MATH 101 TESTING

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Math 101 Testing

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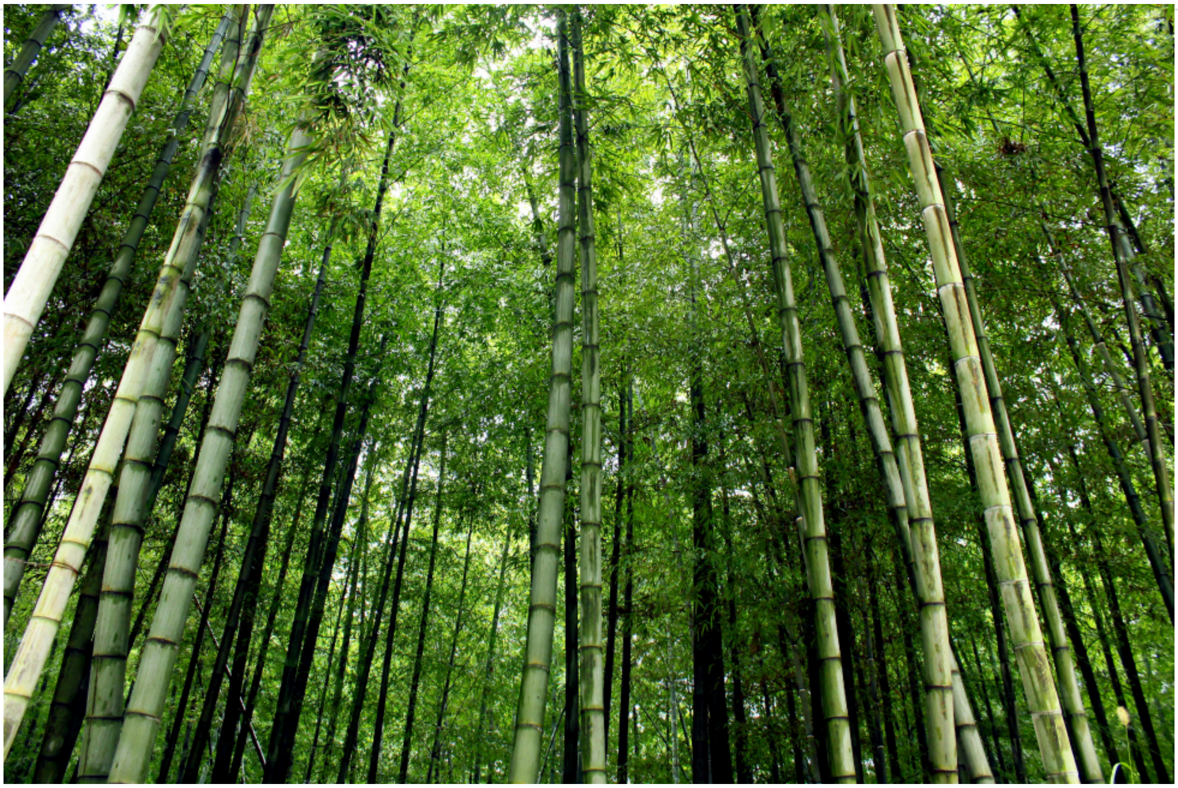
CHAPTER OVERVIEW

1: Linear Functions

Recall that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

1.1: Introduction to Linear Functions



A bamboo forest in China (credit: “JFXie”/Flickr)

Chapter Outline

Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain types of bamboo species. These members of the grass family are the fastest-growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour.[1] In a twenty-four hour period, this bamboo plant grows about 36 inches, or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function.

Recall from Functions and Function Notation that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

Footnotes

- 1http://www.guinnessworldrecords.com/...growing-plant/

1.2: The Rectangular Coordinate Systems and Graphs

Learning Objectives

- Plot ordered pairs in a Cartesian coordinate system.
- Graph equations by plotting points.
- Graph equations with a graphing utility.
- Find x-intercepts and y-intercepts.
- Use the distance formula.
- Use the midpoint formula.

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in Figure $1.2.1$. Laying a rectangular coordinate grid over the map, we can see that each stop aligns with an intersection of grid lines. In this section, we will learn how to use grid lines to describe locations and changes in locations.

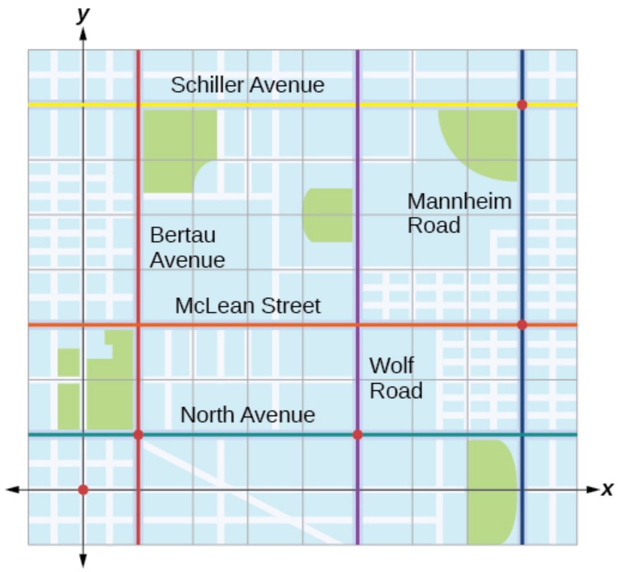


Figure 1.2.1

Plotting Ordered Pairs in the Cartesian Coordinate System

An old story describes how seventeenth-century philosopher/mathematician René Descartes invented the system that has become the foundation of algebra while sick in bed. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly's location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers—the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes' grid system existed centuries earlier, it was Descartes who introduced the components that comprise the Cartesian coordinate system, a grid system having perpendicular axes. Descartes named the horizontal axis the x-axis and the vertical axis the y-axis.

The Cartesian coordinate system, also called the *rectangular coordinate system*, is based on a two-dimensional plane consisting of the x-axis and the y-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a quadrant; the quadrants are numbered counterclockwise as shown in Figure $1.2.2$.

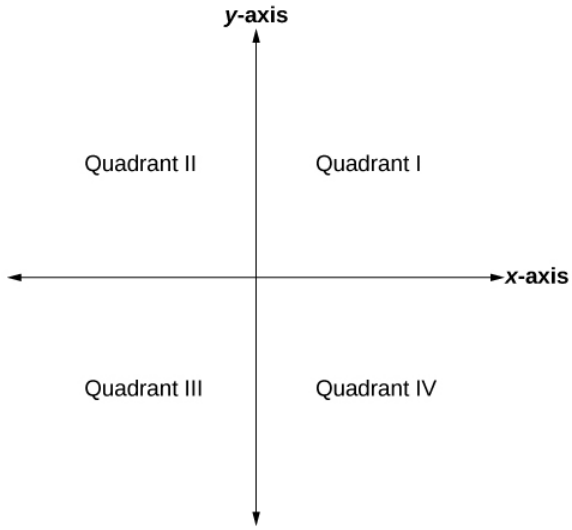


Figure 1.2.2

The center of the plane is the point at which the two axes cross. It is known as the origin, or point $(0, 0)$. From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the x-axis and up the y-axis; decreasing, negative numbers to the left on the x-axis and down the y-axis. The axes extend to positive and negative infinity as shown by the arrowheads in Figure $1.2.3$.

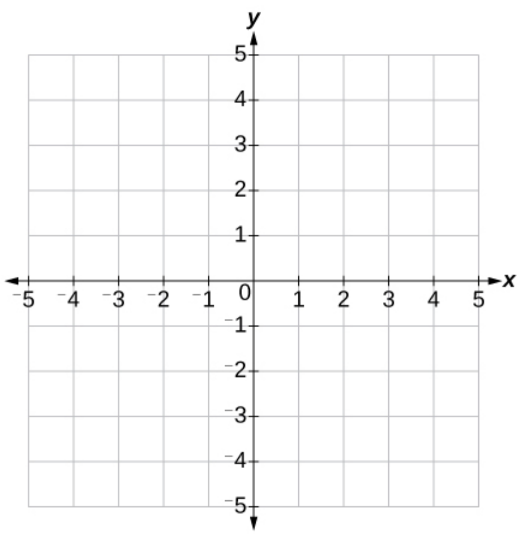


Figure 1.2.3

Each point in the plane is identified by its x-coordinate, or horizontal displacement from the origin, and its y-coordinate, or vertical displacement from the origin. Together, we write them as an ordered pair indicating the combined distance from the origin in the form (x, y). An ordered pair is also known as a coordinate pair because it consists of x- and y-coordinates. For example, we can represent the point $(3, -1)$ in the plane by moving three units to the right of the origin in the horizontal direction, and one unit down in the vertical direction. See Figure $1.2.4$.

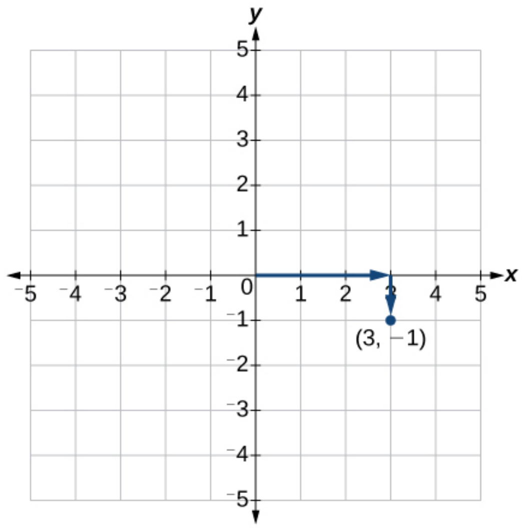


Figure 1.2.4

When dividing the axes into equally spaced increments, note that the x-axis may be considered separately from the y-axis. In other words, while the x-axis may be divided and labeled according to consecutive integers, the y-axis may be divided and labeled by increments of 2, or 10, or 100. In fact, the axes may represent other units, such as years against the balance in a savings account, or quantity against cost, and so on. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

Cartesian Coordinate System

A two-dimensional plane where the

- x-axis is the horizontal axis
- y-axis is the vertical axis

A point in the plane is defined as an ordered pair, (x, y), such that x is determined by its horizontal distance from the origin and y is determined by its vertical distance from the origin.

Example 1.2.1: Plotting Points in a Rectangular Coordinate System

Plot the points $(-2, 4)$, $(3, 3)$, and $(0, -3)$ in the plane.

Solution

To plot the point $(-2, 4)$, begin at the origin. The x-coordinate is -2, so move two units to the left. The y-coordinate is 4, so then move four units up in the positive y direction.

To plot the point $(3, 3)$, begin again at the origin. The x-coordinate is 3, so move three units to the right. The y-coordinate is also 3, so move three units up in the positive y direction.

To plot the point $(0, -3)$, begin again at the origin. The x-coordinate is 0. This tells us not to move in either direction along the x-axis. The y-coordinate is -3, so move three units down in the negative y direction. See the graph in Figure $1.2.5$.

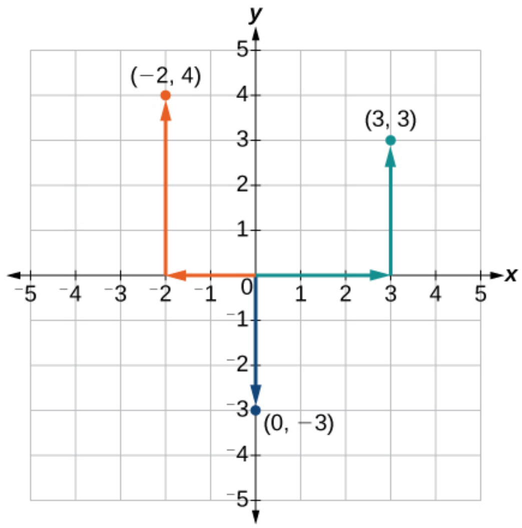


Figure 1.2.5

Analysis

Note that when either coordinate is zero, the point must be on an axis. If the x-coordinate is zero, the point is on the y-axis. If the y-coordinate is zero, the point is on the x-axis.

Graphing Equations by Plotting Points

We can plot a set of points to represent an equation. When such an equation contains both an x variable and a y variable, it is called an **equation in two variables**. Its graph is called a **graph in two variables**. Any graph on a two-dimensional plane is a graph in two variables.

