- Let a, b, a_i, b_i, c be some constants.

(I) key results about variance.

(a) definition

$$\text{Var}(Y) = E\{(Y-E(Y))^2\} = \underline{E(Y^2) - (EY)^2}$$

$$\hookrightarrow \quad E(Y^2) = \text{Var}(Y) + (EY)^2$$

(b) $\text{Var}(c) = 0 \Rightarrow \text{Var}(c+Y) = \text{Var}(Y)$.

(c) $\text{Var}(aX+bY) = \text{Var}(aX) + \text{Var}(bY) + 2\text{cov}(aX, bY)$
$= a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2ab\,\text{cov}(X, Y)$

$\swarrow$ $a=1,\ b=\pm 1$

(c') $\text{Var}(X \pm Y) = \text{Var}(X) \pm 2\text{cov}(X, Y) + \text{Var}(Y)$

Useful example:

$Y_1, \cdots Y_n$: $E(Y_i) = \mu, < \infty$ $\quad \text{Var}(Y_i) = \sigma^2 < \infty$

$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

$$\Rightarrow \boxed{E(\bar{Y}) = \mu \qquad \text{Var}(\bar{Y}) = \text{Var}\left(\tfrac{1}{n}\textstyle\sum_{i=1}^{n} Y_i\right) = \frac{\sigma^2}{n}}$$

(II) key results about covariance.

(a) Definition

$$\text{cov}(X, Y) = E\{(X-EX)(Y-EY)\}$$
$$= E(XY) - (EX)(EY)$$

(b) $\text{cov}(Y, X) = \text{cov}(Y, X)$

(c) $\text{cov}(X, X) = \text{Var}(X) \leftarrow$ by def'n of $\text{cov}(X, X) = E(X^2) - (EX)^2$

(d) $\text{cov}(aX, Y) = \text{cov}(X, aY) = a\,\text{cov}(X, Y)$

Proof: $\text{cov}(aX, Y) = E(aXY) - E(aX)E(Y) = \underline{a\,E(XY) - a\,E(X)E(Y)}$

$\text{cov}(X, aY) = E(XaY) - E(X)E(aY) = \underline{a\,E(XY) - a\,E(X)E(Y)}$

$a\,\text{cov}(X, Y) = a\,[E(XY) - E(X)E(Y)] = \underline{a\,E(XY) - a\,E(X)E(Y)}$

(e) $cov(X_1+X_2, Y) = cov(X_1, Y) + cov(X_2, Y)$

$\Downarrow$

$cov(\sum_1^n X_i, Y) = \sum_{i=1}^n cov(X_i, Y)$ ← $\boxed{e-2}$

proof:

$cov(X_1+X_2, Y) = E((X_1+X_2)Y) - E(X_1+X_2)E(Y)$

$= E(X_1Y + X_2Y) - [E(X_1)+E(X_2)]E(Y)$

$= \underbrace{E(X_1Y)} + E(X_2Y) - \underbrace{E(X_1)E(Y)} - E(X_2)E(Y)$

$cov(X_1, Y) + cov(X_2, Y)$

$= cov(X_1, Y) + cov(X_2, Y)$

(f) $cov(\sum_{i=1}^m X_i, \sum_{j=1}^n Y_j) = \sum_{i=1}^m \sum_{j=1}^n cov(X_i, Y_j)$

proof: $cov(\sum_{i=1}^m X_i, \sum_{j=1}^n Y_j)$

$= cov(\sum_{i=1}^m X_i, Y_1) + \cdots + cov(\sum_{i=1}^m X_i, Y_n)$

$= \sum_{i=1}^m cov(X_i, Y_1) + \cdots + \sum_{i=1}^m cov(X_i, Y_n)$ ↓ $\boxed{e-2}$

$= \sum_{i=1}^m [cov(X_i, Y_1) + \cdots + cov(X_i, Y_n)]$

$= \sum_{i=1}^m [\sum_{j=1}^n cov(X_i, Y_j)]$

$= \sum_{i=1}^m \sum_{j=1}^n cov(X_i, Y_j)$

(g) $cov(\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j \, cov(X_i, Y_j)$

↑ Applying (d) + (f)

(g) $cov(a+X, b+Y) = cov(X, Y)$.

proof:

$$
\begin{aligned}
cov(a+X, b+Y) &= E((a+X)(b+Y)) - E(a+X)E(b+Y) \\
&= E(ab + aY + bX + XY) - (a + E(X))(b + E(Y)) \\
&= ab + aE(Y) + bE(X) + E(XY) - ab - bE(X) - aE(Y) - E(X)E(Y) \\
&= E(XY) - (EX)(EY) \\
&= cov(X, Y)
\end{aligned}
$$

QED ^^

Exercise.

From class. we have $b_1 = \frac{S_{XY}}{S_{XX}} = \sum_{i=1}^{n} k_i Y_i$, $k_i = \frac{X_i - \bar{X}}{S_{XX}}$

① show $\boxed{cov(\bar{Y}, b_1) = 0}$

hint: $cov(\bar{Y}, b_1) = cov(\sum_{i=1}^{n} \frac{1}{n} Y_i, \sum_{j=1}^{n} k_j Y_j)$

$= \cdots$

② show $cov(b_0, b_1) = -\sigma^2 \frac{\bar{X}}{S_{XX}}$

③ show $cov(b_0, \bar{Y}) = \frac{\sigma^2}{n}$. (hint: $b_0 = \bar{Y} - b_1\bar{X}$)