Part 1 MCQ (50%)

Directions: Circle the letter that corresponds to the correct answer. There is only one correct answer for each question. You do not need to show your work.

$(5pts)$**Problem 1**

Let a_n be the sequence defined by

$$\begin{cases} a_1 = \dfrac{2}{3} \\ a_{n+1} = \dfrac{a_n + 1}{2a_n + 1} \end{cases}$$

$a_4 \quad =$

$(a)\ \frac{29}{41} \qquad (b)\ \frac{3}{7} \qquad (c)\ \frac{21}{61} \qquad (d)\ \frac{1}{2} \qquad (e)$ Does not exist

$(5pts)$**Problem 2**

$$\lim_{n\to\infty} \left(4 - 2\frac{n!}{n^n}\right)\left(\frac{3n+1}{4n}\right) \quad =$$

$(a)\ 3 \qquad (b)\ \frac{3}{4} \qquad (c)\ -\frac{1}{8} \qquad (d)\ \frac{3}{16} \qquad (e)\ \frac{1}{4}$

$(5pts)$**Problem 3**

Which of the following is TRUE about the series $\displaystyle\sum_{n=1}^{\infty} \left(\tfrac{-3n}{n+1}\right)^{2n}$

(a) It is divergent

(b) It is absolutely convergent

(c) It is conditionally convergent

(d) It is a geometric series

(e) It is a divergent geometric series

$(5pts)$**Problem 4**

The series $\sum_{n=1}^{\infty} 2^{n+1} \cdot 3^{1-2n}$ is

(a) convergent and its sum is $\frac{12}{7}$

(b) convergent and its sum is 6

(c) convergent and its sum is $\frac{4}{9}$

(d) convergent and its sum is $\frac{3}{8}$

(e) divergent

$(5pts)$**Problem 5**
If

$$a_1 + a_2 + a_3 + ... + a_N = \pi - 4e^{-N},$$

then $\sum_{n=1}^{\infty} a_n =$

$(a)\ \pi \qquad (b)\ \pi - 4 \qquad (c)\ 4 \qquad (d)\ \pi - 1 \qquad (e)\ \ 4e$

$(5pts)$**Problem 6**

Let S and S_N be respectively the sum and the $N-th$ partial sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{n^2}$. The smallest number of terms N such that $|S - S_N| < 0.01$ is equal to

(a) 20

(b) 25

(c) 30

(d) 50

(e) 15

$(5pts)$**Problem 7**

The sum of the series $\sum_{n=1}^{\infty} (-1)^n \ \frac{\pi^{2n}}{n!}$ is

(a) $\frac{1}{e^{\pi^2}}$

(b) $e^{\sqrt{\pi}}$

(c) $e^{2\pi}$

(d) $-\pi^2$

(e) ∞

$(5pts)$**Problem 8**

The series $\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n}\right)$ is

(a) convergent and its sum is $\frac{-3}{2}$

(b) convergent and its sum is $\frac{1}{3}$

(c) convergent and its sum is -1

(d) convergent and its sum is $\frac{1}{2}$

(e) divergent

$(5pts)$**Problem 9** The series $\sum_{n=0}^{\infty}\frac{e^n}{n!}$

(a) converges by the ratio test

(b) diverges by the ratio test

(c) diverges by the limit comparison test

(d) diverges by the divergence test

(e) diverges by the integral test

$(5pts)$**Problem 10**
Use the basic Maclaurin series formulas to evaluate the improper integral

$$\int_0^{\infty}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}dx$$

(a) 1

(b) -1

(c) 0

(d) π

(e) diverges

Part 2 Written Questions (50%)

$(10pts)$**Problem 1**
Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-3)^n (x-1)^n}{\sqrt{n+1}}$$

