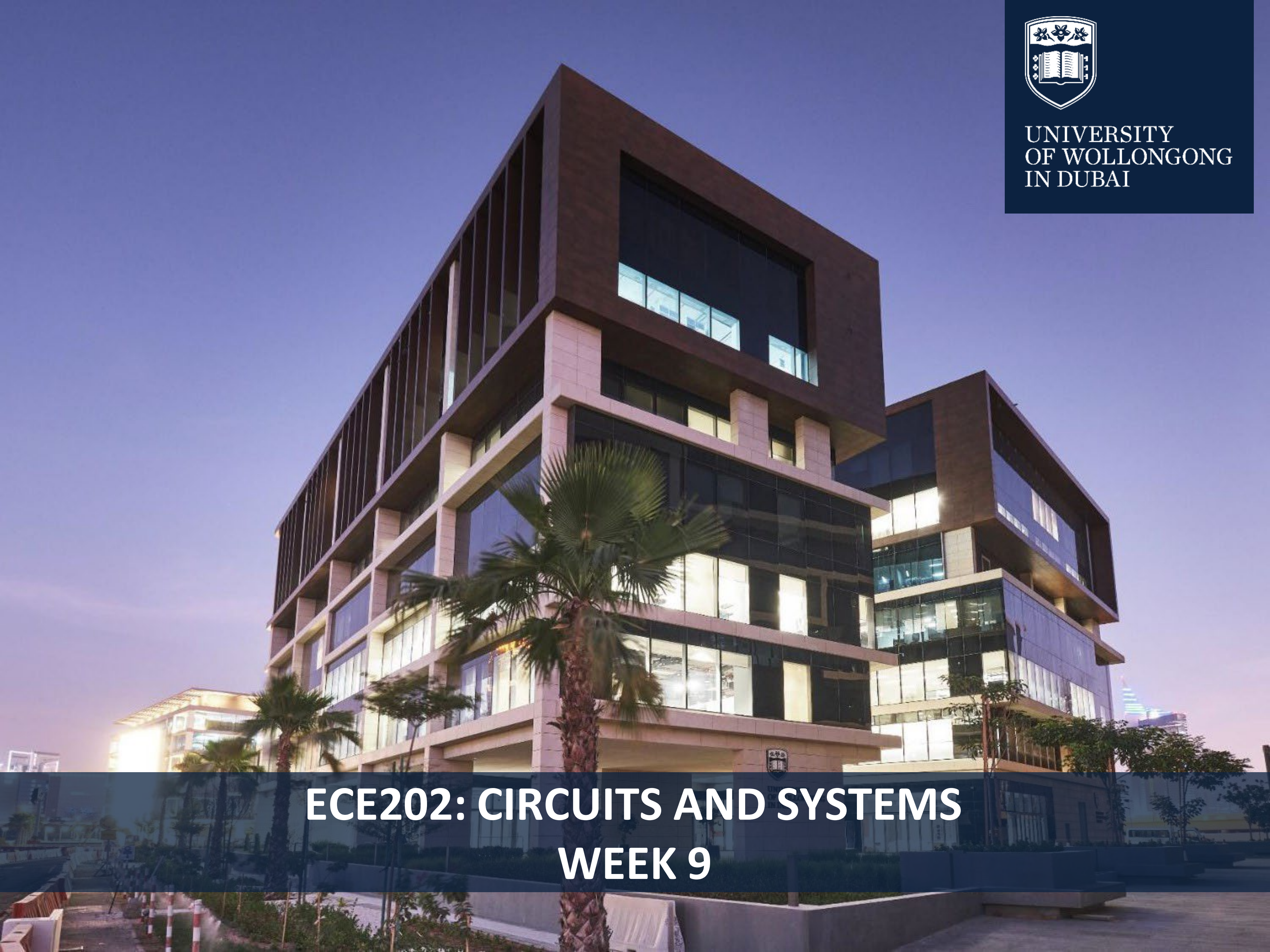

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ECE202: CIRCUITS AND SYSTEMS
WEEK 9

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Laplace Transform and its Applications (Chapter 15 & 16)

