Math 142 Final Exam Review Problems
Spring 2020

Problem 1
Give an explicit formula for the nth term a_n of the sequence
(a)
$$\{1, 4/5, 6/8, 8/11, 10/14, 12/17, ...\}.$$
(b)
$$\{-1/3, 1/9, -1/27, 1/81, ...\}.$$

Solution
(a)
$$a_n = \frac{2n}{3n-1}$$
(b)
$$a_n = \frac{(-1)^n}{3^n}$$

Problem 2

Write the first six terms of the sequence given by
(a)
$$a_1 = 1;\ a_{n+1} = 2a_n - n(n+1)$$
(b)
$$a_1 = 1;\ a_{n+1} = 1 - 2a_n$$

Solution

(a)
$$1, 0, -6, -24, -68, -166$$
(b)
$$1, -1, 3, -5, 11, -21$$

Problem 3
State whether or not the sequence converges as $n \to \infty$, if it does, find the limit.

1. $a_n = n \ln\left(1 + \frac{1}{n}\right)$
2. $b_n = \dfrac{74n - 9n^7}{5n^7 + 34n + 100}$
3. $c_n = \ln\sqrt{5n+2} - \ln\sqrt{9n+1}$

Solution

1.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \text{“}\frac{0}{0}\text{”}.$$

We now use the l'Hospital's rule by replacing n by x.

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{-\frac{1}{x^2\left(\frac{1}{x}+1\right)}}{\frac{-1}{x^2}} \\ &= \lim_{x\to\infty} \frac{1}{\left(\frac{1}{x}+1\right)} = 1. \end{aligned}$$

We conclude that

$$\lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1.$$

2.

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{74n - 9n^7}{5n^7 + 34n + 100} \\ &= \lim_{n\to\infty} \frac{-9n^7}{5n^7} \\ &= \lim_{n\to\infty} \frac{-9}{5} = \frac{-9}{5}. \end{aligned}$$

3.

$$\begin{aligned} \lim_{n\to\infty} c_n &= \lim_{n\to\infty} \ln\sqrt{5n+2} - \ln\sqrt{9n+1} \\ &= \lim_{n\to\infty} \ln\frac{\sqrt{5n+2}}{\sqrt{9n+1}} \\ &= \lim_{n\to\infty} \ln\sqrt{\frac{5n+2}{9n+1}} \\ &= \lim_{n\to\infty} \frac{1}{2}\ln\frac{5n+2}{9n+1} \\ &= \frac{1}{2}\ln\frac{5}{9}. \end{aligned}$$

Problem 4
Determine whether the series converges or diverges. If it converges, find the sum.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{7^n}$

Solution
1. Put

$$S_n = \sum_{k=1}^{n} \frac{1}{(k+3)(k+4)}.$$

Now doing the partial fraction decomposition, we get

$$\frac{1}{(k+3)(k+4)} = \frac{1}{k+3} - \frac{1}{k+4}.$$

The series is telescoping and S_n is given by

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \left(\frac{1}{k+3} - \frac{1}{k+4} \right) \\ &= \left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) + \left(\frac{1}{6} - \frac{1}{7} \right) + \cdots + \left(\frac{1}{n+3} - \frac{1}{n+4} \right) \\ &= \frac{1}{4} - \frac{1}{n+4}. \end{aligned}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{4} - \frac{1}{n+4} \right) = \frac{1}{4}.$$

We conclude that the series converges and its sum is $\boxed{\frac{1}{4}}$.

2.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-3)^n}{7^n} &= \sum_{n=1}^{\infty} \left(\frac{-3}{7} \right)^n \\ &= \sum_{n=1}^{\infty} \left(\frac{-3}{7} \right)^{n-1+1} \\ &= \sum_{n=1}^{\infty} \left(\frac{-3}{7} \right) \left(\frac{-3}{7} \right)^{n-1}. \end{aligned}$$

This is a geometric series with $a_1 = \frac{-3}{7}$ and $r = \frac{-3}{7}$.

$$\left| \frac{-3}{7} \right| = \frac{3}{7} < 1.$$

The series converges and its sum is

$$S = \frac{a_1}{1-r} = \frac{\frac{-3}{7}}{1+\frac{3}{7}} = -\frac{3}{10}.$$

Problem 5

(a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the value of the sum is accurate to within 0.0005?

