Tutorial 8

Q1.

(a). Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first ten terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the value of the sum is accurate within 0.0005.

Solution

Note that the function $f(x) = \frac{1}{x^3}$ does satisfy the conditions of the integral test (you must verify the three conditions by yourself!). By the integral test itself we know that this series converges (it is a p-series with p = 3). Recall that if S is the sum of the series (its true value!), then for any n we have

$$R_n = S - S_n$$

where the remainder R_n can be estimated by using the formula

$$\int_{n+1}^{\infty} \frac{dx}{x^3} \leq R_n \leq \int_{n}^{\infty} \frac{dx}{x^3}$$

(a) Using $n = 10$,we have

$$\int_{11}^{\infty} \frac{dx}{x^3} \leq S - S_{10} \leq \int_{10}^{\infty} \frac{dx}{x^3}$$

$\Rightarrow$

$$S_{10} + \int_{11}^{\infty} \frac{dx}{x^3} \leq S \leq S_{10} + \int_{10}^{\infty} \frac{dx}{x^3}$$

$$S_{10} = \sum_{k=1}^{10} \frac{1}{k^3} = 1 + \frac{1}{2^2} + ... + \frac{1}{10^3} = 1.1975$$

$$\int_{10}^{\infty} \frac{dx}{x^3} = 0.005, \quad \text{and} \quad \int_{11}^{\infty} \frac{dx}{x^3} = 4.1322 \times 10^{-3}.$$

We have

$$1.1975 + 4.1322 \times 10^{-3} \leq S \leq 1.1975 + 0.005$$

$$1.2016 \leq S \leq 1.2025$$

(b) To ensure that the value of the sum is accurate to within 0.0005 we must find a positive integer n such that $R_n \leq 0.0005$. Since

$$R_n \leq \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2},$$

it suffices to find a positive integer n satisfying

$$\frac{1}{2n^2} < 0.0005$$

Solving this inequality we get

$$n^2 > \frac{1}{0.001} = 1000 \qquad \text{or} \qquad n > \sqrt{1000} \approx 31.6$$

Thus, if we take $n = 32$ terms, it is certain that we have the desired accuracy.

Q2. Determine whether the series converges absolutely or conditionally.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$

2. $\sum_{n=2}^{\infty}\frac{(-1)^n}{nlnn}$

Solution:

1. $\sum_{n=1}^{\infty}\frac{n+3}{n(n+1)}$ diverges by the limit comparison test and $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$ converges by the alternating series. Hence, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$ converges conditionally.

2. $\sum_{n=2}^{\infty}\frac{1}{nlnn}$ diverges by the integral test and $\sum_{n=2}^{\infty}\frac{(-1)^n}{nlnn}$ converges by the alternating series test. Hence, $\sum_{n=2}^{\infty}\frac{(-1)^n}{nlnn}$ converges conditionally.

Q3. Determine whether the series converges or diverges. Justify your answer by citing a relevant test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$ 2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{13n+1}\right)^n$ 3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+20}$

Solution

1. Put

$$a_n = \frac{e^n}{n!}.$$

Using the ratio test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{e^n e}{(n+1)\,n!}\cdot\frac{n!}{e^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{e}{(n+1)}\right| = 0 < 1.\end{aligned}$$

We conclude that the series converges.

2. Put

$$a_n = \left(\frac{2n}{13n+1}\right)^n.$$

Using the root test, we have

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{2n}{13n+1} = \frac{2}{13} < 1.$$

We conclude that the series converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+20}$ is of the form $\displaystyle\sum_{n=1}^{\infty} (-1)^n b_n$ where $b_n = \dfrac{1}{n+20}$ is positive, decreasing and $\lim_{n\to\infty} \dfrac{1}{n+20} = 0$.We conclude that the series converges by the alternating series test.

Q4. Find the interval of convergence for

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}(x+1)^n$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

1.

Solution Letting $u_n = (-1)^n(x+1)^n/2^n$ produces

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}}}{\dfrac{(-1)^n(x+1)^n}{2^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2^n(x+1)}{2^{n+1}}\right|$$

$$= \left|\frac{x+1}{2}\right|.$$

By the Ratio Test, the series converges if $|(x+1)/2| < 1$ or $|x+1| < 2$. So, the radius of convergence is $R = 2$. Because the series is centered at $x = -1$, it will converge in the interval $(-3, 1)$. Furthermore, at the endpoints you have

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$$ Diverges when $x = -3$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$$ Diverges when $x = 1$

both of which diverge. So, the interval of convergence is $(-3,\ 1)$.

2.

Solution Let $u_n = (-1)^n x^{2n+1}/(2n+1)!$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}}{\dfrac{(-1)^n x^{2n+1}}{(2n-1)!}}\right|$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)}.$$

For any value of this limit is 0. So, the radius of convergence is ∞ and the interval of convergence is $(-\infty\ ,\ \infty)$.