Laplace Transform

Definition of Laplace Transform

- The Laplace Transform is an integral transformation of a function *f(t)* from the time domain to the complex frequency domain, giving *F(s)*.
- The two-sided Laplace transform of a signal *f*(*t*) is defined as:

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

- The single-sided (unilateral) Laplace transform of a signal f(t) is defined as:

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt$$

Example Laplace Transforms

2. Find LT of exponential function e^{-at}

$$LT(e^{-at}) = \int_0^{\infty} e^{-at}\, e^{-st} dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty} = \frac{1}{s+a}$$

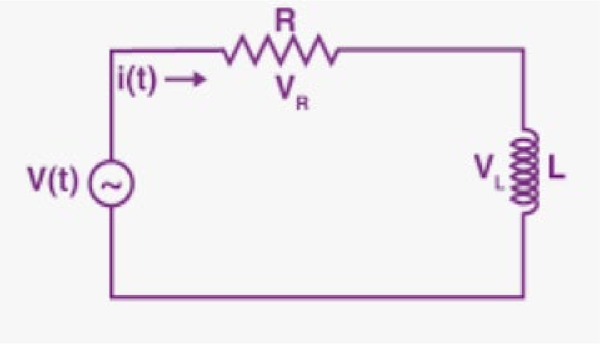
3. Find LT of the delta function δ(t):

$$LT[\delta(t)] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-0} = 1$$

Since $\delta(t)$ is 1 at $t = 0$, and zero everywhere else.

Simple RL Circuit Example

- A simple differential equation can be derived to describe the system



$$L\frac{di(t)}{dt}+Ri(t)=v(t)$$

$$=Vu(t), \qquad t>0$$

- Input: voltage, $v(t)$ (in this case, a unit step)
- Output: current, $i(t)$
- One method: Solve the DE for different input voltages

Alternative solution

- Take Laplace transforms of both sides and then solve

$$L\left(sI(s)-i(0)\right)+RI(s)=V(s)=\frac{V}{s}$$

- A linear equation
- Assuming $i(0)=0$, simplification yields:

$$I(s)=\frac{V/L}{s(s+R/L)}$$

- To find $i(t)$, take the inverse transform

Hence: Laplace transforms convert differential equations to algebraic equations, which are simpler to solve.

Laplace Transform Tables

- Describe transforms of different functions
- Also describe various properties
- Much of this is covered in MATH283

Example LT pairs

$$(ii) \quad \delta(t) \leftrightarrow 1$$

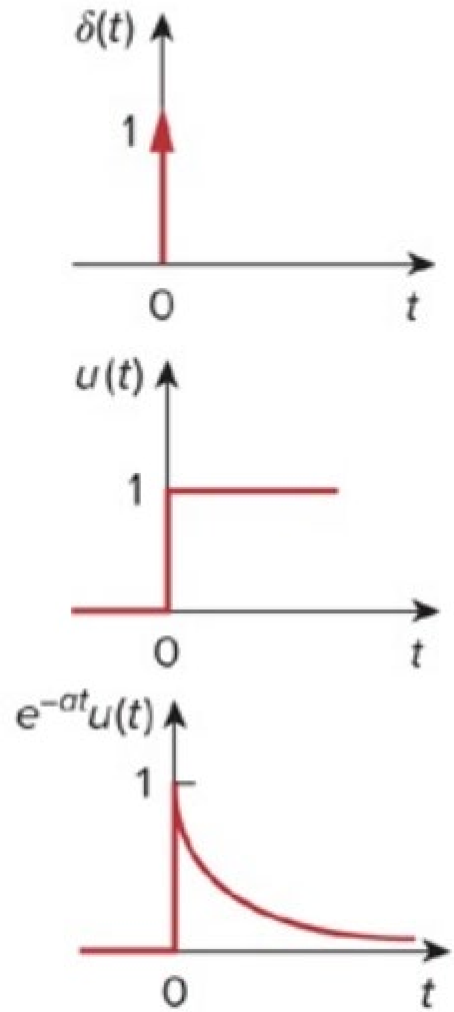
$$u(t) \leftrightarrow \frac{1}{s}$$

$$u(t)e^{-at} \leftrightarrow \frac{1}{s+a}$$

$$(iii) \quad u(t)\sin\omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}$$

$$(iii) \quad u(t)\cos\omega t \leftrightarrow \frac{s}{s^2+\omega^2}$$

$$(iv) \quad u(t)t^n \leftrightarrow \frac{n!}{s^{n+1}}$$



LT pairs

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{s}$
e^{-at}	$\dfrac{1}{s+a}$
t	$\dfrac{1}{s^2}$
t^n	$\dfrac{n!}{s^{n+1}}$
te^{-at}	$\dfrac{1}{(s+a)^2}$
$t^n e^{-at}$	$\dfrac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\dfrac{s \sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\dfrac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