Solution

Note that the function $f(x) = \dfrac{1}{x^3}$ does satisfy the conditions of the integral test (you must verify the three conditions by yourself!). By the integral test itself we know that this series converges (it is a p-series with p = 3). Recall that if S is the sum of the series (its true value!), then for any n we have

$$R_n = S - S_n$$

where the remainder R_n can be estimated by using the formula

$$\int_{n+1}^{\infty} \frac{dx}{x^3} \le R_n \le \int_{n}^{\infty} \frac{dx}{x^3}$$

(a) Using $n = 10$,we have

$$\int_{11}^{\infty} \frac{dx}{x^3} \le S - S_{10} \le \int_{10}^{\infty} \frac{dx}{x^3}$$

$\Rightarrow$

$$S_{10} + \int_{11}^{\infty} \frac{dx}{x^3} \le S \le S_{10} + \int_{10}^{\infty} \frac{dx}{x^3}$$

$$\begin{aligned} S_{10} &= \sum_{k=1}^{10} \frac{1}{k^3} = 1 + \frac{1}{2^2} + ... + \frac{1}{10^3} = 1.1975 \\ \int_{10}^{\infty} \frac{dx}{x^3} &= 0.005, \quad \text{and} \quad \int_{11}^{\infty} \frac{dx}{x^3} = 4.1322 \times 10^{-3}. \end{aligned}$$

We have

$$1.1975 + 4.1322 \times 10^{-3} \le S \le 1.1975 + 0.005$$

$$1.2016 \le S \le 1.2025$$

(b) To ensure that the value of the sum is accurate to within 0.0005 we must find a positive integer n such that $R_n \leq 0.0005$. Since

$$R_n \leq \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2},$$

it suffices to find a positive integer n satisfying

$$\frac{1}{2n^2} < 0.0005$$

Solving this inequality we get

$$n^2 > \frac{1}{0.001} = 1000 \qquad \text{or} \qquad n > \sqrt{1000} \approx 31.6$$

Thus, if we take $n = 32$ terms, it is certain that we have the desired accuracy.

Problem 6

(1)

Which one of the following statements is true?

(a) If $\sum_{n=1}^{\infty} a_n$ converges to s and $s_n = \sum_{i=1}^{n} a_i$, then $\lim_{n\to\infty} s_n = s$

(b) If $a_n > 0$ and $\left|\frac{a_{n+1}}{a_n}\right| < 1$ for all $n \geq 1$, then $\{a_n\}$ is increasing

(c) If $\lim_{n\to\infty} a_n = 3$, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges to $\frac{1}{3}$

(d) If $\lim_{n\to\infty} a_n = 10$, then the $\sum_{n=1}^{\infty} a_n$ is a convergent series.

(e) If $\sum_{n=1}^{\infty} a_n$ converges to 1, then $\lim_{n\to\infty} a_n = 1$

<u>**Solution**</u>

(b)-(e) are all false. Only (a) is true.

(2) $\sum_{n=1}^{\infty} n^{-e}$ is

(a) a convergent, p - series.

(b) a divergent, p - series.

(c) a convergent, geometric series.

(d) a divergent, geometric series.

(e) a convergent, alternating series.

Solution

$$\sum_{n=1}^{\infty} n^{-e} = \sum_{n=1}^{\infty} \frac{1}{n^e}.$$

This is a p-series with $p = e = 2.7183 > 1$.
Answer is (a).

Problem 7

Determine whether the series converges or diverges. Justify your answer by citing a relevant test.

1. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ 2. $\sum_{n=1}^{\infty} \left(\frac{2n}{13n+1}\right)^n$ 3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+20}$

Solution

1. Put

$$a_n = \frac{e^n}{n!}.$$

Using the ratio test, we have

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{e^n e}{(n+1)\,n!} \cdot \frac{n!}{e^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{e}{(n+1)}\right| = 0 < 1.
\end{aligned}$$

We conclude that the series converges.

