Tutorial 6

Question 1

Show that the differential equation is homogeneous and solve it.

$$xy' = \frac{y^2}{x} + y$$

$$x\frac{dy}{dx} = \frac{y^2}{x} + y \rightarrow \frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x}$$

$$\boxed{\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)} \quad (I)$$

$$= F\left(\frac{y}{x}\right) \quad \text{Hom.}$$

where $F(u) = u^2 + u$

Let $v = \frac{y}{x} \; (1) \rightarrow y = xv$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad (2)$$

substitute equations (1) and (2) in (I)

$$\cancel{v} + x\frac{dv}{dx} = v^2 + \cancel{v}$$

$$x\frac{dv}{dx} = v^2$$

$$\int \frac{dv}{v^2} = \int \frac{dx}{x}$$

$$-\frac{1}{v} = \frac{\ln|x| + C}{1}$$

$$v = \frac{-1}{\ln|x| + C}$$

$$\frac{y}{x} = \frac{-1}{\ln|x| + C}$$

$$\therefore \; y = \frac{-x}{\ln|x| + C}$$

Question 2

Show that the differential equation is homogeneous and solve it.

$$(x^2 - 3y^2)dx + 2xydy = 0$$

$$2xy\,dy = -(x^2 - 3y^2)\,dx$$

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$$

$$= \frac{3y^2}{2xy} - \frac{x^2}{2xy}$$

$$= \frac{3y}{2x} - \frac{x}{2y}$$

$$= \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{x}{y}\right)$$

$$\frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{y}{x}\right)^{-1} \quad (I)$$

$$= F\left(\frac{y}{x}\right)$$

where $F(u) = \frac{3}{2}u - \frac{1}{2}u^{-1}$

let $v = \frac{y}{x} \rightarrow y = xv$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{3}{2}(v) - \frac{1}{2}v^{-1}$$

$$x\frac{dv}{dx} = \frac{1}{2}v - \frac{1}{2}v^{-1}$$

$$x\frac{dv}{dx} = \frac{1}{2}\left(v - \frac{1}{v}\right)$$

$$x\frac{dv}{dx} = \frac{1}{2}\left(\frac{v^2 - 1}{v}\right)$$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\frac{dv}{\frac{v^2-1}{2v}} = \frac{dx}{x}$$

$$\int \frac{2v\,dv}{v^2 - 1} = \int \frac{dx}{x}$$

$$\ln|v^2 - 1| = \ln|x| + C_1$$

$$\ln\left|\frac{v^2 - 1}{x}\right| = C_1$$

$$\frac{v^2 - 1}{x} = C$$

$$v^2 = Cx + 1$$

$$\frac{y^2}{x^2} = Cx + 1$$

$$y^2 = Cx^3 + x^2$$

Question 3

Show that the differential equation is homogeneous and solve it.

$$\frac{dy}{dx} = \frac{x-2y}{x}$$

$$\frac{dy}{dx} = \frac{x}{x} - \frac{2y}{x}$$

$$\frac{dy}{dx} = 1 - 2\left(\frac{y}{x}\right)$$

Let $v = \frac{y}{x} \rightarrow y = xv$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = 1 - 2v$$

$$x\frac{dv}{dx} = 1 - 3v$$

$$-\frac{1}{3}\int \frac{-3\,dv}{1-3v} = \int \frac{dx}{x}$$

$$-\frac{1}{3}\ln|1-3v| = \ln|x| + C_1$$

$$\ln|1-3v| = -3\ln|x| - 3C_1$$

$$\ln|1-3v| + 3\ln|x| = -3C_1$$

$$\ln|1-3v| + \ln|x^3| = -3C_1$$

$$\ln\left|(1-3v)x^3\right| = -3C_1$$

$$(1-3v)x^3 = C$$

$$1-3v = \frac{C}{x^3}$$

$$-3v = \frac{C}{x^3} - 1$$

$$-3v = \frac{C-x^3}{x^3}$$

$$v = \frac{C-x^3}{-3x^3}$$

$$\frac{y}{x} = \frac{C-x^3}{-3x^3}$$

$$y = \frac{C-x^3}{-3x^2}$$

OR

$$y = \frac{x^3+C}{3x^2}$$

Question 4

Show that the following equation is exact then solve the following IVP.

