Tutorial 3

Question 1

Use partial fraction decomposition to evaluate

$$\int \frac{3x-4}{x^2-2x+1}dx$$

Solution

$$\frac{3x-4}{x^2-2x+1} = \frac{3x-4}{(x-1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}$$

$$A_1 = 3 \text{ and } A_2 = -1$$

$$\begin{aligned}\int \frac{3x-4}{x^2-2x+1}dx &= \int \left(\frac{3}{x-1} - \frac{1}{(x-1)^2}\right) dx \\ &= 3\ln|x-1| + \frac{1}{x-1} + C\end{aligned}$$

Question 2

Evaluate the following integrals

1. $\int \frac{\sqrt{x}dx}{x-4}$,
2. $\int \frac{x^3+2x^2-4}{x^2-x}dx$

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Solution

1. $\displaystyle\int \frac{\sqrt{x}dx}{x-4}$?

$$\text{Put } u=\sqrt{x}, \quad u^2 = x \Rightarrow dx = 2udu \ \text{ and } x = u^2.$$

The integral becomes

$$\begin{aligned}
\int \frac{\sqrt{x}dx}{x-4} &= \int \frac{2u^2du}{u^2-4} \\
&= 2\int \frac{u^2du}{u^2-4} = 2\int \frac{(u^2-4+4)\,du}{u^2-4} \\
&= 2\int \left(1+\frac{4}{u^2-4}\right) du
\end{aligned}$$

$$\frac{4}{u^2-4} = \frac{4}{(u-2)(u+2)} = \frac{1}{u-2} - \frac{1}{u+2}$$

$$\begin{aligned}
\int \frac{\sqrt{x}dx}{x-4} &= 2\int \left(1+\frac{4}{u^2-4}\right) du \\
&= 2\int \left(1+\frac{1}{u-2}-\frac{1}{u+2}\right) du \\
&= 2\left[u + \ln|u-2| - \ln|u+2|\right] + C \\
&= 2\left[\sqrt{x} + \ln\left|\sqrt{x}-2\right| - \ln\left|\sqrt{x}+2\right|\right] + C
\end{aligned}$$

2. $\displaystyle\int \frac{x^3+2x^2-4}{x^2-x}dx$?

$$\frac{x^3+2x^2-4}{x^2-x} = x+3+\frac{3x-4}{x^2-x}$$

$$\frac{3x-4}{x^2-x} = \frac{3x-4}{x(x-1)} = \frac{4}{x} - \frac{1}{x-1}.$$

Thus,

$$\begin{aligned}
\int \frac{x^3+2x^2-4}{x^2-x}dx &= \int \left(x+3+\frac{4}{x}-\frac{1}{x-1}\right) dx \\
&= \frac{x^2}{2} + 3x + 4\ln|x| - \ln|x-1| + C
\end{aligned}$$

Question 3

Determine the convergence or divergence the following improper integrals. If the integral is convergent, then find its value.

1. $\displaystyle\int_0^{12} \frac{9}{\sqrt{12-x}}dx,$ 2. $\displaystyle\int_0^{\infty} \frac{e^x}{1+e^x}dx$

Solution

1. $\int_0^{12} \frac{9}{\sqrt{12-x}}dx?$

$$\begin{aligned}
\int_0^{12} \frac{9}{\sqrt{12-x}}dx &= \lim_{t\to 12^-} \int_0^{t} \frac{9}{\sqrt{12-x}}dx \\
&= \lim_{t\to 12} \left(36\sqrt{3} - 18\sqrt{12-t}\right) \\
&= 36\sqrt{3} = 62.354.
\end{aligned}$$

The improper integral converges to $36\sqrt{3} = 62.354$

2. $\int_0^{\infty} \frac{e^x}{1+e^x}dx?$

$$\begin{aligned}
\int_0^{\infty} \frac{e^x}{1+e^x}dx &= \lim_{t\to\infty} \int_0^{t} \frac{e^x}{1+e^x}dx \\
&= \lim_{t\to\infty} \ln\left(1+e^x\right)\Big|_0^t \\
&= \lim_{t\to\infty} \left[1+e^t - \ln 2\right] = \infty.
\end{aligned}$$

The improper integral diverges.

