$\mathrm{cov}(\bar{Y}, b_1) = 0$

proof:

$$\mathrm{cov}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i,\ \sum_{j=1}^{n} k_j Y_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{n} k_j \,\mathrm{cov}(Y_i, Y_j)$$

$$= \underbrace{\sum_{i=1}^{n} \frac{1}{n} k_i \,\mathrm{cov}(Y_i, Y_i)}_{i=j} + \sum_{i=1}^{n}\sum_{j\neq i}^{n} \frac{1}{n} k_j \underbrace{\mathrm{cov}(Y_i, Y_j)}_{0}$$

under GM or under Normal error model.

$$= \frac{\sigma^2}{n}\sum_{i=1}^{n} k_i$$

$$= \frac{\sigma^2}{n}\sum_{i=1}^{n} \frac{X_i - \bar{X}}{S_{XX}}$$

$$= \frac{\sigma^2}{n}\frac{1}{S_{XX}} \underbrace{\sum_{i=1}^{n} (X_i - \bar{X})}_{0}$$

$$= 0$$

show $\mathrm{var}(b_0) = \sigma^2\left(\frac{1}{n} + \frac{\bar{X}}{S_{XX}}\right)$ $\qquad S_{XX} = \sum_{i=1}^{n} (X_i - \bar{X})^2$

proof:

$$b_0 = \bar{Y} - b_1\bar{X}$$

$$\Rightarrow \mathrm{var}(b_0) = \mathrm{var}(\bar{Y} - b_1\bar{X})$$

$$= \mathrm{var}(\bar{Y}) + \bar{X}^2\,\mathrm{var}(b_1) - 2\bar{X}\underbrace{\mathrm{cov}(\bar{Y}, b_1)}_{0} \quad \text{by above}$$

$$= \frac{\sigma^2}{n} + \bar{X}^2\frac{\sigma^2}{S_{XX}}$$

$$= \sigma^2\left(\frac{1}{n} + \frac{\bar{X}^2}{S_{XX}}\right)$$

Show $\mathrm{cov}(b_0, b_1) = -\sigma^2\bar{X}/S_{XX}$

proof:

$$\mathrm{cov}(b_0, b_1) = \mathrm{cov}(\bar{Y} - b_1\bar{X}, b_1)$$

$$= \mathrm{cov}(\bar{Y}, b_1) - \mathrm{cov}(b_1\bar{X}, b_1)$$

$$= 0 - \bar{X}\,\mathrm{cov}(b_1, b_1)$$

$$= -\bar{X}\frac{\sigma^2}{S_{XX}} = -\sigma^2\frac{\bar{X}}{S_{XX}}$$

VIII Show $cov(b_0, \bar{Y}) = \sigma^2/n$.

proof:

$$cov(\bar{Y} - b_1\bar{X}, \bar{Y}) = cov(b_0, \bar{Y}). \quad \text{since } b_0 = \bar{Y} - b_1\bar{X}$$

$$= cov(\bar{Y}, \bar{Y}) - cov(b_1\bar{X}, \bar{Y})$$

$$= Var(\bar{Y}) - \bar{X}\underbrace{cov(b_1, \bar{Y})}_{0 \text{ (proved)}}$$

$$= \frac{\sigma^2}{n}$$

Assume: $Y_i = \beta_1 X_i + \varepsilon_i$. $\varepsilon_i \sim iid\ N(0, \sigma^2)$

(1) Find LSE of $\beta_1$, say $b_1$

(2) $var(b_1)$

(3) find an unbiased estimator of $var(b_1)$

(1) Let the LSE of $\beta_1$ be $b_1$. so, we want to minimize

$$Q = \Sigma_{i=1}^n (Y_i - b_1 X_i)^2$$

$$\frac{\partial Q}{\partial b_1} = \Sigma_{i=1}^n -2X_i(Y_i - b_1 X_i)$$

$$= -\Sigma 2X_i Y_i + 2b_1 \Sigma_{i=1}^n X_i^2 = 0 \Rightarrow \boxed{b_1 = \frac{\Sigma_i^n (X_i Y_i)}{\Sigma_{i=1}^n X_i^2}}$$

(2) $\varepsilon_i \sim N(0, \sigma^2) \Rightarrow Y_i \overset{indep.}{\sim} N(\beta_1 X_i, \sigma^2)$

$$\Rightarrow Var(b_1) = Var\left(\Sigma_{i=1}^n \boxed{\frac{X_i}{\Sigma_i^n X_i^2}} Y_i\right)$$

$$= Var\left(\Sigma_{i=1}^n \tilde{K}_i Y_i\right)$$

$$= \Sigma_{i=1}^n \tilde{K}_i^2 var(Y_i)$$

$$= \sigma^2 \Sigma_{i=1}^n \left(\frac{X_i}{\Sigma X_i^2}\right)^2 = \sigma^2 \frac{1}{(\Sigma X_i^2)^2} \Sigma X_i^2 = \boxed{\frac{\sigma^2}{\Sigma X_i^2}}$$

