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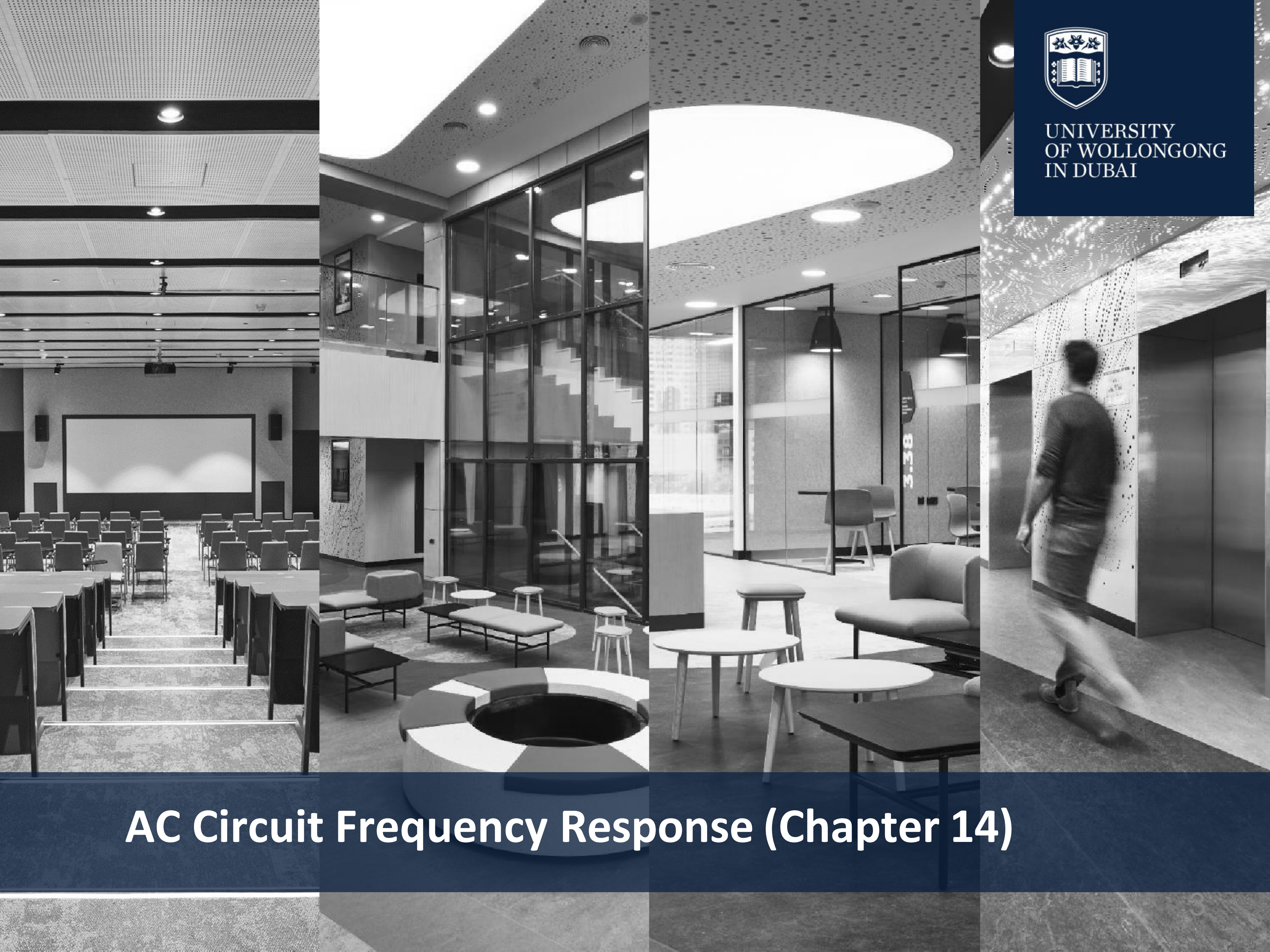
ECE202: CIRCUITS AND SYSTEMS
WEEK 6



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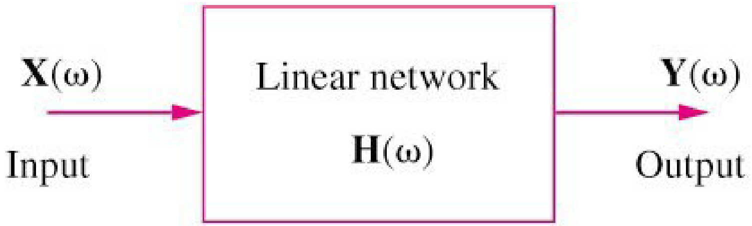
AC Circuit Frequency Response (Chapter 14)

AC Circuit Frequency Response

- Last lecture we looked at AC circuit analysis techniques for circuits with a constant frequency,
- This week we look at the magnitude and phase of V or I in circuits when the frequency varies – i.e. when ω changes from $0 \rightarrow \infty$ (rad/s),
- We define the function $H(\omega)$ to be a complete description of a circuit's behavior as a function of frequency,
- $H(\omega)$ is known as the "Transfer Function" and can be plotted to show a circuit's "Frequency Response".
- The frequency response of a circuit is the variation in its behavior in response to changes in signal frequency.

Transfer Function

- H(ω) = frequency dependant ratio of a circuit's Output "Y(ω)" to Input "X(ω)"



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$\textit{being the ratio of} \rightarrow \frac{\textit{phasor output } \{\textit{voltage } V_o(\omega), \textit{ or current } I_o(\omega)\}}{\textit{phasor input } \{\textit{voltage } V_i(\omega), \textit{ or current } I_i(\omega)\}}$$

- This relationship was implicit in the impedance and admittance concepts (i.e. ratio of voltage and current)

Transfer Function Examples

Assuming zero initial conditions:

$$H(\omega) = \frac{V_0(\omega)}{V_i(\omega)} \rightarrow \textit{voltage gain}$$

$$H(\omega) = \frac{I_0(\omega)}{I_i(\omega)} \rightarrow \textit{current gain}$$

$$H(\omega) = \frac{V_0(\omega)}{I_i(\omega)} \rightarrow \textit{transfer impedance}$$

$$H(\omega) = \frac{I_0(\omega)}{V_i(\omega)} \rightarrow \textit{transfer admittance}$$

Some texts use $H(j\omega)$ *rather than* $H(\omega)$

Methodology:

- Convert all circuit elements to phasor or frequency domain
- Obtain required circuit relationship ratio
- Analyse and plot magnitude and phase w.r.t. varying frequency

Transfer Function

The transfer function can be represented in terms of its Numerator Polynomial, N(ω) and Denominator Polynomial, D(ω)

$$\mathrm{H}(\omega) = \frac{\mathrm{N}(\omega)}{\mathrm{D}(\omega)}$$

where $\mathrm{N}(\omega)$ *and* $\mathrm{D}(\omega)$ *are not necessarily the same expressions as the input and ouput functions*

i.e. $\mathrm{H}(\omega)$ is reduced to lowest terms – common numerator & denominator factors have cancelled.

Transfer Function

$$H(\omega) = \frac{N(\omega)}{D(\omega)}$$

Particularly interested in the roots of both N(ω) and D(ω) implying significant behavioural response of H(ω).

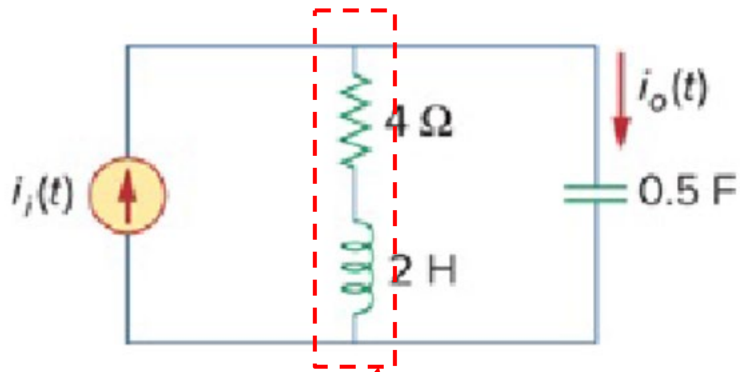
$$\textbf{roots of } N(\omega) \rightarrow \textbf{zeros}\,;\; H(\omega) = 0$$

$$\textbf{roots of } D(\omega) \rightarrow \textbf{poles}\,;\; H(\omega) = \infty$$

zeros may also be regarded as values of $s = j\omega$ that force H(s) to become zero, and poles as the values of $s = j\omega$ that make H(s) infinite. "s" is used to avoid complex algebra.

Transfer Function Example Problem:

Calculate the gain $I_o(\omega)/I_i(\omega)$ for the following circuit, and its poles and zeros.