$$\cos x - 2xy + (e^y - x^2)y' = 0 \qquad \boxed{y(1) = 4}$$

$$\cos x - 2xy + (e^y - x^2)\frac{dy}{dx} = 0$$

$$\underbrace{(\cos x - 2xy)}_{M(x,y)}dx + \underbrace{(e^y - x^2)}_{N(x,y)}dy = 0$$

$$\frac{\partial M}{\partial y} = 0 - 2x = -2x \qquad \frac{\partial N}{\partial x} = 0 - 2x = -2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{Exact}$$

$$f(x,y) = \int M(x,y)\,dx + h(y)$$

$$= \int (\cos x - 2xy)\,dx + h(y)$$

$$= \sin x - \frac{2x^2}{2}y + h(y)$$

$$\boxed{f(x,y) = \sin x - x^2 y + h(y)}$$

$$\frac{\partial f}{\partial y} = 0 - x^2 + h'(y) = N(x,y)$$

$$\cancel{-x^2} + h'(y) = e^y \cancel{- x^2}$$

$$h'(y) = e^y$$

$$h(y) = e^y$$

$$f(x,y) = \sin x - x^2 y + e^y$$

$$\boxed{f(x,y) = C}$$

$$\sin x - x^2 y + e^y = C$$

$$\sin(1) - 4 + e^4 = C$$

$$\therefore \quad \sin x - x^2 y + e^y = \sin(1) - 4 + e^4$$

Question 5

Show that the following equation is exact then solve it.

$$(x + siny)dx + (xcosy - 2y)dy = 0$$

$M(x,y)$ $N(x,y)$

$\frac{\partial M}{\partial y} = \cos y$ $\frac{\partial N}{\partial x} = \cos y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Exact!

$f(x,y) = \int M(x,y)\,dx + h(y)$

$= \int (x + \sin y)\,dx + h(y)$

$= \frac{x^2}{2} + x\sin y + h(y)$

$\frac{\partial f}{\partial y} = \cancel{x\cos y} + h'(y) = N = \cancel{x\cos y} - 2y$

$h'(y) = -2y$

$h(y) = -y^2$

$\therefore f(x,y) = c$

$$\boxed{\frac{x^2}{2} + x\sin y - y^2 = c}$$

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Question 6

Find the integrating factor then solve the following equation.

$$\frac{y}{x^2}+1+\frac{1}{x}\frac{dy}{dx}=0 \rightarrow \left(\frac{y}{x^2}+1\right)dx+\frac{1}{x}dy=0 \quad (\mathrm{I})$$

$M(x,y)$ $\qquad N(x,y)$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x) \rightarrow \frac{\frac{1}{x^2}-\left(-\frac{1}{x^2}\right)}{\frac{1}{x}}=\frac{\frac{2}{x^2}}{\frac{1}{x}}=\frac{2}{x}=f(x)$$

OR

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=g(y)$$

$$\mu(x)=e^{\int f(x)\,dx}=e^{\int \frac{2}{x}dx}=e^{2\ln x} \quad x>0$$

$$=\cancel{e}^{\cancel{\ln} x^2}=x^2$$

Multiply (I) by x^2

$$(y+x^2)dx+x\,dy=0$$

M' $\qquad N'$

$$f(x,y)=\int M'\,dx+h(y)=\int (y+x^2)\,dx+h(y)=yx+\frac{x^3}{3}+h(y)$$

$$\frac{\partial f}{\partial y}=x+h'(y)=N'$$

$$x+h'(y)=x$$

$$h'(y)=0 \rightarrow h(y)=c_1$$

$$f(x,y)=c_2 \rightarrow yx+\frac{x^3}{3}+c_1=c_2$$

$$\therefore\ yx+\frac{x^3}{3}=c \quad (c=c_2-c_1)$$

Question 7

Find the integrating factor then solve the following equation.

