(12pts)Problem 1.
Evaluate the following integrals

1. $\int x^{10} \ln x dx$ 2. $\int \cos 2x \sin 5x dx$.

Solution

1. By parts: put

$$\begin{aligned} u &= \ln x, & u' = \frac{1}{x} \\ v' &= x^{10}, & v = \frac{x^{11}}{11} \end{aligned}$$

$$\begin{aligned} \int x^{10} \ln x dx &= \frac{x^{11}}{11} \ln x - \frac{1}{11} \int x^{10} dx \qquad (\mathbf{3pts}) \\ &= \frac{x^{11}}{11} \ln x - \frac{1}{121} x^{11} + C \qquad (\mathbf{3pts}) \end{aligned}$$

2.
$$\int \cos 2x \sin 5x dx = -\frac{1}{6} \cos 3x - \frac{1}{14} \cos 7x + C \qquad (\mathbf{6pts})$$

(12pts) Problem 2.
Find the area of the region bounded by the graphs of

$$f(x) = 2 - x^2 \quad \text{and} \quad g(x) = x.$$

Solution

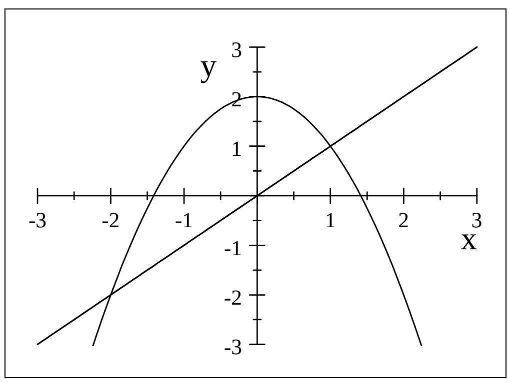
$$A = \int_a^b |f(x) - g(x)|\, dx. \qquad \textbf{(2pts)}$$

$$2 - x^2 = x \Rightarrow x = -2 \text{ or } x = 1.$$

$$a = -2 \text{ and } b = 1. \qquad \textbf{(4pts)}$$

$$\begin{aligned} A &= \int_{-2}^{1} \left|-x^2 - x + 2\right| dx \\ &= \int_{-2}^{1} \left(-x^2 - x + 2\right) dx \quad \text{(opposite sign of a inside of the root)} \qquad \textbf{(4pts)} \\ &= \frac{9}{2} = 4.5. \qquad \textbf{(2pts)} \end{aligned}$$

or by graphing without using absolute value



You see from the grapgh that the parabola is on top of the line between -2 and 1. So,

$$\begin{aligned} A &= \int_{-2}^{1} \left[\left(2 - x^2\right) - x\right] dx \\ &= \int_{-2}^{1} \left(-x^2 - x + 2\right) dx \\ &= \frac{9}{2} = 4.5. \end{aligned}$$

(12pts) Problem 3
Find the arclength of the function

$$f(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

on the interval $[0,2]$.
Solution

$$L = \int_0^2 \sqrt{1 + [f\ '(x)]^2} dx \qquad \textbf{(2pts)}$$

$$f\ '(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\begin{aligned} [f\ '(x)]^2 &= \left[\frac{1}{2}\left(e^{x} - e^{-x}\right)\right]^2 \\ &= \frac{1}{4e^{2x}} + \frac{1}{4}e^{2x} - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 1 + [f\ '(x)]^2 &= \frac{1}{4e^{2x}} + \frac{1}{4}e^{2x} + \frac{1}{2} \\ &= \left[\frac{1}{2}\left(e^{x} + e^{-x}\right)\right]^2. \qquad \textbf{(4pts)} \end{aligned}$$

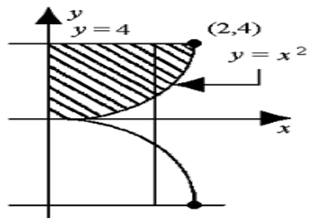
Hence,

$$\begin{aligned} L &= \int_0^2 \sqrt{\left[\frac{1}{2}\left(e^{x} + e^{-x}\right)\right]^2} dx \\ &= \int_0^2 \frac{1}{2}\left(e^{x} + e^{-x}\right) dx \qquad \textbf{(4pts)} \\ &= \frac{1}{2}\left(e^{2} + e^{-2}\right) \qquad \textbf{(2pts)} \\ &= 3.7622 \end{aligned}$$

(**10pts**) **Problem 4**

Sketch the region bounded by $y = x^2, y = 4$,and $x = 0$, and use the disc method to find the volume of the solid generated by revolving the region about the $x-$axis.

