1. a) $\int_0^{\pi/4} x \sin 2x \; dx$

D	I
x	$\sin 2x$
1	$\frac{1}{2}\cos 2x$
0	$-\frac{1}{4}\sin 2x$

$$I = \left[\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{8} \cos \frac{\pi}{2} + \frac{1}{4} \sin \frac{\pi}{2} \right) - \left(0 - \frac{1}{4} \sin 0 \right)$$

$$= 0 + \frac{1}{4}(1)$$

$$= \frac{1}{4}$$

b) $\int \cos(\ln x) \; dx$

$X = \ln x$

$dX = \frac{1}{x} dx$

$x = e^X$

$dx = e^X dX$

$$I = \int \cos X \, e^X \, dX$$

$u = \cos X \qquad u' = -\sin X$

$v' = e^X \qquad v = e^X$

$$I = e^X \cos X + \int e^X \sin X \, dX$$

$u = \sin X \qquad u' = \cos X$

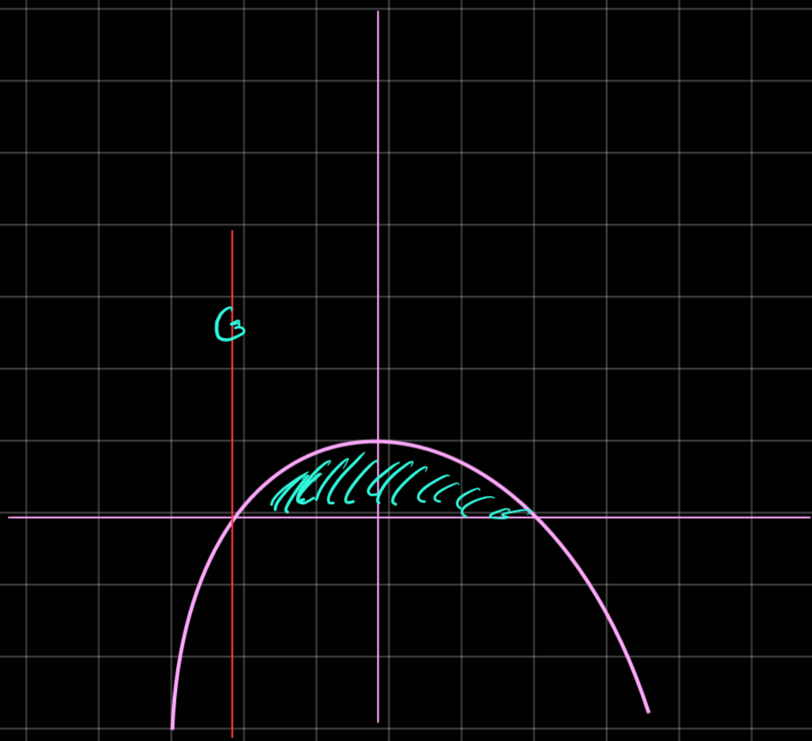
$v' = e^x \qquad v = e^x$

$$I = e^x \cos x + e^x \sin x - \int e^x \cos x$$
$$= e^x \cos x + e^x \sin x - I$$

$$2I = e^x \cos x + e^x \sin x$$
$$I = \frac{1}{2}\left[e^x \cos x + e^x \sin x\right]$$
$$= \frac{1}{2}\left[e^{\ln x} \cos(\ln x) + e^{\ln x} \sin(\ln x)\right]$$
$$= \frac{1}{2}\left[x\cos(\ln x) + x \sin(\ln x)\right]$$

3. $V = 2\pi \int_a^b \text{Avg radius} \cdot \text{Height } dx$

$$1 - x^2 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$



$$V = 2\pi \int_{-1}^{1} (x+1)(1-x^2)\, dx$$
$$= 2\pi \int_{-1}^{1} (x - x^3 + 1 - x^2)\, dx$$
$$= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4} + x - \frac{x^3}{3}\right)\Bigg|_{-1}^{1}$$
$$= \frac{8\pi}{3}$$

