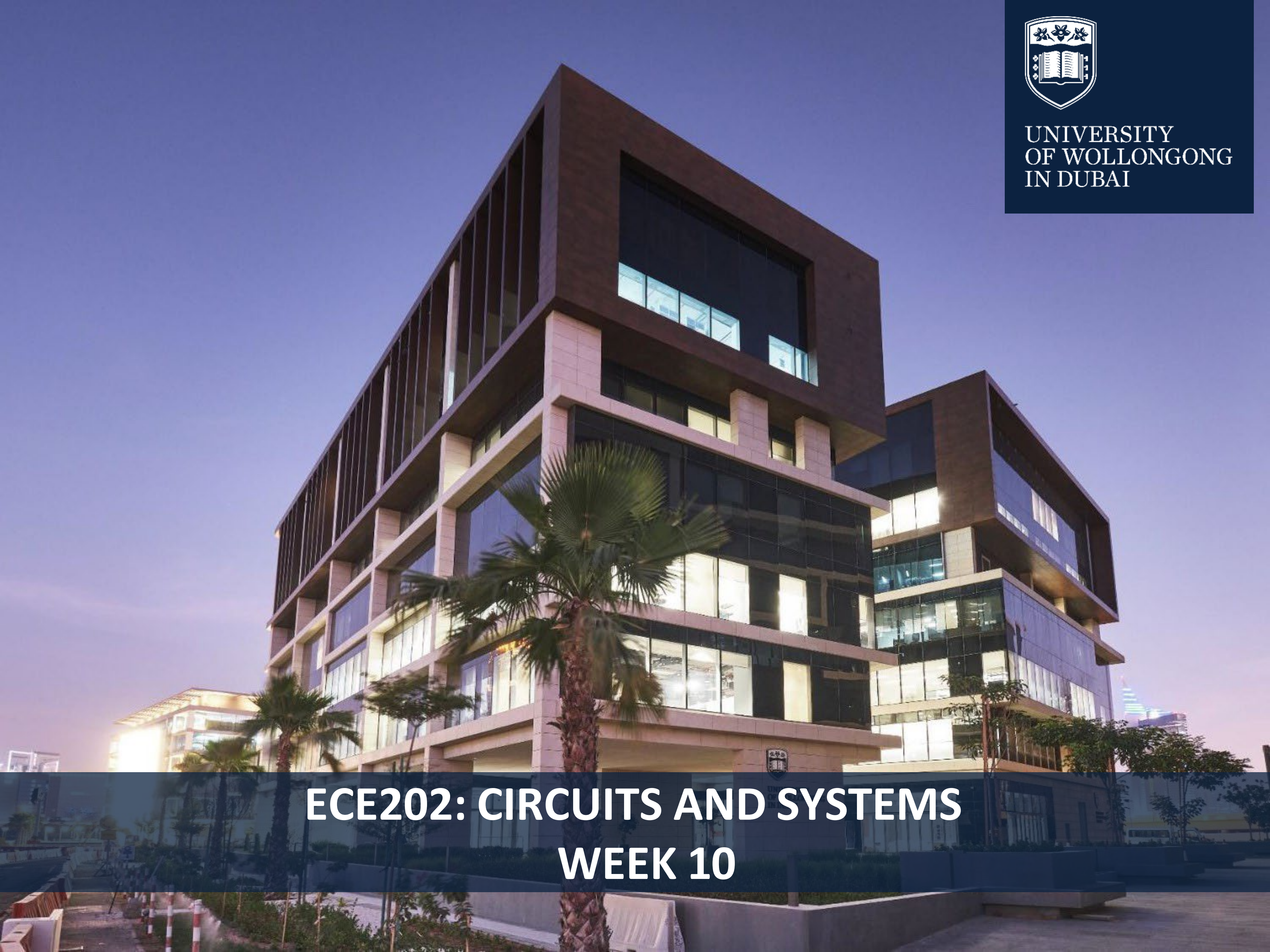

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ECE202: CIRCUITS AND SYSTEMS
WEEK 10

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Fourier Series and Magnetically Coupled Circuits (Chapter 17 & 13)

Fourier Series

Fourier Series

- **Periodic Functions**

  $f(t) = f(t+NT)$, where T is the period

- **Fundamental frequency:**

  $\omega_o = 2\pi/T$

- **Harmonic frequencies:**

  Multiples of ω_o i.e. $n\omega_o$ for $n = 2, 3, 4, \ldots$ where n is the harmonic order

- **Fourier's theorem:**

  Any periodic function can be represented by the sum of sinewaves of fundamental & harmonic frequencies

Fourier Series cont.

