Differential Equations

→ Linear Equations

A first order Differential Equation of the form

$$a(x)\frac{dy}{dx} + b(x)\,y = c(x)$$

Example

Which of the following equations are linear?

1. $e^{x}\frac{dy}{dx} + x^{2}y = \sin x$ Linear

2. $y\frac{dy}{dx} + (2x+1)y = e^{x}$ Non-linear

3. $\cos x\frac{dy}{dx} + xy^{2} = 2$ Non-linear

4. $x\frac{dy}{dx} + e^{x}y = xy^{2}$ Non-linear

Solving Linear Equations

To solve a first order linear equation, do the following:

1. Write the equation in the form

$$\frac{dy}{dx} + P(x)\,y = Q(x) \Rightarrow \text{Standard form}$$

2. Compute Integrating Factor (IF) $= e^{\int P(x)\,dx}$

3. Solution is

$$y = \frac{1}{IF}\int Q(x)\, IF\, dx$$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$\boxed{e^{\int P(x)\,dx}}$ ⟶ Integrating factor

$$e^{\int P(x)\,dx}\frac{dy}{dx} + P(x)\,e^{\int P(x)\,dx}\,y = Q\,e^{\int P(x)\,dx}$$

$$\frac{dy}{dx}$$

$$\frac{d}{dx}\left[y e^{\int P(x)\,dx}\right] = Q(x) e^{\int P(x)\,dx}$$

$$y e^{\int P(x)\,dx} = \int Q(x) e^{\int P(x)\,dx}\,dx$$

$$y = \frac{\int Q(x) e^{\int P(x)\,dx}\,dx}{e^{\int P(x)\,dx}}$$

Example

Show that the equation is linear and solve it.

1. $\frac{dy}{dx} - 3y = 6$

$P(x) = -3$

$IF = e^{\int -3\,dx} = e^{\int -3\,dx}$

$= e^{-3x}$

$$y = \frac{1}{e^{-3x}} \int 6e^{-3x}\,dx$$

$$= e^{3x}\left[-\frac{6}{3} e^{-3x} + c\right]$$

$$= e^{3x}\left[-2e^{-3x} + c\right]$$

$$= -2 + e^{2x} + c$$

2. $x\frac{dy}{dx} - 4y = x^6 e^x$

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

$P(x) = -\frac{4}{x}$ $\qquad IF = e^{\int P(x)\,dx} = e^{-\int 4/x\,dx}$

$= e^{-4\ln x}$

$= x^{-4}$

$$y = \frac{1}{x^{-4}} \int \frac{x^5 e^x}{x^4}\,dx$$

$$= x^4 \int x e^x\,dx$$

$$= x^4\left[xe^x - e^x + c\right]$$

$$= x^5 e^x - x^4 e^x + x^4 c$$

Example

Show that the equation is linear and solve the initial value problem.

$$(x^2 - 9)\frac{dy}{dx} + xy = 0, \quad y(4) = 1$$

$$\frac{dy}{dx} + \frac{x}{x^2 - 9} y = 0$$

$$P(x) = \frac{x}{x^2 - 9}$$

$$IF = e^{\int P(x)\,dx}$$

$$= e^{\frac{1}{2}\ln|x^2 - 9|}$$

$$= \sqrt{x^2 - 9}$$

$$y = \frac{1}{\sqrt{x^2 - 9}} \int 0 \cdot \sqrt{x^2 - 9}\,dx$$

$$y = \frac{c}{\sqrt{x^2 - 9}}$$

$$1 = \frac{c}{\sqrt{16 - 9}}$$

$$c = \sqrt{7}$$

$$y = \frac{\sqrt{7}}{\sqrt{x^2 - 9}}$$

Exact Equations

Function of Two variables

A function of two variables (x,y) is a relation that assigns a unique number $z = f(x,y)$ to each ordered pair (x,y)

Example

$f(x,y) = x+y-1$

$f(x,y) = \cos(x-y)$

$f(x,y) = xe^{y} - ye^{x} + xy + 1$

Let $z = f(x,y)$ be a function of two variables

The partial derivative with respect to x, denoted by

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$$

$\frac{d}{dx} \rightarrow$ Full derivative

is calculated by taking y as constant and compute the usual derivative with respect to x.