Substituting $s = j\omega$, impedances become

$$Z_C = \frac{1}{j\omega C} = \frac{2}{j\omega} = \frac{2}{s}$$

$Z_L = j\omega L = j\ \omega\ 2 = 2s,$ $\qquad Z_2 = 2s + 4$

$$I_o(s) = \frac{2s+4}{2s+4+\frac{2}{s}} I_i(s) = \frac{2\ s^2 + 4s}{2s^2+4s+2} I_i(s)$$

$$= \frac{s+2}{s^2+2s+1} I_i(s)$$

$$\frac{I_o(s)}{I_i(s)} = \frac{(s+2)}{s^2+2s+1}$$

$$\boxed{\begin{matrix} \boldsymbol{P_1 = P_2 = -1} \\ \boldsymbol{Z_1 = -2} \end{matrix}}$$

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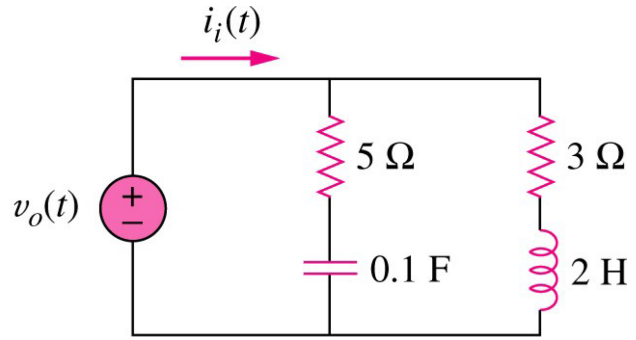
Transfer Function Example Problem:

Find the transfer function $V_o(\omega)/I_i(\omega)$ for the following circuit, obtaining its poles and zeros.



Substituting $s = j\omega$, impedances become

$$Z_c = \frac{1}{j\omega C} = \frac{10}{j\omega} = \frac{10}{s}$$

$$Z_L = j\omega L = j\,\omega\,2 = 2s$$

The desired transfer function is the input impedance

$$Z(s) = 3 + 2s \,//\, 5 + \frac{10}{s}$$

$$Z(s) = \frac{5(s+2)(s+1.5)}{s^2 + 4s + 5}$$

$$\boxed{\begin{gathered} \boldsymbol{P_1 = -2 \pm j} \\ \boldsymbol{Z_1 = -2\,, Z_2 = -1.5} \end{gathered}}$$

Frequency Response

- This is a plot of **H**(ω) versus frequency
- Typically, magnitude and phase of **H**(ω) are plotted separately,

  where:

$$H = \left| \mathbf{H}(\omega) \right| \qquad\qquad \phi = \angle \mathbf{H}(\omega)$$

Magnitude | Phase

- Consider the transfer function: $\mathbf{H}(\omega) = \dfrac{j\omega}{\omega_0 + j\omega}$

$$H = \left| \mathbf{H}(\omega) \right| = \frac{\omega}{\sqrt{\omega_0^2 + \omega^2}}$$

$$\phi = \angle \mathbf{H}(\omega) = \frac{\pi\pi}{2} - t^{-1} \frac{\omega\omega}{\omega\omega_0}$$

at	$\omega = 0$	$H = 0$	$\phi = 90°$
at	$\omega \to \infty$	$H = 1$	$\phi = 0°$
at	$\omega = \omega_o$	$H = \frac{1}{\sqrt{2}}$	$\phi = 90° - 45° = 45°$

Bode Plots & The Decibel Scale

- Sometimes it is difficult to determine a transfer function's magnitude and phase quickly — a more systematic way to obtain the frequency response is through Bode Plots.

- Bode Plots represent frequency on a logarithmic scale and also use a logarithmic formulation, the Decibel, to record magnitude.

$$G = number\ of\ bels = \log_{10} \frac{P_2}{P_1}$$

$$G_{dB} = 10 \log_{10} \frac{P_2}{P_1}$$

$$\therefore\ if\ P_2 = P_1\ gain in\ dB = \quad 0\ dB.$$

$$if\ P_2 = 2 \times P_1\ gain in\ dB = \quad 3\ dB.$$

$$if\ P_2 = 0.5 \times P_1\ gain in\ dB = -3\ dB.$$

The Decibel Scale

- Decibels (dB) is a logarithmic units for power gain.
- While 10 $\log_{10}$ is used for power, 20 $\log_{10}$ is used for voltage or current because of the square relationship between them – i.e.

  $P = V^2/R = I^2R.$

- Hence:

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1}$$

$$G_{dB} = 20\log_{10}\frac{V_2}{V_1}$$

$$G_{dB} = 20\log_{10}\frac{I_2}{I_1}$$

Power in Decibel Scale

Example:

An amplifier has two stages with power gains of 15 and 25 dB

(a) What is its total power gain in dB?

(b) What is the output power if the input power in 10 mW?

Solution:

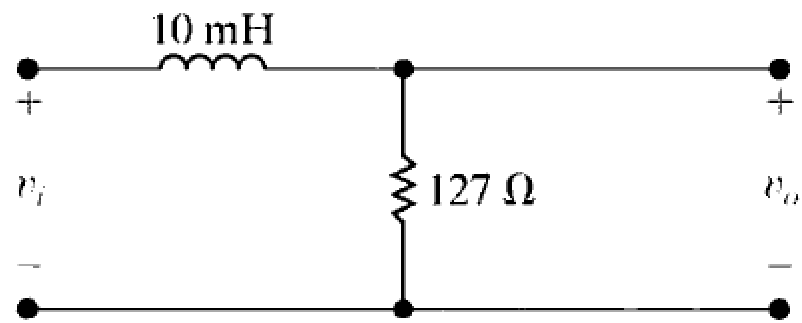
(a) Total power gain is $15 + 25 = 40$ dB

(b) So $10 \log(P_o/10\text{mW}) = 40$

giving $P_o = 10^4 \times 10\text{mW} = 100$ W

Example 1

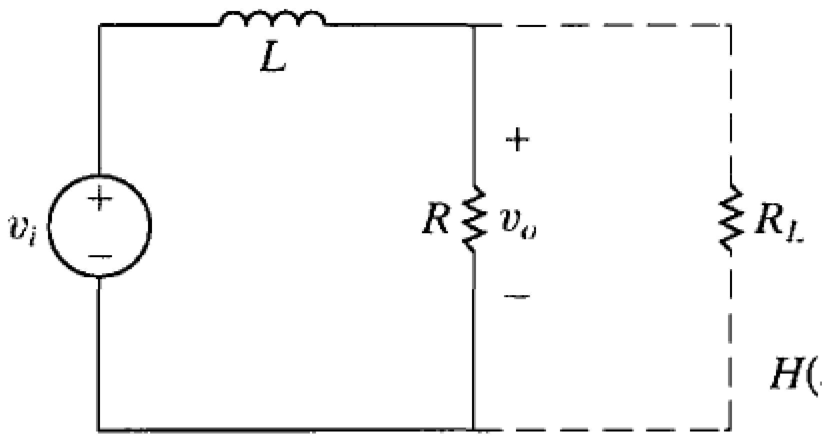
Calculate the transfer function for the following circuit



Example 2

Study the circuit shown in the below figure without the load resistor.

- What will be the value of the output voltage if the frequency goes to zero
- What will be the output voltage if the frequency goes to an infinite value
- What is the transfer function of the circuit
- Suppose we wish to add a load resistor, determine the transfer function after we include the load resistor

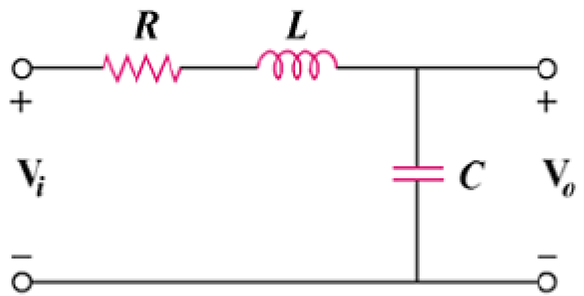


$$H(s) = \frac{V_o(s)}{V_i(s)}$$

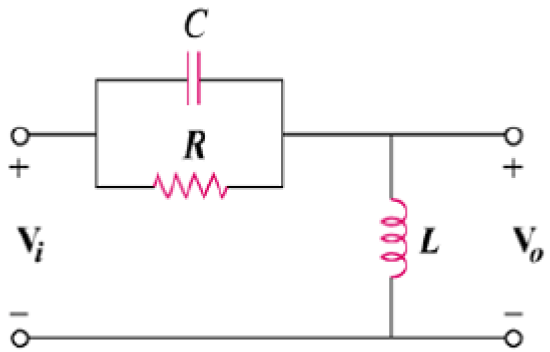
Example 3

Find the transfer function for the following two circuits

(a)



(b)



Bode plots

Are semilog plots of the magnitude (in dB) and the phase (in degrees) of the transfer function versus frequency.

$$Consider\ a\ transfer\ \ function \quad \mathbf{H} = H\angle\phi = He^{j\phi}$$

Bode plots consist of two plots:

- Magnitude (or gain) versus frequency
- Phase versus frequency

Gain is plotted in decibels:

$$\boxed{H_{dB} = 20\log_{10} H}$$

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Bode plots

In the general case a transfer function such as:

$$H(\omega) = \frac{N(\omega)}{D(\omega)}$$ *may be expressed in terms of factors by dividing out the poles and zeros, having real and imaginary parts such as :*

$$\mathbf{H}(\omega) = \frac{\boldsymbol{K}(j\omega)^{\pm 1}(1+j\omega/\boldsymbol{z}_{\boldsymbol{1}})[1+j2\zeta_1\omega/\omega_k+(j\omega/\omega_k)^2]}{(1+j\omega/\boldsymbol{p}_{\boldsymbol{1}})[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2]}$$

Which has several factors such as

a gain K, *a pole* $(j\omega)^{-1}$ *or zero* $(j\omega)^{+1}$ *at the origin,*

a simple (1st order) pole $= 1/(1+j\omega/\boldsymbol{p_1})$ *or zero at* $1/(1+j\omega/\boldsymbol{z_1})$,

a quadratic pole $= 1/[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2]$, *or zero at* $1/[1+j2\zeta_1\omega/\omega_k+(j\omega/\omega_k)^2]$

