* Horizontal and Vertical tangents

$y = f(x)$.

→ horizontal tangent.

$$f'(x) = 0$$

→ Vertical tangent

$$f'(x) = \pm \infty.$$

Consider a set of parametric equations

$$x = f(t) \quad \text{and} \quad y = g(t).$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

* If $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$, then the curve given by

$$x = f(t) \quad \text{and} \quad y = g(t)$$

has a horizontal tangent at t.

* If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$, then the curve has a vertical tangent at t.

Exple A curve C is defined by the parametric equations

$x = t^2$ and $y = t^3 - 3t$

A) Show that the curve has two tangents at the point (3, 0) and find the equations

B) Find the points on C where the tangent is horizontal or vertical

c) Determine where the curve is Concave up or down

D) Sketch the curve.

A) $x = t^2$ and $y = t^3 - 3t$

$$(3, 0) \longrightarrow \begin{cases} x = 3 \\ y = 0 \end{cases}$$

$$\begin{cases} t^2 = 3 \\ t^3 - 3t = 0 \end{cases} \Leftrightarrow \begin{cases} t = \pm\sqrt{3} \\ t(t^2 - 3) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} t = \pm\sqrt{3} \\ t = 0 \text{ or } t = \pm\sqrt{3} \end{cases}$$

$$(3, 0) < \begin{matrix} t = \sqrt{3} \\ t = -\sqrt{3} \end{matrix}$$

slope $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3t^2 - 3}{2t}$

* At $t = \sqrt{3}$ slope $= \frac{6}{2\sqrt{3}} = \frac{\cancel{6}\sqrt{3}}{\cancel{6}} = \sqrt{3}$
* At $t = -\sqrt{3}$ slope $= -\sqrt{3}$.

Equations of tangent lines at $(3,0)$

$y = \sqrt{3}(x-3) + 0$, $y = -\sqrt{3}(x-3) + 0$

$y = \sqrt{3}x - 3\sqrt{3}$ $y = -\sqrt{3}x + 3\sqrt{3}$

B) Horizontal tangents

$\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$3t^2 - 3 = 0$ | $2t = 0$

$3(t-1)(t+1) = 0$ | $t = 0$

$t = 1$ or $t = -1$

The curve has horizontal tangents at $t = 1$ and $t = -1$.

The corresponding points are

$t = 1 \longrightarrow (1, -2)$ -

$t = -1 \longrightarrow (1, 2)$ -

Vertical tangents

$\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$t = 0$ $\frac{dy}{dt} = 0 \iff t = 1$ or $t = -1$

$\Rightarrow$ The curve has a vertical tangent at $t = 0$.