Suppose we want to graph the equation $y = 2x - 1$. We can begin by substituting a value for x into the equation and determining the resulting value of y. Each pair of x- and y-values is an ordered pair that can be plotted. Table $1.2.1$ lists values of x from -3 to 3 and the resulting values for y.

Table 1.2.1

x	$y=2x-1$	(x,y)
-3	$y=2(-3)-1=-7$	$(-3,-7)$
-2	$y=2(-2)-1=-5$	$(-2,-5)$
-1	$y=2(-1)-1=-3$	$(-1,-3)$
0	$y=2(0)-1=-1$	$(0,-1)$
1	$y=2(1)-1=1$	$(1,1)$
2	$y=2(2)-1=3$	$(2,3)$
3	$y=2(3)-1=5$	$(3,5)$

We can plot the points in the table. The points for this particular equation form a line, so we can connect them (Figure $1.2.6$). This is not true for all equations.

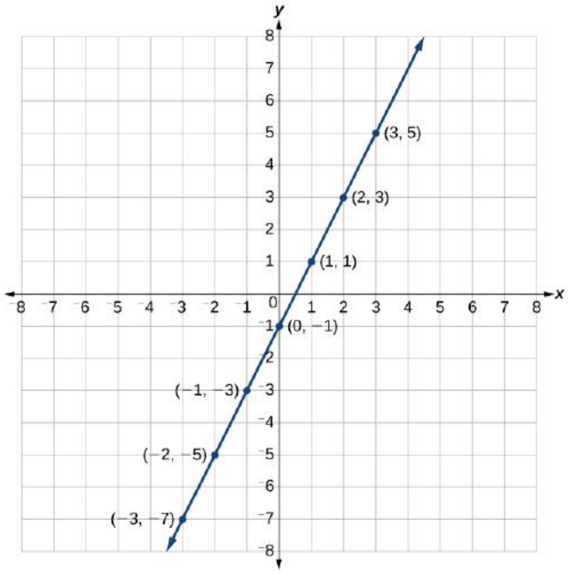


Figure 1.2.6

Note that the x-values chosen are arbitrary, regardless of the type of equation we are graphing. Of course, some situations may require particular values of x to be plotted in order to see a particular result. Otherwise, it is logical to choose values that can be calculated easily, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two to graph a line. Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

Howto: Given an equation, graph by plotting points

1. Make a table with one column labeled x, a second column labeled with the equation, and a third column listing the resulting ordered pairs.
2. Enter x-values down the first column using positive and negative values. Selecting the x-values in numerical order will make the graphing simpler.
3. Select x-values that will yield y-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

Example 1.2.2: Graphing an Equation in Two Variables by Plotting Points

Graph the equation $y=-x+2$ by plotting points.

Solution

First, we construct a table similar to Table $1.2.2$. Choose x values and calculate y.

Table 1.2.2

x	$y=-x+2$	(x,y)
-5	$y=-(-5)+2=7$	$(-5,7)$
-3	$y=-(-3)+2=5$	$(-3,5)$
-1	$y=-(-1)+2=3$	$(-1,3)$
0	$y=-(0)+2=2$	$(0,2)$
1	$y=-(1)+2=1$	$(1,1)$
3	$y=-(3)+2=-1$	$(3,-1)$
5	$y=-(5)+2=-3$	$(5,-3)$

Now, plot the points. Connect them if they form a line. See Figure 1.2.7.

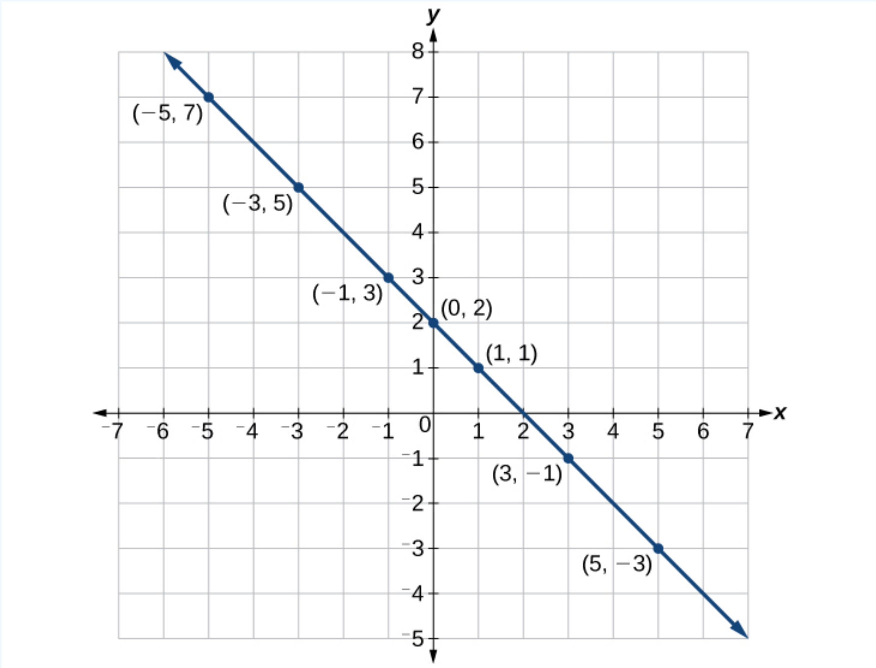


Figure 1.2.7

? Exercise 1.2.1

Construct a table and graph the equation by plotting points: $y = \frac{1}{2}x + 2$.

Answer

Please see Table 1.2.3and graph below.

Table 1.2.3

x	$y = 12x + 2$	(x, y)
-2	$y = 12(-2) + 2 = 1$	$(-2, 1)$
-1	$y = 12(-1) + 2 = 32$	$(-1, 32)$
0	$y = 12(0) + 2 = 2$	$(0, 2)$
1	$y = 12(1) + 2 = 52$	$(1, 52)$
2	$y = 12(2) + 2 = 3$	$(2, 3)$

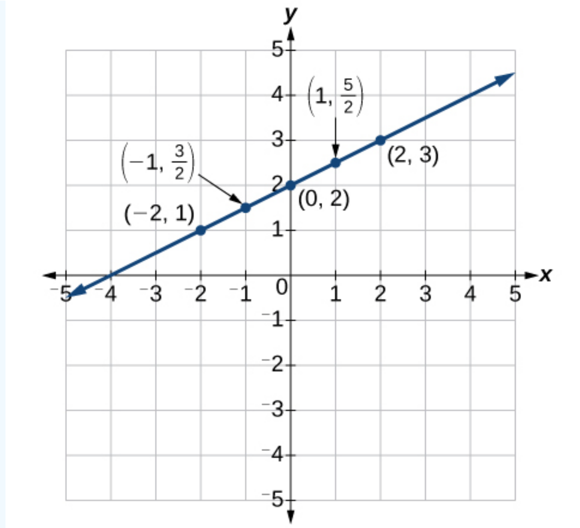


Figure 1.2.8

Graphing Equations with a Graphing Utility

Most graphing calculators require similar techniques to graph an equation. The equations sometimes have to be manipulated so they are written in the style $y =$_____ . The TI-84 Plus, and many other calculator makes and models, have a mode function, which allows the window (the screen for viewing the graph) to be altered so the pertinent parts of a graph can be seen.

For example, the equation $y = 2x - 20$ has been entered in the TI-84 Plus shown in Figure $1.2.9a$. In Figure $1.2.9b$, the resulting graph is shown. Notice that we cannot see on the screen where the graph crosses the axes. The standard window screen on the TI-84 Plus shows $-10 \leq x \leq 10$, and $-10 \leq y \leq 10$. See Figure \(\PageIndex{9c}\).

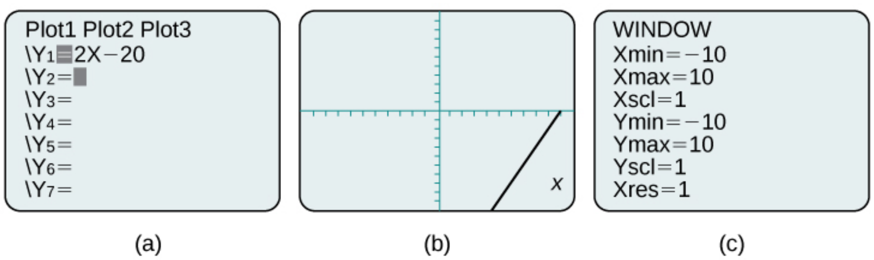


Figure 1.2.9: a. Enter the equation. b. This is the graph in the original window. c. These are the original settings.

By changing the window to show more of the positive x-axis and more of the negative y-axis, we have a much better view of the graph and the x- and y-intercepts. See Figure $1.2.10a$ and Figure $1.2.10b$

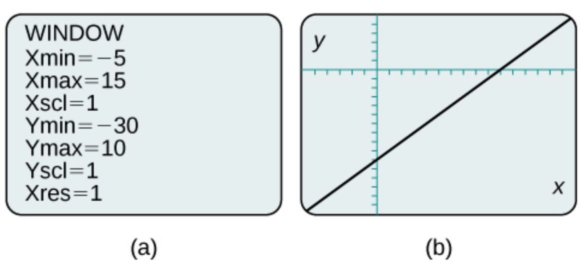


Figure 1.2.10: a. This screen shows the new window settings. b. We can clearly view the intercepts in the new window.

✓ Example 1.2.3: Using a Graphing Utility to Graph an Equation

Use a graphing utility to graph the equation: $y = -\frac{2}{3}x - \frac{4}{3}$.

Solution

Enter the equation in the $y=$ function of the calculator. Set the window settings so that both the x- and y- intercepts are showing in the window. See Figure 1.2.11.

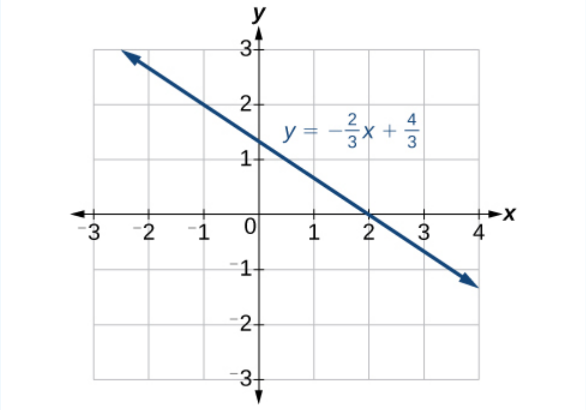


Figure 1.2.11

Finding x-intercepts and y-intercepts

The **intercepts** of a graph are points at which the graph crosses the axes. The x-intercept is the point at which the graph crosses the \(x\)-axis. At this point, the y-coordinate is zero. The y-intercept is the point at which the graph crosses the y-axis. At this point, the x-coordinate is zero.

To determine the x-intercept, we set y equal to zero and solve for x. Similarly, to determine the y-intercept, we set x equal to zero and solve for y. For example, lets find the intercepts of the equation $y=3x-1$.

To find the x-intercept, set $y=0$.

$$\begin{aligned} y &= 3x-1 \\ 0 &= 3x-1 \\ 1 &= 3x \\ \frac{1}{3} &= x \end{aligned}$$

x−intercept: $\left(\frac{1}{3}, 0\right)$

To find the y-intercept, set $x=0$.

$$\begin{aligned} y &= 3x-1 \\ y &= 3(0)-1 \\ y &= -1 \end{aligned}$$

y−intercept: $(0, -1)$

We can confirm that our results make sense by observing a graph of the equation as in Figure 1.2.12. Notice that the graph crosses the axes where we predicted it would.

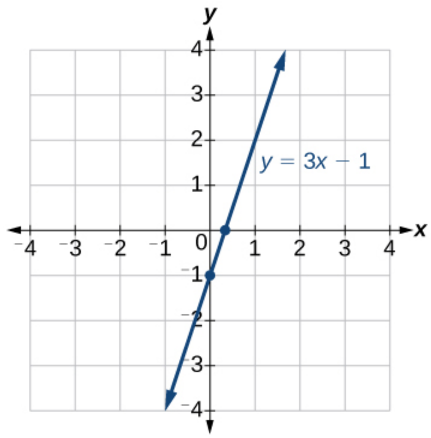


Figure 1.2.12

Howto: GIVEN AN EQUATION, FIND THE INTERCEPTS

1. Find the x-intercept by setting $y=0$ and solving for x.
2. Find the y-intercept by setting $x=0$ and solving for y.

Example 1.2.4: Finding the Intercepts of the Given Equation

Find the intercepts of the equation $y=-3x-4$. Then sketch the graph using only the intercepts.

