Integration of Rational Functions Using Partial Fractions

(1) Partial Fractions – Distinct Linear Factors

$$\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

Ex. $\displaystyle\int \frac{dx}{x^2+x-2}$.

$$\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)}$$

$$\frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

1st way $A(x-1)+B(x+2)=1$

$x=1 \rightarrow 3B=1$

$B=\frac{1}{3}$

$x=-2 \rightarrow -3A=1$

$A=-\frac{1}{3}$

$$\int \frac{1}{x^2+x-2}\,dx = \int\left(\frac{-1/3}{x+2}+\frac{\frac{1}{3}}{x-1}\right)dx$$

$$= -\frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + c$$

2nd way $A(x-1)+B(x+2)=1$

$(A+B)x+(-A+2B)=1$

$A+B=0$ and $-A+2B=1$

$A=-B \rightarrow B+2B=1$

$3B=1$

$B=1/3$ and $A=-\frac{1}{3}$

3rd way Cover-up Method. (only for linear factors)

$$\frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$x=-2 \quad x=1$

$$A = \frac{1}{\cancel{(x+2)}(-2-1)} = -1/3 \qquad B = \frac{1}{(1+2)\cancel{(x-1)}} = \frac{1}{3}$$

Ex. Evaluate $\displaystyle\int \frac{3x^2-7x-2}{x^3-x}\,dx$

$$\frac{3x^2-7x-2}{x^3-x} = \frac{3x^2-7x-2}{x(x^2-1)}$$

$$= \frac{3x^2-7x-2}{x(x-1)(x+1)}$$

$$\Rightarrow \frac{3x^2-7x-2}{x(x-1)(x+1)} = \underset{x=0}{\frac{A}{x}} + \underset{x=1}{\frac{B}{x-1}} + \underset{x=-1}{\frac{C}{x+1}}$$

Cover-up Method

$$A = \frac{-2}{(-1)(1)} = 2$$

$$B = \frac{3-7-2}{(1)(2)} = -\frac{6}{2} = -3$$

$$C = \frac{3+7-2}{(-1)(-2)} = \frac{8}{2} = 4$$

$$\int \left(\frac{2}{x} + \frac{-3}{x-1} + \frac{4}{x+1}\right)dx = 2\ln|x| - 3\ln|x-1| + 4\ln|x+1| + C.$$

(2) Partial Fractions – Repeated Linear Factors

$$\frac{P(x)}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

Ex. $\displaystyle\int \frac{2x+4}{x^3-2x^2}\,dx$

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$Ax(x-2) + B(x-2) + Cx^2 = 2x+4$$

$x=0 \to -2B = 4 \to B = -2$

$x=2 \to 4C = 8 \to C = 2$

compare coefficients $\rightarrow A + C = 0 \rightarrow A = -2$

$$= \int \left(\frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2} \right) dx$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C$$

<u>Ex.</u> $\displaystyle\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\, dx$

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{5x^2 + 20x + 6}{x(x^2 + 2x + 1)}$$

$$= \frac{5x^2 + 20x + 6}{x(x+1)(x+1)}$$

$$= \frac{5x^2 + 20x + 6}{x(x+1)^2}$$

$$\Rightarrow \frac{5x^2 + 20x + 6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$A(x+1)^2 + Bx(x+1) + Cx = 5x^2 + 20x + 6$$

$x = 0 \rightarrow A = 6$

$x = -1 \rightarrow -C = 5 - 20 + 6$

$-C = -9 \rightarrow C = 9$

compare coeff. $A + B = 5$

$6 + B = 5$

$B = -1$

$$I = \int \left(\frac{6}{x} + \frac{-1}{x+1} + \frac{9}{(x+1)^2} \right) dx = 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + C$$

(3) Partial Fractions - Irreducible Quadratic Factors

↳ (can't be factored using integers)

$$\frac{P(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)+\cdots+(a_nx^2+b_nx+c_n)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{a_2x^2+b_2x+c_2} + \cdots + \frac{A_nx+B_n}{a_nx^2+b_nx+c_n}$$

