§ Linear Equations

A first order differential equation of the form

$$a(x)\frac{dy}{dx} + b(x)y = c(x)$$

is called linear.

<u>Exple</u> which of the following equations are linear.

1.) $e^{x}\frac{dy}{dx} + x^{2}y = \sin x$ ✓

2) $y\frac{dy}{dx} + (2x+1)y = e^{x}$ ✗

3.) $\cos x\frac{dy}{dx} + xy^{2} = 2$ ✗

4) $x\frac{dy}{dx} + e^{x}y = xy^{2}$. ✗

* <u>Solving linear equation</u>

To solve a first order linear equation, do the following:

1.) Write the equation in the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

(standard form)

2.) Compute $IF = e^{\int P(x)\,dx}$

3.) the Solution is

$$y = \frac{1}{IF}\int Q(x)\, IF\, dx$$

Exple Show that the equation is linear and Solve it.

1.) $\dfrac{dy}{dx} - 3y = 6$

2.)

$$\rightarrow \frac{dy}{dx} + P(x)\,y = Q(x)$$

$$\underbrace{e^{\int P(x)\,dx}}$$

$$\underbrace{e^{\int P(x)\,dx} \cdot \frac{dy}{dx} + P(x)\,e^{\int P(x)\,dx}\,y} = Q(x) \cdot e^{\int P(x)\,dx}$$

$$\downarrow$$

$$\frac{d}{dx}\left[y e^{\int P(x)\,dx}\right] = Q(x)\,e^{\int P(x)\,dx}.$$

$$y\,e^{\int P(x)\,dx} = \int Q(x)\,e^{\int P(x)\,dx}\,dx$$

$$y = \frac{1}{IF}\int Q(x)\,IF\,dx$$

$$IF = e^{\int P(x)\,dx} = \text{Integrating factor}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

1.) $\frac{dy}{dx} - 3y = 6$

$P(x) = -3$. $IF = e^{\int P(x)dx} = e^{\int -3\,dx}$

$= e^{-3x}$

$$y = \frac{1}{e^{-3x}} \int 6 \cdot e^{-3x}\,dx$$

$$y = e^{3x}\left[\frac{6}{-3} e^{-3x} + c\right]$$

$$y = e^{3x}\left(-2e^{-3x} + c\right)$$

$$y = -2 + ce^{3x}$$

$$x\frac{dy}{dx} - 4y = x^6 e^x.$$

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x.$$

$P(x) = -\frac{4}{x}$. $IF = e^{\int p(x)dx} = e^{\int -\frac{4}{x}dx}$

$= e^{-4\int \frac{dx}{x}}$ $= e^{-4\ln x}$ $\underline{x > 0}$

$= e^{\ln x^{-4}} = x^{-4} = \frac{1}{x^4}$

Solution

$$y = \frac{1}{\frac{1}{x^4}} \int \left(\cancel{x^{5}}\, e^{x}\right)\left(\frac{1}{\cancel{x^4}}\right) dx$$

$$= x^4 \underbrace{\int x e^{x}\, dx}$$

$u = x \qquad u' = 1$

$v' = e^{x} \qquad v = e^{x}$

$$\int x e^{x}\, dx = x e^{x} - \int e^{x}\, dx = \underline{x e^{x} - e^{x} + C}$$

$$y = x^4 \left(x e^{x} - e^{x} + C \right)$$

$$y = x^5 e^{x} - x^4 e^{x} + C x^4 .$$

Exple Show that the equation is linear and solve the initial-value problem.

$$(x^2 - 9) \frac{dy}{dx} + x y = 0 , \quad y(4) = 1 .$$

$$\frac{dy}{dx} + \frac{x}{x^2 - 9} y = 0$$

$$IF = e^{\frac{1}{2} \int \frac{2x}{x^2-9} dx} = e^{\frac{1}{2} \ln (x^2 - 9)} \qquad \underline{x^2 - 9 > 0}$$

$$= e^{\ln \sqrt{x^2 - 9}} = \sqrt{x^2 - 9}$$

$$y = \frac{1}{\sqrt{x^2-9}}\left[\int 0 \cdot dx\right] = \frac{c}{\sqrt{x^2-9}}$$

