(12pts)Problem 1.
Evaluate the following integrals

1. $\int x \sin x dx$ 2. $\int 3 \sec^3 x dx.$

Solution
1.By parts

$$\begin{aligned} u &= x, \quad u' = 1 \\ v' &= \sin x \quad v = -\cos x \end{aligned}$$

$$\int x \sin x dx = \sin x - x \cos x + C \qquad (6pts)$$

2.

$$\int 3 \sec^3 x dx = \int \sec x \sec^2 x dx$$

Use integration by parts. Let $u = \sec x$, $dv = \sec^2 x dx$. Then $du = \sec x \tan x dx$ and $v = \tan x$:

$$\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x dx$$

Notice on the right side we have the same integral as what we started with, so move it over to the left side:

$$2 \int \sec^3 x dx = \sec x \tan x + \ln|\sec x + \tan x|$$

Divide by 2 and add C:

$$\int \sec^3 x dx = \frac{1}{2} (\sec x \tan x + \ln|\sec x + \tan x|) + C$$

$$\int 3 \sec^3 x dx = \frac{3}{2} (\sec x \tan x + \ln|\sec x + \tan x + \tan x|) + C \qquad (6pts)$$

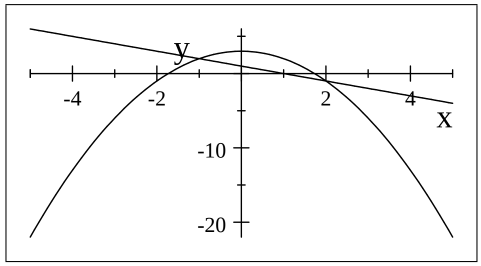
(12pts) Problem 2.

Find the area of the region bounded by the graphs of

$$f(x) = 3 - x^2 \quad \text{and} \quad g(x) = -x + 1$$

from $x = 0$ to $x = 2$.

Solution



$(4pts)$

$$\begin{aligned} A &= \int_0^2 \left[\left(3 - x^2\right) - (-x + 1)\right] dx \\ &= \int_0^2 \left(-x^2 + x + 2\right) dx \\ &= \frac{10}{3} = 3.333\,3. \qquad (8pts) \end{aligned}$$

(12pts) Problem 3
Find the arclength of the function

$$f(x)=\frac{x^3}{6}+\frac{1}{2x}$$

on the interval $[1,3]$.
Solution

$$L=\int_1^3\sqrt{1+[f\ '(x)]^2}dx. \qquad (4pts)$$

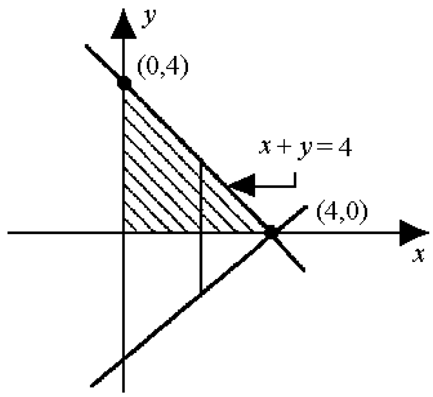
$$f\ '(x)=\frac{1}{2}x^2-\frac{1}{2x^2}$$

$$\begin{aligned}1+[f\ '(x)]^2 &= 1+\left(\frac{1}{2}x^2-\frac{1}{2x^2}\right)^2\\ &= \frac{1}{4x^4}+\frac{1}{4}x^4+\frac{1}{2}\\ &= \left(\frac{1}{2}x^2+\frac{1}{2x^2}\right)^2\end{aligned}$$

$$\begin{aligned}L &= \int_1^3\sqrt{\left(\frac{1}{2}x^2+\frac{1}{2x^2}\right)^2}dx\\ &= \int_1^3\left(\frac{1}{2}x^2+\frac{1}{2x^2}\right)dx\\ &= \frac{14}{3}=4.6667 \qquad (8pts)\end{aligned}$$

(10pts) Problem 4
Sketch the region bounded by $x + y = 4,\ y = 0$ and $x = 0$, and use the disc method to find the volume of the solid generated by revolving the region about the $x-$axis.

