Ex. A curve C is defined by the parametric equations

$x = t^2$ and $y = t^3 - 9t$

a) Show that C has two tangents at the point $(9,0)$ and find their equations.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-9}{2t}$$

$(9,0) \longrightarrow x = 9 \quad y = 0$

$t^2 = 9 \Rightarrow \boxed{t = \pm 3}$

$$\left.\frac{dy}{dx}\right|_{t=3} = \frac{18}{6} = 3 \qquad \left.\frac{dy}{dx}\right|_{t=-3} = \frac{18}{-6} = -3$$

$\therefore$ 2 slopes $\Rightarrow$ 2 tangent lines

$y - y_1 = m(x - x_1)$ $\qquad \underset{x_1,\ y_1}{(9,0)}$

$y - 0 = \pm 3(x-9)$

$$\boxed{y = \pm 3(x-9)}$$

b) Find all points on C where we have horizontal and vertical tangents

Horizontal $\quad dy/dt = 0$ and $dx/dt \neq 0$

$3t^2 - 9 = 0 \qquad 2t \neq 0$

$t^2 - 3 = 0$

$t^2 = 3$

$\boxed{t = \pm\sqrt{3}}$

$t = \sqrt{3} \Rightarrow (3, -6\sqrt{3})$ $\quad -10.2$

$t = -\sqrt{3} \Rightarrow (3, 6\sqrt{3})$ $\quad 10.2$

Vertical $\quad dx/dt = 0$ and $dy/dt \neq 0$

$$2t = 0 \qquad 3t^2 - 9 \neq 0$$

$$\boxed{t=0} \Rightarrow (0,0)$$

c) Determine when the curve is concave up or down.

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{\frac{6t(2t) - 2(3t^2-9)}{(2t)^2}}{2t}$$

$$= \frac{12t^2 - 6t^2 + 18}{8t^3}$$

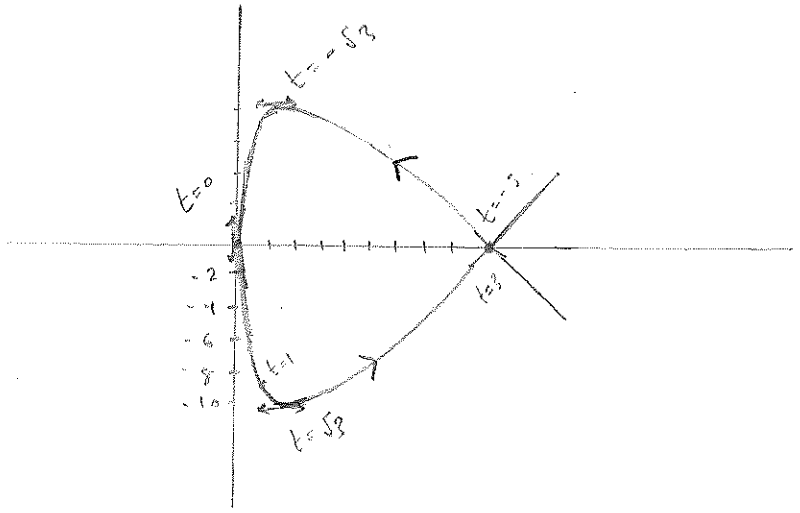
$$= \frac{6t^2+18}{8t^3}$$

$$= \frac{6(t^2+3)}{8t^3}$$

$$= \frac{3(t^2+3)}{4t^3}$$

C is concave up when $t > 0$ and down when $t < 0$

d) Sketch C



Arc length in Parametric Form

$C: \quad x = f(t) \qquad y = g(t) \qquad a \le t \le b$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Ex. find the arc length of the curve over the stated interval.