Ex. $\displaystyle\int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx$

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

$$A(x^2+1) + (Bx+C)(3x-1) = x^2+x-2$$

$$x = \frac{1}{3} \rightarrow \quad \frac{10}{9}A = \frac{1}{9} + \frac{1}{3} - 2$$

$$\frac{10}{9}A = -\frac{14}{9}$$

$$A = \frac{-14}{10} = -\frac{7}{5}$$

Compare coeff. $\rightarrow$ $A + 3B = 1$ (coeff. of x^2)

$$-\frac{7}{5} + 3B = 1$$

$$3B = 1 + \frac{7}{5}$$

$$3B = \frac{12}{5} \rightarrow B = \frac{4}{5}$$

$A - C = -2$ (constant term)

$$-\frac{7}{5} - C = -2$$

$$-C = -2 + \frac{7}{5}$$

$$-C = -\frac{3}{5} \rightarrow C = \frac{3}{5}$$

$$= \int \left(\frac{-7/5}{3x-1} + \frac{4/5\,x + 3/5}{x^2+1} \right) dx$$

$$= -\frac{7}{5}\int \frac{1}{3x-1}\,dx + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{1}{x^2+1}\,dx$$

$$= -\frac{7}{5}\cdot\frac{1}{3}\int \frac{3}{3x-1}\,dx + \frac{4}{5}\cdot\frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \frac{3}{5}\int \frac{1}{x^2+1}\,dx$$

$$= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln(x^2+1) + \frac{3}{5}\tan^{-1}x + C$$

Ex. $\displaystyle\int \frac{5x^2+6x+2}{(x+2)(x^2+2x+5)}\,dx$ (challenging!)

$$\frac{5x^2+6x+2}{(x+2)(x^2+2x+5)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+5}$$

$$A(x^2+2x+5) + (Bx+C)(x+2) = 5x^2+6x+2$$

$x = -2 \longrightarrow 5A = 10$

$A = 2$

Compare coefficients $\longrightarrow$ $A + B = 5$ (coeff. of x^2)

$2 + B = 5$

$B = 3$

$5A + 2C = 2$

$5(2) + 2C = 2$

$2C = 2 - 10$

$2C = -8$

$C = -4$

$$= \int \left(\frac{2}{x+2} + \frac{3x-4}{x^2+2x+5} \right) dx$$

$$= \int \left(\frac{2}{x+2} + \frac{3x+3-3-4}{x^2+2x+5} \right) dx$$

$$= \int \left(\frac{2}{x+2} + \frac{3(x+1)-7}{x^2+2x+5} \right) dx$$

$$= \int \frac{2}{x+2} + \frac{3}{2} \int \frac{2(x+1)}{x^2+2x+5}\, dx - 7 \int \frac{1}{x^2+2x+5}\, dx$$

completing the square Method

$$x^2+2x+5$$
$$= x^2+2x+1-1+5$$
$$= (x+1)^2+4$$

$$= 2\int \frac{dx}{x+2} + \frac{3}{2} \int \frac{2(x+1)\,dx}{x^2+2x+5} - 7 \int \frac{1}{(x+1)^2+4}\, dx$$

$$= 2\ln|x+2| + \frac{3}{2}\ln|x^2+2x+5| - \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

Note $\int \frac{1}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

More on Integrals

Ex. Evaluate

$$I = \int \frac{x}{x^2-4x+8}\,dx$$

$$x^2-4x+8 = x^2-4x+4-4+8$$
$$= (x-2)^2+4$$

Completing the Square Method

$$I = \int \frac{x}{(x-2)^2+4}\,dx$$

let $u = x-2 \rightarrow x = u+2$
$du = dx$

$$= \int \frac{u+2}{u^2+4}\,du$$

$$= \int \frac{u}{u^2+4}\,du + 2\int \frac{1}{u^2+4}\,du$$

$$= \frac{1}{2}\int \frac{2u}{u^2+4}\,du + 2\int \frac{1}{u^2+4}\,du$$

Recall $\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

$$= \frac{1}{2}\ln(u^2+4) + 2 \cdot \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\ln\left[(x-2)^2+4\right] + \tan^{-1}\left(\frac{x-2}{2}\right) + C.$$

Tutorial (4)

Improper Integrals

Definition of Improper Integrals with Infinite Integration limits

(1) f is continuous on $[a, \infty)$

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

(2) f is continuous on $(-\infty, a]$

$$\int_{-\infty}^a f(x)\,dx = \lim_{t \to -\infty} \int_t^a f(x)\,dx$$

(3) f is continuous on $(-\infty, \infty)$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Ex. Evaluate the integrals that converge.

