UNIVERSITY
OF WOLLONGONG
IN DUBAI

Faculty of Engineering
and Information Sciences

MATH142

Essentials of Engineering Mathematics

First-Order ODEs

I. Separable
II. Homogeneous
III. Linear
IV. Exact
V. Bernoulli

IV. Exact Equations

Prerequisite -First Partial Derivative of a function of two variables $z = f(x, y)$

The first partial derivatives of f with respect to x and y are denoted by $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$

Example 1: Find the first partial derivatives of the following functions.

a. $f(x, y) = 4x + 2y + 2x^3y^2$

b. $h(x, y) = 3x - x^2y^2 + 2x^3y$

c. $g(x, y) = xe^{x^2 y}$

d. $z = x^4 sin(xy^3)$

Definition: A first-order ODE of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (**test for exactness**).
In this case, there exists a function $f(x,y)$ satisfying:

$\frac{\partial f}{\partial x} = M(x,y)$, $\frac{\partial f}{\partial y} = N(x,y)$ and hence y is implicitly defined by $f(x,y) = c$.

Steps for Solving Exact Equations

1. Write the given DE in differential form $M(x,y)dx + N(x,y)dy = 0 \quad (I)$
2. Test for exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If the partial derivatives are equal, proceed to the following steps.
3. Write $\frac{\partial f}{\partial x} = M(x,y)$ and integrate w.r.t $x \Rightarrow f(x,y) = \int M(x,y)dx + h(y) \quad (II)$
4. Take the partial derivative w.r.t y of $f(x,y)$ in (II) and equate it to $N(x,y)$
5. Find $h'(y)$ from step 4 and integrate to get $h(y)$
6. Replace $h(y)$ in (II) to find $f(x,y)$
7. Finally, y is implicitly defined by $f(x,y) = c$. Solve for y if possible.

Example 2: Solve the following IVP.

$$cosx - 2xy + (e^y - x^2)y' = 0 \qquad\qquad y(1) = 4$$

Example 3: Solve $(x + siny)dx + (xcosyy - 2y)dy = 0$

Example 4: $y' = \frac{2+ye^{xy}}{2y-xe^{xy}}$

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Special Integrating Factors

Given $M(x,y)dx + N(x,y)dy = 0 \quad (I)$ and suppose $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Find:

1. $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \Rightarrow \mu(x) = e^{\int f(x)dx}$

2. $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y) \Rightarrow \mu(y) = e^{\int g(y)dy}$

and multiply (I) by one of the integrating factors to change it to an exact DE.

Example 5: Solve $\frac{y}{x^2} + 1 + \frac{1}{x}\frac{dy}{dx} = 0$

Example 6: Solve $2xydx + (y^2 - 3x^2)dy = 0$