Trigonometric Substitution

Integral involving $\sqrt{a^2 - x^2}$, $a > 0$

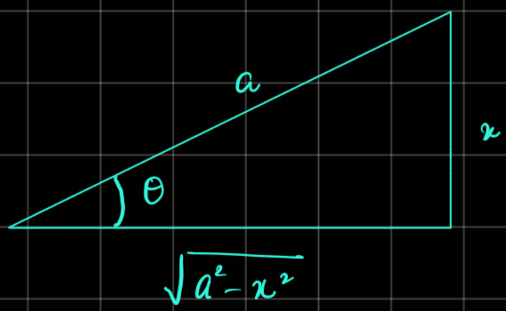
For integrals involving $\sqrt{a^2 - x^2}$, you always do the substitution

$$x = a \sin\theta \quad \text{with} \quad \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2\sin^2\theta} \\
&= \sqrt{a^2(1 - \sin^2\theta)} \\
&= \sqrt{a^2\cos^2\theta} \\
&= \sqrt{(a\cos\theta)^2} \\
&= |a\cos\theta| \\
&= a\,|\cos\theta| \quad \text{since } a > 0 \\
&= a\cos\theta \quad \text{since } \cos\theta > 0 \text{ in } \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]
\end{aligned}$$

$x = a \sin\theta$

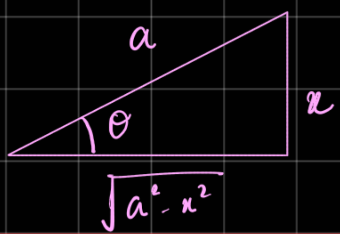
$\sin\theta = \dfrac{x}{a}$



$\sqrt{a^2 - x^2}$

$x = a\sin\theta \qquad \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$dx = a\cos\theta\, d\theta$



Example

Evaluate the integral

$$\int \frac{dx}{x^2\sqrt{4 - x^2}}$$

$a = 2$

$x = 2\sin\theta \qquad -\pi \leq \theta \leq \pi$

$$dx = 2\cos\theta \, d\theta$$

$$\sqrt{4-x^2} = 2\cos\theta$$

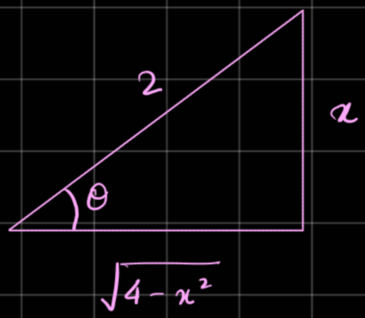
$$\int \frac{\cancel{2\cos\theta}\, d\theta}{4\sin^2\theta\, \cancel{(2\cos\theta)}}$$

$$= \frac{1}{4}\int \operatorname{cosec}^2 d\theta$$

$$= -\frac{1}{4}\cot\theta + c$$

CONVERT USING TRIANGLE !!

$$= -\frac{\sqrt{4-x^2}}{4x} + c$$



Integral involving $\sqrt{a^2+x^2}$, $a > 0$

For integrals involving $\sqrt{a^2+x^2}$, you always do the substitution

$$x = a\tan\theta \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = a\sec^2\theta \, d\theta$$

$$\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta}$$

$$= \sqrt{a^2(1+\tan^2\theta)}$$

$$= \sqrt{a^2\sec^2\theta}$$

$$= \sqrt{(a\sec\theta)^2}$$

$$= a\,|\sec\theta| \quad \text{since } a > 0$$

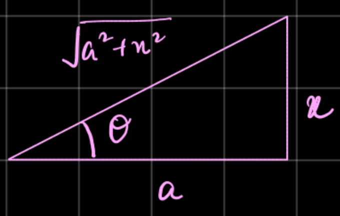
$$= a\sec\theta \quad \text{since } \sec\theta > 0 \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\sqrt{a^2+x^2}$

$$x = a\tan\theta \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$dx = a\sec^2\theta \, d\theta$$

