Tutorial 1

Question 1

Find the area enclosed by the graphs of $f(x) = x^2$ *and* $g(x) = 2 - x^2, 0 \leq x \leq 2$.

Solution

$$\begin{aligned} A &= \int_0^2 |f(x) - g(x)|\, dx \\ &= \int_0^2 \left|x^2 - (2 - x^2)\right| dx \\ &= \int_0^2 |2x^2 - 2|\, dx \\ &= 2\int_0^2 |x^2 - 1|\, dx. \end{aligned}$$

To remove the absolute value, we need to check the sign of $x^2 - 1$ between 0 and 2.

x	$-\infty$	-1		0		1		2		$+\infty$
x^2-1	+		$-$		$-$	0	+		+	
$\lvert x^2-1\rvert$	x^2-1		$-(x^2-1)$		$-(x^2-1)$		x^2-1		x^2-1	

$0 \leq x \leq 2$

$$\begin{aligned} A &= 2\left[\int_0^1 -\left(x^2 - 1\right) dx + \int_1^2 \left(x^2 - 1\right) dx\right] \\ &= 2\left(\frac{2}{3} + \frac{4}{3}\right) = 4 \end{aligned}$$

Question 2

Find the area enclosed by the graphs of $f(x) = x^3 - 2x^2$ *and* $g(x) = 2x^2 - 3x$.

Solution

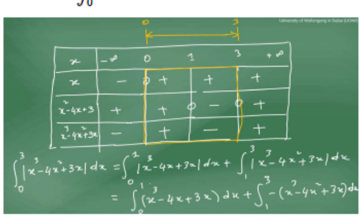
$$\begin{aligned} A &= \int_a^b |f(x) - g(x)|\, dx = \int_a^b \left|\left(x^3 - 2x^2\right) - \left(2x^2 - 3x\right)\right| dx \\ &= \int_a^b \left|x^3 - 4x^2 + 3x\right| dx. \end{aligned}$$

To find a and b, we solve the equation

$$\begin{aligned} x^3 - 2x^2 &= 2x^2 - 3x \Leftrightarrow \\ x^3 - 4x^2 + 3x &= 0 \Rightarrow x = 1 \text{ or } x = 0 \text{ or } x = 3. \end{aligned}$$

$$a = 0 \text{ and } b = 3.$$

$$A = \int_0^3 |x\,(x-1)\,(x-3)|\, dx$$



$$\begin{aligned} A &= \int_0^1 x\,(x-1)\,(x-3)\, dx - \int_1^3 x\,(x-1)\,(x-3)\, dx \\ &= \frac{5}{12} - \left(-\frac{8}{3}\right) = \frac{37}{12} = 3.0833 \end{aligned}$$

Question 3

Find the area enclosed by the graphs of $x = y^2 + 2$ *and* $y = x - 8$

Solution

$$x = y^2 + 2 \text{ and } \quad y = x - 8 \Leftrightarrow x = y + 8.$$

$$\begin{aligned} A &= \int_a^b \left|(y^2+2) - (y+8)\right| dy \\ &= \int_a^b \left|y^2 - y - 6\right| dy \end{aligned}$$

To find a and b, we set

$$y^2 - y - 6 = 0 \text{ and solve to get}$$

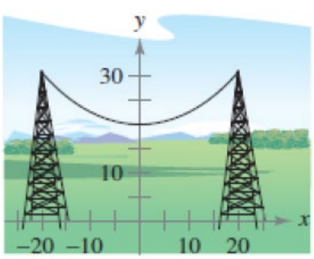
$$a = -2 \text{ and } b = 3.$$

equation Now using the fact that the quadratic $y^2 - y - 6$ has the opposite sign of $+1$ inside the roots, we have

$$\begin{aligned} A &= \int_{-2}^{3} -\left(y^2 - y - 6\right) dy \\ &= \frac{125}{6} = 20.833 \end{aligned}$$

Question 4

Electrical wires suspended between two towers from a catenary (see figure) modeled by the equation $y = 10\left(e^{\frac{x}{20}} + e^{\frac{-x}{20}}\right), -20 \leq x \leq 20$



Where x and y are measured in meters. The two towers are 40 meters apart. Find the length of the suspended cable.

