EXAMINATION COVERSHEET

Spring 2023 Quiz 1

UNIVERSITY OF WOLLONGONG IN DUBAI

THIS EXAMINATION CONTENT IS STRICTLY CONFIDENTIAL Students must comply with requirements stated in the Examination Policy & Procedures	
Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	**05/18/2023**
Subject Code:	**Math 142**
Subject Title:	**Essentials of Engineering Mathematics**
Time Permitted to Write Exam:	1 Hour
Total Number of Questions:	**5 written questions**
Total Number of Pages *(including this page):*	**6**

Problem 1 (8 points)
Find the area of the region bounded by

$$y = x^2 - 4x + 5 \text{ and } \; y = 2x - 3.$$

Solution
Method 1 without graphing

$$\begin{aligned} A &= \int_a^b \left| \left(x^2 - 4x + 5\right) - (2x - 3) \right| dx \\ &= \int_a^b \left| x^2 - 6x + 8 \right| dx. \qquad \textbf{(2pts)} \end{aligned}$$

Next, we find a and b by solving

$$x^2 - 4x + 5 = 2x - 3$$

$$\begin{aligned} x^2 - 6x + 8 &= 0 \Leftrightarrow \\ (x-2)(x-4) &= 0 \end{aligned}$$

$$a = 2 \text{ and } b = 4. \quad \textbf{(2pts)}$$

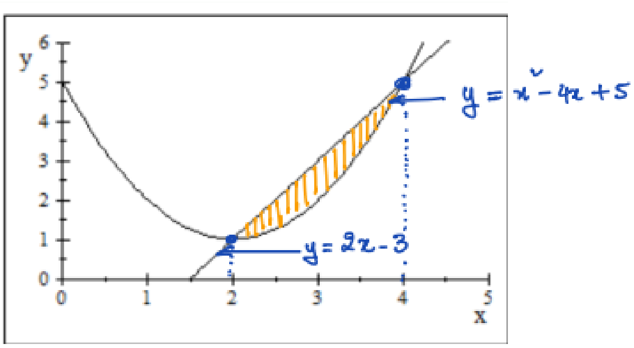
Now using the fact that $x^2 - 6x + 8$ has the opposite sign of $a = 1$, we conclude that

$$\left| x^2 - 6x + 8 \right| = -\left(x^2 - 6x + 8\right) \text{ for } 2 \le x \le 4. \quad \textbf{(2pts)}$$

Hence,

$$\begin{aligned} A &= -\int_2^4 \left(x^2 - 6x + 8\right) dx \\ &= \left[\frac{-x^3}{3} + 3x^2 - 8x \right]_2^4 \\ &= \frac{4}{3} = 1.3333 \quad \textbf{(2pts)} \end{aligned}$$

Method 2 with graphing
We graph both functions on the same window



(4pts)

You can see that between 2 and 4, the line is on top of the parabola. Thus,

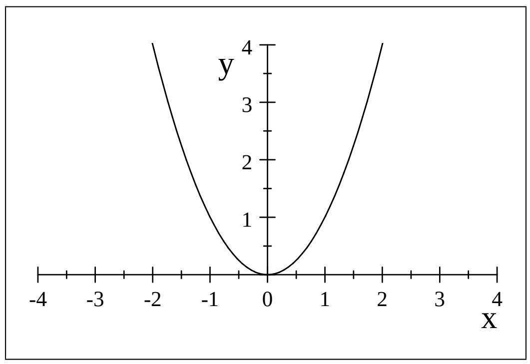
$$\begin{aligned} &\int_a^b (2x - 3) - \left(x^2 - 4x + 5\right) dx \\ &= \int_2^4 \left(-x^2 + 6x - 8\right) dx \\ &= \frac{4}{3} = 1.3333 \qquad \textbf{(4pts)} \end{aligned}$$

Problem 2 (8 points)

A lampshade is constructed by rotating the curve $y = x^2$ around the y-axis from $(1, 1)$ to $(3, 9)$ as seen in the picture below.

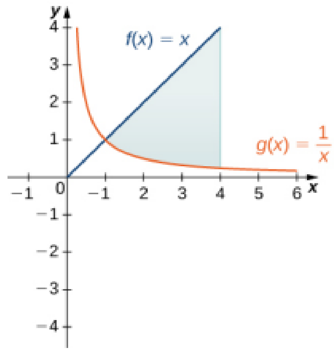
Determine how much material you would need to construct this lampshade—that is, the surface area.