Winter 23 Mid

1. $\int^1 x^2\, dx + \int^2 (2-x)\, dx$

$$= \frac{x^3}{3}\Big|_0^1 + 2x - \frac{x^2}{2}\Big|_1^2$$

$$= 5/6$$

2. $9\int \sqrt{x} \ln x \, dx$

$u = \ln x \qquad u' = 1/x$

$v' = x^{1/2} \qquad v = \frac{2}{3} x^{3/2}$

$$\frac{2x^{3/2} \ln x}{3} - \frac{2}{3}\int x^{1/2}$$

$$= 9\left(\frac{2x^{3/2} \ln x}{3} - \frac{4}{9} x^{3/2} + C\right)$$

$$= 6x^{3/2} \ln x - 4x^{3/2} + C$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-5e^{-t}}{3e^{t}}$$

@ $t = 0$

$$= \frac{-5}{3}$$

$x = 3, \; y = 5$

$$y = \frac{-5}{3}(x - 3) + 5$$

$$= \frac{-5}{3}x + 5 + 5$$

$$y = 10 - \frac{5}{3}x$$

$$3y = 30 - 5x$$

$5x + 3y = 30$

$r = 4\cos\theta - 5\sin\theta$

$x^2 + y^2 = 4x - 5y$

$x^2 - 4x + y^2 + 5y = 0$

$(x-2)^2 - 4 + \left(y + \frac{5}{2}\right)^2 - \frac{25}{4} = 0$

Centre at $(2, -5/2)$ with radius $\sqrt{41}/4$

$3 - y^2 = y + 1$

$-y^2 - y - 1 + 3 = 0$

$-y^2 - y + 2 = 0$

$y^2 + y - 2 = 0$

$y = \frac{-1 \pm \sqrt{1 - 4(1)(-2)}}{2}$

$= \frac{-1 \pm 3}{2}$

$= -2$ or 1

$A = \int_{-2}^{1} y^2 + y - 2 \, dy$

$= \frac{y^3}{3} + \frac{y^2}{2} - 2y \Big|_{-2}^{1}$

$= \left(\frac{1}{3} + \frac{1}{2} - 2\right) - \left(\frac{-8}{3} + \frac{4}{2} + 4\right)$

$x + 1 = Ax(x-1) + B(x-1) + Cx^2$

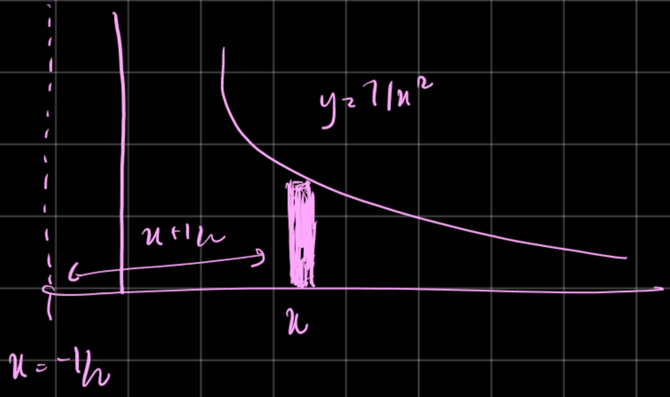
$x + 1 = Ax^2 - Ax + Bx - B + Cx^2$

$A + C = 0 \qquad -A + B = 1 \qquad -B = 1$

$C = 2 \qquad -A - 1 = 1 \qquad B = -1$

$-A = 2$

$A = 2$



2.

