Exact Equations

$$\frac{d}{dx}(k) = 0$$

If $f'(x) = 0 \ \forall x$
then $f(x)$ is constant

If $df(x,y) = 0 \ \forall x, y$
then $f(x,y)$ is constant

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$df(x,y) = f_x\,dx + f_y\,dy$$

$$df(x,y) = 0 \implies f(x,y) = c$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If $M_y = N_x$, then the equation is exact.
Meaning you can find a function $f(x,y)$ such that

$$M(x,y)\,dx + N(x,y)\,dy = df(x,y) = f_x\,dx + f_y\,dy$$

$$f_x = M(x,y)$$
$$f_y = N(x,y)$$

Example

Show that the equation is exact and solve it.

$$\underbrace{2xy}_{M}\,dx + \underbrace{(x^2-1)}_{N}\,dy = 0$$

$M_y = 2x$

$M_y = 2x$

$N_x = 2x$

$M_y = N_x$

$\therefore$ Equation is exact

$$f_x = M(x,y)$$
$$= 2xy$$

$$\boxed{\begin{aligned} f_y &= N(x,y) \\ &= x^2 - 1 \end{aligned}}$$

$$f_x = 2xy$$

$$f(x,y) = \int 2xy \, dx + c(y)$$
$$= x^2 y + c(y)$$

$$f(x,y) = x^2 y + c(y)$$

$$f_y = f_y$$

$$\cancel{x^2} - 1 = \cancel{x^2} + c'(y)$$
$$c'(y) = -1$$
$$c(y) = -y + c$$

$$f(x,y) = x^2 y - y + c$$

$$\underline{f(x,y) = x^2 y - y}$$

Solution

$$f(x,y) = c$$
$$x^2 y - y = c$$

$$y(x^2-1) = c$$

$$y = \frac{c}{x^2-1}$$

Example

Show that the equation is exact and solve the initial value problem.

$$\frac{dy}{dx} = \frac{xy^2 - \sin x \cos x}{y(1-x^2)} \qquad y(0) = 2$$

$$y(1-x^2)\,dy - (xy^2 - \sin x \cos x)\,dx = 0$$

$$(\sin x \cos x - xy^2)\,dx + (y(1-x^2))\,dy = 0$$

$$M_y = 2xy$$

$$N_x = 2xy$$

$$M_y = N_x$$

$\therefore$ Equation is exact

$$f_x = \sin x \cos x - xy^2$$

$$f_y = y - x^2 y$$

$$f(x,y) = \int (y - x^2 y)\,dy + c(x)$$

$$= \frac{y^2}{2} - \frac{x^2y^2}{2} + c(x)$$

$$\sin x \cos x - \cancel{xy^2} = 0 - \cancel{xy^2} + c'(x)$$

$$c'(x) = \sin x \cos x$$

$$c'(x) = \frac{1}{2} \sin 2x$$

$$c(x) = \frac{1}{2}\int \sin 2x \, dx$$

$$= -\frac{\cos 2x}{4} + c$$

$$f(x,y) = \frac{y^2}{2} - \frac{x^2y^2}{2} - \frac{\cos 2x}{4}$$

$$f(x,y) = c$$

$$f(0,2) = 2 - 0 - \frac{1}{4}$$

$$c = \frac{7}{4}$$

$$\frac{y^2}{2} - \frac{x^2y^2}{2} - \frac{\cos 2x}{4} = \frac{7}{8}$$

If constant is $c(x)$, the entire equation will be a function of x.
If constant is $c(y)$, the entire equation will be a function of y.

Special Integrating Factor

Case I

Consider the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$M_y \neq N_x \Rightarrow$ Equation is not exact.

Compute $\dfrac{M_y - N_x}{N}$

If this a function of x alone, then

$a(x) = e^{\int \frac{M_y - N_x}{N} dx}$ is a Special Integrating Factor.

Meaning, if we multiply now the original equation by $a(x)$ then the resulting equation will be exact.

Case II

Otherwise compute $\dfrac{N_x - M_y}{M}$

If this a function of y alone, then

$$b(x) = e^{\int \frac{N_x - M_y}{M} dx}$$ is a Special Integrating Factor.

Meaning, if we multiply now the original equation by $b(x)$ then the resulting equation will be exact.

Example

Show that the equation is not exact, make it exact and solve it.