LT properties

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)dx$	$\frac{1}{s} F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds} F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$

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Examples

Use the LT tables and properties to find the find the Laplace Transform of the following functions

1. $Cos(2t+30°)$
2. $\delta(t) + t$
3. te^{-2t} u(t)
4. t^2e^{-2t} u(t)
5. cos(5t)u(t)
6. $e^{-4t}\,\mathrm{sin}(\mathrm{t})$ u(t)
7. $\delta(t)$+2 u(t)-3 e^{-5t} u(t)
8. $\cos(10t - 10)u(t-1)$
9. $t^2 \sin 2t\, u(\mathrm{t})$
10. $e^{-3t}u(t-2)$

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Inverse Laplace Transform

$$\text{Let } F(s) = \frac{N(s)}{D(s)} \text{ where } N \ \& \ D \text{ are polynomials in } s \text{ of order } m \text{ and } n \textbf{ respectively.}$$

The roots of N(s) & D(s) are called zeros & poles respectively. F(s) can always be reduced to a ratio with m < n by polynomial division.

Example

$$\text{Find the ILT of } F(s) = \frac{s^2 + s + 1}{(s+1)(s+2)}$$

$$= \boxed{\frac{s^2 + s + 1}{(s^2 + 3s + 2)}} = 1 - \boxed{\frac{2s + 1}{(s^2 + 3s + 2)}}$$

Inverse Laplace Transform

$$F(s) = \frac{2s+1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

Method 1 : Multiply by denominator & equate powers of s

$$2s + 1 = A(s+2) + B(s+1)$$

$$\therefore \qquad 2s + 1 = (A + B)s + (2A + B)$$

equate like coefficients of s

$$2 = A + B \quad \& \quad 1 = 2A + B$$

$$\textit{giving} \quad A = -1 \quad \& \quad B = 3$$

$$\textit{hence} \quad F(s) = 1 - \frac{-1}{(s+1)} + \frac{3}{(s+2)} \qquad \therefore$$

$$f(t) = \delta(t) + e^{-t} - 3e^{-2t}$$

General solution for simple poles

$$F(s) = \sum_{j=1}^{n} \frac{k_j}{\left(s + p_j\right)}$$

$$k_j = (s + p_j)F(s)\Big|_{s=-p_j}, \quad j = 1, 2, + \; n$$

Example, repeated poles

We need a term for each power of the repeated root

$$F(s) = \frac{S+1}{(s+2)^2} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2}$$

Soln : Multiply both sides by the denominator, D(s)

$$s + 1 = As + (2A + B)$$

equating terms $\quad A = 1,\ B = -1$

$$\&\ thus \quad F(s) = \frac{1}{(s+2)} - \frac{1}{(s+2)^2}$$

$$f(t) = e^{-2t} - te^{-2t} = (1-t)e^{-2t}$$

Example, repeated poles

$$Reactive \;\; circuit \;\; example:$$

$$e^{-t} = x + \frac{dx}{dt} \quad \text{and} \quad x(0^-) = 0$$

$$Ans. \quad LT \rightarrow \quad \frac{1}{s+1} = X(s) + sX(s) - x(0^-) = (s+1)X(s) \rightarrow X(s) = \frac{1}{(s+1)^2}$$

$$using\; ILT \; pairs \rightarrow \quad x(t) = te^{-t}$$

Complex poles

Leads to a partial fraction term with a quadratic denominator. These can be found by *completing the square* in denominator re-expressed in frequency-shift form.