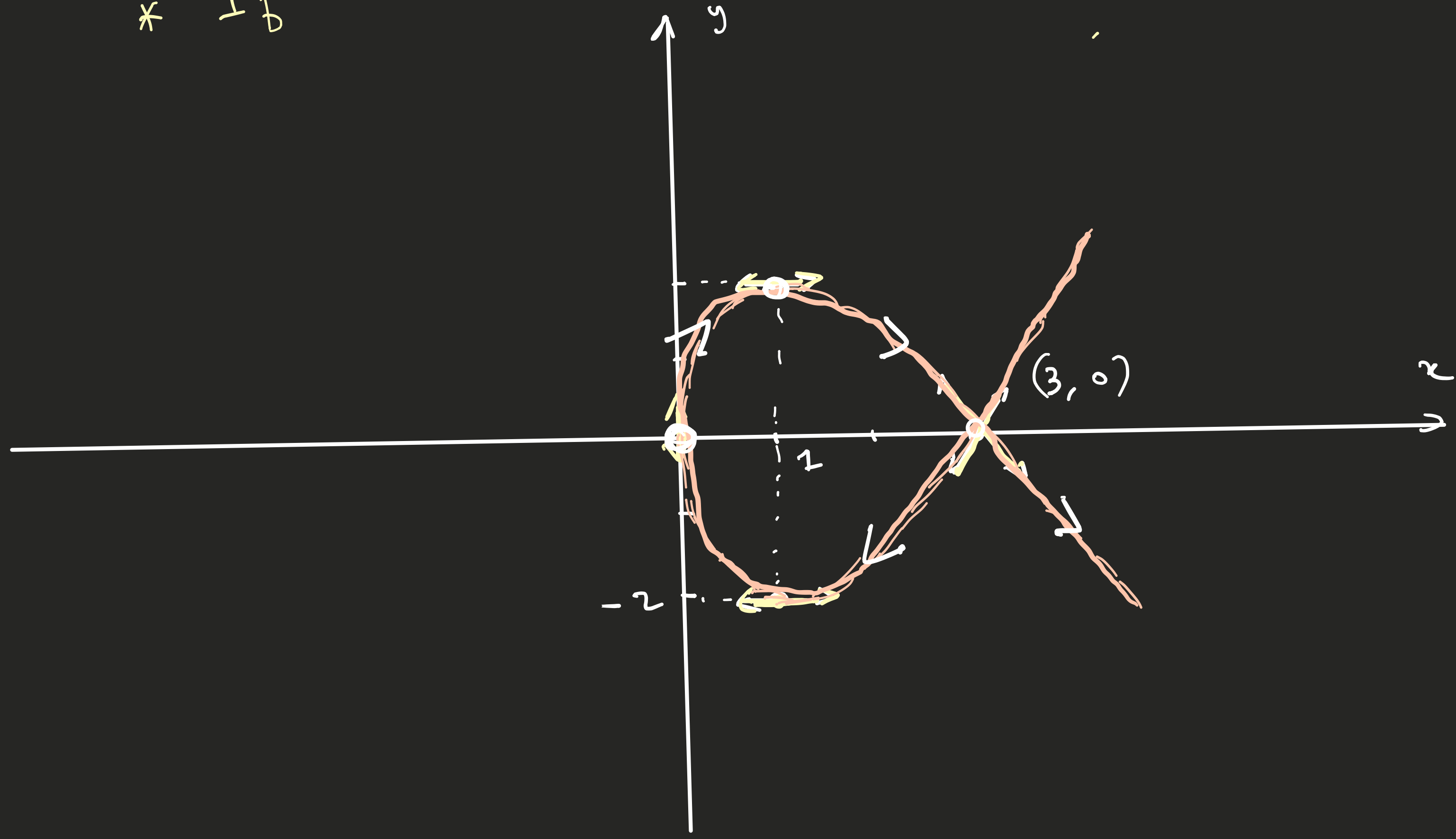
$t = 0 \longrightarrow (0, 0)$

c) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

$$= \frac{\frac{(6t)(2t) - (3t^2-3)(2)}{(2t)^2}}{2t}$$

$$= \frac{12t^2 - 6t^2 + 6}{8t^3} = \frac{6t^2+6}{8t^3}$$

* If $t > 0 \longrightarrow$ Concave up
* If $t < 0 \longrightarrow$ Concave down.



$y = f(x) \qquad a \le x \le b$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

§ Arclength of Parametrized Curve.

If a curve C is given by

$x = f(t)$ and $y = g(t)$, $a \le t \le b$

where both f' and g' are continuous then the arclength of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Exple Find the arclength of the parametrized curve given by

$x = 5\cos t - \cos 5t$

$y = 5\sin t - \sin 5t \qquad 0 \le t \le \frac{\pi}{2}$

$$\frac{dx}{dt} = -5\sin t + 5\sin 5t$$

$$\frac{dy}{dt} = 5\cos t - 5\cos 5t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \underline{(-5\sin t + 5\sin 5t)^2} + (5\cos t - 5\cos 5t)^2$$

$$25\sin^2 t - 50\sin t \sin 5t + 25\sin^2 5t + 25\cos^2 t - 50\cos t\cos 5t + 25\cos^2 5t$$

$$= 25 + 25 - 50(\sin t\sin 5t + \cos t\cos 5t)$$

$$= 50 - 50\cos 4t$$

$$= 50(1-\cos 4t)$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= 1-\sin^2\alpha - \sin^2\alpha$$
$$= 1 - 2\sin^2\alpha$$

$\alpha = 2t$

$$1-\cos 4t = 1-(1-2\sin^2 2t) = \cancel{1}-\cancel{1}+2\sin^2 2t$$
$$= 2\sin^2 2t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 50(1-\cos 4t)$$
$$= (50)(2\sin^2 2t)$$
$$= 100\sin^2 2t$$
$$= (10\sin 2t)^2$$

$$\sqrt{\theta^2} = |\theta|$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{(10\sin 2t)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} 10\sin 2t \, dt$$

$$= 10\left[-\frac{1}{2}\cos 2t\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{10}{2}(\cos\pi - \cos 0)$$

$$= -\frac{10}{2}(-2) = 10$$

§ Area of Surfaces of Revolution.

