Winter 22 Final

1. $a_n = \ln\left(\frac{n+1}{n}\right)$

$\lim\limits_{n\to\infty} a_n$

$= \lim\limits_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$

$= \lim\limits_{n\to\infty} \ln\left(\frac{n}{n}\right)$

$= \lim\limits_{n\to\infty} \ln(1)$

$= 0$

$\therefore$ c) a_n converges to 0

2. $r = \frac{-1}{4} = \frac{-1/16}{1/4}$ $\qquad r = \frac{a_{n+1}}{a_n}$

$a_1 = 4$

$|r| < 1 \quad \therefore$ converges

$S = \frac{a_1}{1-r} = \frac{4}{5/4} = \frac{16}{5}$

$\therefore$ e) $\frac{16}{5}$

3. $\sum\limits_{n=1}^{\infty} \left| \frac{(-1)^n}{n\sqrt{n}} \right| = \sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$p = 3/2 > 1$

$\therefore$ converges absolutely

$\therefore$ a) converges absolutely

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n+1}$

$$\lim_{n\to\infty} \left| \frac{\cancel{2^n}\cdot 2 \cdot \cancel{x^n}\cdot x}{n+2} \times \frac{n+1}{\cancel{2^n}\,\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2x(n+1)}{n+2} \right|$$

$$= \lim_{n\to\infty} \frac{2(n+1)}{n+2}\, |x|$$

$$= |x| \lim_{n\to\infty} \frac{2(n+1)}{n+2}$$

$$= |x| \times 2$$

$$= 2|x|$$

Center = 0

Radius of convergance = $\frac{1}{2}$

$\therefore$ b) $1/2$

5. $\displaystyle\frac{1}{4-x^2}$

$$= \frac{1}{4\left(1-\frac{x^2}{4}\right)}$$

$$= \frac{1}{4\left(1-\left(\frac{x}{2}\right)^2\right)}$$

$$= \frac{1}{4} \cdot \frac{1}{1-\left(\frac{x}{2}\right)^2}$$

$$= \frac{1}{4} \sum_{n=0} \left(\frac{x}{2}\right)^{2n} \quad \text{if } \left|\frac{x^2}{4}\right| < 1 \rightarrow |x^2| < 4 \rightarrow |x| < 2$$

$$= \sum_{n=0}^{\infty} \frac{1}{4} \cdot \frac{x^{2n}}{4^n}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{4^{n+1}} \quad \text{if } |x| < 2$$

$$= \text{b)} \sum_{n=0}^{\infty} \frac{x^{2n}}{4^{n+1}} \quad \text{if } |x| < 2$$

6. $\ln(1-x) = \sum_{n=0}^{\infty} x - \frac{x^2}{2} - \frac{x^3}{3}$

a) $-1/3$

Maclaurin series

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^3(0)}{3!} x^3$$

$f(x) = \ln(1-x)$

$f'(x) = \frac{-1}{1-x} = -(1-x)^{-1}$

$f''(x) = -(1-x)^{-2}$

$f'''(x) = (-1)(-2)(-1)(1-x)^{-3}$

$= -2(1-x)^{-3}$

$f'''(0) = -2$

$\frac{-2}{3!} = \frac{-2}{6} = \frac{-1}{3}$ ✓

Written

1. $\sum_{n=1}^{\infty} \frac{x^n}{n2^n} \Rightarrow a = 0$

a)

$$\lim_{n\to\infty} \left| \frac{\cancel{x^n}\, x}{(n+1)\cancel{2^n} \cdot 2} \cdot \frac{n \cancel{2^n}}{\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \frac{n}{2n+2} |x|$$

$$= |x| \lim_{n\to\infty} \frac{n}{2n}$$

$= \frac{1}{2} |x|$

Radius $= R = \frac{1}{1/2} = 2$

Endpoints: $a - r = -2$

$a + r = 2$

@ $x = -2$

$\sum_{n=1}^{\infty} \frac{(-2)^n}{n 2^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{2^n}}{n \cancel{2^n}}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$b_n = \frac{1}{n} \Rightarrow$ positive, decreasing

and $\lim_{n\to\infty} b_n = 0$

$\therefore$ Converges by AST

@ $x = 2$

$\sum_{n=1}^{\infty} \frac{\cancel{2^n}}{n \cancel{2^n}}$

$= \sum_{n=1}^{\infty} \frac{1}{n}$

$p = 1 \leq 1$

$\therefore$ diverges by p-series test

$\therefore$ IC $= [-2, 2)$

1. b) $\sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}$ $\quad a = -2$

$\lim_{n\to\infty} \left| \frac{\cancel{(x+2)^n}(x+2)}{(n+1)\cancel{n!}} \times \frac{\cancel{n!}}{\cancel{(x+2)^n}} \right|$

$= \lim_{n\to\infty} \frac{1}{n+1} |x+2|$

$$= |x+2| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0$$

$\therefore$ Radius of convergence $= \infty$

$IC = (-\infty, \infty)$

2. $$\frac{dy}{dx} = 3x^2y^2 \qquad y(0) = \frac{1}{2}$$

$$\frac{1}{y^2}\,dy = 3x^2\,dx$$

$$\int y^{-2}\,dy = \int 3x^2\,dx$$

$$-y^{-1} = x^3 + c$$

$$-\frac{1}{y} = x^3 + c$$

$$-\frac{1}{1/2} = 0 + c$$

$$c = -2$$

$$-\frac{1}{y} = x^3 - 2$$

$$y = \frac{-1}{x^3-2}$$

3. $$(\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy = 0$$

$$M_y = -\sin y + \cos x$$

$$N_x = \cos x - \sin y$$

$$M_y = N_x$$

$\therefore$ Equation is exact

$f_x = \cos y + y \cos x$

$f_y = \sin x - x \sin y$

$f(x,y) = \int f_y \, dy + c(x)$

$= \int (\sin x - x \sin y) \, dy + c(x)$

$= y \sin x + x \cos y + c(x)$

$f_x = f_x$

$y \cancel{\cos} x + \cancel{\cos} y = \cancel{\cos} y + y \cancel{\cos} x + h'(x)$

$h'(x) = 0$

$h(x) = c$

$f(x,y) = y \sin x + x \cos y + c$

4. $x \dfrac{dy}{dx} - 2y = 4x^3 y^{1/2}$

$\dfrac{dy}{dx} - \dfrac{2}{x} y = 4x^2 y^{1/2}$

$P(x) = -\dfrac{1}{x}$

$IF = e^{\int (1-n) P(x) dx}$ $\quad \dfrac{dy}{dx} + P(x) \cdot y = Q(x) y^n$

$y^{1-n} = \dfrac{1}{[\ldots]} \left[\int ((1-n)) Q(x) \cdot IF \, dx + C \right]$

If