(8pts)**Problem 1.**
Consider the sequence a_n given by

$$\frac{2}{4}, \frac{4}{5}, \frac{6}{6}, \frac{8}{7}, \ldots$$

(a) Find a formula for a_n.
(b) Find $\lim_{n\to\infty} a_n$

Solution
(a)

$$a_n = \frac{2n}{3+n}. \qquad (\textbf{4pts})$$

(b)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3+n} = 2 \qquad (\textbf{4pts})$$

(9pts)**Problem 2.**
Find $\lim_{n\to\infty} a_n$.

1. $a_n = n\sin\frac{1}{n}$ 2. $a_n = \frac{n2^n+1}{n^3+4}$ 3. $a_n = \frac{\sqrt{\frac{1}{n}+1}-1}{\frac{1}{n}}$

Solution
1.
$$\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1. \qquad (\mathbf{3pts})$$
2.
$$\lim_{n\to\infty}\frac{n2^n+1}{n^3+4} = \lim_{n\to\infty}\frac{2^n}{n^2} = \infty. \qquad (\mathbf{3pts})$$
3.
$$\lim_{n\to\infty}\frac{\sqrt{\frac{1}{n}+1}-1}{\frac{1}{n}} = \frac{1}{2} \qquad (\mathbf{3pts})$$

(10pts)**Problem 3.**

Determine the sum of the series

$$1.\ \sum_{n=1}^{\infty} \frac{2}{4n^2-1} \qquad\qquad 2.\ \sum_{n=1}^{\infty} 2\left(-\frac{2}{3}\right)^{n-1}$$

Solution

1. This is a telescoping series.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2}{4n^2-1} &= \sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right) = 1 \qquad (\mathbf{5pts})\end{aligned}$$

2.

$$\sum_{n=1}^{\infty} 2\left(-\frac{2}{3}\right)^{n-1} \text{ is a geometric series with } a=2 \text{ and common ratio } r=-\frac{2}{3}$$

$$\left|-\frac{2}{3}\right|<1 \Rightarrow \sum_{n=1}^{\infty} 2\left(-\frac{2}{3}\right)^{n-1} = \frac{2}{1-\left(-\frac{2}{3}\right)} = \frac{6}{5}.$$

(20pts)**Problem 4.**
Determine convergence or divergence of the series. Justify your answer by applying the appropriate test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ 2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+1}\right)^n$

3. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ 4. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^4}$

Solution
1.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \mathrm{e}^{-1} \approx 0.36788 \quad \text{as } n \to \infty.$$

Hence, the series $\sum_{n=1}^{\infty} a_n$ (absolutely) converges via the Ratio Test since $\rho = \mathrm{e}^{-1} < 1$. (**5pts**)

2.

$$\lim_{n\to\infty} \left|\left(\frac{2n+1}{3n+1}\right)^n\right|^{1/n} = \lim_{n\to\infty} \left(\frac{2n+1}{3n+1}\right) = \frac{2}{3} < 1.$$

Hence, the series converges by the root test. (**5pts**)

3.

$$ne^{-n^2} = f(n) \quad \text{where } f(x) = xe^{-x^2} \text{ is continuous positive and decreasing on } [1,\ \infty).$$

The convergence of the series is determined by the convergence of the improper integral

$$\int_1^{\infty} xe^{-x^2}dx$$

$$\int_1^{\infty} xe^{-x^2}dx = \lim_{t\to\infty}\int_1^{t} xe^{-x^2}dx = \frac{1}{2}e^{-1}.$$

Hence, the series converges by the integral test. (**5pts**)

4.

$$\lim_{n\to\infty} \frac{e^n}{n^4} = \infty \neq 0.$$

Hence, the series diverges by the divergent series test. (**5pts**)

(8pts)**Problem 5.**

Using the Integral Test Remainder Estimate for the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ to find that the smallest number of terms needed to ensure that the sum is accurate to within 0.005.

Solution

$$\frac{1}{n^4} = f(n) \quad \text{where } f(x) = \frac{1}{x^4}, \quad \text{positive, continuous and decreasing on } [1,\ \infty).$$

Using the remainder estimate formula, we get

$$\begin{aligned} R_n &\leq \int_n^{\infty} \frac{dx}{x^4} = \lim_{t\to\infty} \int_n^t x^{-4} dx \\ &= \frac{1}{3n^3} \qquad\qquad \textbf{(4pts)} \end{aligned}$$

$$\begin{aligned} \frac{1}{3n^3} < \frac{5}{1000} &\Rightarrow n > \sqrt[3]{\frac{1000}{15}} = 4.0548 \\ n &\geq 5 \qquad\qquad \textbf{(4pts)} \end{aligned}$$

(10pts)**Problem 6.**
(a) The geometric Series

$$\sum_{n=1}^{\infty}(e-2)^{n-1}$$

(a) Converges to $\dfrac{1}{e-3}$

(b) Converges to $\dfrac{1}{3-e}$

(c) Converges to $\dfrac{1}{e-3}$

(d) Diverges to $+\infty$

(e) Diverges to $-\infty$

Correct answer is (b) (**5pts**)

(b)