Mathematically, a Fourier series can be described by:

$$f(t) = a_0 + a_1 \cos \omega_o t + b_1 \sin \omega_o t + \cdots + a_n \cos n\omega_o t + b_n \sin n\omega_o t \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right)$$

dc term

ac components

A sum of a dc component and an infinite sum of harmonic sinusoids

Determining the Fourier coefficients

a_0, a_n and b_n are known as Fourier coefficients and can be found as:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_o t\,dt \ , \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_o t\,dt$$

A proof is provided in the textbook.

Note: a_0 is a measure of the average value of the signal

Alternatives to find the Fourier coefficients

- Some textbooks will calculate the integrals for the coefficients over the interval from $t=0$ to T (rather than $t = -T/2$ to $T/2$)
- Either method is fine – the key is to evaluate the integral over 1 period of the function
  - The choice is often made depending on which interval is most convenient for the function being analysed

Magnitude Phase Form

An alternative representation of the Fourier Series

$$f(t) = a_0 + \sum^{\infty} A_n \cos(n\omega_0 t + \varphi_n)$$

dc *ac*

Where:

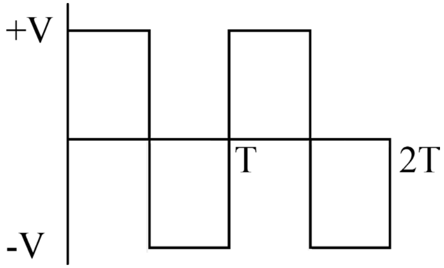
$$A_n = \sqrt{a_n^2 + b_n^2}, \quad \varphi_{\mathrm{n}} = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Amplitude

Phase

Plots of the amplitude and phase terms versus frequency are known as the *magnitude spectrum* and *phase spectrum*, respectively

Fourier Series Example 1



$Ans.$

$$By\ observation,\ a_o = 0.$$

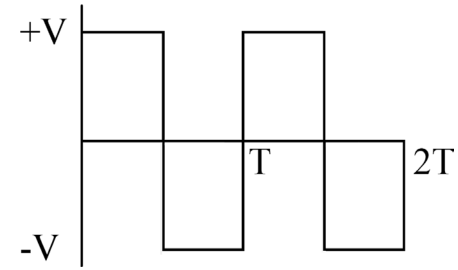
We can also prove this using the earlier integral equation definition. Finding the ac coefficients:

$$a_n = \frac{2}{\mathrm{T}} \int_{-\mathrm{T}/2}^{\mathrm{T}/2} f(t) \cos n\omega_o t dt = 0$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_o t dt$$

$$= \begin{cases} 0, \ n\ even \\ \\ \frac{4}{T} \int_0^{T/2} V \sin n\omega_o t dt = \frac{2V}{n\pi} \left[-\cos n\omega_o t \right]_0^{T/2} = \frac{4V}{n\pi}, \quad n\ odd \end{cases}$$

Fourier Series, Example 1 cont.



Hence, the Fourier series is:

$$f(t) = \frac{4V}{\pi}\left(\sin\omega_o t + \frac{1}{3}\sin 3\omega_o t + \frac{1}{5}\sin 5\omega_o t + \frac{1}{7}\sin 7\omega_o t + \cdots\right)$$

$$= \frac{4V}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\omega_0 t$$

Truncated Fourier Series

FS generally has ∞ terms, but for practical reasons, we ignore all A_n for n sufficiently high.

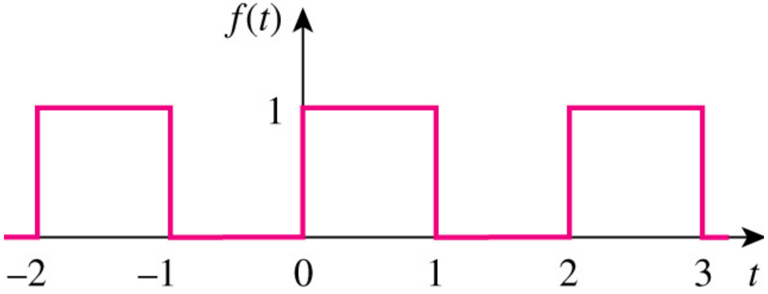
The terms kept are the Truncated FS.