$$y = \frac{c}{\sqrt{x^2-9}}$$

$$y(4) = 1$$

$$1 = \frac{c}{\sqrt{16-9}}$$

$$1 = \frac{c}{\sqrt{7}} \Rightarrow c = \sqrt{7}$$

$$y = \frac{\sqrt{7}}{\sqrt{x^2-9}}.$$

§ Exact Equations

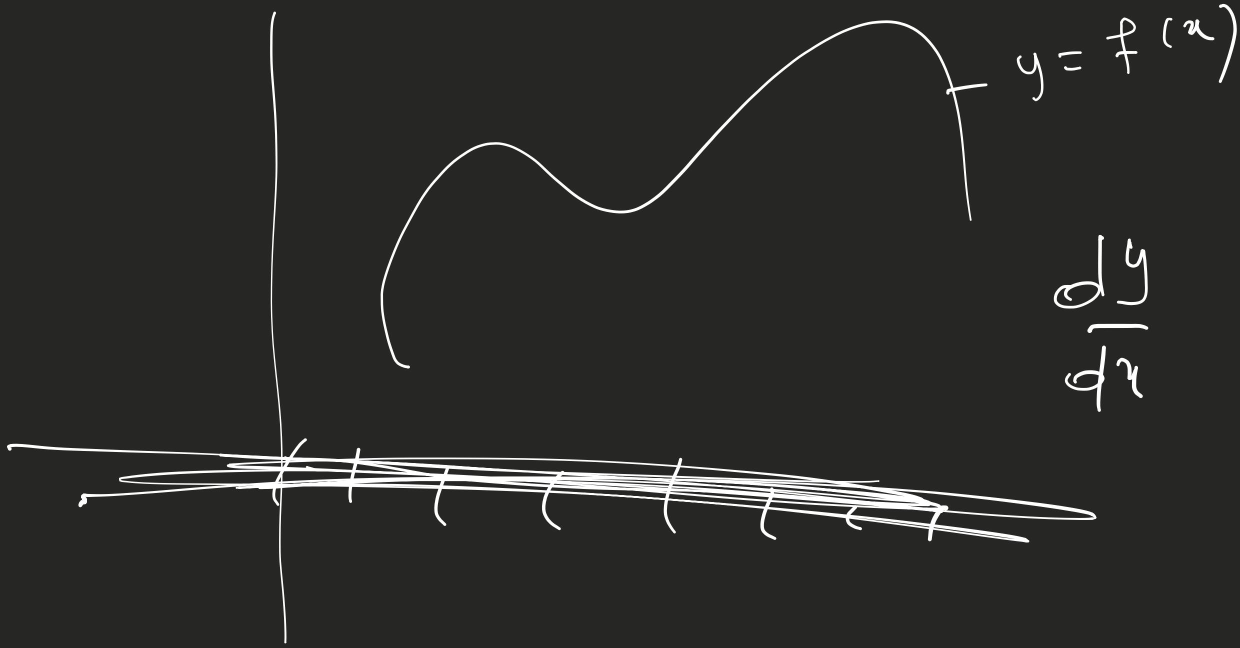
Functions of Two Variables

A function f of two variables (x, y) is a relation that assigns a unique number $z = f(x, y)$ to each ordered pair (x, y).

