1. $x \dfrac{dy}{dx} = \dfrac{y^2}{x} + y$

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}$$

$$= \frac{y^2}{x^2} + \frac{xy}{x^2}$$

$$= \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

$\therefore$ It is homogeneous

$$u = y/x$$

$$y = xu$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\cancel{u} + x\frac{du}{dx} = u^2 + \cancel{u}$$

$$x\frac{du}{dx} = u^2$$

$$\frac{1}{x}\,dx = u^{-2}\,du$$

$$\int \frac{1}{x}\,dx = \int u^{-2}\,du$$

$$\ln|x| + c = \frac{-1}{u}$$

$$u = \frac{-1}{\ln|x| + c}$$

$$y = \frac{-x}{\ln|x| + c}$$

2. $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

$$(x^2 - 3y^2)\,dx = -2xy\,dy$$

$$\frac{dy}{dx} = \frac{x^2 - 3y^2}{2xy}$$

$$= \frac{3y^2 - x^2}{2xy}$$

$$= \frac{3y^{\cancel{2}}}{2x\cancel{y}} - \frac{x^{\cancel{2}}}{2\cancel{x}y}$$

$$= \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{x}{y}\right)$$

$$= \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right)$$

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

$\therefore$ Homogeneous

$$u = \frac{y}{x}$$

$$y = xu$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{3}{2}u - \frac{1}{2u}$$

$$x\frac{du}{dx} = \frac{u}{2} - \frac{1}{2u}$$

$$x\frac{du}{dx} = \frac{u^2 - 1}{2u}$$

$$\frac{2u}{u^2-1}\,du = \frac{1}{x}\,dx$$

Integrating both sides

$$\ln|u^2-1| = \ln|x| + c_1$$

$$\ln\left|\frac{u^2-1}{\cdots}\right| = c_1$$

$$\left|\frac{u^2-1}{x}\right| = e^{c_1}$$

$$\frac{u^2-1}{x} = \pm e^{c_1}$$

$$u^2-1 = xc$$

$$\frac{y^2}{x^2} = xc+1$$

$$y^2 = x^3c + x^2$$

$$y = \sqrt{x^3c+x^2}$$

$$= x\sqrt{xc+1}$$

8. $$\frac{dy}{dx} + y = e^x y^{-2}$$

$$y^2\frac{dy}{dx} + y^3 = e^x$$

$$u = y^3$$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx}$$

$$\frac{1}{3}\frac{du}{dx} = y^2\frac{dy}{dx}$$

$$\frac{1}{3}\frac{du}{dx} + u = e^x$$

$$\frac{du}{dx} + 3u = 3e^x$$

$$IF = e^{\int 3\,dx} = e^{3x}$$

$$u = \frac{1}{e^{3x}}\int 3e^x \cdot e^{3x}\,dx$$

$$= 3e^{-3x}\int e^{4x}\,dx$$

$$= \frac{3}{4} e^{-3x} e^{4x} + c_1 e^{-3x}$$

$$= \frac{3}{4} e^{x} + c_1 e^{-3x}$$

$$y^3 = \frac{3}{4} e^{x} + c_1 e^{-3x}$$

$$c = c_1 e^{-3x}$$

$$y = \sqrt[3]{\frac{3e^{x}}{4} + c}$$

Tutorial 7

1. $a_n = \frac{(-1)^{n+1}}{n}$

a)

$$\lim_{n \to \infty} |a_n|$$

$$= \lim_{n \to \infty} \frac{1}{n}$$

$$= \frac{1}{\infty}$$

$$= 0$$

If $\lim_{n \to \infty} |a_n| = 0$

then $\lim_{n \to \infty} a_n = 0$

b) $a_n = 3n \sin \frac{\pi}{2n}$

$$= \frac{\sin \pi/2n}{1/3n}$$

$$= 3 \frac{\sin \pi/2n}{1/n}$$

$$= \frac{3}{2} \frac{\sin \pi/2n}{1/2n}$$

$$= \frac{3}{2} \pi \frac{\sin \pi/2n}{\pi/2n}$$

$$\lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} \frac{3}{2}\pi \boxed{\frac{\sin \pi/2n}{\pi/2n}}$$

$$= \frac{3}{2}\pi$$

c) $a_n = \ln\left(\frac{n+1}{n}\right)$

$$\lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$$

$$\lim_{x\to\infty} \ln(f(x)) = \ln\left(\lim_{x\to\infty} f(x)\right)$$

$$= \ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right)$$

$$= \ln\left(\lim_{n\to\infty} 1 + \boxed{\frac{1}{n}}\right) \quad \to 0$$

$$= \ln\left(\lim_{n\to\infty} 1\right)$$

$$= \ln(1)$$

$$= \underline{0}$$

2. $a_n = n \sin\left(\pi + \frac{3}{n}\right)$

$$= \frac{\sin\left(\pi + \frac{3}{n}\right)}{n}$$

$$\lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} \frac{\sin(\pi + 3/n)}{1/n}$$

$$= \lim_{n\to\infty} \frac{\sin\left(\pi + 3/\infty\right)}{0} \Rightarrow \text{Indeterminate}$$

$$= \lim_{n\to\infty} \frac{\cos\left(\pi + \frac{3}{n}\right)\left(\cancel{-}\frac{3}{\cancel{n^2}}\right)}{\cancel{-}1/\cancel{n^2}}$$

$$= \lim_{n\to\infty} \cos(\pi + 0)\,(3)$$

$$= (-1)\,3$$

$$= -3$$

b) Num $d = a_2 - a_1 = 7 - 4 = 3$

$$a_n = 4 + (n-1)\,3$$

$$= 3n + 1$$

Den $d = a_2 - a_1$

$$a_n = 1 + (n-1)\,2$$

$$= 2n - 1$$

$$a_n = \frac{3n+1}{2n-1}$$

$$\lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} \frac{3n+1}{2n-1}$$

$$= \lim_{n\to\infty} \frac{3\cancel{n}}{2\cancel{n}}$$

$$= \frac{3}{2}$$