Week 4 Workshop Example Solutions

Example 1:

$$i_C(0) = 0; \qquad v_o(0) = 50\,\mathrm{V}$$

$$\alpha = \frac{R}{2L} = \frac{8000}{2(160 \times 10^{-3})} = 25{,}000\,\mathrm{rad/s}$$

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{(160 \times 10^{-3})(10 \times 10^{-9})} = 625 \times 10^6$$

$$\therefore \quad \alpha^2 = \omega_o^2; \qquad \text{critical damping}$$

$$v_o(t) = V_f + D_1' t e^{-25{,}000t} + D_2' e^{-25{,}000t}$$

$$V_f = 250\,\mathrm{V}$$

$$v_o(0) = 250 + D_2' = 50; \qquad D_2' = -200\,V$$

$$\frac{dv_o}{dt}(0) = -25{,}000 D_2' + D_1' = 0$$

$$D_1' = 25{,}000 D_2' = -5 \times 10^6\ \mathrm{V/s}$$

$$v_o = 250 - 5 \times 10^6 t e^{-25{,}000t} - 200 e^{-25{,}000t}\,\mathrm{V}, \quad t \geq 0$$

Example 2:

$$\alpha = \frac{R}{2L} = 2000\,\mathrm{rad/s}$$

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{(62.5 \times 10^{-3})(6.25 \times 10^{-6})} = 256 \times 10^4$$

$$s_{1,2} = -2000 \pm \sqrt{4 \times 10^6 - 256 \times 10^4} = -2000 \pm j1200\,\mathrm{rad/s}$$

$$v_o = V_f + A_1' e^{-800t} + A_2' e^{-3200t}$$

$$v_o(0) = 0 = V_f + A_1' + A_2'$$

$$v_o(\infty) = 60\,\mathrm{V}; \qquad \therefore \quad A_1' + A_2' = -60$$

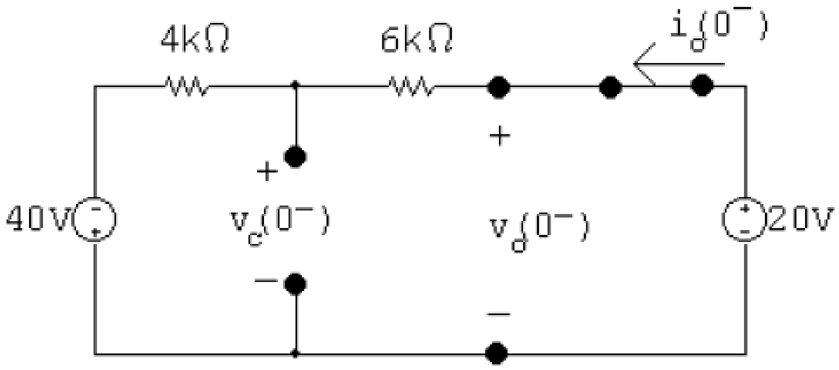
$$\frac{dv_o(0)}{dt} = 0 = -800A_1' - 3200A_2'$$

$$\therefore \quad A_1' = -80\,\mathrm{V}; \qquad A_2' = 20\,\mathrm{V}$$

$$v_o = 60 - 80e^{-800t} + 20e^{-3200t}\,\mathrm{V}, \quad t \geq 0$$

Example 3:

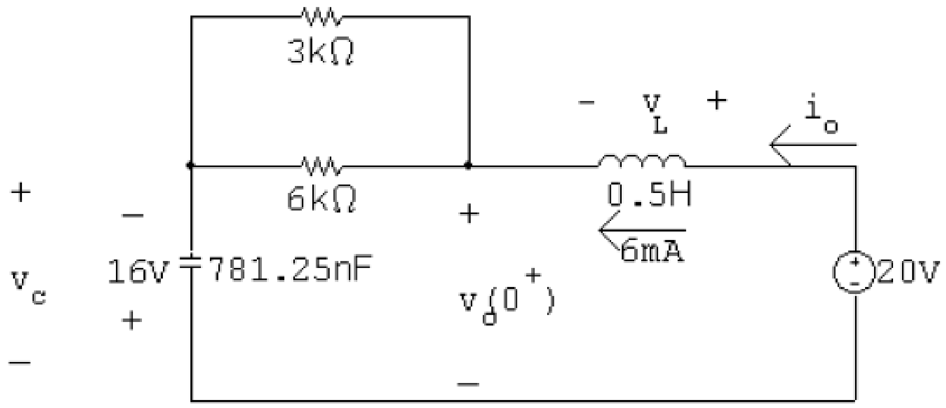
[a] $t < 0$:



$$i_o(0^-) = \frac{60}{10{,}000} = 6\,\text{mA}$$

$$v_C(0^-) = 20 - (6000)(0.006) = -16\,\text{V}$$

$t = 0^+$:



