Lecture 12

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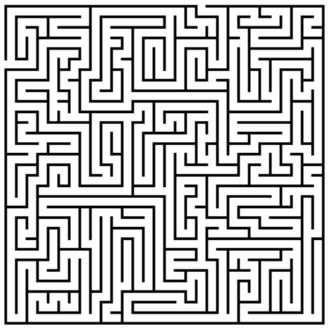
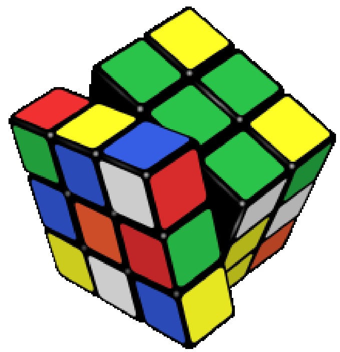
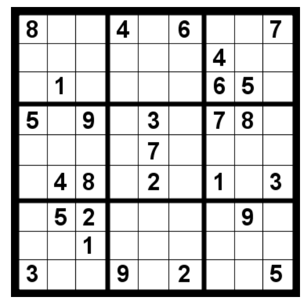
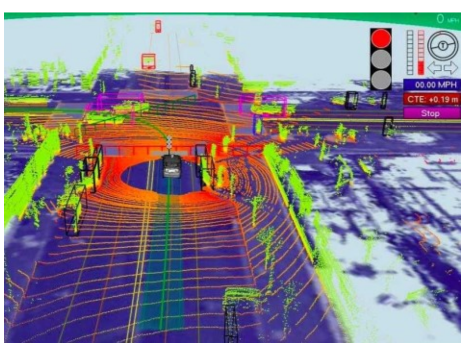
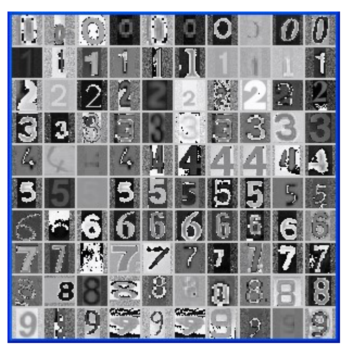
Where are we in CSE 365?

- Now leaving: sequential, deterministic reasoning

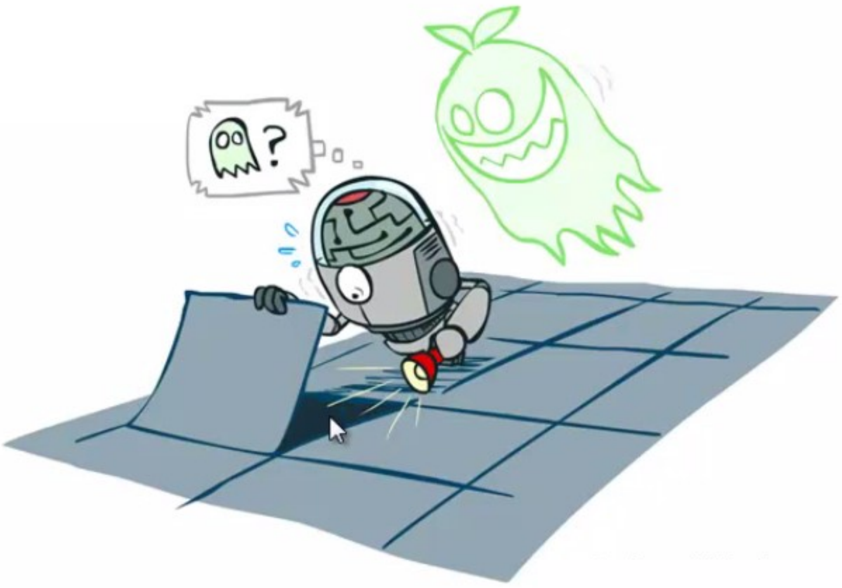





- Entering: probabilistic reasoning and machine learning

- We're done with Part I Search and Planning!

- Part II: Probabilistic Reasoning
  - Diagnosis
  - Speech recognition
  - Tracking objects
  - Robot mapping
  - Genetics
  - Error correcting codes
  - … lots more!



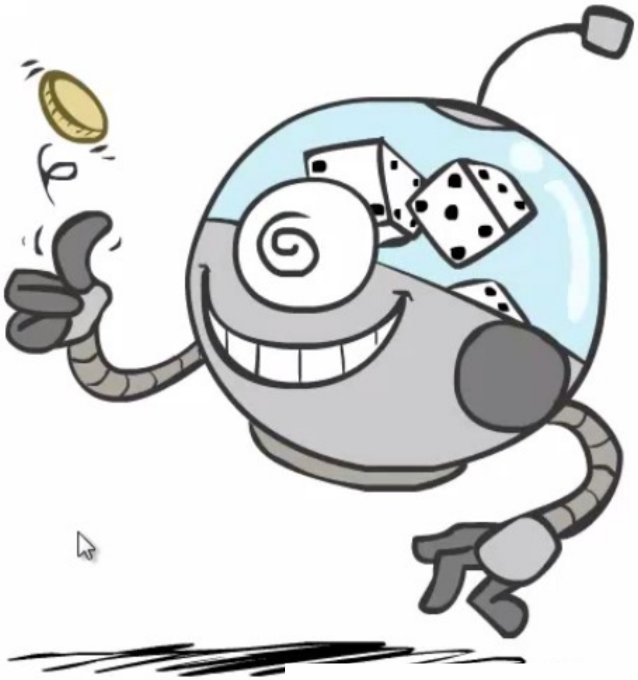
- Part III: Machine Learning

Probability: Review of main concepts (Chapter 13)

Today

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Chain Rule, Bayes' Rule
  - Inference
  - Independence

- You'll need all this stuff A LOT for the next few weeks, so make sure you go over it now!



Where do probabilities come from?

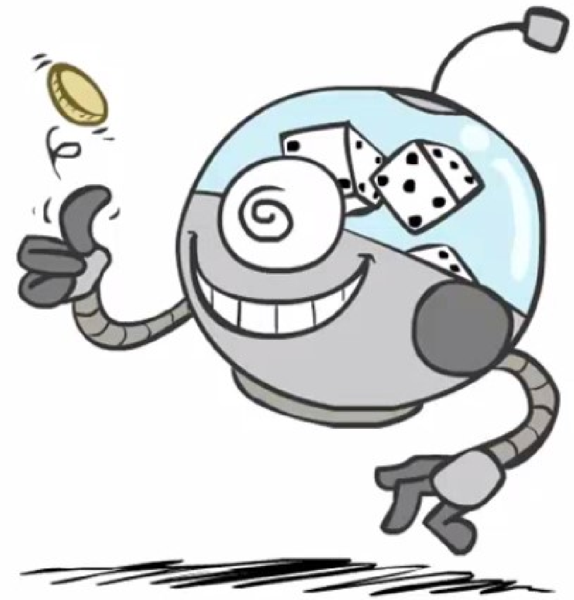
- **Frequentism**
  - Probabilities are relative frequencies
  - For example, if we toss a coin many times, P(heads) is the proportion of the time the coin will come up heads
  - But what if we're dealing with events that only happen once?
    - E.g., what is the probability that Team X will win the Superbowl this year?
    - "Reference class" problem
- **Subjectivism**
  - Probabilities are degrees of belief
  - But then, how do we assign belief values to statements?
  - What would constrain agents to hold consistent beliefs?

Probabilities and rationality

- Why should a rational agent hold beliefs that are consistent with axioms of probability?
  - For example, $P(A) + P(\neg A) = 1$
- If an agent has some degree of belief in proposition A, he/she should be able to decide whether or not to accept a bet for/against A (De Finetti, 1931):
  - If the agent believes that $P(A) = 0.4$, should he/she agree to bet \$4 that A will occur against \$6 that A will not occur?
- **Theorem:** An agent who holds beliefs inconsistent with axioms of probability can be convinced to accept a combination of bets that is guaranteed to lose them money

Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?
- We denote random variables with capital letters



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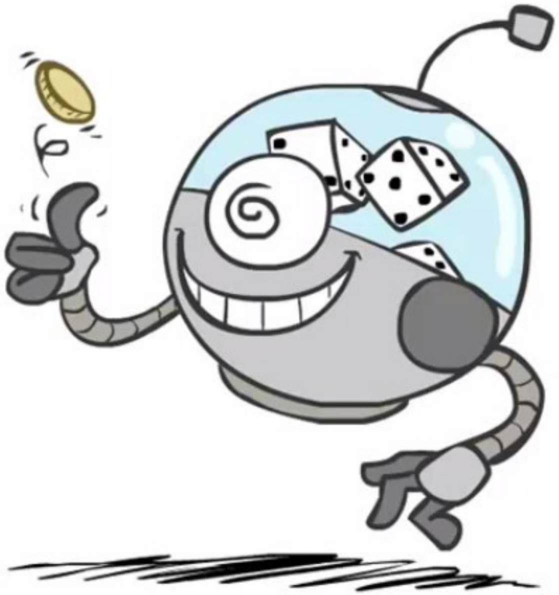
  - L = Where is the ghost?
- We denote random variables with capital letters
- Like variables in a CSP, random variables have domains
  - R in {true, false}   (often write as {+r, -r})
  - T in {hot, cold}
  - D in $[0, \infty)$
  - L in possible locations, maybe {(0,0), (0,1), …}



