

Tutorial 4

Question 1

Use partial fraction decomposition to evaluate

$$\int \frac{3x - 4}{x^2 - 2x + 1} dx$$

Solution

$$\frac{3x - 4}{x^2 - 2x + 1} = \frac{3x - 4}{(x - 1)^2} = \frac{A_1}{x - 1} + \frac{A_2}{(x - 1)^2}$$

$$A_1 = 3 \text{ and } A_2 = -1$$

$$\begin{aligned} \int \frac{3x - 4}{x^2 - 2x + 1} dx &= \int \left(\frac{3}{x - 1} - \frac{1}{(x - 1)^2} \right) dx \\ &= 3 \ln |x - 1| + \frac{1}{x - 1} + C \end{aligned}$$

Question 2

Evaluate the following integrals

1. $\int \frac{\sqrt{x} dx}{x - 4},$

2. $\int \frac{x^3 + 2x^2 - 4}{x^2 - x} dx$



Solution

1. $\int \frac{\sqrt{x} dx}{x-4}$?

Put $u = \sqrt{x}$, $u^2 = x \Rightarrow dx = 2u du$ and $x = u^2$.

The integral becomes

$$\begin{aligned} \int \frac{\sqrt{x} dx}{x-4} &= \int \frac{2u^2 du}{u^2-4} \\ &= 2 \int \frac{u^2 du}{u^2-4} = 2 \int \frac{(u^2-4+4) du}{u^2-4} \\ &= 2 \int \left(1 + \frac{4}{u^2-4}\right) du \\ \frac{4}{u^2-4} &= \frac{4}{(u-2)(u+2)} = \frac{1}{u-2} - \frac{1}{u+2} \end{aligned}$$

$$\begin{aligned} \int \frac{\sqrt{x} dx}{x-4} &= 2 \int \left(1 + \frac{4}{u^2-4}\right) du \\ &= 2 \int \left(1 + \frac{1}{u-2} - \frac{1}{u+2}\right) du \\ &= 2 \left[u + \ln|u-2| - \ln|u+2| \right] + C \\ &= 2 \left[\sqrt{x} + \ln|\sqrt{x}-2| - \ln|\sqrt{x}+2| \right] + C \end{aligned}$$

2. $\int \frac{x^3+2x^2-4}{x^2-x} dx$?

$$\begin{aligned} \frac{x^3+2x^2-4}{x^2-x} &= x+3 + \frac{3x-4}{x^2-x} \\ \frac{3x-4}{x^2-x} &= \frac{3x-4}{x(x-1)} = \frac{4}{x} - \frac{1}{x-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \int \frac{x^3+2x^2-4}{x^2-x} dx &= \int \left(x+3 + \frac{4}{x} - \frac{1}{x-1} \right) dx \\ &= \frac{x^2}{2} + 3x + 4 \ln|x| - \ln|x-1| + C \end{aligned}$$



Question 3

Determine the convergence or divergence the following improper integrals. If the integral is convergent, then find its value.

1. $\int_0^{12} \frac{9}{\sqrt{12-x}} dx,$

2. $\int_0^{\infty} \frac{e^x}{1+e^x} dx$

Solution

1. $\int_0^{12} \frac{9}{\sqrt{12-x}} dx?$

$$\begin{aligned} \int_0^{12} \frac{9}{\sqrt{12-x}} dx &= \lim_{t \rightarrow 12^-} \int_0^t \frac{9}{\sqrt{12-x}} dx \\ &= \lim_{t \rightarrow 12} (36\sqrt{3} - 18\sqrt{12-t}) \\ &= 36\sqrt{3} = 62.354. \end{aligned}$$

The improper integral converges to $36\sqrt{3} = 62.354$

2. $\int_0^{\infty} \frac{e^x}{1+e^x} dx?$

$$\begin{aligned} \int_0^{\infty} \frac{e^x}{1+e^x} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{e^x}{1+e^x} dx \\ &= \lim_{t \rightarrow \infty} \ln(1+e^x) \Big|_0^t \\ &= \lim_{t \rightarrow \infty} [1+e^t - \ln 2] = \infty. \end{aligned}$$

The improper integral diverges.

Question 4

Use partial fraction decomposition to evaluate

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$$



Solution

$$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{x^2 + 2x - 1}{x(x+2)(2x-1)}$$

Next, you do the partial fraction decomposition

$$\frac{x^2 + 2x - 1}{x(x+2)(2x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}$$

After solving, you get

$$A = \frac{1}{2}, \quad B = \frac{-1}{10} \quad \text{and} \quad C = \frac{1}{5}$$

$$\frac{x^2 + 2x - 1}{x(x+2)(2x-1)} = \frac{1}{2x} - \frac{1}{10(x+2)} + \frac{1}{5(2x-1)}$$

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx = \frac{1}{2} \ln |x| - \frac{1}{10} \ln |x+2| + \frac{1}{10} \ln |2x-1| + C$$

Question 5

Evaluate the following integrals

1. $\int \frac{dx}{2\sqrt{x} + 2x},$

2. $\int \frac{3x^3 - 3x^2 + 4}{x^2 - x} dx$



Solution

1.

$$\int \frac{dx}{2\sqrt{x} + 2x}.$$

We rationalize by putting

$$u = \sqrt{x}$$

$$x = u^2 \text{ and } dx = 2u du$$

The integral becomes

$$\begin{aligned} \int \frac{dx}{2\sqrt{x} + 2x} &= \int \frac{2u du}{2u + 2u^2} \\ &= \int \frac{\cancel{2}u du}{\cancel{2}u(1 + u)} \\ &= \int \frac{du}{1 + u} = \ln|1 + u| + C \\ &= \ln|1 + \sqrt{x}| + C \end{aligned}$$

2.