If a smooth curve is given by

$x = f(t)$ and $y = g(t)$ $\quad a \le t \le b$

where both f' and g' are continuous and the does not cross itself, then the area of the surface of revolution obtained is given by

1.) $$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

if the rotation is about the x-axis with $g(t) > 0$.

2.) $$S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

if the rotation is about the y-axis with $f(t) > 0$.

Expl Find the area obtained by rotating the curve about the x-axis

$x = 3\cos t$ and $y = 3\sin t$ $\quad 0 \le t \le \frac{\pi}{3}$.

$$S = 2\pi \int_0^{\frac{\pi}{3}} 3\sin t \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt$$

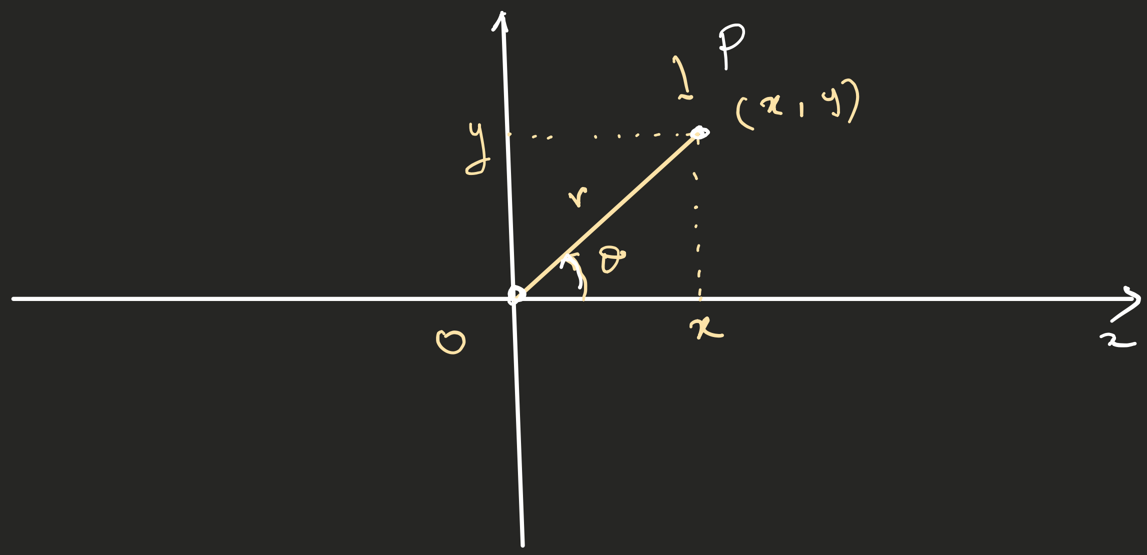
$$= 6\pi \int_0^{\frac{\pi}{3}} \sin t \sqrt{9}\, dt$$

$$= 18\pi \int_0^{\frac{\pi}{3}} \sin t\, dt$$

$$= 18\pi \left[-\cos t\right]_0^{\frac{\pi}{3}}$$

$$= 18\pi\left(-\frac{1}{2}+1\right) = \frac{18\pi}{2} = 9\pi.$$

§ Polar coordinates and Polar curves



* In the polar coordinate system, a point P is represented by the pair (r, θ), where

