(14pts)**Problem 1.**

Determine convergence or divergence of the following improper integrals,

1. $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$ 2. $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^3}$

Solution

1.

Find $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$, if it converges.

Solution: We might think just to do

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \Big[3(x-1)^{1/3}\Big]_0^3,$$

but this is not okay: The function $f(x) = \frac{1}{(x-1)^{2/3}}$ is undefined when $x = 1$, so we need to split the problem into two integrals.

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx.$$

The two integrals on the right hand side both converge and add up to $3[1+2^{1/3}]$, so $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3[1+2^{1/3}]$. $(7pts)$

2.

$$\begin{aligned}
\int_2^\infty \frac{dx}{x(\ln x)^3} &= \lim_{t\to\infty} \int_2^t \frac{1}{x}(\ln x)^{-3}\,dx \\
&= \lim_{t\to\infty} \left[\frac{(\ln x)^{-2}}{-2}\right]_2^t \\
&= \lim_{t\to\infty} \left[\frac{-1}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2}\right] \\
&= \frac{1}{2(\ln 2)^2} = 1.0407 \qquad (7pts)
\end{aligned}$$

(15pts)**Problem 2.**
Solve the following differential equation

$$\frac{dy}{dx} = \frac{(y+3)(x-1)}{(y-2)(x+4)}$$

Solution
The equation is separable

$$\frac{y-2}{y+3}dy = \frac{x-1}{x+4}dx$$

$\Rightarrow$

$$\int \frac{y-2}{y+3}dy = \int \frac{x-1}{x+4}dx$$

$\Rightarrow$

$$\int \left(1 - \frac{5}{y+3}\right) dy = \int \left(1 - \frac{5}{x+4}\right) dx \qquad (7pts)$$

$\Rightarrow$

$$y - 5\ln|y+3| = x - 5\ln|x+4| + C$$

or

$$(y-x) + 5\ln\left|\frac{x+4}{y+3}\right| = C \qquad (8pts)$$

(15pts)**Problem 3.**
Show that the equation is linear and solve the initial value problem

$$x\frac{dy}{dx} - 2y = x^2, \qquad y(1) = 3$$

SOLUTION Begin by writing the equation in standard form.

$$y' + \left(-\frac{2}{x}\right)y = x \qquad \text{Standard form, } y' + P(x)y = Q(x)$$

In this form, you can see that $P(x) = -2/x$ and $Q(x) = x$. So,

$$\begin{aligned}\int P(x)\,dx &= -\int \frac{2}{x}\,dx \\ &= -2\ln x \\ &= -\ln x^2\end{aligned}$$

which implies that the integrating factor is

$$\begin{aligned}u(x) &= e^{\int P(x)\,dx} \\ &= e^{-\ln x^2} \\ &= \frac{1}{e^{\ln x^2}} \\ &= \frac{1}{x^2}. && \text{Integrating factor}\end{aligned}$$

This implies that the general solution is

$$\begin{aligned}y &= \frac{1}{u(x)}\int Q(x)u(x)\,dx && \text{Form of general solution} \\ &= \frac{1}{1/x^2}\int x\left(\frac{1}{x^2}\right)dx && \text{Substitute.} \\ &= x^2\int \frac{1}{x}\,dx && \text{Simplify.} \\ &= x^2(\ln x + C). && \text{General solution}\end{aligned}$$

$(10pts)$

$$y(1) = 3 \text{ gives } C = 3$$

The solution is

$$y = x^2(\ln x + 3) \qquad (5pts)$$

(14pts)**Problem 4.**
Show that the differential equation is exact and solve the equation.

$$\left(1+2x-y^3\right)dx+\left(2y-3xy^2\right)dy=0$$

Solution

$$M(x,y)=1+2x-y^3 \quad \text{and} \quad N(x,y)=2y-3xy^2 \qquad\qquad (4pts)$$

$$M_y=-3y^2=N_x \qquad \therefore \text{ The differential equation is exact.}$$

So, there is a function f such that

$$\frac{\partial f}{\partial x}=1+2x-y^3 \quad \text{and} \quad \frac{\partial f}{\partial y}=2y-3xy^2$$

$$\frac{\partial f}{\partial x}=1+2x-y^3 \Rightarrow f(x,y)=x+x^2-xy^3+g(y). \qquad\qquad (4pts)$$