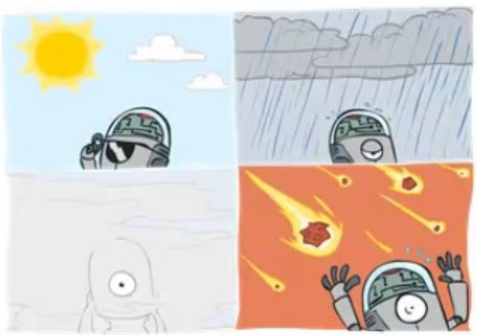
Random variables

- We describe the (uncertain) state of the world using ***random variables***
  - Denoted by capital letters
  - **R**: *Is it raining?*
  - **W**: *What's the weather?*
  - **D**: *What is the outcome of rolling two dice?*
  - **S**: *What is the speed of my car (in MPH)?*
- Just like variables in CSPs, random variables take on values in a *domain*
  - Domain values must be *mutually exclusive* and *exhaustive*
  - **R** in {True, False}
  - **W** in {Sunny, Cloudy, Rainy, Snow}
  - **D** in {(1,1), (1,2), … (6,6)}
  - **S** in [0, 200]

Probability Distributions

- Associate a probability with each value
  - Temperature:

  - Weather:



Probability Distributions

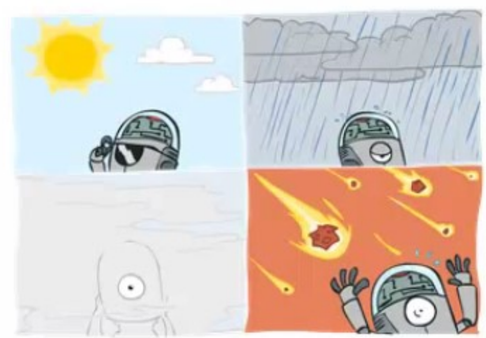
- Associate a probability with each value

  - Temperature:

$$P(T)$$

T	P
hot	0.5
cold	0.5

  - Weather:



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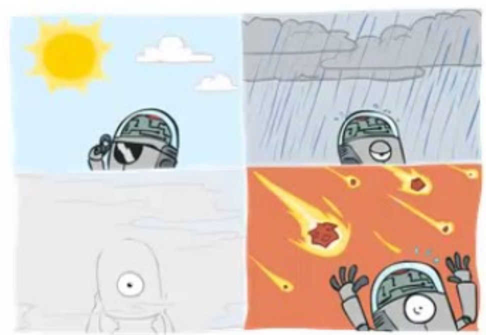

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

- A distribution is a TABLE of probabilities of values

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

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cold	0.5

$P(W)$

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- A distribution is a TABLE of probabilities of values

- A probability (lower case value) is a single number

$$P(W = rain) = 0.1$$

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
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$P(W)$

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- Must have: $\forall x \;\; P(X = x) \geq 0$ and $\sum_x P(X = x) = 1$

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

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hot	0.5
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$P(W)$

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$$P(W = rain) = 0.1$$

- Must have: $\forall x \;\; P(X = x) \geq 0$ and $\sum_x P(X = x) = 1$

Shorthand notation:

$$P(hot) = P(T = hot),$$
$$P(cold) = P(T = cold),$$
$$P(rain) = P(W = rain),$$
$$\ldots$$

OK *if* all domain entries are unique

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

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$$P(x_1, x_2, \ldots x_n)$$

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

  - Must obey: $$P(x_1, x_2, \ldots x_n) \geq 0$$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Joint probability distributions

- A ***joint distribution*** is an assignment of probabilities to every possible atomic event
- Suppose we have a joint distribution of *n* random variables with domain sizes *d*
  - What is the size of the probability table?
  - Impossible to write out completely for all but the smallest distributions

Events

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

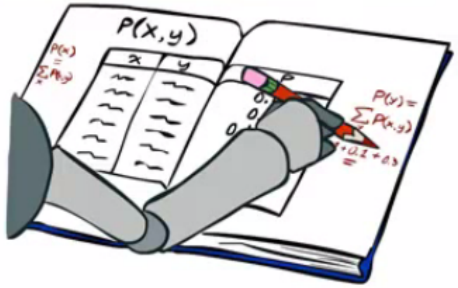
Quiz: Events

- P(+x, +y) ?
- P(+x) ?
- P(-y OR +x) ?

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions



- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$$P(T, W)$$

T	W	P
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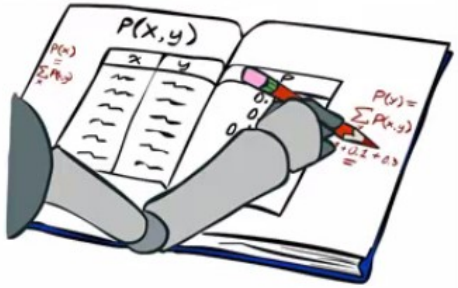

$P(T,W)$

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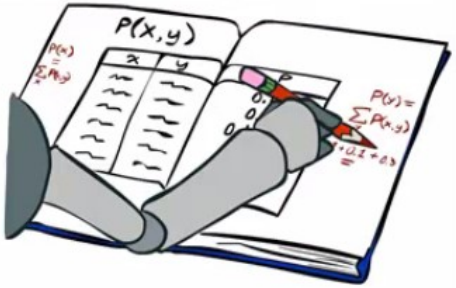
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_s P(t,s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

Marginal Distributions



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$$P(t) = \sum_s P(t, s)$$

$$P(s) = \sum_t P(t, s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.4

Quiz: Marginal Distributions

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$P(X)$

Quiz: Marginal Distributions

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x,y)$$

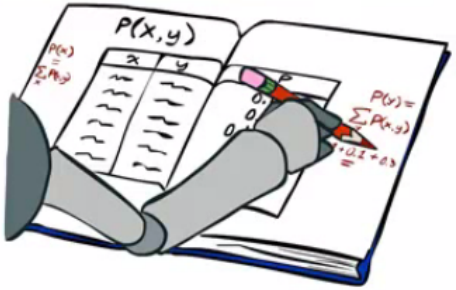
$$P(y) = \sum_x P(x,y)$$

$P(X)$

X	P
+x	0.5
-x	0.5

$P(Y)$

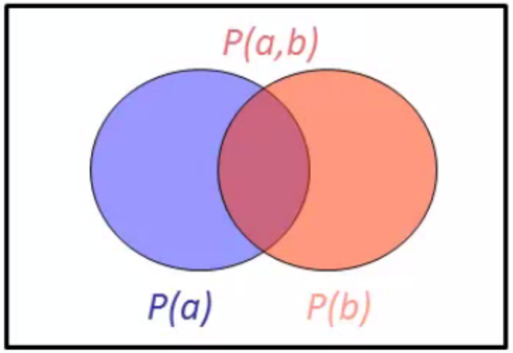
Y	P
+y	0.6
-y	0.4



Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



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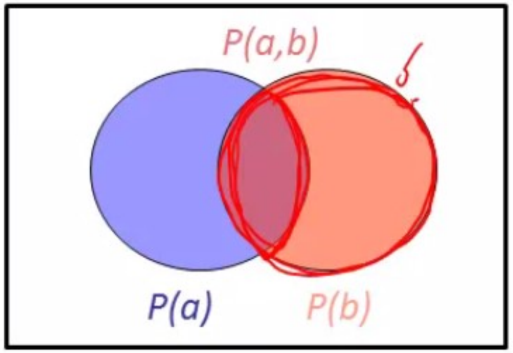

$$P(T, W)$$

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$$P(W = s | T = c) = ???$$

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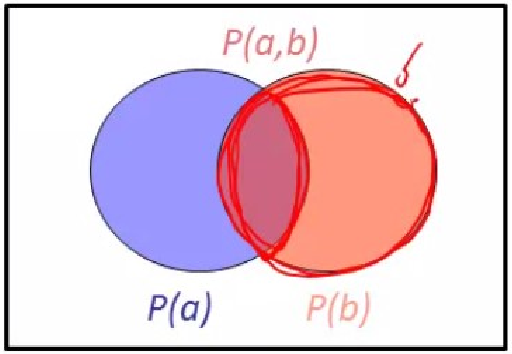

$$P(T,W)$$

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cold	rain	0.3

$$P(W=s|T=c) = ???$$

$$\frac{P(s,c)}{P(c)} = \frac{0.2}{0.5}$$

$= 0.4$

Quiz: Conditional Probabilities

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

Joint Distribution

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$$P(W|T = hot)$$

W	P
sun	0.8
rain	0.2

Joint Distribution

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$\longrightarrow$

$P(W|T = c)$

Normalization Trick

$$P(W = s|T = c) = \frac{P(W = s, T = c)}{P(T = c)}$$

$$P(T, W)$$

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$$\longrightarrow$$

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$$P(T, W)$$

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Normalization Trick

$$P(T, W)$$

T	W	P
hot	sun	0.4
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cold	rain	0.3

$$\begin{aligned} P(W = s|T = c) &= \frac{P(W = s, T = c)}{P(T = c)} \\ &= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.2}{0.2 + 0.3} = 0.4 \end{aligned}$$

$$\longrightarrow$$

$$P(W|T = c)$$

Normalization Trick

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$$P(W=r|T=c) = \frac{P(W=r, T=c)}{P(T=c)}$$

$$\longrightarrow \quad P(W|T=c)$$

Normalization Trick

$$P(T, W)$$

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$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$
$$= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.2}{0.2+0.3} = 0.4$$

$$\longrightarrow \quad P(W|T=c)$$

$$P(W=r|T=c) = \frac{P(W=r,T=c)}{P(T=c)}$$
$$= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.3}{0.2+0.3} = 0.6$$