Solution

We need to find the arclength of the curve.

$$L = \int_{-20}^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{1}{2} e^{\frac{1}{20}x} - \frac{1}{2} e^{-\frac{1}{20}x}$$

$$\begin{aligned}\left(\frac{dy}{dx}\right)^2 &= \left(\frac{1}{2} e^{\frac{1}{20}x} - \frac{1}{2} e^{-\frac{1}{20}x}\right)^2 \\ &= \frac{1}{4} e^{\frac{1}{10}x} - \frac{1}{2} + \frac{1}{4} e^{-\frac{1}{10}x}\end{aligned}$$

$$\begin{aligned}1 + \left(\frac{dy}{dx}\right)^2 &= \frac{1}{4} e^{\frac{1}{10}x} + \frac{1}{2} + \frac{1}{4} e^{-\frac{1}{10}x} \\ &= \left(\frac{1}{2} e^{\frac{1}{20}x} + \frac{1}{2} e^{-\frac{1}{20}x}\right)^2\end{aligned}$$

$$\begin{aligned}L &= \int_{-20}^{20} \sqrt{\left(\frac{1}{2} e^{\frac{1}{20}x} + \frac{1}{2} e^{-\frac{1}{20}x}\right)^2}\, dx \\ &= \int_{-20}^{20} \left(\frac{1}{2} e^{\frac{1}{20}x} + \frac{1}{2} e^{-\frac{1}{20}x}\right) dx \\ &= 10 e^{\frac{1}{20}x} - 10 e^{\frac{-1}{20}x}\Big|_{-20}^{20} \\ &= 20\left(e - \frac{1}{e}\right) = 47.008\end{aligned}$$

Question 5

Find the arc length of the graph of $f(x) = \frac{x^6+8}{16x^2}$ on the interval [2,3].

Solution

$$\begin{aligned}f(x) &= \frac{x^6+8}{16x^2} = \frac{x^6}{16x^2} + \frac{8}{16x^2} \\ &= \frac{x^4}{16} + \frac{1}{2x^2}\end{aligned}$$

$$L = \int_2^3 \sqrt{1 + (f'(x))^2}\, dx$$

$$f'(x) = \frac{1}{4}x^3 - \frac{1}{x^3}$$

$$(f'(x))^2 = \left(\frac{1}{4}x^3 - \frac{1}{x^3}\right)^2 = \frac{1}{x^6} + \frac{1}{16}x^6 - \frac{1}{2}$$

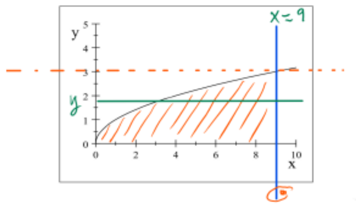
$$\begin{aligned}1 + (f'(x))^2 &= 1 + \frac{1}{x^6} + \frac{1}{16}x^6 - \frac{1}{2} \\ &= \frac{1}{x^6} + \frac{1}{16}x^6 + \frac{1}{2} \\ &= \left(\frac{1}{4}x^3 + \frac{1}{x^3}\right)^2.\end{aligned}$$

$$\begin{aligned}L &= \int_2^3 \sqrt{1 + (f'(x))^2}\, dx = \int_2^3 \sqrt{\left(\frac{1}{4}x^3 + \frac{1}{x^3}\right)^2}\, dx \\ &= \int_2^3 \left(\frac{1}{4}x^3 + \frac{1}{x^3}\right) dx = \frac{595}{144} = 4.1319.\end{aligned}$$

Question 6

Sketch the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$, and use the disc method to find the volume of the solid generated by revolving the region about the line $x = 9$.

Solution

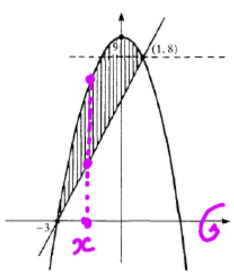


$$
\begin{aligned}
V &= \int_0^3 \pi \left(9 - y^2\right)^2 dy \\
&= \frac{648}{5}\pi = 407.15.
\end{aligned}
$$

Question 7

Sketch the region bounded by $y = 9 - x^2$, $y = 2x + 6$, and use the disc method to find the volume of the solid generated by revolving the region about the x-axis.

Solution



$$
\begin{aligned}
V &= \int_{-3}^{1} \left[\pi \left(9 - x^2\right)^2 - \pi \left(2x + 6\right)^2\right] dx \\
&= \pi \int_{-3}^{1} \left[\left(9 - x^2\right)^2 - \left(2x + 6\right)^2\right] dx \\
&= \pi \int_{-3}^{1} \left(x^4 - 22x^2 - 24x + 45\right) dx \\
&= \frac{1792}{15}\pi = 375.32
\end{aligned}
$$

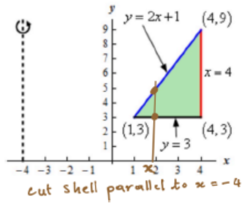
Question 8

Find the volume of the solid formed by revolving the region bounded by $y = 2x + 1$, $x = 4$ *and* above $y = 3$ *about the line* $x = -4$.