Solution



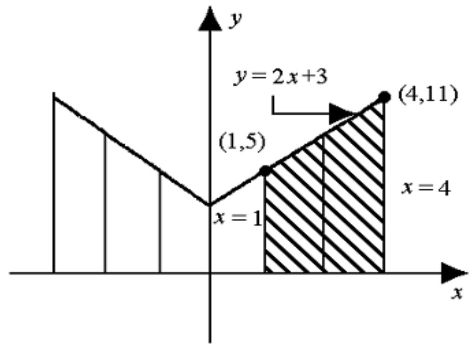
(**2pts**)

$$\begin{aligned} V &= \int_0^2 \left[\pi 4^2 - \pi \left(x^2\right)^2\right] dx \quad (\mathbf{4pts}) \\ &= \int_0^2 \left(16\pi - \pi x^4\right) dx \\ &= \frac{128}{5}\pi = 80.425 \quad (\mathbf{4pts}) \end{aligned}$$

(10pts) Problem 5

Sketch the region bounded by $y = 2x + 3$, $x = 1$, and $x = 4$,and $y = 0$, and use the shell method to find the volume of the solid generated by revolving the region about the y-axis.

Solution



(**2pts**)

$$
\begin{aligned}
V &= \int_1^4 2\pi \cdot (x) \cdot (2x+3)\, dx && (\textbf{4pts}) \\
&= 129\pi = 405.27. && (\textbf{4pts})
\end{aligned}
$$

(**12pts**) **Problem 6**
Use trigonometric substitution to evaluate the integral

$$\int \frac{dx}{x^2\sqrt{4-x^2}}.$$

Solution
Put

$$x = 2\sin\theta, \qquad dx = 2\cos\theta d\theta \qquad (\mathbf{3pts})$$

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4-x^2}} &= \int \frac{2\cos\theta d\theta}{4\sin^2\theta 2\cos\theta} \\
&= \int \frac{d\theta}{4\sin^2\theta} \qquad (\mathbf{4pts}) \\
&= \frac{1}{4}\int \csc^2\theta d\theta \\
&= -\frac{1}{4}\cot\theta + C \\
&= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C \qquad (\mathbf{5pts})
\end{aligned}$$

(12pts) Problem 7

Use partial fraction decomposition to evaluate the integrals

$$1. \int \frac{(2x+6)\,dx}{(x-1)(x-2)^2}$$

Solution

1.

$$\frac{x+3}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B_1}{x-2} + \frac{B_2}{(x-2)^2} \qquad (\mathbf{3pts})$$

$\Rightarrow$

$$\frac{x+3}{(x-1)(x-2)^2} = \frac{4}{x-1} - \frac{4}{x-2} + \frac{5}{(x-2)^2} \qquad (\mathbf{6pts})$$

Hence,

$$\begin{aligned} 2\int \frac{(x+3)\,dx}{(x-1)(x-2)^2} &= \int \frac{8}{x-1} - \frac{8}{x-2} + \frac{10}{(x-2)^2}dx \\ &= 8\ln|x-1| - 8\ln|x-2| - \frac{10}{x-2} + C. \qquad (\mathbf{3pts}) \end{aligned}$$

(10pts) Problem 8
Find the area of the surface obtained by rotating the graph of

$$f(x) = 2\sqrt{x+1}\ , \qquad -1 \le x \le 1$$

about the x-axis.
Solution

$$\begin{aligned}
\text{S} &= 2\pi \int_{-1}^{1} 2\sqrt{x+1}\sqrt{1+[f\,'(x)]^2}dx && (\textbf{3pts}) \\
&= 2\pi \int_{-1}^{1} 2\sqrt{x+1}\sqrt{1+\left(\frac{1}{\sqrt{x+1}}\right)^2}dx && (\textbf{4pts}) \\
&= 2\pi\left(4\sqrt{3}-\frac{4}{3}\right) \\
&= 35.154. \qquad (\textbf{3pts})
\end{aligned}$$

(10pts)Problem 9

Evaluate the integral

$$\int \frac{\sqrt{x}}{1+x} dx$$

Solution: Well, in order to eliminate the "square root" here it would be nice to try out the substitution $x = z^2$, $dx = 2z\,dz$. This is because

$$\begin{aligned}
\int \frac{\sqrt{x}}{1+x}\,dx &= \int \frac{z}{1+z^2}\,2z\,dz \\
&= \int \frac{2z^2}{1+z^2}\,dz \\
&= 2\int \left(1 - \frac{1}{1+z^2}\right)\,dz \\
&= 2z - 2\,\mathrm{Arctan}\,(z) + C \\
&= 2\sqrt{x} - 2\,\mathrm{Arctan}\,(\sqrt{x}) + C,
\end{aligned}$$

where C is the usual constant of integration. Note that the guessed substitution gave us a rational function in z which, coupled with the method of partial fractions, allowed for an easy integration. (**10pts**)