Solution
Center c = 1

$$\begin{aligned} & \lim_{n\to\infty} \left| \frac{(-3)^{n+1}(x-1)^{n+1}}{\sqrt{n+2}} \frac{\sqrt{n+1}}{(-3)^n (x-1)^n} \right| \\ = \quad & 3\,|x-1| \end{aligned}$$

$$R = \frac{1}{3} \qquad (\mathbf{3pts})$$

$$c - R = 1 - \frac{1}{3} = \frac{2}{3}, \qquad c + R = 1 + \frac{1}{3} = \frac{4}{3} \qquad (\mathbf{2pts})$$

- When $x = \frac{2}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{2}{3} - 1\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \quad \text{diverges} \qquad (\mathbf{2pts})$$

- When $x = \frac{2}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{4}{3} - 1\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \quad \text{converges by the alternating series test} \qquad (\mathbf{2pts})$$

$$\text{Interval of convergence } = (\frac{2}{3}, \frac{4}{3}] \qquad (\mathbf{1pt})$$

$(10pts)$**Problem 2**
Find the Maclaurin series of $f(x) = x\cos(x^3)$.

Solution

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x. \quad (\mathbf{3pts})$$

$$\begin{aligned} x\cos\left(x^3\right) &= x\sum_{n=0}^{\infty} \frac{(-1)^n \left(x^3\right)^{2n}}{(2n)!} && (\mathbf{4pts}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!} && (\mathbf{3pts}) \end{aligned}$$

$(10pts)$**Problem 3**

Find the power series representation of $f(x) = \dfrac{x^2}{4+x^3}$ and the corresponding interval of convergence.

Solution

$$
\begin{aligned}
\frac{1}{4+x^3} &= \frac{1}{4}\left(\frac{1}{1+\frac{x^3}{4}}\right) \\
&= \frac{1}{4}\left(\frac{1}{1-(-\frac{x^3}{4})}\right) \\
&= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x^3}{4}\right)^n \quad \text{when} \quad \left|\left(-\frac{x^3}{4}\right)\right| < 1 \quad (\textbf{5pts}) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{3n} \quad \text{when} \quad \left|x^3\right| < 4
\end{aligned}
$$

$$
f(x) = \frac{x^2}{4+x^3} = x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{3n} \quad \text{when} \quad |x| < \sqrt[3]{4}
$$

$$
f(x) = \frac{x^2}{4+x^3} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{3n+2} \quad \text{when} \quad |x| < \sqrt[3]{4} \qquad ((\textbf{3pts}) + (\textbf{2pts}))
$$

$(10pts)$**Problem 4**
Find the sum of the first three terms of the Taylor series of $f(x) = \ln(1 + x^2)$ centered at $c = 2$.

Solution
The Taylor series of $f(x)$ centered at 2 is given by

$$f(2) + \frac{f\ '(2)}{1!}(x-2) + \frac{f\ ''(2)}{2!}(x-2)^2 + ... \quad (\mathbf{1pt})$$

$$f(x) = \ln(1+x^2), \quad f(2) = \ln 5$$
$$f\ '(x) = \frac{2x}{1+x^2}, \quad \frac{f\ '(2)}{1!} = \frac{4}{5}$$

$$\begin{aligned} f\ ''(x) &= \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} \\ &= \frac{2-2x^2}{(1+x^2)^2}, \quad \frac{f\ ''(2)}{2!} = \frac{1}{2}\frac{-6}{25} = \frac{-3}{25} \end{aligned}$$

The sum of the first three terms of the Taylor series of $f(x) = \ln(1+x^2)$ centered at $c = 2$ is

$$\ln 5 + \frac{4}{5}(x-2) - \frac{3}{25}(x-2)^2 \qquad ((\mathbf{3pts}) + (\mathbf{3pts}) + (\mathbf{3pts}))$$

($10pts$)**Problem 5**
Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Solution
Center $c = 0$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(n+1)!} \frac{n!}{x^{2n}} \right| &= \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(n+1)\,n!} \frac{n!}{x^{2n}} \right| \quad (\textbf{3pts}) \\ &= \lim_{n\to\infty} \left| \frac{x^2}{(n+1)} \right| = 0 \quad (\textbf{2pts}) \end{aligned}$$

$$R = \infty \qquad (\textbf{5pts})$$