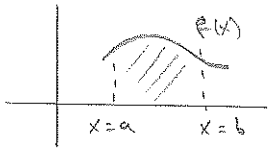
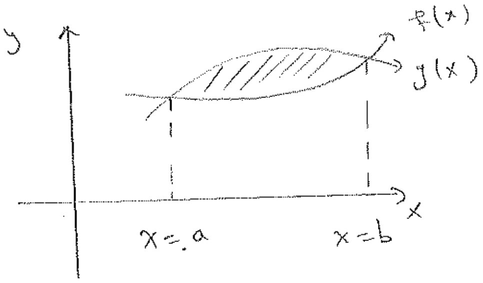
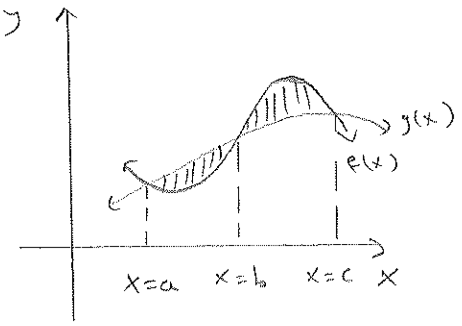
(2) ﻟ (1)

Area Between Curves



$$\text{Area} = \int_a^b f(x)\,dx$$





$$\text{Area} = \left| \int_a^b [f(x) - g(x)]\,dx \right|$$

$$\text{Area} = \left| \int_a^b [f(x) - g(x)]\,dx \right| + \left| \int_b^c [f(x) - g(x)]\,dx \right|$$

Ex. 1 Find the area of the region bounded by the graphs of $f(x) = 3 - x^2$ and $g(x) = -x + 1$ between $x = 0$ and $x = 2$

$f(x) = g(x)$

$3 - x^2 = -x + 1$

$x^2 - x - 2 = 0 \longrightarrow (x-2)(x+1) = 0$

$x = 2$ or $x = -1$

0 ——— 2

$$\text{Area} = \left| \int_0^2 [(3 - x^2) - (-x+1)]\,dx \right|$$

$$= \left| \int_0^2 (-x^2 + x + 2)\,dx \right|$$

$$= \left| \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^2 \right|$$

$$= \frac{10}{3}$$

Ex. 2 Find the area of the region enclosed by the curves $y = x^3 - 2x$ and $y = 2x$

$y = y \Rightarrow x^3 - 2x = 2x$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x = 0 \quad x = \pm 2$

$$\text{Area} = \left| \int_{-2}^{0} (x^3 - 2x - 2x)\,dx \right| + \left| \int_{0}^{2} (x^3 - 2x - 2x)\,dx \right|$$

$$= \left| \int_{-2}^{0} (x^3 - 4x)\,dx \right| + \left| \int_{0}^{2} (x^3 - 4x)\,dx \right|$$

$$= \left| \left[\frac{x^4}{4} - 2x^2 \right]_{-2}^{0} \right| + \left| \left[\frac{x^4}{4} - 2x^2 \right]_{0}^{2} \right|$$

$$= 4 + 4$$

$$= 8$$

Ex 3. Find the area bounded by $y = x^3 - 3x^2 + 2x$ and the x-axis. Set up the integral only.

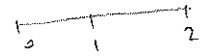
$(\text{x-axis} \to y = 0)$

$x^3 - 3x^2 + 2x = 0$

$x(x^2 - 3x + 2) = 0$

$x = 0$ or $x^2 - 3x + 2 = 0 \to (x-2)(x-1) = 0$

$x = 2$ or $x = 1$



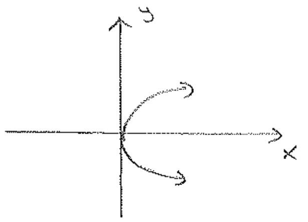
$$\text{Area} = \left| \int_{0}^{1} (x^3 - 3x^2 + 2x)\,dx \right| + \left| \int_{1}^{2} (x^3 - 3x^2 + 2x)\,dx \right|$$

(4)

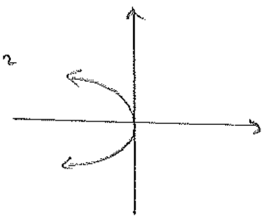
Area Computed by Integrating with Respect to y.

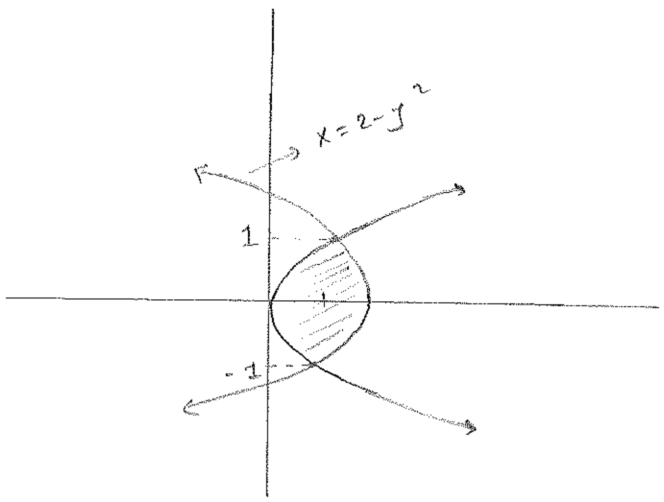
Ex. 4 Find the area bounded by the graphs of $x = y^2$ and $x = 2 - y^2$