Solution

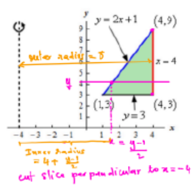
By Shell Method

The shell method is more convenient for this problem.



$$
\begin{aligned}
V &= \int_a^b 2\pi \cdot \text{average radius} \cdot \text{height } dx \\
&= \int_1^4 2\pi (x+4)\left[(2x+1)-3\right] dx \\
&= \int_1^4 2\pi (x+4)(2x-2)\, dx \\
&= 126\pi = 395.84
\end{aligned}
$$

By the Disc Method



$$\text{Inner Radius} = 4 + \frac{1}{2}(y-1) = \frac{1}{2}y + \frac{7}{2} \qquad \text{Outer Radius} = 4 + 4 = 8$$

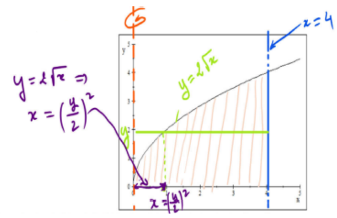
$$
\begin{aligned}
A(x) &= \pi\left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] \\
&= \pi\left[(8)^2 - \left(\frac{1}{2}y + \frac{7}{2}\right)^2\right] = \pi\left(\frac{207}{4} - \frac{7}{2}y - \frac{1}{4}y^2\right)
\end{aligned}
$$

$$V = \int_3^9 \pi\left(\frac{207}{4} - \frac{7}{2}y - \frac{1}{4}y^2\right) dy = \pi\left(\frac{207}{4}y - \frac{7}{4}y^2 - \frac{1}{12}y^3\right)\Bigg|_3^9 = \boxed{126\pi}$$

Question 9

Sketch the region bounded by the curve $y = 2\sqrt{x}$, the x-axis and the line x = 4, and use the disc method to find the volume obtained by rotating the region about the y-axis.

Solution

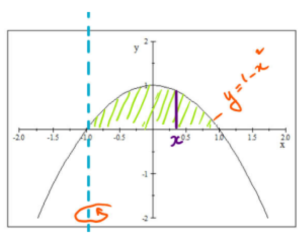


$$
\begin{aligned}
V &= \int_0^4 \pi\left[4^2 - \left(\frac{y^2}{4}\right)^2\right] dy \\
&= \int_0^4 \pi\left(16 - \frac{1}{16}y^4\right) dy \\
&= \frac{256}{5}\pi = 160.85.
\end{aligned}
$$

Question 10

Sketch the region bounded by the curves $y = 1 - x^2$ *and* $y = 0$, and use the method of cylindrical shells to find the volume obtained by rotating the region about the line $x = -1$.

Solution



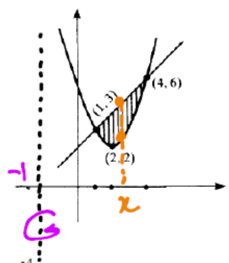
(**3pts**)

$$
\begin{aligned}
V &= \int_{-1}^{1} 2\pi \cdot (x+1) \cdot \left(1 - x^2\right) dx \qquad \textbf{(6pts)} \\
&= \int_{-1}^{1} 2\pi \left(x+1\right)\left(1 - x^2\right) dx \\
&= \frac{8\pi}{3} = 8.3776. \qquad \textbf{(1pt)}
\end{aligned}
$$

Question 11

Sketch the region bounded by $y = x^2 - 4x + 6, y = x + 2,$ *and* use the shell method to find the volume of the solid generated by revolving the region about the line $x = -1$.

Solution

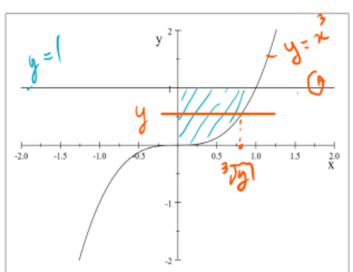


$$\begin{aligned} V &= \int_1^4 2\pi (x+1) \left[(x+2) - (x^2 - 4x + 6)\right] dx \\ &= \int_1^4 2\pi (x+1) \left(-x^2 + 5x - 4\right) dx \\ &= 2\pi \int_1^4 \left(-x^3 + 4x^2 + x - 4\right) dx \\ &= \frac{63}{2}\pi = 98.96 \end{aligned}$$

Question 12

Sketch the region bounded by $y = x^3, y = 1$ *and* $x = 0$, and use the shell method to find the volume of the solid generated by revolving the region about the line $y = 1$.

Solution



$$\begin{aligned} V &= \int_0^1 2\pi \cdot (1 - y) \cdot \sqrt[3]{y} dy \\ &= 2.0196 \end{aligned}$$