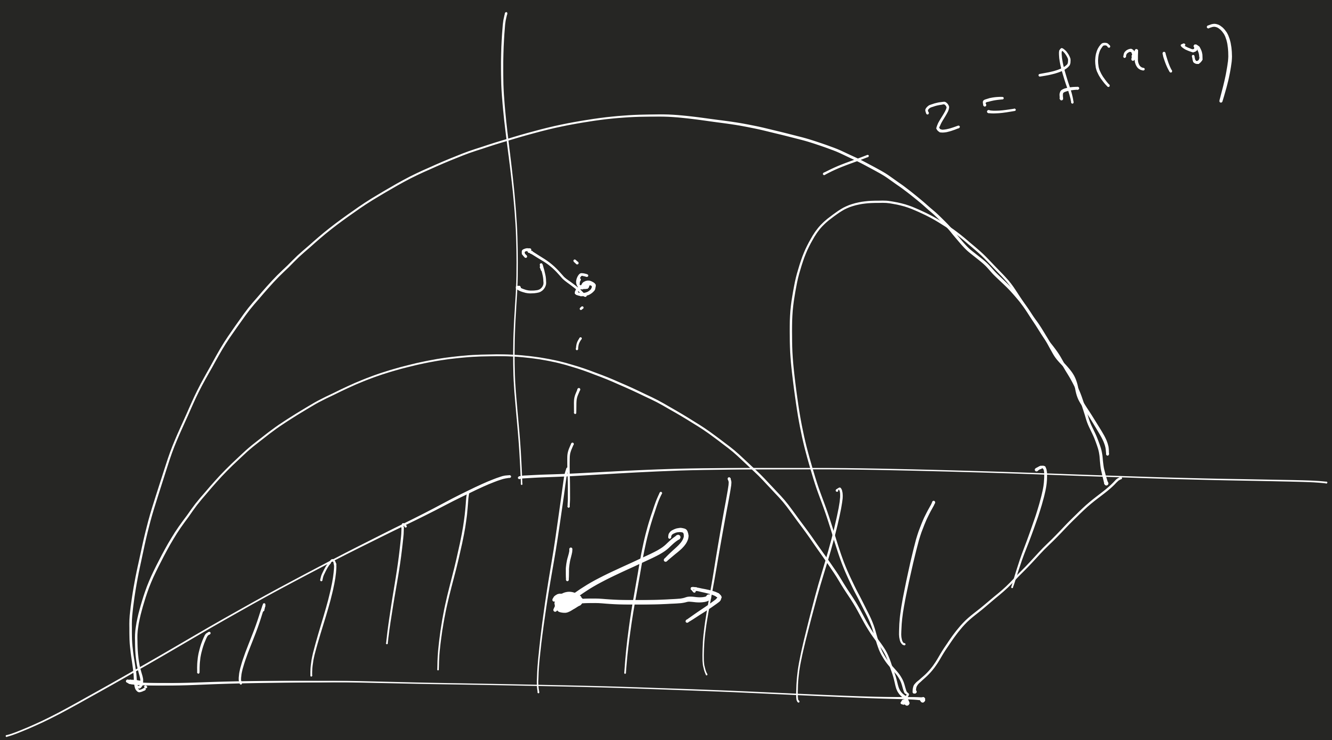
Expl $\quad f(x, y) = x + y - 1$

$$f(x,y) = \cos(x-y)$$

$$f(x,y) = xe^{y} - ye^{x} + xy + 1$$



$$\frac{dy}{dx}$$



let $z = f(x,y)$ be a function of two variables.

* The partial derivative with respect to x denoted by

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x},$$

is calculated by taking y as constant and computing the usual derivative with respect to x

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$$f(x,y) = x^2 y + e^{xy} + 1 \leftarrow$$

$$f_x = 2xy + y e^{xy} \qquad f_y = x^2 + x e^{xy}$$

$$f(x,y) = \frac{y}{x} + xy^2 \leftarrow$$

$$f_x = -\frac{y}{x^2} + y^2, \qquad f_y = \frac{1}{x} + 2xy$$

* The partial derivative with respect to y, denoted by

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

is calculated by taking x as constant and computing the usual derivative with respect to y

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$f(x,y)$ $\qquad$ f_x

f_y

* If f is a functions of two variables, then the differential of f is given by

$$df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy$$

* dx is a variable that expresses an infinitesimal in x.
* dy is a variable that expresses an infinitesimal change in y.

<u>Exple</u> Find the differential of

$$f(x,y) = xy^2 + 3x^2y - 1$$

at the point $(1,-2)$.

$$df(x,y) = f_x\,dx + f_y\,dy$$

$$f_x(x,y) = y^2 + 6xy$$

$$f_y(x,y) = 2xy + 3x^2$$

$$df(x,y) = (y^2 + 6xy)\,dx + (2xy + 3x^2)\,dy$$

$$df(1,-2) = (4-12)\,dx + (-4+3)\,dy$$

$$= -8\,dx - 1\,dy$$

$$\frac{d}{dx}(\text{constant}) = 0$$

If $\frac{d}{dx} f(x) = 0$ for all x

then $f(x) = \text{constant}$

* If $df(x,y) = 0$ for all (x,y)

then $f(x,y) = C = \text{Constant}$

* A differential Equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be exact if