$x = \cos 3t \qquad y = \sin 3t \qquad 0 \le t \le \pi$

$$s = \int_0^{\pi} \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2}\, dt$$

$$s = \int_0^{\pi} \sqrt{9\sin^2 3t + 9\cos^2 3t}\, dt \qquad \sin^2 3t + \cos^2 3t = 1$$

$$x^2 + y^2 = 1$$

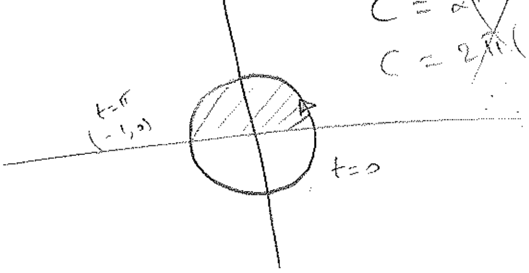
$$s = \int_0^{\pi} \sqrt{9}\, dt$$

$$= 3\int_0^{\pi} dt$$

$$= 3[t]_0^{\pi}$$

$$= 3\pi - 0$$

$$= 3\pi$$



$C = 2\pi \cancel{r}$ wrong

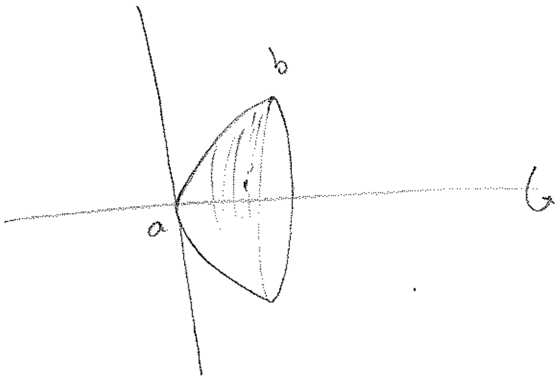
$C = 2\pi(1) = 2\pi$

Area of a Surface of Revolution

$C: \quad x = f(t) \qquad y = g(t) \qquad a \le t \le b$

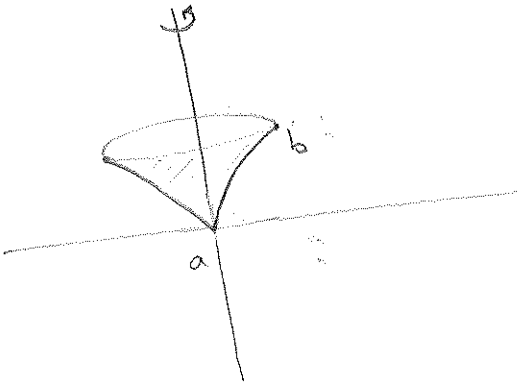
$$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

revolution about the x-axis
$g(t) \geq 0$



$$S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

revolution about the y-axis
$f(t) \geq 0$



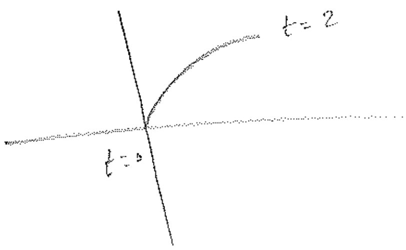
Ex. Find the area of the surface generated by revolving $x = t^2$, $y = 5t$ $\quad 0 \le t \le 2$ about the x-axis.

t	0	1	2
x	0	1	4
y	0	5	10

$x = t^2 \qquad y = 5t \Rightarrow$

$t = \frac{y}{5} \Rightarrow x = \frac{y^2}{25}$

$y^2 = 25x$



$$S = 2\pi \int_0^2 5t \sqrt{(2t)^2 + (5)^2}\, dt$$

$$S = 2\pi \int_0^2 5t \sqrt{4t^2 + 25}\, dt$$

$u = 4t^2 + 25 \Rightarrow du = 8t\,dt$

$dt = \frac{du}{8t}$

$t = 0 \Rightarrow u = 25$

$t = 2 \Rightarrow u = 41$

$$S = 2\pi \int_{25}^{41} 5t \sqrt{u} \frac{du}{8t}$$

$$= \frac{5\pi}{4} \int_{25}^{41} \sqrt{u}\, du$$

$u^{1/2}$

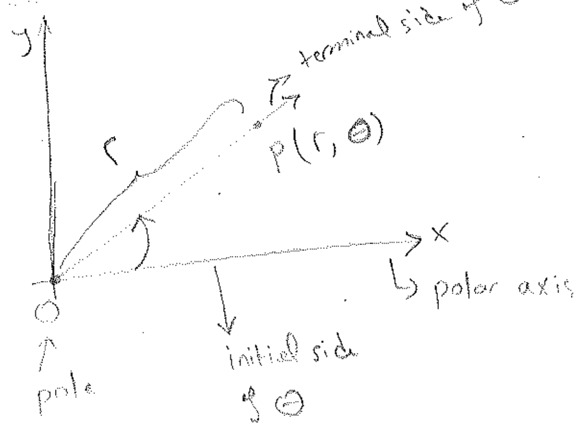
$$= \frac{5\pi}{4} \cdot \frac{2}{3} \left[u^{\frac{3}{2}} \right]_{25}^{41}$$