Solution



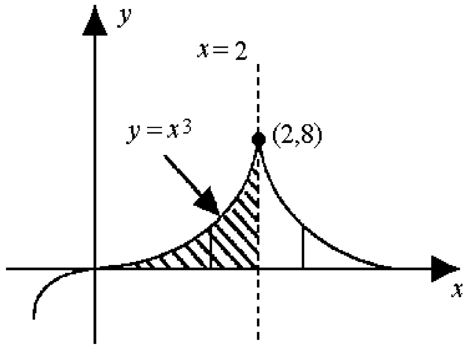
(3pts)

$$
\begin{aligned}
V &= \int_0^4 \pi\,(4-x)^2\,dx \\
\frac{64}{3}\pi &= 67.021 \qquad (7pts)
\end{aligned}
$$

(10pts) Problem 5

Sketch the region bounded by $y = x^3$, $x = 2$, and $y = 0$, and use the shell method to find the volume of the solid generated by revolving the region about the line $x = 2$.

Solution



$(3pts)$

$$\begin{aligned} V &= \int_0^2 2\pi\,(2-x)\left(x^3\right)dx \\ &= \frac{16}{5}\pi = 10.053 \qquad (7pts) \end{aligned}$$

(12pts) Problem 6

Use trigonometric substitution to evaluate the integral

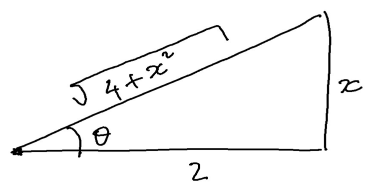
$$\int \frac{dx}{(x^2+4)\sqrt{x^2+4}}.$$

Solution

Put

$$\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta d\theta \qquad\qquad (2pts) \end{aligned}$$

$$\begin{aligned} \int \frac{dx}{(x^2+4)\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta d\theta}{(4\tan^2\theta+4)\sqrt{4\tan^2\theta+4}} \\ &= \int \frac{2\sec^2\theta d\theta}{4\sec^2\theta\,(2\sec\theta)} \\ &= \int \frac{d\theta}{2\sec\theta} \\ &= \frac{1}{2}\sin\theta + C \qquad\qquad (4pts) \end{aligned}$$



$\sin\theta = \dfrac{x}{\sqrt{4+x^2}}$

(2pts)

$$\int \frac{dx}{(x^2+4)\sqrt{x^2+4}} = \frac{x}{2\sqrt{x^2+4}} + C. \qquad (2pts)$$

(12pts) Problem 7
Use partial fraction decomposition to evaluate the integrals

$$1.\ \int \frac{(x+1)\,dx}{x^2\,(x-1)}$$

Solution

$$\frac{x+1}{x^2\,(x-1)} = -\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1} \qquad (6pts)$$

$$\int \frac{(x+1)\,dx}{x^2\,(x-1)} = -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C \qquad (6pts)$$

(10pts) Problem 8
Find the area of the surface obtained by rotating the graph of

$$f(x) = \sqrt{x+1}\,, \qquad -1 \le x \le 1$$

about the x-axis.
Solution

$$\begin{aligned}
\mathrm{S} &= 2\pi \int_{-1}^{1} \sqrt{x+1}\sqrt{1+[f\,'(x)]^2}dx && \textbf{(3pts)} \\
&= 2\pi \int_{-1}^{1} \sqrt{x+1}\sqrt{1+\left(\frac{1}{\sqrt{x+1}}\right)^2}\,dx && \textbf{(4pts)} \\
&= \pi\left(4\sqrt{3}-\frac{4}{3}\right) \\
&= 17.577. && \textbf{(3pts)}
\end{aligned}$$

(10pts)Problem 9
Evaluate the integral

$$\int \frac{1}{x + x\sqrt{x}} dx$$

Solution
Put

$$u = \sqrt{x}$$

$$x = u^2 \quad \text{and} \quad dx = 2udu \qquad (3pts)$$

$$\begin{aligned} \int \frac{1}{x + x\sqrt{x}} dx &= \int \frac{2udu}{u^2 + u^3} \\ &= \int \frac{2du}{u(1+u)} \end{aligned}$$

$$\frac{2}{u(1+u)} = \frac{2}{u} - \frac{2}{u+1}$$

$$\int \frac{2du}{u(1+u)} = 2\ln|u| - 2\ln|u+1| + C \qquad (5pts)$$

$$\int \frac{1}{x + x\sqrt{x}} dx = 2\ln\sqrt{x} - 2\ln\left|\sqrt{x} + 1\right| + C \qquad (2pts)$$