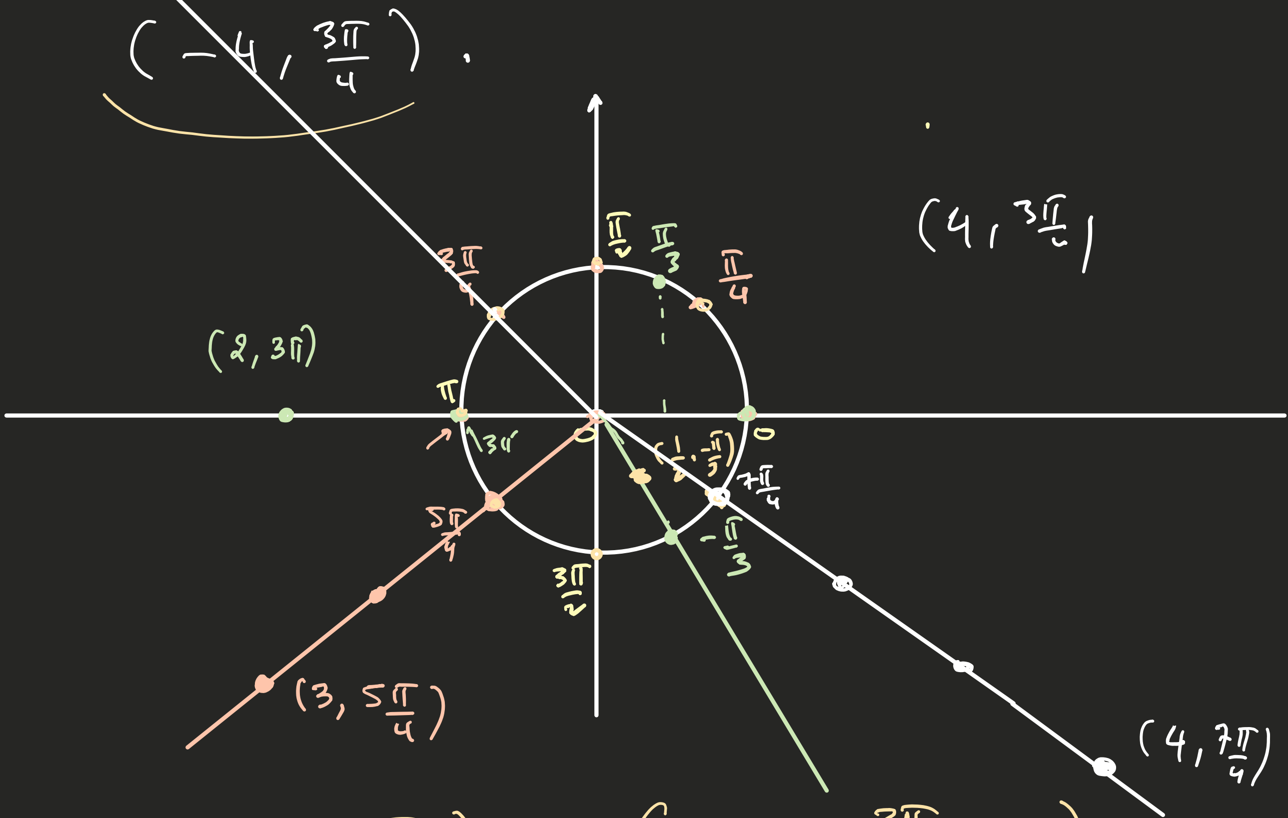
* r is the distance from the origin to the point P.
* θ is the angle counterclockwise from the positive x-axis to the line segment joining O and P.

Remark If $r > 0$ by $(-r, \theta)$ we mean $(r, \theta + \pi)$

$$\boxed{(-r, \theta) = (r, \theta + \pi) \quad (r > 0)}$$

Exple Plot the following points whose polar coordinates are

$\left(3, \frac{5\pi}{4}\right)$, $(2, 3\pi)$, $\left(\frac{1}{2}, -\frac{\pi}{3}\right)$

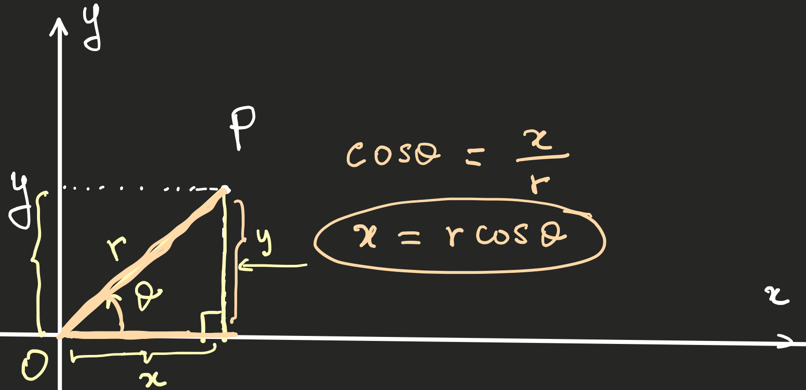


$$\left(-4, \frac{3\pi}{4}\right) = \left(4 + \frac{3\pi}{4} + \pi\right)$$
$$= \left(4, \frac{7\pi}{4}\right)$$

