Example

Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2+n}+2^{3-n}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2+n}+\sum_{n=1}^{\infty}2^{3-n}$$

$$=1+8$$

$$=9$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n}$$

$$S_N=\sum_{n=1}^{N}\frac{1}{n(n+1)}$$

$$=\sum_{n=1}^{N}\left(\frac{\cancel{n+1}}{n(\cancel{n+1})}-\frac{\cancel{n}}{\cancel{n}(n+1)}\right)$$

$$=\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)\cdots\cdots\left(\cancel{\frac{1}{N}}+\frac{1}{N+1}\right)$$

$$=1-\frac{1}{N+1}$$

$$\lim_{N\to\infty}S_N$$

$$=\lim_{N\to\infty}1-\frac{1}{N+1}$$

$$=1$$

$$\sum_{n=1}^{\infty}2^{3-n}$$

$$=\sum_{n=1}^{\infty}\frac{2^3}{2^n}$$

$$=\sum_{n=1}^{\infty}8\left(\frac{1}{2}\right)^n$$

$$=\sum_{n=1}^{\infty}8\left(\frac{1}{2}\right)^{n-1+1}$$

$$= \sum_{n=1}^{\infty} 4\left(\frac{1}{2}\right)^{n-1}$$

$|r| = \frac{1}{2} < 1$ ∴ series converges

$$\text{Sum} = \sum_{n=1}^{\infty} 2^{3-n} = \frac{a_1}{1-r} = \frac{4}{1-1/2} = \frac{4}{1/2} = \frac{8}{2}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2+n} + 2^{3-n}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2+n} + \sum_{n=1}^{\infty} 2^{3-n}$$

$$= 1 + 8$$

$$= 9$$

Integral Test

Consider the series

$$\sum_{n=1}^{\infty} a_n \qquad \text{with } a_n = f(n)$$

where f is continuous, positive and decreasing on $[1, \infty)$

$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge

The p-test

Recall that

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \text{converging} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

The p-Series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

$\frac{1}{\ }= f(n)$ where $f(x) = \frac{1}{\ }$

n^p x^p

Example

Determine convergence or divergence of the following series

1. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$$a_n = \frac{n}{n^2+1} = f(n)$$

where $f(x) = \frac{x}{x^2+1}$

On $[1, \infty)$

$f(x) = \frac{x}{x^2+1}$ is continuous

f is decreasing

f is positive

$$\int_1^{\infty} \frac{x}{x^2+1} \, dx$$

$$= \lim_{t \to \infty} \frac{1}{2} \int_1^t \frac{2x}{x^2+1}$$

$$= \lim_{t \to \infty} \frac{1}{2} \ln|x^2+1| \Big|_1^t$$

$$= \lim_{t \to \infty} \frac{1}{2} \left[\ln(t^2+1) - \ln 2 \right]$$

$$= +\infty$$

$\because f$ diverges

$\therefore \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges

2. $\sum^{\infty}$

2. $\sum_{n=2} \frac{1}{n\ln(n)}$

$a_n = f(n)$

where $f(x) = \frac{1}{x\ln(x)}$ with $x \in [2, \infty)$

On $[2, \infty)$

$f(x)$ is continuous, decreasing and positive

$$\int_2^\infty \frac{dx}{x\ln x}$$

$$= \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln x}$$

$$= \lim_{t\to\infty} \ln|\ln x| \Big|_2^t$$

$$= \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2|$$

$$= +\infty$$

Diverges $\Rightarrow \sum_{n=2}^\infty \frac{1}{n\ln(n)}$ diverges

5. $\sum_{n=1}^\infty ne^{-n}$

$$\sum_{n=1}^\infty ne^{-n}$$

$ne^{-n} = f(n)$ where

$f(x) = xe^{-x}$

On $[1, \infty)$

$f(x) = xe^{-x} = \frac{x}{e^x}$ is positive, continuous and decreasing

$$\int_1^\infty xe^{-x}\,dx$$

$$= \lim \int^t xe^{-x}\,dx$$

$t \to \infty$

$u = x \quad u' = 1$

$v' = e^{-x} \quad v = -e^{-x}$

$$= \lim_{t \to \infty} \left[-xe^{-x} \Big|_1^t + \int_1^t e^{-x}\, dx \right]$$

$$= \lim_{t \to \infty} \left[-te^{-t} + e^{-1} - e^{-t} + e^{-1} \right]$$

$$= 2e^{-1}$$

$$\lim_{t \to \infty} \frac{-t}{e^t} = \frac{\infty}{\infty} \qquad \lim e^{-t} = 0$$

Using L'hopitals

$$\lim_{t \to \infty} \frac{-1}{e^t} = 0$$

$\therefore$ Converges

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$p = \frac{3}{2} > 1$$

$\therefore$ Converges by p-series test

4. $\displaystyle\sum_{n=1}^{\infty} \frac{7}{n}$

$$\sum_{n=1}^{\infty} 7 \frac{1}{n}$$

$$\frac{1}{n} \Rightarrow p = 1$$

$\therefore$ Diverges by p-series test

Estimating the sum of the series (through the integral test)

Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = f(n)$ with f continuous, positive and decreasing.