Why ignore high frequency terms?

- **They are usually small.**
- **There is usually the practical rationale in that a part of the circuit has a low pass characteristic & will make high frequency components insignificantly small. Eg the response of the human ear is negligible above 25 kHz so ignore all components higher than this in audio amplifier analysis.**

Error resulting from truncated Fourier Series

For an example, examine the Fourier series of a square wave of period 2:



Fourier Series of a Square Wave

Solution for a square wave:

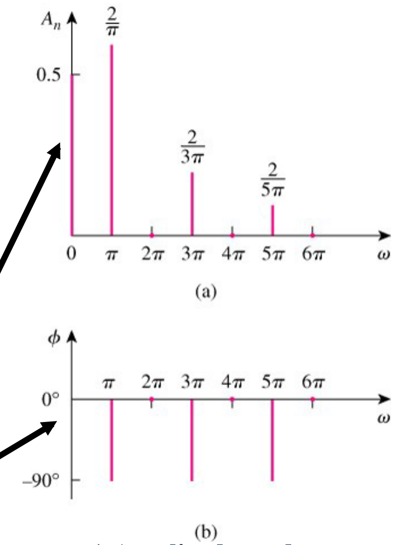
$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases} \text{ and } f(t) = f(t+2)$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt = 0 \text{ and}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt = \begin{cases} 2/n\pi, & \text{n} = \text{odd} \\ 0, & \text{n} = \text{even} \end{cases}$$

$$A_n = \begin{cases} 2/n\pi, & \text{n} = \text{odd} \\ 0, & \text{n} = \text{even} \end{cases}$$

$$\varphi_n = \begin{cases} -90°, & \text{n} = \text{odd} \\ 0, & \text{n} = \text{even} \end{cases}$$



a) Amplitude and
b) Phase spectrum

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin(n\pi t), \quad n = 2k-1$$

Notice the limits chosen for the integrals

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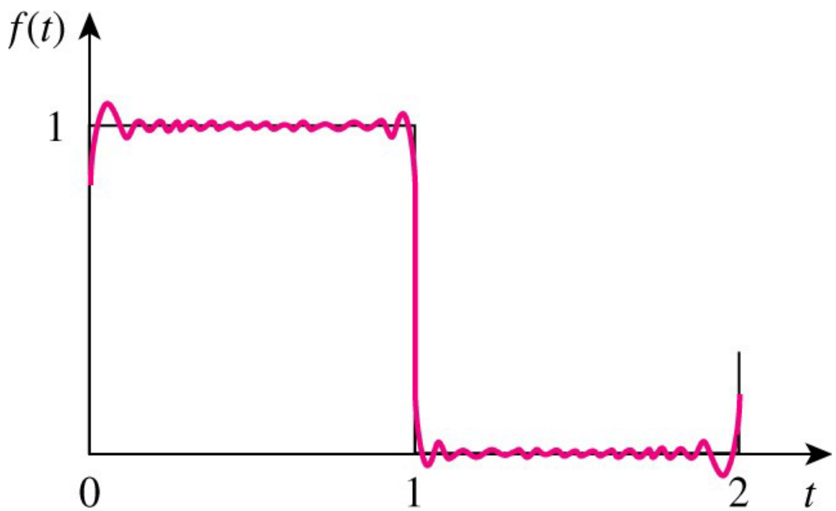
Truncated Fourier Series of a Square Wave

Truncating the series at N=11:



Original Square wave

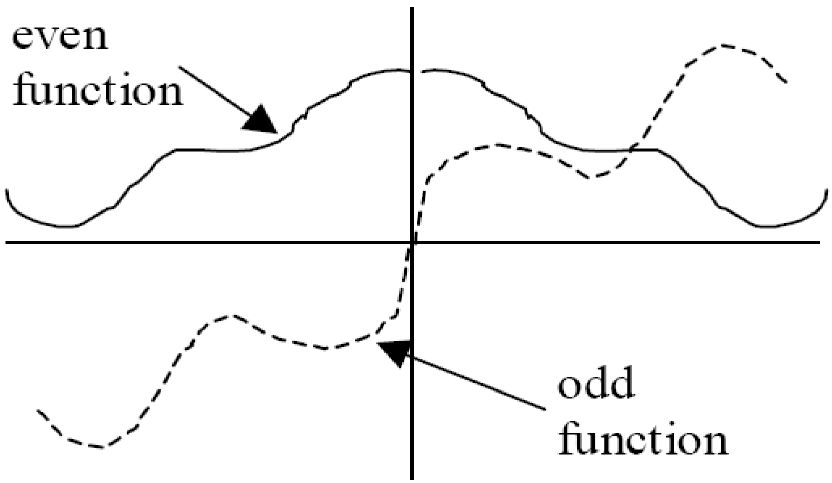
Fourier series approximation

The ripple effect is known as *Gibbs phenomenon*

Symmetry Properties

Even function: f(t) = f(-t); no sine terms in the Fourier series

Odd function: f(t) = -f(-t): no cosine terms in the Fourier series



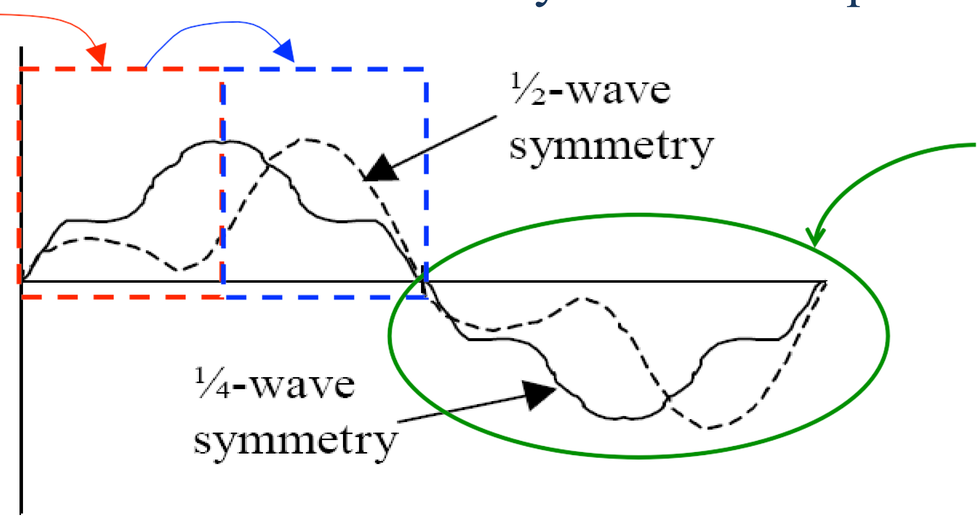
We can make use of these properties to simplify calculations of FS.

Symmetry Properties cont.

Half-wave symmetry: 2nd half-cycle is negative of the first; no even harmonics; integration can be made over ½ cycle.

Quarter-wave symmetry: Has half-wave symmetry & 2nd quarter is reflection of the 1st quarter; as for ½-wave symmetry & integration can be made over ¼ cycle and multiplied by 4!

This quarter of the solid curve is reflected in the next quarter



This half of the dotted curve is the negative of the first half

Can make use of these properties to simplify calculations of FS.

Circuits & Fourier Series

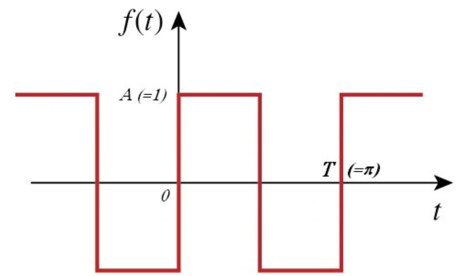
Fourier Series (FS) approaches can be used where a repetitive **non-sinusoidal** forcing function occurs and the Natural Response (NR) is not important.

The steps are

1. Decide on the upper limit of n: N_{max}
2. Find the FS of the forcing function for n = 0 - N_{max}
3. Solve for each component of the Forced Response (FR) using phasor methods
4. Combine the responses to each harmonic frequency to give the overall response.

Example 1

A square wave of amplitude 1 and period π seconds is applied to an RC low pass filter with R = 1 Ω, C = 2 F. Determine the value of the output considering only the lowest four components.



