Dimensionality Reduction

Question 1: Note: In this question, all columns will be written in their transposed form, as rows, to make the typography simpler. Matrix M has three rows and three columns, and the columns form an orthonormal basis. One of the columns is [2/7,3/7,6/7], and another is [6/7, 2/7, -3/7]. Let the third column be [x,y,z]. Since the length of the vector [x,y,z] must be 1, there is a constraint that $x^2+y^2+z^2 = 1$. However, there are other constraints, and these other constraints can be used to deduce facts about the ratios among x, y, and z. Compute these ratios.

Sol:

Q-1 Assign - 8

for orthonormal column, their dotproduct is equal to 0

$col(i) * col(j) == 0$ for $i \ne j$

$c_1 = [2/7, 3/7, 6/7]$ $c_2 = [6/7, 2/7, -3/7]$

$c_3 = [x, y, z]$

$c_1 * c_2 = [2/7, 3/7, 6/7] [6/7, 2/7, -3/7]$

$= 12/49 + 6/49 + (-18)/49 = 0$

$c_1 * c_3 = [2/7, 3/7, 6/7] [x, y, z]$

$= 2/7 x + 3/7 y + 6/7 z$

$c_2 * c_3 = [6/7, 2/7, -3/7] (x, y, z]$

$= \{ 6/7 x + 2/7 y - 3/7 z$

Equation $c_1 * c_3 == c_2 * c_3$

$2/7 x + 3/7 y + 6/7 z = 0$

$2x + 3y + 6z = 0$ (divide by 2)

$x + 3/2 y + 3z = 0$ (divide by 3) — A

$x/3 + 1/2 y + z = 0$ — B

$c_2 * c_3$

$6/7 x + 2/7 y - 3/7 z = 0$

$x + 1/3y - 1/2z = 0$ (multiply by 2) - (C)

$2x + 2/3y - z = 0$ — (D)

Eq (A) — eq (C)

$(\cancel{x} + 3/2y + 3z) - (\cancel{x} + 1/3y - 1/2z) = 0$

$(3/2 - 1/3)y + (3 + 1/2)z = 0$

$7/6y + 7/2z = 0$

$7/6y = -7/2z$

$y = -\cancel{7}/2 \times \cancel{6}^{3}/\cancel{7}$ 2

$\boxed{y = -3z}$ or $\boxed{z = -1/3y}$

Eq (B) + (D)

$(x/3 + 1/2y + \cancel{z}) + (2x + 2/3y - \cancel{z}) = 0$

$(1/3 + 2)x + (1/2 + 2/3)y = 0$

$7/3x + 7/6y = 0$

$$\frac{14x + 7y}{6} = 0$$

$14x + 7y = 0$

$7y = -14x$

$\boxed{y = -2x}$ or $\boxed{x = -1/2y}$

X:y:z = -2:1:-3

Question 2: Find the eigenvalues and eigenvectors of the following matrix:

2	3
3	10

You should assume the first component of an eigenvector is 1. Then, find out One eigenvalue and One eigenvector.

Sol:

(3) let $A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$

if λ is eigen value of A then let $\lambda I - A = 0$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda - 2 & -3 \\ -3 & \lambda - 10 \end{bmatrix} = 0$$

$$(\lambda - 2)(\lambda - 10) - 9 = 0$$

$$\lambda^2 - 10\lambda - 2\lambda + 20 - 9 = 0$$

$$\lambda^2 + 12\lambda + 11 = 0$$

$$\lambda^2 \cancel{+} - 11\lambda - \lambda + 11 = 0$$

$$\lambda(\lambda - 1) - 1(\lambda - 11) = 0$$

$$\lambda = 1 \text{ or } \lambda = 11$$

when $\lambda = 11$

$$\begin{bmatrix} 11-2 & -3 \\ -3 & 11-10 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} = B$$

$B \cdot \bar{x} = 0$ $\therefore \bar{x}$ is eigen vector

$$\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

using row reduction

$R_2 = 3R_2 + R_1$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\begin{bmatrix} 9 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if $x_1 = 1$ $x_2 = 3$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ = eigen vectors

when $\lambda = 1$

$$\begin{bmatrix} \lambda-2 & -3 \\ -3 & \lambda-10 \end{bmatrix} = 0$$

$$B = \begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix} = 0$$

$\therefore B \cdot \bar{x} = 0$

$$\begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

using row reduction $R_2 = 3R_1 + R_2$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\begin{bmatrix} -1 & -3 \\ -6 & 0 \end{bmatrix}$$

$-x_1 - 3x_2 = 0$

$-x_1 = 3x_2$

if $x_1 = 1$ $x_2 = -1/3$

$\therefore \begin{bmatrix} 1 \\ -1/3 \end{bmatrix}$ = eigen vectors

Question 3: Suppose [1,3,4,5,7] is an eigenvector of some matrix. What is the unit eigenvector in the same direction? Find out the components of the unit eigenvector.

Sol:

Q3: Given eigen vector = [1, 3, 4, 5, 7]

to find out unit eigen vector in same direction

$$D = \sqrt{1+9+16+25+49}$$

$$= \sqrt{100} = 10$$

unit eigen vector = [$1/10$, $3/10$, $4/10$, $5/10$, $7/10$]

Distance to the unit eigen vector should always be 1 so unit eigen vector of [1, 3, 4, 5, 7] is

[$1/10$, $3/10$, $4/10$, $5/10$, $7/10$]

Question 4: Suppose we have three points in a two dimensional space: (1,1), (2,2), and (3,4). We want to perform PCA on these points, so we construct a 2-by-2 matrix, call it N, whose eigenvectors are the directions that best represent these three points. Construct the matrix N and identify, its elements.

Sol:

Qu:

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

$M^T \times M$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 1+4+12 \\ 1+4+12 & 1+4+16 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 17 \\ 17 & 21 \end{bmatrix}$$

Question 5: Consider the diagonal matrix M =

1	0	0
0	2	0
0	0	0

Compute its Moore-Penrose pseudoinverse.

Sol:

Assign - 8

(Q5)

given diagonal matrix = $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

To compute the moore penrose pseudo-inverse of above diagonal matrix.

Moore penrose pseudo-inverse of a diagonal matrix Σ i.e, if the ith diagonal element of Σ is $\sigma \neq 0$, then replace it by $1/\sigma$

pseudo inverse matrix

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 6: When we perform a CUR dcomposition of a matrix, we select rows and columns by using a particular probability distribution for the rows and another for the columns. Here is a matrix that we wish to decompose:

1	2	3
4	5	6
7	8	9
10	11	12

Calculate the probability distribution for the rows.

Sol:

Q6: row probability $p_i = \sum_j \frac{M_{ij}^2}{f}$

$$f = \sum_{ij} M_{ij}^2$$

$$\text{Matrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Row Sum values

14

77

194

365

$f = 650$

The probabilities are $\Rightarrow$ $P_1 = \frac{14}{650} = 0.022$

$P_2 = 77/650 = 0.118$

$P_3 = 194/650 = 0.298$

$P_4 = 365/650 = 0.562$