(3). If we could find an unb. est. of $\sigma^2$.
then we are done.

claim: $MSE = SSE/(n-1)$

$$= \frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{n-1} = \frac{\sum_{i=1}^{n}(Y_i - b_1X_i)^2}{n-1}$$

$E(MSE) = \sigma^2 \iff E(SSE) = (n-1)\sigma^2$

Proof:

- $b_1 = \frac{\sum X_i Y_i}{\sum X_i^2} = \sum_{i=1}^{n} \frac{X_i}{S_{XX}} Y_i$, $S_{XX} = \sum_{j=1}^{n} X_j^2$

$$= \sum_{i=1}^{n} \tilde{k}_i Y_i$$

- $E(SSE) = E\{\sum_{i=1}^{n}(Y_i - b_1X_i)^2\} = \sum_{i=1}^{n} E((Y_i - b_1X_i)^2)$

$$= \sum_{i=1}^{n}\{var(Y_i - b_1X_i) + (\underbrace{E(Y_i - b_1X_i)}_{\to 0})^2\} \quad (*)$$

- $E(Y_i - b_1X_i) = E(Y_i) - X_i E(b_1)$

$$= \beta_1 X_i - X_i E(\textstyle\sum_i^n \tilde{k}_i Y_i)$$

$$= \beta_1 X_i - X_i (\textstyle\sum_{i=1}^{n} \tilde{k}_i X_i \beta_1)$$

$$= \beta_1 X_i - \beta_1 X_i (\textstyle\sum_{i=1}^{n} \tilde{k}_i X_i)$$

$$= \beta_1 X_i - \beta_1 X_i (\underbrace{\textstyle\sum_{i=1}^{n} \frac{X_i}{S_{XX}} X_i}) \leftarrow \frac{S_{XX}}{S_{XX}} = 1$$

$$= 0$$

- $var(Y_i - b_1X_i) = var(Y_i) - 2X_i cov(Y_i, \ldots) + X_i^2 var(b_1)$

$$= \sigma^2 - 2X_i cov(Y_i, \textstyle\sum_{j=1}^{n} \tilde{k}_j Y_j) + X_i^2 var(b_1).$$

Exercise ④

$$= \sigma^2 - 2X_i \, cov(Y_i, \tilde{k}_i Y_i) + X_i^2 \, var(b_1)$$

$$= \sigma^2 - 2X_i \tilde{k}_i \sigma^2 + X_i^2 \, var(\textstyle\sum_{i=1}^n \tilde{k}_i Y_i)$$

$$= \sigma^2 - 2\underbrace{X_i \tilde{k}_i}\sigma^2 + X_i^2 \underbrace{\textstyle\sum_{i=1}^n \tilde{k}_i^2}\sigma^2$$

$$X_i \tilde{k}_i = X_i \cdot \frac{X_i}{S_{XX}} = \frac{X_i^2}{S_{XX}} \qquad \sum_{i=1}^n \left(\frac{X_i}{S_{XX}}\right)^2 = \frac{\sum X_i^2}{(S_{XX})^2} = \frac{1}{S_{XX}}$$

$$= \sigma^2 - 2\frac{X_i^2}{S_{XX}}\sigma^2 + \frac{\sigma^2 X_i^2}{S_{XX}}$$

$$= \sigma^2 - \sigma^2 \frac{X_i^2}{S_{XX}}$$

So $\sum_{i=1}^n Var(Y_i - b_1 X_i) = \sum_{i=1}^n \sigma^2 - \sigma^2 \sum_{i=1}^n \frac{X_i^2}{S_{XX}}$

$$= n\sigma^2 - \sigma^2 \frac{\sum X_i^2}{S_{XX}}$$

$$= n\sigma^2 - \sigma^2.$$

$$= (n-1)\sigma^2 \qquad (**)$$

put $(**)$ into $(*)$ we have.

$$E(SSE) = (n-1)\sigma^2$$

$\Rightarrow \quad \hat{\sigma}^2 = MSE = \frac{SSE}{n-1}$ is an unb. est of $\sigma^2$

$\Rightarrow$ an unb est of $var(b_1)$ is $\boxed{\frac{MSE}{\sum X_i^2} = \frac{MSE}{S_{XX}}}$