Upper and Lower Bounds

A real-valued function f is bounded from above on a set D, if there exists a number N such that $f(x) \le D$ for all $x \in D$ We call N, when it exists, an upper bound for f on D.

For example, suppose that the global maximum of a function is 17 on the interval $[0,10]$, then 17 is an upper bound of the function on that interval. However, 20, 50, 1,000 are also upper bounds of the function on that interval. The best upper bound is 17 because if you go below 17, it is no longer an upper bound. There will be another value above it in the interval.

In a similar manner, we say f is bounded from below on a set D, if there exists a number M such that $f(x) \ge M$ for all $x \in D$ We call M, when it exists, a lower bound for f on D.

For example, suppose that the global minimum of a function is -80 on the interval $[0,10]$, then -80 is a lower bound of the function on that interval. However, -200, -500, -1,000 are also lower bounds of the function on that interval. The best lower bound is -80 because if you go above -80, it is no longer a lower bound. There will be another value below it in the interval.

The global maximum is the best upper bound (also know as the least upper bound). The global minimum is the best lower bound (also know as the greatest lower bound). Any value above the global maximum is also an upper bound – just not the best. Any value below the global minimum is also a lower bound – just not the best.

Example 1: Suppose that a function f has a global maximum of 200 and a global minimum of 6 on the interval $[0,10]$, then 200 is an upper bound of the function and 6 is the lower bound of the function. We could also say that 400 is an upper bound of the function and -10 is a lower bound of the function because the all the function values on the interval $[0,10]$ will fall between these two values. We could also say that 4,000 is an upper bound of the function and -1,000 is a lower bound of the function because the all the function values on the interval $[0,10]$ will fall between these two values.

The upper bound of 200 is the best upper bound and 6 is the best lower bound because if you value the function at 199, it will be less than 200 on the interval and therefore, not an upper bound, and if you value the function at 10, it will be more than 6 on the interval, and therefore, not a lower bound.

Example 2: Find the best upper and lower bounds (also called least upper bound and greatest lower bound) for the function $f(x)=x^2-15$ on the interval $[-2,10]$.

Solution: We do this by finding the global maximum and global minimum values of the function.

Step 1. Find the value of the function at the end points.

$$f(-2)=f(x)=(-2)^2-15=-11$$

$$f(10)=f(x)=(100)^2-15=85$$

Step 2:Find the critical points of the function.

$$f(x)=x^2-15$$

$$f'(x)=2x=0$$

$$x=0$$

There is one critical point at $x=0$.

Step 3:Find the critical intervals.

The critical intervals are $[-2,0)$ and $(0,10]$.

Step 4:Find the Local Extrema using either the First Derivative Test or the Second Derivative Test.

First Derivative Test: Find the sign of the first derivative in the critical domains to determine whether the function is increasing or decreasing in that interval.

$-1 \in [-2,0)$ and $f'(-1) = 2(-1) < 0$ means the function is decreasing on the interval $[-2,0)$.

$5 \in (0,10]$ and $f'(5) = 2(5) > 0$ means the function is increasing on the interval $(0,10]$.

Since the function is decreasing to the left of $x=0$ and increasing to the right of $x=0$, then $x=0$ is the location of a local minimum.

Second Derivative Test: Find the sign of the second derivative at each critical point to determine whether the function is concave or concave down. If the function is concave down at a critical point, then the critical point is the location of a local maximum. If the function is concave up at a critical point, then the critical point is the location of a local minimum.

$$f'(x) = 2x$$
$$f''(x) = 2 > 0$$

This means the graph of the function is concave up everywhere. Therefore, $x=0$ is the location of a local minima.

Step 5:Evaluation the local minima:

$$f(0) = (0)^2 - 15 = -15$$

Step 6: Compare the values of the function at the endpoints and local extrema.

The value of the function at the left endpoint is -11.

The value of the function at the right endpoint is 85.

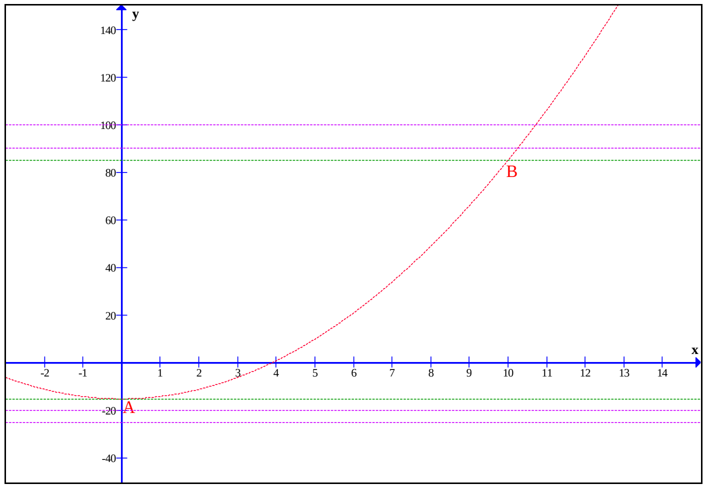
The value of the local minima is -15.

Therefore, the global maximum value of the function on this interval is 85, and the global mininmum value of the function on this interval is -15.

These are the upper and lower bounds on the interval $[-2,10]$ – the best upper and lower bounds. However, there are an infinite number of upper bounds greater than 85 and an infinite number of lower bounds less than -15.

Step 7: Use a graph to verify your analysis.

On the following graph, **A** represents the global minimum value of the function and **B** represents the global maximum value of the function on the interval $[-2,10]$. The green horizontal lines represent the best upper and lower bound of the function: $y_L = -15$ and $y_U = 85$. The purple horizontal lines also represent upper and lower bounds, but not the best. Note that if you sketched in horizontal lines between A and B, they would not be bounds of the function on this interval. Note: the graph has been extended beyond the domain of $[-2,10]$ for illustration purposes.



If you find a mistake or want to make a comment or suggest, contact Dr. Nancy Marcus at nancymarcus@utep.edu.

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