(5pts) Problem 1.
If

$$L = \int_e^{\infty} \frac{dx}{x(\ln x)^2},$$

then

(a) $L = 2e$ $\boxed{\text{(b)}}$ $L = 1$ (c) $L = \infty$ (d) $L = -1$ (e) $L = 0$

Solution

Solution: Use the definition of improper integral and make the substitution $u = \ln x$ with $dx = x du$. Then

$$\int_e^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_1^{\ln t} \frac{1}{u^2} du$$
$$= \lim_{t\to\infty} \left[-\frac{1}{u}\right]_1^{\ln t} = \lim_{t\to\infty} \left(-\frac{1}{\ln t} + 1\right) = 1.$$

(5pts) Problem 2. If

$$L = \int_{-2}^{2} \frac{dx}{x+1},$$

then

(a) $L = \frac{8}{9}$ (b) $L = \frac{1}{2}\ln 3$ (c) $L = 0$ (d) $L = \ln 3$ $\boxed{\text{(e)}}$ $L = -\infty$

Solution

Solution: Function $\frac{1}{x+1}$ has an infinite discontinuity at the point $x = -1$. Therefore

$$\int_{-2}^{2} \frac{1}{x+1} dx = \int_{-2}^{-1} \frac{1}{x+1} dx + \int_{-1}^{2} \frac{1}{x+1} dx,$$

where each of the integrals is improper. Compute the first integral as follows

$$\int_{-2}^{-1} \frac{1}{x+1} dx = \lim_{t\to -1} \int_{-2}^{t} \frac{1}{x+1} dx = \lim_{t\to -1} [\ln|x+1|]_{-2}^{t} = \lim_{t\to -1} \ln|t+1| - \ln 1 = -\infty.$$

Since $\int_{-2}^{-1} \frac{1}{x+1} dx$ diverges, then the initial integral diverges as well.

Remark. The correct answer should be diverges L does not exist. Thus I will take $L = -\infty$ which is closest to the correct answer.

(5pts) Problem 3. Evaluate the improper integral

$$L = \int_0^{\infty} x e^{-x^2} dx$$

then

$\boxed{\text{(a)}}$ $\frac{1}{2}$ (b) 1 (c) $2e$ (d) divergent (e) e

Solution

Put $u = x^2$ then $dx = \frac{1}{2}du$. Also $x = 0 \Rightarrow u = 0$ and $x = +\infty \Rightarrow u = +\infty$. So,

$$\int_0^{+\infty} xe^{-x^2}\,dx = \frac{1}{2}\int_0^{+\infty} e^{-u}\,du.$$

By definition

$$\frac{1}{2}\int_0^{+\infty} e^{-u}\,du = \lim_{a\to+\infty}\frac{1}{2}\int_0^{a} e^{-u}\,du = \lim_{a\to+\infty}\frac{1}{2}\left[-e^{-u}\right]_0^a = \frac{1}{2}$$

(5pts) Problem 4. Evaluate the improper integral

$$L = \int_0^2 \frac{dx}{x-1}$$

then

(a) 0 $\boxed{\text{(b)}}$ diverges (c) 4 (d) -2 (e) e

Solution

Note that this integral is an improper integral as $\frac{1}{x-1}$ is not defined at $x = 1$. Now,

$$\int_0^2 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x-1}\,dx.$$

By definition,

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to1^-}\int_0^t \frac{1}{x-1}\,dx.$$

$$\begin{aligned}\lim_{t\to1^-}\int_0^t \frac{1}{x-1}\,dx &= \lim_{t\to1^-}\left[\ln|x-1|\right]_0^t \\ &= \lim_{t\to1^-}\ln|t-1| = +\infty.\end{aligned}$$

This means that $\int_0^2 \frac{1}{x-1}\,dx$ diverges.

Note: You could equally consider $\int_1^2 \frac{1}{x-1}\,dx$ and get the same conclusion. The point is that one of these integrals is divergent is enough to conclude $\int_0^2 \frac{1}{x-1}\,dx$ is divergent.

(12pts)Problem 5.
Show that the equation is separable and find the general solution.

$$2\frac{dy}{dx} = \left(y^2 - 1\right)\sin x.$$

Solution
This is a separable equation.

$$2\frac{dy}{y^2-1} = \sin x dx \qquad (\mathbf{4pts})$$

$$\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy = \int \sin x dx$$

$$\ln\left|\frac{y-1}{y+1}\right| = -\cos x + \ln C \qquad (\mathbf{6pts})$$

$$\frac{y-1}{y+1} = Ce^{-\cos x}.$$

Expanding and solving for y, we get

$$y = \frac{1 + Ce^{-\cos x}}{1 - Ce^{-\cos x}} \qquad (\mathbf{2pts})$$

(12pts)Problem 6.
Solve the initial value problem for the linear equation below

$$\frac{dy}{dx} = -\frac{1}{x}y + \sin x, \qquad y(\pi) = 1.$$

Solution
This is a linear first order equation.

$$\frac{dy}{dx} + \frac{1}{x}y = \sin x. \qquad \textbf{(4pts)}$$

The integrating factor is

$$e^{\int \frac{1}{x} dx} = |x| \qquad x > 0 \text{ near } \pi.$$

$$\frac{d}{dx}\left[|x|\, y\right] = x \sin x.$$

$$|x|\, y = \int x \sin x dx.$$

Integrating by parts, we get

$$|x|\, y = -x\cos x + \sin x + C$$

So

$$y = \frac{-x\cos x}{|x|} + \frac{\sin x}{|x|} + \frac{C}{|x|}. \qquad \textbf{(6pts)}$$