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s)$$

Remaining portion of F(s) not containing quadratic forms

f-shift

$$s^2 + as + b \quad \rightarrow \quad s^2 + 2\alpha s + \alpha^2 + \beta^2 \quad \rightarrow \quad (s+\alpha)^2 + \beta^2$$

$$A_1 s + A_2 \quad \rightarrow \quad A_1(s+\alpha) + B_1\beta$$

$$F(s) = \frac{A_1(s+\alpha)}{(s+\alpha)^2 + \beta^2} + \frac{B_1\beta}{(s+\alpha)^2 + \beta^2} + F_1(s)$$

$$\sin \omega t \quad \leftrightarrow \quad \frac{\omega}{s^2 + \omega^2}$$

$$\cos \omega t \quad \leftrightarrow \quad \frac{s}{s^2 + \omega^2}$$

$$e^{-\alpha t} f(t)u(t) \leftrightarrow F(s+\alpha)$$

$$f(t) = (A_1 e^{-\alpha t} \cos \beta t + B_1 e^{-\alpha t} \sin \beta t)u(t) + f_1(t)$$

Example, complex poles

$$F(s) = \frac{s+1}{s^2+6s+10} \quad \rightarrow \quad = \frac{(s+3)}{(s+3)^2+1} - \frac{2}{(s+3)^2+1}$$

$$\boxed{f(t) = e^{-3t}(\cos t - 2\sin t)}$$

$$F(s) = \frac{10}{(s+2)(s^2+6s+10)} \quad \rightarrow \quad \frac{A}{(s+2)} + \frac{Bs+C}{(s^2+6s+10)}$$

$$\therefore \qquad 10 = (A+B)s^2 + (6A+2B+C)s + (10A+2C)$$

$$giving \qquad A=5, \quad B=-5, \quad C=-20$$

equating coefficients

$0 = (A+B)$

$0 = (6A+2B+C)$

$10 = (10A+2C)$

$$\therefore \qquad F(s) = \frac{5}{(s+2)} - \frac{5s+20}{(s^2+6s+10)} = \frac{5}{(s+2)} - \frac{5(s+3)+5}{(s+3)^2+1}$$

$$\rightarrow \qquad f(t) = e^{-2t} - 5e^{-3t}(\cos t + \sin t)$$

Initial & final value theorems

The initial- & final-value properties allow us to determine the initial value *f(0)* and the final value *f(∞)* of *f(t)* directly from the Laplace Transform. If one only wants the initial or final value behaviour there is no need to find the ILT.

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0^-)}$$

$$sF(s) - f(0) = \mathcal{L}\left[\frac{df}{dt}\right] = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt$$

$$\lim_{s\to\infty}[sF(s) - f(0)] = 0$$

$$\boxed{f(0) = \lim_{s\to\infty} sF(s)}$$

Example, initial value

Find the initial value of: $F(s) = \dfrac{s}{s^2 + \omega^2}$ $\quad (LT\ of\ \cos\omega t)$

$$Ans. \quad f(0) = \lim_{s \to \infty} \frac{s^2}{s^2 + \omega^2} = \lim_{s \to \infty} \frac{1}{1 + \dfrac{\omega^2}{s^2}} = 1$$

Find the initial value of: $F(s) = \dfrac{\omega}{s^2 + \omega^2}$ $\quad (LT\ of\ \sin\omega t)$

$$Ans. \quad f(0) = \lim_{s \to \infty} \frac{s\omega}{s^2 + \omega^2} = \lim_{s \to \infty} \frac{\dfrac{\omega}{s}}{1 + \dfrac{\omega^2}{s^2}} = 0$$

Final value theorem

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0^-)}$$

$$sF(s) - f(0) = \mathcal{L}\left[\frac{df}{dt}\right] = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt$$

$$\lim_{s \to 0}[sF(s) - f(0^-)] = \int_{0^-}^{\infty} \frac{df}{dt} e^{0t}\, dt = \int_{0^-}^{\infty} df = f(\infty) - f(0^-)$$

$$\boxed{f(\infty) = \lim_{s \to 0} sF(s)}$$

Example, Final value

Obtain the initial and final values of $G(s)=\dfrac{s^3+2s+6}{s(s+1)^2(s+3)}$

$$g(0)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty}\frac{s^3+2s+6}{(s^2+2s+1)(s+3)}$$

$$g(0)=\lim_{s\to\infty}\frac{1+\frac{2}{s^2}+\frac{6}{s^3}}{\left(1+\frac{2}{s}+\frac{1}{s^2}\right)\left(1+\frac{3}{s}\right)}=\underline{\mathbf{1}}$$