$$\sqrt{a^2+x^2} = a\sec\theta$$



Example

Evaluate $\displaystyle\int \frac{dx}{(x^2+1)\sqrt{x^2+1}}$

$a = 1$

$x = \tan\theta \qquad -\pi/2 < \theta < \pi/2$

$dx = \sec^2\theta \, d\theta$

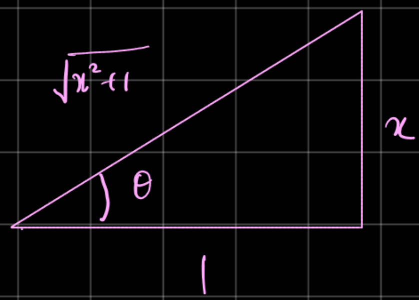
$\sqrt{1+x^2} = \sec\theta$

$$\int \frac{dx}{(x^2+1)\sqrt{x^2+1}} = \int \frac{\cancel{\sec^2\theta}\, d\theta}{\cancel{\sec^2\theta} \cdot \sec\theta}$$

$$= \int \cos\theta \, d\theta$$

$$= \sin\theta + c$$

$$= \frac{x}{\sqrt{x^2+1}} + c$$



Integral involving $\sqrt{x^2-a^2}$, $a > 0$

For integrals involving $\sqrt{x^2-a^2}$, you always do the substitution

$x = a\sec\theta \quad$ with $\quad 0 \leq \theta < \frac{\pi}{2} \quad$ OR $\quad \pi \leq \theta < \frac{3\pi}{2}$

$dx = a\sec\theta\tan\theta \, d\theta$

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$$

$$= \sqrt{a^2(\sec^2\theta - 1)}$$

$$= \sqrt{a^2\tan^2\theta}$$

$$= \sqrt{(a\tan\theta)^2}$$

$= a\,|\tan\theta| \quad$ since $a > 0$

$= a\tan\theta \quad$ since $\tan\theta > 0$ in $\left[0, \frac{\pi}{2}\right)$ OR $\left[\pi, \frac{3\pi}{2}\right)$

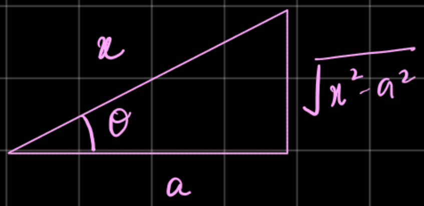
$\sqrt{x^2-a^2}$

$x = a\sec\theta \qquad 0 \leq \theta < \pi/2 \quad$ OR $\quad \pi \leq \theta < 3\pi$

$x = a\sec\theta \qquad 0 < \theta < \frac{\pi}{2}$

$dx = a\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2 - a^2} = a\tan\theta$



Example

Evaluate $\displaystyle\int \frac{\sqrt{x^2 - 2}}{x}\, dx$

$a = \sqrt{2}$

$x = \sqrt{2}\sec\theta$

$dx = \sqrt{2}\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2 - 2} = \sqrt{2}\tan\theta$

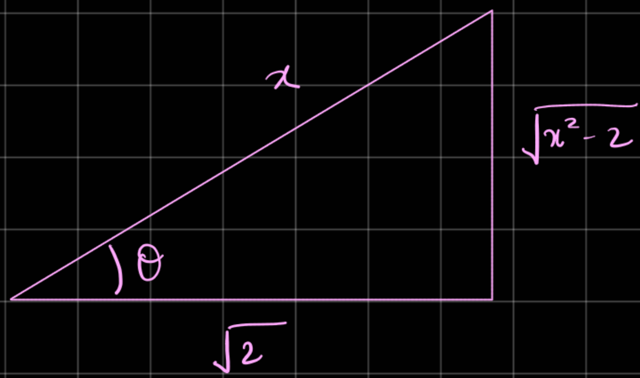
$$\int \frac{\sqrt{2}\tan\theta}{\cancel{\sqrt{2}}\,\cancel{\sec\theta}} \times \cancel{\sqrt{2}}\,\cancel{\sec\theta}\tan\theta\, d\theta$$

$$= \sqrt{2}\int \tan^2\theta\, d\theta$$

$$= \sqrt{2}\int (\sec^2\theta - 1)\, d\theta$$

$$= \sqrt{2}\,(\tan\theta - \theta) + C$$

$$= \sqrt{2}\left[\sqrt{\frac{x^2 - 2}{2}} - \cos^{-1}\left(\frac{\sqrt{2}}{x}\right)\right] + C$$