$M(x,y)\,dx + N(x,y)\,dy$ is the differential of some function $f(x,y)$

Exple $x^2y^3\,dx + x^3y^2\,dy = 0$

$$d\left(\frac{1}{3}x^3y^3\right) = x^2y^3\,dx + x^3y^2\,dy$$

$$d\left(\frac{1}{3}x^3y^3\right) = 0$$

$$\frac{1}{3}x^3y^3 = c$$

$$x^3y^3 = 3c$$

$$y^3 = \frac{3c}{x^3}$$

$$y = \frac{\sqrt[3]{3c}}{x}$$

$$x^2y^3\,dx + x^3y^2\,dy = 0$$

$$df = f_x\,dx + f_y\,dy$$

$$\begin{cases} f_x = x^2y^3 \\ f_y = x^3y^2 \end{cases}$$

$$f_x = x^2y^3 \rightarrow f(x,y) = \int x^2y^3\,dx + C(y)$$

$$f(x,y) = y^3\int x^2\,dx + C(y)$$

$$f(x,y) = \frac{x^3y^3}{3} + C(y)$$

$$f_y = x^3y^2$$

$$f(x,y) = \frac{x^3y^3}{3} + C$$

$$f(x,y) = \frac{x^3y^3}{3}$$

$$\cancel{x^3y^2} + c'(y) = \cancel{x^3y^2} \qquad \frac{x^3y^3}{3} + c = c'$$

$$c'(y) = 0$$

$$c(y) = c$$

Test for exactness

$$\underbrace{M(x,y)\,dx + N(x,y)\,dy} = 0$$

$$df(x,y) = 0$$

$$f(x,y) = c$$

Test for exactness.

Consider the differential equation

$$\underline{M(x,y)}\,dx + \underline{N(x,y)}\,dy = 0$$

If $M_y = N_x$ then the equation is exact.

Exple which of the following equation is exact.

1) $\underbrace{2xy}_{M}\,dx + \underbrace{(x^2-1)}_{N}\,dy = 0$

2.) $\underbrace{x \sin y}_{M}\, dx + \underbrace{x e^{y}}_{N}\, dy = 0$

$M = 2xy$, $N = x^2 - 1$

$M_y = 2x$ $N_x = 2x$

$M_y = N_x \Rightarrow$ Equation is exact

2.) $M = x \sin y$ $N = x e^{y}$

$M_y = x \cos y$ $N_x = e^{y}$

$M_y \neq N_x \Rightarrow$ Equation is not exact.

Exact Equations

$df(x,y) = f_x\, dx + f_y\, dy$

$\frac{d}{dx}(\text{Constant}) = 0$

If $f'(x) = 0$ for all x, then $f(x) = \text{Constant}$

If $df(x, y) = 0$ then $f(x, y) = C$

$M(x, y)\, dx + N(x, y)\, dy = 0$

f_x f_y

$$df(x,y) = 0 \Rightarrow f(x,y) = c$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If $M_y = N_x$, then the equation is exact. Meaning you can find a function $f(x,y)$ such that

$$M(x,y)\,dx + N(x,y)\,dy = df = f_x\,dx + f_y\,dy$$

$$\begin{cases} f_x = M(x,y) \\ f_y = N(x,y) \end{cases}$$

<u>Exple</u> show that the equation is exact and solve it.

$$\underbrace{2xy}_{M}\,dx + \underbrace{(x^2-1)}_{N}\,dy = 0$$

$M_y = N_x$?

$M = 2xy$

$N = x^2 - 1$

$M_y = 2x$ $\qquad$ $N_x = 2x$

$M_y = N_x \Rightarrow$ Equation is exact

$$\begin{cases} f_x = 2xy \\ f_y = x^2 - 1 \end{cases}$$

$$f_x = 2xy$$

$$f(x,y) = \int 2xy\,dx + C(y)$$

$$= y\int 2x\,dx + C(y)$$

$$= x^2 y + C(y)$$

$$f(x,y) = x^2 y + C(y)$$

$$f_y = x^2 - 1$$

$$\cancel{x^2} + C'(y) = \cancel{x^2} - 1$$

$$C'(y) = -1$$

$$C(y) = \int -1\,dy = -y + C$$

$$f(x,y) = x^2 y - y + C$$

$$f(x,y) = x^2y - y$$

Solution is $f(x,y) = c$

$$x^2y - y = c$$

$$y(x^2-1) = c$$

$$y = \frac{c}{x^2-1}$$

Exple show that the equation is exact and solve the initial-value problem.