Solution

Set $y=0$ to find the x-intercept.

$$\begin{aligned} y &= -3x-4 \\ 0 &= -3x-4 \\ 4 &= -3x \\ \frac{4}{3} &= x \end{aligned}$$

x−intercept: $\left(-\frac{4}{3},0\right)$

Set $x=0$ to find the y-intercept.

$$\begin{aligned} y &= -3x-4 \\ y &= -3(0)-4 \\ y &= -4 \end{aligned}$$

y−intercept: $(0,-4)$

Plot both points, and draw a line passing through them as in Figure 1.2.13.

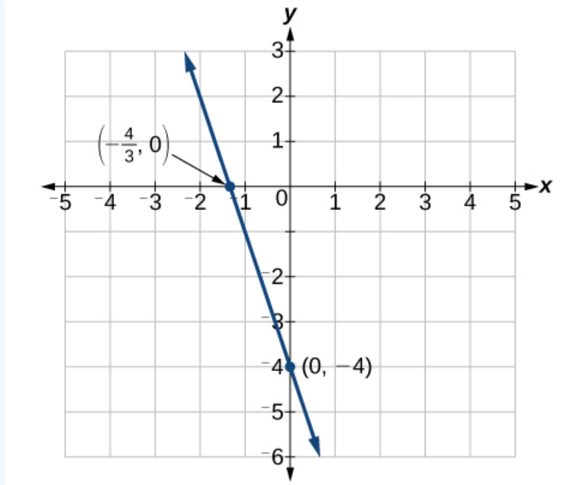


Figure 1.2.13

? Exercise 1.2.2

Find the intercepts of the equation and sketch the graph: $y=-\frac{3}{4}x+3$.

Answer

x-intercept is $(4,0)$; y-intercept is $(0,3)$

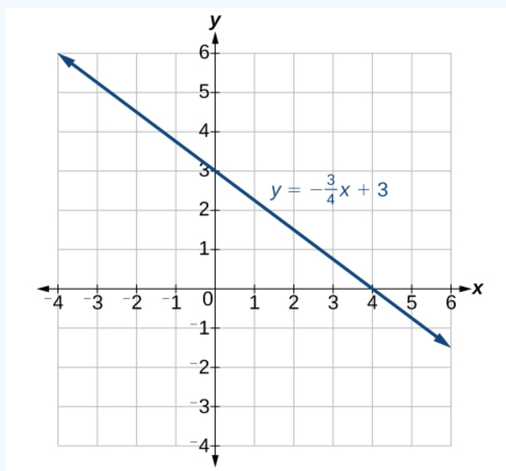


Figure 1.2.14

Using the Distance Formula

Derived from the **Pythagorean Theorem**, the **distance formula** is used to find the distance between two points in the plane. The Pythagorean Theorem, $a^2+b^2=c^2$, is based on a right triangle where a and b are the lengths of the legs adjacent to the right angle, and c is the length of the hypotenuse. See Figure $1.2.15$.

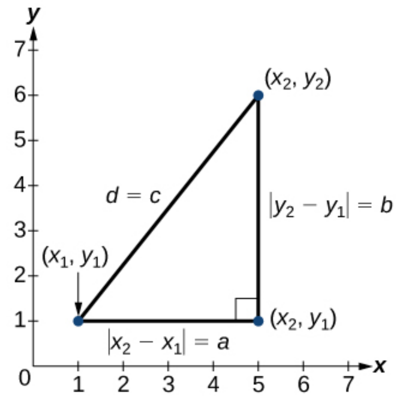


Figure 1.2.15

The relationship of sides $|x_2 - x_1|$ and $|y_2 - y_1|$ to side d is the same as that of sides a and b to side c. We use the absolute value symbol to indicate that the length is a positive number because the absolute value of any number is positive. (For example, $|-3| = 3$.) The symbols $|x_2 - x_1|$ and $|y_2 - y_1|$ indicate that the lengths of the sides of the triangle are positive. To find the length c, take the square root of both sides of the Pythagorean Theorem.

$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2} \tag{1.2.1}$$

It follows that the distance formula is given as

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \rightarrow d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.2.2}$$

We do not have to use the absolute value symbols in this definition because any number squared is positive.

distance between two points

Given endpoints (x_1, y_1) and (x_2, y_2), the distance between two points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.2.3}$$

Example 1.2.5: Finding the Distance between Two Points

Find the distance between the points $(-3, -1)$ and $(2, 3)$.

Solution

Let us first look at the graph of the two points. Connect the points to form a right triangle as in Figure 1.2.16

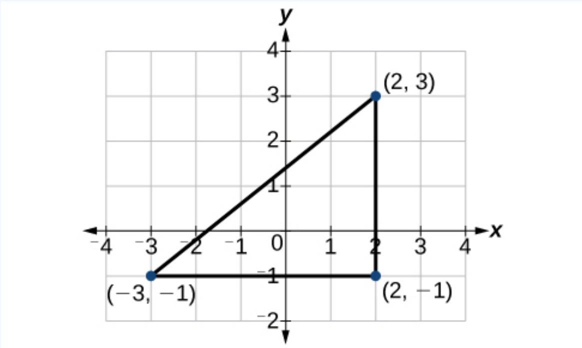


Figure 1.2.16

Then, calculate the length of d using the distance formula.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2-(-3))^2 + (3-(-1))^2} \\ &= \sqrt{(5)^2 + (4)^2} \\ &= \sqrt{25+16} \\ &= \sqrt{41} \end{aligned}$$

? Exercise 1.2.3

Find the distance between two points: $(1,4)$ and $(11,9)$.

Answer

$\sqrt{125} = 5\sqrt{5}$

✔ Example 1.2.6: Finding the Distance between Two Locations

Let's return to the situation introduced at the beginning of this section.

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in Figure 1.2.1. Find the total distance that Tracie traveled. Compare this with the distance between her starting and final positions.

Solution

The first thing we should do is identify ordered pairs to describe each position. If we set the starting position at the origin, we can identify each of the other points by counting units east (right) and north (up) on the grid. For example, the first stop is 1 block east and 1 block north, so it is at $(1,1)$. The next stop is 5 blocks to the east, so it is at $(5,1)$. After that, she traveled 3 blocks east and 2 blocks north to $(8,3)$. Lastly, she traveled 4 blocks north to $(8,7)$. We can label these points on the grid as in Figure 1.2.17.

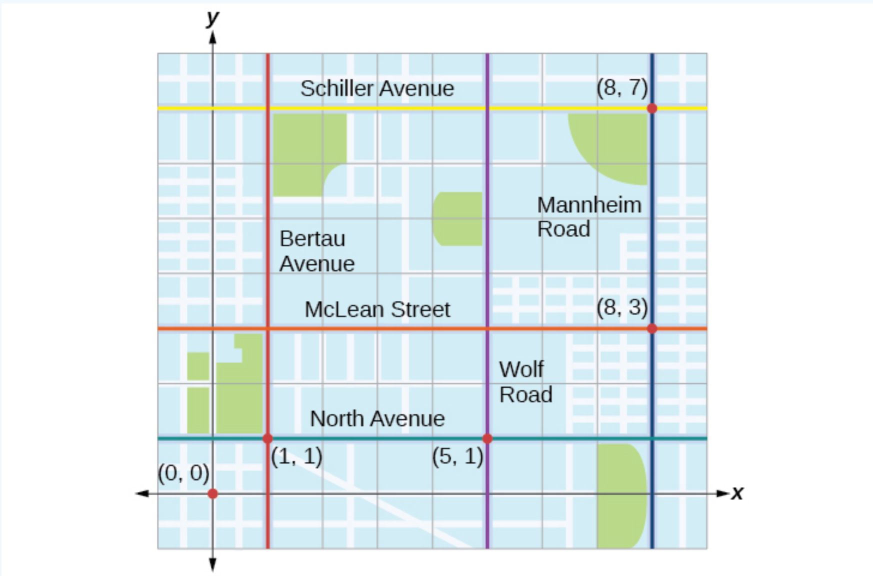


Figure 1.2.17

Next, we can calculate the distance. Note that each grid unit represents $1,000$ feet.

- From her starting location to her first stop at $(1,1)$, Tracie might have driven north $1,000$ feet and then east $1,000$ feet, or vice versa. Either way, she drove $2,000$ feet to her first stop.
- Her second stop is at $(5,1)$. So from $(1,1)$ to $(5,1)$, Tracie drove east $4,000$ feet.

- Her third stop is at $(8,3)$. There are a number of routes from $(5,1)$ to $(8,3)$. Whatever route Tracie decided to use, the distance is the same, as there are no angular streets between the two points. Let's say she drove east $3,000$ feet and then north $2,000$ feet for a total of $5,000$ feet.
- Tracie's final stop is at $(8,7)$. This is a straight drive north from $(8,3)$ for a total of $4,000$ feet.

Next, we will add the distances listed in Table $1.2.4$.

Table 1.2.4

From/To	Number of Feet Driven
$(0,0)$ to $(1,1)$	$2,000$
$(1,1)$ to $(5,1)$	$4,000$
$(5,1)$ to $(8,3)$	$5,000$
$(8,3)$ to $(8,7)$	$4,000$
Total	$15,000$

The total distance Tracie drove is $15,000$ feet, or 2.84 miles. This is not, however, the actual distance between her starting and ending positions. To find this distance, we can use the distance formula between the points $(0,0)$ and $(8,7)$.

$$\begin{aligned} d &= \sqrt{(0-8)^2+(7-0)^2} \\ &= \sqrt{64+49} \\ &= \sqrt{113} \\ &= 10.63 \text{ units} \end{aligned}$$

At $1,000$ feet per grid unit, the distance between Elmhurst, IL, to Franklin Park is $10,630.14$feet, or 2.01 miles. The distance formula results in a shorter calculation because it is based on the hypotenuse of a right triangle, a straight diagonal from the origin to the point $(8,7)$. Perhaps you have heard the saying "as the crow flies," which means the shortest distance between two points because a crow can fly in a straight line even though a person on the ground has to travel a longer distance on existing roadways.

Using the Midpoint Formula

When the endpoints of a line segment are known, we can find the point midway between them. This point is known as the midpoint and the formula is known as the **midpoint formula**. Given the endpoints of a line segment, (x_1,y_1) and (x_2,y_2), the midpoint formula states how to find the coordinates of the midpoint M.

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \tag{1.2.4}$$

A graphical view of a midpoint is shown in Figure $1.2.18$. Notice that the line segments on either side of the midpoint are congruent.

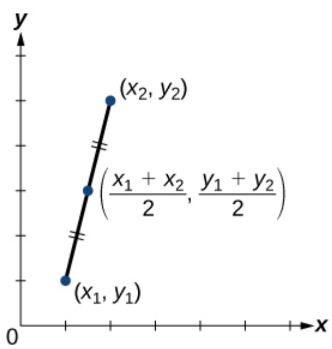


Figure 1.2.18

✓ Example 1.2.7: Finding the Midpoint of the Line Segment

Find the midpoint of the line segment with the endpoints $(7,-2)$ and $(9,5)$.

Solution

Use the formula to find the midpoint of the line segment.

$$\begin{aligned}\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) &= \left(\frac{7+9}{2},\frac{-2+5}{2}\right)\\ &= \left(8,\frac{3}{2}\right)\end{aligned}$$

? Exercise 1.2.4

Find the midpoint of the line segment with endpoints $(-2,-1)$ and $(-8,6)$.

Answer

$\left(-5,\frac{5}{2}\right)$

✓ Example 1.2.8: Finding the Center of a Circle

The diameter of a circle has endpoints $(-1,-4)$ and $(5,-4)$. Find the center of the circle.

Solution

The center of a circle is the center, or midpoint, of its diameter. Thus, the midpoint formula will yield the center point.

$$\begin{aligned}\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) &= \left(\frac{-1+5}{2},\frac{-4-4}{2})\right)\\ &= \left(\frac{4}{2},-\frac{8}{2}\right)\\ &= (2,4)\end{aligned}$$

Media

Access these online resources for additional instruction and practice with the Cartesian coordinate system.

