(12pts)**Problem 1**

Find the area of the region bounded by $y = x^3 - 3x$ and $y = x$.
Solution of Problem 1
Method 1 (without graphing):

$$
\begin{aligned}
A &= \int_a^b \left| x^3 - 3x - x \right| dx \\
&= \int_a^b \left| x^3 - 4x \right| dx. \qquad [\textbf{2 points}]
\end{aligned}
$$

To find a and b, we do

$$x^3 - 3x = x \Leftrightarrow x\left(x^2 - 4\right) = 0.$$

$$x = 0, \quad \text{or} \quad x = 2 \quad \text{or } x = -2.$$

$$a = -2 \quad \text{and} \quad b = 2.$$

$$A = \int_{-2}^{2} \left| x^3 - 4x \right| dx. \qquad [\textbf{3 points}]$$

To evaluate this, we need to know the sign of the quantity inside the absolute value.

x	$-\infty$	-2		0		2	$+\infty$
x	$-$		$-$	0	$+$		$+$
x^2-4	$+$	0	$-$		$-$	0	$+$
x^3-4x	$-$		$+$		$-$		$+$
$\lvert x^3-4x \rvert$							

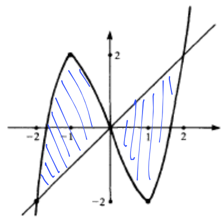
x^3-4x $\qquad$ $-(x^3-4x)$

$$
\begin{aligned}
\int_{-2}^{2} |x^3-4x|\,dx &= \int_{-2}^{0} |x^3-4x|\,dx + \int_0^2 |x^3-4x|\,dx \\
&= \int_{-2}^{0} (x^3-4x)\,dx + \int_0^2 -(x^3-4x)\,dx
\end{aligned}
$$

[**4 points**]

$$
\begin{aligned}
A &= \int_{-2}^{0} \left(x^3 - 4x\right) dx - \int_0^2 \left(x^3 - 4x\right) dx \\
&= 4 - (-4) = 8 \qquad [\textbf{3 points}]
\end{aligned}
$$

Method 1 (with graphing):



[**4 points**]

From the graph, you can see that the function $y = x^3 - 3x$ is on top of the line $y = 3$ on the interval $[-2,\ 0]$. The corresponding area is

$$\int_{-2}^{0} \left[\left(x^3 - 3x\right) - x\right] dx = \int_{-2}^{0} \left(x^3 - 4x\right) dx = 4. \quad [\mathbf{3\ points}]$$

On the interval $[0,\ 2]$, the line $y = x$ is on top and the corresponding area is

$$\int_{0}^{2} \left[x - \left(\left(x^3 - 3x\right)\right)\right] dx = \int_{0}^{2} \left(-x^3 + 4x\right) dx = 4. \qquad [\mathbf{3\ points}]$$

Hence, the area bounded by the two curves is

$$\begin{aligned} A &= \int_{-2}^{0} \left(x^3 - 4x\right) dx + \int_{0}^{2} \left(-x^3 + 4x\right) dx \\ &= 4 + 4 \\ &= 8. \qquad [\mathbf{2\ points}] \end{aligned}$$

$(14pts)$**Problem 2**
Consider the curve given by

$$x^{2/3} + y^{2/3} = 4, \qquad 1 \le x \le 8.$$

(a) Find the arclength of the curve.
(b) Find the area of the surface obtained by rotating the curve about the x-axis.
Hint: Use implicit differentiation to find $\frac{dy}{dx}$.