$$= \frac{5\pi}{6} \left[u^{3/2} \right]_{25}^{41}$$

$$= \frac{5\pi}{6} \left(41\sqrt{41} - 125 \right)$$

Polar Coordinates & Polar Graphs

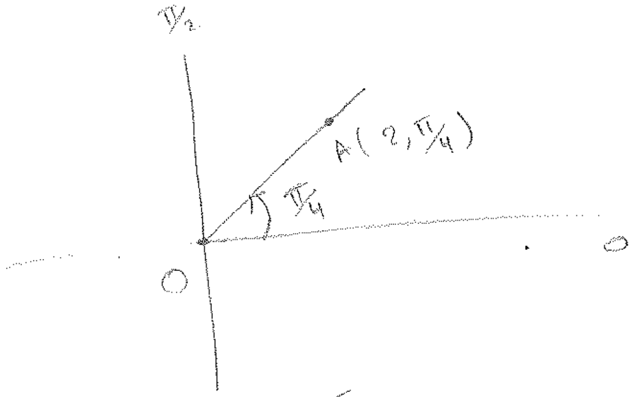
Polar Coordinate System



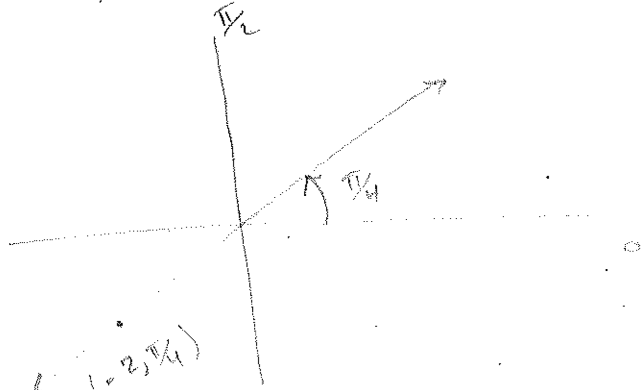
- If r is +ve, then P lies on the terminal side of Θ
- If r is −ve, then P lies on the ray opposite the terminal side of Θ.

Graph each point

1) $A\left(2, \frac{\pi}{4}\right)$

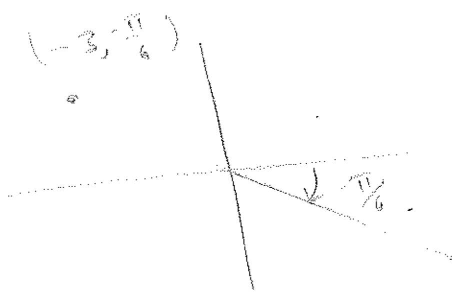


2) $B\left(-2, \frac{\pi}{4}\right)$



3) $C\left(3, -\frac{\pi}{6}\right)$

4) $D\left(-3, -\frac{\pi}{6}\right)$

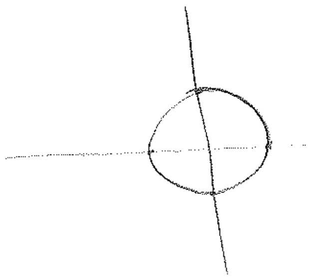


5) $\left(-3, \frac{5\pi}{4}\right)$

Polar Graphs (Basic Graphs)

Graph each polar equation.

1) $r = 2$



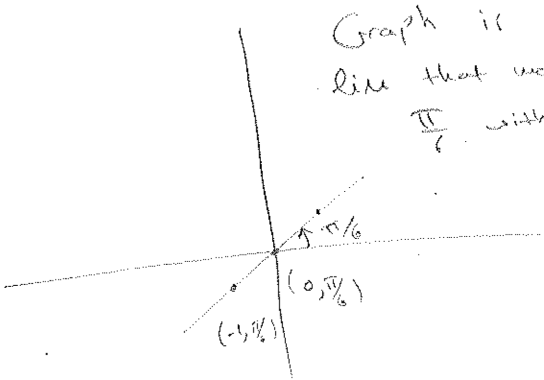
θ = free $\quad (2, 0)$, $(2, \frac{\pi}{2})$ ... $(2, 2\pi)$