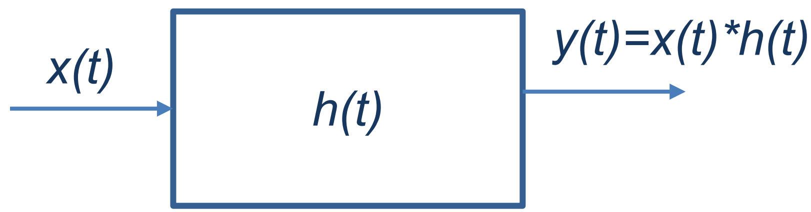
Since all poles $s=0,-1,-1,-3$ the left-hand s-plane, we can apply the final-value theorem.

$$g(\infty)=\lim_{s\to 0} sF(s)=\lim_{s\to 0}\frac{s^3+2s+6}{(s+1)^2(s+3)}\qquad g(\infty)=\lim_{s\to 0}\frac{6}{(1)^2(3)}=\underline{\mathbf{2}}$$

Convolution

Convolution is used to find the response y(t) of a system to an excitation x(t), when the impulse response h(t) of the system is known.

$$y(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda \quad \text{or} \quad y(t) = x(t) * h(t)$$



$$F_1(s)F_2(s) = L\left[f_1(t) * f_2(t)\right]$$

$$L\left[x(t) * h(t)\right] = X(s)H(s)$$

Convolution in the t-domain ≡ Multiplication in the s-domain

Example

A system has the following response to an impulse:

$$h(t) = 4e^{-t}$$

Find the output y(t), when the input is

$$x(t) = 5e^{-2t}$$

$solution$

$$y(t) = x(t) * h(t)$$

$$H(s) = \frac{4}{s+1}, \qquad X(s) = \frac{5}{s+2}$$

$$Y(s) = X(s).H(s) = \frac{4}{s+1}\frac{5}{s+2} = \frac{20}{(s+1)(s+2)}$$

$$y(t) = 20(e^{-t} - e^{-2t})$$

Differential Equations

$$\frac{d^2v}{dt^2}+4v=1 \quad were\ v(0^-)=10\mathrm{V}\ \&\ \mathrm{d}v/dt(0^-)=100\mathrm{V}/s$$

Soln. *LT of the equation above gives*

$$s^2\mathrm{V}(s)-10s-100+4\mathrm{V}(s)=\frac{1}{s} \quad \rightarrow \quad (s^2+4)\mathrm{V}(s)=\frac{1}{s}+10s+100=\frac{10s^2+100s+1}{s}$$

$$\therefore \quad \mathrm{V}(s)=\frac{10s^2+100s+1}{s(s^2+4)} \qquad expand\ by\ P.F. \quad \mathrm{V}(s)=\frac{\mathrm{A}}{s}+\frac{\mathrm{B}s+\mathrm{C}}{(s^2+4)}$$

$$\times by\ D(s), \rightarrow \quad 10s^2+100s+1=(\mathrm{A}+\mathrm{B})\mathrm{s}^2+\mathrm{C}s+4\mathrm{A}$$

$$whereby\ equating\ coefficients\ we\ find: \mathrm{A}=0.25, \mathrm{B}=9.75, \mathrm{C}=100$$

$$Hence \quad \mathrm{V}(s)=\frac{0.25}{s}+\frac{9.75s+100}{(s^2+4)} \qquad ILT \Rightarrow v(t)=0.25+9.75\cos(2t)+50\sin(2t)$$

Check i.c: *subs. into de?*

$$v'(t) = -19.5\sin(2t)+100\cos(2t)$$

$$v''(t) = -39\cos(2t)-200\sin(2t)$$

$$\boxed{\begin{aligned} v''+4v &= -39\cos(2t)-200\sin(2t) \\ &+39\cos(2t)+200\sin(2t)+1=1 \end{aligned}}$$

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Applying the Laplace Transform to Circuit Problems

1. **Transform the circuit:** from the time domain to the s-domain
2. **Solve the circuit:** using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.
3. **Take the inverse transform:** of the solution and thus obtain the solution in the time domain.