Normalization Trick

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned} P(W=s|T=c) &= \frac{P(W=s,T=c)}{P(T=c)} \\ &= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.2}{0.2+0.3} = 0.4 \end{aligned}$$

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Normalization Trick

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$$P(W|T=c)$$

W	P
sun	0.4
rain	0.6

Normalization Trick

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$
$$= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.2}{0.2+0.3} = 0.4$$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence →

$P(c,W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one) →

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

$$P(W=r|T=c) = \frac{P(W=r,T=c)}{P(T=c)}$$
$$= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.3}{0.2+0.3} = 0.6$$

Quiz: Normalization Trick

- P(X | Y=-y) ?

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence

$\longrightarrow$

NORMALIZE the selection (make it sum to one)

$\longrightarrow$

Quiz: Normalization Trick

- P(X | Y=-y) ?

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence

P(X, -y)

X	P
+x	0.3
-x	0.1

0.4

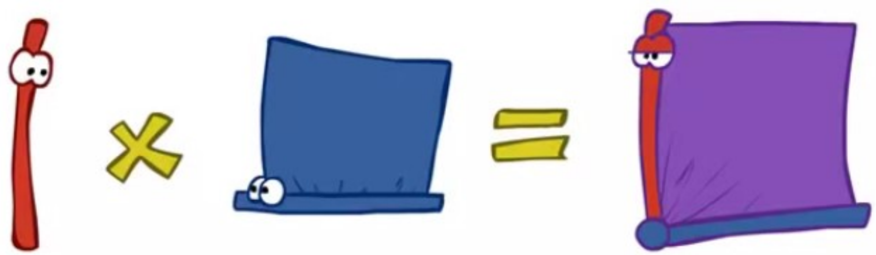
NORMALIZE the selection (make it sum to one)

X	P(X|-y)
+x	.3/.4
-x	.1/.4

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x,y) \quad \Longleftrightarrow \quad P(x|y) = \frac{P(x,y)}{P(y)}$$



The Product Rule

$$P(y)P(x|y) = P(x,y)$$

- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$P(D,W)$

The Product Rule

$$P(y)P(x|y) = P(x,y)$$

- Example:

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dry	rain	0.3

$P(D,W)$

D	W	P
wet	sun	0.08
dry	sun	
wet	rain	
dry	rain	

P(sun) P(wet|sun)
0.8 * 0.1

The Product Rule

$$P(y)P(x|y) = P(x,y)$$

- Example:

b

b

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$P(D,W)$

D	W	P
wet	sun	0.08
dry	sun	0.72
wet	rain	0.14
dry	rain	0.06

P(sun) P(wet|sun)

0.8 * 0.1

The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

- Why is this always true?

The Chain Rule

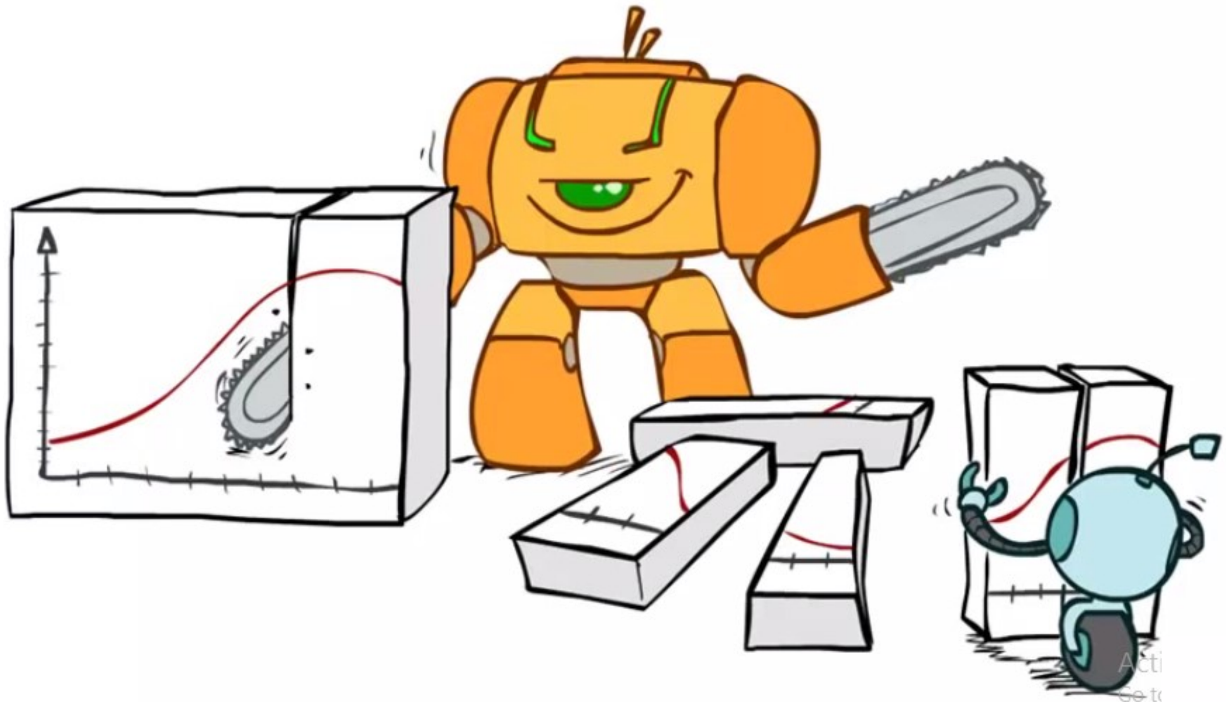
- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2) \leftarrow \cancel{P(x_1)} \cdot \frac{\cancel{P(x_2, x_1)}}{\cancel{P(x_1)}} \frac{P(x_3, x_2, x_1)}{\cancel{P(x_2, x_1)}}$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

- Why is this always true?

Bayes Rule



Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

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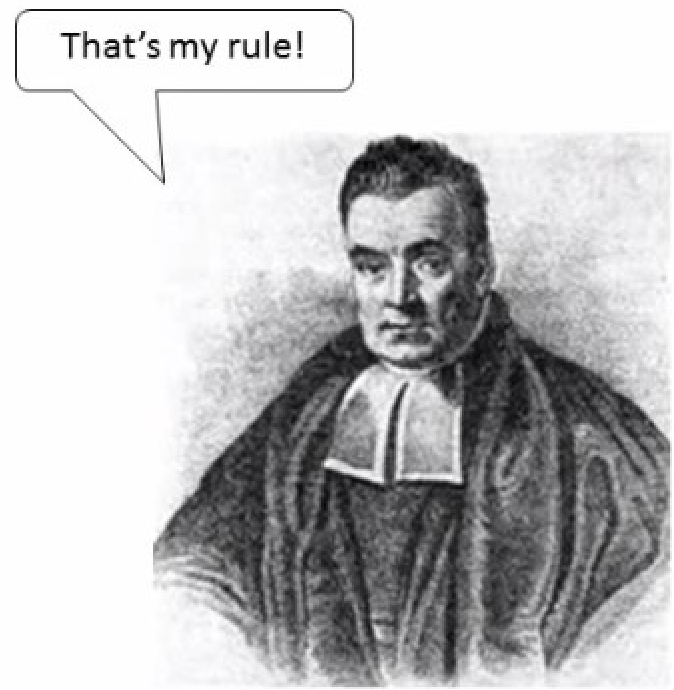

Quiz: Bayes' Rule

- Given:

$$P(W)$$

R	P
sun	0.8
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$$P(D|W)$$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is P(W | dry) ?

Quiz: Bayes' Rule

- Given:

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$P(D|W)$

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wet	sun	0.1
dry	sun	0.9
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- What is P(W | dry) ?

