Improper Integrals

Type I

If f is continuous on $[a, +\infty)$, then the integral

$$\int_a^{\infty} f(x)\,dx$$

is called an improper integral of type I.

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx$$

If the limit exists, then we say that the improper integral "converges".
Otherwise it diverges.

Example

Determine conversions or diversions of the following improper integrals.

a) $\displaystyle\int_1^{\infty} xe^{-x^2}\,dx$

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty} \int_1^{t} xe^{-x^2}\,dx$$

$$\frac{-1}{2}\int_1^{t} -2xe^{-x^2}\,dx = -\frac{1}{2} e^{-x^2}\Big|_1^{t}$$

$$= \frac{-1}{2}\left(e^{-t^2} - e^{-1}\right)$$

$$\lim_{t\to\infty}\int_1^{t} xe^{-x^2}\,dx = \lim_{t\to\infty} \frac{-1}{2}\left(e^{-t^2} - e^{-1}\right)$$

$$= \lim$$

$$\lim_{t\to\infty} \frac{-1}{2}\left(\frac{1}{e^{t^2}} - \frac{1}{e}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{e}\right) \quad \because \frac{1}{\infty} = 0$$

$$= \frac{1}{2e}$$

Converges to $\frac{1}{2e}$

b) $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln x}$$

$$= \lim_{t\to\infty} \int_2^t \frac{1/x}{\ln x}\,dx$$

$$= \lim_{t\to\infty} \ln|\ln x|\Big|_2^t$$

$$= \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2|$$

$$= \infty - \ln|\ln 2|$$

$$= \infty$$

Diverges

If f is continuous on $(-\infty, b]$ then the integral

$$\int_{-\infty}^{b} f(x)\,dx$$

is an improper integral of type I.

b b

$$\int_{-\infty} f(x)\,dx = \lim_{t \to -\infty} \int_{-\infty} f(x)\,dx$$

Example

Evaluate $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x^2}$

$$\int_{-\infty}^{-1} \frac{dx}{x^2} = \lim_{t \to -\infty} \int^{-1} \frac{dx}{x^2}$$

$$= \lim_{t \to -\infty} \left(-x^{-1}\right)\Big|_{t}^{-1}$$

$$= 1 - \frac{1}{t}$$

$$= 1$$

Converges to 1.

$$\int_{-\infty}^{-2} \frac{dx}{x}$$

$$\int_{-\infty}^{-2} \frac{dx}{x} = \lim_{t \to -\infty} \int_{-t}^{-2} \frac{dx}{x}$$

$$= \lim_{t \to -\infty} \ln|x|\Big|_{-t}^{-2}$$

$$= \lim_{t \to -\infty} \ln 2 - \ln|t|$$

$$= \ln 2 - \ln|-\infty|$$

$$= \ln 2 - \ln \infty$$

$$= -\infty$$

$\therefore$ Diverges

If f is now continuous on $(-\infty, \infty)$

then $\int_{-\infty}^{\infty} f(x)\,dx$

is also an improper integral of type I.

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{a} f(x)\,dx}_{\text{Case II}} + \underbrace{\int_{a}^{\infty} f(x)\,dx}_{\text{Case I}}$$

For this to converge, both integrals must converge

Example

Evaluate

a) $\int_{-\infty}^{\infty} e^{x}\,dx$

$$\int_{-\infty}^{\infty} e^{x}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{x}\,dx$$

$$\int_{-\infty}^{0} e^{x}\,dx = \lim_{t\to-\infty} \int_{t}^{0} e^{x}\,dx$$

$$= \lim_{t\to-\infty} (1-e^{t})$$

$$= 1-\boxed{e^{-\infty}} \longrightarrow \lim_{t\to-\infty} e^{t} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

$$= 1-0$$

$$= 1$$

Converges to 1

$$\int_{0}^{\infty} e^{x}\,dx = \lim_{t\to\infty} \int_{0}^{t} e^{x}\,dx$$

$$= \lim_{t\to\infty} e^{t} - e^{0}$$

$$= \lim_{t \to \infty} e^{t} - 1$$

$$= e^{\infty} - 1$$

$$= \infty - 1$$

$$= \infty$$

Diverges

$$\therefore \int_{-\infty}^{\infty} e^{x}\,dx \Rightarrow \text{diverges}$$

b) $\int_{-\infty}^{\infty} \sin x \, dx$

$$\int_{-\infty}^{\infty} \sin x = \lim_{t \to \infty} \int_{-t}^{t} \sin x \, dx$$

$$= \lim_{t \to \infty} 0$$

$$= \underline{0}$$

The p-test for improper integrals of Type I

If $a > 0$ $\int_{a}^{\infty} \frac{dx}{x^{p}}$ converges if $p > 1$ and diverges if $p < 1$