$$3\,\text{k}\Omega \| 6\,\text{k}\Omega = 2\,\text{k}\Omega$$

$$\therefore \quad v_o(0^+) = (0.006)(2000) - 16 = 12 - 16 = -4\,\text{V}$$

$$\text{and } v_L(0^+) = 20 - (-4) = 24\,\text{V}$$

[b] $v_o(t) = 2000 i_o + v_C$

$$\frac{dv_o}{dt}(t) = 2000\frac{di_o}{dt} + \frac{dv_C}{dt}$$

$$\frac{dv_o}{dt}(0^+) = 2000\frac{di_o}{dt}(0^+) + \frac{dv_C}{dt}(0^+)$$

$$v_L(0^+) = L\frac{di_o}{dt}(0^+)$$

$$\frac{di_o}{dt}(0^+) = \frac{v_L(0^+)}{L} = \frac{24}{0.5} = 48 \text{ A/s}$$

$$C\frac{dv_c}{dt}(0^+) = i_o(0^+)$$

$$\therefore \quad \frac{dv_c}{dt}(0^+) = \frac{6 \times 10^{-3}}{781.25 \times 10^{-9}} = 7680$$

$$\therefore \quad \frac{dv_o}{dt}(0^+) = 2000(48) + 7680 = 103{,}680 \text{ V/s}$$

[c] $\omega_o^2 = \frac{1}{LC} = 2.56 \times 10^6; \qquad \omega_o = 1600 \text{ rad/s}$

$$\alpha = \frac{R}{2L} = 2000 \text{ rad/s}$$

$$\alpha^2 > \omega_o^2 \qquad \text{overdamped}$$

$$s_{1,2} = -2000 \pm j1200 \text{ rad/s}$$

$$v_o(t) = V_f + A_1' e^{-800t} + A_2' e^{-3200t}$$

$$V_f = v_o(\infty) = 20\,\text{V}$$

$$20 + A_1' + A_2' = -4; \qquad -800A_1' - 3200A_2' = 103{,}680$$

$$\text{Solving} \qquad A_1' = 11.2; \qquad A_2' = -35.2$$

$$\therefore \quad v_o(t) = 20 + 11.2e^{-800t} - 35.2e^{-3200t}\,\text{V}, \qquad t \geq 0^+$$

Example 4:

[a] Let i be the current in the direction of the voltage drop $v_o(t)$. Then by hypothesis

$$i = i_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t$$

$$i_f = i(\infty) = 0, \qquad i(0) = \frac{V_g}{R} = B_1'$$

Therefore $\quad i = B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t$

$$L\frac{di(0)}{dt} = 0, \qquad \text{therefore} \quad \frac{di(0)}{dt} = 0$$

$$\frac{di}{dt} = [(\omega_d B_2' - \alpha B_1') \cos \omega_d t - (\alpha B_2' + \omega_d B_1') \sin \omega_d t]\, e^{-\alpha t}$$

Therefore $\quad \omega_d B_2' - \alpha B_1' = 0; \qquad B_2' = \dfrac{\alpha}{\omega_d} B_1' = \dfrac{\alpha}{\omega_d}\dfrac{V_g}{R}$

Therefore

$$v_o = L\frac{di}{dt} = -\left\{ L\left(\frac{\alpha^2 V_g}{\omega_d R} + \frac{\omega_d V_g}{R}\right) \sin \omega_d t \right\} e^{-\alpha t}$$

$$= -\left\{ \frac{LV_g}{R}\left(\frac{\alpha^2}{\omega_d} + \omega_d\right) \sin \omega_d t \right\} e^{-\alpha t}$$

$$= -\frac{V_g L}{R}\left(\frac{\alpha^2 + \omega_d^2}{\omega_d}\right) e^{-\alpha t} \sin \omega_d t$$

$$= -\frac{V_g L}{R}\left(\frac{\omega_o^2}{\omega_d}\right) e^{-\alpha t} \sin \omega_d t$$

$$= -\frac{V_g L}{R\omega_d}\left(\frac{1}{LC}\right) e^{-\alpha t} \sin \omega_d t$$

$$v_o = -\frac{V_g}{RC\omega_d} e^{-\alpha t} \sin \omega_d t \,\mathrm{V}, \quad t \geq 0$$

[b] $$\frac{dv_o}{dt} = -\frac{V_g}{\omega_d RC}\{\omega_d \cos\omega_d t - \alpha \sin\omega_d t\}e^{-\alpha t}$$