Recall $\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$

a) $$\int_{-\infty}^2 \frac{dx}{x^2+4} = \lim_{t \to -\infty} \int_t^2 \frac{dx}{x^2+4}$$

$$= \lim_{t \to -\infty} \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_t^2$$

$$= \lim_{t \to -\infty} \frac{1}{2}\left[\tan^{-1} 1 - \tan^{-1}\left(\frac{t}{2}\right)\right].$$

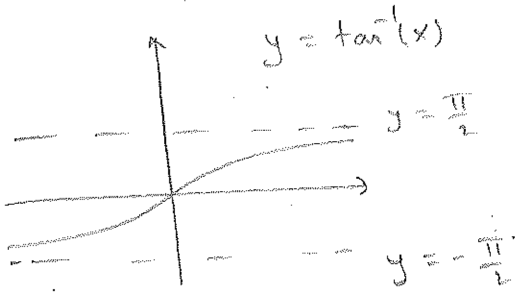
$$= \lim_{t \to -\infty} \frac{1}{2}\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{t}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{\pi}{2}\right]$$

$$= \frac{1}{2}\left(\frac{3\pi}{4}\right)$$

$$= \frac{3\pi}{8}.$$



b) $$\int_3^\infty \frac{3}{x^2-1}\,dx = \lim_{t\to\infty}\int_3^t \frac{3}{x^2-1}\,dx$$

$$= \lim_{t\to\infty}\int_3^t \left[\frac{\frac{3}{2}}{x-1} + \frac{-\frac{3}{2}}{x+1}\right]dx$$

$$\frac{3}{x^2-1} = \frac{3}{(x-1)(x+1)}$$

$$\frac{3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$x=1 \qquad x=-1$

$A = \frac{3}{2} \qquad B = -\frac{3}{2}$

$$= \lim_{t\to\infty}\left[\frac{3}{2}\ln(x-1) - \frac{3}{2}\ln(x+1)\right]_3^t$$

$$= \lim_{t\to\infty}\frac{3}{2}\left[\ln(x-1) - \ln(x+1)\right]_3^t$$

$$= \lim_{t\to\infty}\frac{3}{2}\left[\ln\left(\frac{x-1}{x+1}\right)\right]_3^t$$

$$= \lim_{t\to\infty}\frac{3}{2}\left[\ln\left(\frac{t-1}{t+1}\right) - \ln\left(\frac{2}{4}\right)\right]$$

$$= \frac{3}{2}\lim_{t\to\infty}\ln\left(\frac{t-1}{t+1}\right) - \frac{3}{2}\ln\left(\frac{1}{2}\right)$$

$$= \frac{3}{2}\cdot\cancel{\ln(1)}^{\,0} - \frac{3}{2}\ln\left(\frac{1}{2}\right) = -\frac{3}{2}\left[\ln(1)^{0} - \ln(2)\right]$$

$$= \frac{3}{2}\ln(2)$$

c) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x}}{1+e^{-2x}}\,dx = \int_{-\infty}^{0} \frac{e^{-x}}{1+e^{-2x}}\,dx + \int_{0}^{\infty} \frac{e^{-x}}{1+e^{-2x}}\,dx$

We can choose any number different than zero.

$$= \lim_{t\to-\infty}\int_{t}^{0} \frac{e^{-x}}{1+e^{-2x}}\,dx + \lim_{t\to\infty}\int_{0}^{t} \frac{e^{-x}}{1+e^{-2x}}\,dx$$

$$= \lim_{t\to-\infty}\Big[-\tan^{-1}(e^{-x})\Big]_{t}^{0} + \lim_{t\to\infty}\Big[-\tan^{-1}(e^{-x})\Big]_{0}^{t}$$

$$= \lim_{t\to-\infty}\Big[-\tan^{-1}(1)+\tan^{-1}(e^{-t})\Big] + \lim_{t\to\infty}\Big[-\tan^{-1}(e^{-t})+\tan^{-1}(1)\Big]$$

$$= -\frac{\pi}{4}+\frac{\pi}{2}-0+\frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