Bode plots

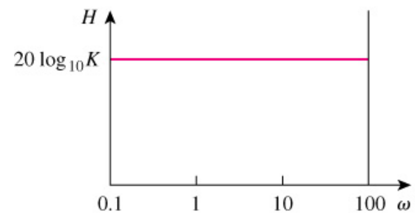
In order to construct a Bode plot:

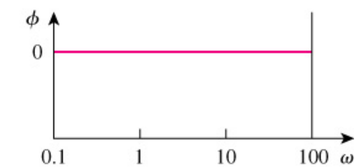
1. **Plot each factor separately then later combine them**
2. **Straight line approximations for all TFs factors are possible**

for the gain

$$\text{the magnitude is} = 20\log_{10} K$$

$$\text{and, its phase response is } 0°$$

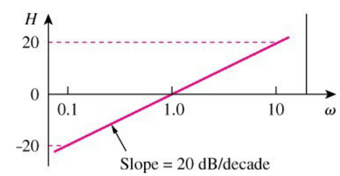


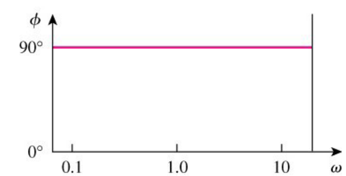


for the zero at the origin $(j\omega)$

$$\text{the magnitude is} = 20\log_{10}\omega$$

$$\text{and, its phase is } 90°$$

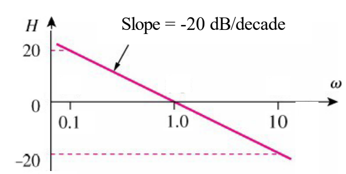


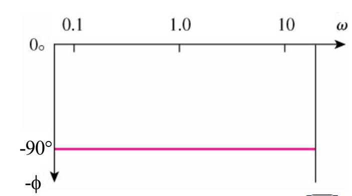


for the pole at the origin $(j\omega)^{-1}$

$$\text{the magnitude is} = -20\log_{10}\omega$$

$$\text{and, its phase is } -90°$$



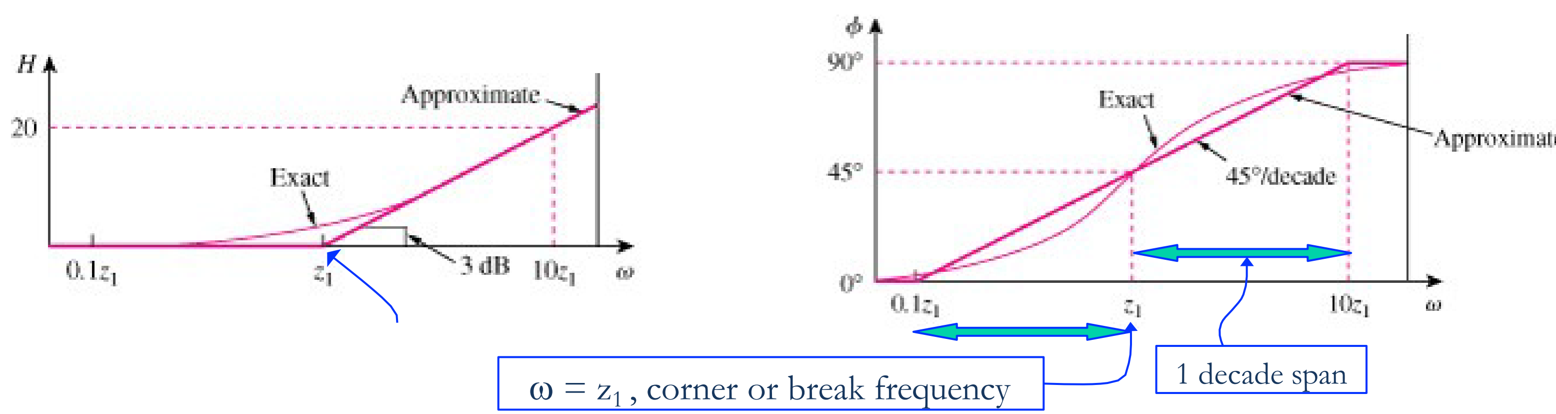


Bode plots

For the simple zero $1+\frac{j\omega}{z}$

The magnitude is $=20\log_{10}\left|1+\frac{j\omega}{z}\right| = \begin{cases} 20\log_{10}1 = 0 & \omega \to 0 \\ 20\log_{10}\frac{\omega}{z} & \omega \to \infty \end{cases}$

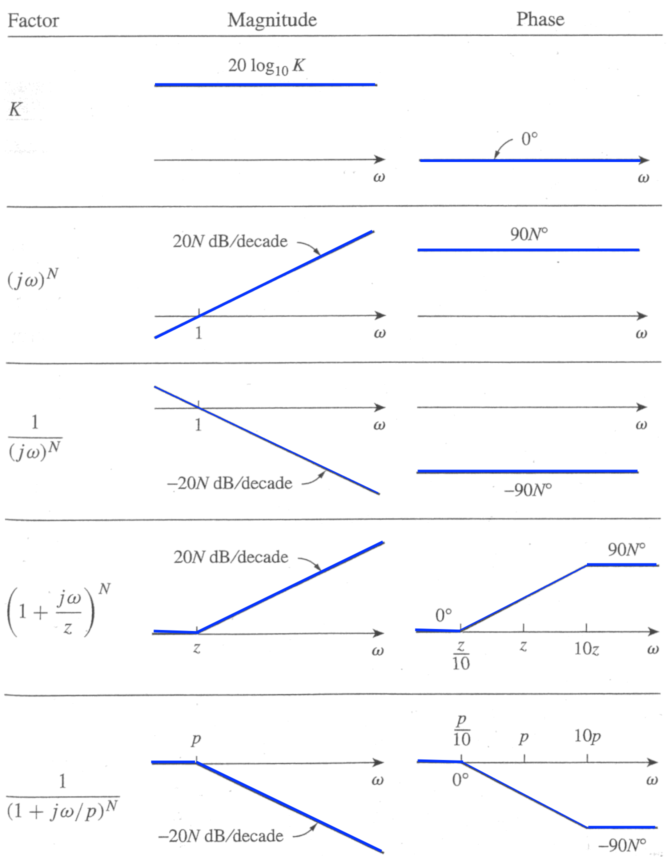
The phase $\emptyset = \tan^{-1}\left(\frac{\omega}{z}\right) = \begin{cases} 0 & \omega \to 0 \\ 45° & \omega = z \\ 90° & \omega \to \infty \end{cases} = \begin{cases} 0 & \omega < z/10 \\ 45° & \omega = z \\ 90° & \omega > 10z \end{cases}$

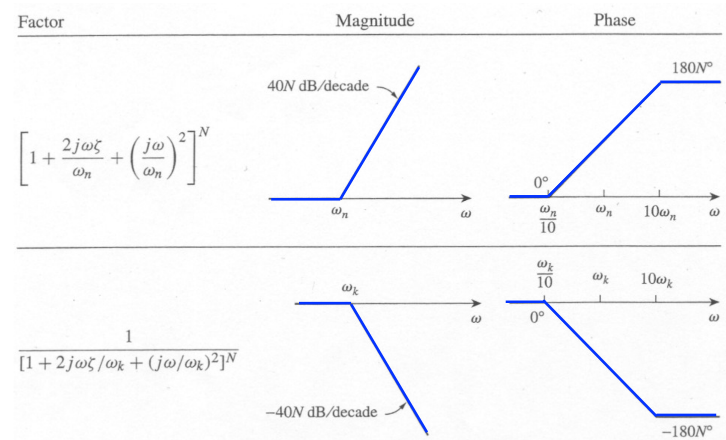


Asymptotic Bode Plot

- This is the easiest method in most cases
- If H has s as a factor on either the top or the bottom line, we need to take special measures

