Tutorial 4

Question 1

Find the equation of the tangent line to the curve $x = 3e^{t}, y = 5e^{-t}$ at $t = 0$.

Solution

$$\text{Slope } m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-5}{3}e^{-2t}. \text{ At } t = 0, \quad m = \frac{-5}{3}, \; x = 3 \text{ and } y = 5.$$

The equation is $y = \frac{-5}{3}(x-3) + 5 \Leftrightarrow y = 10 - \frac{5}{3}x \Leftrightarrow 5x + 3y = 30$

Question 2

Find the graph of the polar equation $r = 4cos\theta - 5sin\theta$.

Solution

$$\begin{aligned} r &= 4\cos\theta - 5\sin\theta \Leftrightarrow r^2 = 4r\cos\theta - 5r\sin\theta \\ &\Leftrightarrow x^2 + y^2 = 4x - 5y \Leftrightarrow x^2 - 4x + y^2 + 5y = 0 \\ &\Leftrightarrow (x-2)^2 - 4 + \left(y + \frac{5}{2}\right)^2 - \frac{25}{4} = 0 \\ &\Leftrightarrow (x-2)^2 + \left(y + \frac{5}{2}\right)^2 = 4 + \frac{25}{4} = \frac{41}{4} \end{aligned}$$

A circle centered at $\left(2, -\frac{5}{2}\right)$ with radius $\frac{\sqrt{41}}{2}$

Question 3

Find the area of the surface generated by revolving the parametric curve

$$x = \frac{1}{2}t^2 \quad \text{and} \quad y = \frac{1}{3}(2t+1)^{3/2}, \qquad 0 \le t \le 1.$$

about the y-axis.

Solution

$$\text{Area } = 2\pi \int_0^1 \frac{1}{2}t^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\left(\frac{dx}{dt}\right)^2 = t^2, \qquad \left(\frac{dy}{dt}\right)^2 = 2t+1$$

$$\begin{aligned}
\text{Area} &= 2\pi \int_0^1 \frac{1}{2}t^2 \sqrt{t^2+2t+1}dt \\
&= 2\pi \int_0^1 \frac{1}{2}t^2 \sqrt{(t+1)^2}dt \\
&= 2\pi \int_0^1 \frac{1}{2}t^2 \, (t+1)\, dt \\
&= \frac{7}{12}\pi = 1.832\,6
\end{aligned}$$

Question 4

Find the slope of the line that is tangent to the polar curve

$$r = 3\sin\theta$$

at $\theta = \frac{\pi}{2}$.

Solution
We have

$$x = 3\sin\theta\cos\theta = \frac{3}{2}\sin 2\theta \quad \text{and} \quad y = 3\sin^2\theta$$

$$\begin{aligned}
\text{Slope} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\
&= \frac{6\cos\theta\sin\theta}{3\cos 2\theta}
\end{aligned}$$

The slope at $\theta = \frac{\pi}{2}$ is

$$\text{Slope } = \frac{0}{-1} = 0.$$

Question 5

Find the area of the surface generated by revolving the parametric curve

$$x = e^t - t \quad \text{and} \quad y = 4e^{t/2}, \qquad 0 \leq t \leq 1.$$

about the x-axis.

Solution

$$S = 2\pi \int_0^1 y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \left(e^t - 1\right)^2 + \left(2e^{t/2}\right)^2 \\
&= e^{2t} - 2e^t + 4e^t + 1 \\
&= e^{2t} + 2e^t + 1 \\
&= \left(e^t + 1\right)^2
\end{aligned}$$

$$\begin{aligned}
S &= 2\pi \int_0^1 4e^{t/2}\sqrt{\left(e^t+1\right)^2} dt \\
&= 2\pi \int_0^1 4e^{t/2}\left(e^t + 1\right) dt \\
&= 8\pi \int_0^1 \left(e^{3t/2} + e^{t/2}\right) dt \\
&= 8\pi \left(2e^{\frac{1}{2}} + \frac{2}{3}e^{\frac{3}{2}} - \frac{8}{3}\right) = 90.945
\end{aligned}$$

Question 6

Find the slope and the equation of the tangent line to the graph of the polar curve

$$r = e^{2\theta}$$

at $\theta = 0$.