The series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n+1}}$

(a) converges conditionally

(b) converges absolutely

(c) diverges

(d) converges as a geometric series

(e) converges as a $p-$ series with $p = \dfrac{1}{2}$

Correct answer is (a) (**5pts**)

Correct Answer is (a) (**5pts**)

(10pts)**Problem 7.**
(a)

How many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ do we need to add so that $|\text{error}| < 0.0001$

(a) 100

(b) 98

(c) 96

(d) 94

(e) 92

Correct answer is (a) $(5pts)$

(b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n \pi^n}{(2n)!} =$$

(a) 0

(b) -1

(c) 1

(d) $\frac{1}{\sqrt{2}}$

(e) 0.5

$$\sum_{n=1}^{\infty} \frac{(-1)^n \pi^n}{(2n)!} = \cos\pi = -1$$

Correct answer is (b) (**5pts**)

(15pts)**Problem 8.**
Determine the radius and the interval of convergence of the power series:

1. $\sum_{n=1}^{\infty} \frac{(-3)^n \left(x - \frac{1}{3}\right)^n}{n 2^n}$ 2. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ 3. $\sum_{n=0}^{\infty} n^n x^n$

Solution

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n \left(x-\frac{1}{3}\right)^n}{n2^n}$, center is $\dfrac{1}{3}$.

Let $a_n = (-1)^n \frac{(3x-1)^n}{n2^n}$.

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}(3x-1)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(-1)^n(3x-1)^n}\right| \\ &= \frac{n}{2(n+1)}|3x-1| \to \frac{1}{2}|3x-1| \quad \text{as } n \to \infty \text{ regardless of the value of } x.\end{aligned}$$

Therefore, if $\frac{1}{2}|3x-1| < 1$, then this power series converges *absolutely* (and hence converges) by the Ratio Test. Since

$$\frac{1}{2}|3x-1| < 1 \Longleftrightarrow \left|x-\frac{1}{3}\right| < \frac{2}{3} \Longleftrightarrow -\frac{2}{3} < x - \frac{1}{3} < \frac{2}{3} \Longleftrightarrow -\frac{1}{3} < x < 1,$$

this means that $\boxed{\text{the radius of convergence is } R = \frac{2}{3}.}$

(2pts)

As for the interval of convergence, we need to check the end points of the interval $-\frac{1}{3} < x < 1$. If $x = -\frac{1}{3}$, then

$$\begin{aligned} &= \sum_{n=1}^{\infty} (-1)^n \frac{(-2)^n}{n2^n} \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n 2^n}{n2^n} \\ &= \sum_{n=1}^{\infty} (-1)^{2n} \frac{1}{n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\end{aligned}$$

This is the harmonic series. Hence, it diverges . Hence, $x = \dfrac{-1}{3}$ cannot be included in the interval of convergence. On the other hand, if $x = 1$, then

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n2^n} \\
&= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \\
&= -\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n},
\end{aligned}
$$

which is a negative of the alternating harmonic series. Hence it converges (to $-\ln 2$). Hence, $x = 1$ must be included in the interval of convergence.

Therefore, $\boxed{\text{the interval of convergence is } -\frac{1}{3} < x \le 1, \text{ or } x \in \left(-\frac{1}{3}, 1\right].}$

(3pts)

2. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, center is 0.

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Hence, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \ \infty)$.

3. $\sum_{n=0}^{\infty} n^n x^n$ center is 0.

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{n^n x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n} \right| |x| = \infty.$$

Hence, the radius of convergence is $R = 0$ and the interval of convergence is $[0, 0] = \{0\}$.

(10pts)**Problem 9.**
(a)

Using the definition of Taylor series, the first three nonzero terms of the series for $f(x) = \sin x$ centered at $a = \dfrac{\pi}{6}$ is

(a) $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right) - \dfrac{1}{4}\left(x - \dfrac{\pi}{6}\right)^2$

(b) $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right) + \dfrac{1}{2}\left(x - \dfrac{\pi}{6}\right)^2$

(c) $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right) + \dfrac{1}{8}\left(x - \dfrac{\pi}{6}\right)^2$

(d) $\dfrac{1}{2} - \dfrac{\sqrt{3}}{4}\left(x - \dfrac{\pi}{6}\right) + \dfrac{1}{2}\left(x - \dfrac{\pi}{6}\right)^2$

(e) $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right) - \dfrac{1}{8}\left(x - \dfrac{\pi}{6}\right)^2$

Correct answer is (a). **(5pts)**

(b)

The Maclaurin Series for the function

$$f(x) = \frac{x^2}{3 - x}$$

is

(a) $\sum_{n=0}^{\infty} \dfrac{x^{n+2}}{3^{n+1}}$, $|x| < 3$

(b) $\sum_{n=0}^{\infty} \dfrac{x^{n+1}}{3^{n-1}}$, $|x| < 3$

(c) $\sum_{n=0}^{\infty} \dfrac{x^{2n}}{3^{n}}$, $|x| < 3$

(d) $\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{3^{n+1}}$, $|x| < 3$

(e) $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{n+1}}{3^{n+1}}$

Correct answer is (a) (**5pts**).