Impedance of an element in the s-domain.*

Element	$Z(s) = V(s)/I(s)$
Resistor	R
Inductor	sL
Capacitor	$1/sC$

* Assuming zero initial conditions

Difference between Phasor domain and Laplace Transform

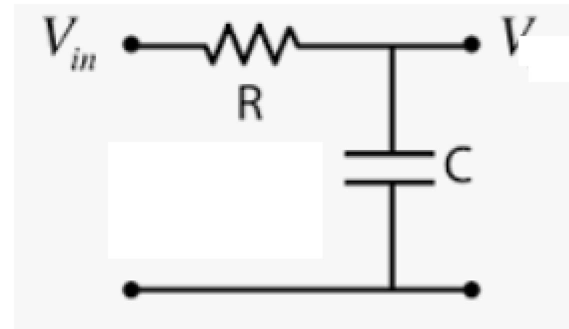
- A similar procedure was followed when analysing circuits in the phasor domain
- Key assumptions for phasors was **sinusoidal inputs** and **steady-state** (i.e. zero initial conditions)
- Laplace domain can be applied to a wider range of inputs
  - E.g. exponentials, unit steps etc.
- Laplace transforms can include effects caused by non-zero initial conditions

Definition: Zero-State & Zero-Input Response

1. ZERO-State: ZSR with zero initial conditions
2. Zero – Input: ZIR with no input

Example Zero-State & Zero-Input Response

For the RC circuit, Find the ZSR and ZIR for the voltage across the capacitor ?



$$v_i = RC\frac{dv}{dt} + v$$

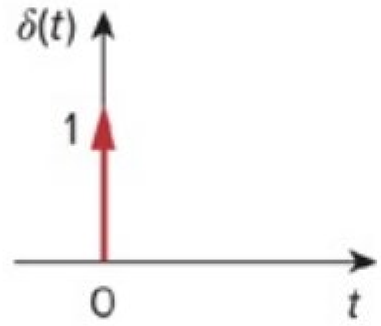
$$LT \qquad V_i(s) = RCsV(s) - RCv(0) + V(s)$$

$$\therefore \quad V(s) = \frac{\boxed{V_i(s)} + \boxed{RCv(0)}}{\boxed{RCs+1}} = \boxed{ZSR} + \boxed{ZIR}$$

Impulse response and step response

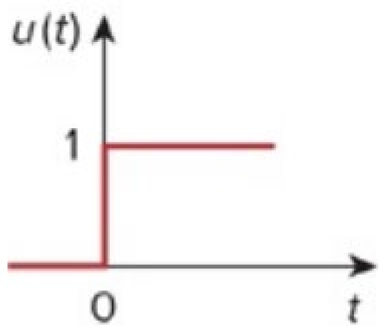
Impulse response: is the **zero-state response** for an impulsive input $\delta(t)$

With an impulsive input, $Vi(s) = 1$



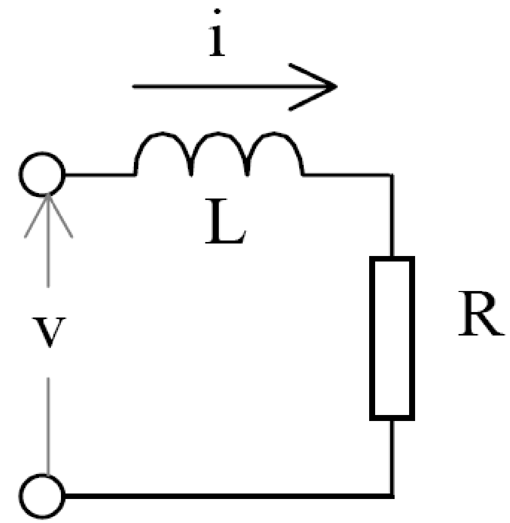
Step response: is the **system response** when a unit step is the input **u**(t)