1. Plotting points on the coordinate plane
2. Find x and y intercepts based on the graph of a line

Key Concepts

- We can locate, or plot, points in the Cartesian coordinate system using ordered pairs, which are defined as displacement from the x-axis and displacement from the y-axis. See Example.
- An equation can be graphed in the plane by creating a table of values and plotting points. See Example.
- Using a graphing calculator or a computer program makes graphing equations faster and more accurate. Equations usually have to be entered in the form $y=$_____. See Example.
- Finding the x- and y-intercepts can define the graph of a line. These are the points where the graph crosses the axes. See Example.
- The distance formula is derived from the Pythagorean Theorem and is used to find the length of a line segment. See Example and Example.
- The midpoint formula provides a method of finding the coordinates of the midpoint dividing the sum of the x-coordinates and the sum of the y-coordinates of the endpoints by 2. See Example and Example.

1.2.1: The Rectangular Coordinate Systems and Graphs (Exercises)

For the following exercises, find the x -intercept and the y -intercept without graphing.

1. $4x-3y=12$
2. $2y-4=3x$

For the following exercises, solve for y in terms of x, putting the equation in slope-intercept form.

3. $5x=3y-12$
4. $2x-5y=7$

For the following exercises, find the distance between the two points.

5. (-2,5)(4,-1)
6. (-12,-3)(-1,5)
7. Find the distance between the two points (-71,432) and (511,218) using your calculator, and round your answer to the nearest thousandth.

For the following exercises, find the coordinates of the midpoint of the line segment that joins the two given points.

8. (-1,5) and (4,6)
9. (-13,5) and (17,18)

For the following exercises, construct a table and graph the equation by plotting at least three points.

10. $y=\frac{1}{2}x+4$
11. $4x-3y=6$

1.3: Linear Functions

Learning Objectives

In this section, you will:

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Calculate and interpret slope.
- Write the point-slope form of an equation.
- Write and interpret a linear function.

Figure 1 Shanghai MagLev Train (credit: "kanegen"/Flickr)

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (Figure 1). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes.[2]

Suppose a maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train's distance from the station at a given point in time.

Representing Linear Functions

The function describing the train's motion is a **linear function**, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train's motion as a function using each method.

Representing a Linear Function in Word Form

Let's begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the slope-intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

$$\begin{array}{ll} \text{Equation form} & y = mx + b \\ \text{Equation notation} & f(x) = mx + b \end{array}$$

In the example of the train, we might use the notation $D(t)$ in which the total distance D is a function of the time t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D(t) = 83t + 250$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

1 second 1 second 1 second

t	0	1	2	3
$D(t)$	250	333	416	499

83 meters 83 meters 83 meters

Figure 2 Tabular representation of the function D showing selected input and output values

Q&A

Can the input in the previous example be any real number?

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph, represented in Figure 3. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or **slope** of the line. The point at which the input value is zero is the vertical intercept, or ***y*-intercept**, of the line. We can see from the graph in Figure 3 that the y-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

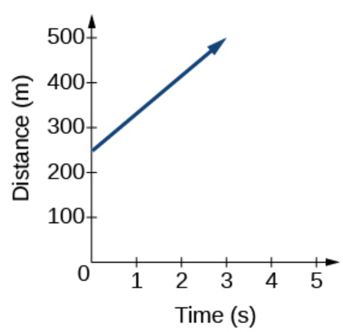


Figure 3 The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x+1$. Ask yourself what numbers can be input to the function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Linear Function

A linear function is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or slope of the function. The y-intercept is at $(0, b)$.

Example 1

Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in Figure 4 depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.

Figure 4 (credit: Ilse Reijs and Jan-Noud Hutten)

Answer

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an increasing function has a positive slope. A line with a positive slope slants upward from left to right as in Figure 5**(a)**. For a decreasing function, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 5**(b)**. If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 5**(c)**.

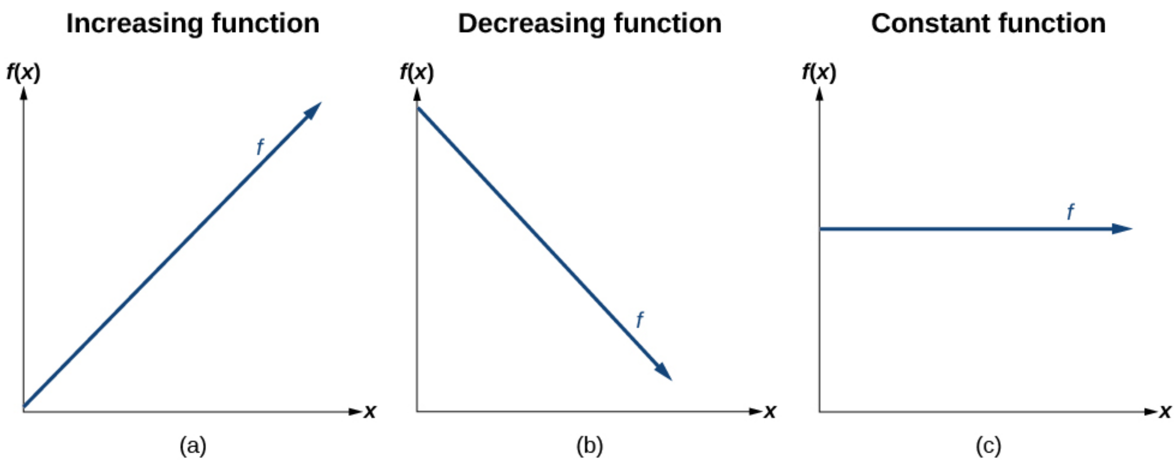


Figure 5

Increasing and Decreasing Functions

The slope determines if the function is an increasing linear function, a decreasing linear function, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is an decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

Example 2

Deciding whether a Function Is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day.[3] For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

1. ⓐ The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
2. ⓑ A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
3. ⓒ A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Answer

Analyze each function.

1. ⓐ The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
2. ⓑ The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
3. ⓒ The cost function can be represented as $f(x) = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2 — which can be represented by a set of points, (x_1, y_1) and (x_2, y_2) —we can calculate the slope m, as follows

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where Δy is the vertical displacement and Δx is the horizontal displacement. Note in function notation two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure 6 indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

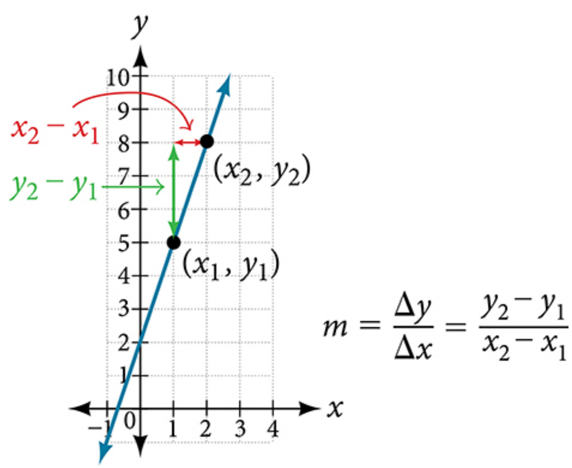


Figure 6 The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1), as long as each calculation is started with the elements from the same coordinate pair.

Q&A

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Calculate Slope

The slope, or rate of change, of a function m can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

How To

Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

Example 3

Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3,-2)$ and $(8,1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Answer

The coordinate pairs are $(3,-2)$ and $(8,1)$. To find the rate of change, we divide the change in output by the change in input.

$$m=\frac{\text{change in output}}{\text{change in input}}=\frac{1-(-2)}{8-3}=\frac{3}{5}$$

We could also write the slope as $m=0.6$. The function is increasing because $m>0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used.

Try It #1

If $f(x)$ is a linear function, and $(2,\quad 3)$ and $(0,\quad 4)$ are points on the line, find the slope. Is this function increasing or decreasing?

Example 4

Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

Answer

The rate of change relates the change in population to the change in time. The population increased by $27,800-23,400=4{,}400$people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4{,}400\text{people}}{4\text{years}}=1{,}100\quad\frac{\text{people}}{\text{year}}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

Try It #2

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the point-slope form.

$$y-y_1=m\left(x-x_1\right)$$

The point-slope form is derived from the slope formula.

$$\begin{array}{ll} m=\frac{y-y_1}{x-x_1} & \text{assuming} x\neq x_1 \\ m\left(x-x_1\right)=\frac{y-y_1}{x-x_1}\left(x-x_1\right) & \text{Multiply both sides by } \left(x-x_1\right). \\ m\left(x-x_1\right)=y-y_1 & \text{Simplify.} \\ y-y_1=m\left(x-x_1\right) & \text{Rearrange.} \end{array}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y-4=-\frac{1}{2}(x-6)$. We can convert it to the slope-intercept form as shown.

$$\begin{array}{ll} y-4=-\frac{1}{2}(x-6) & \\ y-4=-\frac{1}{2}x+3 & \text{Distribute the} -\frac{1}{2}. \\ y=-\frac{1}{2}x+7 & \text{Add 4 to each side.} \end{array}$$

Therefore, the same line can be described in slope-intercept form as $y=-\frac{1}{2}x+7$.

Point-Slope Form of a Linear Equation

The **point-slope form** of a linear equation takes the form

$$y-y_1=m\left(x-x_1\right)$$

where m is the slope, x_1 and y_1 are the x-and$y$- coordinates of a specific point through which the line passes.

Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4,1)$. We know that $m=2$ and that $x_1=4$ and $y_1=1$. We can substitute these values into the general point-slope equation.

$$\begin{gathered} y-y_1=m\left(x-x_1\right) \\ y-1=2\left(x-4\right) \end{gathered}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$\begin{array}{ll} y-1=2(x-4) & \\ y-1=2x-8 & \text{Distribute the} 2. \\ y=2x-7 & \text{Add 1 to each side.} \end{array}$$

Both equations, $y-1=2\left(x-4\right)$ and $y=2x-7$, describe the same line. See Figure 7.

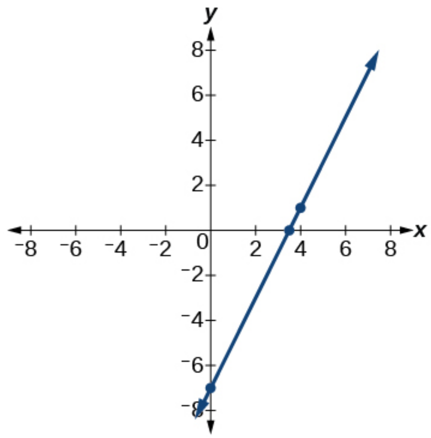


Figure 7

Example 5

Writing Linear Equations Using a Point and the Slope

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point $(6, -1)$. Then rewrite it in the slope-intercept form.

Answer

Let's figure out what we know from the given information. The slope is 3, so $m = 3$. We also know one point, so we know $x_1 = 6$ and $y_1 = -1$. Now we can substitute these values into the general point-slope equation.

$$\begin{array}{ll} y - y_1 = m(x - x_1) & \\ y - (-1) = 3(x - 6) & \text{Substitute known values.} \\ y + 1 = 3(x - 6) & \text{Distribute} -1 \text{to find point-slope form.} \end{array}$$

Then we use algebra to find the slope-intercept form.

$$\begin{array}{ll} y + 1 = 3(x - 6) & \\ y + 1 = 3x - 18 & \text{Distribute 3.} \\ y = 3x - 19 & \text{Simplify to slope-intercept form.} \end{array}$$

Try It #3

Write the point-slope form of an equation of a line with a slope of -2 that passes through the point $(-2,\ \ 2)$. Then rewrite it in the slope-intercept form.

Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0,\ \ 1)$ and $(3,\ \ 2)$. We can use the coordinates of the two points to find the slope.

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{2-1}{3-0} \\ &= \frac{1}{3} \end{aligned}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use (0, 1) for our point.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-1 &= \frac{1}{3}(x-0) \end{aligned}$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{array}{ll} y-1=\frac{1}{3}(x-0) & \\ y-1=\frac{1}{3}x & \text{Distribute the}\,\frac{1}{3}. \\ y=\frac{1}{3}x+1 & \text{Add 1 to each side.} \end{array}$$

Both equations describe the line shown in Figure 8.

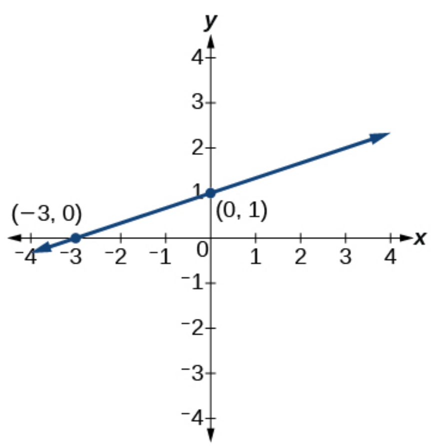


Figure 8

Example 6

Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points (5, 1) and (8, 7). Then rewrite it in the slope-intercept form.