Asymptotic Bode Plot





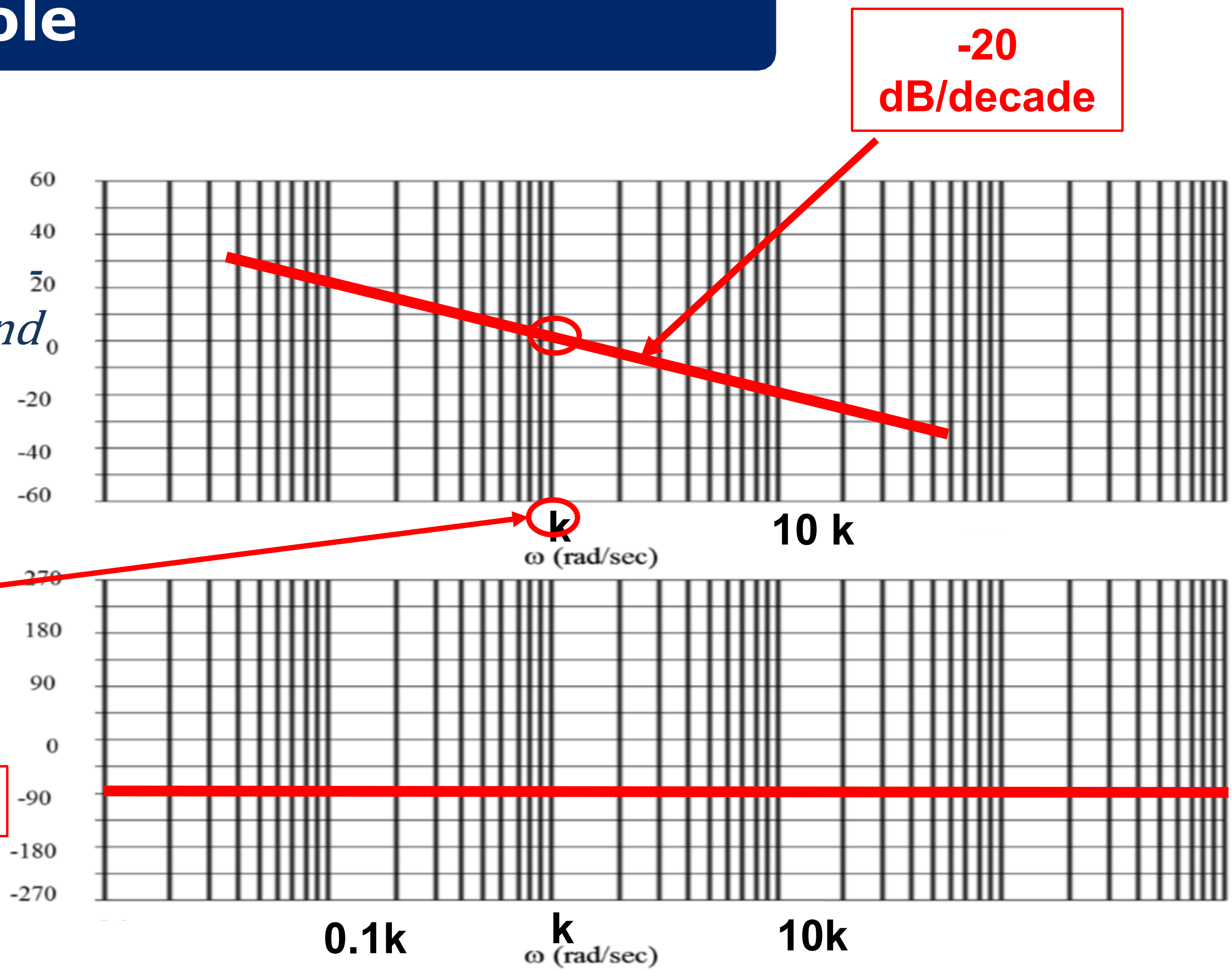
Summary of Bode straight-line magnitude & phase plots – (Alexander & Sadiku, Table 14.3)

DC Pole

the gain slope -20dB/decade and the phase at -90

$$\frac{1}{s/k}$$

-90^o



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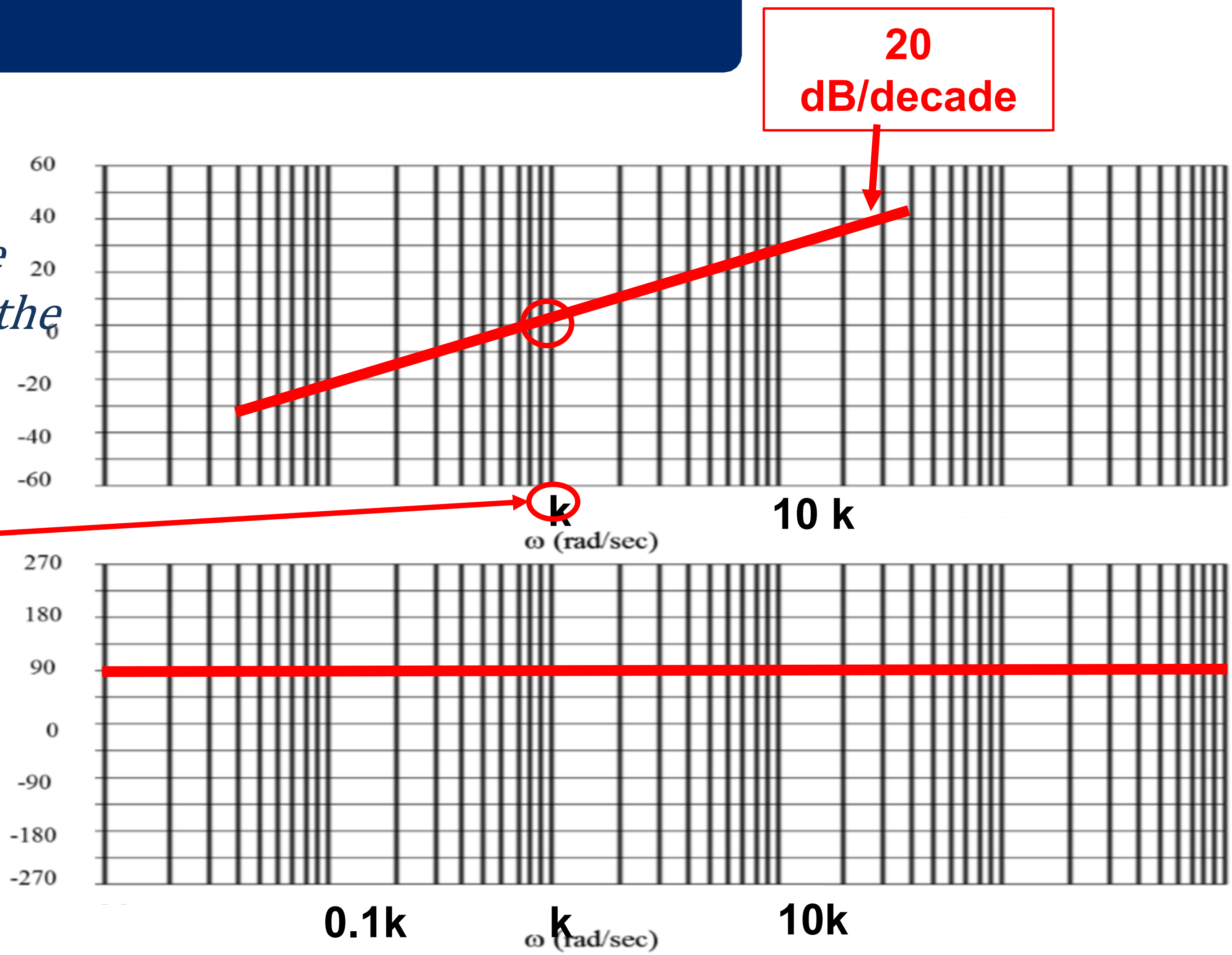
DC Zero