W	P(W|dry)
sun	
rain	

Quiz: Bayes' Rule

- Given:

$P(W)$

R	P
sun	0.8
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$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is P(W | dry) ?

$$P(sun|dry) = \frac{P(dry|sun)\,P(sun)}{P(dry)} \leftarrow$$

$$= \frac{P(dry|sun)\,P(sun)}{P(dry|sun)\,P(sun) + P(dry|rain)\,P(rain)}$$

W	P(W|dry)
sun	
rain	

Independence

- Two variables are *independent* in a joint distribution if:

$$P(X,Y) = P(X)P(Y)$$

$$\forall x, y \, P(x,y) = P(x)P(y)$$

Example: Independence?

$$P_1(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Example: Independence?

$$P_1(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(T)$$

T	P
hot	0.5
cold	0.5

$$P(W)$$

W	P
sun	0.6
rain	0.4

Example: Independence?

$P(T)$

T	P
hot	0.5
cold	0.5

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

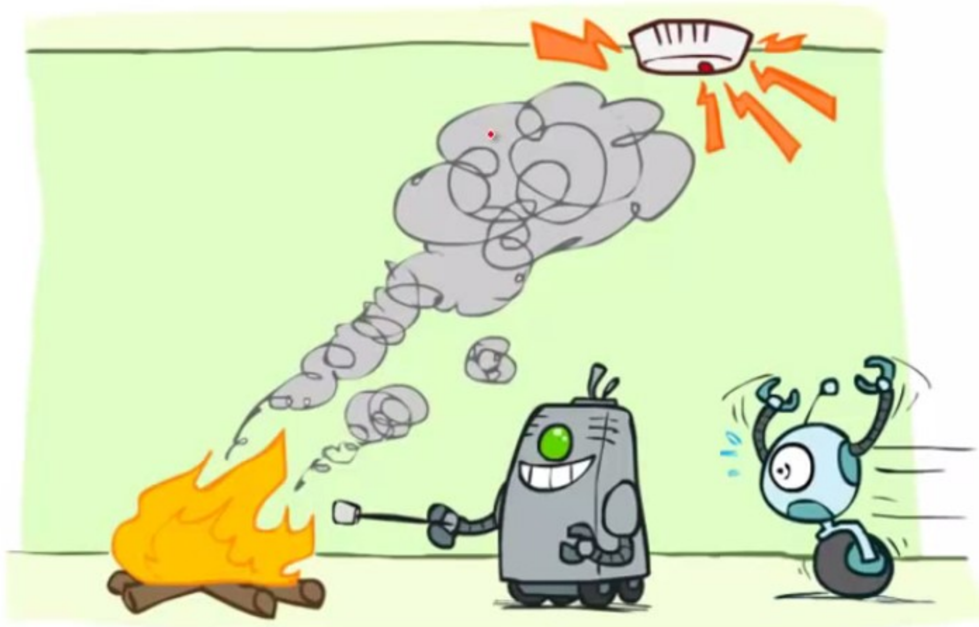
$P_2(T, W) = P(T)P(W)$

T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$P(W)$

W	P
sun	0.6
rain	0.4

Conditional Independence



Conditional Independence

- P(Toothache, Cavity, Catch)

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  - P(+catch | +toothache, +cavity) = P(+catch | +cavity)

Conditional Independence

- Unconditional (absolute) independence very rare (why?)
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- X is conditionally independent of Y given Z $X \perp\!\!\!\perp Y | Z$

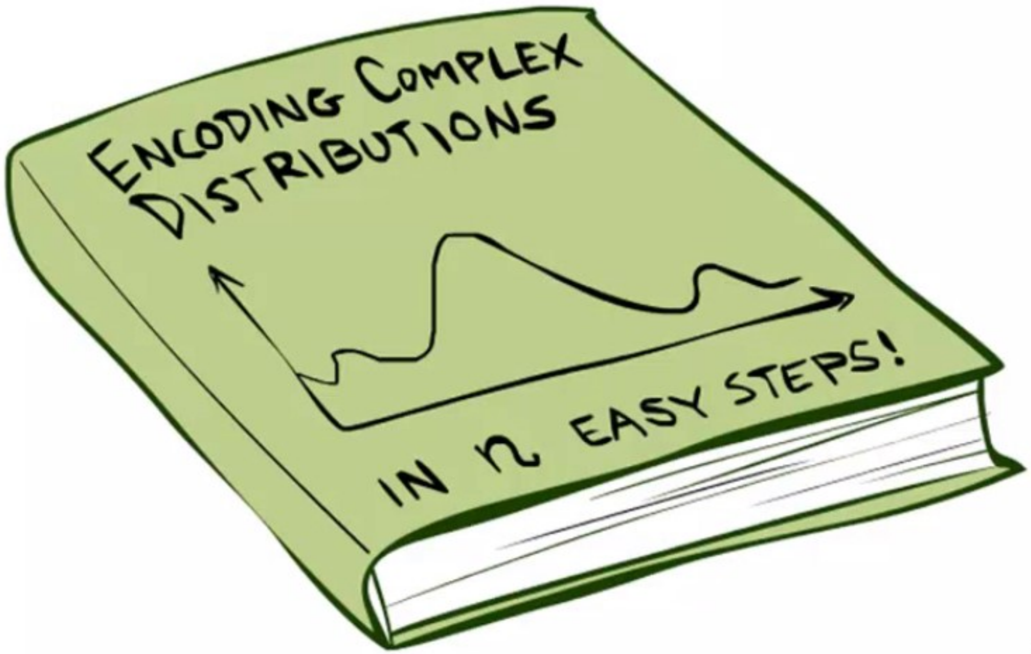
  if and only if:

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

  or, equivalently, if and only if

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

Bayes' Nets: Big Picture



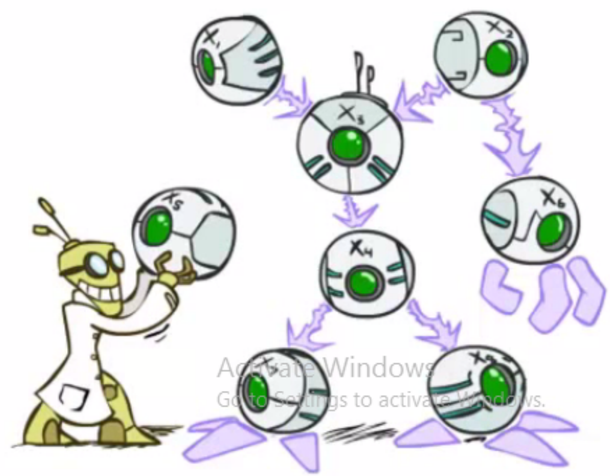
Bayes' Nets: Big Picture

- Two problems with using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time

