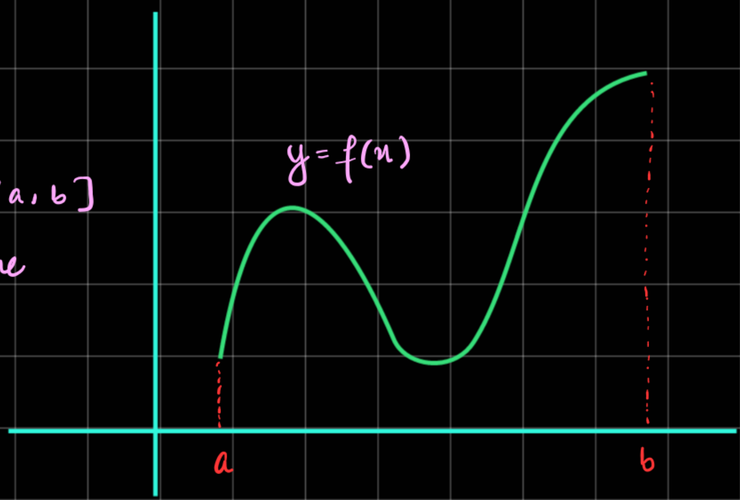
Application of Integrals

2. Using definite integrals to find arc length

$$L = b - a$$

If f and f' are both continuous on $[a, b]$ then the length of the curve given by the graph of $y = f(x)$ is equal to

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$



Examples

Find the arc length of $y = \ln \cos x$ from $x = 0$ to $x = \pi/4$

$$y = \ln \cos x$$

$$y' = \frac{1}{\cos x} \times -\sin x$$

$$= -\tan x$$

$$L = \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/4} |\sec x|\, dx$$

$$\int_0 |\sec x|\, dx$$

$$= \int_0^{\pi/4} \sec x \; dx$$

$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

$$= \ln|\sqrt{2}+1| - \ln|1+0|$$

$$= \ln|\sqrt{2}+1| - \ln 1 \quad \hookrightarrow 0$$

$$= \underline{\ln|\sqrt{2}+1|}$$

Example

Find the arc length of $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ from $x=1$ to $x=2$

$$f'(x) = \frac{3x^2}{6} - \frac{1}{2x^2}$$

$$= \frac{x^2}{2} - \frac{1}{2x^2}$$

$$[f'(x)]^2 = \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2$$

$$= \frac{x^4}{4} - \frac{\cancel{2x^2}}{\cancel{4x^2}_{\,2}} + \frac{1}{4x^4}$$

$$= \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$$

$$1 + [f'(x)]^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$$

$$= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}$$

opposite of this

$$= \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

$$L = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\; dx$$

$$= \int_1^2 \left[\frac{x^2}{2} + \frac{1}{2x^2}\right] dx$$

$$= \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2$$

$$= \frac{8}{6} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2}$$

$$= \frac{7}{6} + \frac{1}{4}$$

$$= \frac{14+3}{12}$$

$$= \frac{17}{12}$$

Example

Find the arc length of the curve given by the equation $(y-1)^3 = x^2$ on the interval $[0,8]$

$$(y-1)^3 = x^2$$
$$(y-1)^{3/2} = x$$

at $x = 0$ $\quad y = 1$

at $x = 8$ $\quad y = 5$

$$L = \int_1^5 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{3}{2}(y-1)^{1/2}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{9}{4}(y-1)$$

$$L = \int_1^5 \sqrt{\frac{9}{4}y - \frac{5}{4}}\, dy$$

$$= \int_1^5 \sqrt{\frac{1}{4}(9y-5)}\, dy$$

$$= \frac{1}{2}\int_1^5 \sqrt{9y-5}\, dy$$

$$= \frac{1}{18}\int_1^5 9\sqrt{9y-5}\,dy$$

$$= \frac{1}{18}\left[\frac{2}{3}(9y-5)^{3/2}\right]_1^5$$

$$= \frac{1}{18}\left[\frac{2}{3}\left[40^{3/2} - 4^{3/2}\right]\right]$$

$$= \frac{1}{27}\left(40\sqrt{40} - 4\sqrt{4}\right)$$

$$= \frac{1}{27}\left(80\sqrt{10} - 8\right)$$

$$= \frac{80\sqrt{10} - 8}{27}$$

Application of Integrals

3. Area of Surface of Revolution

Surface of Revolution

The surface generated by rotating a curve is called a surface of revolution.

If f and f' are both continuous on $[a,b]$ then the area S of the surface of revolution obtained is given by

1. $S = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$ if rotation is about x-axis with $f(x) \geq 0$ on $[a,b]$

2. $S = 2\pi \int_a^b x\sqrt{1+[f'(x)]^2}\,dx$ if rotation is about y-axis

Example

Find the area of the surface obtained by revolving the graph of

A) $f(x) = x^3$ on the interval $[0,1]$ about the x-axis

A) $S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \, dx$

$= 2\pi \int_0^1 x^3 \sqrt{9x^4 + 1} \, dx$

$= \frac{2\pi}{36} \int_0^1 36x^3 (9x^4 + 1)^{1/2} \, dx$

$= \frac{\pi}{18} \left[\frac{2}{3} (9x^4 + 1)^{3/2} \right]_0^1$

$= \frac{\pi}{18} \left[\frac{2}{3} \times (9+1)^{3/2} - \frac{2}{3} (1)^{3/2} \right]$

$= \frac{\pi}{27} [10\sqrt{10} - 1]$ sq units

B) $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis

$S = 2\pi \int_a^b x \sqrt{1 + [f'(x)]^2} \, dx$

$= 2\pi \int_0^{\sqrt{2}} x \sqrt{1 + 4x^2} \, dx$

$= \frac{\pi}{4} \int_0^{\sqrt{2}} 8x (1 + 4x^2)^{1/2} \, dx$

$= \frac{\pi}{4} \left[\frac{2}{3} (1 + 4x^2)^{3/2} \right]_0^{\sqrt{2}}$

$= \frac{\pi}{6} \left[(1 + 8)^{3/2} - (1) \right]$

$= \frac{\pi}{6} (27 - 1)$

$= \frac{26\pi}{6}$

$= \frac{13\pi}{3}$ sq units

Application of Integrals

4. Volume of Solids of Revolution

The solid that is obtained by rotating a surface is called a solid of revolution

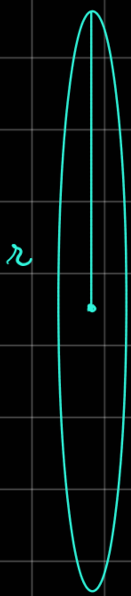
To find the volume of a solid of revolution, we will use two methods

- Disc method
- Shell method

$$\text{Volume of Solids of Revolution} = \int_a^b \text{Area of slice } dx \text{ or } dy$$

Disc method / washer method

In the disc method, the slice is always a disc and is chosen perpendicular to the axis of revolution.



Step 1 : Find whether dx or dy

Step 2 : If $\perp$ to x, then dx else dy

Step 3 : Get upper and lower bounds

Step 4 : Get radius given by $f(x)$

Step 5 : Get area by πr^2

Step 6 : Integrate !

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

Example

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from $x=0$ to $x=1$.

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

$$V = \int_0^1 \pi (\sqrt{x})^2 \, dx$$

$$= \int_0^1 \pi x \, dx$$

$$= \frac{\pi}{2}$$