$b_0$ : best

$b_0 = \bar{Y} - b_1\bar{X}$, where $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$, $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$

(iii) Want to show $b_0$ is best, i.e. has the smallest variance among all <u>linear</u> <u>unbiased</u> estimators

proof:

- In lecture, we shown $b_0 = \sum_{i=1}^n W_i Y_i$, where $w_i = \frac{1}{n} - k_i\bar{X}$

$$\begin{cases} E(b_0) = \beta_0 \\ V(b_0) = \sum_{i=1}^n W_i^2 \, Var(Y_i) = \sigma^2 \underbrace{\sum_{i=1}^n W_i^2} \end{cases}$$

- Now assume $\tilde{b}_0$ is another linear unbiased estimator of $\beta_0$

$$\tilde{b}_0 = \sum_{i=1}^n a_i Y_i, \text{ where } Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

$\tilde{b}_0$ is unbiased

$$\Rightarrow \boxed{\begin{aligned} E(\tilde{b}_0) &= E(\sum_{i=1}^n a_i(\beta_0 + \beta_1 X_i + \varepsilon_i)) \\ &= \beta_0 \sum_{i=1}^n a_i + \beta_1 \sum_{i=1}^n a_i X_i + 0 \\ &= \beta_0 \end{aligned}}$$

$$\Updownarrow$$

$$\boxed{\sum_1^n a_i = 1 \quad \text{and} \quad \sum_1^n a_i X_i = 0}$$

that is $\tilde{b}_0 = \beta_0 + \sum_1^n a_i \varepsilon_i$

$$\Rightarrow Var(\tilde{b}_0) = E\{(\tilde{b}_0 - E(\tilde{b}_0))^2\} = E((\tilde{b}_0 - \beta_0)^2)$$

$$= E\{(\sum_1^n a_i \varepsilon_i)^2)$$

$$= \sum_{i=1}^n a_i^2 E(\varepsilon_i^2) + \sum_{i \neq j} 2a_i a_j \underbrace{E(\varepsilon_i \varepsilon_j)}_{\substack{\text{since uncor.} \\ =0}}$$

$$= \sigma^2 \sum_{i=1}^n a_i^2$$

since crossed term $= 0$

or $V(\tilde{b}_0) = Var(\sum_1^n a_i Y_i) = \sum_1^n a_i^2 \sigma^2$.

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$b_0$ : best

VII Now we will show $b_0$ is best. i.e.

$$\underset{\sigma^2 \sum_{i=1}^n w_i^2}{\underset{\|}{V(b_0)}} \le \underset{\sigma^2 \sum_{i=1}^n a_i^2}{\underset{\|}{V(\tilde{b}_0)}}$$

proof:

since $a_i$ is arbitrary, so $a_i = w_i + d_i$

$$\Rightarrow \sum_1^n a_i^2 = \sum_{i=1}^n (w_i + d_i)^2$$
$$= \sum_{i=1}^n (w_i^2 + 2w_i d_i + d_i^2)$$
$$= \sum_{i=1}^n (w_i^2 + d_i^2) \quad \text{by } (*)$$
$$\ge \sum_{i=1}^n w_i^2 \quad \text{Done.}$$

(*) show $\sum_1^n w_i d_i = 0$, $w_i = \frac{1}{n} - k_i \bar{x}$

$$\sum_{i=1}^n \left(\frac{1}{n} - k_i \bar{x}\right) d_i$$
$$= \frac{1}{n} \sum_{i=1}^n d_i - \sum_{i=1}^n \frac{(x_i - \bar{x})\bar{x}}{S_{xx}} d_i$$
$$= \frac{1}{n} \sum_{i=1}^n d_i - \frac{\bar{x}}{S_{xx}} \sum_{i=1}^n x_i d_i + \frac{\bar{x}^2}{S_{xx}} \sum_{i=1}^n d_i$$
$$= 0 - \frac{\bar{x}}{S_{xx}} \cdot 0 + \frac{\bar{x}^2}{S_{xx}} 0 \quad \text{by } (**)$$
$$= 0$$

$$(**) \begin{cases} \sum a_i = 1 = \sum (w_i + d_i) = \overset{1}{\overset{\|}{\sum w_i}} + \sum d_i \iff \sum d_i = 0 \\ \sum a_i x_i = 0 \iff \sum (w_i + d_i) x_i = \underset{0}{\underset{\|}{\sum w_i x_i}} + \underbrace{\sum d_i x_i}_{0} = 0 \iff \sum d_i x_i = 0 \end{cases}$$

$b_1$: best.

$$b_1 = \frac{\Sigma(X_i-\bar{X})(Y_i-\bar{Y})}{\Sigma(X_i-\bar{X})^2} = \frac{S_{XY}}{S_{XX}} = \Sigma_{i=1}^n k_i Y_i.$$

where $k_i = \frac{X_i-\bar{X}}{\Sigma(X_i-\bar{X})^2}$

and $\Sigma k_i = 0$ & $\Sigma k_i X_i = 1$

Want to show $b_1$ is best: $V(b_1) \le V(\tilde{b}_1)$ where $\tilde{b}_1$ is another linear unbiased estimator.

$$\begin{cases} b_1 = \Sigma_1^n k_i Y_i \\ E(b_1) = \beta_1 \\ V(b_1) = V(\Sigma_1^n k_i Y_i) = \sigma^2 \Sigma_1^n k_i^2 \end{cases}$$ , since $V(Y_i) = \sigma^2$