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y(0) = 2.$$

$$y(1-x^2)\,dy = \left(xy^2 - \frac{\sin 2x}{2}\right)dx$$

$$\underbrace{\left(xy^2 - \frac{\sin 2x}{2}\right)}_{M}dx \underbrace{-y(1-x^2)}_{N}dy = 0$$

$$M = xy^2 + \frac{1}{2}\sin 2x, \quad N = -y + x^2y$$

$$M_y = 2xy \qquad N_x = 2xy$$

$M_y = N_x \Rightarrow$ Equation is exact.

$$\begin{cases} f_x = xy^2 + \frac{1}{2}\sin 2x \\ f_y = -y + x^2 y \end{cases}$$

$$f_y = y(x^2-1) \Rightarrow$$

$$f(x,y) = \int y(x^2-1)\,dy + c(x)$$

$$f(x,y) = \frac{y^2}{2}(x^2-1) + c(x)$$

$$\frac{\cancel{2}xy^2}{\cancel{2}} + c'(x) = xy^2 + \frac{1}{2}\sin 2x$$

$$c'(x) = \frac{1}{2}\sin 2x$$

$$c(x) = \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) + C$$

$$= -\frac{1}{4}\cos 2x + C$$

$$f(x,y) = \frac{y^2}{2}(x^2-1) - \frac{1}{4}\cos 2x$$

Solution is $f(x,y) = C$

$$\frac{y^2}{2}(x^2-1) - \frac{1}{4}\cos 2x = C$$

$y(0) = 2$ when $x = 0$, $y = 2$

$$\frac{2^2}{2}(0-1) - \frac{1}{4}\cos 0 = c$$

$$-2 - \frac{1}{4} = c$$

$$c = \frac{-9}{4}$$

Solution : $\frac{y^2}{2}(x^2-1) - \frac{1}{4}\cos 2x = -\frac{9}{4}$

Special Integrating Factors

Consider the differential Equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$M_y \neq N_x \Rightarrow$ the equation is not exact.

* Compute $\dfrac{M_y - N_x}{N}$

If this is a function of x alone then

$$a(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

is a special integrating factor meaning if multiply now the original equation by $a(x)$, then the resulting equation will be exact.

* otherwise compute

$$\frac{N_x - M_y}{M}.$$

If this is a ~~function of~~ y alone

then

$$b(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

is a special integrating factor.

Exple Show that the equation is not exact. Make it exact and solve.

$$\underbrace{xy}_{M}\, dx + (\underbrace{2x^2 + 3y^2 - 20}_{N})\, dy = 0$$

$M = xy$, $\quad N = 2x^2 + 3y^2 - 20$

$M_y = x \quad N_x = 4x$

$M_y \neq N_x \Rightarrow$ Equation is Not exact.

$$* \quad \frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

$$* \quad \frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3\cancel{x}}{\cancel{x}y} = \frac{3}{y}$$

$$SIF = e^{\int \frac{3}{y} dy} = e^{3\int \frac{dy}{y}} = e^{3\ln y} = e^{\ln y^3} = y^3$$

Now multiply the original equation by $SIF = y^3$

$$\underbrace{xy^4}_{M}dx + \underbrace{(2x^2y^3 + 3y^5 - 20y^3)}_{N}dy = 0$$

$$M_y = 4xy^3, \quad N_x = 4xy^3$$

$$M_y = N_x$$

$$\begin{cases} f_x = xy^4 \\ f_y = 2x^2y^3 + 3y^5 - 20y^3 \end{cases}$$

$$f_x = xy^4 \Rightarrow f(x,y) = \int xy^4\,dx + c(y)$$

$$f(x,y) = \frac{x^2y^4}{2} + c(y)$$

$$\cancel{\frac{4x^2y^3}{2}} + c'(y) = \cancel{2x^2y^3} + 3y^5 - 20y^3$$

$$c'(y) = 3y^5 - 20y^3$$

$$c(y) = \int (3y^5 - 20y^3)\,dy$$

$$= \frac{3y^6}{6} - \frac{20y^4}{4} + c$$

$$= \frac{y^6}{2} - 5y^4 + c$$

$$f(x,y) = \frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4$$

Solution $f(x,y) = C$

$$\frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4 = C$$

§ Homogeneous Equations.

A first order differential equation

$$\frac{dy}{dx} = f(x,y) \quad \text{is}$$

said to homogeneous if

$$f(x,y) = F\left(\frac{y}{x}\right)$$

Exple which of the following equations are homogeneous

1.) $(x^2+y^2)\,dx + (x^2 - xy)\,dy = 0$

2.)