With unit step input, $Vi(s) = 1/s$



Example: Impulse response and step response

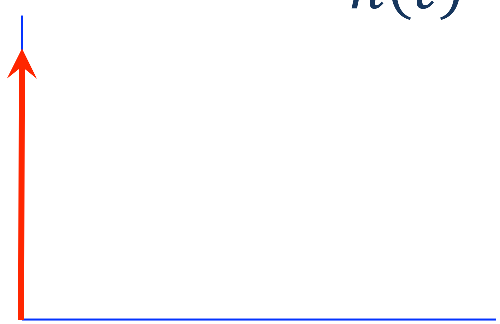
What is the impulse & step response of the system shown where **v** is the *input* & **i** is the *output* ?



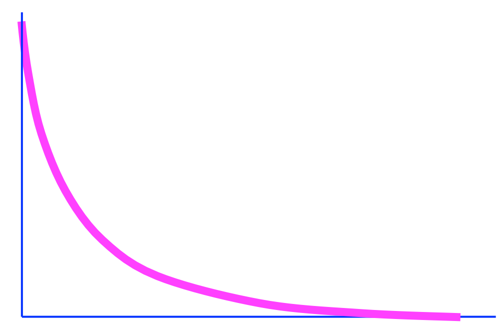
Impulse response

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{(Ls + R)} = \frac{1/L}{(s + R/L)}$$

$$h(t) = \frac{1}{L} e^{-\frac{R}{L}t}$$



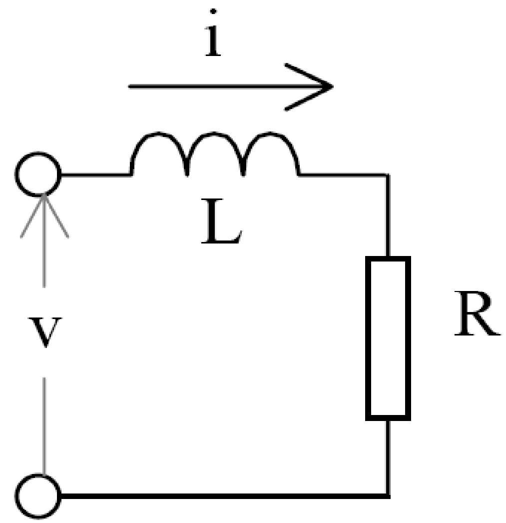
Input



Impulse response

Example: Impulse response and step response

What is the impulse & step response of the system shown where **v** is the *input* & **i** is the *output* ?

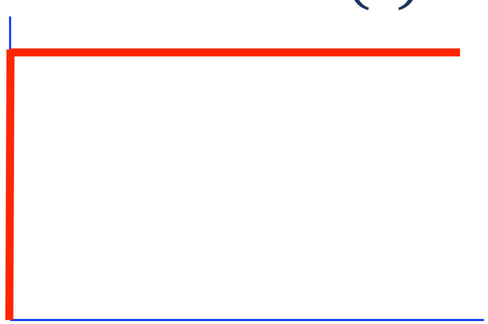


Step response

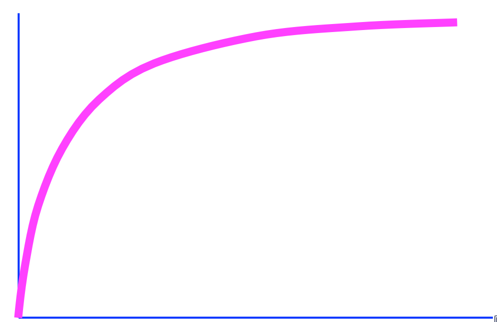
$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{s(Ls+R)} = \frac{1/L}{s(s+R/L)}$$

$$H(s) = \frac{1}{Rs} - \frac{1}{R(s+R/L)}$$

$$h(t) = \frac{1}{R}(1 - e^{-\frac{R}{L}t})$$



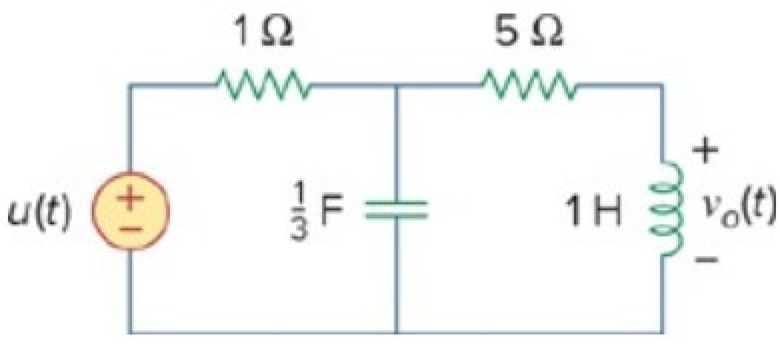
Input



Step response

Example:

