§ Slope of Polar Curves

Consider the polar curve

$$r = f(\theta)$$

$x = r\cos\theta$, $y = r\sin\theta$

$x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$

Slope $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

$$\boxed{\text{slope} = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Ex ple Find the slope of the polar cure $r = e^{\cos\theta}$ at $\theta = 0$, $\theta = \frac{\pi}{2}$

$x = r\cos\theta$, $y = r\sin\theta$

$x = e^{\cos\theta}\cos\theta$ and $y = e^{\cos\theta}\sin\theta$

slope $= \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{-\sin\theta e^{\cos\theta}\cdot\sin\theta + e^{\cos\theta}\cos\theta}{-\sin\theta e^{\cos\theta}\cdot\cos\theta - e^{\cos\theta}\sin\theta}$

$$= \frac{0+e}{0-0} = \frac{e}{0} = \infty$$

At $\theta = \frac{\pi}{2}$

$$\text{Slope} = \frac{-1+0}{0-1} = 1$$

* Arclength of Polar curves

If $f(\theta)$ has continuous derivative on the interval $[\alpha, \beta]$ $(\alpha \le \theta \le \beta)$, then the length of the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Exple Find the arclength of the polar curve $r = 2 - 2\cos\theta$ from $\theta = 0$ to $\theta = 2\pi$.

$$\frac{dr}{d\theta} = 2\sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4\sin^2\theta + 4(1-\cos\theta)^2$$

$$= 4\sin^2\theta + 4(1 - 2\cos\theta + \cos^2\theta)$$

$$= \underbrace{4\sin^2\theta} + 4 - 8\cos\theta + \underbrace{4\cos^2\theta}$$

$$= 4 + 4 - 8\cos\theta$$

$$= 8 - 8\cos\theta$$

$$= 8(\underbrace{1-\cos\theta}) = 8\left(2\sin^2\frac{\theta}{2}\right)$$

$$= 16\sin^2\frac{\theta}{2}$$

$$= \left(4\sin\frac{\theta}{2}\right)^2$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\cos\theta = \cos 2\cdot\left(\frac{\theta}{2}\right) = 1 - 2\sin^2\frac{\theta}{2}$$

$$-\cos\theta = -1 + 2\sin^2\frac{\theta}{2}$$

$$1 - \cos\theta = \cancel{1} - \cancel{1} + 2\sin^2\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}$$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \sqrt{\theta^2} = |\theta|$$

$$= \int_0^{2\pi} \sqrt{\left(4\sin\frac{\theta}{2}\right)^2}\, d\theta \qquad \int \sin ax\, dx = -\frac{1}{a}\cos ax + C$$

$$= \int_0^{2\pi} \left|4\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 4\sin\frac{\theta}{2}\, d\theta$$

$$= -8\cos\frac{\theta}{2}\Big|_0^{2\pi}$$

$$= -8\cos\pi + 8\cos 0$$

$$= 8 + 8 = 16$$

$\int$ Differential Equations

$x\cos x + e^x = 1$

$x^2 + 3x - 1 = 0$

* A differential Equation is an equation that involve a function and its derivatives

* $x\frac{dy}{dx} + \cos y = e^x \longrightarrow$ 1st order differential Equation.

* $e^x\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 2$

* We will be interested in solving first order differential equations of the following types:

- Separable differential Equations
- Linear ______
- Exact ______
- Non Exact ______
- homogeneous ______
- Bernouilli ______

§ Separable differential Equations

A first-order differential Equation is said to separable, if it is of the form

$$\frac{dy}{dx} = g(x) \cdot h(y).$$

Exple

Which of the following equations are separable?

1.) $\frac{dy}{dx} - y^2 x e^{3x+4y} = 0$ $\qquad e^{A+B} = e^A \cdot e^B$

2) $\frac{dy}{dx} - y = \sin x$

1.) $\frac{dy}{dx} = y^2 x e^{3x+4y} = y^2 x e^{3x} \cdot e^{4y}$

$= \underbrace{\left(x e^{3x}\right)}_{g(x)} \underbrace{\left(y^2 e^{4y}\right)}_{h(y)}$

yes

2.) $\frac{dy}{dx} - y = \sin x$

$\frac{dy}{dx} = \sin x + y \neq g(x) \cdot h(y)$

Not seperable.

Exple Show that the equation is separable and solve it.

$(1+x)\,dy - y\,dx = 0$

$(1+x)\,dy = y\,dx$

$$(1+x)\frac{dy}{dx} = y \longrightarrow \frac{dy}{dx} = \frac{y}{1+x} = \left(\frac{1}{1+x}\right)\cdot(y)$$

$g(x)\,.\ h(y)$

Separable.

$\cancel{\ln} a = \cancel{\ln} b \Rightarrow a = b$

$$\frac{dy}{dx} = \frac{y}{1+x} \iff \frac{dy}{y} = \frac{dx}{1+x}$$

$$\ln A + \ln B = \ln(A\cdot B)$$

$\Rightarrow$

$C = \ln A$

$A > 0$

$$\ln|y| = \ln|1+x| + C$$

$$\ln|y| = \ln|1+x| + \ln A$$

$$\ln|y| = \ln|A(1+x)|$$

$$|y| = |A(1+x)|$$

$$y = (\pm A)(1+x)$$

$\pm A = K$

$$y = K(1+x)$$

K arbitrary.

* A differential Equation with an initial Condition is called an initial-value problem.

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|1+x| + C$$

$$|y| = e^{\ln|1+x| + C} = e^{C} \cdot \underbrace{e^{\ln|1+x|}}$$

$$|y| = e^{C}|1+x|$$

$$|y| = \left| e^{C}(1+x) \right|$$

$$y = \pm e^{C}(1+x)$$

$$K = \pm e^{C}$$

$$y = K(1+x)$$

Exple Show that the equation is separable and solve the initial value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = -3$$

$$\frac{dy}{dx} = (-x)\cdot\left(\frac{1}{y}\right) \leftarrow \text{Separable}$$

$$y\,dy = -x\,dx$$

$$\Rightarrow \int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C \qquad 2C = R$$

$$y^2 = -x^2 + 2C$$

$$y^2 = -x^2 + R$$

When $x = 4$, $y = -3$ $\quad (y(4) = -3)$

$$(-3)^2 = -(4)^2 + R$$

$$9 = -16 + R$$

$$R = 9 + 16 = 25$$

the solution is

$$y^2 = -x^2 + 25 \iff x^2 + y^2 = 25$$