Answer

Let's begin by finding the slope.

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{7-1}{8-5} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

So $m=2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5,1)$.

$$y - y_1 = m\left(x - x_1\right)$$
$$y - 1 = 2\left(x - 5\right)$$

The point-slope equation of the line is $y_2 - 1 = 2(x_2 - 5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$y - 1 = 2(x - 5)$$
$$y - 1 = 2x - 10$$
$$y = 2x - 9$$

The slope-intercept equation of the line is $y = 2x - 9$.

Try It #4

Write the point-slope form of an equation of a line that passes through the points $(-1, 3)$ and $(0, 0)$. Then rewrite it in the slope-intercept form.

Writing and Interpreting an Equation for a Linear Function

Now that we have written equations for linear functions in both the slope-intercept form and the point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in Figure 9.

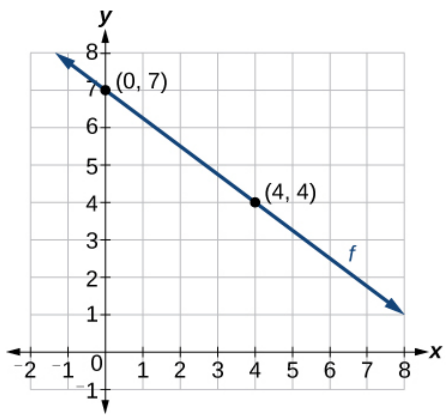


Figure 9

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0,\ 7)$ and $(4,\ 4)$. We can use these points to calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4-7}{4-0}$$
$$= -\frac{3}{4}$$

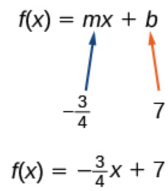
Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

$$y - y_1 = m(x - x_1)$$
$$y - 4 = -\frac{3}{4}(x - 4)$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$y - 4 = -\frac{3}{4}(x - 4)$$
$$y - 4 = -\frac{3}{4}x + 3$$
$$y = -\frac{3}{4}x + 7$$

If we wanted to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the y-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.



So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

How To

Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the y-axis to identify the y-intercept by visual inspection.
4. Substitute the slope and y-intercept into the slope-intercept form of a line equation.

Example 7

Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in Figure 10.

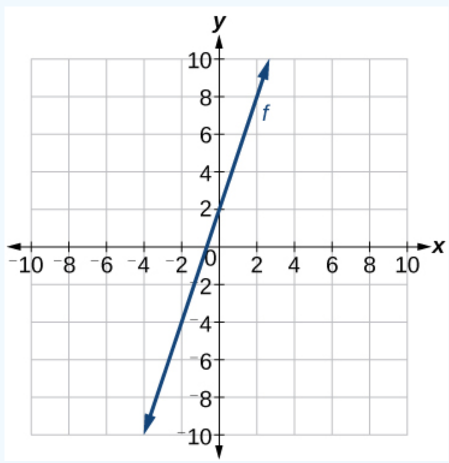


Figure 10

Answer

Identify two points on the line, such as $(0, \quad 2)$ and $(-2, -4)$. Use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4-2}{-2-0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$y - y_1 = m(x - x_1)$$
$$y - (-4) = 3(x - (-2))$$
$$y + 4 = 3(x + 2)$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$y + 4 = 3(x + 2)$$
$$y + 4 = 3x + 6$$
$$y = 3x + 2$$

Analysis

This makes sense because we can see from Figure 11 that the line crosses the y-axis at the point $(0, \quad 2)$, which is the y-intercept, so $b = 2$.

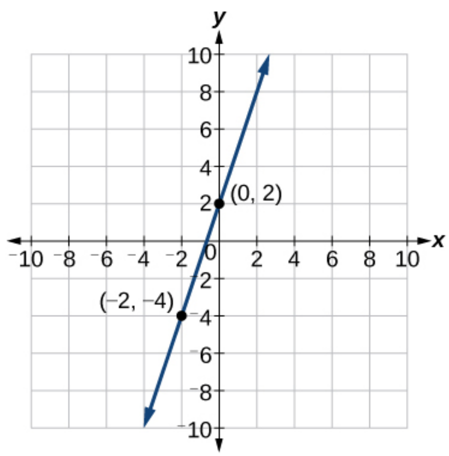


Figure 11

Example 8

Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

Answer

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50 for Ben. The variable cost, called the marginal cost, is represented by 37.5.The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is represented by

$$C(100) = 1250 + 37.5(100)$$
$$= 5000$$

So his monthly cost would be $5,000.

Example 9

Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

Answer

We can write the given points using coordinates.

$$f(3)=-2\rightarrow(3,-2)$$
$$f(8)=1\rightarrow(8,1)$$

We can then use the points to calculate the slope.

$$\begin{aligned}m&=\frac{y_2-y_1}{x_2-x_1}\\&=\frac{1-(-2)}{8-3}\\&=\frac{3}{5}\end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$y-y_1=m(x-x_1)$$
$$y-(-2)=\frac{3}{5}(x-3)$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$y+2=\frac{3}{5}(x-3)$$
$$y+2=\frac{3}{5}x-\frac{9}{5}$$
$$y=\frac{3}{5}x-\frac{19}{5}$$

Try It #5

If $f(x)$ is a linear function, with $f(2)=-11$, and $f(4)=-25$, find an equation for the function in slope-intercept form.

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

How To

Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x)=mx+b$.
3. Evaluate the function at $x=c$.

Example 10

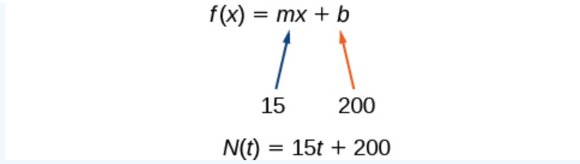
Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months. How many songs will he own in a year?

Answer

The initial value for this function is 200 because he currently owns 200 songs, so $N(0)=200$, which means that $b=200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m=15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.



We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have in 1 year (12 months). In other words, we can evaluate the function at $t = 12$.

$$\begin{aligned} &N(12) = 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

Example 11

Using a Linear Function to Calculate Salary Plus Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income, I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Answer

The given information gives us two input-output pairs: $(3,760)$ and $(5,920)$. We start by finding the rate of change.

$$\begin{aligned} &m = \frac{920-760}{5-3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} &I(n) = 80n + b \\ &760 = 80(3) + b \quad \text{when} n = 3, \quad I(3) = 760 \\ &760 - 80(3) = b \\ &520 = b \end{aligned}$$

The value of b is the starting value for the function and represents Ilya's income when $n = 0$, or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya’s base salary is \$520 per week and he earns an additional \$80 commission for each policy sold.

Example 12

Using Tabular Form to Write an Equation for a Linear Function

Table 1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

***w*, number of weeks**	0	2	4	6
***P*(*w*), number of rats**	1000	1080	1160	1240

Table 1

Answer

We can see from the table that the initial value for the number of rats is 1000, so $b=1000$.

Rather than solving for m, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w)=40w+1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2,1080)$and $(6,1240)$

$$\begin{aligned} m &= \frac{1240-1080}{6-2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

Q&A

Is the initial value always provided in a table of values like Table 1?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x)=mx+b$, and solve for b.

Try It #6

A new plant food was introduced to a young tree to test its effect on the height of the tree. Table 2 shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

x	0	2	4	8	12
$H(x)$	12.5	13.5	14.5	16.5	18.5

Table 2

Media

Access this online resource for additional instruction and practice with linear functions.

- Linear Functions

2.1 Section Exercises

Verbal

1.

Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after t seconds is given by $E(t) = 3000 - 70t$. Write a complete sentence describing Terry's starting elevation and how it is changing over time.

2.

Maria is climbing a mountain. Maria's elevation, $E(t)$, in feet after t minutes is given by $E(t) = 1200 + 40t$. Write a complete sentence describing Maria's starting elevation and how it is changing over time.

3.

Jessica is walking home from a friend's house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate in miles per hour?

4.

Sonya is currently 10 miles from home and is walking farther away at 2 miles per hour. Write an equation for her distance from home *t* hours from now.

5.

A boat is 100 miles away from the marina, sailing directly toward it at 10 miles per hour. Write an equation for the distance of the boat from the marina after *t* hours.

6.

Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding n rides?

Algebraic

For the following exercises, determine whether the equation of the curve can be written as a linear function.

7.

$y = \frac{1}{4}x + 6$

8.

$y = 3x - 5$

9.

$y = 3x^2 - 2$

10.

$3x + 5y = 15$

11.

$3x^2 + 5y = 15$

12.

$3x + 5y^2 = 15$

13.

$-2x^2 + 3y^2 = 6$

14.

$-\frac{x-3}{5} = 2y$

For the following exercises, determine whether each function is increasing or decreasing.

15.

$f(x) = 4x + 3$

16.

$g(x) = 5x + 6$

17.

$a(x) = 5 - 2x$

18.

$b(x) = 8 - 3x$

19.

$h(x) = -2x + 4$

20.

$k(x) = -4x + 1$

21.

$j(x) = \frac{1}{2}x - 3$

22.

$p(x) = \frac{1}{4}x - 5$

23.

$n(x) = -\frac{1}{3}x - 2$

24.

$m(x) = -\frac{3}{8}x + 3$

For the following exercises, find the slope of the line that passes through the two given points.

25.

$(2, \quad 4)$ and $(4, 10)$

26.

$(1, 5)$ and $(4, 11)$

27.

$(-1, 4)$ and $(5, 2)$

28.

$(8, -2)$ and $(4, 6)$

29.

$(6, \quad 11)$ and $(-4, \quad 3)$

For the following exercises, given each set of information, find a linear equation satisfying the conditions, if possible.

30.

$f(-5) = -4$, and $f(5) = 2$

31.

$f(-1) = 4$ and $f(5) = 1$

32.

$(2, 4)$ and $(4, 10)$

33.

Passes through $(1, 5)$ and $(4, 11)$

34.

Passes through $(-1, 4)$ and $(5, 2)$

35.

Passes through $(-2, 8)$ and $(4, 6)$

36.

x intercept at $(-2, 0)$ and y intercept at $(0, -3)$

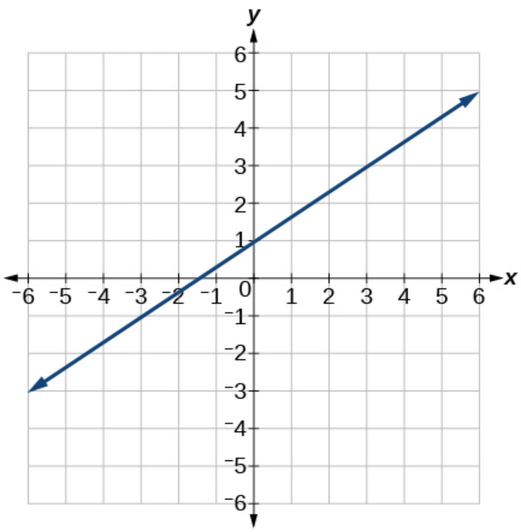
37.

x intercept at $(-5, 0)$ and y intercept at $(0, 4)$

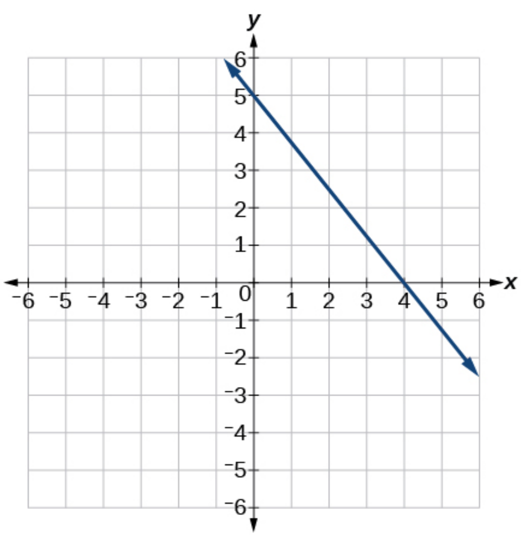
Graphical

For the following exercises, find the slope of the lines graphed.