Solution of Problem 2

$$x^{2/3} + y^{2/3} = 4 \Leftrightarrow y^{2/3} = 4 - x^{2/3}$$

(a)

$$L = \int_1^8 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Next, we use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} = -\frac{y^{1/3}}{x^{1/3}} \qquad \textbf{[2 points]}$$

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{y^{2/3}}{x^{2/3}} \\ &= 1 + \frac{4 - x^{2/3}}{x^{2/3}} \\ &= \frac{4}{x^{\frac{2}{3}}} = \left(\frac{2}{x^{1/3}}\right)^2 \qquad \textbf{[4 points]} \end{aligned}$$

Thus,

$$\begin{aligned} L &= \int_1^8 \sqrt{\left(\frac{2}{x^{1/3}}\right)^2}\, dx \\ &= \int_1^8 \frac{2}{x^{1/3}} dx \\ &= 9. \qquad \textbf{[3 points]} \end{aligned}$$

(b)

$$\begin{aligned} S &= 2\pi \int_1^8 \left(4 - x^{2/3}\right)^{3/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_1^8 \left(4 - x^{2/3}\right)^{3/2} \frac{2}{x^{1/3}} dx. \qquad \textbf{[2 points]} \end{aligned}$$

Now put

$$u = 4 - x^{2/3}, \qquad du = \frac{1}{3}\frac{-2}{x^{1/3}}dx \Rightarrow \frac{2}{x^{1/3}}dx = -3du.$$

$$\text{when } x = 1, \quad u = 3 \ \text{ and when } x = 8, \quad u = 0.$$

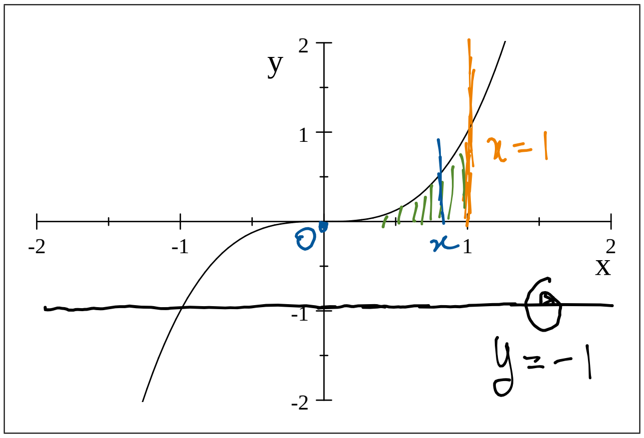
Thus,

$$\begin{aligned} S &= 2\pi \int_3^0 u^{3/2}\,(-3du) \\ &= 6\pi \int_0^3 u^{3/2} du \\ &= \frac{108}{5}\sqrt{3}\pi = 37.412\pi = 117.53. \qquad [\textbf{3 points}] \end{aligned}$$

(12pts)**Problem 3**

Sketch the region bounded by $y = x^3$, $x = 1$ and $y = 0$ and use the **disc method** to find the volume of the solid obtained by rotating the region about the line $y = -1$.

Solution of Problem 3



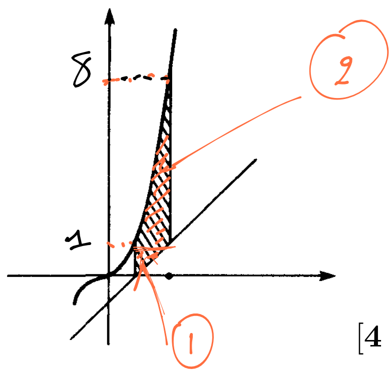
[**4 points**]

$$
\begin{aligned}
V &= \int_0^1 \left(\pi \left[x^3 - (-1)\right]^2 - \pi 1^2\right) dx \qquad [\textbf{4 points}] \\
&= \pi \int_0^1 \left[\left(x^3 + 1\right)^2 - 1\right] dx \\
&= \pi \int_0^1 \left(x^6 + 2x^3\right) dx \\
&= \frac{9}{14}\pi = 0.642\,86\pi = 2.019\,6 \qquad [\textbf{4 points}]
\end{aligned}
$$

(12pts)**Problem 4**

Sketch the region bounded by $y = x^3$, $x = 1$, $x = 2$, and $y = x - 1$ and use the **cylindrical shell** method to find the volume of the solid generated by rotating the region about the x-axis.

