Alternating Series

A series of the form

$$\sum_{n=1}^{\infty} (-1)^n b_n \quad \text{OR} \quad \sum_{n=1}^{\infty} (-1)^{n\pm 1} b_n$$

where $b_n > 0$ is called an Alternating Series

Alternating Series Test

Any alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ where b_n satisfies the following

- $b_n > 0$ for $n \geq 1$
- b_n is decreasing $\Rightarrow b_{n+1} < b_n$
- $\lim_{n\to\infty} b_n = 0$

will converge.

Example

Determine convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$$

$b_n = \frac{1}{\sqrt{2}} \Rightarrow$ positive, decreasing and $\lim_{n\to\infty} b_n = 0$

$\therefore$ Converges by AST

2. $\sum_{n=1}^{\infty} \frac{n}{(-3)^{n-1}}$

$$\sum_{n=1}^{\infty} \frac{n}{(-1)^{n-1} \cdot 3^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n}{\underbrace{(-1)^{n-1}(-1)^{n-1}}_{(-1)^2 \text{ OR } (1)^2 = 1} 3^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n}{3^{n-1}}$$

$b_n = \dfrac{3n}{3^n} \Rightarrow$ positive, decreasing and $\lim_{n\to\infty} b_n = 0$

$$\hookrightarrow \lim_{x\to\infty} \frac{x}{3^{x-1}} = \frac{1}{3^{x-1}\ln 3} = \frac{1}{\infty} = 0$$

$\therefore$ Converges by AST

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$$

$b_n = \dfrac{n^2}{n^3+1} \Rightarrow$ positive, decreasing and $\lim_{n\to\infty} b_n = 0$

$$\hookrightarrow \lim_{x\to\infty} \frac{x^2}{x^3+1} = \frac{x^2}{x^3} = \frac{1}{x} = \frac{1}{\infty} = 0$$

$\therefore$ Converges by AST

Estimating the Sum of Alternating Series

Consider the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

where $\begin{cases} b_n > 0 \quad \text{for } n \geq 1 \\ b_n \text{ decreasing} \\ \lim_{n\to\infty} b_n = 0 \end{cases}$

In the approximation of S by

$$S_n = \sum_{k=1}^{n} (-1)^{k-1} b_k$$

the absolute error

$$|S - S_n| < b_{n+1}$$

@ $n=10$ $\quad |S - S_{10}| < b_{11}$

$$-b_{11} \leq S - S_{10} \leq b_{11}$$

$$S_{10} - b_{11} \leq S \leq S_{10} + b_{11}$$

Example

A) Use $n=6$ to approximate the sum of the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$

B) How many terms are required to ensure the sum is accurate to within 0.001

A) $|S - S_6| \leq b_7$ $\qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$

$$S_6 - b_7 \leq S \leq S_6 + b_7$$

$$S_6 = \sum_{k=1}^{6} \frac{(-1)^{k-1}}{k^2+1} = \frac{1}{2} - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} + \frac{1}{26} - \frac{1}{37}$$

$$= 0.35261$$

$$b_7 = \frac{(-1)^{7+1}}{50} \approx \frac{1}{50} = 0.02$$

$$0.3326 \leq S \leq 0.3726$$

B) $$\underbrace{|S - S_n|}_{R} \leq b_{n+1} = \frac{1}{(n+1)^2+1} \leq 10^{-3}$$

$$(n+1)^2 + 1 \geq 10^3$$

$$\sqrt{(n+1)^2} \geq \sqrt{999}$$

$$n \geq \sqrt{999} - 1$$

$$n \geq 30.6$$

$$\therefore \underline{n \geq 31}$$

Limit Comparision Test

Suppose $a_n > 0$ and $b_n > 0$

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$

then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

both converge or both diverge

Example

Determine convergence or diverge

1. $\sum_{n=1}^{\infty} \frac{2n}{n^3+4n+1}$

$$\sum_{n=1}^{\infty} \frac{2n}{n^3+4n+1}$$

$$a_n = \frac{2n}{n^3+4n+1}$$

$$b_n = \frac{2n}{n^3} = \frac{2}{n^2}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^2}$$

Converges by p-series test $\because p = 2 > 1$

$$\therefore \sum_{n=1}^{\infty} \frac{2n}{n^3+4n+1}$$

not required in exam

$$\boxed{\lim_{n\to\infty} \frac{\frac{2n}{n^3+4n+1}}{\frac{2}{n^2}} = \frac{2n^3}{2n^3+8n+1} = \frac{2n^3}{2n^3} = 1}$$

2. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+7}$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+7} \sim \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2}$$

$$a_n = \frac{\sqrt{n}}{n^2+7}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$p = 3/2 > 1$