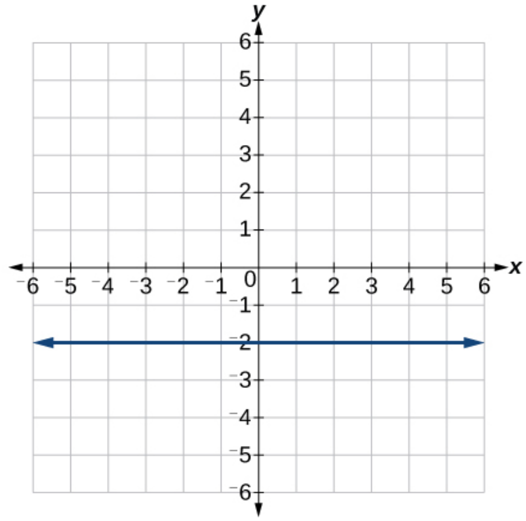
38.



39.

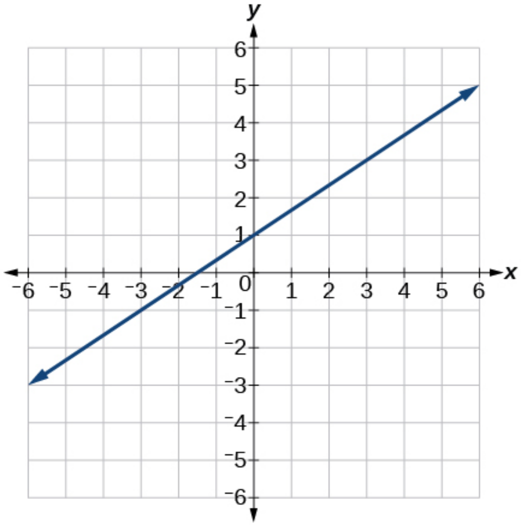


40.



For the following exercises, write an equation for the lines graphed.

41.



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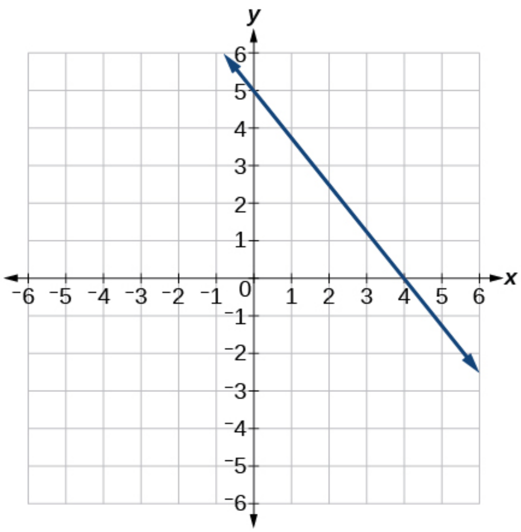

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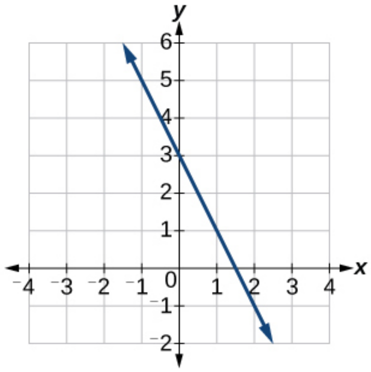

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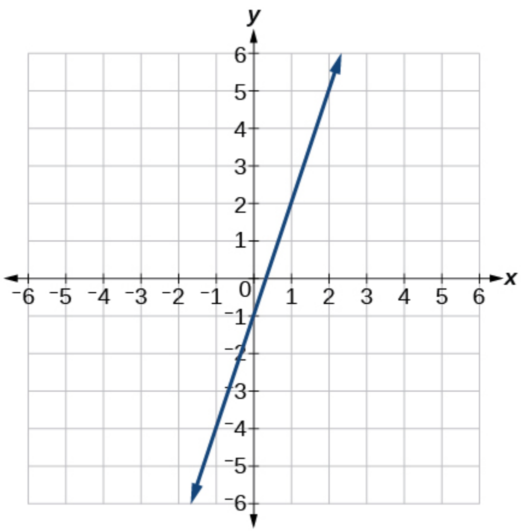
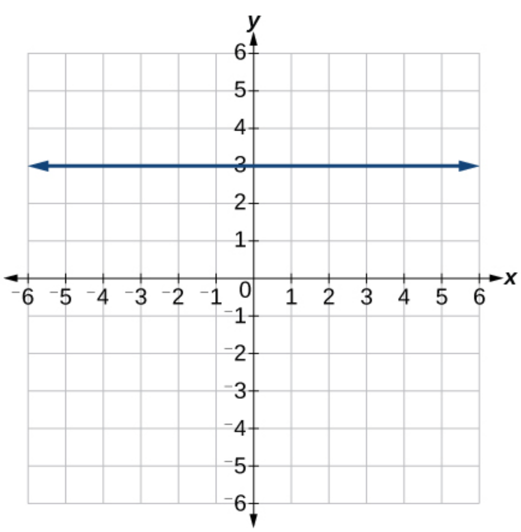
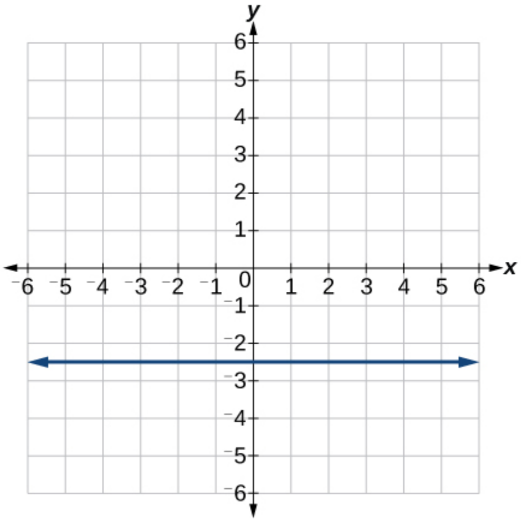


45.



46.



Numeric

For the following exercises, which of the tables could represent a linear function? For each that could be linear, find a linear equation that models the data.

47.

x	0	5	10	15
$g(x)$	5	–10	–25	–40

48.

x	0	5	10	15
$h(x)$	5	30	105	230

49.

x	0	5	10	15
$f(x)$	–5	20	45	70

50.

x	5	10	20	25
$k(x)$	13	28	58	73

51.

x	0	2	4	6
$g(x)$	6	–19	–44	–69

52.

x	2	4	6	8
$f(x)$	–4	16	36	56

53.

x	2	4	6	8
$f(x)$	–4	16	36	56

54.

x	0	2	6	8
$k(x)$	6	31	106	231

Technology

55.

If f is a linear function, $f(0.1)=11.5, \quad \text{and} \quad f(0.4)=-5.9$,find an equation for the function.

56.

Graph the function f on a domain of $[-10, \quad 10]: \quad f(x)=0.02x-0.01$.Enter the function in a graphing utility. For the viewing window, set the minimum value of x to be -10 and the maximum value of x to be 10.

57.

Graph the function f on a domain of $[-10, \quad 10]: fx)=2,500x+4,000$

58.

Table 3 shows the input, w, and output, k, for a linear function k. a. Fill in the missing values of the table. b. Write the linear function k, round to 3 decimal places.

w	–10	5.5	67.5	b
k	30	–26	a	–44

Table 3

59.

Table 4 shows the input, p, and output, q, for a linear function q. a. Fill in the missing values of the table. b. Write the linear function k.

p	0.5	0.8	12	b
q	400	700	a	1,000,000

Table 4

60.

Graph the linear function f on a domain of $[-10,10]$ for the function whose slope is $\frac{1}{8}$ and y-intercept is $\frac{31}{16}$. Label the points for the input values of -10 and 10.

61.

Graph the linear function f on a domain of $[-0.1,0.1]$for the function whose slope is 75 and y-intercept is -22.5. Label the points for the input values of -0.1 and 0.1.

62.

Graph the linear function f where $f(x)=ax+b$ on the same set of axes on a domain of $[-4,4]$ for the following values of a and b.

a. $a=2; \quad b=3$
b. $a=2; \quad b=4$
c. $a=2; \quad b=-4$
d. $a=2; \quad b=-5$

Extensions

63.

Find the value of x if a linear function goes through the following points and has the following slope: $(x,2),(-4,6), \quad m=3$

64.

Find the value of y if a linear function goes through the following points and has the following slope: $(10,y),(25,100), \quad m=-5$

65.

Find the equation of the line that passes through the following points: $(a, \quad b)$ and $(a, \quad b+1)$

66.

Find the equation of the line that passes through the following points: $(2a, \quad b)$ and $(a, \quad b+1)$

67.

Find the equation of the line that passes through the following points: $(a, \quad 0)$ and $(c, \quad d)$

Real-World Applications

68.

At noon, a barista notices that she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

69.

A gym membership with two personal training sessions costs $125, while gym membership with five personal training sessions costs $260. What is cost per session?

70.

A clothing business finds there is a linear relationship between the number of shirts, n, it can sell and the price, p, it can charge per shirt. In particular, historical data shows that 1,000 shirts can be sold at a price of $\$30$, while 3,000 shirts can be sold at a price of $22. Find a linear equation in the form $p(n)=mn+b$ that gives the price p they can charge for n shirts.

71.

A phone company charges for service according to the formula: $C(n)=24+0.1n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

72.

A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationship in the form $y=mn+b$ that gives the yield when n stalks are planted.

73.

A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the rate of growth of the population and make a statement about the population rate of change in people per year.

74.

A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1,700 people each year. Write an equation, $P(t)$, for the population t years after 2003.

75.

Suppose that average annual income (in dollars) for the years 1990 through 1999 is given by the linear function: $I(x)=1054x+23,286,$ where x is the number of years after 1990. Which of the following interprets the slope in the context of the problem?

a. As of 1990, average annual income was $23,286.
b. In the ten-year period from 1990–1999, average annual income increased by a total of $1,054.
c. Each year in the decade of the 1990s, average annual income increased by $1,054.
d. Average annual income rose to a level of $23,286 by the end of 1999.

76.

When temperature is 0 degrees Celsius, the Fahrenheit temperature is 32. When the Celsius temperature is 100, the corresponding Fahrenheit temperature is 212. Express the Fahrenheit temperature as a linear function of C, the Celsius temperature, $F(C)$.

a. Find the rate of change of Fahrenheit temperature for each unit change temperature of Celsius.
b. Find and interpret $F(28)$.
c. Find and interpret $F(-40)$.

Footnotes

- 2http://www.chinahighlights.com/shang...glev-train.htm
- 3http://www.cbsnews.com/8301-501465_1...ay-study-says/

1.4: Graphs of Linear Functions

Learning Objectives

In this section, you will:

- Graph linear functions.
- Write the equation for a linear function from the graph of a line.
- Given the equations of two lines, determine whether their graphs are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.
- Solve a system of linear equations.

Two competing telephone companies offer different payment plans. The two plans charge the same rate per long distance minute, but charge a different monthly flat fee. A consumer wants to determine whether the two plans will ever cost the same amount for a given number of long distance minutes used. The total cost of each payment plan can be represented by a linear function. To solve the problem, we will need to compare the functions. In this section, we will consider methods of comparing functions using graphs.

Graphing Linear Functions

In Linear Functions, we saw that that the graph of a linear function is a straight line. We were also able to see the points of the function as well as the initial value from a graph. By graphing two functions, then, we can more easily compare their characteristics.

There are three basic methods of graphing linear functions. The first is by plotting points and then drawing a line through the points. The second is by using the *y*-intercept and slope. And the third is by using transformations of the identity function $f(x)=x$.

Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x)=2x$, we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1,2)$. Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2,4)$. Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

How To

Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

Example 1

Graphing by Plotting Points

Graph $f(x)=-\frac{2}{3}x+5$ by plotting points.

Answer

Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.

$$x = 0 \qquad f(0) = -\tfrac{2}{3}(0) + 5 = 5 \Rightarrow (0, 5)$$
$$x = 3 \qquad f(3) = -\tfrac{2}{3}(3) + 5 = 3 \Rightarrow (3, 3)$$
$$x = 6 \qquad f(6) = -\tfrac{2}{3}(6) + 5 = 1 \Rightarrow (6, 1)$$

Plot the coordinate pairs and draw a line through the points. Figure 1 represents the graph of the function $f(x) = -\frac{2}{3}x + 5$.