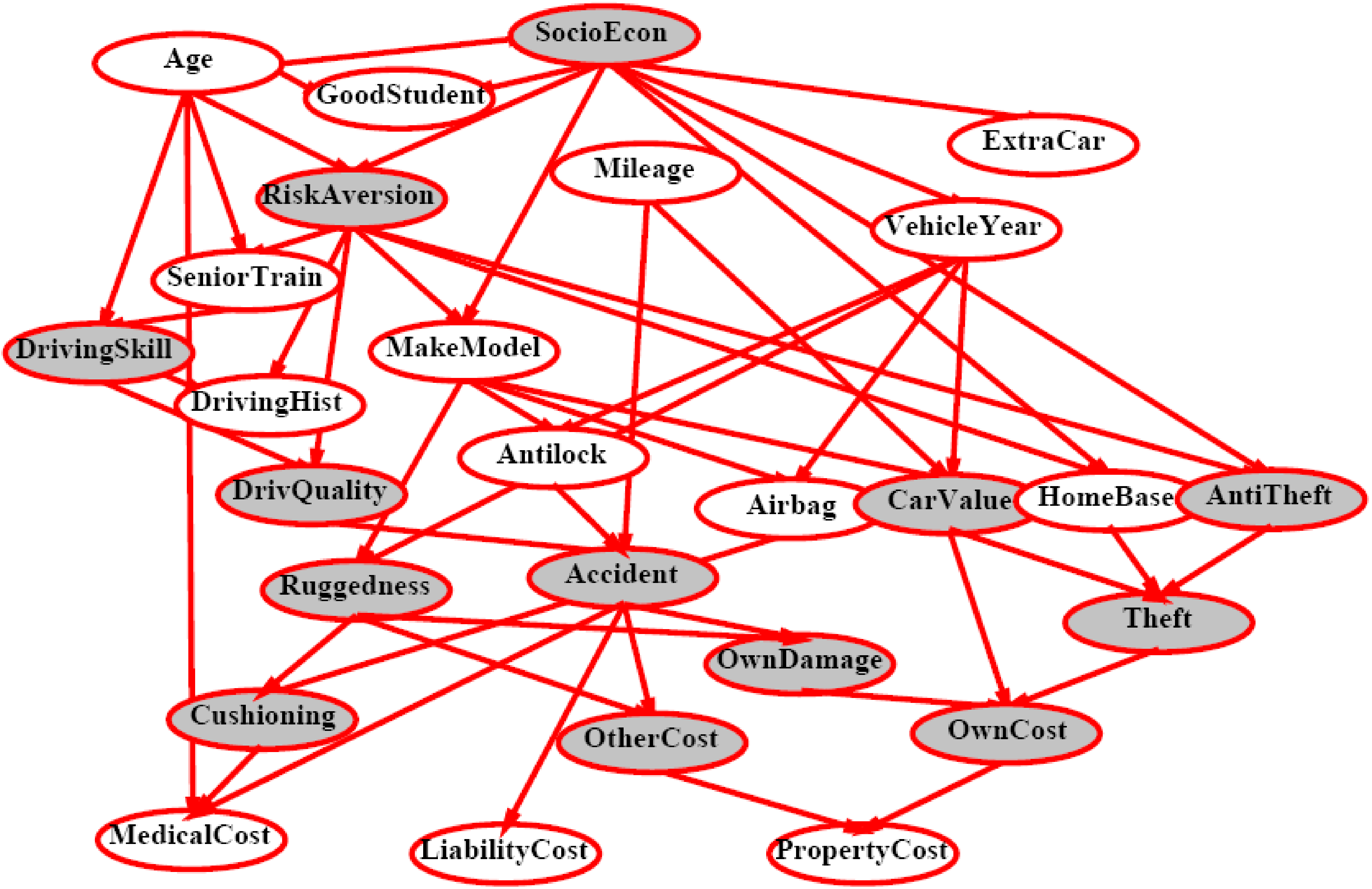
Bayes' Nets: Big Picture

- Two problems with using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time

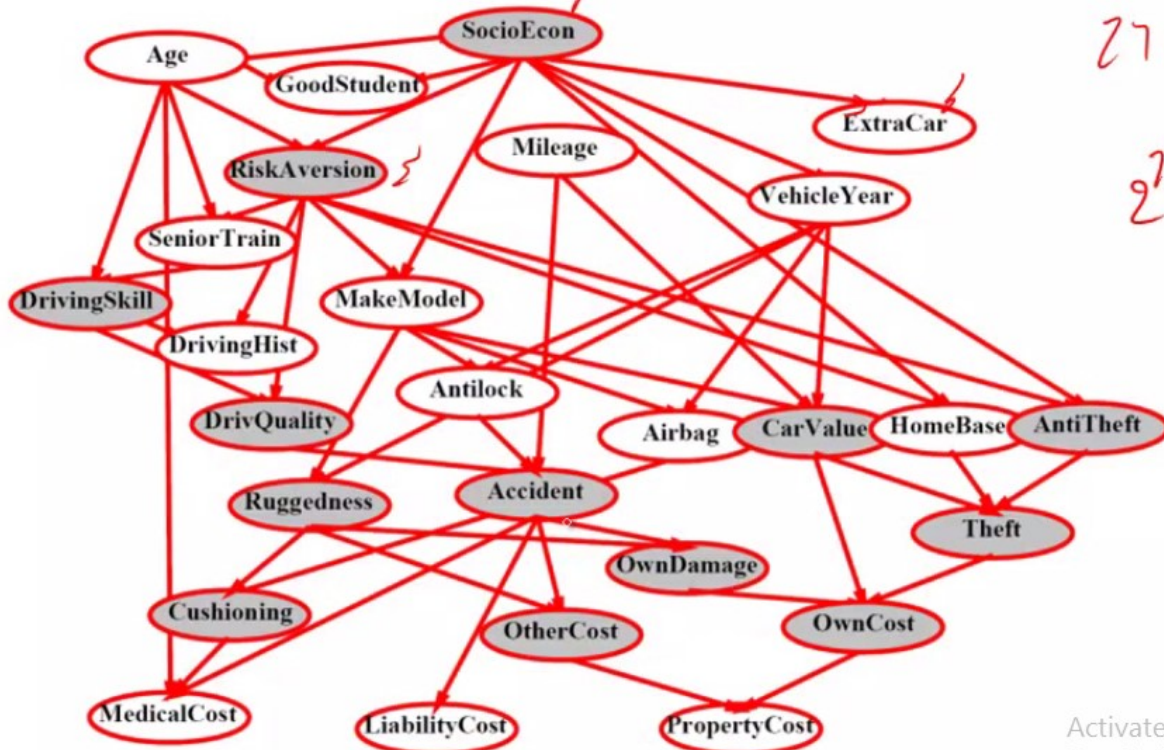
- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - More properly called graphical models
  - We describe how variables locally interact
  - Local interactions chain together to give global, indirect interactions
  - For about 10 min, we'll be vague about how these interactions are specified