$r = x + 1/2$

$h = 7/x^2$

$$V = 2\pi \int_1^5 \left(x + 1/2\right)\left(\frac{7}{x^2}\right)$$

$$2\pi \int_1^5 \left(\frac{7}{x} + \frac{7}{2x^2}\right) dx$$

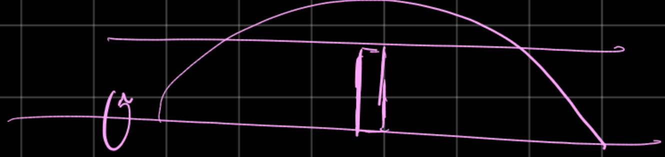
$$2\pi \int_1^5 \frac{7}{x} + \frac{7}{2} x^{-2} \, dx$$

$$= 2\pi \left[7 \ln|x| - \frac{7}{2} x^{-1} \right|_1^5$$

$= 2\pi$

$= 88.3797$

b)



$\sqrt{25 - x^2} = 3$

$25 - x^2 = 9$

$x = \pm 4$

$$\int^4 \pi\left(\sqrt{25 - x^2}\right)^2 - \pi(3)^2 \, dx$$

-4

$$= \pi \int_{-4}^{4} 16 - x^2$$

$$= \pi \left(16x - \frac{x^3}{3} \right)_{-4}^{4}$$

$$= \pi \left(\frac{128}{3} + \frac{128}{3} \right)$$

$$= \frac{256\pi}{6}$$

$$= 268.083$$

$$\int \frac{x^2+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$\begin{aligned}
x^2+1 &= A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx \\
&= A(x^3 - 1 - 3x^2 + 3x) + Bx(x^2 - 2x + 1) + Cx^2 - Cx + Dx \\
&= Ax^3 - A - 3Ax^2 + 3Ax + Bx^3 - 2Bx^2 + Bx + Cx^2 - Cx + Dx \\
&= x^3(A+B) + x^2(-3A - 2B + C) + x(3A + B - C + D) - A
\end{aligned}$$

$$A + B = 0$$

$$-3A - 2B + C = 1$$

$$3A + B - C + D = 0$$

$$-A = 1$$

$$A = -1$$

$$B = 1$$

$$C = 0$$

$$D = 2$$

$$= \int \frac{-1}{x}\,dx + \int \frac{1}{x-1} + 2\int \frac{1}{(x-1)^{-3}}\,dx$$

$$= -\ln|x| + \ln|x-1| - (x-1)^{-2} + C$$

5. $$\int \frac{4}{x^2\sqrt{x^2+4}}\,dx$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{\cancel{8}^{2} \sec^2\theta\, d\theta}{\cancel{4}\tan^2\theta \sqrt{4(\tan^2\theta+1)}}$$

$$\int \frac{\sec^{\cancel{2}}\theta\, d\theta}{\tan^2\theta \times \cancel{2\sec\theta}}$$

$$= \int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \int \frac{1}{\cancel{\cos\theta}} \times \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cos\theta \sin^{-2}\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