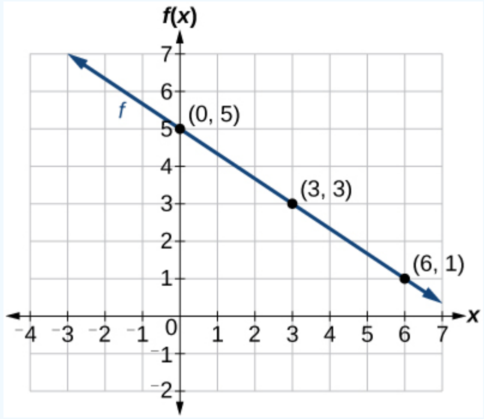


Figure 1 The graph of the linear function $f(x) = -\frac{2}{3}x + 5$.

Analysis

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative constant rate of change in the equation for the function.

Try It #1

Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.

Graphing a Function Using *y*-intercept and Slope

Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its *y*-intercept, which is the point at which the input value is zero. To find the *y*-intercept, we can set $x = 0$ in the equation.

The other characteristic of the linear function is its slope m, which is a measure of its steepness. Recall that the slope is the rate of change of the function. The slope of a function is equal to the ratio of the change in outputs to the change in inputs. Another way to think about the slope is by dividing the vertical difference, or rise, by the horizontal difference, or run. We encountered both the *y*-intercept and the slope in Linear Functions.

Let’s consider the following function.

$$f(x) = \frac{1}{2}x + 1$$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The *y*-intercept is the point on the graph when $x = 0$. The graph crosses the *y*-axis at $(0, 1)$. Now we know the slope and the *y*-intercept. We can begin graphing by plotting the point $(0, 1)$ We know that the slope is rise over run, $m = \frac{\text{rise}}{\text{run}}$. From our example, we have $m = \frac{1}{2}$, which means that the rise is 1 and the run is 2. So starting from our *y*-intercept $(0, 1)$, we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in Figure 2.

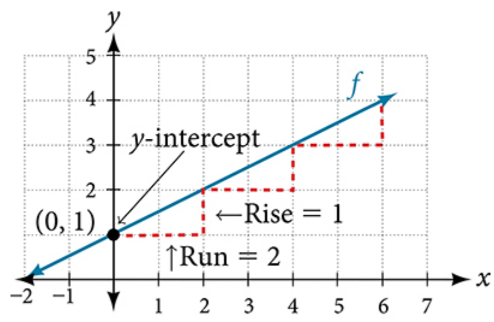


Figure 2

Graphical Interpretation of a Linear Function

In the equation $f(x) = mx + b$

- b is the *y*-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the *y*-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Q&A

Do all linear functions have *y*-intercepts?

Yes. All linear functions cross the y-axis and therefore have y-intercepts. (Note: *A vertical line parallel to the y-axis does not have a y-intercept, but it is not a function.*)

How To

Given the equation for a linear function, graph the function using the *y*-intercept and slope.

1. Evaluate the function at an input value of zero to find the *y*-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the *y*-intercept.
4. Use $\frac{\text{rise}}{\text{run}}$ to determine at least two more points on the line.
5. Sketch the line that passes through the points.

Example 2

Graphing by Using the *y*-intercept and Slope

Graph $f(x) = -\frac{2}{3}x + 5$ using the *y*-intercept and slope.

Answer

Evaluate the function at $x = 0$ to find the *y*-intercept. The output value when $x = 0$ is 5, so the graph will cross the *y*-axis at $(0, 5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$. This tells us that for each vertical decrease in the "rise" of -2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the y-intercept on the graph in Figure 3. From the initial value $(0,5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then draw a line through the points.

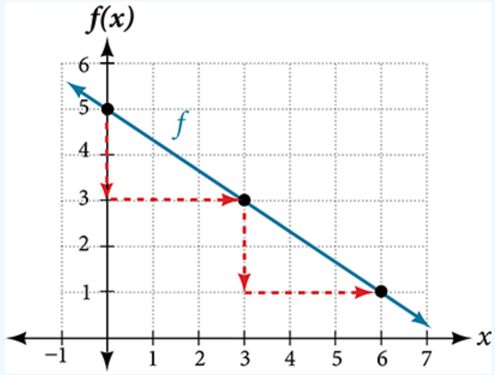


Figure 3

Analysis

The graph slants downward from left to right, which means it has a negative slope as expected.

Try It #2

Find a point on the graph we drew in Example 2 that has a negative x-value.

Graphing a Function Using Transformations

Another option for graphing is to use transformations of the identity function $f(x)=x$. A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

Vertical Stretch or Compression

In the equation $f(x)=mx$, the m is acting as the vertical stretch or compression of the identity function. When m is negative, there is also a vertical reflection of the graph. Notice in Figure 4 that multiplying the equation of $f(x)=x$ by m stretches the graph of f by a factor of m units if $m>1$ and compresses the graph of f by a factor of m units if $0<m<1$. This means the larger the absolute value of m, the steeper the slope.

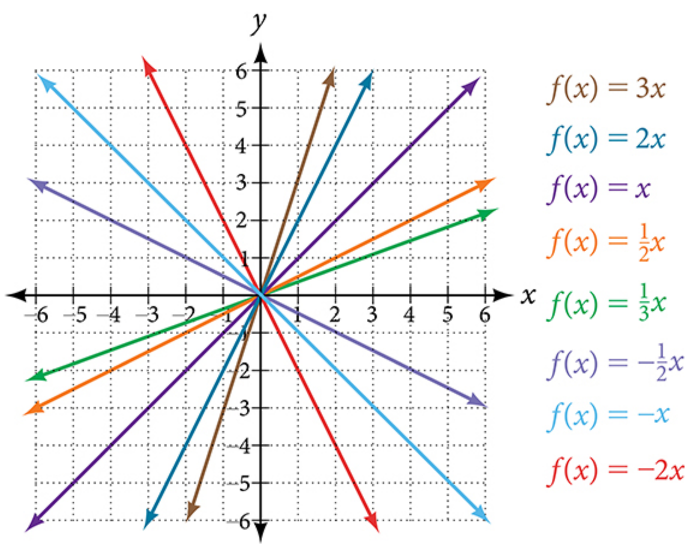


Figure 4 Vertical stretches and compressions and reflections on the function $f(x)=x$.

Vertical Shift

In $f(x) = mx + b$, the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Notice in Figure 5 that adding a value of b to the equation of $f(x) = x$ shifts the graph of f a total of b units up if b is positive and $|b|$ units down if b is negative.

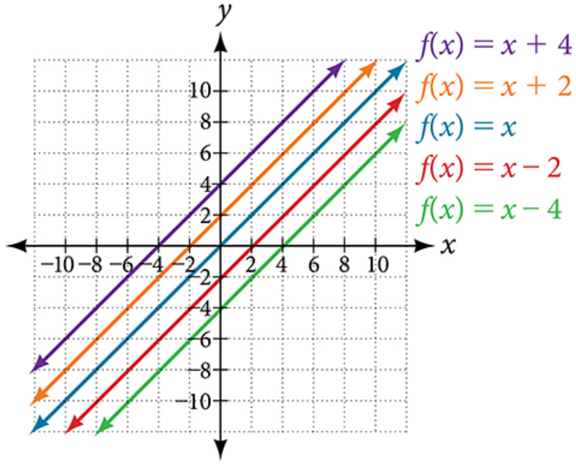


Figure 5 This graph illustrates vertical shifts of the function $f(x) = x$.

Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

How To

Given the equation of a linear function, use transformations to graph the linear function in the form $f(x) = mx + b$.

1. Graph $f(x) = x$.
2. Vertically stretch or compress the graph by a factor m.
3. Shift the graph up or down b units.

Example 3

Graphing by Using Transformations

Graph $f(x) = \frac{1}{2}x - 3$ using transformations.

Answer

The equation for the function shows that $m = \frac{1}{2}$ so the identity function is vertically compressed by $\frac{1}{2}$. The equation for the function also shows that $b = -3$ so the identity function is vertically shifted down 3 units. First, graph the identity function, and show the vertical compression as in Figure 6.

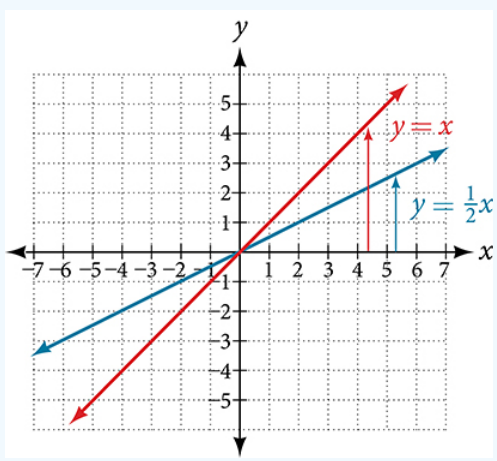


Figure 6 The function, $y = x$, compressed by a factor of $\frac{1}{2}$.

Then show the vertical shift as in Figure 7.

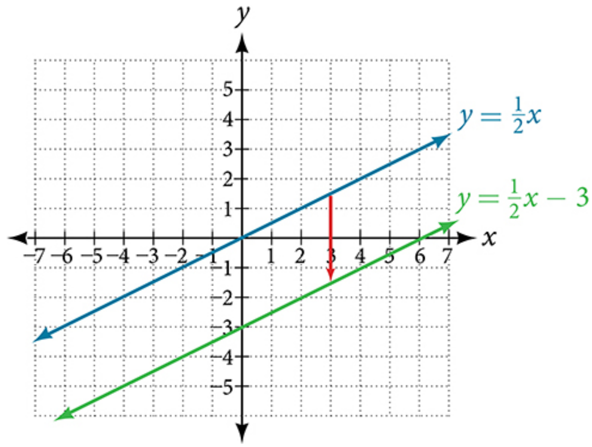


Figure 7 The function $y = \frac{1}{2}x$, shifted down 3 units.

Try It #3

Graph $f(x) = 4 + 2x$, using transformations.

Q&A

In Example 3, could we have sketched the graph by reversing the order of the transformations?

No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.

$$\begin{aligned} f(2) &= \tfrac{1}{2}(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Writing the Equation for a Function from the Graph of a Line

Recall that in Linear Functions, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at Figure 8. We can see right away that the graph crosses the *y*-axis at the point $(0, 4)$ so this is the *y*-intercept.

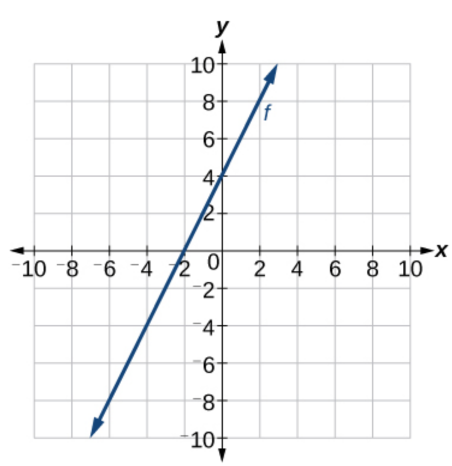


Figure 8

Then we can calculate the slope by finding the rise and run. We can choose any two points, but let's look at the point $(-2, 0)$. To get from this point to the *y*-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2$$

Substituting the slope and *y*-intercept into the slope-intercept form of a line gives

How To

Given a graph of linear function, find the equation to describe the function.

1. Identify the *y*-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the *y*-intercept and slope into the slope-intercept form of a line.

Example 4

Matching Linear Functions to Their Graphs

Match each equation of the linear functions with one of the lines in Figure 9.

1. ⓐ $f(x) = 2x + 3$
2. ⓑ $g(x) = 2x - 3$
3. ⓒ $h(x) = -2x + 3$
4. ⓓ $j(x) = \frac{1}{2}x + 3$

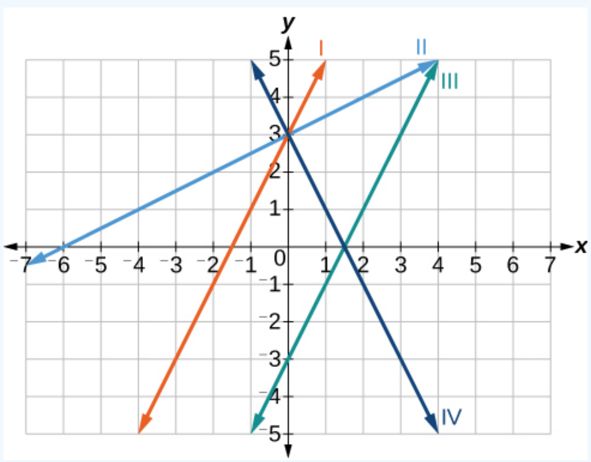


Figure 9

Answer

Analyze the information for each function.