1. Determine N_{max}: First 4 components of square wave have harmonic order 1, 3, 5 & 7 giving N_{max} = 7

$$f(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\omega_0 t$$

Fourier series, square wave (Table 17.3)

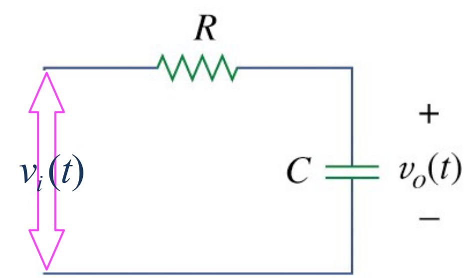
2. Find the Fourier Series of the forcing function: The fundamental frequency is:

$$\omega_o = 2\pi / \mathrm{T} = 2\text{rad/s}.$$

Symmetry	a_0	a_n	b_n
~~Even~~	$\cancel{a_0 \neq 0}$	$\cancel{a_n \neq 0}$	$\cancel{b_n = 0}$
Odd	$a_0 = 0$	$a_n = 0$	$b_n \neq 0$
Half-wave	$\cancel{a_0 = 0}$	$\cancel{a_{2n} = 0}$	$b_{2n} = 0$

Using the first 4 terms of the Fourier series, the square wave input is approximated by:

$$v_i(t) = \frac{4}{\pi}\left(\left(\sin 2t + \frac{1}{3}\sin 6t + \frac{1}{5}\sin 10t + \frac{1}{7}\sin 14t\right)\right)$$



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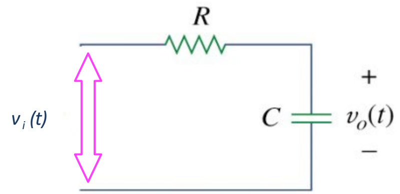
Example 1 cont.

3. Find the transfer function of the circuit:



$$V_o / V_i = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j4n}$$

Setting n = 1,3,5,7 and finding the transfer function for each n:

$$\rightarrow 0.24\angle{-76^\circ}, \quad 0.08\angle{-85^\circ}, \quad 0.05\angle{-87^\circ}, \quad 0.04\angle{-88^\circ}$$

Notice that the magnitude of the higher harmonic components is reduced much more than the lower harmonic components.

This is because it is a low pass filter i.e. low frequencies are passed and high frequencies are attenuated.

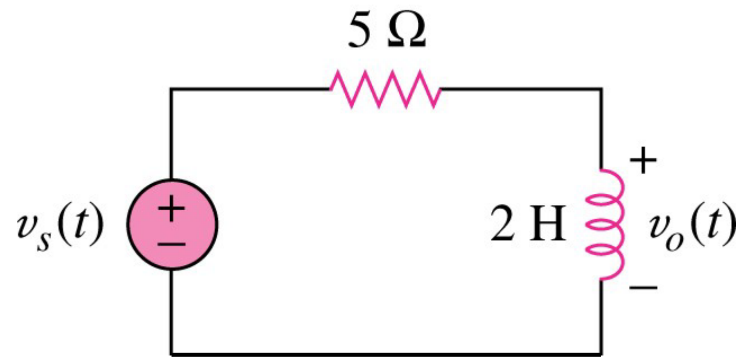
4. Adding the outputs resulting from each of the 4 inputs passing through the circuit:

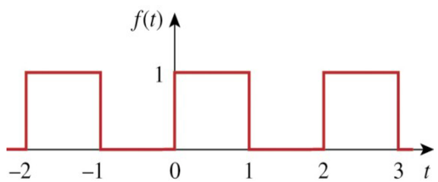
$$v_o(t) = 0.31\,\mathbf{sin}\,(2t - 76^\circ) + 0.035\,\mathbf{sin}\,(6t - 85^\circ)$$
$$+ 0.013\,\mathbf{sin}\,(10t - 87^\circ) + 0.007\,\mathbf{sin}\,(14t - 88^\circ)$$

Example 2

Find the response $v_0(t)$ of the circuit below when the voltage source $v_s(t)$ is given by the following square wave.

$$v_s(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin\left(n\pi\omega t\right), n = 2k-1$$



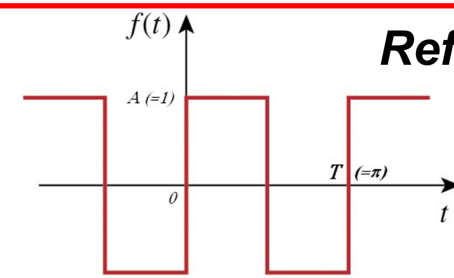


$$f(t) = f(t+T) \; where \; T = 2$$

$$\therefore \; \omega_0 = 2\pi / T = \pi$$

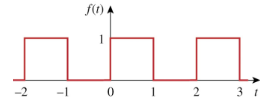
$$and \;\; \omega_n = n\omega_0 = n\pi$$

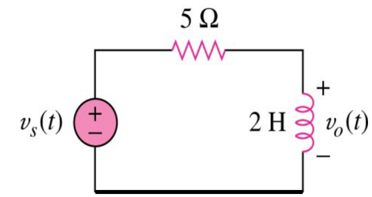
Refence



$$f(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\omega_0 t$$

Example 2 cont.