Solution

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$x = e^{2\theta}\cos\theta \quad \text{and} \quad y = e^{2\theta}\sin\theta$$

$$\text{Slope } = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2e^{2\theta}\sin\theta + e^{2\theta}\cos\theta}{2e^{2\theta}\cos\theta - e^{2\theta}\sin\theta}$$

At $\theta = 0$, the slope is

$$m = \frac{0+1}{2-0} = \frac{1}{2}.$$

When $\theta = 0$, $x = 1$ and $y = 0$.
The equation of the tangent line is

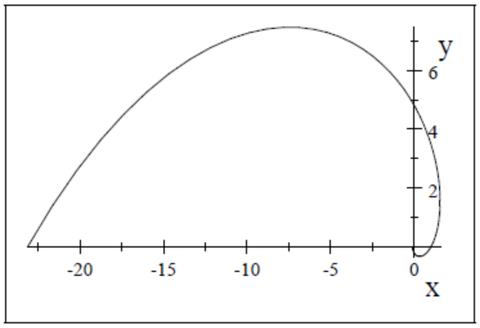
$$y = \left(\frac{1}{2}\right)(x-1) + 0$$

$$y = \frac{1}{2}x - \frac{1}{2}$$

Question 7

Find the arc length of the polar curve $r = e^{\theta}$ from $\theta = 0$ to $\theta = \ln 2$.

Solution



$$
\begin{aligned}
L &= \int_0^{\ln 2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= \int_0^{\ln 2} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta \\
&= \sqrt{2} \int_0^{\ln 2} e^{\theta}\, d\theta \\
&= \sqrt{2} = 1.4142
\end{aligned}
$$

Question 8

Consider the curve given by

$$x^{2/3} + y^{2/3} = 4, \qquad 1 \le x \le 8.$$

(a) Find the arclength of the curve.
(b) Find the area of the surface obtained by rotating the curve about the x-axis.
Hint: Use implicit differentiation to find $\dfrac{dy}{dx}$.

Solution

$$x^{2/3} + y^{2/3} = 4 \Leftrightarrow y^{2/3} = 4 - x^{2/3}$$

(a)

$$L = \int_1^8 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Next, we use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{y^{2/3}}{x^{2/3}} \\ &= 1 + \frac{4 - x^{2/3}}{x^{2/3}} \\ &= \frac{4}{x^{\frac{2}{3}}} = \left(\frac{2}{x^{1/3}}\right)^2 \end{aligned}$$

Thus,

$$\begin{aligned} L &= \int_1^8 \sqrt{\left(\frac{2}{x^{1/3}}\right)^2}\, dx \\ &= \int_1^8 \frac{2}{x^{1/3}} dx \\ &= 9. \end{aligned}$$

(b)

$$\begin{aligned} S &= 2\pi \int_1^8 \left(4 - x^{2/3}\right)^{3/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_1^8 \left(4 - x^{2/3}\right)^{3/2} \frac{2}{x^{1/3}} dx. \end{aligned}$$

Now put

$$u = 4 - x^{2/3}, \qquad du = \frac{1}{3}\frac{-2}{x^{1/3}}dx \Rightarrow \frac{2}{x^{1/3}}dx = -3du.$$

when $x = 1$, $u = 3$ and when $x = 8$, $u = 0$.

Thus,

$$\begin{aligned} S &= 2\pi \int_3^0 u^{3/2}\,(-3du) \\ &= 6\pi \int_0^3 u^{3/2} du \\ &= \frac{108}{5}\sqrt{3}\pi = 37.412\pi = 117.53. \end{aligned}$$

Question 9

Find the area of the surface obtained by rotating the graph of

$$f(x) = 2\sqrt{x+1}\ , \qquad 0 \leq x \leq 1$$

about the x-axis.

Solution

$$\begin{aligned} S &= 2\pi \int_0^1 2\sqrt{x+1}\sqrt{1 + [f\,'(x)]^2} dx \\ &= 2\pi \int_0^1 2\sqrt{x+1}\sqrt{1 + \left(\frac{1}{\sqrt{x+1}}\right)^2} dx \\ &= 2\pi \int_0^1 2\frac{\sqrt{x+2}}{\sqrt{x+1}}\sqrt{x+1} dx \\ &= 4\pi \int_0^1 \sqrt{x+2} dx \\ &= 4\pi\left(2\sqrt{3} - \frac{4}{3}\sqrt{2}\right) = 19.836. \end{aligned}$$