Tutorial 7

3. i) $\sum_{n=1}^{\infty} 4^n 5^{1-n}$

$= \sum_{n=1}^{\infty} \frac{4^n \cdot 5}{5^n}$

$= \sum_{n=1}^{\infty} 5\left(\frac{4}{5}\right)^n$

$= \sum_{n=1}^{\infty} 5\left(\frac{4}{5}\right)^{n-1+1}$

$= \sum_{n=1}^{\infty} 4\left(\frac{4}{5}\right)^{n-1}$

$|r| = \frac{4}{5} < 1$ $\therefore$ Converges

$\text{Sum} = \frac{a_1}{1-r} = \frac{4}{1-4/5} = \frac{4}{1/5} = 20$

4. i) $\sum_{n=1}^{\infty} \left[(-0.2)^n + (0.6)^{n-1}\right]$

$\sum_{n=1}^{\infty} (-0.2)^n$

$|r| = 0.2 < 1$ $\therefore$ converges

$a = -0.2$

$\text{Sum} = \frac{-0.2}{1-(-0.2)} = \frac{-0.2}{1.2} = \frac{-2/10}{12/10} = -\frac{1}{6}$

$\sum_{n=1}^{\infty} (0.6)^{n-1}$

$|r| = 0.6 < 1$ $\therefore$ converges

$a = 1$

$\text{Sum} = \frac{a}{1-r} = \frac{1}{0.4} = \frac{10}{4}$

Total sum $= \frac{10}{4} - \frac{1}{6}$

$= \frac{60-4}{24}$

$= \frac{56}{24} = \frac{7}{3} = 2.33$

Extra Example

$$\sum_{n=1}^{\infty} 5 \cdot 7^{1-n}$$

$$= \sum_{n=1}^{\infty} \frac{5 \times 7}{7^n}$$

$$= \sum_{n=1}^{\infty} 35 \times \left(\frac{1}{7}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{35}{7} \left(\frac{1}{7}\right)^{n-1}$$

$|r| = \frac{1}{7} < 1 \quad \therefore$ Converges

$$S = \frac{a}{1-r} = \frac{35/7}{6/7} = \frac{35}{6}$$

$\rightarrow \sum_{k=1}^{\infty} 3^{2k} \cdot 5^{1-k}$

$\rightarrow \sum_{n=1}^{\infty} 3\left(\frac{11}{8}\right)^{n+1}$

Extra Questions

Diverges $\because |r| > 1$

3\. ii) $\sum_{n=1}^{\infty} \frac{1}{(n+1)!\,n!} \left((n+1)! - n!\right)$

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$$= \sum_{n=1} \frac{(n+1)! - n!}{(n+1)!\, n!}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} - \frac{1}{(n+1)!}$$

$$S_N = \sum_{n=1}^{N} \frac{1}{n!} - \frac{1}{(n+1)!}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n!} - \frac{1}{(n+1)!}$$

$$= \lim_{N\to\infty} \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{24}}\right) - - - - - \left(\cancel{\frac{1}{N!}} - \frac{1}{(N+1)!}\right)$$

$$= \lim_{N\to\infty} 1 - \frac{1}{(N+1)!}$$

$$= 1 - 0$$

$$= \underline{1}$$

$\therefore$ Converges & its value is $L = 1$

4. ii) $$\sum_{n=0}^{\infty} \left[6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right) \right]$$

$$S_N = \sum_{n=0}^{N} \left[6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right) \right]$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} \left[6\tan^{-1}\left(\frac{\sqrt{3}}{n+1}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{n+2}\right) \right]$$

$$= \lim_{N\to\infty} \left(6\tan^{-1}(\sqrt{3}) - 6\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) + \left(6\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right) +$$

$$- - - + \left(6\tan^{-1}\left(\frac{\sqrt{3}}{N+1}\right) - 6\tan^{-1}\left(\frac{\sqrt{3}}{N+2}\right)\right)$$

$$= \lim_{N\to\infty} 6\tan^{-1}(\sqrt{3}) - 6\tan^{-1}\left(\frac{\sqrt{3}}{N+2}\right)$$

$$= 6\tan^{-1}(\sqrt{3}) - 6\tan^{-1}(0)$$

$$= 6\left[\frac{\pi}{3} - 0\right]$$

$$= \frac{2\pi}{2}$$

Extra Example

Find the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^2+7k+12}$ $\qquad \sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)$

$$\sum_{k=1}^{\infty} \frac{A}{k+3} + \frac{B}{k+4}$$

$$1 = A(k+4) + B(k+3)$$

@ $k=-4$ $\qquad$ @ $k=-3$

$B=-1$ $\qquad$ $A=1$

$$\sum_{k=1}^{\infty} \left[\frac{1}{k+3} - \frac{1}{k+4}\right]$$

5. i) $\sum_{n=1}^{\infty} \sqrt[3]{\frac{-8n}{n+4}}$

$$\lim_{n\to\infty} \sqrt[3]{\frac{-8n}{n+4}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{-8\cancel{n}}{\cancel{n}(1+4/n)}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{-8}{1+4/n}}$$

$$= \sqrt[3]{\frac{-8}{1+0}}$$

$$= \sqrt[3]{-8}$$

$$= \underline{-2} \neq 0$$

$\therefore$ Series diverges

5. ii) $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

$\lim_{n\to\infty} \frac{e^n}{n^3}$

$= \lim_{n\to\infty} \frac{e^n}{3n^2}$

$= \lim_{n\to\infty} \frac{e^n}{6n}$

$= \lim_{n\to\infty} \frac{e^n}{6}$

Using L'Hopital's Rule

$= \frac{\infty}{6}$

$= \infty \neq 0$

$\therefore$ Series diverges

Extra Examples

$\rightarrow \sum_{k=1}^{\infty} \frac{k}{e^k}$

$\rightarrow \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n/2}$

$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n/2}$

$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{n}\right)^n\right]^{1/2}$

$\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k$

$= (e^1)^{1/2}$

$= e^{1/2}$

Determine convergence or divergence $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$

$p = \frac{1}{2} < 1$

$\therefore$ Diverges

6. i) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln(n)}$

$\frac{1}{3 \ln 3} < \frac{1}{2 \ln 2}$

$\therefore$ a_n is decreasing

$\lim_{n \to \infty} \frac{1}{n \ln n}$

$= \frac{1}{\infty \ln \infty}$

$= 0$

$\because$ Both conditions are met

$\therefore \sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln(n)}$ is converging

(by alternating series test)

6. ii) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

$\frac{1}{n \ln(n)} = f(n)$ where $f(x) = \frac{1}{x \ln x}$

$f(x)$ is decreasing, positive and continuous

$\int_2^{\infty} \frac{dx}{x \ln x}$

$= \lim_{t \to \infty} \int_2^t \frac{dx}{x \ln x}$

Alternating Series Test

General Form $\sum_{n=1}^{\infty} (-1)^n a_n$

Conditions

→ $a_{n+1} \leq a_n$ (decreasing)

→ $\lim_{n \to \infty} a_n = 0$

If two conditions are satisfied then convergent. Otherwise diverges.

$$= \lim_{t\to\infty} \ln|\ln x| \Big|_2^t$$

$$= \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2|$$

$$= \infty$$

$\therefore$ Integral diverges

$\therefore$ The series diverges

Extra Example

$\rightarrow \displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

$\rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$