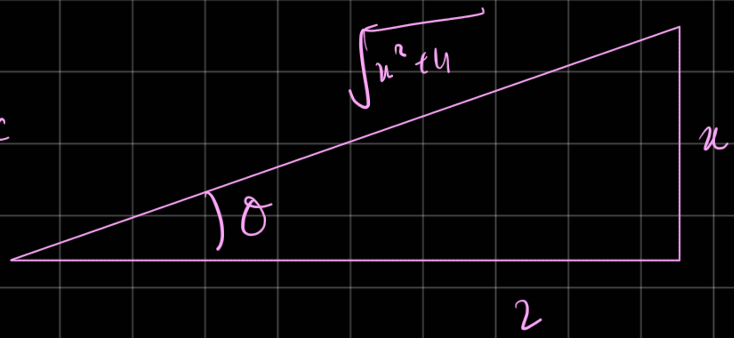
$$\int u^{-2}\,du$$

$$= -u^{-1} + C$$

$= -\sin^{-1}\theta$

$= -\csc\theta + c$

$= -\dfrac{\sqrt{x^2+4}}{x} + c$



$$\int \frac{\sqrt{x}}{2(1+x)}\,dx$$

$\sqrt{x} = u$

$x = u^2$

$2u\,du = dx$

$$\int \frac{\cancel{2}u^2\,du}{\cancel{2}(1+u^2)}$$

$$\int \frac{u^2}{u^2+1}\,du$$

$$= \int \frac{u^2+1}{u^2+1} - \frac{1}{u^2+1}\,du$$

$= u - \tan^{-1}(u) + c$

$= \sqrt{x} + \tan^{-1}\sqrt{x} + c$

1. $\displaystyle\int_0^1 (x-1)e^{-x}\,dx$

$$\begin{array}{ll} D & I \\ x-1 & e^{-x} \\ 1 & -e^{-x} \\ 0 & e^{-x} \end{array}$$