Note: $x = y^2$



$x = -y^2$





$x = x$

$y^2 = 2 - y^2$

$2y^2 = 2$

$y^2 = 1 \rightarrow y = \pm 1$

$$\text{Area} = \int_{-1}^{1} (2 - y^2 - y^2)\, dy$$

$$= \int_{-1}^{1} (-2y^2 + 2)\, dy$$

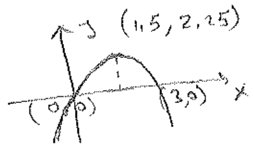
$$= \left[-\frac{2}{3}y^3 + 2y \right]_{-1}^{1}$$

$$= \frac{8}{3}$$

Ex 5. Sketch the region bounded by $x = 3y - y^2$ and $x + y = 3$ and find its area.

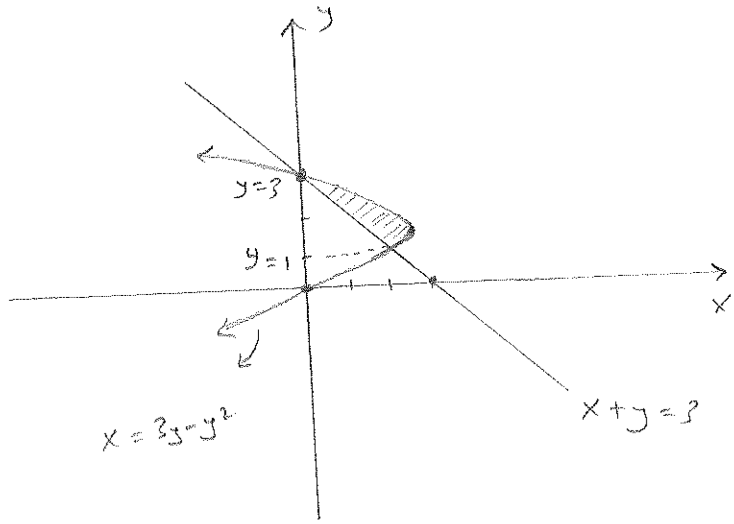
Note $y = 3x - x^2$

$y = -x^2 + 3x$



$x + y = 3$

$y = 3 - x$



$$x = x \longrightarrow 3y - y^2 = 3 - y$$

$$4y - y^2 - 3 = 0$$

$$-y^2 + 4y - 3 = 0$$

$$y^2 - 4y + 3 = 0$$

$$(y-3)(y-1) = 0$$

$$y = 3 \text{ or } y = 1$$

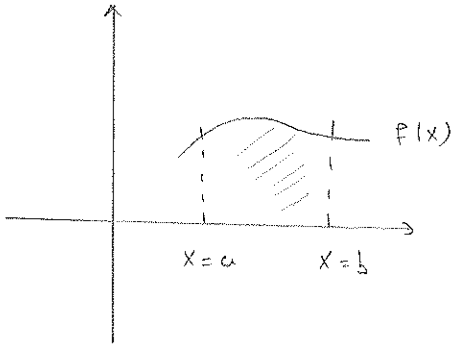
$$\text{Area} = \int_1^3 \left[(3y - y^2) - (3 - y)\right] dy$$

$$= \int_1^3 (-y^2 + 4y - 3)\, dy$$

$$= \left[-\frac{y^3}{3} + 2y^2 - 3y \right]_1^3$$

$$= \frac{4}{3}$$

Arc length



$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$$

Ex. 6 Find the arc length of the curve $f(x) = x^{2/3}$, $x \in [0,8]$

set up the integral and don't evaluate.

$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$$

$$= \frac{2}{3x^{1/3}}$$

$$L = \int_0^8 \sqrt{1+\left(\frac{2}{3x^{1/3}}\right)^2}\, dx$$

$$= \int_0^8 \sqrt{1+\frac{4}{9x^{2/3}}}\, dx$$

Ex. 7 Given $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ and $x \in [1,3]$. Find the arc length of $f(x)$.

$$f'(x) = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$[f'(x)]^2 = \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}$$ and $$1+[f'(x)]^2 = \frac{x^4}{4} + \frac{1}{4x^4} + 1$$

$$L = \int_1^3 \sqrt{\frac{x^4}{4} + \frac{1}{4x^4} + 1}\, dx$$

$$L = \int_1^3 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx$$

$$= \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx \quad \frac{1}{2x^2} \to \frac{1}{2}x^{-2}$$

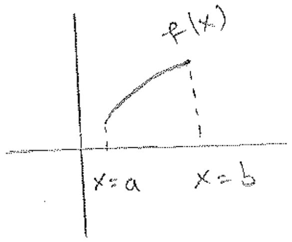
$$= \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^3$$

$$= \frac{14}{3}$$

Surface Area

Revolution about the x-axis

$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$$



Revolution about the y-axis

$$S = 2\pi \int_c^d g(y) \sqrt{1 + [g'(y)]^2}\, dy$$