$\tilde{b}_1 = \Sigma_1^n c_i Y_i$ is another unbiased estimator of $\beta_1$

$$\Leftrightarrow E(\tilde{b}_1) = E(\Sigma_1^n c_i(\beta_0+\beta_1 X_i+\varepsilon_i))$$

$$= \beta_0 \Sigma_1^n c_i + \beta_1 \Sigma_1^n c_i X_i + 0$$

$$= \beta_1$$

$$\Leftrightarrow \boxed{\Sigma_1^n c_i = 0 \text{ and } \Sigma_1^n c_i X_i = 1} \ *$$

$c_i$ is arbitrary, so we let $c_i = k_i + d_i$. $(*)$ implies

$$\begin{cases} \Sigma(k_i+d_i) = 0 \Leftrightarrow \underbrace{\Sigma k_i}_{=0} + \Sigma d_i = 0 \Leftrightarrow \boxed{\Sigma d_i = 0} \text{ (A)} \\ \Sigma_1^n c_i X_i = 1 \Leftrightarrow \underbrace{\Sigma k_i X_i}_{=1} + \Sigma d_i X_i = 1 \Leftrightarrow \boxed{\Sigma d_i X_i = 0} \text{ (B)} \end{cases}$$

Now show $V(b_1) \le V(\tilde{b}_1) \Leftrightarrow \sigma^2 \Sigma k_i^2 \le \sigma^2 \Sigma c_i^2$
$\Leftrightarrow \Sigma k_i^2 \le \Sigma c_i^2$

Proof: $\Sigma c_i^2 = \Sigma(k_i+d_i)^2 = \Sigma k_i^2 + \Sigma k_i d_i + \Sigma d_i^2$

$= \Sigma k_i^2 + \Sigma d_i^2 \ge \Sigma k_i^2$ since $\Sigma k_i d_i = 0$

Done.

$b_1$ : best

show $\Sigma k_i d_i = 0$

proof:

$$\Sigma k_i d_i = \Sigma_1^n \frac{X_i - \bar{X}}{S_{XX}} d_i$$

$$= \frac{1}{S_{XX}} \underbrace{\Sigma X_i d_i}_{=0 \text{ by } (A)} - \frac{\bar{X}}{S_{XX}} \underbrace{\Sigma_1^n d_i}_{=0 \text{ by } (B)}$$

$$= 0.$$

[III] $E(SSE) = E(\sum_i^n (Y_i - \hat{Y}_i)^2) = (n-2)\sigma^2$

proof: $E(\sum_i^n (Y_i - \hat{Y}_i)^2)$

$= \sum_{i=1}^n E((Y_i - \hat{Y}_i)^2)$

$= \sum_i^n Var(Y_i - \hat{Y}_i) + \sum_i^n (E(Y_i - \hat{Y}_i))^2$

$\boxed{\begin{array}{c} Var(X) = E(X^2) - (EX)^2 \\ \Updownarrow \\ E(X^2) = Var(X) + (EX)^2 \end{array}}$

$(E(Y_i - \hat{Y}_i))^2 = 0$ since $\begin{cases} E(Y_i) = \beta_0 + \beta_1 X_i \\ E(\hat{Y}_i) = E(b_0) + E(b_1) X_i = \beta_0 + \beta_1 X_i \end{cases}$

$= \sum_{i=1}^n Var(Y_i - \bar{Y} - b_1(X_i - \bar{X}))$

since $\hat{Y}_i = b_0 + b_1 X_i$
$= (\bar{Y} - b_1\bar{X}) + b_1 X_i$
$= \bar{Y} + b_1(X_i - \bar{X})$

$= \sum_i^n [Var(Y_i - \bar{Y}) - 2cov(Y_i - \bar{Y}, b_1(X_i - \bar{X})) + (X_i - \bar{X})^2 Var(b_1)]$

$\downarrow$ ①

$= \sum_i^n [V(\varepsilon_i - \bar{\varepsilon}) - 2cov((Y_i - \bar{Y})(X_i - \bar{X}), b_1) + (X_i - \bar{X})^2 \frac{\sigma^2}{S_{XX}}]$

$\downarrow$ ②

$= (n-1)\sigma^2 - 2cov(\sum_i^n (X_i - \bar{X})(Y_i - \bar{Y}), b_1) + \frac{\sigma^2}{S_{XX}} \sum (X_i - \bar{X})^2$

$\downarrow$ Ⓐ

$= (n-1)\sigma^2 - 2cov(b_1 S_{XX}, b_1) + \frac{\sigma^2}{S_{XX}} S_{XX}$

$\downarrow$ ①

$= (n-1)\sigma^2 - S_{XX}\, 2 Var(b_1) + \sigma^2$

$= (n-1)\sigma^2 - S_{XX} \frac{2\sigma^2}{S_{XX}} + \sigma^2$

$= (n-1)\sigma^2 - \sigma^2$

$= (n-2)\sigma^2$

⓪ $cov(X, Y) = E(XY) - E(X)E(Y)$

① $cov(aX, Y) = cov(X, aY)$

② $\sum_i^n cov(X_i, Y) = cov(X_1, Y) + \cdots + cov(X_n, Y)$
$= cov(\sum X_i, Y)$

$\boxed{\begin{array}{l} Ⓐ\ b_1 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{S_{XX}} \\ \Rightarrow \sum (X_i - \bar{X})(Y_i - \bar{Y}) = b_1 S_{XX} \end{array}}$