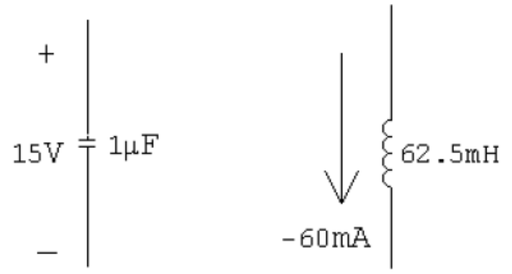
$$\frac{dv_o}{dt} = 0 \quad \text{when} \quad \tan\omega_d t = \frac{\omega_d}{\alpha}$$

$$\text{Therefore} \quad \omega_d t = \tan^{-1}(\omega_d/\alpha) \quad (\text{smallest } t)$$

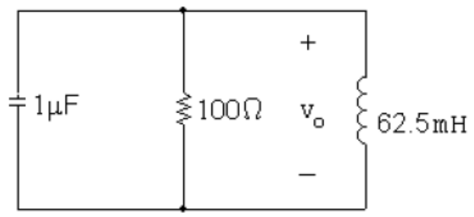
$$t = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{\alpha}\right)$$

Example 5:

$$t < 0: \qquad V_o = 15\,\mathrm{V}, \qquad I_o = -60\,\mathrm{mA}$$



$t > 0$:



$$i_R(0) = \frac{15}{100} = 150\,\mathrm{mA}; \qquad i_\mathrm{L}(0) = -60\,\mathrm{mA}$$

$$i_\mathrm{C}(0) = -150 - (-60) = -90\,\mathrm{mA}$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2(100)(10^{-6})} = 5000 \text{ rad/s}$$

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{(62.5 \times 10^{-3})(10^{-6})} = 16 \times 10^6$$

$$s_{1,2} = -5000 \pm \sqrt{25 \times 10^6 - 16 \times 10^6} = -5000 \pm 3000$$

$$s_1 = -2000 \text{ rad/s}; \qquad s_2 = -8000 \text{ rad/s}$$

$$\therefore \quad v_o = A_1 e^{-2000t} + A_2 e^{-8000t}$$

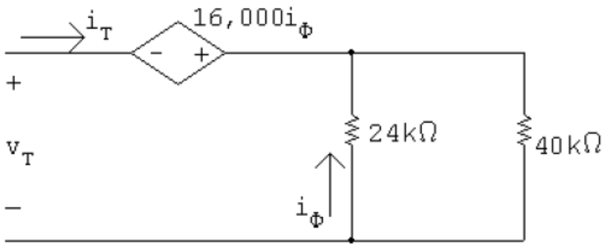
$$A_1 + A_2 = v_o(0) = 15$$

$$\frac{dv_o}{dt}(0) = -2000A_1 - 8000A_2 = \frac{-90 \times 10^{-3}}{10^{-6}} = -90{,}000$$

$$\text{Solving,} \qquad A_1 = 5\,\mathrm{V}, \qquad A_2 = 10\,\mathrm{V}$$

$$\therefore \quad v_o = 5e^{-2000t} + 10e^{-8000t}\,\mathrm{V}, \qquad t \geq 0$$

Example 6:



$$v_T = -16{,}000 i_\phi + i_T(15{,}000) = -16{,}000\frac{-i_T(40)}{64} + i_t(15{,}000)$$

$$\frac{v_T}{i_T} = 10{,}000 + 15{,}000 = 25\,\text{k}\Omega$$

$$V_o = \frac{4000}{5000}(7.5) = 6\,\text{V}; \qquad I_o = 0$$

$$i_\text{C}(0) = -i_R(0) - i_\text{L}(0) = -\frac{6}{25{,}000} = -240\,\mu\text{A}$$

$$\frac{i_\text{C}(0)}{C} = \frac{-240\times 10^{-6}}{4\times 10^{-9}} = -60{,}000$$

$$\omega_o^2 = \frac{1}{LC} = \frac{10^9}{(4)(15.625)} = 16\times 10^6; \qquad \omega_o = 4000\ \text{rad/s}$$

$$\alpha = \frac{1}{2RC} = \frac{10^9}{(2)(4)(25\times 10^3)} = 5000\ \text{rad/s}$$

$$\alpha^2 > \omega_0^2 \qquad \text{so the response is overdamped}$$

$$s_{1,2} = -5000 \pm \sqrt{5000^2 - 4000^2} = -5000 \pm 3000\ \text{rad/s}$$

$$v_o = A_1 e^{-2000t} + A_2 e^{-8000t}$$

$$v_o(0) = A_1 + A_2 = 6\,\text{V}$$

$$\frac{dv_o}{dt}(0) = -2000A_1 - 8000A_2 = -60{,}000$$

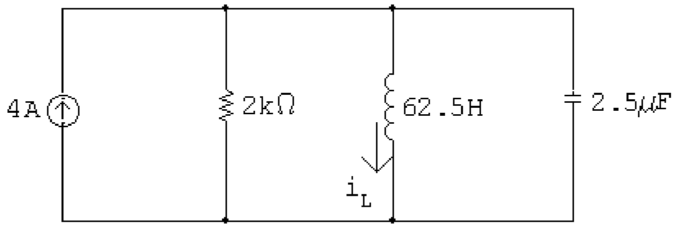
$$\therefore\quad A_1 = -2\,\text{V}; \qquad\qquad A_2 = 8\,\text{V}$$

$$v_o = 8e^{-8000t} - 2e^{-2000t}\,\text{V}, \qquad t \ge 0$$

Example 7:

$t < 0: \qquad i_L(0^-) = \dfrac{-15}{3000} = -5\,\mathrm{mA}; \qquad\qquad v_C(0^-) = 0\,\mathrm{V}$

The circuit reduces to:



$$i_L(\infty) = 4\,\mathrm{mA}$$

$$\omega_o^2 = \frac{1}{LC} = \frac{10^6}{(62.5)(2.5)} = 6400; \qquad \omega_o = 80\ \mathrm{rad/s}$$

$$\alpha = \frac{1}{2RC} = \frac{10^6}{(4000)(2.5)} = 100$$

$$s_{1,2} = -100 \pm \sqrt{100^2 - 80^2} = -100 \pm 60$$

$$s_1 = -40\ \mathrm{rad/s}; \qquad s_2 = -160\ \mathrm{rad/s}$$

$$i_L = I_f + A_1' e^{-40t} + A_2' e^{-160t}$$

$$i_L(\infty) = I_f = 4\mathrm{mA}$$

$$i_L(0) = A_1' + A_2' + I_f = -5\,\mathrm{mA}$$

$$\therefore \quad A_1' + A_2' + 4 = -5 \quad \text{so} \quad A_1' + A_2' = -9\,\mathrm{mA}$$

$$\frac{di_L}{dt}(0) = 0 = -40A_1 - 160A_2'$$

$$\text{Solving,} \qquad A_1' = -12\,\mathrm{mA}, \qquad A_2' = 3\,\mathrm{mA}$$

$$i_L = 4 - 12e^{-40t} + 3e^{-160t}\,\mathrm{mA}, \qquad t \geq 0$$

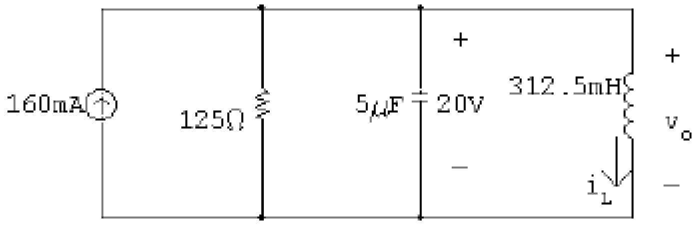
Example 8:

$t < 0$:

$$v_o(0^-) = v_o(0^+) = \frac{625}{781.25}(25) = 20\,\mathrm{V}$$

$$i_\mathrm{L}(0^-) = i_\mathrm{L}(0^+) = 0$$

$t > 0$:



$$-160 \times 10^{-3} + \frac{20}{125} + i_\mathrm{C}(0^+) + 0 = 0; \qquad \therefore \quad i_\mathrm{C}(0^+) = 0$$

$$\frac{1}{2RC} = \frac{1}{2(125)(5 \times 10^{-6})} = 800\,\mathrm{rad/s}$$

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{(312.5 \times 10^{-3})(5 \times 10^{-6})} = 64 \times 10^4$$

$$\therefore \quad \alpha^2 = \omega_o^2 \quad \text{critically damped}$$

[a] $v_o = V_f + D_1' t e^{-800t} + D_2' e^{-800t}$

$$V_f = 0$$

$$\frac{dv_o(0)}{dt} = -800 D_2' + D_1' = 0$$

$$v_o(0^+) = 20 = D_2'$$

$$D_1' = 800 D_2' = 16{,}000\,\mathrm{V/s}$$

$$\therefore \quad v_o = 16{,}000 t e^{-800t} + 20 e^{-800t}\,\mathrm{V}, \quad t \geq 0^+$$

[b] $i_L = I_f + D_3' t e^{-800t} + D_4' e^{-800t}$

$$i_L(0^+) = 0; \qquad I_f = 160\,\text{mA}; \qquad \frac{di_L(0^+)}{dt} = \frac{20}{312.5 \times 10^{-3}} = 64 \text{ A/s}$$

$$\therefore \quad 0 = 160 + D_4'; \qquad D_4' = -160\,\text{mA};$$

$$-800D_4' + D_3' = 64; \qquad D_3' = -64\,\text{A/s}$$

$$\therefore \quad i_L = 160 - 64{,}000te^{-800t} - 160e^{-800t}\,\text{mA} \qquad t \geq 0$$