2. Put

$$a_n = \left(\frac{2n}{13n+1}\right)^n.$$

Using the root test, we have

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{2n}{13n+1} = \frac{2}{13} < 1.$$

We conclude that the series converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+20}$ is of the form $\displaystyle\sum_{n=1}^{\infty} (-1)^n b_n$ where $b_n = \dfrac{1}{n+20}$ is positive, decreasing and $\lim_{n\to\infty} \dfrac{1}{n+20} = 0$.We conclude that the series converges by the alternating series test.

Problem 8

Find the interval of convergence for

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} (x+1)^n$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

1.

Solution Letting $u_n = (-1)^n (x+1)^n / 2^n$ produces

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}}}{\dfrac{(-1)^n (x+1)^n}{2^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2^n (x+1)}{2^{n+1}} \right|$$

$$= \left| \frac{x+1}{2} \right|.$$

By the Ratio Test, the series converges if $|(x+1)/2| < 1$ or $|x+1| < 2$. So, the radius of convergence is $R = 2$. Because the series is centered at $x = -1$, it will converge in the interval $(-3, 1)$. Furthermore, at the endpoints you have

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$$ Diverges when $x = -3$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$$ Diverges when $x = 1$

both of which diverge. So, the interval of convergence is $(-3,\ 1)$.

2.

Solution Let $u_n = (-1)^n x^{2n+1} / (2n+1)!$. Then

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}}{\dfrac{(-1)^n x^{2n+1}}{(2n-1)!}} \right|$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)}.$$

For any value of this limit is 0. So, the radius of convergence is ∞ and the interval of convergence is $(-\infty\ ,\ \infty)$.

Problem 9
Find the Mac-Laurin series of the following functions. (Your answer should include the interval of convergence)

1. $g(x) = e^{\sqrt{\frac{x}{2}}}$ 2. $h(x) = \tan^{-1}(5x)$

Solution
Here we use the formulas

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x.$$

and

$$\tan^{-1} x = \arctan x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| \leq 1.$$

1.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x.$$

Thus

$$e^{\sqrt{\frac{x}{2}}} = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n/2}}{n!} \qquad \text{for all } x. \text{ Interval of convergence is } (-\infty,\ \infty).$$

2.

$$\tan^{-1} x = \arctan x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \text{for } |x| \leq 1.$$

Thus

$$h(x) = \tan^{-1}(5x) = \sum_{n=0}^{\infty} \frac{(5x)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{5^{2n+1} x^{2n+1}}{2n+1} \text{for } |5x| \leq 1 \Leftrightarrow |x| \leq \frac{1}{5}$$

$$\text{Interval of convergence is } \left[-\frac{1}{5},\ \frac{1}{5}\right].$$

Problem 10

Decide whether the statement is true or false. (Label the statement **T** if it is true and **F** if it is false)

(a) $\sum_{n=1}^{\infty} n^{-n}$ converges.

(b) If the sequence $\{a_n\}$ diverges and the sequence $\{b_n\}$ diverges, then the sequence $\{a_n + b_n\}$ diverges.

(c) If $\lim_{n\to\infty} a_n$ exists, then $\sum_{n=1}^{\infty} a_n$ converges.

(d) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

(e) If the sequence $\{a_n\}$ converges, then $\lim_{n\to\infty}(a_n - a_{n+1}) = 0$.

(f) If $a_n \neq 0$ for all $n \geq 1$ and $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(g) If $0 < a_n \leq b_n$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

(h) If $|r| < 1$, then $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$.

(i) If $f(x)$ is a positive, continuous and decreasing function for $x \geq 1$ and if $a_n = f(n)$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n = \int_1^{\infty} f(x)\,dx$.

(j) If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} (-a_n)$ converge, then so does $\sum_{n=1}^{\infty} |a_n|$.

1 point each

(a) T (b) F (c) F (d) T (e) T (f) F (g) F (h) T (i) F (j) F

True (a) Since $n^{-n} = \frac{1}{n^n} \le \frac{1}{n^2}$ for $n \ge 1$ and the *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by the Direct Comparison Test $\sum_{n=1}^{\infty} n^{-n}$ also converges. Alternatively, since $n^{-n} = \left(\frac{1}{n}\right)^n \le \left(\frac{1}{2}\right)^n$ for $n \ge 2$ and the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges, then by the Direct Comparison Test $\sum_{n=1}^{\infty} n^{-n}$ also converges. The Ratio Test (or Root Test) can also be used to show the series converges.

