(11pts)**Problem 1**

Find the area of the region bounded by $y = x^2$ and $y = -x^2 + 6x$.
Solution
Method 1(without graphing)

$$\begin{aligned} A &= \int_a^b \left|\left(x^2 - \left(-x^2 + 6x\right)\right)\right| dx \\ &= \int_a^b \left|2x\left(x-3\right)\right| dx \qquad [3\ \textbf{points}] \end{aligned}$$

To find a and b, put

$$x^2 = -x^2 + 6x \Leftrightarrow 2x\left(x-3\right)$$
$$x = 0 \quad \text{or } x = 3.$$
$$a = 0 \quad \text{and} \quad b = 3 \qquad [4\ \textbf{points}]$$
$$A = \int_0^3 \left|2x\left(x-3\right)\right| dx$$

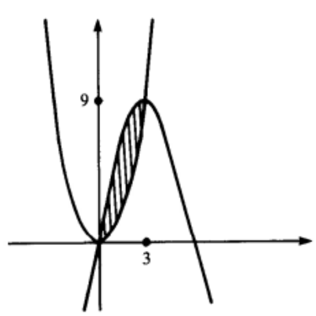
Since the quadratic function has the sign of $a = 2$ outside of the root $\left((-\infty,\ 0) \cup (3,\ \infty)\right)$ and the opposite sign of $a = 2$ inside of the root, then

$$\left|2x\left(x-3\right)\right| = -2x\left(x-3\right).$$

Thus

$$\begin{aligned} A &= \int_0^3 -2x\left(x-3\right) dx \\ &= 9. \qquad [4\ \textbf{points}] \end{aligned}$$

Method 2(with graphing)
You start by graphing both functions on the same window to get



[4 **points**]

The graph shows that $y = -x^2 + 6x$ is on top of $y = x^2$ on the interval $[0,\ 3]$. Hence,

$$\begin{aligned} A &= \int_0^3 \left[\left(-x^2 + 6x\right) - \left(x^2\right)\right] dx \qquad [3\ \textbf{points}] \\ &= \int_0^3 \left(6x - 2x^2\right) dx = 9 \qquad [4\ \textbf{points}] \end{aligned}$$

$(11pts)$**Problem 2**

Find the arc length of the curve

$$x = \ln \sin t, \qquad y = t, \qquad \frac{\pi}{6} \le t \le \frac{\pi}{2}.$$

Solution

$$L = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \cot^2 t + 1 = \csc^2 t \qquad [4 \textbf{ points}]$$

$$\begin{aligned} L &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\csc^2 t} dt \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc t dt \qquad [4 \textbf{ points}] \\ &= \ln |\csc t - \cot t| \Big|_{\pi/6}^{\pi/2} \\ &= -\ln \left(2 - \sqrt{3}\right) = 1.3170 \qquad [3 \textbf{ points}] \end{aligned}$$

$(11pts)$**Problem 3**
Find an equation of the tangent line to the curve $x = 3e^t$, $y = 5e^{-t}$ at $t = 0$.
Solution

$$\text{Slope} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-5e^{-t}}{3e^t}. \qquad [4\ \textbf{points}]$$

$$\text{Slope} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=0} = \frac{-5}{3}. \qquad [2\ \textbf{points}]$$

$$t = 0 \Rightarrow \quad x = 3 \ \text{ amd } \ y = 5. \qquad [2\ \textbf{points}]$$

The equation of the tangent line is

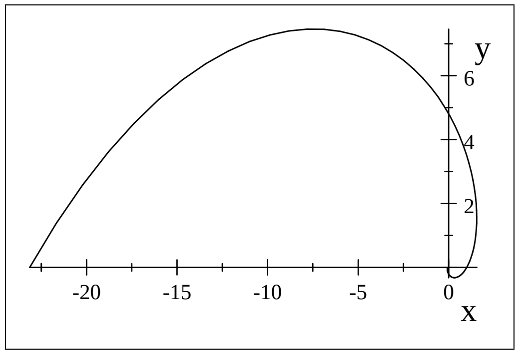
$$y = \frac{-5}{3}(x-3) + 5$$

$$y = 10 - \frac{5}{3}x. \qquad [3\ \textbf{points}]$$

$(11pts)$**Problem 4**

Find the arc length of the polar curve $r = e^{\theta}$ from $\theta = 0$ to $\theta = \ln 2$.

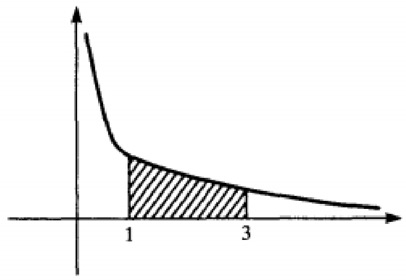
Solution



$$\begin{aligned} L &= \int_0^{\ln 2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{\ln 2} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta && [5 \textbf{ points}] \\ &= \sqrt{2} \int_0^{\ln 2} e^{\theta}\, d\theta \\ &= \sqrt{2} = 1.4142 && [6 \textbf{ points}] \end{aligned}$$

(12*pts*)**Problem 5**

The region bounded by $y = \dfrac{1}{x}$, $x = 1$, $x = 3$ is shown below



(a) Use the disk method to find the volume of the solid obtained by rotating the region about the x-axis.
(b) Use the cylindrical shell method to find the volume obtained by rotating the region about the y-axis.