Plugging this into $\frac{\partial f}{\partial y}=2y-3xy^2$, we obtain

$$-3xy^2+g'(y)=2y-3xy^2 \Rightarrow g'(y)=2y.$$

Thus

$$g(y)=y^2+C_1$$

and

$$f(x,y)=x+x^2-xy^3+y^2+C_1$$

The solution of the differential equation is

$$x+x^2-xy^3+y^2+C_1=C_2 \quad \text{or} \quad x+x^2-xy^3+y^2=C \qquad\qquad (6pts)$$

(14pts)**Problem 5.**
Show that the differential equation is homogeneous and solve it.

$$(x - 2y)dx + xdy = 0.$$

Solution
Simplifying we get

$$\frac{dy}{dx} = \frac{2y - x}{x} = 2\frac{y}{x} - 1 = F(\frac{y}{x})$$

This shows that the equation is homogeneous.
Put

$$u = \frac{y}{x}$$

The equation becomes

$$\frac{dy}{u-1} = \frac{dx}{x}$$

Integrating, we get

$$\ln|u - 1| = \ln|x| + \ln C \qquad \text{with } C > 0.$$

Thus

$$|u - 1| = C\,|x| \qquad (8pts)$$

$$u - 1 = Ax \quad \text{where } A = \pm C$$

$$\frac{y}{x} = Ax + 1$$

and

$$y = Ax^2 + x. \qquad (6pts)$$

(14pts)**Problem 6**
Solve the following Bernoulli differential equation

$$x\frac{dy}{dx} + y = x^2y^2$$

The equation can be re-written in form (1) simply dividing by x:

$$y' + \frac{1}{x}y = xy^2$$

The substitution to be used in this case is $u = y^{1-2} = 1/y$, or $y = 1/u$. Given that,

$$y' = -\frac{1}{u^2}u'$$

the initial equation becomes,

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = x\frac{1}{u^2}$$

$$\Downarrow$$

$$u' - \frac{1}{x}u = -x$$

We have, thus, obtained a first order linear differential equation with $P(x) = -1/x$ and $Q(x) = -x$. The general solution of this equation is:

$$u = -x^2 + cx$$

To conclude, given that $y = 1/u$, we have the following general solution for the given Bernoulli's equation:

$$y = \frac{1}{-x^2 + cx} \qquad (14pts)$$

(14pts)**Problem 7**

Show that the differential equation is not exact, find the special integrating factor, make it exact and solve the equation.

$$\left(y^2 - x\right) dx + 2y dy = 0.$$

Solution The given equation is not exact because $M_y(x, y) = 2y$ and $N_x(x, y) = 0$. However, because

$$\frac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \frac{2y - 0}{2y} = 1 = h(x)$$

it follows that $e^{\int h(x)\, dx} = e^{\int dx} = e^x$ is an integrating factor. Multiplying the given differential equation by e^x produces the exact differential equation

$$(y^2 e^x - xe^x)\, dx + 2ye^x\, dy = 0$$

(6pts)

whose solution is obtained as follows.

$$f(x, y) = \int N(x, y)\, dy = \int 2ye^x\, dy = y^2 e^x + g(x)$$

$$f_x(x, y) = y^2 e^x + g'(x) = \overbrace{y^2 e^x - xe^x}^{M(x,y)}$$

$$\boxed{g'(x) = -xe^x}$$

(4pts)

Therefore, $g'(x) = -xe^x$ and $g(x) = -xe^x + e^x + C_1$, which implies that

$$f(x, y) = y^2 e^x - xe^x + e^x + C_1.$$

The general solution is $y^2 e^x - xe^x + e^x = C$, or $y^2 - x + 1 = Ce^{-x}$.

(4pts)