False (b) For a counterexample, let $a_n = n$ and $b_n = -n$ for all $n \ge 1$. Then $\lim_{n\to\infty} a_n = \infty$ (diverges) and $\lim_{n\to\infty} b_n = -\infty$ (diverges); however, $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} 0 = 0$ (converges).

False (c) For a counterexample, let $a_n = 1$ for all $n \ge 1$. Then $\lim_{n\to\infty} a_n = 1$ (converges) while $\sum_{n=1}^{\infty} a_n = \infty$ (diverges). Note that even if $\lim_{n\to\infty} a_n$ exists and equals 0, the series $\sum_{n=1}^{\infty} a_n$ may still diverge as in the harmonic series, where $a_n = \frac{1}{n}$ for $n \ge 1$.

True (d) This statement is logically equivalent to the contrapositive statement "if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges," which we know to be true by Theorem 9.16. Recall that if $\sum_{n=1}^{\infty} |a_n|$ converges, then we say $\sum_{n=1}^{\infty} a_n$ converges absolutely.

True (e) Since $\{a_n\}$ converges, let $L = \lim_{n\to\infty} a_n$. Then it's also true that $\lim_{n\to\infty} a_{n+1} = L$. By properties of limits (Theorem 1.2), $\lim_{n\to\infty}(a_n - a_{n+1}) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} a_{n+1} = L - L = 0$.

False (f) The limit inequality $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| < 1$ is equivalent to saying $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.
Therefore, the conclusion according to the Ratio Test should be that the series $\sum_{n=1}^{\infty} a_n$ diverges.

False (g) Knowing that $\sum_{n=1}^{\infty} a_n$ converges says nothing about the convergence of $\sum_{n=1}^{\infty} b_n$. For instance, if $a_n = b_n = \frac{1}{2^n}$ for $n \ge 1$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge to 1. On the other hand if $a_n = \frac{1}{2^n}$ but $b_n = 1$ (so that $0 < a_n \le b_n$ for all *n*), then $\sum_{n=1}^{\infty} a_n$ converges, however $\sum_{n=1}^{\infty} b_n$ diverges. Note that if $\sum_{n=1}^{\infty} a_n$ had diverged, then $\sum_{n=1}^{\infty} b_n$ would have diverged also by the Direct Comparison Test.

True (h) For $|r| < 1$, $\sum_{n=1}^{\infty} r^n$ is just the geometric series $\sum_{n=0}^{\infty} r^n$, which converges to $\frac{1}{1-r}$, except it is missing the first term $r^0 = 1$ when $n = 0$. Therefore $\sum_{n=1}^{\infty} r^n = \frac{1}{1-r} - 1 = \frac{r}{1-r}$. To see this another way, write out the terms of the series and factor out an *r*:
$\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + r^4 + \cdots = r(1 + r + r^2 + r^3 + \cdots) = r\sum_{n=0}^{\infty} r^n = r \cdot \frac{1}{1-r} = \frac{r}{1-r}$. For a third approach notice that the first term of the geometric series $r + r^2 + r^3 + r^4 + \cdots$ is $a = r$ and its common ratio is *r* and so the sum is $\frac{a}{1-r} = \frac{r}{1-r}$.

False (i) According to the Integral Test, the conclusion should be that the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge. If they both converge, they will not converge to the same number. In fact the integral $\int_1^{\infty} f(x)\,dx$ will be *less than* $\sum_{n=1}^{\infty} a_n$ (but be greater than $\sum_{n=2}^{\infty} a_n$). For a counterexample, let $f(x) = e^{-x}$. Then $\sum_{n=1}^{\infty} e^{-n} = \frac{1}{e-1} \approx 0.582$ while $\int_1^{\infty} e^{-x}\,dx = \frac{1}{e} \approx 0.368$.