Note

$\displaystyle\int \frac{e^{-x}}{1+(e^{-x})^2}\,dx$

let $u = e^{-x}$

$du = -e^{-x}dx$

$e^{-x}dx = -du$

$\displaystyle\int \frac{-du}{1+u^2} = -\tan^{-1}u$

$= -\tan^{-1}(e^{-x})$

Special Type of Improper Integral

$$\int_{1}^{\infty}\frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } p\le 1 \end{cases}$$

Ex. Evaluate

a) $\displaystyle\int_{1}^{\infty}\frac{dx}{\sqrt[3]{x^2}} = \int_{1}^{\infty}\frac{dx}{x^{2/3}}$ $\quad p=\frac{2}{3}\le 1$

$=$ Diverges

b) $\int_1^\infty 3x^{-4}\,dx = 3\int \frac{1}{x^4}\,dx \qquad p=4>1$

$= 3 \cdot \frac{1}{4-1}$

$= 3 \cdot \frac{1}{3}$

$= 1$.

Definition of Improper Integrals with Infinite Discontinuities

① f is continuous on $[a,b)$ and has infinite discontinuity at b.

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

② f is continuous on $(a,b]$ and has an infinite discontinuity at a

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

③ f is continuous on $[a,b]$, except for $c \in (a,b)$ at which f has an infinite discontinuity

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex. Evaluate

(a) $$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{t \to 1^-} \int_0^t \frac{dx}{\sqrt{1-x}}$$

Infinite discontinuity at $x = 1$

$$= \lim_{t \to 1^-} -\int_0^t -(1-x)^{-1/2}\,dx$$

$$= \lim_{t \to 1^-} -\left[2(1-x)^{+\frac{1}{2}}\right]_0^t$$

$$= \lim_{t \to 1^-} -2\left[(1-t)^{\frac{1}{2}} - 1\right]$$

$$= 2$$

(b) $$\int_1^2 \frac{dx}{1-x} = \lim_{t \to 1^+} \int_t^2 \frac{dx}{1-x}$$

Infinite discontinuity at $x = 1$

$$= \lim_{t \to 1^+} -\int_t^2 \frac{-dx}{1-x}$$

$$= \lim_{t \to 1^+} -\left[\ln|1-x|\right]_t^2$$

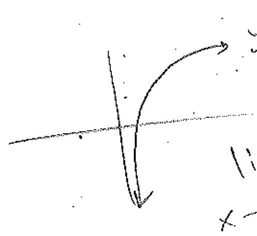
$$= \lim_{t \to 1^+} -\left[\ln|-1| - \ln|1-t|\right]$$

$$= \lim_{t \to 1^+} -\left[0 - \ln|1-t|\right]$$

$$= \lim_{t \to 1^+} \ln|1-t|$$

$$= \ln 0^+$$

$$= -\infty \quad \text{Diverges}$$



$y = \ln x$

$\lim_{x \to 0^+} \ln x = -\infty$

⑥

c) $\displaystyle\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$

Infinite discontinuity at $x = 2$

$$= \lim_{t \to 2^-} \int_1^t \frac{dx}{(x-2)^{2/3}} + \lim_{t \to 2^+} \int_t^4 \frac{dx}{(x-2)^{2/3}}$$

$\displaystyle\int \frac{dx}{(x-2)^{2/3}}$

$= \displaystyle\int (x-2)^{-2/3} dx$

$= 3(x-2)^{\frac{1}{3}}$

$$= \lim_{t \to 2^-} \left[3(x-2)^{\frac{1}{3}}\right]_1^t + \lim_{t \to 2^+} \left[3(x-2)^{\frac{1}{3}}\right]_t^4$$

$$= \lim_{t \to 2^-} \left[3(t-2)^{\frac{1}{3}} + 3\right] + \lim_{t \to 2^+} \left[3(2)^{\frac{1}{3}} - 3(t-2)^{\frac{1}{3}}\right]$$

$$= 0 + 3 + 3(2)^{1/3} - 0$$

$$= 3 + 3(2)^{1/3}.$$

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