Circle centered at $(0,0)$
& radius 2

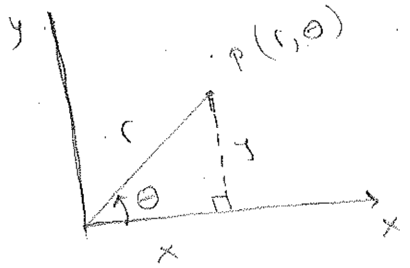
2) $\theta = \frac{\pi}{6}$

r = free $\quad (0, \frac{\pi}{6})$, $(-1, \frac{\pi}{6})$, $(1, \frac{\pi}{6})$ ...

Graph is a straight line that makes an angle of $\frac{\pi}{6}$ with the +ve polar axis.



Coordinate Conversion



$\tan\theta = \frac{y}{x}$ $\quad \sin\theta = \frac{y}{r}$ $\quad \cos\theta = \frac{x}{r}$

$r^2 = x^2 + y^2$

Ex. Find the rectangular coordinates of the following points given polar coordinates.

a) $A\left(4, \frac{\pi}{6}\right)$ $r = 4$ $\theta = \frac{\pi}{6}$

$$\left.\begin{aligned} x &= r\cos\theta \\ &= 4\cos\frac{\pi}{6} \\ &= 4\left(\frac{\sqrt{3}}{2}\right) \\ &= 2\sqrt{3} \end{aligned}\right\} \quad \begin{aligned} y &= r\sin\theta \\ y &= 4\sin\left(\frac{\pi}{6}\right) \\ &= 4\left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

$$\boxed{(2\sqrt{3}, 2)}$$

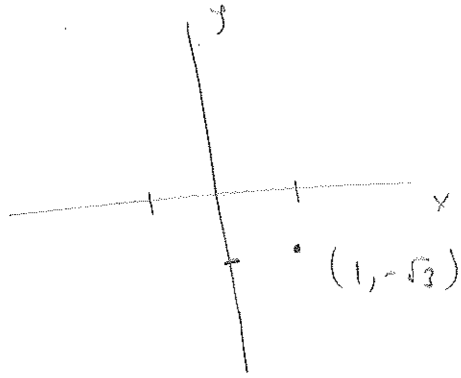
b) $B\left(3, \frac{2\pi}{3}\right)$

$r = 3$ $\theta = \frac{2\pi}{3}$

$$\begin{aligned} x = r\cos\theta &= 3\cos\left(\frac{2\pi}{3}\right) \\ &= -3\cos\left(\frac{\pi}{3}\right) \\ &= -3\left(\frac{\sqrt{3}}{2}\right) \\ &= -\frac{3\sqrt{3}}{2} \end{aligned} \quad \left\{ \begin{aligned} y &= r\sin\theta \\ &= 3\sin\left(\frac{2\pi}{3}\right) \\ &= 3\sin\left(\frac{\pi}{3}\right) \\ &= 3\left(\frac{1}{2}\right) \\ &= \frac{3}{2} \end{aligned}\right.$$

$$\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$$

Ex. find ______ polar coordinates for the following point in rectangular coordinates.



a) $(1, -\sqrt{3})$

$x = 1 \qquad y = -\sqrt{3}$

$\tan\theta = \left(-\frac{\sqrt{3}}{1}\right) = -\sqrt{3}$

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \Rightarrow \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\boxed{\text{or since } x>0 \quad \theta = \tan^{-1}(-\sqrt{3}) = -\pi/3}$

