n+1)!}$.

   Hence $x\sin(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x)^{6n+4}}{(2n+1)!}$.

2. Recall the Maclaurin series for $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$

   Therefore, $\frac{\ln(1+x)}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n+1}$

- **Solution**

1. Recall the Maclaurin series for $\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$

   Therefore, $\sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{6n+3}}{(2n+1)!}$.

   Hence $x \sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{6n+4}}{(2n+1)!}$.

2. Recall the Maclaurin series for $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

   Therefore, $\frac{\ln(1+x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

3. Recall the Maclaurin series for $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

   Therefore, $x - \arctan(x) = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$

   Hence $\frac{x - \arctan x}{x^3} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{2n+1}$

- **Problem 3** Find the sum of the following series

- **Problem 3** Find the sum of the following series

1. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)}$$

- **Problem 3** Find the sum of the following series

1. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)}$$

2. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{100^{n+1}(n+1)}$$

- **Problem 3** Find the sum of the following series

1. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)}$$

2. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{100^{n+1}(n+1)}$$

3. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{10^{2n+1}(2n+1)!}$$

- **Solution**

- **Solution**

1. $$\arctan\frac{1}{2}$$

   Notice that the Maclaurin series $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$

- **Solution**

1. $$\arctan\frac{1}{2}$$
   Notice that the Maclaurin series $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$

2. $$\ln(1.01)$$
   Notice that the Maclaurin series $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$

- **Solution**

1. $$\arctan\frac{1}{2}$$
   Notice that the Maclaurin series $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$

2. $$\ln(1.01)$$
   Notice that the Maclaurin series $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$

3. $$\sin\left(\frac{\pi}{10}\right)$$
   Notice that the Maclaurin series $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Power Series Formulas

$\dfrac{1}{1-x}$	$=$	$1 + x + x^2 + x^3 + x^4 + \dots$	NOTE THIS IS THE GEOMETRIC SERIES. JUST THINK OF x AS r
	$=$	$\displaystyle\sum_{n=0}^{\infty} x^n$	$x \in (-1, 1)$
e^x	$=$	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dots$	SO: $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ $e^{(17x)} = \sum_{n=0}^{\infty} \frac{(17x)^n}{n!} = \displaystyle\sum_{n=0}^{\infty} \frac{17^n x^n}{n!}$
	$=$	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in \mathbb{R}$
$\cos x$	$=$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \dots$	NOTE $y = \cos x$ IS AN <u>EVEN</u> FUNCTION (I.E., $\cos(-x) = +\cos(x)$) AND THE TAYLOR SERIS OF $y = \cos x$ HAS ONLY <u>EVEN</u> POWERS.
	$=$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$\sin x$	$=$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \dots$	NOTE $y = \sin x$ IS AN <u>ODD</u> FUNCTION (I.E., $\sin(-x) = -\sin(x)$) AND THE TAYLOR SERIS OF $y = \sin x$ HAS ONLY <u>ODD</u> POWERS.
	$=$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$
$\ln(1+x)$	$=$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dots$	QUESTION: IS $y = \ln(1+x)$ EVEN, ODD, OR NEITHER?
	$=$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} \overset{\text{or}}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$x \in (-1, 1]$
$\tan^{-1} x$	$=$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dots$	QUESTION: IS $y = \arctan(x)$ EVEN, ODD, OR NEITHER?
	$=$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{2n-1} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x \in [-1, 1]$