Car insurance



Example Bayes' Net: Insurance

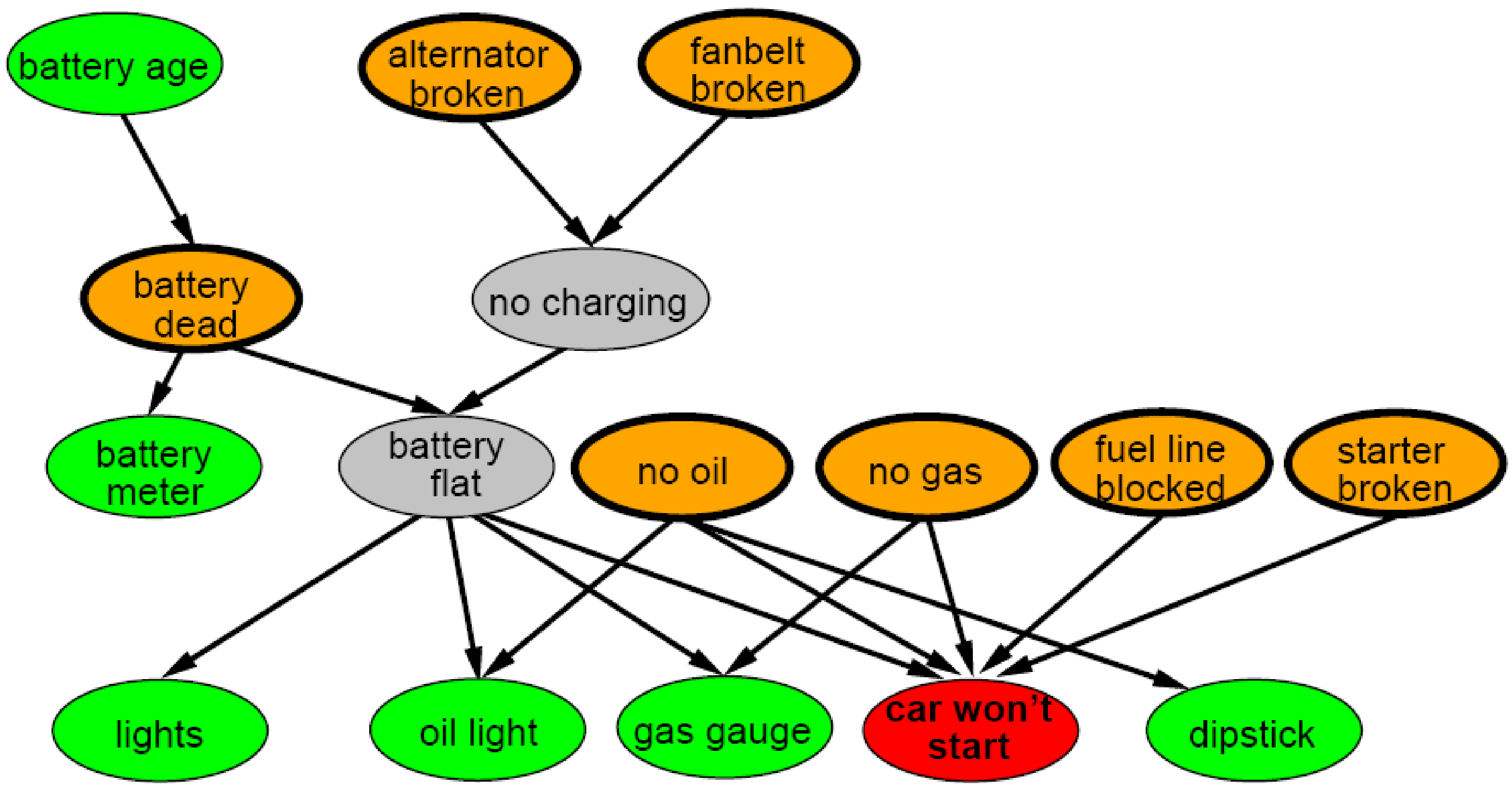


27 vars

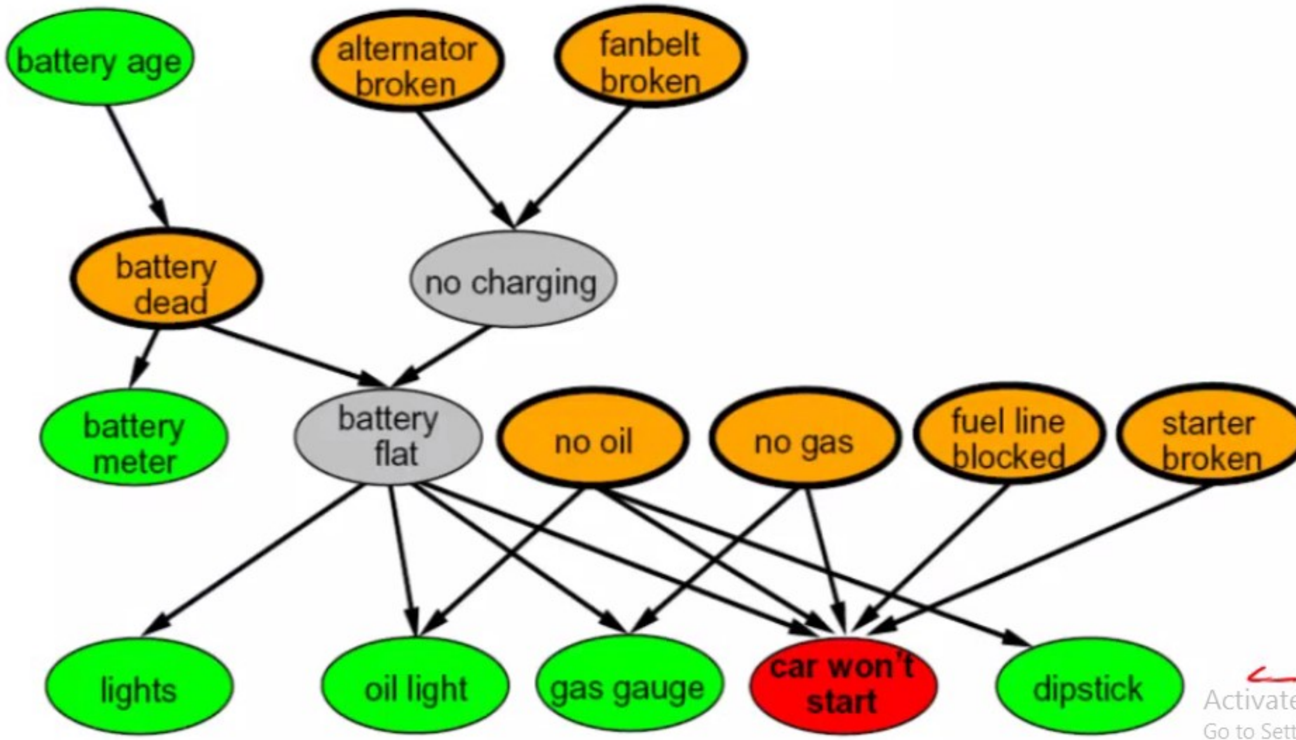
$2^{27} \approx 134M$

A more realistic Bayes Network: Car diagnosis

- Initial observation: car won't start
- Orange: "broken, so fix it" nodes
- Green: testable evidence
- Gray: "hidden variables" to ensure sparse structure, reduce parameters



Example Bayes' Net: Car

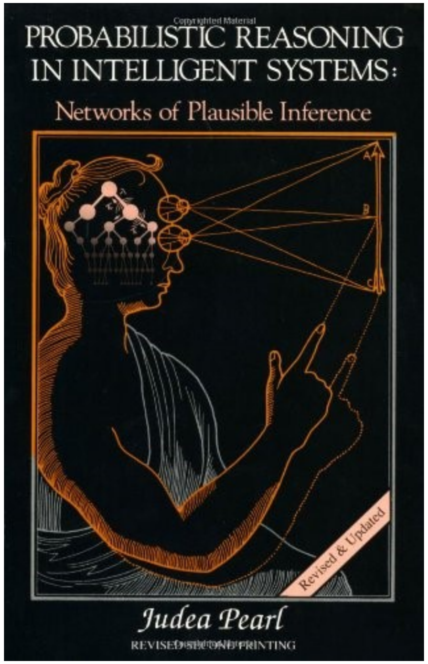


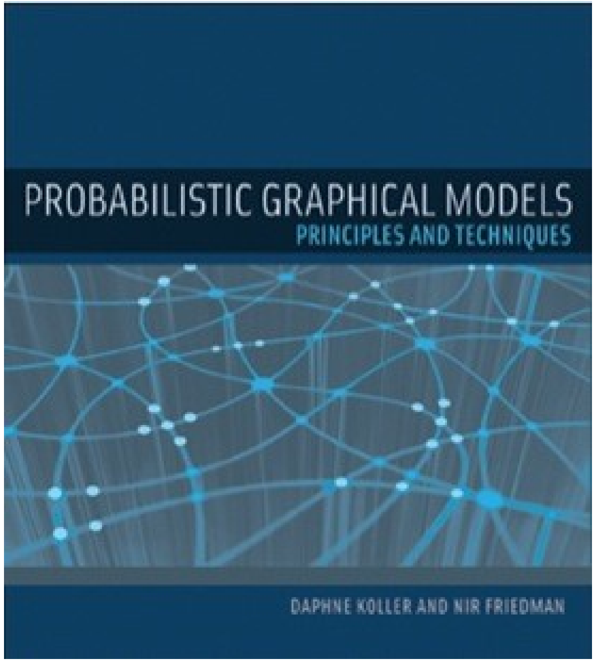
16 vars

$2^{16} \simeq 65$!

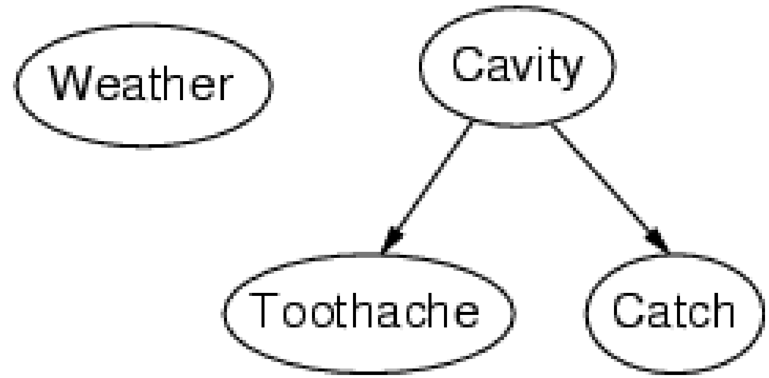
Bayesian networks

- More commonly called *graphical models*
- A way to depict conditional independence relationships between random variables
- A compact specification of full joint distributions





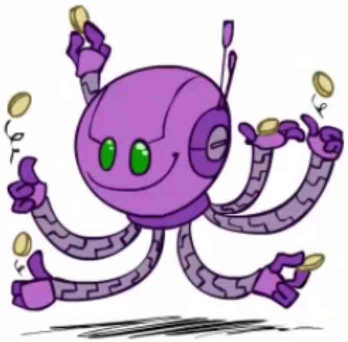
Bayesian networks: Structure



- **Nodes:** random variables
- **Arcs:** interactions
  - An arrow from one variable to another indicates direct influence
  - Must form a directed, *acyclic* graph

Example: Coin Flips

- N independent coin flips



X_1 $\quad$ X_2 $\quad \cdots \quad$ X_n

- No interactions between variables: absolute independence

Example: Traffic

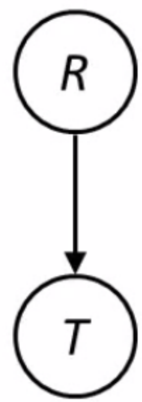
- Variables:
  - R: It rains
  - T: There is traffic

Example: Traffic

- Variables:
  - R: It rains
  - T: There is traffic

- Model 1: independence

- Model 2: rain causes traffic



- Why is an agent using model 2 better?

Example: Traffic II

- Let's build a causal graphical model!

