Example

2. $f(x) = \frac{1}{x}$ with centre 1

$$\frac{1}{1-\square} = \sum_{n=0}^{\infty} \square^n \quad \text{if } |\square| < 1$$

$$\frac{1}{x} = \frac{1}{1-1+x} = \frac{1}{1-(1-x)}$$

$$= \sum_{n=0}^{\infty} (1-x)^n \quad \text{if } |1-x| < 1$$

$$= \sum_{n=0}^{\infty} [-(x-1)]^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$\therefore \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad \text{if } |x-1| < 1$$

$$-1 < x-1 < 1$$

$$0 < x < 2$$

Operation of Power Series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

1. $f(kx) = \sum_{n=0}^{\infty} a_n (kx)^n$

$$= \sum_{n=0}^{\infty} a_n k^n x^n$$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n (x^N)^n$

$$= \sum_{n=0}^{\infty} a_n x^{nN}$$

3. $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$

Example

Find a power series centered at 0 for $f(x) = \frac{3x-1}{x^2-1}$

$$\frac{3x-1}{x^2-1} = \frac{3x-1}{(x-1)(x+1)}$$

$$\frac{3x-1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{1+x}$$

$$= \frac{2}{1+x} + \frac{B}{x-1}$$

$$= \frac{2}{1-(-x)} - \frac{1}{1-x}$$

$$= 2\sum_{n=0}^{\infty} (-x)^n - \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} 2(-1)^n x^n - \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} x^n \left[2(-1)^n - 1\right] \quad \text{if } |x| < 1$$

Example

Find a power series for $f(x) = \ln x$ centered at 1.

$f'(x) = \frac{1}{x}$. Using what we did above,

$$f'(x) = \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad \text{if } |x-1| < 1$$

$$f(x) = \int \frac{1}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n\,dx \quad \text{if } |x-1| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n \int (x-1)^n\,dx$$

$$= \sum_{n=0} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

Taylor & Maclaurin Series

If f has derivative of all orders (f', f'', f''' --- all exist) then the Taylor series of f centered at $x = a$ is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

when $a = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Example

Find the coefficient of x^4 in the Taylor series expansion of $f(x) = \cos x$ centered at $\pi/2$

$$f(x) = f(\pi/2) + \frac{f'(\pi/2)}{1!}\left(x - \frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2!}(x - \pi/2)^2 + \frac{f'''(\pi/2)}{3!}(x - \pi/2)^3$$

$$+ \frac{f''''(\pi/2)}{4!}(x - \pi/2)^4$$

$$\text{Coefficient} = \frac{f^{(4)}(\pi/2)}{4!}$$

$$= \frac{\cos \pi/2}{12}$$

$$= 0$$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^4(x) = \cos x$

Basic Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\infty < x < \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad -\infty < x < \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad -\infty < x < \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad [-1, 1]$$

Example

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$