§ Coordinate Conversion

$\sin\theta = \frac{y}{r}$

$y = r\sin\theta$



$\cos\theta = \frac{x}{r}$

$x = r\cos\theta$

Exple Find the rectangular coordinates of the point $\left(\frac{3}{2}, \frac{\pi}{4}\right)$ given in polar coordinates

$r = \frac{3}{2}$, $\theta = \frac{\pi}{4}$

$x = ?$ $y = ?$

$x = \frac{3}{2}\cos\frac{\pi}{4} = \frac{3}{2} \cdot \frac{1}{\sqrt{2}} = \frac{3}{2\sqrt{2}}$

* If you know r and θ then

$x = r\cos\theta$
$y = r\sin\theta$

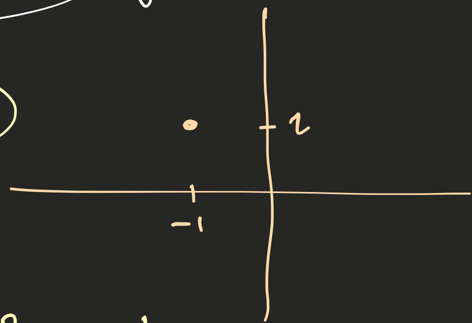
$y = \frac{3}{2} \sin \frac{\pi}{4} = \frac{3}{2} \cdot \frac{1}{\sqrt{2}} = \frac{3}{2\sqrt{2}}$

* If you know the rectangular coordinates x and y, then

$\longrightarrow r^2 = x^2 + y^2 \longrightarrow r = \pm\sqrt{x^2+y^2}$

$\longrightarrow \tan\theta = \frac{y}{x} \longrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Exple Find the polar coordinates of the points $(-1, 1)$ and $(0, 2)$ given in rectangular coordinates.



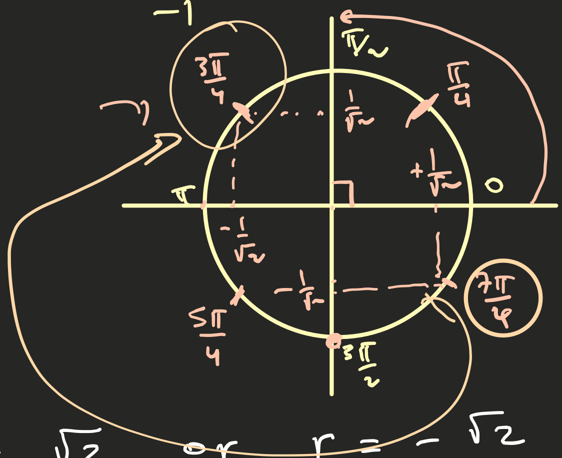
* $(-1, 1)$ $\quad x = -1$, $y = 1$

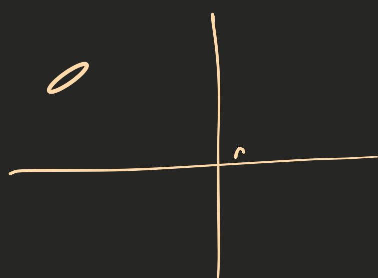
$r = \pm\sqrt{(-1)^2 + 1^2} = \pm\sqrt{2}$

* $\tan\theta = \frac{1}{-1} = -1$ $\qquad \tan\theta = -1$

$\theta = \frac{3\pi}{4} \leftarrow$

$\theta = \frac{7\pi}{4}$





* $r = \sqrt{2}$ or $r = -\sqrt{2}$

* $\theta = \frac{3\pi}{4}$ or $\theta = \frac{7\pi}{4}$

* If $r = \sqrt{2}$, $\theta = \frac{3\pi}{4}$

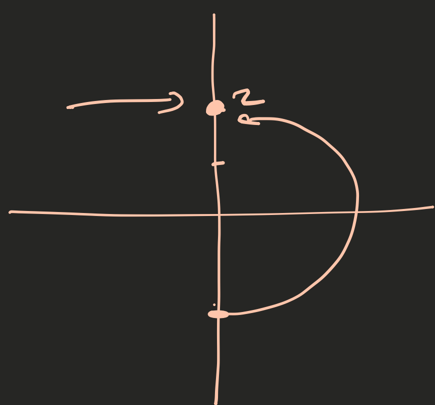
* If $r = -\sqrt{2}$, $\theta = \frac{7\pi}{4}$

