Week 1 Workshop Example Solutions

Example 1:

$$\textbf{[a]}\ \omega_o^2 = \frac{1}{LC} = \frac{10^9}{(125)(0.32)} = 25 \times 10^6$$

$$\alpha = \frac{R}{2L} = \omega_o = 5000\,\text{rad/s}$$

$$\therefore \quad R = (5000)(2)L = 1250\,\Omega$$

$$\textbf{[b]}\ i(0) = i_\text{L}(0) = 6\,\text{mA}$$

$$v_\text{L}(0) = 15 - (0.006)(1250) = 7.5\,\text{V}$$

$$\frac{di}{dt}(0) = \frac{7.5}{0.125} = 60\ \text{A/s}$$

$$\textbf{[c]}\ v_\text{C} = D_1 te^{-5000t} + D_2 e^{-5000t}$$

$$v_\text{C}(0) = D_2 = 15\,\text{V}$$

$$\frac{dv_\text{C}}{dt}(0) = D_1 - 5000D_2 = \frac{i_\text{C}(0)}{C} = \frac{-i_\text{L}(0)}{C} = -18{,}750$$

$$\therefore \quad D_1 = 56{,}250\ \text{V/s}$$

$$v_\text{C} = 56{,}250te^{-5000t} + 15e^{-5000t}\,\text{V}, \qquad t \geq 0$$

Example 2:

[a] $-\alpha + \sqrt{\alpha^2 - \omega_0^2} = -4000; \qquad -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -16{,}000$

$$\therefore \quad \alpha = 10{,}000 \text{ rad/s}, \qquad \omega_0^2 = 64 \times 10^6$$

$$\alpha = \frac{R}{2L} = 10{,}000; \qquad R = 20{,}000L$$

$$\omega_o^2 = \frac{1}{LC} = 64 \times 10^6; \qquad L = \frac{10^9}{64 \times 10^6 (31.25)} = 0.5\,\text{H}$$

$$R = 10{,}000\ \Omega$$

[b] $i(0) = 0$

$$L\frac{di(0)}{dt} = v_c(0); \qquad \frac{1}{2}(31.25) \times 10^{-9} v_c^2(0) = 9 \times 10^{-6}$$

$$\therefore \quad v_c^2(0) = 576; \qquad v_c(0) = 24\,\text{V}$$

$$\frac{di(0)}{dt} = \frac{24}{0.5} = 48\,\text{A/s}$$

[c] $i(t) = A_1 e^{-4000t} + A_2 e^{-16{,}000t}$

$$i(0) = A_1 + A_2 = 0$$

$$\frac{di(0)}{dt} = -4000A_1 - 16{,}000A_2 = 48$$

Solving,

$$\therefore \quad A_1 = 4\,\text{mA}; \qquad A_2 = -4\,\text{mA}$$

$$i(t) = 4e^{-4000t} - 4e^{-16{,}000t}\,\text{mA}, \qquad t \geq 0$$

[d] $\dfrac{di(t)}{dt} = -16e^{-4000t} + 64e^{-16{,}000t}$

$$\frac{di}{dt} = 0 \text{ when } 64e^{-16{,}000t} = 16e^{-4000t}$$

$$\text{or } e^{12{,}000t} = 4$$

$$\therefore \quad t = \frac{\ln 4}{12{,}000} = 115.52\,\mu\text{s}$$

Example 3:

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{(10)(4 \times 10^{-3})} = 25$$

$$\alpha = \frac{R}{2L} = \frac{80}{2(10)} = 4; \qquad \alpha^2 = 16$$

$$\alpha^2 < \omega_o^2 \quad \therefore \qquad \text{underdamped}$$

$$s_{1,2} = -4 \pm j\sqrt{9} = -4 \pm j3 \text{ rad/s}$$

$$i = B_1 e^{-4t} \cos 3t + B_2 e^{-4t} \sin 3t$$

$$i(0) = B_1 = -240/100 = -2.4 \text{ A}$$

$$\frac{di}{dt}(0) = 3B_2 - 4B_1 = 0$$

$$\therefore \quad B_2 = -3.2 \text{ A}$$

$$i = -2.4e^{-4t} \cos 3t - 3.2 \sin 3t \text{ A}, \qquad t \geq 0$$

Example 4:

[a] $\alpha = \dfrac{1}{2RC} = \dfrac{10^{12}}{(4000)(10)} = 25{,}000$

$$\omega_o^2 = \frac{1}{LC} = \frac{10^{12}}{(250)(10)} = 4 \times 10^8$$

$$s_{1,2} = -25{,}000 \pm \sqrt{625 \times 10^6 - 400 \times 10^6} = -25{,}000 \pm 15{,}000$$

$$s_1 = -10{,}000 \text{ rad/s}$$

$$s_2 = -40{,}000 \text{ rad/s}$$

[b] overdamped

[c] $\omega_d = \sqrt{\omega_o^2 - \alpha^2}$

$$\therefore \quad \alpha^2 = \omega_o^2 - \omega_d^2 = 4 \times 10^8 - 144 \times 10^6 = 256 \times 10^6$$

$$\alpha = 16 \times 10^3 = 16{,}000$$

$$\frac{1}{2RC} = 16{,}000; \qquad \therefore \quad R = \frac{10^9}{(32{,}000)(10)} = 3125\,\Omega$$