$\tan\theta = \sqrt{\dfrac{x^2 - 2}{2}}$

$\theta = \cos^{-1}\left(\dfrac{\sqrt{2}}{x}\right)$

Rationalization Technique

Example

Evaluate the following integrals

a) $\int \frac{\sqrt{x}}{2+x}\,dx$

$$\int \frac{\sqrt{x}}{2+x}\,dx$$

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$$\int \frac{\sqrt{x}}{2+x}\,dx = \int \frac{2u^2}{2+u^2}\,du$$

$$= 2\int \frac{u^2}{2+u^2}\,du$$

$$= 2\int \frac{u^2+2-2}{2+u^2}\,du$$

$$= 2\int \frac{u^2+2}{u^2+2}\,du - 4\int \frac{1}{u^2+2}\,du$$

$$= 2u - 2\sqrt{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= 2\sqrt{x} - 2\sqrt{2}\tan^{-1}\left(\sqrt{\frac{x}{2}}\right) + C$$

b) $\int \frac{dx}{x + x\sqrt{x}}$

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$$\int \frac{dx}{x+x\sqrt{x}} = \int \frac{2u}{u^2+u^3}\,du$$

$$= \int \frac{2}{u + u^2}\, du$$

$$= 2 \int \frac{du}{u(u+1)} \Rightarrow 2 \int \frac{u+1-u}{u(u+1)}\, du$$

$$= 2 \int \frac{\cancel{u+1}}{u\cancel{(u+1)}}\, du - 2 \int \frac{\cancel{u}}{\cancel{u}(u+1)}\, du$$

$$= 2 \int \frac{du}{u} - 2 \int \frac{du}{u+1}$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu$$

@ $u = 0$ @ $u = -1$

$A = 1$ $B = -1$

$$2 \int \frac{du}{u(u+1)} = 2 \int \frac{du}{u} - 2 \int \frac{du}{u+1}$$

$$= 2 \ln|u| - 2 \ln|u+1| + C$$

$$= 2 \ln|\sqrt{x}| - 2 \ln|\sqrt{x}+1| + c$$

$$= \ln\left|\frac{\sqrt{x}}{\sqrt{x}+1}\right|^2 + C$$

c) $\displaystyle\int \frac{dx}{\sqrt{x} + \sqrt[4]{x}}$

$u = \sqrt[4]{x}$

$u^2 = \sqrt{x}$

$u^4 = x$

$4u^3\, du = dx$

$$\int \frac{dx}{\sqrt{x} + \sqrt[4]{x}} = \int \frac{4u^3\, du}{u^2 + u}$$

$$= \int \frac{4u^2}{u+1}\, du \Rightarrow 4\int \frac{u^2+1-1}{u+1}$$

$$= 4\int \frac{u^2-1}{u+1}\, du + 4\int \frac{du}{u+1}$$

$$= 4\int u-1\, du + 4\ln|u+1|$$

$$= 2u^2 - 4u + 4\ln|u+1|$$

$$= \int 4u\, du - \int \frac{4u}{u+1}$$

$$= 2u^2 - 4\int \frac{u}{u+1}$$

$$= 2u^2 - 4\left[\int \frac{u+1}{u+1}\, du - \int \frac{du}{u+1}\right]$$

$$= 2u^2 - 4\left[u - \ln|u+1|\right] + c$$

$$= 2u^2 + 4\ln|u+1| - 4u + c$$

$$= 2\sqrt{x} + \ln|\sqrt[4]{x}+1|^4 - 4\sqrt[4]{x} + c$$