False (j) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-a_n) = -\sum_{n=1}^{\infty} a_n$ must also converge, but this does not imply $\sum_{n=1}^{\infty} |a_n|$ converges. It is possible for $\sum_{n=1}^{\infty} |a_n|$ to diverge (in which case we would say that $\sum_{n=1}^{\infty} a_n$ converges conditionally). For a counterexample, let $a_n = \frac{(-1)^n}{n}$ for $n \ge 1$. Then both $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge; however $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Problem 11

Determine whether the series converges or diverges. Justify your answer by citing a relevant test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{13^n}{2n!}$ 2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{6n+7}{15n+14}\right)^{\frac{n}{2}}$ 3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2}$

Solution

1. $\displaystyle\sum_{n=1}^{\infty} \frac{13^n}{2n!}$. Using the ratio test with $a_n = \dfrac{13^n}{2n!}$ we have

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{13^{n+1}}{2\,(n+1)!} \cdot \frac{2n!}{13^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{13}{(n+1)}\right| = 0 < 1.
\end{aligned}$$

The series converges.

2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{6n+7}{15n+14}\right)^{\frac{n}{2}}$. Using the root test with $a_n = \left(\dfrac{6n+7}{15n+14}\right)^{\frac{n}{2}}$, we have

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \sqrt{\frac{6n+7}{15n+14}} = \sqrt{\frac{6}{15}} < 1$$

The series converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2} = \sum_{n=1}^{\infty} (-1)^n\, b_n \quad \text{where } b_n = \frac{1}{n^2+2}$$

Because b_n is positive, decreasing and $\lim_{n\to\infty} b_n = 0$, we conclude that the series converges by the alternating series test.

Problem 12
Find the interval of convergence for

$$1.\ \sum_{n=1}^{\infty} \frac{n}{3^n}(x-1)^n \qquad\qquad 2.\ \sum_{n=1}^{\infty} \frac{x^n}{2n!}$$

Solution
1. This is a power series with $a = 1$.

$$a_n = \frac{n}{3^n}(x-1)^n$$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)}{3^n 3}(x-1)(x-1)^n \frac{3^n}{n(x-1)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)}{3}(x-1)\frac{1}{n}\right| = \frac{1}{3}|x-1|
\end{aligned}$$

$$R = 3$$

$$a - R = 1 - 3 = -2 \quad \text{and} \quad a + R = 4.$$

When $x = -2$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{3^n}(x-1)^n &= \sum_{n=1}^{\infty} \frac{n(-1)^n}{3^n} 3^n \\
&= \sum_{n=1}^{\infty} n(-1)^n \qquad \text{diverges.}
\end{aligned}$$

When $x = 4$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{3^n}(x-1)^n &= \sum_{n=1}^{\infty} \frac{n}{3^n} 3^n \\
&= \sum_{n=1}^{\infty} n \qquad \text{diverges.}
\end{aligned}$$

The interval of convergence is

$$(-2,\ 4).$$

2.

$$a_n = \sum_{n=1}^{\infty} \frac{x^n}{2n!}$$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^n x}{2(n+1)!}\cdot\frac{2n!}{x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^n x}{2n!(n+1)}\cdot\frac{2n!}{x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x}{(n+1)}\right| = 0
\end{aligned}$$

$$R = \infty \quad \text{and} \quad \text{the interval of convergence is } (-\infty,\ \infty)$$

Problem 13

Find a power series representation for the following functions. (Your answer should include the interval of convergence)

1. $g(x) = \cos\sqrt{x}$ 2. $h(x) = \tan^{-1}\left(\dfrac{x}{5}\right)$

Solution

1. We know that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} \quad \text{for all } x.$$

$$\cos\sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{n}}{(2n)!} \quad \text{with interval of convergence } (-\infty,\ \infty)\,.$$

2. We know that

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| \le 1.$$

Hence

$$\tan^{-1}\left(\frac{x}{5}\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{5}\right)^{2n+1}}{2n+1} \quad \text{for } \left|\frac{x}{5}\right| \le 1.$$

$$\tan^{-1}\left(\frac{x}{5}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\,5^{2n+1}} \quad \text{for } |x| \le 5.$$

$$\tan^{-1}\left(\frac{x}{5}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\,5^{2n+1}} \quad \text{with interval of convergence } [-5,\ \ 5]\,.$$