[d] $s_1 = -16{,}000 + j12{,}000 \text{ rad/s}; \qquad s_2 = -16{,}000 - j12{,}000 \text{ rad/s}$

[e] $\alpha = 4 \times 10^4 = \dfrac{1}{2RC}; \qquad \therefore \quad R = \dfrac{1}{2C(4 \times 10^4)} = 2500\,\Omega$

Example 5:

[a] $i_{\mathrm{R}}(0) = \dfrac{15}{200} = 75\,\mathrm{mA}$

$i_{\mathrm{L}}(0) = -45\,\mathrm{mA}$

$i_{\mathrm{C}}(0) = -i_{\mathrm{L}}(0) - i_{\mathrm{R}}(0) = 45 - 75 = -30\,\mathrm{mA}$

[b] $\alpha = \dfrac{1}{2RC} = \dfrac{1}{2(200)(0.2 \times 10^{-6})} = 12{,}500$

$\omega_o^2 = \dfrac{1}{LC} = \dfrac{1}{(50 \times 10^{-3})(0.2 \times 10^{-6})} = 10^8$

$s_{1,2} = -12{,}500 \pm \sqrt{1.5625 \times 10^8 - 10^8} = -12{,}500 \pm 7500$

$s_1 = -5000 \text{ rad/s}; \qquad s_2 = -20{,}000 \text{ rad/s}$

$v = A_1 e^{-5000t} + A_2 e^{-20{,}000t}$

$v(0) = A_1 + A_2 = 15$

$\dfrac{dv}{dt}(0) = -5000A_1 - 20{,}000A_2 = \dfrac{-30 \times 10^{-3}}{0.2 \times 10^{-6}} = -15 \times 10^4 \mathrm{V/s}$

Solving, $\quad A_1 = 10; \quad A_2 = 5$

$v = 10e^{-5000t} + 5e^{-20{,}000t}\,\mathrm{V}, \qquad t \geq 0$

[c] $i_{\mathrm{C}} = C\dfrac{dv}{dt}$

$= 0.2 \times 10^{-6}[-50{,}000e^{-5000t} - 100{,}000e^{-20{,}000t}]$

$= -10e^{-5000t} - 20e^{-20{,}000t}\,\mathrm{mA}$

$i_{\mathrm{R}} = 50e^{-5000t} + 25e^{-20{,}000t}\,\mathrm{mA}$

$i_{\mathrm{L}} = -i_{\mathrm{C}} - i_{\mathrm{R}} = -40e^{-5000t} - 5e^{-20{,}000t}\,\mathrm{mA}, \quad t \geq 0$

Example 6:

$$\frac{1}{2RC} = \frac{1}{2(312.5)(0.2 \times 10^{-6})} = 8000$$

$$\frac{1}{LC} = \frac{1}{(50 \times 10^{-3})(0.2 \times 10^{-6})} = 10^8$$

$$s_{1,2} = -8000 \pm \sqrt{8000^2 - 10^8} = -8000 \pm j6000\,\text{rad/s}$$

$\therefore$ response is underdamped

$$v(t) = B_1 e^{-8000t}\cos 6000t + B_2 e^{-8000t}\sin 6000t$$

$$v(0^+) = 15\,\text{V} = B_1; \qquad i_\text{R}(0^+) = \frac{15}{312.5} = 48\,\text{mA}$$

$$i_\text{C}(0^+) = [-i_\text{L}(0^+) + i_\text{R}(0^+)] = -[-45 + 48] = -3\,\text{mA}$$

$$\frac{dv(0^+)}{dt} = \frac{-3 \times 10^{-3}}{0.2 \times 10^{-6}} = -15{,}000\,\text{V/s}$$

$$\frac{dv(0)}{dt} = -8000B_1 + 6000B_2 = -15{,}000$$

$$6000B_2 = 8000(15) - 15{,}000; \qquad \therefore \quad B_2 = 17.5\,\text{V}$$

$$v(t) = 15e^{-8000t}\cos 6000t + 17.5e^{-8000t}\sin 6000t\,\text{V}, \qquad t \geq 0$$

Example 7:

[a] $-\alpha + \sqrt{\alpha^2 - \omega_o^2} = -250$

$-\alpha - \sqrt{\alpha^2 - \omega_o^2} = -1000$

Adding the above equations, $-2\alpha = -1250$

$\alpha = 625\,\text{rad/s}$

$$\frac{1}{2RC} = \frac{1}{2R(0.1 \times 10^{-6})} = 625$$

$R = 8\,\text{k}\Omega$

$2\sqrt{\alpha^2 - \omega_o^2} = 750$

$4(\alpha^2 - \omega_o^2) = 562{,}500$

$\therefore \quad \omega_o = 500\,\text{rad/s}$

$$\omega_o^2 = 25 \times 10^4 = \frac{1}{LC}$$

$$\therefore \quad L = \frac{1}{(25 \times 10^4)(0.1 \times 10^{-6})} = 40\,\text{H}$$

[b] $i_\text{R} = \dfrac{v(t)}{R} = -1e^{-250t} + 4e^{-1000t}\,\text{mA}, \qquad t \geq 0^+$

$i_\text{C} = C\dfrac{dv(t)}{dt} = 0.2e^{-250t} - 3.2e^{-1000t}\,\text{mA}, \qquad t \geq 0^+$

$i_\text{L} = -(i_\text{R} + i_\text{C}) = 0.8e^{-250t} - 0.8e^{-1000t}\,\text{mA}, \qquad t \geq 0$

Example 8:

[a] $\left(\frac{1}{2RC}\right)^2 = \frac{1}{LC} = (500)^2$

$$\therefore \quad C = \frac{1}{(500)^2(4)} = 1\,\mu\text{F}$$

$$\frac{1}{2RC} = 500$$

$$\therefore \quad R = \frac{1}{2(500)(10^{-6})} = 1\,\text{k}\Omega$$

$$v(0) = D_2 = 8\,\text{V}$$

$$i_\text{R}(0) = \frac{8}{1000} = 8\text{mA}$$

$$i_\text{C}(0) = -8 + 10 = 2\,\text{mA}$$

$$\frac{dv}{dt}(0) = D_1 - 500D_2 = \frac{2 \times 10^{-3}}{10^{-6}} = 2000\,\text{V/s}$$

$$\therefore \quad D_1 = 2000 + 500(8) = 6000\ \text{V/s}$$

[b] $v = 6000te^{-500t} + 8e^{-500t}\,\text{V}, \quad t \geq 0$

$$\frac{dv}{dt} = [-3 \times 10^6 t + 2000]e^{-500t}$$

$$i_\text{C} = C\frac{dv}{dt} = (-3000t + 2)e^{-500t}\,\text{mA}, \quad t \geq 0^+$$

Example 9:

[a] $\alpha = \dfrac{1}{2RC} = 800 \text{ rad/s}$

$\omega_o^2 = \dfrac{1}{LC} = 10^6$

$\omega_d = \sqrt{10^6 - 800^2} = 600 \text{ rad/s}$

$\therefore \quad v = B_1 e^{-800t}\cos 600t + B_2 e^{-800t}\sin 600t$

$v(0) = B_1 = 30$

$i_R(0^+) = \dfrac{30}{5000} = 6\,\text{mA}; \qquad i_C(0^+) = -12\,\text{mA}$

$\therefore \quad \dfrac{dv}{dt}(0^+) = \dfrac{-0.012}{125 \times 10^{-9}} = -96{,}000 \text{ V/s}$

$-96{,}000 = -\alpha B_1 + \omega_d B_2 = -(800)(30) + 600B_2$

$\therefore \quad B_2 = -120$

$\therefore \quad v = 30e^{-800t}\cos 600t - 120e^{-800t}\sin 600t\,\text{V}, \qquad t \geq 0$

[b] $\dfrac{dv}{dt} = 6000e^{-800t}(13\sin 600t - 16\cos 600t)$

$\dfrac{dv}{dt} = 0 \quad \text{when} \quad 16\cos 600t = 13\sin 600t \quad \text{or} \quad \tan 600t = \dfrac{16}{13}$

$\therefore \quad 600t_1 = 0.8885, \qquad t_1 = 1.48\,\text{ms}$

$600t_2 = 0.8885 + \pi, \qquad t_2 = 6.72\,\text{ms}$

$600t_3 = 0.8885 + 2\pi, \qquad t_3 = 11.95\,\text{ms}$

[c] $t_3 - t_1 = 10.47\,\text{ms}; \qquad T_d = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{600} = 10.47\,\text{ms}$