Example

$f(x,y) = x^2y + e^{xy} + 1$

$f_x = 2xy + ye^{xy}$

$f(x,y) = \frac{y}{x} + xy^2$

$f_x = -\frac{y}{x^2} + y^2$

The partial derivative with respect to y, denoted by

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

is calculated by taking x as constant and computing the usual derivative with respect to y.

Example

$$f(x,y) = x^2 y + e^{xy} + 1$$

$$f_y = x^2 + xe^{xy}$$

$$f(x,y) = \frac{y}{x} + xy^2$$

$$f_y = \frac{1}{x} + 2xy$$

The Differential

If f is a function of two variables, then the differential of f is given by

$$df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy$$

dx is a variable that expresses an infinitesimal change in x.
dy is a variable that expresses an infinitesimal change in y.

Example

Find the differential of $f(x,y) = xy^2 + 3x^2 y + 1$ at the point $(1,-2)$

$$df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$f_x = y^2 + 6xy$

$f_y = 2xy + 3x^2$

$\therefore \; df(x,y) = \left(y^2 + 6xy\right) dx + \left(2xy + 3x^2\right) dy$

at $(1,-2)$

$$df(1,-2) = (4-12)\,dx + (-4+3)\,dy$$
$$= -8\,dx - dy$$

$$\frac{d}{dx} k = 0$$

If $\frac{d}{dx} f(x) = 0 \;\forall x$

then $f(x) =$ constant

If $df(x,y) = 0 \;\forall (x,y)$ then $f(x,y) = c =$ constant

$$\underbrace{M(x,y)\,dx + N(x,y)\,dy}_{\downarrow} = 0$$

$$df(x,y) = 0$$
$$f(x,y) = c$$

A differential equation of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if $M(x,y)\,dx + N(x,y)\,dy$ is the differential of some function $f(x,y)$

Example

$$\underbrace{x^2y^3\,dx + x^3y^2\,dy}_{d\left(\frac{1}{3}x^3y^3\right)} = 0$$

$$f_x = \frac{\cancel{3}x^2y^3}{\cancel{3}}\,dx$$

$$f_y = \frac{\cancel{3}x^3y^2}{\cancel{3}}\,dy$$

$$d\left(\frac{x^3y^3}{3}\right) = 0$$

$$\frac{x^3y^3}{3} = c$$

$$x^3y^3 = 3c$$

$$y = \left(3c\,x^{-3}\right)^{1/3}$$

$$= \frac{\sqrt[3]{3c}}{x}$$

Example

Find the function $f(x,y)$

$$\underbrace{x^2y^3\,dx + x^3y^2\,dy}_{df = f_x\,dx + f_y\,dy} = 0$$

$$f_x = x^2y^3$$

$$f_y = x^3y^2$$

$$f_x = x^2y^3 \Rightarrow f(x,y) = \int x^2y^3\,dx + c(y)$$

$$f(x,y) = y^3\int x^2\,dx + c(y)$$

$$= \frac{1}{3}x^3y^3 + c(y)$$

take f_y, equate to original f_y

$$\cancel{x^3y^2} + c'(y) = \cancel{x^3y^2}$$

$$c'(y) = 0$$

$$c(y) = c$$

$$f(x,y) = \frac{x^3y^3}{3} + c$$

$$f(x,y) = \frac{x^3y^3}{3}$$

Test for exactness

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$df(x,y) = 0$$

$$f(x,y) = c$$

Consider the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If $M_y = N_x$ then the equation is exact.

Example

Which of the following equations is exact?

1. $2xy\,dx + (x^2 - 1)\,dy = 0$

$M_y = 2x$

$N_x = 2x$

The equation is exact

2. $x \sin y \, dx + xe^y \, dy = 0$

$M_y = x \cos y$

$N_x = e^y$

$M_y \neq N_x$

The equation is not exact.