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- Let's build a causal graphical model!
- Variables
  - T: Traffic
  - R: It rains
  - L: Low pressure
  - D: Roof drips
  - B: Ballgame
  - C: Cavity

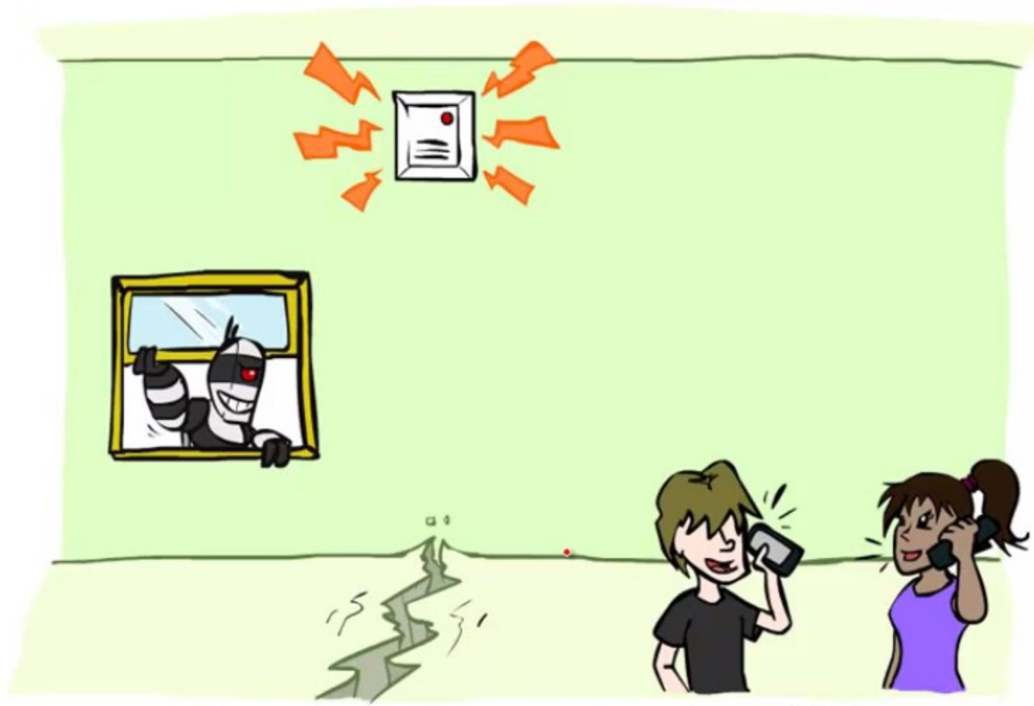
Example: Traffic II

- Let's build a causal graphical model!
- Variables
  - T: Traffic
  - R: It rains
  - L: Low pressure
  - D: Roof drips
  - B: Ballgame
  - C: Cavity

Example: Burglar Alarm

- I have a burglar alarm that is sometimes set off by minor earthquakes. My two neighbors, John and Mary, promised to call me at work if they hear the alarm
  - Example inference task: suppose Mary calls and John doesn't call. What is the probability of a burglary?
- What are the random variables?
  - *Burglary*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*
- What are the direct influence relationships?
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call

Example: Alarm Network



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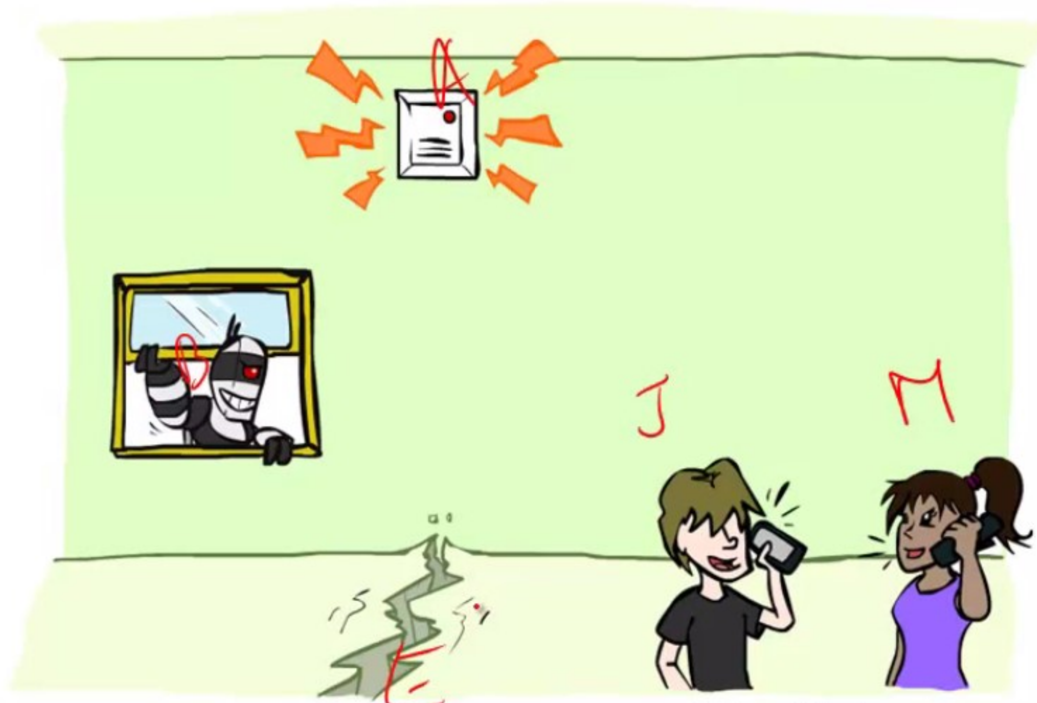
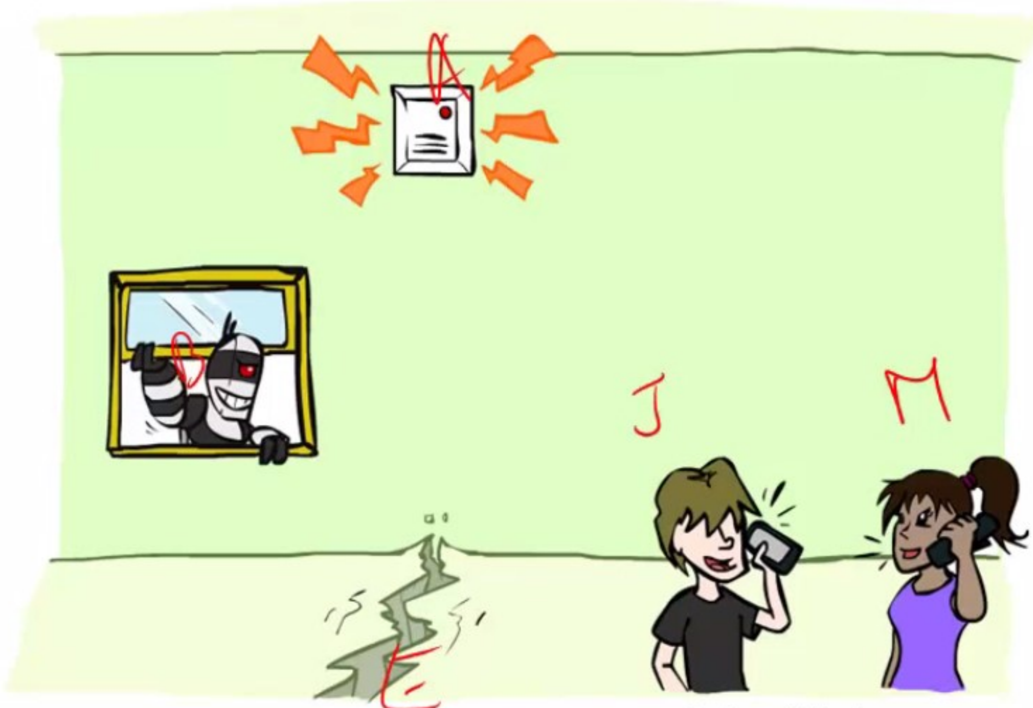


Example: Alarm Network

- Variables
  - B: Burglary
  - A: Alarm goes off
  - M: Mary calls
  - J: John calls
  - E: Earthquake!



Example: Alarm Network

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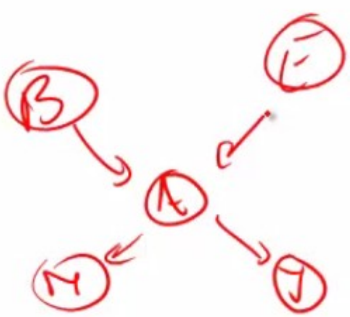
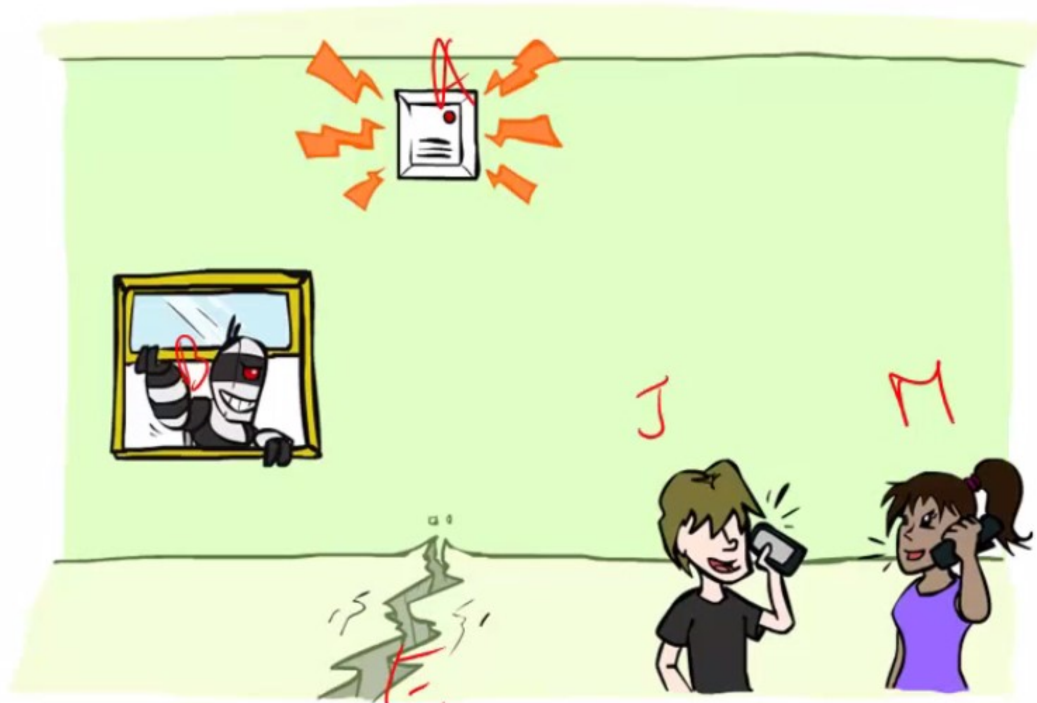