Solution

$$V_0 = \frac{j2n\pi}{5 + j2n\pi} V_s$$

$$v_s(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin(n\pi\omega t),\ n = 2k-1$$

Phasor of the circuit

For dc component, ($\omega_n = 0$ for n=0), V_s = ½ => $V_o = 0$

For n^{th} harmonic:

$$V_o = \frac{2}{n\pi} < -90° \times \frac{2n\pi}{\sqrt{25+4n^2\pi^2}} < (90 - tan^{-1}\frac{2n\pi}{5})$$

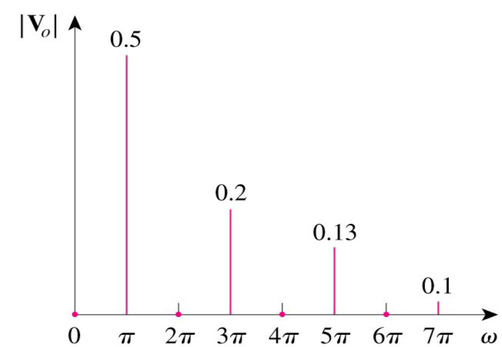
$$= \frac{4}{\sqrt{25+4n^2\pi^2}} < (-tan^{-1}\frac{2n\pi}{5})$$

Recall, when converting each harmonic to a phasor:

$$\sin(\omega t) = \cos(\omega t - 90)$$

In the time domain:

$$v_0(t) = \sum_{k=1}^{\infty}\frac{4}{\sqrt{25+4n^2\pi^2}}cos(n\pi t - \tan^{-1}\frac{2n\pi}{5})$$



Amplitude spectrum of the output voltage

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Fourier Series, Average power and RMS Values

Given:

$$v(t) = \mathrm{V}_{\mathrm{dc}} + \sum_{n=1}^{\infty} \mathrm{V}_{\mathrm{n}} \cos(n\omega_0 t - \theta_n) \text{ and } i(t) = \mathrm{I}_{\mathrm{dc}} + \sum_{n=1}^{\infty} \mathrm{I}_n \cos(n\omega_0 t - \varphi_n)$$

The average power is:

$$\mathrm{P} = \mathrm{V}_{\mathrm{dc}}\mathrm{I}_{\mathrm{dc}} + \frac{1}{2}\sum_{n=1}^{\infty} \mathrm{V}_{\mathrm{n}}\mathrm{I}_{\mathrm{n}} \cos(\theta_n - \varphi_n)$$

Note: Here, voltages and currents *must* be for the same harmonic frequency

The rms value is:

$$F_{rms} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2}$$

Example

The voltage across the terminals of a circuit is given as:

$$\mathbf{v}(\mathbf{t}) = 30 + 20\cos(60\pi\,\mathbf{t} + 45°) + 10\cos(120\pi\,\mathbf{t} - 45°)\ \text{V}$$

If the current entering the terminal at higher potenUal is:

$$\mathbf{i}(\mathbf{t}) = 6 + 4\cos(60\pi\,\mathbf{t} + 10°) - 2\cos(120\pi\,\mathbf{t} - 60°)\ \text{A}$$

$$\boxed{F_{rms} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}A_n^2}}$$

find:

(a) the rms value of the voltage,

$$\sqrt{30^2 + \frac{1}{2}(20^2 + 10^2)}$$

(b) the rms value of the current,

$$\sqrt{6^2 + \frac{1}{2}(4^2 + 2^2)}$$

(c) the average power absorbed by the circuit.

$$\text{P} = \text{V}_{\text{dc}}\text{I}_{\text{dc}} + \frac{1}{2}\sum_{n=1}^{\infty}\text{V}_{\text{n}}\text{I}_{\text{n}}\cos(\theta_n - \varphi_n)$$