Suppose that the series converges and call its value S.

$$S = \sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_n}_{S_n} + \underbrace{a_{n+1} + \cdots}_{R_n}$$

Let $S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{\infty} a_k$

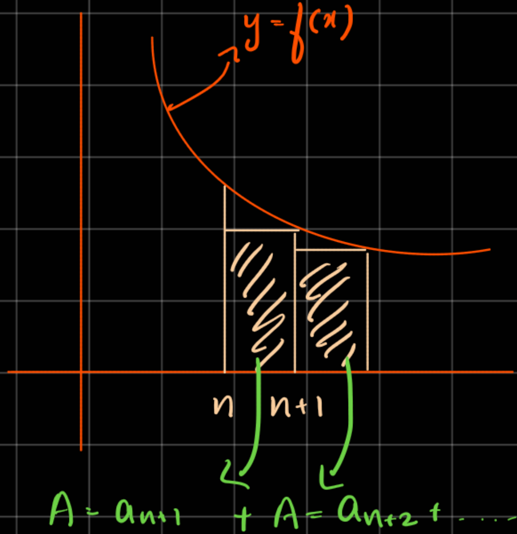
$$S = S_n + R_n \qquad R_n = a_{n+1} + a_{n+2} + \cdots$$

$$R_n = S - S_n$$

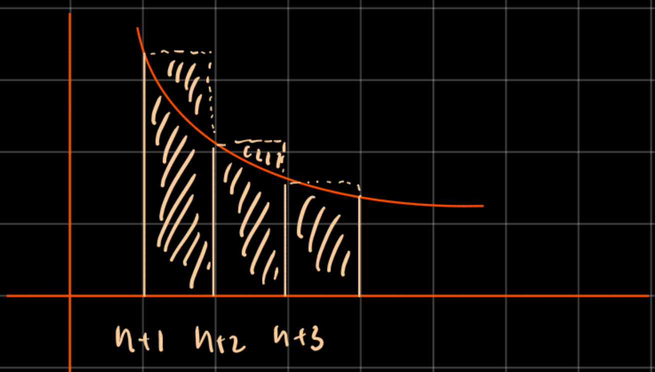
$\hookrightarrow$ Error when estimating the sum of the series

We have the following formula:

$$\int_{n+1}^{\infty} f(x)\,dx \leq S - S_n \leq \int_{n}^{\infty} f(x)\,dx$$



$$a_{n+1} + a_{n+2} + \cdots \leq \int_{n}^{\infty} f(x)\,dx$$



$$\int_{n+1}^{\infty} f(x)\,dx \leq a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

$$\therefore \int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Example

A) Use $n = 10$ to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$

A) $n = 10$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = f(n)$$

where $f(x) = \frac{1}{x^3}$

$$\int_{11}^{\infty} \frac{dx}{x^3} \leqslant S - S_{10} \leqslant \int_{10}^{\infty} \frac{dx}{x^3}$$

$$\int_{11}^{\infty} \frac{dx}{x^3} + S_{10} \leqslant S \leqslant \int_{10}^{\infty} \frac{dx}{x^3} + S_{10}$$

$$S_{10} = \sum_{k=1}^{10} \frac{1}{k^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} - - - - + \frac{1}{10^3}$$

$$= 1.1975$$

$$\int_{11}^{\infty} x^{-3}\,dx = \lim_{t \to \infty} \int_{11}^{t} x^{-3}\,dx$$

$$= \lim_{t \to \infty} \left. \frac{-1}{2x^2} \right|_{11}^{t}$$

$$= \lim_{t \to \infty} \frac{-1}{2t^2} + \frac{1}{242}$$

$$= 0 + \frac{1}{242}$$

$$= \frac{1}{242}$$

$$\int_{10}^{\infty} x^{-3}\,dx = \lim_{t \to \infty} \int_{11}^{t} x^{-3}\,dx$$

$$= \lim_{t \to \infty} \left. \frac{-1}{2x^2} \right|_{10}^{t}$$

$$= \lim_{t \to \infty} \frac{-1}{2t^2} + \frac{1}{200}$$

$$= 0 + \frac{1}{200}$$

$$= \frac{1}{200}$$

$$\frac{1}{242} + 1.1975 \leqslant S \leqslant \frac{1}{200} + 1.1975$$

$$1.201664 \leqslant S \leqslant 1.202532$$

B) How many terms are required to ensure that the sum is accurate to within 0.0005?

$$R_n = S - S_n \leq \underbrace{\int_n^\infty \frac{dx}{x^3}}$$

$$\downarrow$$

$$S - S_n \leq \frac{1}{2n^2} \leq 0.0005$$

$$\frac{1}{2n^2} \leq 0.0005$$

$$\frac{1}{n^2} \leq 0.001$$

$$n^2 \geq 1000$$

$$n \geq \sqrt{1000}$$

$$n \geq 31.6$$

$$\boxed{n \geq 32}$$