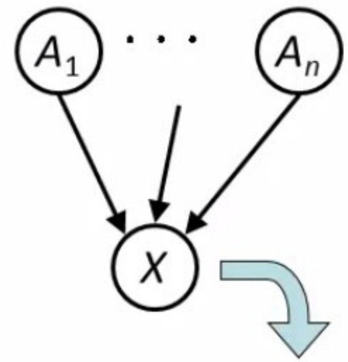


Bayes' Net Semantics

- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents' values

    $$P(X|a_1 \ldots a_n)$$

  - CPT: conditional probability table
  - Description of a noisy "causal" process



$$P(X|A_1 \ldots A_n)$$

A Bayes net = Topology (graph) + Local Conditional Probabilities

Probabilities in BNs

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

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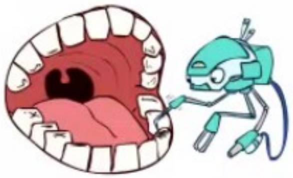
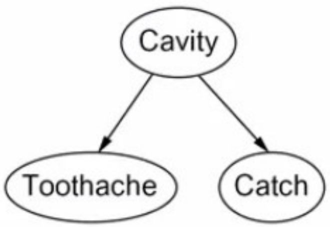
CLAIM

  - Example:





$P(+cavity,\ +catch,\ -toothache)$

Probabilities in BNs

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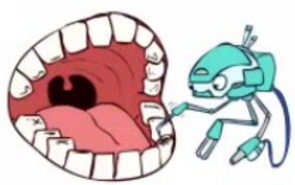
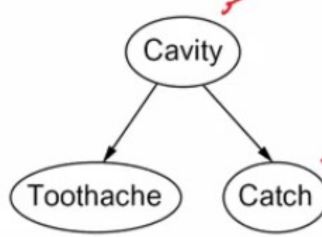
$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i)) \quad \leftarrow \quad \text{CLAIM}$$

  - Example:

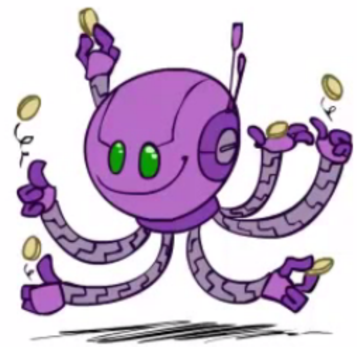




$$P(+cavity,\ +catch,\ -toothache) = P(+cavity) \cdot P(+catch \mid +cavity) \cdot P(-toothache \mid +cavity)$$

Example: Coin Flips

X_1 $\quad$ X_2 $\quad \cdots \quad$ X_n



$P(X_1)$

h	0.5
t	0.5

$P(X_2)$

h	0.5
t	0.5

$\cdots$

$P(X_n)$

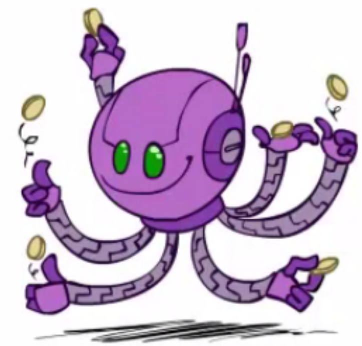
h	0.5
t	0.5

$$P(h, h, t, h) =$$

Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.

Example: Coin Flips

X_1 $\quad$ X_2 $\quad \cdots \quad$ X_n



$P(X_1)$

h	0.5
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t	0.5

$\cdots$

$P(X_n)$

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t	0.5

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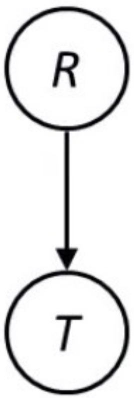
Example: Traffic



$P(R)$

+r	1/4
-r	3/4

$P(T|R)$

+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

$$P(+r, -t) =$$

Example: Traffic

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$P(T|R)$

+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

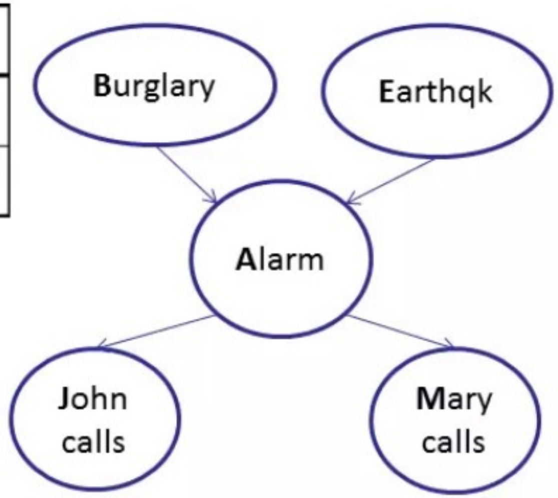
$$P(+r, -t) = P(+r)\ P(-t|+r)$$

$$= \frac{1}{4} \cdot \frac{1}{4}$$

Example: Alarm Network

Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998

A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

Example: Alarm Network

B	P(B)
+b	0.001
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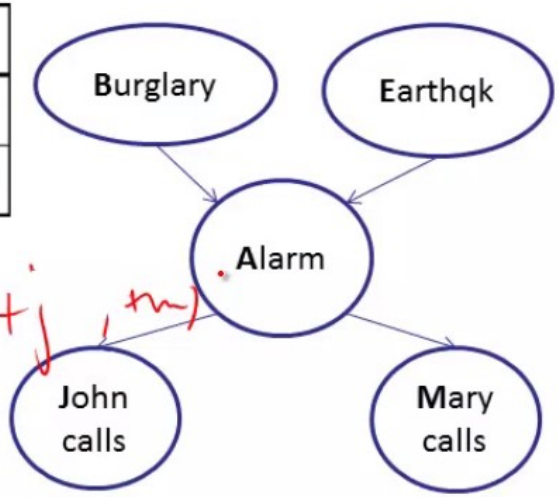

$P(+b, -e, +a, +j, +m)$.

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B	E	A	P(A|B,E)
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+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

Example: Alarm Network

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+b	0.001
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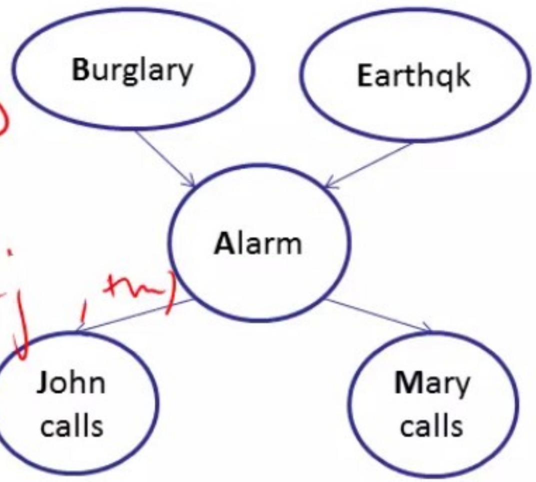

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+e	0.002
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$P(+b, -e, +a, +j, +m)$

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+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

Summary

- Bayesian networks provide a natural representation for (causally induced) conditional independence
- Topology + conditional probability tables
- Generally easy for domain experts to construct

Probability Recap

- Conditional probability $$P(x|y) = \frac{P(x,y)}{P(y)}$$
- Product rule $$P(x,y) = P(x|y)P(y)$$
- Chain rule
$$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots \\ &= \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1}) \end{aligned}$$
- X, Y independent if and only if: $\forall x, y : P(x,y) = P(x)P(y)$
- X and Y are conditionally independent given Z if and only if: $X \perp\!\!\!\perp Y | Z$
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

Thank You