Find Vo in the circuit of the figure assuming Zero initial conditions



Example 1:

Find f(t) if Fs is shown below

a) $F(s) = \frac{6s^2+26s+26}{(s+1)(s+2)(s+3)}$

b) $F(s) = \frac{7s^2+63s+134}{(s+3)(s+4)(s+5)}$

Example 2:

Find f(t) if Fs is shown below

$$F(s) = \frac{100(s+3)}{(s+1)(s^2+6s+25)}$$

Example 3:

Find f(t) if Fs is shown below

$$F(s) = \frac{10(s^2+119)}{(s+5)(s^2+10s+169)}$$

Example 4:

Find the Laplace transform of each of the following functions:

a) $f(t) = te^{-at}$;

b) $f(t) = \sin \omega t$;

c) $f(t) = \sin(\omega t + \theta)$;

d) $f(t) = t$;

e) $f(t) = \cosh(t + \theta)$.

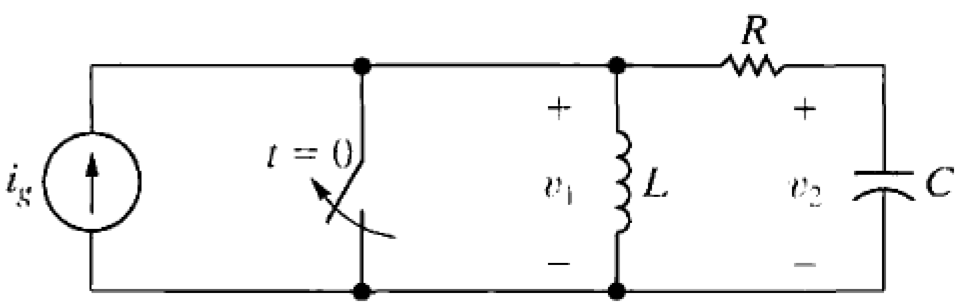
Example 5:

There is no energy stored in the circuit shown in the below circuit at the time the switch is opened.

a) Derive the integrodifferential equations that govern the behavior of the node voltages v1, and v2.

b) Show that

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}.$$



Example 6:

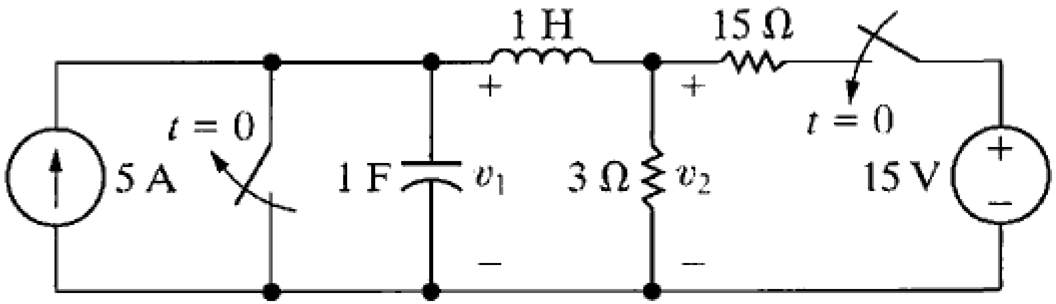
The dc current and voltage sources are applied simultaneously to the circuit shown. No energy is stored in the circuit at the instant of application.

a) Derive the s-domain expressions for *V1* and *V2.*

b) For $t > 0$, derive the time-domain expressions **for** *V1* and *V2.*

c) Calculate V1(0+) **and** *V2(0+).*

d) Compute the steady-state values of *V1 and V2*



Laplace Transform
End