MATH142 Tutorial 1 (New)

1. Find the area enclosed by the graphs of $f(x) = x^2$ and $g(x) = 2 - x^2$, $0 \leq x^2 \leq 2$

$$\text{Area} = \left| \int_a^b [f(x) - g(x)]\, dx \right|$$

$$= \left| \int_0^2 [x^2 - (2 - x^2)]\, dx \right|$$

$$= \left| \int_0^2 [2x^2 - 2]\, dx \right|$$

$$= \left| \left[\frac{2x^3}{3} - 2x \right]_0^2 \right|$$

$$= \left[\frac{16}{3} - 4 \right]$$

$$= \frac{4}{3}$$

2. Find the area enclosed by the graphs of $f(x) = x^3 - 2x^2$ and $g(x) = 2x^2 - 3x$

Finding limits

$x^3 - 2x^2 = 2x^2 - 3x$

$x^3 - 4x^2 + 3x = 0$

$x(x^2 - 4x + 3) = 0$

$x(x^2 - 3x - x + 3) = 0$

$x[x(x-3) - 1(x-3)] = 0$

$x(x-3)(x-1) = 0$

$x = 0$, $x = 1$ OR $x = 3$

$\therefore a = 0$, $b = 3$

Solving integral

$$\text{Area} = \left| \int_a^b [f(x) - g(x)]\, dx \right|$$

$$= \left| \int_0^3 \left[(x^3 - 2x^2) - (2x^2 - 3x) \right] dx \right|$$

$$= \left| \int_0^3 \left[x^3 - 4x^2 + 3x \right] dx \right|$$

$$= \left| \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} \right]_0^3 \right|$$

$$= \left| \frac{81}{4} - 36 + \frac{27}{2} \right|$$

$$= \left| \frac{81 - 144 + 54}{4} \right|$$

$$= \left| -\frac{9}{4} \right|$$

$$= \frac{9}{4}$$

3. Find the area enclosed by the graphs of $x = y^2 + 2$ and $y = x - 8$

Finding limits

$x = y^2 + 2$

$y = x - 8 \Rightarrow x = y + 8$

$y^2 + 2 = y + 8$

$y^2 - y - 6 = 0$

$y^2 + 2y - 3y - 6 = 0$

$y(y+2) - 3(y+2) = 0$

$(y+2)(y-3) = 0$

$y = -2$ OR $y = 3$

$\therefore a = -2, \quad b = 3$

Solving integral

$$\text{Area} = \left| \int_a^b [f(y) - g(y)]\, dy \right|$$

$$= \left| \int_{-2}^{3} [y^2 - y - 6]\, dy \right|$$

$$= \left| \left[\frac{y^3}{3} - \frac{y^2}{2} - 6y \right]_{-2}^{3} \right|$$

$$= \left| \left[9 - \frac{9}{2} - 18 \right] - \left[\frac{-8}{3} - 2 + 12 \right] \right|$$

$$= \left| 9 - \frac{9}{2} - 18 + \frac{8}{3} + 2 - 12 \right|$$

$$= \left| \frac{54}{6} - \frac{27}{6} - \frac{108}{6} + \frac{16}{6} + \frac{12}{6} - \frac{72}{6} \right|$$

$$= \left| \frac{-125}{6} \right|$$

$$= \frac{125}{6}$$

4. Electrical wires suspended between two towers form a catenary (see figure) modeled by the equation $y = 10\left(e^{\frac{x}{20}} + e^{\frac{-x}{20}}\right)$, $-20 \leq x \leq 20$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$f(x) = 10\left(e^{\frac{x}{20}} + e^{\frac{-x}{20}}\right)$$

$$f'(x) = 10\left(\frac{e^{\frac{x}{20}}}{20} - \frac{e^{\frac{-x}{20}}}{20}\right)$$

$$= \frac{1}{2}\left(e^{\frac{x}{20}} - e^{\frac{-x}{20}}\right)$$

$$[f'(x)]^2 = \frac{1}{4}\left(e^{\frac{x}{10}} + e^{\frac{-x}{10}} - 2\right)$$

$$= \frac{e^{\frac{x}{10}}}{4} + \frac{e^{\frac{-x}{10}}}{4} - \frac{1}{2}$$

$$1 + [f'(x)]^2 = \frac{e^{\frac{x}{10}}}{4} + \frac{e^{\frac{-x}{10}}}{4} + \frac{1}{2}$$

$$= \frac{1}{4}\left(e^{\frac{x}{10}} + e^{\frac{-x}{10}} + 2\right)$$

$$= \left[\frac{1}{2}\left(e^{\frac{x}{10}} + e^{\frac{-x}{10}}\right)\right]^2$$

$$\sqrt{1 + [f'(x)]^2} = \frac{1}{2}\left(e^{\frac{x}{10}} + e^{\frac{-x}{10}}\right)$$

$$L = \frac{1}{2}\int_{-20}^{20} \left(e^{\frac{x}{10}} + e^{\frac{-x}{10}}\right) dx$$

$$10\int e^{u}\, du$$

$$u = \frac{x}{10}$$

$$= \frac{1}{2}\left[10 e^{\frac{x}{10}} - 10 e^{\frac{-x}{10}}\right]_{-20}^{20}$$

$$= 5\left[\left(e^{2} - e^{-2}\right) - \left(e^{-2} - e^{2}\right)\right]$$

$$= 10\left(e^{2} - e^{-2}\right)$$

$$= 10\left(e^{2} - \frac{1}{e^{2}}\right)$$

$$\approx 72.537 \text{ m}$$

5. Find the arc length of the graph of $f(x) = \dfrac{x^6 + 8}{16x^2}$ on the interval $[2,3]$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$f(x) = \frac{x^4}{16} + \frac{x^{-2}}{2}$$

$$f'(x) = \frac{4x^3}{16} - \frac{2x^{-3}}{2}$$

$$= \frac{x^3}{4} - \frac{1}{x^3}$$

$$= \frac{x^6 - 4}{4x^3}$$

$$[f'(x)]^2 = \left[\frac{1}{4x^3}(x^6 - 4)\right]^2$$

$$= \frac{1}{16x^6}\left(x^{12} + 16 - 8x^6\right)$$

$$= \frac{x^6}{16} + \frac{1}{x^6} - \frac{1}{2}$$

$$1 + [f'(x)]^2 = \frac{x^6}{16} + \frac{1}{x^6} + \frac{1}{2}$$

$$= \frac{x^{12} + 16}{16x^6} + \frac{1}{2}$$

$$= \frac{x^{12} + 16 + 8x^6}{16x^6}$$

$$= \frac{(x^6 + 4)^2}{(4x^3)^2}$$

$$= \left[\frac{x^6 + 4}{4x^3}\right]^2$$

$$\sqrt{1+[f'(x)]^2} = \frac{x^6+4}{4x^3}$$

$$L = \frac{1}{4}\int_2^3 \left(x^3 + 4x^{-3}\right) dx$$

$$= \frac{1}{4}\left[\frac{x^4}{4} - 2x^{-2}\right]_2^3$$

$$= \frac{1}{4}\left[\frac{81}{4} - \frac{2}{9} - 4 + \frac{1}{2}\right]$$

$$= \frac{595}{144}$$

$$\approx 4.132$$