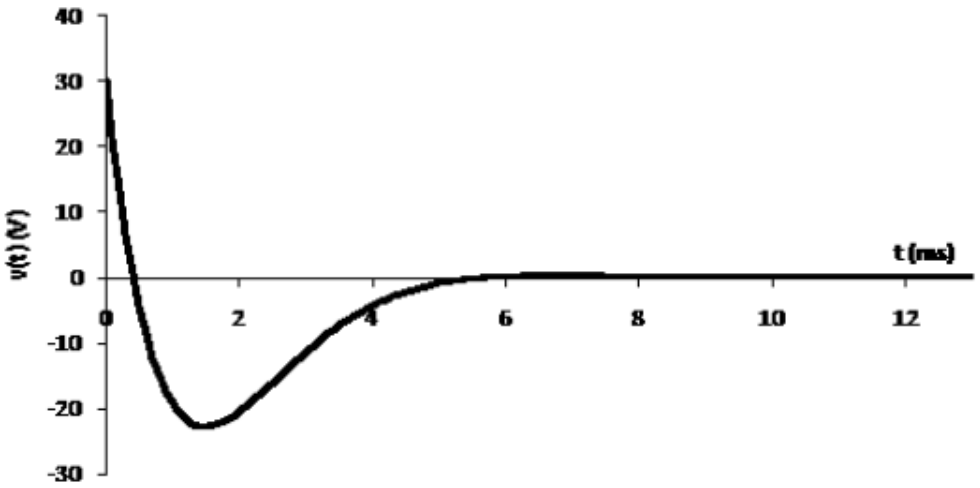
[d] $t_2 - t_1 = 5.24\,\text{ms}; \qquad \dfrac{T_d}{2} = \dfrac{10.48}{2} = 5.24\,\text{ms}$

[e] $v(t_1) = 30e^{-(1.184)}(\cos 0.8885 - 4\sin 0.8885) = -22.7\,\text{V}$

$v(t_2) = 30e^{-(5.376)}(\cos 4.032 - 4\sin 4.032) = 0.334\,\text{V}$

$v(t_3) = 30e^{-(9.56)}(\cos 7.17 - 4\sin 7.17) = -5.22\,\text{mV}$

[f]



Example 10:

[a] $\alpha = 0; \qquad \omega_d = \omega_o = \sqrt{10^6} = 1000\,\text{rad/s}$

$$v = B_1 \cos \omega_o t + B_2 \sin \omega_o t; \qquad v(0) = B_1 = 30$$

$$C\frac{dv}{dt}(0) = -i_{\text{L}}(0) = -0.006$$

$$-48{,}000 = -\alpha B_1 + \omega_d B_2 = -0 + 1000 B_2$$

$$\therefore \quad B_2 = \frac{-48{,}000}{1000} = -48\,\text{V}$$

$$v = 30 \cos 1000t - 48 \sin 1000t\,\text{V}, \qquad t \geq 0$$

[b] $2\pi f = 1000; \qquad f = \dfrac{1000}{2\pi} \cong 159.15\,\text{Hz}$

[c] $\sqrt{30^2 + 48^2} = 56.6\,\text{V}$

Example 11:

[a] $\omega_o^2 = \dfrac{1}{LC} = \dfrac{10^9}{(2.5)(100)} = 4 \times 10^6$

$\omega_o = 2000 \text{ rad/s}$

$\dfrac{1}{2RC} = 2000; \qquad R = \dfrac{1}{4000C} = 2500\,\Omega$

[b] $v(t) = D_1 t e^{-5000t} + D_2 e^{-5000t}$

$v(0) = -15\,\text{V} = D_2$

$i_C(0) = 5 + \dfrac{15}{2.5} = 11\,\text{mA}$

$\dfrac{dv}{dt}(0) = \dfrac{i_C(0)}{C} = \dfrac{11 \times 10^{-3}}{100 \times 10^{-9}} = 110{,}000$

$D_1 - 2000(-15) = 110{,}000 \qquad \text{so} \qquad D_1 = 80{,}000 \text{ V/s}$

$\therefore \quad v(t) = (80{,}000t - 15)e^{-2000t}\,\text{V}, \qquad t \geq 0$

[c] $i_C(t) = 0$ when $\dfrac{dv}{dt}(t) = 0$

$\dfrac{dv}{dt} = (110{,}000 - 160 \times 10^6 t))e^{-2000t}$

$\dfrac{dv}{dt} = 0$ when $160 \times 10^6 t_1 = 110{,}000; \qquad \therefore \quad t_1 = 687.5\,\mu\text{s}$

$v(687.5\mu\text{s}) = (55 - 15)e^{-1.375} = 10.1136\,\text{V}$

[d] $w(0) = \dfrac{1}{2}(100 \times 10^{-9})(15)^2 + \dfrac{1}{2}(2.5)(0.005)^2 = 42.5\,\mu\text{J}$

$w(687.5\,\mu\text{s}) = \dfrac{1}{2}(100 \times 10^{-9})(10.1136)^2 + \dfrac{1}{2}(2.5)\left(\dfrac{10.1136}{2500}\right)^2 = 25.571\,\mu\text{J}$

$\%\text{ remaining } = \dfrac{25.571}{42.5}(100) = 60.17\%$