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

$M_y = x$

$N_x = 4x$

$M_y \neq N_x$

$\therefore$ Equation is not exact

$$a(x) = \frac{x - 4x}{2x^2 + 3y^2 - 20}$$

$a(x)$ is not a function of x alone

$$b(y) = \frac{4x - x}{xy}$$

$$= \frac{3}{y}$$

$$Sp.\ IF = e^{\int \frac{3}{y} dy} = e^{\ln y^3}$$

$= y^3$

$y^3(xy)\,dx + y^3(2x^2 + 3y^2 - 20)\,dy = 0$

$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$

$M_y = 4xy^3$

$N_x = 4xy^3$

$M_y = N_x$

$\therefore$ Equation is exact

$f_x = xy^4$

$f_y = 2x^2y^3 + 3y^5 - 20y^3$

$f(x,y) = \int xy^4\,dx + c(y)$

$= \frac{x^2y^4}{2} + c(y)$

$f_y = f_y$

$\cancel{2x^2y^3} + 3y^5 - 20y^3 = \cancel{2x^2y^3} + c'(y)$

$c'(y) = 3y^5 - 20y^3$

$c(y) = \frac{y^6}{2} - 5y^4 + c$

$f(x,y) = \frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4$

$f(x,y) = c$

$c = \frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4$

Homogeneous Equations

A first order differential equation

$$\frac{dy}{dx} = f(x,y)$$

is said to be homogeneous if

$$f(x,y) = F\left(\frac{y}{x}\right)$$

Example

Which of the following equations are homogeneous?

1. $(x^2+y^2)\,dx + (x^2 - xy)\,dy = 0$

$$(x^2 - xy)\,dy = -(x^2+y^2)\,dx$$

$$\frac{dy}{dx} = -\frac{x^2+y^2}{x^2-xy}$$

$$= -\frac{\cancel{x^2}\left(1+\frac{y^2}{x^2}\right)}{\cancel{x^2}\left(1-\frac{y}{x}\right)}$$

$$= \frac{-\left(1+(y/x)^2\right)}{1-y/x}$$

Homogeneous

2. $(xe^x - y)\,dx + (xy + x^2)\,dy = 0$

$$\frac{dy}{dx} = \frac{y - xe^x}{xy + x^2}$$

$$= \frac{-\cancel{x}\left(-y/x + e^x\right)}{\cancel{x}(y+x)}$$

Not a function of (y/x)

Not Homogeneous

3. $(x - 2y)\,dx + x\,dy = 0$

$$\frac{dy}{dx} = \frac{-x - 2y}{x}$$

$$= 2\frac{y}{x} - 1$$

Homogeneous

Solving Homogeneous Equations

To solve a homogeneous equation do the following

1. Write the equation in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$
2. Do the substitution

$$u = y/x$$

$$y = xu$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

3. Convert now the original equation in y into a separable equation in u and solve.

Example

Show that the equation is homogeneous and solve the initial value problem.

$$\left[x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)\right] dx + x \cos\left(\frac{y}{x}\right) dy = 0 \qquad y(1) = \pi/2$$

$$\frac{dy}{dx} = \frac{y\cos(y/x) - x\sin(y/x)}{x\cos(y/x)}$$

$$= \frac{y/x\,\cos(y/x) - \sin(y/x)}{\cos(y/x)}$$

$$= \frac{y}{x} - \tan(y/x) = F(y/x)$$

$$= \frac{y}{x} - \tan\left(\frac{y}{x}\right)$$

Homogeneous

$$u = y/x$$

$$y = xu$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\cancel{u} + x\frac{du}{dx} = \cancel{u} - \tan(u)$$

$$x\frac{du}{dx} = -\tan u$$

$$\frac{du}{dx} = -\frac{\tan u}{x}$$

$$\frac{1}{\tan u}\, du = \frac{-1}{x}\, dx$$

$$\int \frac{\cos u}{\sin u}\, du = -\ln x + c$$

$$\ln|\sin u| = -\ln x + c$$

$$\ln|\sin u| = \ln x^{-1} + \ln k \qquad c = \ln k$$

$$\cancel{\ln}|\sin u| = \cancel{\ln}\frac{k}{|x|}$$

$$|\sin u| = \left|\frac{k}{x}\right|$$

$$\sin u = \pm\frac{k}{x} \qquad \pm k = A$$

$$\sin u = \frac{A}{x}$$

$$\sin\frac{y}{x} = \frac{A}{x}$$

$$\frac{y}{x} = \sin^{-1}\left(\frac{A}{x}\right)$$

$$x \qquad \left(x\right)$$

$$y = x \sin^{-1}\left(\frac{A}{x}\right)$$

$$y(1) = \pi/2$$

$$y = \sin^{-1}(A)$$

$$\sin\frac{\pi}{2} = A$$

$$A = 1$$

Solution

$$y = x \sin^{-1}\left(\frac{1}{x}\right)$$

Bernoulli Equation

A differential equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

is called a Bernoulli Equation.

If $n=1$ or $n=0$, the equation is linear

Assume that $n \neq 0$, $n \neq 1$

1. Divide by y^n to get

$$y^{-n}\frac{dy}{dx} + P(x)\,y \cdot y^{-n} = Q(x)$$

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$$

2. Do the substitution $u = y^{-n}$