$\therefore$ Converges by p-series

$b_n = \frac{n^{1/2}}{n^2} = n^{-3/2}$

$= \frac{1}{n^{3/2}}$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $\quad p = 3/2 > 1$

$\therefore$ Converges by p series

$\therefore \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+7}$ converges

3. $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$

$$\sum_{n=1}^{\infty} \frac{n2^n}{4n^3} \sim \sum_{n=1}^{\infty} \frac{n2^n}{4n^{\cancel{3}2}}$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{4n^2}$$

$$\lim_{n\to\infty} \frac{2^n}{4n^2} = \lim_{x\to\infty} \frac{2^x}{4x^2}$$

$$= \lim_{x\to\infty} \frac{2^x \ln 2}{8x}$$

$$= \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{8}$$

$$= \infty \neq 0$$

$\therefore$ Series diverges

$\therefore \sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$ diverges

Absolute vs. Conditional Convergence

Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute convergance implies normal convergence.

$\sum^{\infty}$ ... $\sum^{\infty}$

$$\sum_{n=1} |a_n| \text{ converges} \implies \sum_{n=1} a_n \text{ converges}$$

$$a_n \leqslant |a_n|$$

$$\sum_{n=1}^{\infty} a_n \leqslant \sum_{n=1}^{\infty} |a_n|$$

Conditional Convergence

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Rightarrow \text{converges by AST} \qquad \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \frac{1}{n} \Rightarrow \text{diverges by p-series}$$

Example

Determine whether the series converges or diverges and classify any convergence as conditional or absolute.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)}}{9^n}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n(n+1)}}{9^n} \right|$$

$$= \sum_{n=1}^{\infty} \frac{1}{9^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{1}{9}\right)^{n-1}$$

$$|r| = \frac{1}{9} < 1$$

$\therefore$ Series converges absolutely

2. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{n^2+2} \right|$$

$$= \sum_{n=1}^{\infty} \frac{n}{n^2+2} \sim \sum_{n=1}^{\infty} \frac{n}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

$p = 1$

Diverges by p-series test

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{n^2+2} \right| \text{ diverges}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$$

$b_n = \frac{n}{n^2+2} \Rightarrow$ positive, decreasing and $\lim_{n\to\infty} b_n = 0$

$\hookrightarrow \lim_{n\to\infty} \frac{n}{n^2} = \frac{1}{n} = 0$

$\therefore$ Converges by AST

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$ converges conditionally

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right|$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$p = 2 > 1$

$\therefore$ Converges by p-series

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely

Ratio Test

Consider the series $\sum_{n=1}^{\infty} a_n$

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$

then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$

then the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

then the test cannot decide. Use another test.

Example

Determine convergence or divergence.

1. $\sum_{n=0}^{\infty} \frac{5^n}{n!}$

$$a_n = \frac{5^n}{n!}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{\frac{5^{n+1}}{(n+1)!}}{5^n/n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{5\cdot \cancel{5^n}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{5^n}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{5}{n+1}\right|$$

$$= \frac{5}{\infty} = 0 < 1$$

$\therefore \sum_{n=0}^{\infty} \frac{5^n}{n!}$ converges absolutely

2. $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

$$\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$$

$$a_n = \frac{n^2 2^{n+1}}{3^n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2 \, 2^{n+2}}{3^{n+1}} \times \frac{3^n}{n^2 2^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2 \times \cancel{2^{n+1}} \times 2 \times \cancel{3^n}}{\cancel{3^n} \times 3 \times n^2 \times \cancel{2^{n+1}}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2(n+1)^2}{3n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2n^2 + 4n + 2}{3n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2\cancel{n^2}}{3\cancel{n^2}} \right|$$

$$= \frac{2}{3} < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ converges absolutely

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

$$a_n = (-1)^n \frac{n^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (n+1)^{(n+1)}}{(n+1)!} \times \frac{n!}{(-1)^n n^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^n} \cancel{(-1)} (n+1)^n \cancel{(n+1)}}{\cancel{(n+1)} \cancel{n!}} \times \frac{\cancel{n!}}{\cancel{(-1)^n} n^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^n \right|$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$= e > 1$$

Diverges

4. $\sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

$a_n = (-1)^n \frac{\sqrt{n}}{n+1}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \sqrt{n+1}}{n+2} \times \frac{n+1}{(-1)^n \sqrt{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{\sqrt{n}} \times \frac{n+1}{n+2} \times \frac{\cancel{(-1)^n} \cancel{(-1)}}{\cancel{(-1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \sqrt{\frac{n+1}{n}} \; \frac{n+1}{n+2} \right|$$

$$= 1$$

$\therefore$ Not conclusive