$r^2 = x^2 + y^2 = (1)^2 + (-\sqrt{3})^2$
$= 1 + 3$
$= 4 \Rightarrow r = 2$

$(2, -\pi/3)$ or $(2, -\pi/3 + 2\pi) = (2, 5\pi/3)$

b) $(-6, -12)$

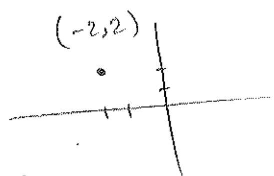
$x = -6 \quad y = -12$

$\tan\theta = \frac{y}{x} = \frac{-12}{-6} = -2$

$\theta = \tan^{-1}(2) + \pi$

$r^2 = 36 + 144 = 180 \Rightarrow r = \sqrt{180} \rightarrow r = 6\sqrt{5}$

$(6\sqrt{5}, \tan^{-1}(2) + \pi)$



c) $(-2, 2) \quad x = -2 \quad y = 2$

$\tan\theta = \frac{y}{x} = \frac{2}{-2} = -1$

$\tan^{-1}(1) = \pi/4$

$\theta = \pi - \pi/4$

$\theta = 3\pi/4$

$r^2 = x^2 + y^2$

$r^2 = 4 + 4$

$r^2 = 8 \Rightarrow \boxed{r = 2\sqrt{2}}$

$(2\sqrt{2}, 3\pi/4)$

6

c) $r = -5\sin\theta$

$r^2 = -5r\sin\theta$

$x^2 + y^2 = -5y$

$x^2 + y^2 + 5y = 0$

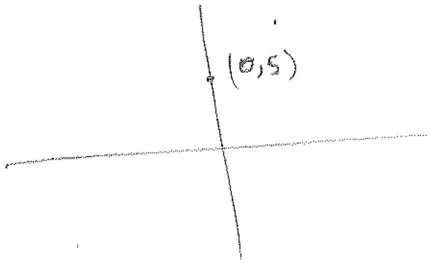
$x^2 + \left(y + \frac{5}{2}\right)^2 - \frac{25}{4} = 0$

$x^2 + \left(y + \frac{5}{2}\right)^2 = \frac{25}{4}$

Circle center $\left(0, -\frac{5}{2}\right)$

and radius $\frac{5}{2}$.

d) $(0, 5)$



$x = 0 \quad y = 5$

$\cos\theta = \frac{x}{r} = 0$

$\Rightarrow \theta = \pi/2 \qquad r^2 = x^2 + y^2$

$r^2 = 5^2 \Rightarrow r = 5$

$(5, \pi/2)$.

Ex. Write each equation in rectangular form, and then identify its graph.

a) $r = 2$

$r^2 = 4 \Rightarrow x^2 + y^2 = 4$ (circle center $(0,0)$ and radius 2)

b) $\theta = \pi/6 \qquad \tan\theta = \frac{y}{x} \Rightarrow \tan\left(\frac{\pi}{6}\right) = \frac{y}{x}$

$\frac{\sqrt{3}}{3} = \frac{y}{x}$

$y = \frac{\sqrt{3}}{3} x$

straight line passing through the origin with slope $= \sqrt{3}/3$

Slope and Tangent lines

$r = f(\theta)$ (polar equation)

$x = r\cos\theta$ and $y = r\sin\theta$

$x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + \cos\theta \cdot f(\theta)}{f'(\theta)\cdot\cos\theta - f(\theta)\sin\theta}$$

Ex.

Find the horizontal and vertical tangent lines of $r = \cos\theta$ $\quad 0 \le \theta \le \pi$

$x = r\cos\theta \qquad y = r\sin\theta$

$x = \cos^2\theta \qquad y = \cos\theta\cdot\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\sin\theta\cdot\sin\theta + \cos\theta\cdot\cos\theta}{-2\cos\theta\cdot\sin\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{-2\sin\theta\cdot\cos\theta}$$

Hori. $\quad \cos^2\theta - \sin^2\theta = 0$ and $-2\sin\theta\cdot\cos\theta \ne 0$

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$ or $2\theta = \frac{3\pi}{2} \Rightarrow \boxed{\theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4}}$

$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right) \quad \left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

Vertical $\quad -2\sin\theta\cdot\cos\theta = 0$ and $\cos^2\theta - \sin^2\theta \ne 0$

$\sin 2\theta = 0$ and $\cos 2\theta \ne 0$

$2\theta = 0$ or $2\theta = \pi \Rightarrow \boxed{\theta = 0}$ or $\boxed{\theta = \pi/2}$

$(1, 0)$

$\left(0, \frac{\pi}{2}\right)$

Ex: Consider the graph of $r = 1 + \sin\theta$

a) Find the slope of the tangent line when $\theta = \frac{\pi}{4}$.

$$\left.\frac{dy}{dx}\right|_{\theta = \pi/4}$$

$x = r\cos\theta \qquad y = r\sin\theta$

$x = (1+\sin\theta)\cos\theta \qquad y = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta$

$= \cos\theta + \sin\theta \cdot \cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\cdot\cos\theta}{-\sin\theta + (\cos^2\theta - \sin^2\theta)}$$

$$= \frac{\cos\theta + 2\sin\theta\cdot\cos\theta}{-\sin\theta + \cos^2\theta - \sin^2\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{\sqrt{2}/2 + 1}{-\sqrt{2}/2} = \frac{\sqrt{2}+2}{-\sqrt{2}}$$

$$= \frac{2 + 2\sqrt{2}}{-2}$$

$$\boxed{= -1 - \sqrt{2}}$$

b) Find the points on the graph where the tangent line is horizontal or vertical.