Now using the initial condition, we get

$$1 = 1 + 0 + \frac{C}{\pi}.$$

$$C = 0. \qquad \textbf{(2pts)}$$

(14pts)Problem 7.
Show that the differential equation is exact and find the general solution.

$$\left(xe^{2y} - x^2\right) dx + \left(x^2 e^{2y} + e^y\right) dy = 0.$$

Solution
Put

$$M = xe^{2y} - x^2 \qquad \text{and} \qquad N = x^2 e^{2y} + e^y$$

$$M_y = 2xe^{2y} = N_x \quad \text{shows that the equation is exact.} \qquad (\mathbf{4pts})$$

There exist a function f such that

$$\begin{cases} \dfrac{\partial f}{\partial x} = xe^{2y} - x^2 \\ \dfrac{\partial f}{\partial y} = x^2 e^{2y} + e^y \end{cases} \qquad (\mathbf{4pts})$$

$$\frac{\partial f}{\partial y} = x^2 e^{2y} + e^y \Rightarrow f(x, y) = \frac{1}{2} x^2 e^{2y} + e^y + g(x).$$

Now using the expression for $\dfrac{\partial f}{\partial x}$, we have

$$xe^{2y} + g'(x) = xe^{2y} - x^2$$

$$g'(x) = -x^2$$

and

$$g(x) = -\frac{x^3}{3} + C.$$

The general solution is

$$\frac{1}{2} x^2 e^{2y} + e^y - \frac{x^3}{3} = C \qquad (\mathbf{6pts})$$

(14pts)Problem 8.
Show that the differential equation is NOT exact and transform it into an exact equation.

$$\left(x^3 y - y\right) dx - x dy = 0$$

Solution
Put

$$M = x^3 y - y \quad \text{and} \quad N = -x$$

$$M_y = x^3 - 1 \neq N_x = -1 \qquad (\textbf{4pts})$$

Therefore, the equation is not exact.

$$\frac{M_y - N_x}{N} = \frac{(x^3 - 1) - (-1)}{-x} = -x^2 \text{ depends only on } x.$$

An integrating factor would be

$$e^{\int -x^2 dx} = e^{\frac{-x^3}{3}}. \qquad (\textbf{6pts})$$

The new exact equation is

$$\left(x^3 y - y\right) e^{\frac{-x^3}{3}} dx - x e^{\frac{-x^3}{3}} dy = 0 \qquad (\textbf{4pts})$$

(14pts)Problem 9.
(a) Solve the homogeneous equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}.$$

(b) Find an explicit solution of the initial value problem

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}, \qquad y(e) = 2e.$$

Solution
(a)

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy} = \left(\frac{y}{x}\right)^{-1} + \frac{y}{x}. \qquad (\mathbf{2pts})$$

Put

$$u = \frac{y}{x} \Rightarrow y = xu \qquad (\mathbf{4pts})$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

The equation becomes

$$u + x\frac{du}{dx} = \frac{1}{u} + u$$

$$x\frac{du}{dx} = \frac{1}{u}$$

$$udu = \frac{1}{x}dx$$

Integrating, we get

$$\frac{u^2}{2} = \ln|x| + C$$

$$\frac{1}{2}\frac{y^2}{x^2} = \ln|x| + C. \qquad (\mathbf{4pts})$$

(b)

$$y^2 = 2x^2\left(\ln|x| + C\right).$$

$$y(e) = 2e \Rightarrow C = 1 \qquad (\mathbf{4pts})$$

$$y^2 = 2x^2\left(\ln|x| + 1\right)$$

(14pts)Problem 10.
Find the general solution of the Bernoulli equation

$$3\left(1+x^2\right)\frac{dy}{dx}=2xy\left(y^3-1\right).$$

Solution
The equation can be written as

$$3y^{-4}\frac{dy}{dx}+\frac{2x}{(1+x^2)}y^{-3}=\frac{2x}{(1+x^2)}.$$

This is a Bernoulli equation with $n=4$.
Put

$$u=y^{1-4}=y^{-3} \qquad (\mathbf{4pts})$$

$$\frac{du}{dx}=-3y^{-4}\frac{dy}{dx}\Rightarrow 3y^{-4}\frac{dy}{dx}=-\frac{du}{dx}.$$

The equation becomes

$$-\frac{du}{dx}+\frac{2x}{(1+x^2)}u=\frac{2x}{(1+x^2)}.$$

$$\frac{du}{dx}-\frac{2x}{(1+x^2)}u=\frac{-2x}{(1+x^2)}. \qquad (\mathbf{5pts})$$

This is a linear equation with integrating factor

$$e^{\int\frac{-2x}{(1+x^2)}dx}=\frac{1}{x^2+1}.$$

$$\frac{d}{dx}\left[u\left(\frac{1}{x^2+1}\right)\right]=-2x\left(1+x^2\right)^{-2}$$

Integrating, we get

$$u\left(\frac{1}{x^2+1}\right)=\int -2x\left(1+x^2\right)^{-2}dx=\frac{1}{x^2+1}+C$$

$$u=1+C\left(x^2+1\right)$$

$$y^{-3}=1+C\left(x^2+1\right). \qquad (\mathbf{5pts})$$