From class, we show both b_0 and b_1 are linear unb. estimators.

$$b_1 = \sum_{i=1}^{n} k_i Y_i, \quad \text{where } k_i = \frac{X_i - \bar{X}}{S_{XX}}, \quad S_{XX} = \sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$b_0 = \sum_{i=1}^{n} w_i Y_i \quad \text{where } w_i = \frac{1}{n} - k_i \bar{X}$$

Now show.

① $\sum k_i = 0$

proof: $\sum_{i=1}^{n} k_i = \frac{1}{S_{XX}} \sum_{i=1}^{n} (X_i - \bar{X}) = \frac{1}{S_{XX}} (\sum X_i - n\bar{X}) = \frac{1}{S_{XX}} (n\bar{X} - n\bar{X})$

$= 0$

② $\sum_{i=1}^{n} k_i X_i = 1$

Proof: $\sum_{i}^{n} k_i X_i = \frac{1}{S_{XX}} \sum_{i=1}^{n} (X_i - \bar{X}) X_i$

$= \frac{1}{S_{XX}} \left(\sum_{i=1}^{n} X_i^2 - \bar{X} \sum_{i=1}^{n} X_i \right)$

$= \frac{1}{S_{XX}} \left(\sum_{i=1}^{n} X_i^2 - \bar{X} n \bar{X} \right)$

$= \frac{1}{S_{XX}} (\underbrace{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}_{= S_{XX}})$

$= 1$

If we show $\sum (X_i - \bar{X}) X_i = S_{XX}$, we are done.

Note that $\boxed{\sum (X_i - \bar{X}) X_i} = \sum (X_i - \bar{X}) X_i - \bar{X} (\underbrace{\sum (X_i - \bar{X})}_{0})$

$= \sum_{i=1}^{n} (X_i - \bar{X}) (X_i - \bar{X})$

$= \sum_{i=1}^{n} (X_i - \bar{X})^2 = \boxed{S_{XX}}$

$\Rightarrow \sum_{i=1}^{n} k_i X_i = \frac{1}{S_{XX}} \sum_{i=1}^{n} (X_i - \bar{X}) X_i$

$= 1$

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Exercise 1 · ②

③ $\Sigma W_i = 1$

proof: $\Sigma W_i = \Sigma\left(\frac{1}{n} - k_i \bar{X}\right)$

$= \frac{1}{n} \cdot n - \bar{X} \underbrace{\Sigma k_i}$

$= 1 - \bar{X} \cdot 0$

$= 1$

④ $\Sigma W_i X_i = 0$

proof: $\Sigma W_i X_i = \Sigma\left(\frac{1}{n} - k_i \bar{X}\right) X_i$

$= \frac{1}{n} \Sigma X_i - \bar{X} \Sigma k_i X_i$

$= \frac{1}{n} \cdot n\bar{X} - \bar{X} \sum_{i=1}^{n} \frac{(X_i - \bar{X}) X_i}{S_{xx}}$

$= \bar{X} - \bar{X} \cdot \frac{1}{S_{xx}} \underbrace{\sum_{i=1}^{n} (X_i - \bar{X}) X_i}_{= S_{xx}}$

$= \bar{X} - \bar{X}$

$= 0$