$$= -e^{-x}(x-1) - e^{-x} \Big|_0^1$$

$$= \left[-e^{-1}(1-1) - e^{-1}\right] - \left[-e^{0}(0-1) - e^{0}\right]$$

$$= \left[-e^{-1}\right] - \left[1-1\right]$$

$$= -e^{-1}$$

$$= -0.368$$

1.b) $\int x^{11} \ln x \, dx$

$u = \ln x \qquad u' = 1/x$

$v' = x^{11} \qquad v = \dfrac{x^{12}}{12}$

$$= \frac{x^{12} \ln x}{12} - \int \frac{x^{11}}{12} \, dx$$

$$= \frac{x^{12} \ln x}{12} - \frac{1}{12}\left[\frac{x^{12}}{12}\right] + C$$

$$= \frac{x^{12}}{12}\left(\ln x - \frac{1}{12}\right) + C$$

2. $y = \dfrac{2}{x} \qquad y = 3 - x$

$$\frac{2}{x} = 3 - x$$

$$2 = 3x - x^2$$

$$x^2 - 3x + 2 = 0$$

$x^2 - x - 2x + 2 = 0$

$x(x-1) - 2(x-1) = 0$

$(x-1)(x-2) = 0$

$x = 1$ or $x = 2$

$$\text{Area} = \int_1^2 \frac{2}{x} - 3 + x \, dx$$

$$= 2\ln|x| - 3x + \frac{x^2}{2} \Big|_1^2$$

$$= 0.113$$

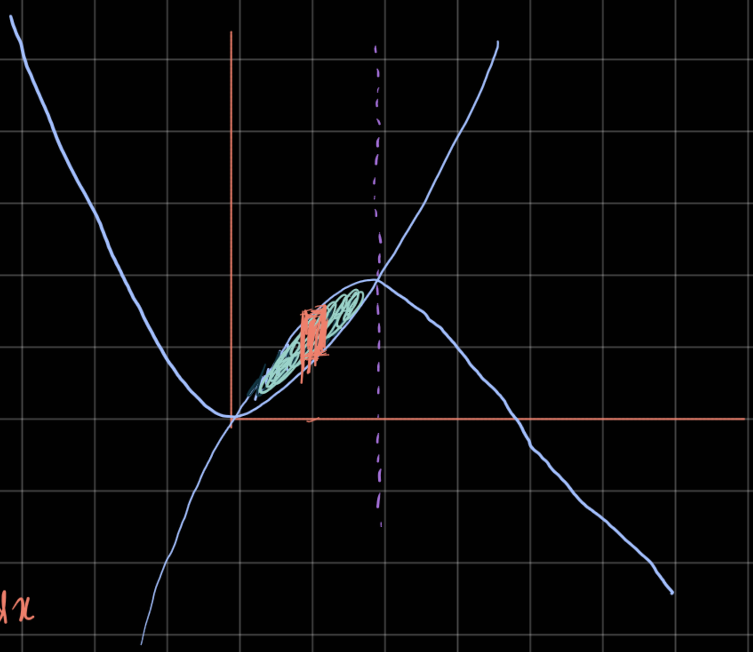
3. $x^2 = 4x - x^2$

$2x^2 = 4x$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

$x = 0$ or $x = 2$



$$V = 2\pi \int_0^2 (2-x)(4x - 2x^2) \, dx$$

$$= 2\pi \int_0^2 \left(8x - 4x^2 - 4x^2 + 2x^3\right) dx$$

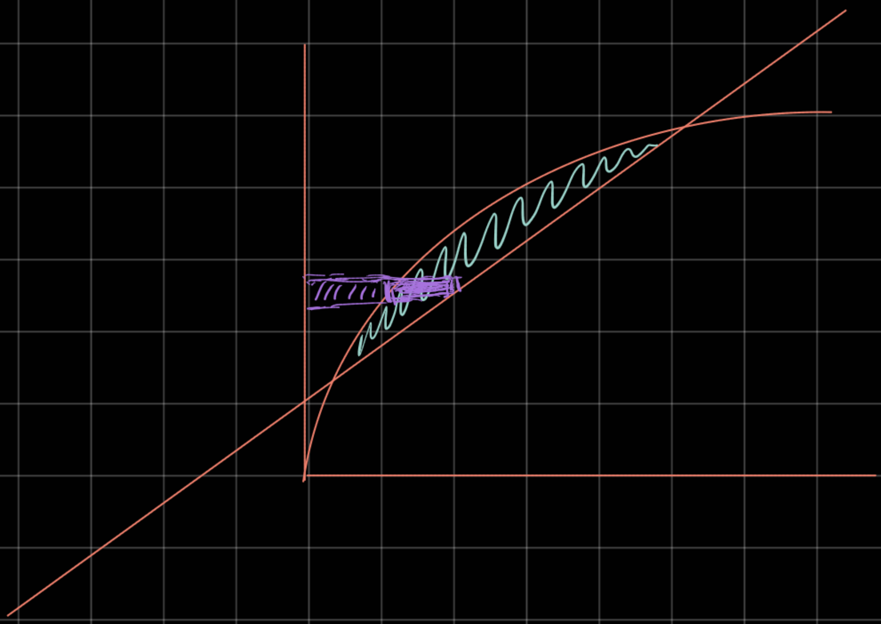
$$= 2\pi \int_0^2 \left(2x^3 - 8x^2 + 8x\right) dx$$

$$= 2\pi \left[\frac{2x^4}{4} - \frac{8x^3}{3} + \frac{8x^2}{2}\right]_0^2$$

$$= 2\pi \left[x^4 \quad 8x^3 \quad x^2 \right]^2$$

$$= 2\pi \left[\frac{x}{2} - \frac{8x}{3} + 4x \right]_0$$

$$= \frac{16\pi}{3}$$



4. $y = \sqrt{x}$ $\quad y = x - 2$

$x = y^2$ $\quad x = y + 2$

$$y^2 = y + 2$$
$$y^2 - y - 2 = 0$$
$$y^2 - 2y + y - 2 = 0$$
$$y(y-2) + 1(y-2) = 0$$
$$(y-2)(y+1) = 0$$
$$y = 2 \text{ OR } y = -1$$

$$\text{Volume} = \int_0^2 \pi (y+2)^2 - \pi (y^2)^2 \, dy$$

$$= \pi \int_0^2 y^2 + 4y + 4 - y^4 \, dy$$

$$= \pi \left[-\frac{y^5}{5} + \frac{y^3}{3} + 2y^2 + 4y \right]^2$$

$$= \frac{184}{15} \pi$$

5. $L = \int \sqrt{1 + [f'(x)]^2} \, dx$

$$f(x) = \frac{3}{2} x^{2/3}$$

$$f'(x) = \frac{3}{2} \times \frac{2}{3} x^{-1/3}$$

$$= x^{-1/3}$$

$$(f'(x))^2 = x^{-2/3}$$

$$1+(f'(x))^2 = x^{-2/3} + 1$$

6. $$\int \frac{x^2}{\sqrt{16-x^2}}\, dx$$

$$x = 4\sin\theta$$

$$dx = 4\cos\theta \, d\theta$$

$$\int \frac{16\sin^2\theta \times 4\cos\theta \, d\theta}{\sqrt{16(1-\sin^2\theta)}}$$