<u>Example</u>

Determine conversions or diversions

$$\int_{2}^{\infty} \frac{dx}{x\sqrt{x}}, \quad \int_{1}^{\infty} \frac{dx}{\sqrt{x}}, \quad \int_{12}^{\infty} \frac{dx}{x^{3}}$$

Conversion Diversion Conversion

<u>Type II</u>

If f is continuous on $[a, b)$

then the integral

$$\int_a^b f(x)\,dx$$

is an improper integral of Type II

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

If the limit exists then the improper integral converges

Otherwise it diverges.

Example

Evaluate

a) $\displaystyle\int_1^2 \frac{dx}{(x-2)^2}$

$$\int_1^2 \frac{dx}{(x-2)^2} = \lim_{t \to 2^-} \int_1^t (x-2)^{-2}\,dx$$

$$= \lim_{t \to 2^-} \left[\frac{-1}{x-2} \right]_1^t$$

$$= \lim_{t \to 2^-} \left[\frac{-1}{t-2} - 1 \right]$$

$$= \frac{-1}{2-2} - 1$$

$$= \pm\infty - 1$$

$$= \pm\infty$$

Diverges

b) $\displaystyle\int_1^2 \frac{dx}{\sqrt{2-x}}$

$$\lim_{t \to 2^-} \int_1^t (2-x)^{-1/2}\, dx$$

$$= \lim_{t \to 2^-} - \int_1^t -(2-x)^{-1/2}\, dx$$

$$= \lim_{t \to 2^-} -2\sqrt{2-x}\,\Big|_1^t$$

$$= \lim_{t \to 2^-} -2\left(\sqrt{2-t} - \sqrt{2-1}\right)$$

$$= \lim_{t \to 2^-} -2\left(\sqrt{2-t} - 1\right)$$

$$= \lim_{t \to 2^-} -2\sqrt{2-t} + 2$$

$$= 0 + 2$$

$$= 2$$

Converges to 2

If f is continuous on $(a, b]$ then the integral

$$\int_a^b f(x)\, dx$$

is an improper integral of type II.

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx$$

If the limit exists then the improper integral converges
Otherwise it diverges.

Example

Evaluate

a) $\int^3 \frac{dx}{}$

$$\int_{1} \frac{}{|x-1|}$$

$$\int_1^3 \frac{dx}{|x-1|}$$

$$= \int_1^3 \frac{dx}{x-1}$$

$$= \lim_{t \to 1^+} \int_t^3 \frac{dx}{x-1}$$

$$= \lim_{t \to 1^+} \ln|x-1| \Big|_t^3$$

$$= \lim_{t \to 1^+} \ln 2 - \ln|t-1|$$

$$= \ln 2 + \infty$$

$$= \infty$$

Diverges

$$\int_0^1 \ln x \, dx$$

$$\int_0^1 \ln x \, dx$$

$$= \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

$$= \lim_{t \to 0^+} x \ln x - x \Big|_t^1$$

$$= \lim_{t \to 0^+} \left(-1 - (t \ln|t| - t)\right)$$

$$= \lim_{t \to 0^+} \left(-1 - t\ln|t| + t\right)$$

$$\lim_{t \to 0^+} t \ln|t| = \lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t^2}} = \frac{1/t}{-1/t^2} = -t$$

$$= \lim_{t \to 0^+} -1 - \cancel{t} + \cancel{t}$$

$$= -1$$

If f is continuous on $[a, b]$ and discontinuous at c in (a, b) then the integral $\int_a^b f(x)\,dx$ is an improper integral of type II.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \Big\}$$ For this to converge, both must converge

The p-test for improper integrals of Type II

If $a > 0$ $\displaystyle\int_0^a \frac{dx}{x^p}$ converges if $p < 1$ and diverges if $p \geq 1$.

Example

Let $L = \displaystyle\int_{-2}^{1} \frac{dx}{x^2}$

$$\int_{-2}^{1} \frac{dx}{x^2} = \int_{-2}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$

Diverges