Solution of Problem 4



[**4 points**]

$V = V_1 + V_2$ [**2 points**]

$$\begin{aligned} V_1 &= \int_0^1 2\pi y\,(y + 1 - 1)\,dy \\ &= \int_0^1 2\pi y^2 dy = \frac{2}{3}\pi \end{aligned}$$ [**3 points**]

$$\begin{aligned} V_2 &= \int_1^8 2\pi y\,(2 - \sqrt[3]{y})\,dy \\ &= 2\pi \int_1^8 \left(2y - y^{4/3}\right) dy \\ &= \frac{120}{7}\pi \end{aligned}$$ [**3 points**]

$$\begin{aligned} V &= \frac{2}{3}\pi + \frac{120}{7}\pi \\ &= \frac{374}{21}\pi = 17.810\pi = 55.95. \end{aligned}$$

$(14pts)$**Problem 5**
Evaluate the following integrals

$$(1) \quad \int_0^{\pi} x\cos(3x-\pi)\,dx, \qquad\qquad (2) \quad \int \ln\left(x^2+1\right)dx$$

Solution of Problem 5
(1)

$$\int_0^{\pi} x\cos(3x-\pi)\,dx = ?$$

Here, we integrate by parts.

$$\begin{aligned} u &= x, \qquad u' = 1 \\ v' &= \cos(3x-\pi), \qquad v = \frac{1}{3}\sin(3x-\pi) \qquad\qquad [\textbf{3 points}] \end{aligned}$$

$$\begin{aligned} \int_0^{\pi} x\cos(3x-\pi)\,dx &= \frac{x}{3}\sin(3x-\pi)\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{3}\sin(3x-\pi)\,dx \\ &= 0-\left(-\frac{2}{9}\right) \\ &= \frac{2}{9} = 0.222\,22 \qquad\qquad [\textbf{4 points}] \end{aligned}$$

(2)

$$\int \ln\left(x^2+1\right)dx = ?$$

Again, we integrate by parts.

$$\begin{aligned} u &= \ln\left(x^2+1\right), \qquad u' = \frac{2x}{x^2+1} \\ v' &= 1, \qquad v = x \qquad\qquad [\textbf{3 points}] \end{aligned}$$

$$\begin{aligned} \int \ln\left(x^2+1\right)dx &= x\ln\left(x^2+1\right) - 2\int\frac{x^2}{x^2+1}dx \\ &= x\ln\left(x^2+1\right) - 2\int\frac{x^2+1-1}{x^2+1}dx \\ &= x\ln\left(x^2+1\right) - 2\left(\int 1-\frac{1}{x^2+1}dx\right) \\ &= x\ln\left(x^2+1\right) - 2x + 2\tan^{-1}x + C \qquad\qquad [\textbf{4 points}]. \end{aligned}$$

$(12pts)$**Problem 6**
Evaluate the following integral

$$\int \frac{x^2+2}{x\left(x^2+5x+6\right)}dx$$

Solution of Problem 6

$$\begin{aligned}
\frac{x^2+2}{x\left(x^2+5x+6\right)} &= \frac{x^2+2}{x\left(x+3\right)\left(x+2\right)} \\
&= \frac{A}{x}+\frac{B}{x+3}+\frac{C}{x+2} \qquad [\textbf{3 points}]. \\
&= \frac{1}{3x}+\frac{11}{3\left(x+3\right)}-\frac{3}{x+2}
\end{aligned}$$

$$A=\frac{1}{3}, \qquad B=\frac{11}{3}, \quad \text{and} \quad C=-3. \qquad [\textbf{6 points}].$$

$$\begin{aligned}
\int \frac{x^2+2}{x\left(x^2+5x+6\right)}dx &= \int \left(\frac{1}{3x}+\frac{11}{3\left(x+3\right)}-\frac{3}{x+2}\right)dx \\
&= \frac{1}{3}\ln|x|-3\ln|x+2+|\frac{11}{3}\ln|x+3|+C \qquad [\textbf{3 points}].
\end{aligned}$$