$$\int \frac{16\sin^2\theta \times \cancel{4\cos\theta}\, d\theta}{\cancel{4\cos\theta}}$$

$$= 16\int \sin^2\theta \, d\theta$$

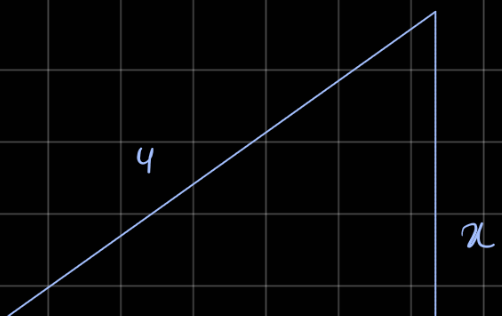
$$= 16\int \frac{1-\cos 2\theta}{2}\, d\theta$$

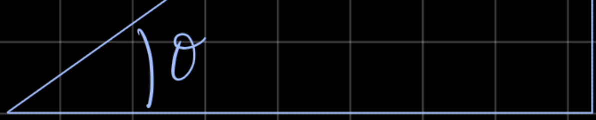
$$= 16\int \frac{1}{2} - \frac{\cos 2\theta}{2}\, d\theta$$

$$= 16\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right] + C$$

$$= 8\theta - 8\sin\theta\cos\theta$$

$$= 8\theta - \frac{1}{2}x\sqrt{16-x^2}$$





$\sqrt{16-x^2}$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C$$

$$\frac{3x-4}{x^2-2x+1} = \frac{3x-4}{(x-1)(x-1)}$$

$$\frac{3x-4}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$3x-4 = A(x-1) + B$$
$$= Ax - A + B$$

$$3x = Ax \qquad -4 = -A + B$$
$$A = 3 \qquad -4 + 3 = B$$
$$B = -1$$

$$\int \frac{3\,dx}{x-1} - \int \frac{dx}{(x-1)^2}$$

$$= 3\ln|x-1| + \frac{1}{x-1} + C$$

$$\frac{\sqrt{x}\,dx}{x-4}$$

$$u = \sqrt{x}$$

$$u^2 = x \qquad 2u\,du = dx$$

$$\int \frac{2u^2\,du}{u^2-4}$$

$$2\int u^2-4\,du \qquad 8\int du$$

$$\int \frac{}{u^2-4} + \int \frac{}{u^2-4}$$

$$\to 2u + \frac{8}{2} \ln|u^2-4|$$

$$= 2\sqrt{x} + 4\ln|x-4| + C$$

$$\int_0^\infty \frac{e^x}{1+e^x}$$

$$\lim_{t\to\infty} \int_0^t \frac{e^x}{e^x+1}\,dx$$

$$u = e^x + 1$$

$$du = e^x\,dx$$

$$= \lim_{t\to\infty} \ln|u| \Big|_0^t$$

$$= \lim_{t\to\infty} \ln|e^x+1| \Big|_0^t$$

$$= \lim_{t\to\infty} \ln|e^t+1| - \ln|2|$$

$$= \infty$$

Diverges

$$\int_0^{12} \frac{9}{\sqrt{12-x}}$$

$$\lim_{t\to 12^-} \int_0^t \frac{9}{\sqrt{12-x}}$$

t

$$\lim_{t \to 12^-} 9 \int_0^t (12-x)^{-1/2} \, dx$$

$$\lim_{t \to 12^-} \frac{9 \times 1}{2} (12-x)^{1/2} \Big|_0^t$$

$$\lim_{t \to 12^-} \frac{9}{2} (12-x)^{3/2} \Big|_0^t$$

$$= \lim_{t \to 12^-} \frac{9}{2} (12-t)^{3/2} - \frac{9}{2} (12-0)^{3/2}$$

$$-6 (12)^{3/2}$$

$$= -249.42$$

$\therefore$ Converges at -249.42