the gain with slope 20dB/decade and the phase at 90

$$\frac{s}{k}$$

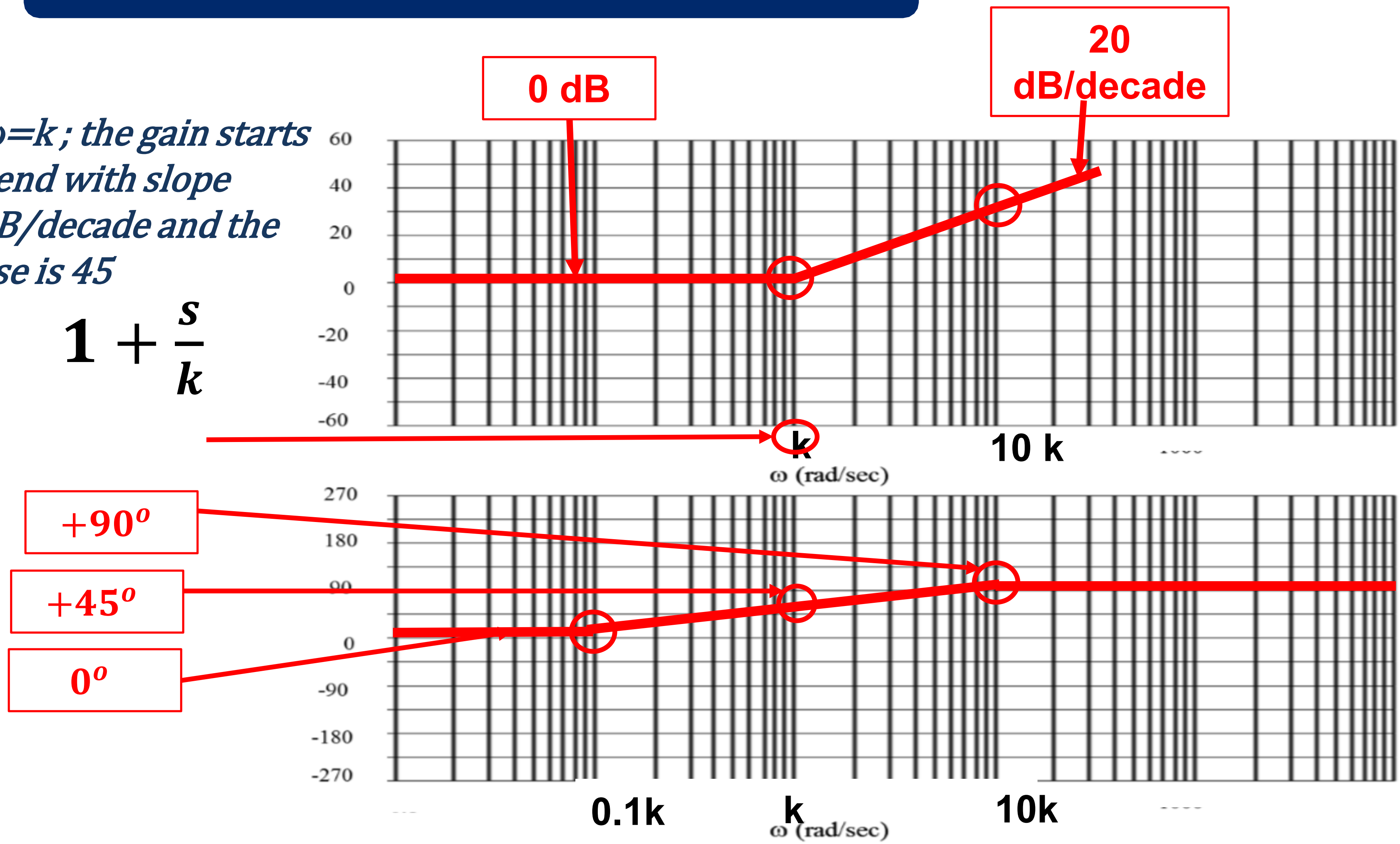


$+90^o$

Simple Zero

At ω=k ; the gain starts to bend with slope 20dB/decade and the phase is 45

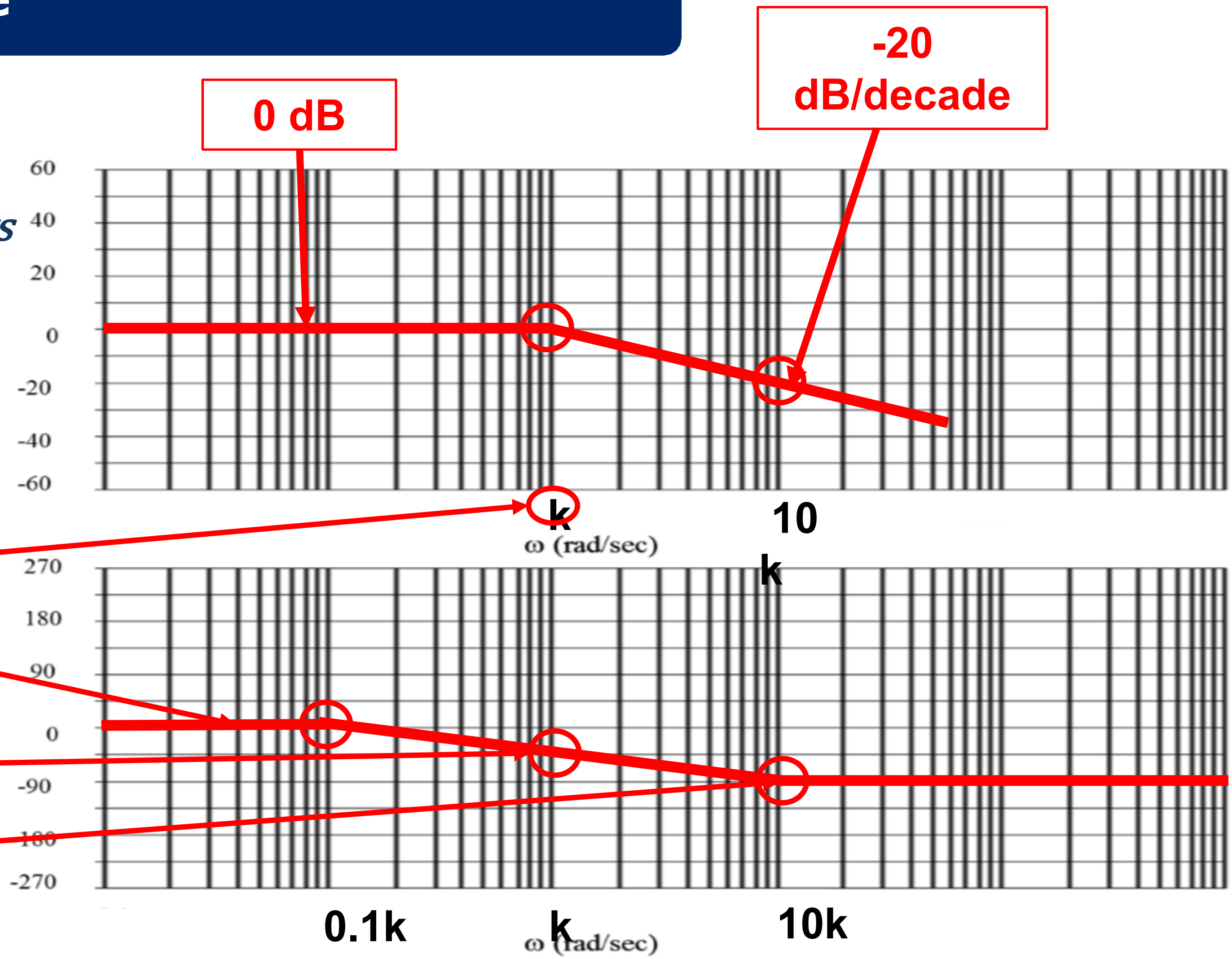
$$1 + \frac{s}{k}$$



Simple Pole

At ω=k ; the gain starts to bend with slope -20dB/decade and the phase is -45

$$\frac{1}{1+\frac{s}{k}}$$



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Asymptotic Bode Plot example

Draw the ABP of

$$H(s) = K\,\frac{(1+s/\omega_1)}{(1+s/\omega_2)} \qquad where \quad s = j\omega$$

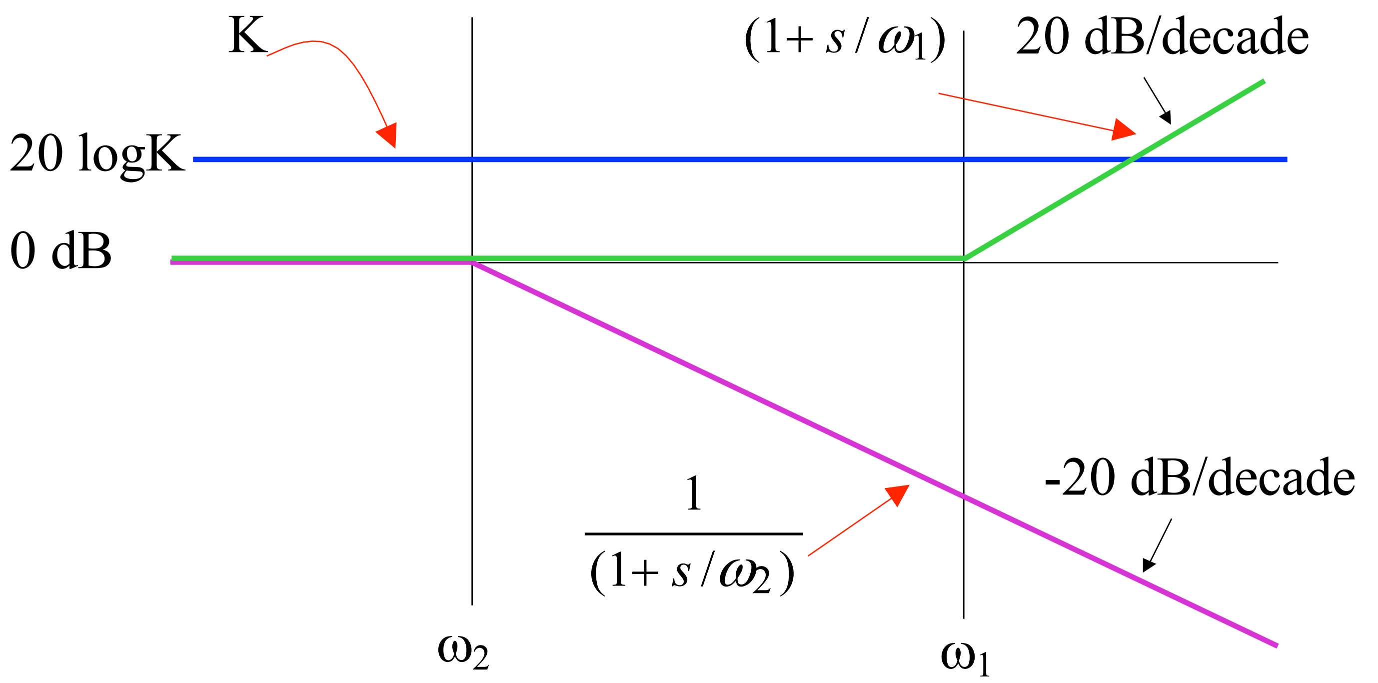
two methods are available:

1. Direct addition
2. Using breakpoints

- Note that log(ABC) = logA + logB + logC
- We draw the ABP for each factor separately
- We can then sum them directly on the graph
- Here we will assume that $K > 1$ and $\omega_1 > \omega_2$