$$\underbrace{2xy}_{M(x,y)}dx + \underbrace{(y^2 - 3x^2)}_{N(x,y)}dy = 0 \qquad (I)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x+6x}{y^2-3x^2} = \frac{8x}{y^2-3x^2}$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x-2x}{2xy} = \frac{-8x}{2xy} = \frac{-4}{y} = g(y)$$

$$\mu(y) = e^{\int g(y)dy} = e^{\int \frac{-4}{y}dy} = e^{-4\ln y} \quad y>0$$
$$= e^{\ln y^{-4}} = y^{-4}$$

$$\underbrace{2xy^{-3}}_{M'}dx + (\underbrace{y^{-2} - 3x^2y^{-4}}_{N'})dy = 0$$

$$f(x,y) = \int M' dx + h(y)$$
$$= \int 2xy^{-3}dx + h(y)$$

$$f(x,y) = x^2y^{-3} + h(y)$$

$$\frac{\partial f}{\partial y} = \cancel{-3x^2y^{-4}} + h'(y) = N' = y^{-2} \cancel{- 3x^2y^{-4}}$$

$$h'(y) = y^{-2} \rightarrow h(y) = \frac{y^{-1}}{-1} = -\frac{1}{y}$$

$$\therefore \boxed{x^2y^{-3} - \frac{1}{y} = c}$$

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Question 8

Solve the following Bernoulli equation.

$$\frac{dy}{dx} + y = e^x y^{-2}$$

$n = -2$

$$(I)\quad \boxed{y^2 \frac{dy}{dx} + y^3 = e^x} \quad (\text{multiply by } y^2)$$

Let $v = y^{1-n}$

$$\boxed{v = y^3}$$

$$\frac{dv}{dx} = \boxed{3y^2 \frac{dy}{dx}}$$

multiply (I) by 3

$$\boxed{3y^2 \frac{dy}{dx}} + 3\boxed{y^3} = 3e^x$$

$$\frac{dv}{dx} + 3P(x) = Q(x)$$

$$\frac{dv}{dx} + 3v = 3e^x \quad \text{Linear equation}$$

$P(x) = 3$

$Q(x) = 3e^x$

$$v = \frac{1}{\mu(x)}\left[\int Q(x) \cdot \mu(x)\, dx + C\right]$$

$$\mu(x) = e^{\int 3\, dx} = e^{3x}$$

$$v = \frac{1}{e^{3x}}\left[\int 3e^x \cdot e^{3x}\, dx + C\right]$$

$$= \frac{1}{e^{3x}}\left[3\int e^{4x}\, dx + C\right]$$

$$= \frac{1}{e^{3x}}\left[\frac{3}{4}e^{4x} + C\right]$$

$$v = \frac{3}{4}e^x + \frac{C}{e^{3x}}$$

$$y^3 = \frac{3}{4}e^x + \frac{C}{e^{3x}}$$

$$= \frac{3}{4}e^x + Ce^{-3x}$$

$$= \frac{3e^x + 4Ce^{-3x}}{4}$$

$$= \frac{3e^x + Ke^{-3x}}{4}$$

Question 9

Solve the following Bernoulli equation.

$$x\frac{dy}{dx} + y = x^3y^2$$

$\frac{dy}{dx} + \frac{1}{x}y = x^2y^2$ (Divide by x) $(n=2)$

(I) $y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x^2$ (multiply by y^{-2})

Let $v = y^{1-n} \rightarrow v = \boxed{y^{-1}}$

$\frac{dv}{dx} = \boxed{-1 \cdot y^{-2} \cdot \frac{dy}{dx}}$

multiply (I) by $(-1) \rightarrow -1 \cdot y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -x^2$

$\frac{dv}{dx} - \frac{1}{x}v = -x^2$ (linear)

$P(x) = -\frac{1}{x}$ $Q(x) = -x^2$

$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x}$ $(x>0)$

$= \cancel{e}^{\cancel{\ln} x^{-1}}$

$= x^{-1}$

$$v = \frac{1}{\mu(x)}\left[\int Q(x)\cdot\mu(x)\,dx + C\right]$$

$= \frac{1}{x^{-1}}\left[\int -x\,dx + C\right]$

$= x\left[\frac{-x^2}{2} + C\right]$

$= \frac{-x^3}{2} + Cx$

$y^{-1} = \frac{-x^3}{2} + Cx$

$y^{-1} = \frac{-x^3 + 2Cx}{2}$

$y^{-1} = \frac{-x^3 + Kx}{2}$

$y = \frac{2}{-x^3 + Kx}$