Solution
(a)

$$\begin{aligned} V &= \int_1^3 \pi \left(\frac{1}{x}\right)^2 dx \\ &= \left.\frac{-\pi}{x}\right|_1^3 \\ &= \frac{2}{3}\pi = 2.0944 \qquad [6 \textbf{ points}] \end{aligned}$$

(b)

$$\begin{aligned} V &= \int_1^3 2\pi \cancel{x} \left(\frac{1}{\cancel{x}}\right) dx \\ &= \int_1^3 2\pi dx \\ &= 4\pi = 12.566 \qquad [6 \textbf{ points}] \end{aligned}$$

(11pts)**Problem 6**
Evaluate the following integrals

$$(1) \quad \int_0^{\pi} x \sin\left(x - \frac{\pi}{2}\right) dx, \qquad\qquad (2) \quad \int x^2 \ln x dx$$

Solution
(1)

$$\int_0^{\pi} x \sin\left(x - \frac{\pi}{2}\right) dx$$

By parts:

$$\begin{aligned} u &= x, \qquad\qquad u' = 1 \\ v' &= \sin\left(x - \frac{\pi}{2}\right), \qquad v = -\cos\left(x - \frac{\pi}{2}\right) \end{aligned}$$

$$\begin{aligned} \int_0^{\pi} x \sin\left(x - \frac{\pi}{2}\right) dx &= -x\cos\left(x - \frac{\pi}{2}\right)\Big|_0^{\pi} + \int_0^{\pi} \cos\left(x - \frac{\pi}{2}\right) dx \qquad\qquad [3 \textbf{ points}] \\ &= -x\cos\left(x - \frac{\pi}{2}\right)\Big|_0^{\pi} + \sin\left(x - \frac{\pi}{2}\right)\Big|_0^{\pi} \\ &= 0 + 2 = 2 \qquad\qquad [3 \textbf{ points}] \end{aligned}$$

(2)

$$\int x^2 \ln x dx$$

By parts:

$$\begin{aligned} u &= \ln x, \qquad\qquad u' = \frac{1}{x} \\ v' &= x^2, \qquad v = \frac{x^3}{3} \qquad\qquad [2 \textbf{ points}] \end{aligned}$$

$$\begin{aligned} \int x^2 \ln x dx &= \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx \\ &= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C \qquad\qquad [3 \textbf{ points}] \end{aligned}$$

(11pts)**Problem 7**
Evaluate the following integral

$$\int \frac{dx}{x^2\sqrt{x^2-9}}.$$

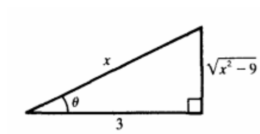
Solution
We will use trigonometric substitution. Put

$$x = 3\sec\theta, \quad \text{with} \quad 0 \le \theta < \frac{\pi}{2} \text{ or } \pi \le \theta < \frac{3\pi}{2}. \qquad [2 \textbf{ points}]$$

$$dx = 3\sec\theta\tan\theta d\theta \text{ and } \sqrt{x^2-9} = 3\tan\theta$$

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{x^2-9}} &= \int \frac{\cancel{3}\sec\theta\tan\theta d\theta}{(9\sec^2\theta)(\cancel{3}\tan\theta)} \\ &= \frac{1}{9}\int \cos\theta d\theta \\ &= \frac{1}{9}\sin\theta + C \qquad [5 \textbf{ points}] \end{aligned}$$

$$x = 3\sec\theta \Leftrightarrow \cos\theta = \frac{3}{x}$$



$$\sin\theta = \frac{\sqrt{x^2-9}}{x}.$$

Hence,

$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \frac{\sqrt{x^2-9}}{9x} + C \qquad [4 \textbf{ points}]$$

(11pts)**Problem 8**
Evaluate the integral

$$\int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4} dx$$

Solution
We will use partial fraction decomposition.
Since the degree of the numerator is greater than the denominator, we will start with a long division.

$$\frac{x^4 - 4x^2 + x + 1}{x^2 - 4} = x^2 + \frac{x+1}{x^2-4}. \qquad [4 \textbf{ points}]$$

Next, we perform a partial fraction decomposition on $\frac{x+1}{x^2-4}$.

$$\begin{aligned} \frac{x+1}{x^2-4} &= \frac{x+1}{(x-2)(x+2)} \\ &= \frac{3}{4(x-2)} + \frac{1}{4(x+2)}. \qquad [4 \textbf{ points}] \end{aligned}$$

Thus,

$$\begin{aligned} \int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4} dx &= \int \left[x^2 + \frac{3}{4(x-2)} + \frac{1}{4(x+2)} \right] dx \\ &= \frac{1}{3}x^3 + \frac{3}{4} \ln|x-2| + \frac{1}{4} \ln|x+2| + C \ . \qquad [3 \textbf{ points}] \end{aligned}$$

(11pts)**Problem 9**
Evaluate the integral

$$\int \frac{dx}{x\left(x^2+5\right)}\ .$$

Solution

$$\frac{1}{x\left(x^2+5\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+5}. \quad [3\ \textbf{points}]$$

So,

$$1 = A\left(x^2+5\right) + Bx^2 + Cx.$$

$$x = 0 \Rightarrow 1 = 5A, \quad A = \frac{1}{5}.$$

Now equate coefficients of x^2 to get

$$0 = A + B \Leftrightarrow B = -A = -\frac{1}{5}.$$

Next, equate the coefficients of x to get

$$C = 0.$$

Hence,

$$\frac{1}{x\left(x^2+5\right)} = \frac{1}{5}\left(\frac{1}{x}\right) - \frac{1}{5}\left(\frac{x}{x^2+5}\right). \qquad [6\ \textbf{points}]$$

$$\int \frac{dx}{x\left(x^2+5\right)} = \frac{1}{5}\ln|x| - \frac{1}{10}\ln\left(x^2+5\right) + C. \qquad [2\ \textbf{points}]$$