Question 4

Use partial fraction decomposition to evaluate

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$$

Solution

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x\left(x+2\right)\left(2x-1\right)}$$

Next, you do the partial fraction decomposition

$$\frac{x^2+2x-1}{x\left(x+2\right)\left(2x-1\right)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}$$

After solving, you get

$$A = \frac{1}{2}, \qquad B = \frac{-1}{10} \text{ and } C = \frac{1}{5}$$

$$\frac{x^2+2x-1}{x\left(x+2\right)\left(2x-1\right)} = \frac{1}{2x} - \frac{1}{10\left(x+2\right)} + \frac{1}{5\left(2x-1\right)}$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \frac{1}{2}\ln|x| - \frac{1}{10}\ln|x+2| + \frac{1}{10}\ln|2x+1| + C$$

Question 5

Evaluate the following integrals

1. $\displaystyle\int \frac{dx}{2\sqrt{x}+2x}$,

2. $\displaystyle\int \frac{3x^3-3x^2+4}{x^2-x}dx$

Solution

1.

$$\int \frac{dx}{2\sqrt{x}+2x}.$$

We rationalize by putting

$$u=\sqrt{x}$$

$$x=u^2 \text{ and } dx=2udu$$

The integral becomes

$$\begin{aligned}\int \frac{dx}{2\sqrt{x}+2x} &= \int \frac{2udu}{2u+2u^2}\\ &= \int \frac{\cancel{2u}du}{\cancel{2u}(1+u)}\\ &= \int \frac{du}{1+u} = \ln|1+u|+C\\ &= \ln\left|1+\sqrt{x}\right|+C\end{aligned}$$

2.

$$\int \frac{3x^3-3x^2+4}{x^2-x}dx$$

Since the degree of the numerator is higher, we first perform a long division.

$$\frac{3x^3-3x^2+4}{x^2-x} = 3x - \frac{4}{x-x^2}.$$

Next, we do a partial fraction decomposition on $\frac{4}{x-x^2}$.

$$\frac{4}{x-x^2} = \frac{4}{-x(x-1)} = \frac{4}{x} - \frac{4}{x-1}$$

$$\begin{aligned}\int \frac{3x^3-3x^2+4}{x^2-x}dx &= \int \left[3x - \left(\frac{4}{x} - \frac{4}{x-1}\right)\right] dx\\ &= \int \left(3x - \frac{4}{x} + \frac{4}{x-1}\right) dx\\ &= \frac{3}{2}x^2 - 4\ln|x| + 4\ln|x-1| + C\end{aligned}$$

Question 6

Determine the convergence or divergence the following improper integrals. If the integral is convergent, then find its value.

1. $\displaystyle\int_0^\infty \frac{x}{1+x^2}dx,$ 2. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

Solution

1.

$$\begin{aligned}
\int_0^\infty \frac{x}{1+x^2}dx &= \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}dx \\
&= \lim_{t\to\infty}\frac{1}{2}\int_0^t \frac{2x}{1+x^2}dx \\
&= \lim_{t\to\infty}\frac{1}{2}\ln\left(1+x^2\right)\Big|_0^t \\
&= \lim_{t\to\infty}\frac{1}{2}\ln\left(1+t^2\right) \\
&= \infty \qquad \text{diverges}
\end{aligned}$$

2.

$$\begin{aligned}
\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} &= \lim_{t\to -2^-}\int_t^{14} \frac{dx}{\sqrt[4]{x+2}} \\
&= \lim_{t\to -2^-}\int_t^{14} (x+2)^{-1/4}\,dx \\
&= \lim_{t\to -2^-}\frac{4}{3}(x+2)^{\frac{3}{4}}\Big|_t^{14} \\
&= \lim_{t\to -2^-}\left[\frac{4}{3}(14+2)^{\frac{3}{4}}-\frac{4}{3}(t+2)^{\frac{3}{4}}\right] \\
&= \frac{32}{3} = 10.667.
\end{aligned}$$