$(-\sqrt{2}, \theta) = (\sqrt{2}, \theta + \pi)$

$(\sqrt{2}, \frac{3\pi}{4})$

$(-1, 1)$

$(-\sqrt{2}, \frac{7\pi}{4})$



$(0, 2)$ $\quad x = 0 \quad , \quad y = 2$

* $r = \pm\sqrt{4} = \pm 2$

* $\tan\theta = \frac{2}{0} = \pm\infty \qquad \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

$r = 2 \quad , \quad \theta = \frac{\pi}{2}$

$r = -2 \quad , \quad \theta = \frac{3\pi}{2}$

$(0, 2)$ — $\left(2, \frac{\pi}{2}\right)$ ✓

$(0, 2)$ — $\left(-2, \frac{3\pi}{2}\right)$ ✓

§ Polar curves

In polar coordinates, the graph of

$$r = f(\theta)$$

is called a polar curve.

Ex ple Sketch the following polar curves

1.) $r = 2$ ✓

2.) $\theta = \frac{\pi}{3}$

3.) $r = \sec\theta$

4.) $r = \sin\theta$ ←

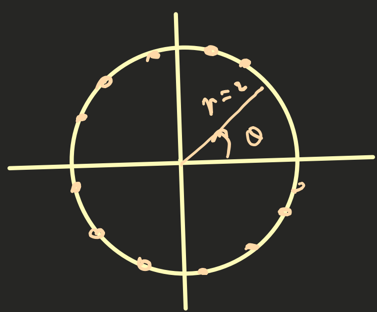
5.) $r = \cos\theta + 2\sin\theta$

1.) $r = 2.$

$r^2 = 4$

$x^2 + y^2 = 4$

this is a circle centered at $(0, 0)$ with radius 2.



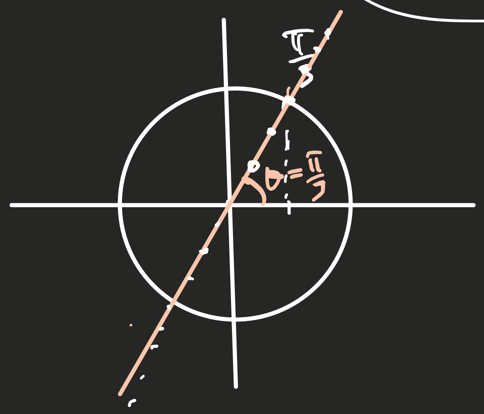
2) $\theta = \frac{\pi}{3}$

$$\tan\theta = \tan\frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\tan\theta = \sqrt{3}$$

$$\frac{y}{x} = \sqrt{3} \longleftrightarrow y = \sqrt{3}\,x$$

$\theta = \frac{\pi}{3}$ is the line $y = \sqrt{3}\,x$



3.) $r = \sec\theta$ $\qquad r = \frac{1}{\cos\theta} \iff \underbrace{r\cos\theta}_{} = 1$

$$x = 1$$

$r = \sec\theta \longleftrightarrow$ the vertical $x = 1$

4) $r = \sin\theta$ $\qquad x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

$$r^2 = r\sin\theta$$

$$x^2 + y^2 = y \iff$$

$$x^2 + \underbrace{y^2 - y}_{} = 0$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

this is a circle centered at $\left(0, \frac{1}{2}\right)$ with radius $\frac{1}{2}$.

5) $r = \cos\theta + 2\sin\theta$.

$$r^2 = r\cos\theta + 2r\sin\theta$$

$$x^2 + y^2 = x + 2y$$

$$x^2 - x + y^2 - 2y = 0$$

$$\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + (y-1)^2 - 1 = 0$$

$$\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = 1 + \frac{1}{4} = \frac{5}{4}$$

This is a circle centered at $\left(\frac{1}{2}, 1\right)$ with radius $\frac{\sqrt{5}}{2}$.