30x6 + 0.5[20x4cos(45º--10º) – 10x2cos(--45º+60º)]

Exponential form of the Fourier Series

Using Eulers idenUty: $\cos n\omega_o t = \dfrac{e^{jn\omega_o t} + e^{-jn\omega_o t}}{2}$, $\sin n\omega_o t = \dfrac{e^{jn\omega_o t} - e^{-jn\omega_o t}}{2j}$

The terms at order n can then be wri\en as:

$$\frac{a_n - jb_n}{2} e^{jn\omega_o t} + \frac{a_n + jb_n}{2} e^{-jn\omega_o t}$$

$Define\ a\ new\ coefficient \quad c_n = (a_n - jb_n)/2\ \ for\ n\ \ positive$

$$c_n = (a_n + jb_n)/2\ for\ n\ negative \qquad \& \quad c_o = a_o$$

FS can be written as: $f(t) = c_o + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$

$$where\ c_n = \frac{1}{\mathrm{T}} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_o t}\, dt$$

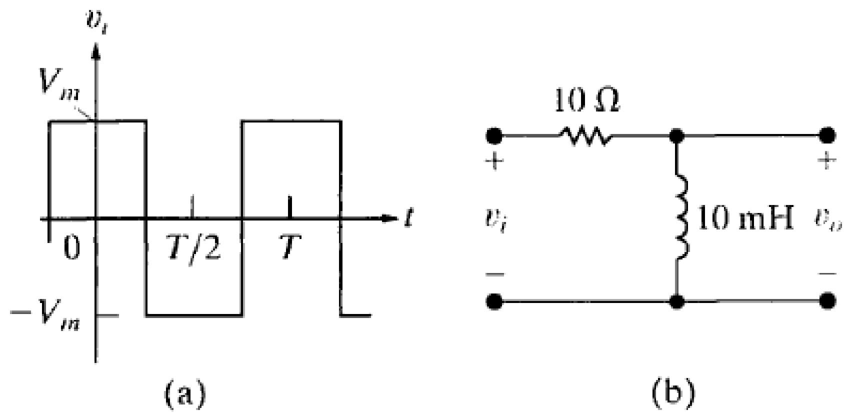
Fourier Series

- c_n can be integrated over any other range of width T Formula includes case of n = 0
- Also known as the complex Fourier Series representation
- a_n, b_n *can be found if required from*

$$a_n = 2Real(c_n), \quad b_n = -2Imag(c_n)$$

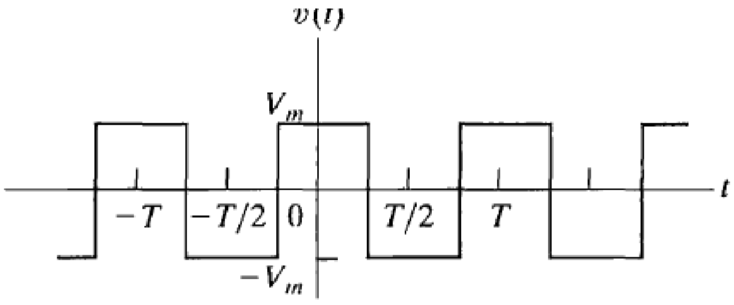
Additional Examples: Problem 1

The periodic square-wave voltage shown below is applied to the circuit.
Derive the first three nonzero terms in the Fourier series that represent the steady-state voltage v0 if Vm = 15π V and the period of the input voltage is 4 π ms?

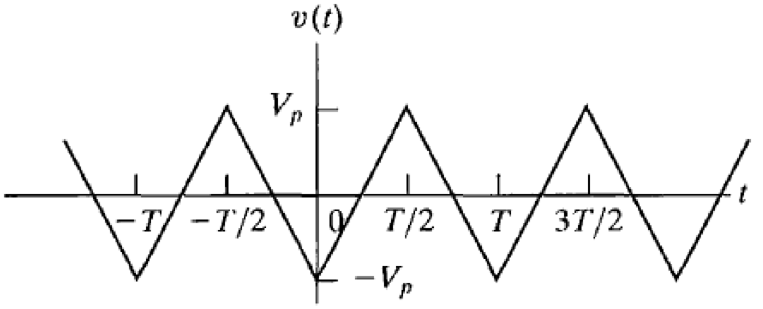


Additional Examples: Problem 2

Find the Fourier series for the following signal?



(a)



(b)

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