(**16pts**)**Problem 1.** Find the general solution of the equation

$$x\left(1+y^2\right)dx = ydy$$

Solution
We have

$$x\left(1+y^2\right)dx = ydy$$

separating the variables, we get

$$xdx = \frac{ydy}{1+y^2} \qquad (\mathbf{3pts})$$

Integrating both sides of this last equation, we get

$$\int xdx = \int \frac{ydy}{1+y^2}$$

or

$$\frac{x^2}{2} = \frac{1}{2}\int \frac{2ydy}{1+y^2}$$

$$\frac{x^2}{2} = \frac{1}{2}\ln\left(1+y^2\right) + c_1 \qquad (\mathbf{10pts})$$

$$x^2 = \ln\left(1+y^2\right) + 2c_1$$

$$x^2 = \ln\left(1+y^2\right) + \ln C, \quad \text{with } C > 0.$$

$$x^2 = \ln\left[C\left(1+y^2\right)\right], \quad \text{with } C > 0.$$

$$C\left(1+y^2\right) = e^{x^2} \qquad (\mathbf{3pts})$$

(16pts)Problem 2. Solve the initial value problem

$$x\frac{dy}{dx} + 2y = x^2, \qquad y(-1) = 2$$

Solution
We have

$$x\frac{dy}{dx} + 2y = x^2$$

Dividing both sides by x we get

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

This is the standard for of a linear differential equation with $P(x) = \dfrac{2}{x}$.

$$IF = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = x^2 \qquad \textbf{(6pts)}$$

$$\frac{d}{dx}\left[x^2 y\right] = x^3$$

$$x^2 y = \int x^3 dx$$

$$x^2 y = \frac{x^4}{4} + C. \qquad \textbf{(6pts)}$$

$$y(-1) = 2 \Rightarrow 2 = \frac{1}{4} + C \Rightarrow C = 2 - \frac{1}{4} = \frac{7}{4} \qquad \textbf{(4pts)}$$

The solution is

$$x^2 y = \frac{x^4}{4} + \frac{7}{4}.$$

(17pts)Problem 3
Show that the differential equation is exact and solve the equation

$$\left(y+2xy^3\right)dx+\left(1+3x^2y^2+x\right)dy=0$$

Solution

$$M=y+2xy^3 \quad \text{and } N=1+3x^2y^2+x$$

$$M_y=1+6xy^2 \quad \text{and } N_x=6xy^2+1$$

$$M_y=N_x \Rightarrow \text{ The equation is exact.} \qquad (\mathbf{4pts})$$

$$\begin{cases} f_x=y+2xy^3 \\ f_y=1+3x^2y^2+x \end{cases}.$$

From $f_x=y+2xy^3$, we get

$$f(x,y)=\int\left(y+2xy^3\right)dx+C(y)$$

$$f(x,y)=yx+x^2y^3+C(y) \qquad (\mathbf{7pts})$$

Using this in the second equation, we get

$$x+3x^2y^2+C'(y)=1+3x^2y^2+x$$

$$C'(y)=1\Rightarrow C(y)=y$$

The solution is given by

$$yx+x^2y^3+y=C \qquad (\mathbf{6pts})$$

(17pts)Problem 4.
Find a special integrating factor and solve the equation

$$\left(x^2+y^2+x\right)dx+xydy=0$$

Solution

$$M=x^2+y^2+x \quad \text{and} \quad N=xy$$

$$M_y=2y \text{ and } N_x=y \Rightarrow \text{The equation is not exact.} \qquad (\mathbf{2pts})$$

$$\frac{M_y-N_x}{N}=\frac{2y-y}{xy}=\frac{1}{x}=f(x) \qquad (\mathbf{5pts})$$

$$SIF=e^{\int \frac{1}{x}dx}=x \qquad (\mathbf{4pts})$$

Multiplying the equation by the integrating factor, we get the exact equation

$$\left(x^3+xy^2+x^2\right)dx+x^2ydy=0$$

Solving this exact equation you get

$$\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2y^2}{2}=c_1 \quad \text{or} \quad 3x^4+4x^3+6x^2y^2=C \qquad (\mathbf{6pts})$$

(17pts)Problem 5.
Solve the initial value problem for the Bernoulli equation

$$\frac{dy}{dx} + xy = xy^2, \qquad y(0) = 2$$

Solution
$n = 2.$
Dividing both sides by y^2, we obtain

$$y^{-2}\frac{dy}{dx} + xy^{-1} = x.$$

Next, we do the substitution

$$u = y^{-1}$$

$$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}.$$

The equation becomes

$$-\frac{du}{dx} + xu = x \quad \text{or} \quad \frac{du}{dx} - xu = -x \quad (\mathbf{8pts})$$

Now solving this linear equation in u, you get

$$u = 1 + ce^{x^2/2}$$

$$y = \frac{1}{u} = \frac{1}{1 + ce^{x^2/2}} \qquad (\mathbf{6pts})$$

$$y(0) = 2 \Rightarrow 2 = \frac{1}{1+c} \Rightarrow c = -\frac{1}{2}. \qquad (\mathbf{3pts})$$

The solution is

$$y = \frac{1}{1 - \frac{e^{x^2/2}}{2}}$$

(17pts)Problem 6.
Show that the equation is homogenous and solve the equation

$$\frac{dy}{dx} = \frac{y+x}{x}$$

Solution

$$\frac{dy}{dx} = \frac{y+x}{x} = \frac{y}{x} + 1 = F(\frac{y}{x}) \qquad (\mathbf{4pts})$$

We do the substitution

$$u = \frac{y}{x} \Rightarrow y = xu$$

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

The equation becomes

$$x\frac{du}{dx} + u = u + 1$$

$$x\frac{du}{dx} = 1$$

$$du = \frac{dx}{x} \qquad (\mathbf{8pts})$$

$$u = \ln|x| + \ln K$$

$$u = \ln|Kx|$$

$$\frac{y}{x} = \ln|Kx|$$

$$y = x\ln|Kx| \quad (\mathbf{5pts})$$