Solution

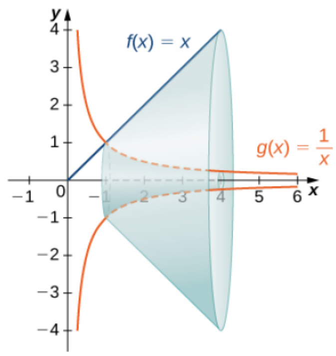


$$\begin{aligned}
S &= 2\pi \int_1^3 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad (\textbf{2pts}) \\
&= 2\pi \int_1^3 x\sqrt{1+4x^2}dx \\
&= \frac{2\pi}{8}\int_1^3 8x\left(1+4x^2\right)^{1/2}dx \qquad (\textbf{3pts}) \\
&= \left[\frac{\pi}{4}\times\frac{2}{3}\left(1+4x^2\right)^{3/2}\right]_1^3 \\
&= \frac{37}{6}\sqrt{37}\pi - \frac{5}{6}\sqrt{5}\pi \\
&= 111.99 \qquad\qquad (\textbf{3pts})
\end{aligned}$$

Problem 3 (8 points)
Use the disc method to find the volume of the solid obtained by rotation the region bounded by the graph of $f(x) = x$ and below the graph of $g(x) = \dfrac{1}{x}$ over the interval $[1, 4]$ around the x-axis.



Solution



$$\begin{aligned}
V &= \int_1^4 Area\ of\ slice\ dx \\
&= \int_1^4 \left[\pi x^2 - \pi\left(\frac{1}{x}\right)^2\right] dx \qquad (\mathbf{4pts}) \\
&= \pi \int_1^4 \left(x^2 - x^{-2}\right) dx \\
&= \pi \left[\frac{x^3}{3} + \frac{1}{x}\right]_1^4 \\
&= \frac{81}{4}\pi = 63.617 \qquad (\mathbf{4pts})
\end{aligned}$$

Problem 4 (8 points)
Find the points (x, y) at which the curve $x = 2t - t^3$, $y = t - 1$ has
(a) a horizontal tangent;
(b) a vertical tangent.

Solution
(a) Horizontal tangent;

$$\frac{dy}{dt} = 0 \text{ and } \frac{dx}{dt} \neq 0.$$

$$\frac{dy}{dt} = 1 \neq 0 \Rightarrow \text{No horizontal tangent.} \quad (\mathbf{4pts})$$

(b) Vertical tangent.

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} \neq 0.$$

$$\begin{aligned} \frac{dx}{dt} &= 0 \Leftrightarrow 2 - 3t^2 = 0 \\ t &= \pm\sqrt{\frac{2}{3}} \end{aligned}$$

Since $\frac{dy}{dt}$ is never zero, the graph has vertical tangent at $t = \pm\sqrt{\frac{2}{3}}$.
The corresponding points are

$$\left(\frac{4}{3}\sqrt{\frac{2}{3}},\ \sqrt{\frac{2}{3}} - 1\right) = (1.0887,\ -0.1835) \qquad (\mathbf{2pts})$$

and

$$\left(\frac{-4}{3}\sqrt{\frac{2}{3}},\ -\sqrt{\frac{2}{3}} - 1\right) = (-1.0887,\ -1.8165) \qquad (\mathbf{2pts})$$

Problem 5 (8 points)
Evaluate the following integrals

$$1. \int_1^2 x^3 \ln x dx, \qquad\qquad 2. \int \frac{dx}{(x^2+4)^{3/2}}.$$

Solution
1. $\int_1^2 x^3 \ln x dx$.
By parts

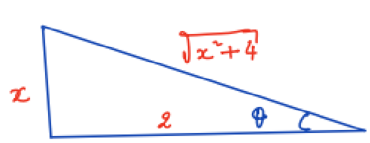
$$\begin{aligned} u &= \ln x, \quad u' = \frac{1}{x} \\ v' &= x^3, \quad v = \frac{x^4}{4} \end{aligned}$$

$$\begin{aligned} \int_1^2 x^3 \ln x dx &= \left[\frac{x^4}{4}\ln x\right]_1^2 - \frac{1}{4}\int_1^2 x^3 dx \qquad (\textbf{2pts}) \\ &= \left[\frac{x^4}{4}\ln x\right]_1^2 - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^2 \\ &= 4\ln 2 - \frac{15}{16} = 1.8351 \qquad (\textbf{2pts}) \end{aligned}$$

2. $\int \frac{dx}{(x^2+4)^{3/2}}$. Here, we use trigonometric substitution.
Put

$$\begin{aligned} x &= 2\tan\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \\ dx &= 2\sec^2\theta d\theta \text{ and } \sqrt{x^2+4} = 2\sec\theta \qquad (\textbf{2pts}) \end{aligned}$$

$$\begin{aligned} \int \frac{dx}{(x^2+4)^{3/2}} &= \int \frac{dx}{(x^2+4)\sqrt{x^2+4}} \\ &= \int \frac{\cancel{2}\sec^2\theta d\theta}{(4\sec^2\theta)(\cancel{2}\sec\theta)} \\ &= \int \frac{d\theta}{4\sec\theta} \\ &= \frac{1}{4}\int \cos\theta d\theta \\ &= \frac{1}{4}\sin\theta + C \end{aligned}$$



$x = 2\tan\theta \iff$

$\tan\theta = \frac{x}{2}$

$\Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}}$

$$\int \frac{dx}{(x^2+4)^{3/2}} = \frac{x}{4\sqrt{x^2+4}} + C \qquad (\textbf{2pts})$$