Week 8 Workshop Example Solutions

Example 1:

$$F(s) = \frac{6s^2 + 26s + 26}{(s+1)(s+2)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_1 = \frac{6 - 26 + 26}{(1)(2)} = 3; \qquad K_2 = \frac{24 - 52 + 26}{(-1)(1)} = 2$$

$$K_3 = \frac{54 - 78 + 26}{(-2)(-1)} = 1$$

$$\text{Therefore} \quad f(t) = [3e^{-t} + 2e^{-2t} + e^{-3t}]\, u(t)$$

$$F(s) = \frac{7s^2 + 63s + 134}{(s+3)(s+4)(s+5)} = \frac{K_1}{s+3} + \frac{K_2}{s+4} + \frac{K_3}{s+5}$$

$$K_1 = \frac{63 - 189 - 134}{1(2)} = 4; \qquad K_2 = \frac{112 - 252 + 134}{(-1)(1)} = 6$$

$$K_3 = \frac{175 - 315 + 134}{(-2)(-1)} = -3$$

$$f(t) = [4e^{-3t} + 6e^{-4t} - 3e^{-5t}]u(t)$$

Problem 2:

We begin by noting that $F(s)$ is a proper rational function. Next we must find the roots of the quadratic term $s^2 + 6s + 25$:

With the denominator in factored form, we proceed as before:

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} \equiv$$

$$\frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}.$$

To find K_1, K_2, and K_3, we use the same process as before:

$$K_1 = \left.\frac{100(s+3)}{s^2+6s+25}\right|_{s=-6} = \frac{100(-3)}{25} = -12,$$

$$K_2 = \left.\frac{100(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4} = \frac{100(j4)}{(3+j4)(j8)}$$

$$= 6 - j8 = 10e^{-j53.13^\circ},$$

$$K_3 = \left.\frac{100(s+3)}{(s+6)(s+3-j4)}\right|_{s=-3-j4} = \frac{100(-j4)}{(3-j4)(-j8)}$$

$$= 6 + j8 = 10e^{j53.13^\circ}.$$

Then

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle -53.13^\circ}{s+3-j4}$$

$$+ \frac{10\angle 53.13^\circ}{s+3+j4}.$$

$$\mathscr{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = (-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t}$$

$$+ 10e^{j53.13^\circ}e^{-(3+j4)t})u(t). \quad ($$

Problem 3:

$$F(s) = \frac{10(s^2 + 119)}{(s+5)(s^2 + 10s + 169)}$$

$$s_{1,2} = -5 \pm \sqrt{25 - 169} = -5 \pm j12$$

$$F(s) = \frac{K_1}{s+5} + \frac{K_2}{s+5-j12} + \frac{K_2^*}{s+5+j12}$$

$$K_1 = \frac{10(25 + 119)}{25 - 50 + 169} = 10$$

$$K_2 = \frac{10[(-5+j12)^2 + 119]}{(j12)(j24)} = j4.17 = 4.17\underline{/90^\circ}$$

Therefore

$$\begin{aligned} f(t) &= [10e^{-5t} + 8.33e^{-5t}\cos(12t + 90^\circ)]\, u(t) \\ &= [10e^{-5t} - 8.33e^{-5t}\sin 12t]\, u(t) \end{aligned}$$

Problem 4:

[a] $\mathcal{L}\{t\} = \frac{1}{s^2}$; therefore $\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$

[b] $\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{j2}$

Therefore

$$\mathcal{L}\{\sin\omega t\} = \left(\frac{1}{j2}\right)\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \left(\frac{1}{j2}\right)\left(\frac{2j\omega}{s^2+\omega^2}\right)$$
$$= \frac{\omega}{s^2+\omega^2}$$

[c] $\sin(\omega t + \theta) = (\sin\omega t\cos\theta + \cos\omega t\sin\theta)$

Therefore

$$\mathcal{L}\{\sin(\omega t+\theta)\} = \cos\theta\mathcal{L}\{\sin\omega t\} + \sin\theta\mathcal{L}\{\cos\omega t\}$$
$$= \frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}$$

[d] $\mathcal{L}\{t\} = \int_0^\infty te^{-st}\,dt = \frac{e^{-st}}{s^2}(-st-1)\Big|_0^\infty = 0 - \frac{1}{s^2}(0-1) = \frac{1}{s^2}$

[e] $f(t) = \cosh t\cosh\theta + \sinh t\sinh\theta$

From Assessment Problem 12.1(a)

$$\mathcal{L}\{\cosh t\} = \frac{s}{s^2-1}$$

From Assessment Problem 12.1(b)

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1}$$

$$\therefore\ \mathcal{L}\{\cosh(t+\theta)\} = \cosh\theta\left[\frac{s}{(s^2-1)}\right] + \sinh\theta\left[\frac{1}{s^2-1}\right]$$
$$= \frac{\sinh\theta + s[\cosh\theta]}{(s^2-1)}$$

Problem 5:

[a] $\frac{1}{L}\int_0^t v_1\,d\tau + \frac{v_1 - v_2}{R} = i_g u(t)$

and

$$C\frac{dv_2}{dt} + \frac{v_2}{R} - \frac{v_1}{R} = 0$$

[b] $\frac{V_1}{sL} + \frac{V_1 - V_2}{R} = I_g$

$$\frac{V_2 - V_1}{R} + sCV_2 = 0$$

or

$$(R+sL)V_1(s) - sLV_2(s) = RLsI_g(s)$$

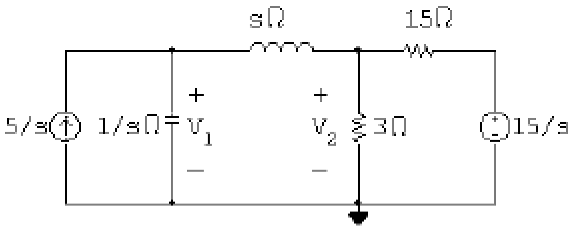
$$-V_1(s) + (RCs+1)V_2(s) = 0$$

Solving,

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}$$

Problem 6

[a]



The two node voltage equations are

$$\frac{V_1 - V_2}{s} + V_1 s = \frac{5}{s} \quad \text{and} \quad \frac{V_2}{3} + \frac{V_2 - V_1}{s} + \frac{V_2 - (15/s)}{15} = 0$$

Solving for V_1 and V_2 yields

$$V_1 = \frac{5(s+3)}{s(s^2+2.5s+1)}, \qquad V_2 = \frac{2.5(s^2+6)}{s(s^2+2.5s+1)}$$

[b] The partial fraction expansions of V_1 and V_2 are

$$V_1 = \frac{15}{s} - \frac{50/3}{s+0.5} + \frac{5/3}{s+2} \quad \text{and} \quad V_2 = \frac{15}{s} - \frac{125/6}{s+0.5} + \frac{25/3}{s+2}$$

It follows that

$$v_1(t) = \left[15 - \frac{50}{3}e^{-0.5t} + \frac{5}{3}e^{-2t}\right] u(t)\,\text{V} \quad \text{and}$$

$$v_2(t) = \left[15 - \frac{125}{6}e^{-0.5t} + \frac{25}{3}e^{-2t}\right] u(t)\,\text{V}$$

[c] $v_1(0^+) = 15 - \dfrac{50}{3} + \dfrac{5}{3} = 0$

$$v_2(0^+) = 15 - \frac{125}{6} + \frac{25}{3} = 2.5\,\text{V}$$

[d] $v_1(\infty) = 15\,\text{V}; \qquad v_2(\infty) = 15\,\text{V}$