Horizontal $\quad \cos\theta + 2\sin\theta\cdot\cos\theta = 0$ and $-\sin\theta + \cos^2\theta - \sin^2\theta \neq 0$

$\cos\theta(1 + 2\sin\theta) = 0$

$\cos\theta = 0$ or $\sin\theta = -1/2$

$\theta = \frac{\pi}{2}, \cancel{\frac{3\pi}{2}}, 7\pi/6, \frac{11\pi}{6}$

$(2, \pi/2) \qquad (1/2, 7\pi/6)$

Vertical

c) $-\sin\theta + \cos^2\theta - \sin^2\theta = 0$ and $\cos\theta + 2\sin\theta\cdot\cos\theta \neq 0$

$$-\sin\theta + (1 - \sin^2\theta) - \sin^2\theta = 0$$

$$-\sin\theta + 1 - \sin^2\theta - \sin^2\theta = 0$$

$$-2\sin^2\theta - \sin\theta + 1 = 0$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$\sin\theta = \frac{1}{2}$ $\quad$ $\sin\theta = -1$

$\theta = \frac{\pi}{6}$ $\quad$ $\theta = \pi - \pi/6 = \frac{5\pi}{6}$ $\quad$ $\theta = 3\pi/2$

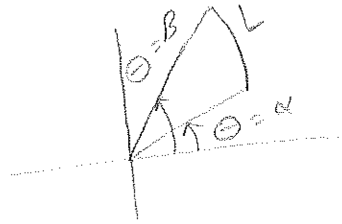
$(1.5, \pi/6)$ $\quad$ $(1.5, 5\pi/6)$

Notice $\left.\frac{dy}{dx}\right|_{\theta = 3\pi/2} = \frac{0}{0}$

Arc length of a Polar Curve

$$r = f(\theta) \qquad \alpha \le \theta \le \beta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$



Ex.

Find the total arc length of $r = 1-\cos\theta$.

Recall that $r = 1-\cos\theta$ is traced once from $\theta = 0$ to $\theta = 2\pi$.

$$L = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta$$

$1 - 2\cos\theta + \cos^2\theta + \sin^2\theta$
$1 - 2\cos\theta + 1 = 2 - 2\cos\theta$

$$L = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta = \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos\theta}\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\theta/2}\, d\theta$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\downarrow$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$0 \le \theta \le 2\pi$$

$$0 \le \frac{\theta}{2} \le \pi$$

$$\downarrow$$

$$\sin\theta/2 = +ve$$

$$= 2\int_0^{2\pi} \sqrt{\sin^2\theta/2}\, d\theta$$

$$= 2\int_0^{2\pi} |\sin\theta/2|\, d\theta$$

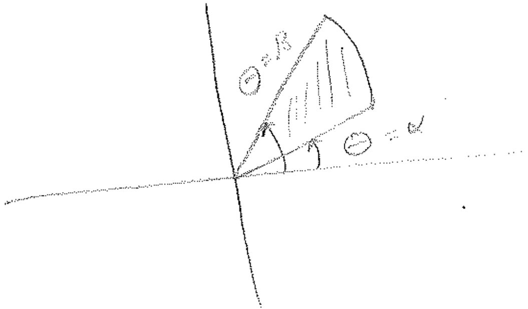
$$= 2\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = 2\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= -4\left[\cos\frac{\theta}{2}\right]_0^{2\pi} = -4(-1-1)$$

$$= \boxed{8}$$

Area in Polar Coordinates

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$



(5)

Ex. Find the entire area within the cardioid.

$r = 1 - \cos\theta$

$$A = \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[(3\pi - 0 + 0) - (0 - 0 - 0)\right]$$

$$= \frac{3\pi}{2}.$$