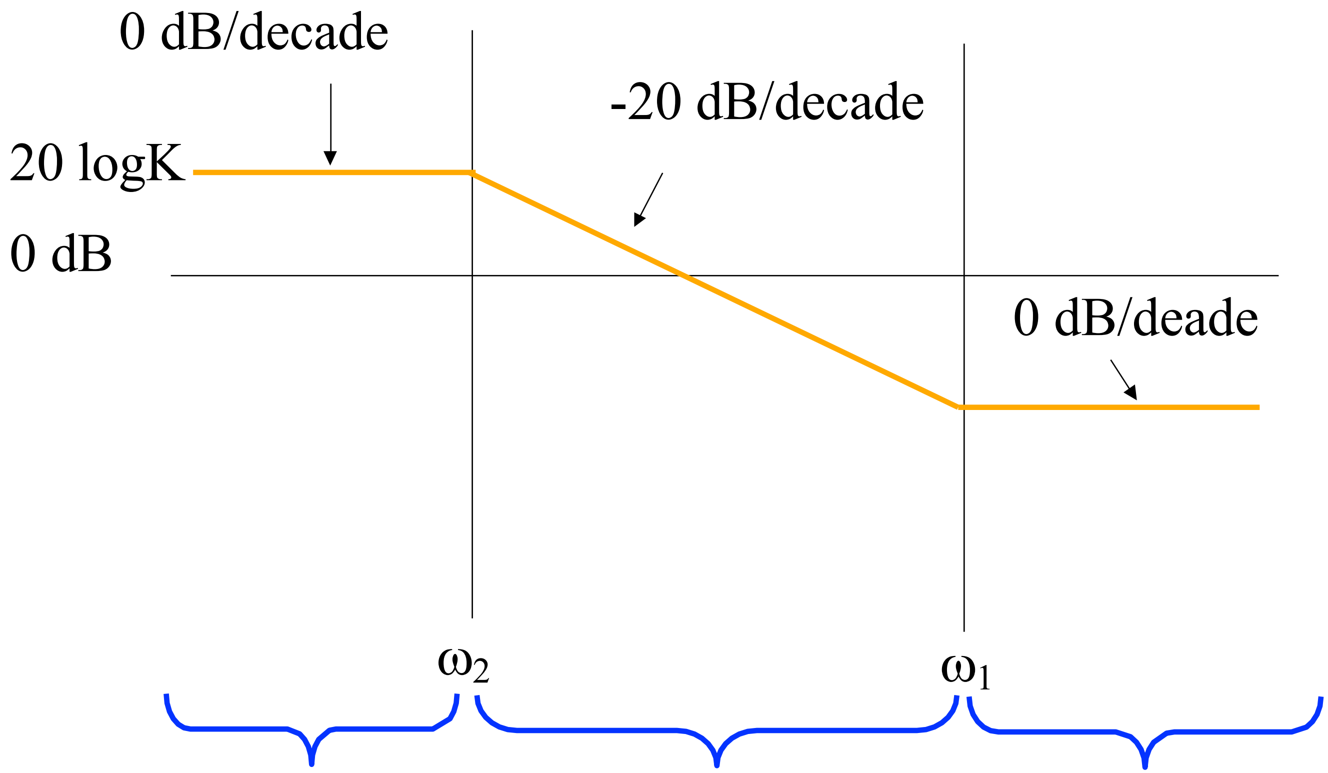
Asymptotic Bode Plot

First, plot each pole and zero separately:



Asymptotic Bode Plot

Next, add each curve together:



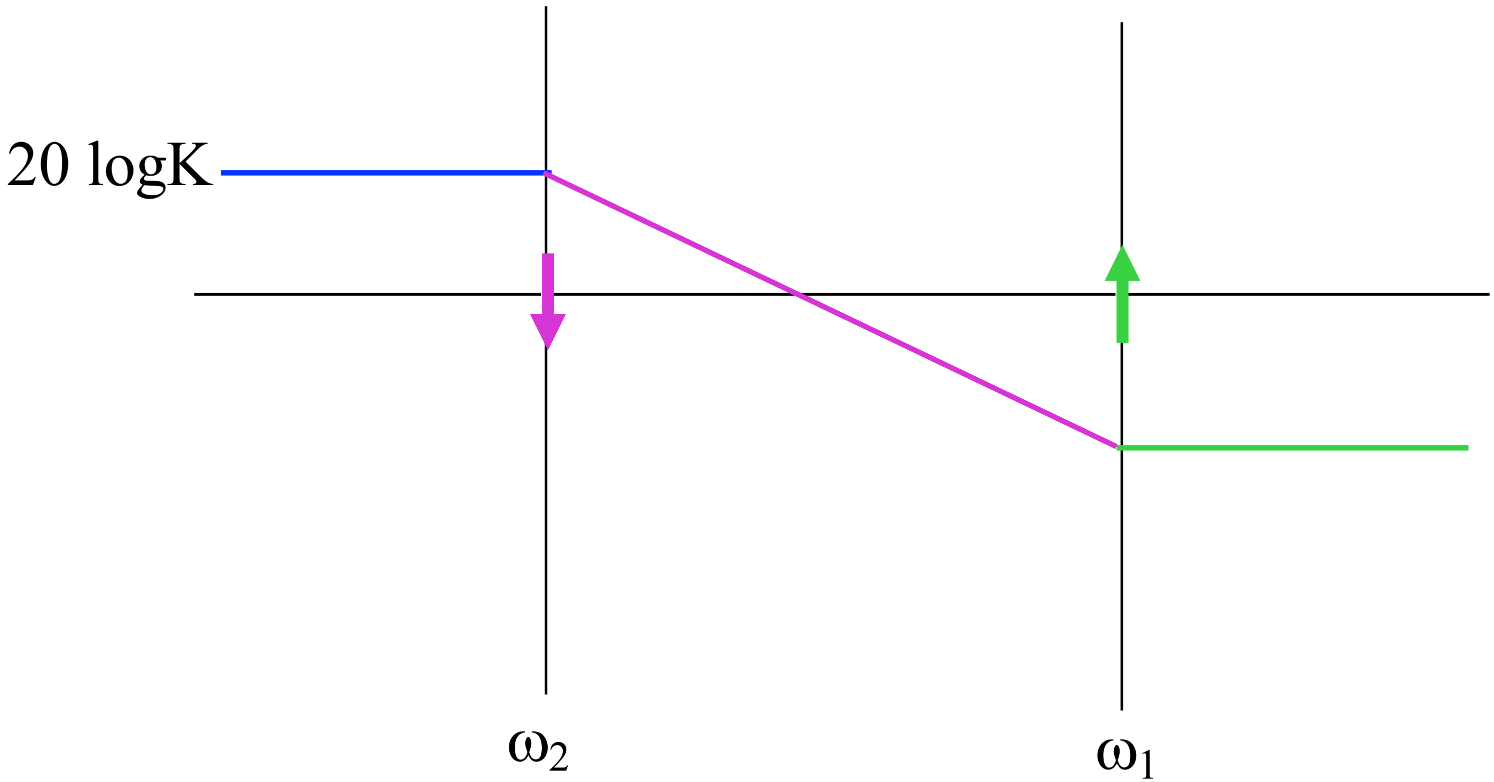
Only the K term contributes to this region

Here, curves for K and s/ω_2 factors add together

Here, curves for all 3 factors are added together

Asymptotic Bode Plot

H has a breakpoint at ω2, where the ABP slope will decrease by 20 dB/decade, and at ω1 where the ABP slope will increase by 20 dB/decade



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Bode Plot, example

Sketch the Bode Plots for

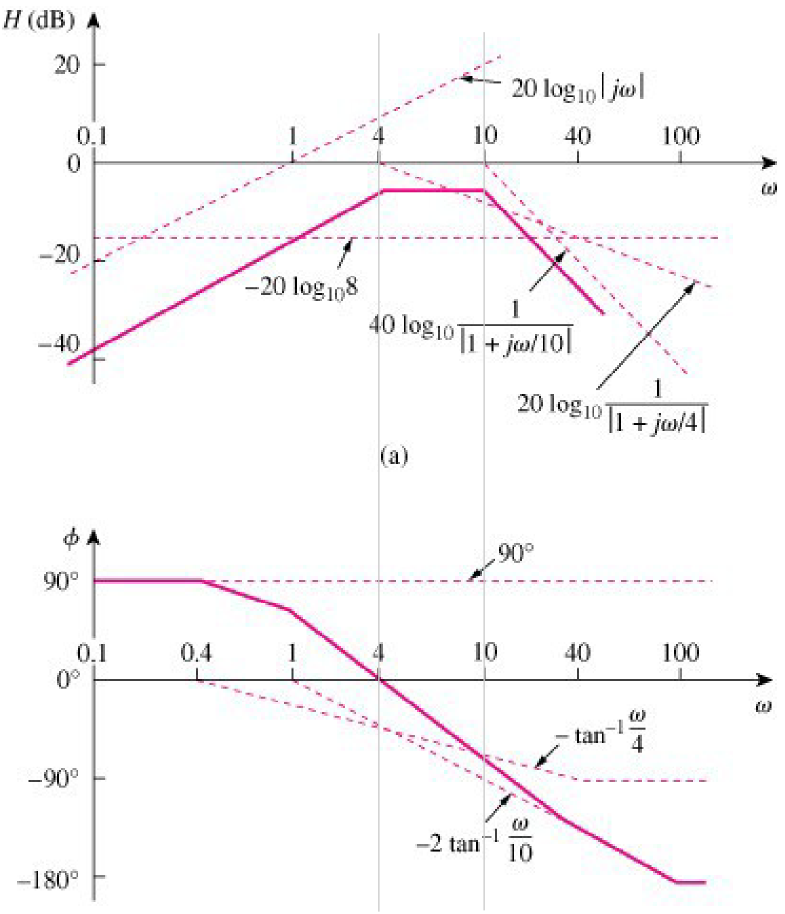
$$H(\omega) = \frac{50\,j\omega}{(j\omega+4)(j\omega+10)^2}$$