(12*pts*)**Problem 7**

Evaluate the integral

$$\int \frac{dx}{(4-x^2)\sqrt{4-x^2}}$$

Solution of Problem 7

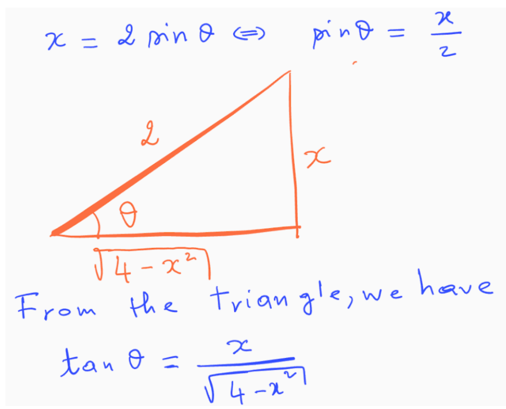
Since the integral involves $\sqrt{a^2-x^2}$ with $a=2$, we do the substitution

$$x = 2\sin\theta.$$

$$dx = 2\cos\theta d\theta \quad \text{and} \quad \sqrt{4-x^2} = 2\cos\theta. \qquad [\textbf{4 points}]$$

The integral becomes

$$\begin{aligned} \int \frac{dx}{(4-x^2)\sqrt{4-x^2}} &= \int \frac{2\cos\theta d\theta}{(4\cos^2\theta)(2\cos\theta)} \\ &= \frac{1}{4}\int \sec^2\theta d\theta \\ &= \frac{1}{4}\tan\theta + C \qquad [\textbf{4 points}] \end{aligned}$$



[**2 points**]

$$\int \frac{dx}{(4-x^2)\sqrt{4-x^2}} = \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C \qquad [\textbf{2 points}]$$

(12*pts*)**Problem 8**
Determine convergence or divergence of the following improper intergals

$$(1)\ \int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx \qquad\qquad (2)\ \int_1^9 \frac{dx}{(x-1)^{2/3}}$$

Solution of Problem 8
(1)

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = ?$$

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{t\to\infty}\int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx \qquad [\textbf{2 points}]$$

Put

$$u=\sqrt{x}, \qquad u^2 = x \text{ and } dx = 2udu$$

$$\begin{aligned}
\int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx &= 2\int_1^{\sqrt{t}} \frac{\cancel{u}e^{-u}}{\cancel{u}}du \\
&= 2\int_1^{\sqrt{t}} e^{-u}du \\
&= 2 -e^{-u}\Big|_1^{\sqrt{t}} \\
&= 2\left(e - e^{-\sqrt{t}}\right) \qquad [\textbf{2 points}]
\end{aligned}$$

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{t\to\infty}\left[2\left(e^{-1} - e^{-\sqrt{t}}\right)\right] = \frac{2}{e}. \quad \text{Converges.} \qquad [\textbf{2 points}]$$

(2)

$$\int_1^9 \frac{dx}{(x-1)^{2/3}} = ?$$

$$\begin{aligned}
\int_1^9 \frac{dx}{(x-1)^{2/3}} &= \lim_{t\to 1^+}\int_t^9 \frac{dx}{(x-1)^{2/3}} \qquad [\textbf{2 points}] \\
&= \lim_{t\to 1^+}\int_t^9 (x-1)^{-2/3}\,dx \\
&= \lim_{t\to 1^+} \left.\frac{(x-1)^{\frac{-2}{3}+1}}{\frac{-2}{3}+1}\right|_t^9 \\
&= \lim_{t\to 1^+} 3\,(x-1)^{\frac{1}{3}}\Big|_t^9 \qquad [\textbf{2 points}] \\
&= \lim_{t\to 1^+}\left[3\left(2-\sqrt[3]{1-t}\right)\right] \\
&= 6. \quad \text{Converges.} \qquad [\textbf{2 points}]
\end{aligned}$$