3.)

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \ ?$$

1.) $(x^2 - xy)\,dy = -(x^2+y^2)\,dx$

$$\frac{dy}{dx} = -\frac{(x^2+y^2)}{x^2-xy} = \frac{-\cancel{x^2}\left(1+\frac{y^2}{x^2}\right)}{\cancel{x^2}\left(1-\frac{y}{x}\right)}$$

$$= -\frac{\left(1+\left(\frac{y}{x}\right)^2\right)}{1-\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right)$$

$\Rightarrow$ equation homogeneous

2.)

$$(xe^x - y)\,dx + (xy + x^2)\,dy = 0$$

$$(xy + x^2)\,dy = -(xe^x - y)\,dx$$

$$\frac{dy}{dx} = -\frac{(xe^x - y)}{xy + x^2} = \frac{-\cancel{x}\left(e^x - \frac{y}{x}\right)}{\cancel{x}(y+x)}$$

not $F\left(\frac{y}{x}\right)$

Not homogeneous.

3.)

$$(x - 2y)\,dx + x\,dy = 0$$

$$x\,dy = (2y - x)\,dx$$

$$\frac{dy}{dx} = \frac{2y - x}{x} = 2\frac{y}{x} - \frac{x}{x} = 2\left(\frac{y}{x}\right) - 1$$

$$= F\left(\frac{y}{x}\right)$$

homogeneous.

* To Solve a homogeneous equation Do the following:

1.) Write the equation in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

2.) Do the substitution

$$u = \frac{y}{x} \iff y = xu$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

3.) Convert now the original equation in y into a separable equation in u and solve.

<u>Exple</u> show that the equation is homogeneous and solve the initial-value problem.

$$\underbrace{\left[x \sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right] dx}_{} + \underbrace{x\cos\left(\frac{y}{x}\right) dy}_{} = 0$$

$$y(1) = \frac{\pi}{2}$$

$$x\cos\left(\frac{y}{x}\right) dy = \left[y\cos\left(\frac{y}{x}\right) - x\sin\left(\frac{y}{x}\right)\right] dx$$

$$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) - x\sin\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}$$

$$= \frac{y\cancel{\cos\left(\frac{y}{x}\right)}}{x\cancel{\cos\left(\frac{y}{x}\right)}} - \frac{\cancel{x}\sin\left(\frac{y}{x}\right)}{\cancel{x}\cos\left(\frac{y}{x}\right)}$$

$$= \left(\frac{y}{x}\right) - \frac{\sin\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right).$$

$\Rightarrow$ Equation is homogeneous.

Put $u = \frac{y}{x} \iff y = xu$

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

The equation

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) - \tan\left(\frac{y}{x}\right)$$

becomes.

$$\cancel{u} + x\frac{du}{dx} = \cancel{u} - \tan u$$

$$\frac{du}{dx} = -\frac{\tan u}{x}$$

$$\frac{du}{\tan u} = -\frac{dx}{x}$$

$$\int \frac{\cos u}{\sin u}\,du = -\int \frac{dx}{x}$$

$$\ln|\sin u| = -\ln|x| + C$$

$C = \ln K$

$K > 0$

$$\ln|\sin u| = \underbrace{\ln|x|^{-1} + \ln K}$$

$$\ln|\sin u| = \ln\left|\frac{K}{x}\right|$$

$$\Rightarrow \quad |\sin u| = \left|\frac{K}{x}\right|$$

$\pm K = A$

$$\Rightarrow \quad \sin u = \frac{\pm K}{x}$$

$$\sin u = \frac{A}{x}$$

$$\sin\frac{y}{x} = \frac{A}{x}$$

$x = 1, \quad y = \frac{\pi}{2}$

$\sin\frac{\pi}{2} = A$

$A = 1$

$$\frac{y}{x} = \sin^{-1}\left(\frac{A}{x}\right)$$

$$y = x\sin^{-1}\left(\frac{A}{x}\right)$$

$A = 1$

$$y = x\sin^{-1}\left(\frac{1}{x}\right).$$

§ Bernoulli Equations

A differential Equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a Bernoulli equation

In $n = 1$ or $n = 0$, the equation is linear.

Assume that $n \neq 0$, $n \neq 1$

divide by y^n to get

$$y^{-n}\frac{dy}{dx} + P(x)\,y\cdot y^{-n} = Q(x)$$

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$$