Q5. Find the interval of convergence for

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}(x-1)^n$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n!}$

Solution

1. This is a power series with $a = 1$.

$$a_n = \frac{n}{3^n}(x-1)^n$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)}{3^n 3}(x-1)(x-1)^n \frac{3^n}{n(x-1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)}{3}(x-1)\frac{1}{n}\right| = \frac{1}{3}|x-1|\end{aligned}$$

$$R = 3$$

$$a - R = 1 - 3 = -2 \quad \text{and} \quad a + R = 4.$$

When $x = -2$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n}{3^n}(x-1)^n &= \sum_{n=1}^{\infty}\frac{n(-1)^n}{3^n}3^n \\ &= \sum_{n=1}^{\infty} n(-1)^n \qquad \text{diverges.}\end{aligned}$$

When $x = 4$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n}{3^n}(x-1)^n &= \sum_{n=1}^{\infty}\frac{n}{3^n}3^n \\ &= \sum_{n=1}^{\infty} n \qquad \text{diverges.}\end{aligned}$$

The interval of convergence is

$$(-2,\ 4).$$

2.

$$a_n = \sum_{n=1}^{\infty}\frac{x^n}{2n!}$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^n x}{2(n+1)!}\cdot\frac{2n!}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^n x}{2n!(n+1)}\cdot\frac{2n!}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{(n+1)}\right| = 0\end{aligned}$$

$R = \infty$ and the interval of convergence is $(-\infty,\ \infty)$

Q6. Find the power series representation of $f(x) = \dfrac{x^2}{4+x^3}$ and the corresponding interval of convergence.

Solution

$$\begin{aligned}
\frac{1}{4+x^3} &= \frac{1}{4}\left(\frac{1}{1+\frac{x^3}{4}}\right) \\
&= \frac{1}{4}\left(\frac{1}{1-(-\frac{x^3}{4})}\right) \\
&= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x^3}{4}\right)^n \quad \text{when} \quad \left|\left(-\frac{x^3}{4}\right)\right| < 1 \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{3n} \quad \text{when} \quad |x^3| < 4
\end{aligned}$$

$$f(x) = \frac{x^2}{4+x^3} = x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{3n} \quad \text{when} \quad |x| < \sqrt[3]{4}$$

$$f(x) = \frac{x^2}{4+x^3} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{3n+2} \quad \text{when} \quad |x| < \sqrt[3]{4}$$

Q7. A) Find the Maclaurin series of

$$f(x) = \frac{x^2}{3-x}. \qquad \text{(Show your work)}$$

Solution

$$\begin{aligned} \frac{1}{3-x} &= \frac{1}{3\left(1-\frac{x}{3}\right)} \\ &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \quad \text{if} \quad \left|\frac{x}{3}\right| < 1. \\ &= \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}} \text{ if } \quad |x| < 3. \end{aligned}$$

$$f(x) = \frac{x^2}{3-x} = \sum_{n=0}^{\infty}\frac{x^{n+2}}{3^{n+1}} \text{ if } \quad |x| < 3$$

B) Find the sum of the series

$$\frac{\pi}{2}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n+1)!}. \qquad \text{(Show your work)}$$

Solution

We know that

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x.$$

$$\begin{aligned} \frac{\pi}{2}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n+1)!} &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!} \\ &= \frac{1}{2}\sin\pi = 0 \end{aligned}$$

Q8. Find the Maclaurin series of $f(x) = x\cos(x^3)$.

Solution

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x.$$

$$\begin{aligned} x\cos(x^3) &= x\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!} \end{aligned}$$

Q 9. Find a power series representation for the following functions. (Your answer should include the interval of convergence)

1. $g(x) = \cos\sqrt{x}$ 2. $h(x) = \tan^{-1}\left(\frac{x}{5}\right)$

3. $g(x) = e^{\sqrt{\frac{x}{2}}}$

Solution

1. We know that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x.$$

$$\cos\sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} \quad \text{with interval of convergence } (-\infty,\ \infty).$$

2. We know that

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| \le 1.$$

Hence

$$\tan^{-1}\left(\frac{x}{5}\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{5}\right)^{2n+1}}{2n+1} \quad \text{for } \left|\frac{x}{5}\right| \le 1.$$

$$\tan^{-1}\left(\frac{x}{5}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\,5^{2n+1}} \quad \text{for } |x| \le 5.$$

$$\tan^{-1}\left(\frac{x}{5}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\,5^{2n+1}} \quad \text{with interval of convergence } [-5,\ \ 5].$$

Here we use the formulas

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x.$$

Thus

$$e^{\sqrt{\frac{x}{2}}} = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n/2}}{n!} \quad \text{for all } x. \text{ Interval of convergence is } (-\infty,\ \infty).$$