$$= \frac{50\,j\omega}{(4)(1+j\omega/4)(100)(1+j\omega/10)^2}$$

$$= \frac{(1/8)\,j\omega}{(1+j\omega/4)(1+j\omega/10)^2}$$

$$H_{db} = -20\log_{10}|8| + 20\log_{10}|j\omega|$$
$$-20\log_{10}|1+j\omega/4| - 40\log_{10}|1+j\omega/10|$$

$$\phi = 90° - \tan^{-1}(\omega/4) - 2\tan^{-1}(\omega/10)$$

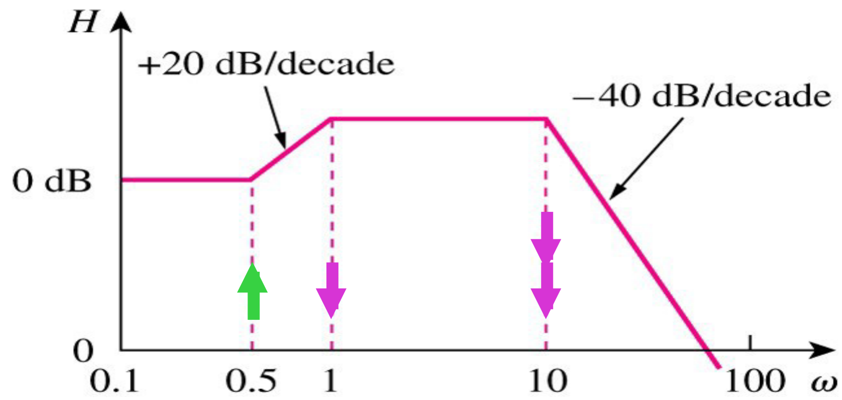


Reading a Bode Plot, example

Obtain the transfer function H(ω) corresponding to the following ABP

initial slope of the gain, here

$20\log_{10}K = 0\ dB$ *suggests* $K = 1$



a single zero (only +20dB/decade)

can be detected at $\omega = 0.5$,

$$\Rightarrow \quad (1 + j\omega/0.5)$$

a single pole (−20dB/decade)

is present at $\omega = 1$

$$\Rightarrow \quad \frac{1}{1 + j\omega/1}$$

a double pole (−40dB/decade)

appears at $\omega = 10$

$$\Rightarrow \quad \frac{1}{(1 + j\omega/10)^2}$$

Hence

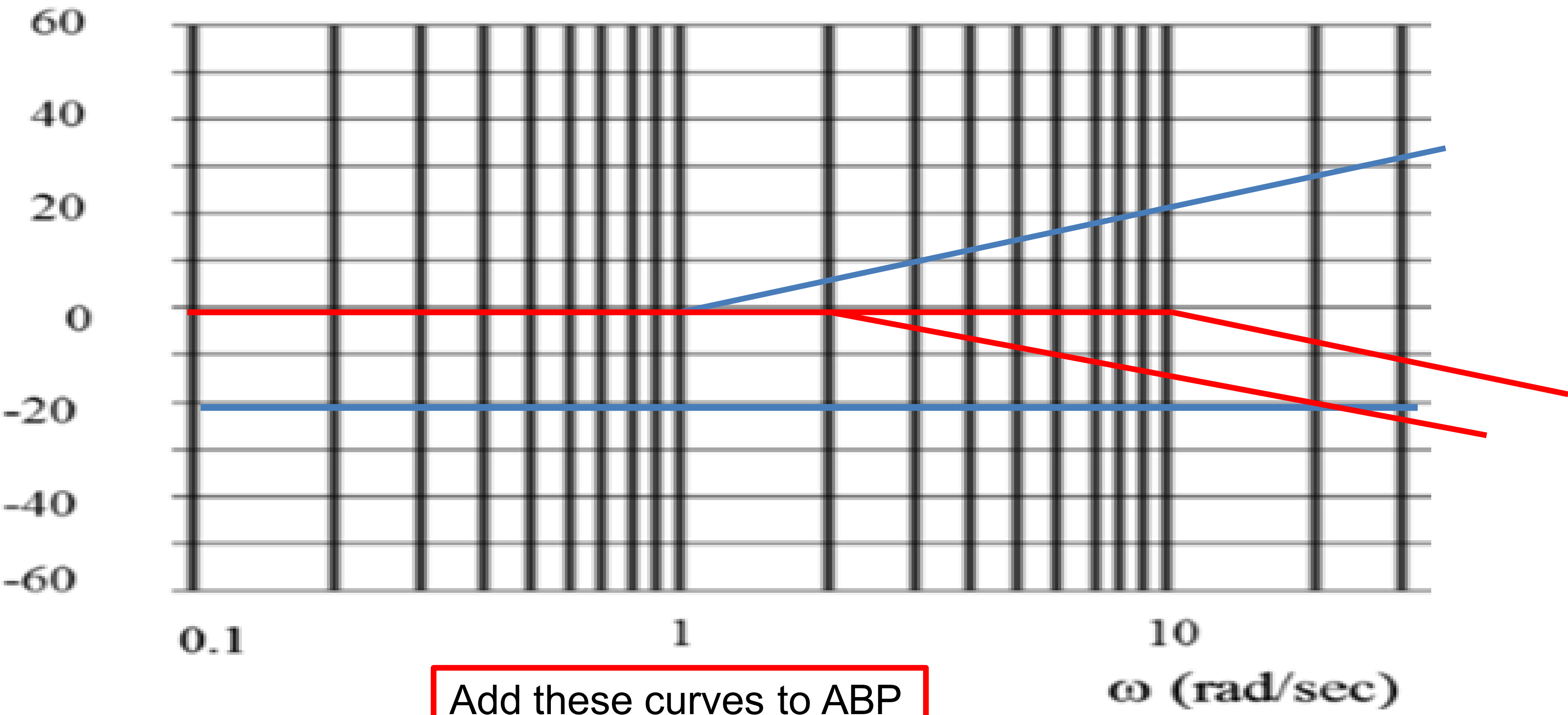
$$\mathbf{H}(\omega) = \frac{1 + j\omega/0.5}{(1 + j\omega/1)(1 + j\omega/10)^2} = \frac{(1/0.5)(0.5 + j\omega)}{(1/100)(1 + j\omega)(10 + j\omega)^2}$$

$$\boxed{\mathrm{H}(\omega) = \frac{200(s + 0.5)}{(s+1)(s+10)^2}}, \quad where\ \ s = j\omega$$

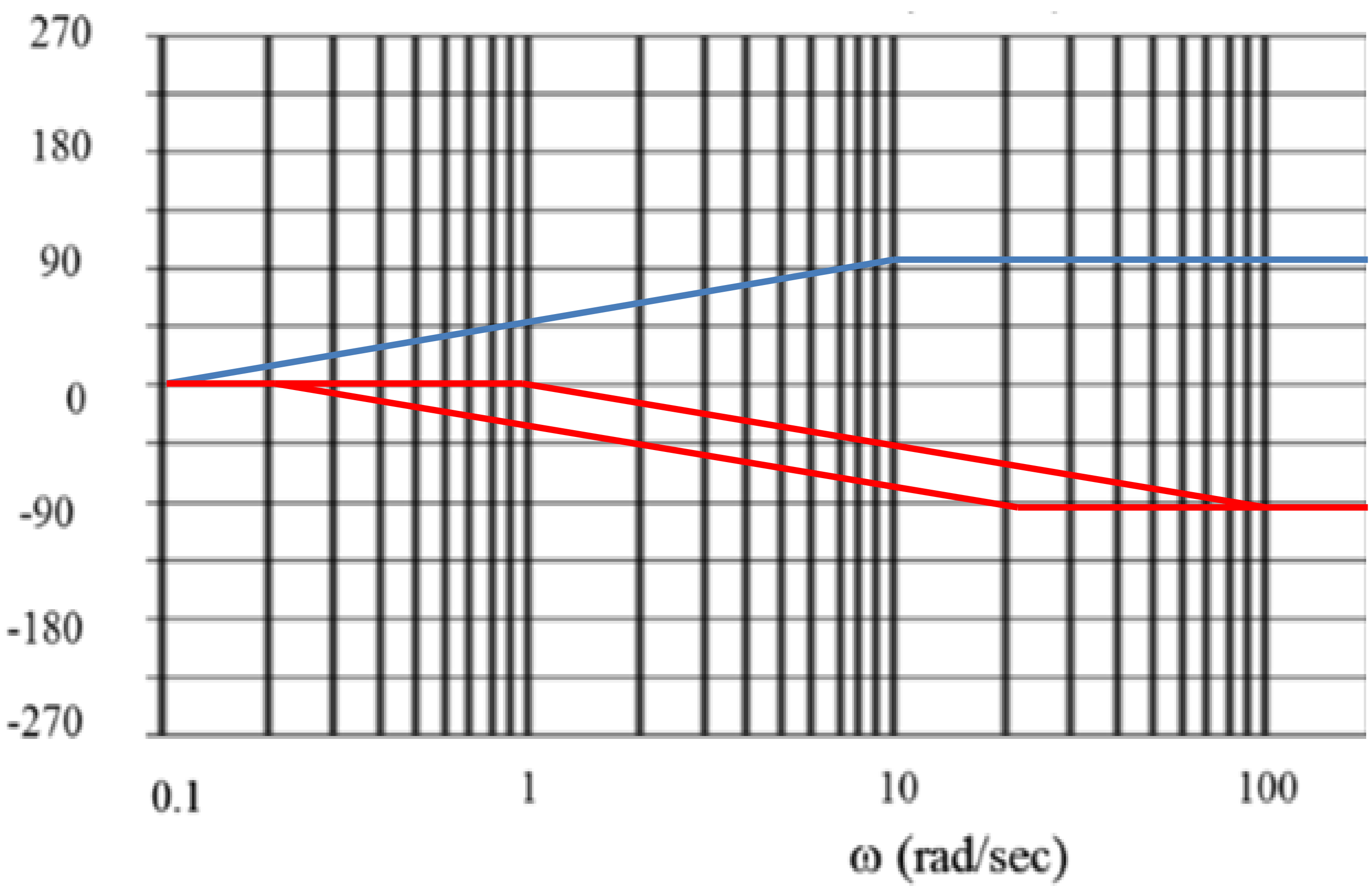
Bode Plot, example 5

Construct the Bode magnitude and phase plots for:

$$H(s) = \frac{2(s+1)}{(s+2)(s+10)}$$



Add these curves to ABP



Add these curves to PHBP

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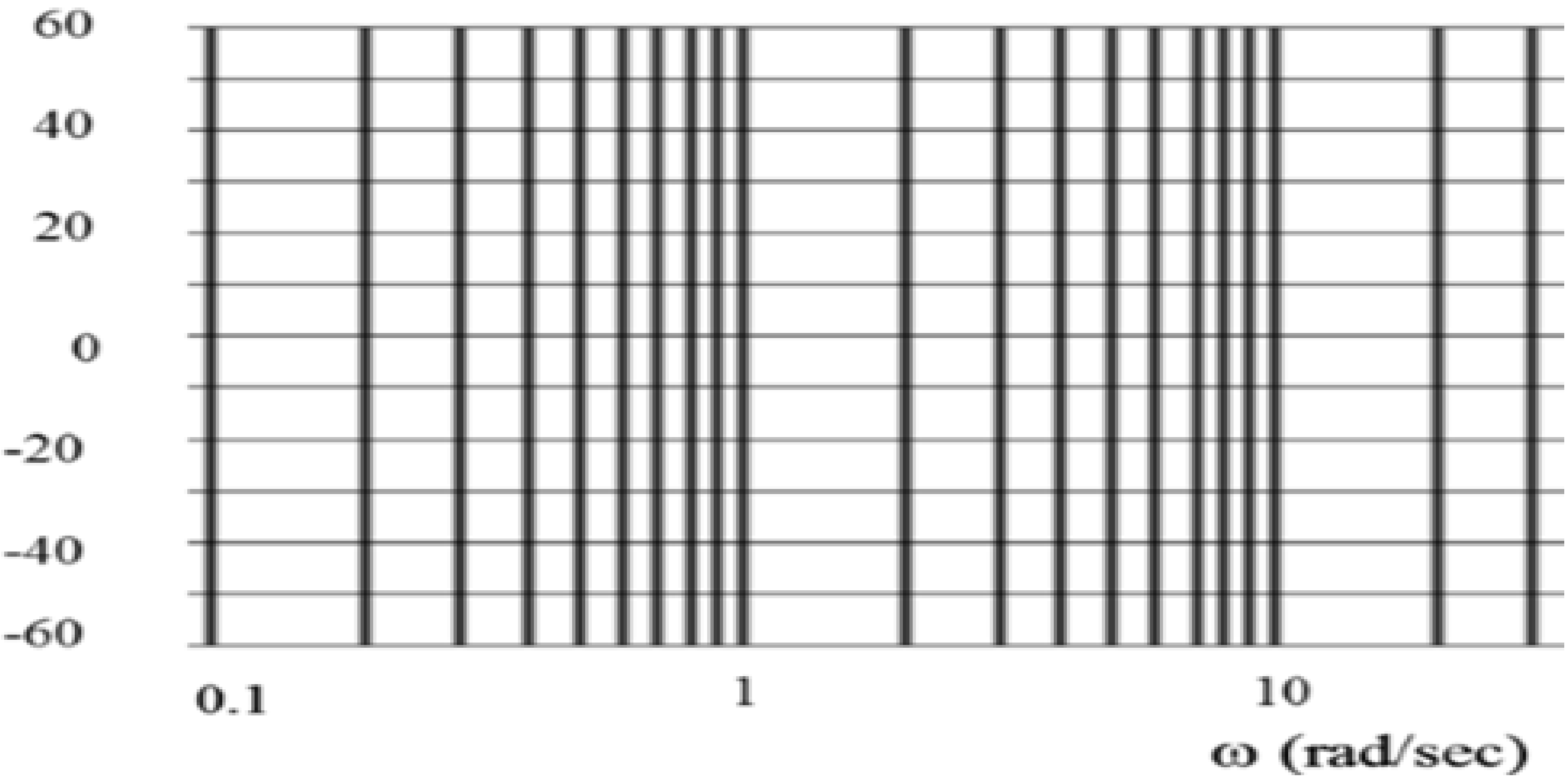
Bode Plot, example 6

Construct the Bode magnitude and phase plots for:

$$H(s) = \frac{1.6}{s(s^2 + 8s + 16)}$$



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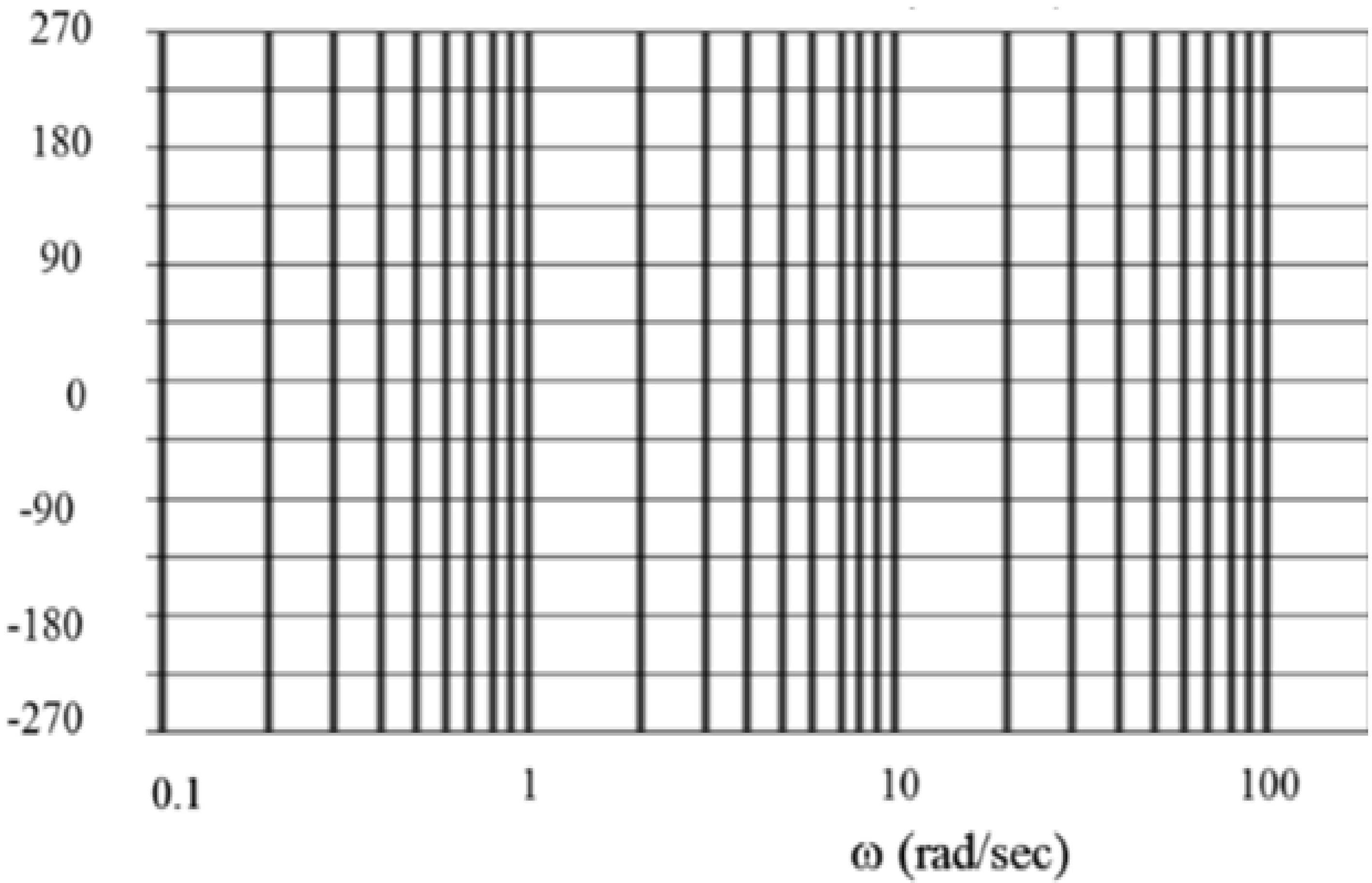
270
180
90
0
-90
-180
-270
0.1
1
10
100
ω (rad/sec)

Bode Plot, example 7

Sketch the Bode plots for

$$G(s) = \frac{s}{(s+2)^2(s+1)}, \qquad s = j\omega$$

Bode Plot, example 8

Find the transfer function $\mathbf{H}(\omega)$ with the Bode magnitude plot shown