Converges and its value is 10.667

Question 7

Evaluate the integral

$$\int \frac{x-3}{x^3+3x}dx$$

Solution

$$\frac{x-3}{x^3+3x} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

Clear the denominators:

$$x-3 = A(x^2+3)+(Bx+C)x$$

Combine like terms:

$$x-3 = (A+B)x^2+Cx+3A$$

Then we get the system of equations: $\begin{cases} A+B=0 \\ C=1 \\ 3A=-3 \end{cases}$

Solving this system we see that $A=-1, B=1, C=1$. Then

$$\frac{x-3}{x(x^2+3)} = -\frac{1}{x}+\frac{x+1}{x^2+3}$$

$$\int \frac{x-3}{x^3+3x}dx = -\ln|x| + \frac{1}{2}\ln\left(x^2+3\right) + \frac{1}{3}\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

Question 8

Evaluate the integral

$$\int \frac{x^4-4x^2+x+1}{x^2-4}dx$$

Solution

We will use partial fraction decomposition.

Since the degree of the numerator is greater than the denominator, we will start with a long division.

$$\frac{x^4-4x^2+x+1}{x^2-4}=x^2+\frac{x+1}{x^2-4}.$$

Next, we perform a partial fraction decomposition on $\frac{x+1}{x^2-4}$.

$$\begin{aligned}\frac{x+1}{x^2-4} &= \frac{x+1}{(x-2)(x+2)}\\ &= \frac{3}{4(x-2)}+\frac{1}{4(x+2)}.\end{aligned}$$

Thus,

$$\begin{aligned}\int \frac{x^4-4x^2+x+1}{x^2-4}dx &= \int \left[x^2+\frac{3}{4(x-2)}+\frac{1}{4(x+2)}\right] dx\\ &= \frac{1}{3}x^3+\frac{3}{4}\ln|x-2|+\frac{1}{4}\ln|x+2|+C\ .\end{aligned}$$

Question 9

Determine convergence or divergence of the following improper intergals

$$(1)\ \int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx \qquad (2)\ \int_1^9 \frac{dx}{(x-1)^{2/3}}$$

Solution

(1)

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = ?$$

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{t\to\infty} \int_1^{t} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$$

Put

$$u = \sqrt{x}, \qquad u^2 = x \text{ and } \ dx = 2udu$$

$$\begin{aligned} \int_1^{t} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx &= 2\int_1^{\sqrt{t}} \frac{\cancel{u}e^{-u}}{\cancel{u}} du \\ &= 2\int_1^{\sqrt{t}} e^{-u} du \\ &= 2 - e^{-u}\Big|_1^{\sqrt{t}} \\ &= 2\left(e - e^{-\sqrt{t}}\right) \end{aligned}$$

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{t\to\infty} \left[2\left(e^{-1} - e^{-\sqrt{t}}\right)\right] = \frac{2}{e}. \quad \text{Converges.}$$

(2)

$$\int_1^{9} \frac{dx}{(x-1)^{2/3}} = ?$$

$$\begin{aligned} \int_1^{9} \frac{dx}{(x-1)^{2/3}} &= \lim_{t\to 1^+} \int_t^{9} \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{t\to 1^+} \int_t^{9} (x-1)^{-2/3} dx \\ &= \lim_{t\to 1^+} \left.\frac{(x-1)^{\frac{-2}{3}+1}}{\frac{-2}{3}+1}\right|_t^9 \\ &= \lim_{t\to 1^+} 3(x-1)^{\frac{1}{3}}\Big|_t^9 \\ &= \lim_{t\to 1^+} \left[3\left(2 - \sqrt[3]{1-t}\right)\right] \\ &= 6. \quad \text{Converges.} \end{aligned}$$