$$\int \frac{3x^3 - 3x^2 + 4}{x^2 - x} dx$$

Since the degree of the numerator is higher, we first perform a long division.

$$\frac{3x^3 - 3x^2 + 4}{x^2 - x} = 3x - \frac{4}{x - x^2}.$$

Next, we do a partial fraction decomposition on $\frac{4}{x - x^2}$.

$$\frac{4}{x - x^2} = \frac{4}{-x(x - 1)} = \frac{4}{x} - \frac{4}{x - 1}$$

$$\begin{aligned} \int \frac{3x^3 - 3x^2 + 4}{x^2 - x} dx &= \int \left[3x - \left(\frac{4}{x} - \frac{4}{x - 1} \right) \right] dx \\ &= \int \left(3x - \frac{4}{x} + \frac{4}{x - 1} \right) dx \\ &= \frac{3}{2}x^2 - 4 \ln|x| + 4 \ln|x - 1| + C \end{aligned}$$



Question 6

Determine the convergence or divergence the following improper integrals. If the integral is convergent, then find its value.

$$1. \int_0^{\infty} \frac{x}{1+x^2} dx, \quad 2. \int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$$

Solution

1.

$$\begin{aligned} \int_0^{\infty} \frac{x}{1+x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{x}{1+x^2} dx \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \int_0^t \frac{2x}{1+x^2} dx \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \ln(1+x^2) \Big|_0^t \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \ln(1+t^2) \\ &= \infty \quad \text{diverges} \end{aligned}$$

2.

$$\begin{aligned} \int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} &= \lim_{t \rightarrow -2^-} \int_t^{14} \frac{dx}{\sqrt[4]{x+2}} \\ &= \lim_{t \rightarrow -2^-} \int_t^{14} (x+2)^{-1/4} dx \\ &= \lim_{t \rightarrow -2^-} \frac{4}{3} (x+2)^{3/4} \Big|_t^{14} \\ &= \lim_{t \rightarrow -2^-} \left[\frac{4}{3} (14+2)^{3/4} - \frac{4}{3} (t+2)^{3/4} \right] \\ &= \frac{32}{3} = 10.667. \end{aligned}$$

Converges and its value is 10.667



Question 7

Evaluate the integral

$$\int \frac{x-3}{x^3+3x} dx$$

Solution

$$\frac{x-3}{x^3+3x} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

Clear the denominators:

$$x-3 = A(x^2+3) + (Bx+C)x$$

Combine like terms:

$$x-3 = (A+B)x^2 + Cx + 3A$$

$$\text{Then we get the system of equations: } \begin{cases} A+B=0 \\ C=1 \\ 3A=-3 \end{cases}$$

Solving this system we see that $A = -1, B = 1, C = 1$. Then

$$\frac{x-3}{x(x^2+3)} = -\frac{1}{x} + \frac{x+1}{x^2+3}$$

$$\int \frac{x-3}{x^3+3x} dx = -\ln|x| + \frac{1}{2} \ln(x^2+3) + \frac{1}{3} \sqrt{3} \tan^{-1} \frac{x}{\sqrt{3}} + C$$

Question 8

Evaluate the integral

$$\int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4} dx$$



Solution

We will use partial fraction decomposition.

Since the degree of the numerator is greater than the denominator, we will start with a long division.

$$\frac{x^4 - 4x^2 + x + 1}{x^2 - 4} = x^2 + \frac{x + 1}{x^2 - 4}.$$

Next, we perform a partial fraction decomposition on $\frac{x + 1}{x^2 - 4}$.

$$\begin{aligned}\frac{x + 1}{x^2 - 4} &= \frac{x + 1}{(x - 2)(x + 2)} \\ &= \frac{3}{4(x - 2)} + \frac{1}{4(x + 2)}.\end{aligned}$$

Thus,

$$\begin{aligned}\int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4} dx &= \int \left[x^2 + \frac{3}{4(x - 2)} + \frac{1}{4(x + 2)} \right] dx \\ &= \frac{1}{3}x^3 + \frac{3}{4} \ln |x - 2| + \frac{1}{4} \ln |x + 2| + C.\end{aligned}$$

Question 9

Determine convergence or divergence of the following improper integrals

$$(1) \int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx \qquad (2) \int_1^9 \frac{dx}{(x - 1)^{2/3}}$$



Solution

(1)

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = ?$$

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$$

Put

$$u = \sqrt{x}, \quad u^2 = x \text{ and } dx = 2u du$$

$$\begin{aligned} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx &= 2 \int_1^{\sqrt{t}} \frac{ue^{-u}}{u} du \\ &= 2 \int_1^{\sqrt{t}} e^{-u} du \\ &= 2 - e^{-u} \Big|_1^{\sqrt{t}} \\ &= 2 \left(e - e^{-\sqrt{t}} \right) \end{aligned}$$

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{t \rightarrow \infty} \left[2 \left(e - e^{-\sqrt{t}} \right) \right] = \frac{2}{e}. \quad \text{Converges.}$$

(2)

$$\int_1^9 \frac{dx}{(x-1)^{2/3}} = ?$$

$$\begin{aligned} \int_1^9 \frac{dx}{(x-1)^{2/3}} &= \lim_{t \rightarrow 1^+} \int_t^9 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{t \rightarrow 1^+} \int_t^9 (x-1)^{-2/3} dx \\ &= \lim_{t \rightarrow 1^+} \frac{(x-1)^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} \Big|_t^9 \\ &= \lim_{t \rightarrow 1^+} 3(x-1)^{\frac{1}{3}} \Big|_t^9 \\ &= \lim_{t \rightarrow 1^+} \left[3 \left(2 - \sqrt[3]{1-t} \right) \right] \\ &= 6. \quad \text{Converges.} \end{aligned}$$