1. ⓐ This function has a slope of 2 and a *y*-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function g has the same slope, but a different *y*-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through $(0, 3)$ so f must be represented by Line I.
2. ⓑ This function also has a slope of 2, but a *y*-intercept of -3. It must pass through the point $(0, -3)$ and slant upward from left to right. It must be represented by Line III.
3. ⓒ This function has a slope of –2 and a *y*-intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.
4. ⓓ This function has a slope of $\frac{1}{2}$ and a *y*-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through $(0, 3)$, but the slope of j is less than the slope of f so the line for j must be flatter. This function is represented by Line II.

Now we can re-label the lines as in Figure 10.

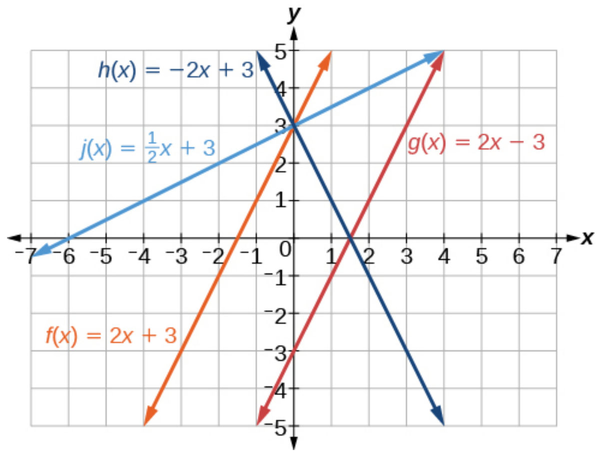


Figure 10

Finding the *x*-intercept of a Line

So far, we have been finding the *y*-intercepts of a function: the point at which the graph of the function crosses the *y*-axis. A function may also have an ***x*-intercept,** which is the *x*-coordinate of the point where the graph of the function crosses the *x*-axis. In other words, it is the input value when the output value is zero.

To find the *x*-intercept, set a function $f(x)$ equal to zero and solve for the value of x. For example, consider the function shown.

$$f(x) = 3x - 6$$

Set the function equal to 0 and solve for x.

$$\begin{aligned} 0 &= 3x - 6 \\ 6 &= 3x \\ 2 &= x \\ x &= 2 \end{aligned}$$

The graph of the function crosses the *x*-axis at the point $(2, 0)$.

Q&A

Do all linear functions have *x*-intercepts?

No. However, linear functions of the form $y = c$, where c is a nonzero real number are the only examples of linear functions with no x-intercept. For example, $y = 5$ is a horizontal line 5 units above the x-axis. This function has no x-intercepts, as shown in Figure 11.

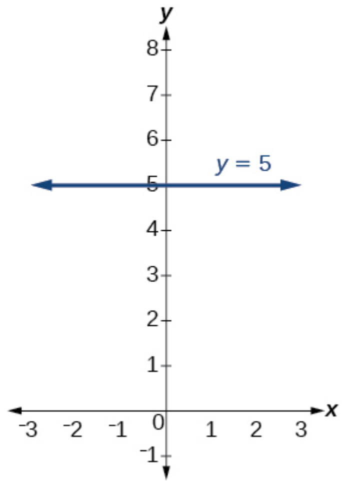


Figure 11

x-intercept

The x-intercept of the function is value of x when $f(x)=0$. It can be solved by the equation $0=mx+b$.

Example 5

Finding an x-intercept

Find the x-intercept of $f(x)=\frac{1}{2}x-3$.

Answer

Set the function equal to zero to solve for x.

$$
\begin{aligned}
0&=\tfrac{1}{2}x-3\\
3&=\tfrac{1}{2}x\\
6&=x\\
x&=6
\end{aligned}
$$

The graph crosses the x-axis at the point $(6,0)$.

Analysis

A graph of the function is shown in Figure 12. We can see that the x-intercept is $(6,0)$ as we expected.

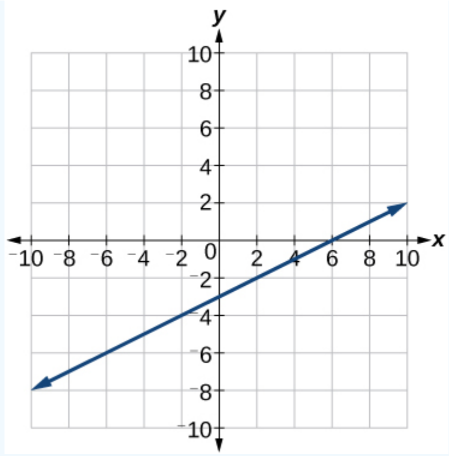


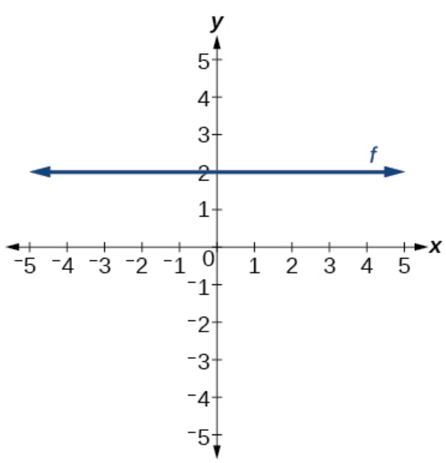
Figure 12 The graph of the linear function $f(x) = \frac{1}{2}x - 3$.

Try It #4

Find the *x*-intercept of $f(x) = \frac{1}{4}x - 4$.

Describing Horizontal and Vertical Lines

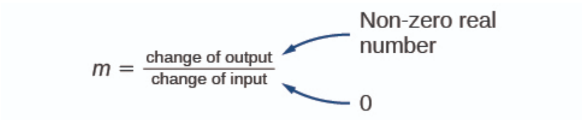
There are two special cases of lines on a graph—horizontal and vertical lines. A **horizontal line** indicates a constant output, or *y*-value. In Figure 13, we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m = 0$ in the equation $f(x) = mx + b$, the equation simplifies to $f(x) = b$. In other words, the value of the function is a constant. This graph represents the function $f(x) = 2$.



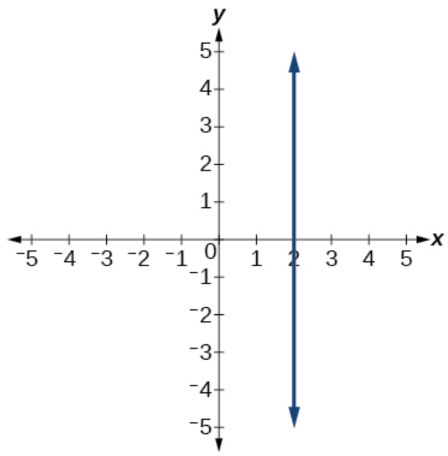
x	−4	−2	0	2	4
y	2	2	2	2	2

Figure 13 A horizontal line representing the function $f(x) = 2$.

A **vertical line** indicates a constant input, or *x*-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.



Notice that a vertical line, such as the one in Figure 14, has an x-intercept, but no y-intercept unless it's the line $x = 0$. This graph represents the line $x = 2$.



x	2	2	2	2	2
y	−4	−2	0	2	4

Figure 14 The vertical line, $x = 2$, which does not represent a function.

Horizontal and Vertical Lines

Lines can be horizontal or vertical.

A horizontal line is a line defined by an equation in the form $f(x) = b$.

A vertical line is a line defined by an equation in the form $x = a$.

Example 6

Writing the Equation of a Horizontal Line

Write the equation of the line graphed in Figure 15.

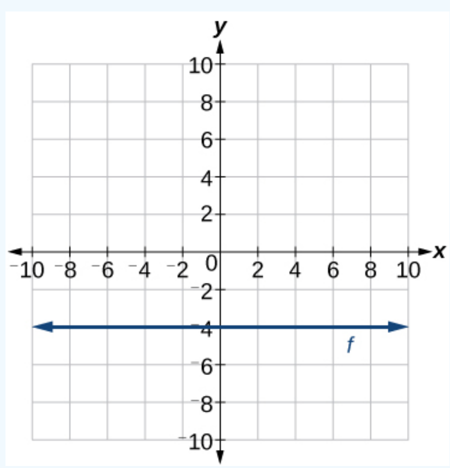


Figure 15

Answer

For any x-value, the y-value is -4, so the equation is $y = -4$.

Example 7

Writing the Equation of a Vertical Line

Write the equation of the line graphed in Figure 16.

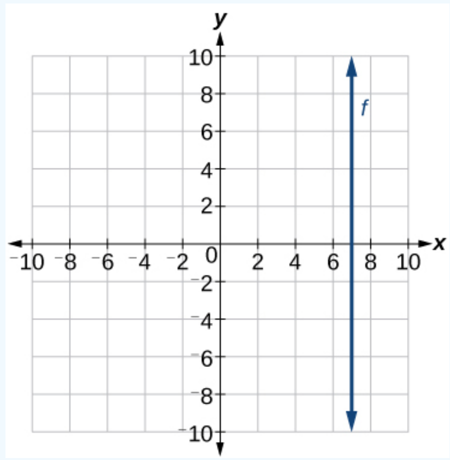


Figure 16

Answer

The constant x-value is 7, so the equation is $x = 7$.

Determining Whether Lines are Parallel or Perpendicular

The two lines in Figure 17 are **parallel lines**: they will never intersect. Notice that they have exactly the same steepness, which means their slopes are identical. The only difference between the two lines is the y-intercept. If we shifted one line vertically toward the y-intercept of the other, they would become the same line.

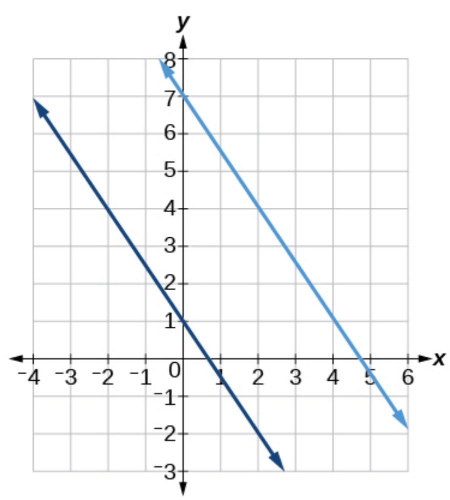


Figure 17 Parallel lines.

We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the y-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

$$\left.\begin{array}{l} f(x) = -2x + 6 \\ f(x) = -2x - 4 \end{array}\right\} \text{ parallel} \qquad \left.\begin{array}{l} f(x) = 3x + 2 \\ f(x) = 2x + 2 \end{array}\right\} \text{ not parallel}$$

Unlike parallel lines, **perpendicular lines** do intersect. Their intersection forms a right, or 90-degree, angle. The two lines in Figure 18 are perpendicular.

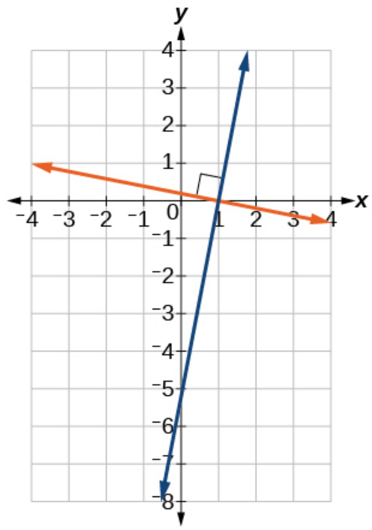


Figure 18 Perpendicular lines.

Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1. So, if m_1 and m_2 are negative reciprocals of one another, they can be multiplied together to yield –1.

$$m_1 m_2 = -1$$

To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is $\frac{1}{8}$, and the reciprocal of $\frac{1}{8}$ is 8. To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes, assuming that the lines are neither horizontal nor vertical. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.

$$\begin{array}{ll} f(x) = \frac{1}{4}x + 2 & \text{negative reciprocal of} \frac{1}{4} \text{is} -4 \\ f(x) = -4x + 3 & \text{negative reciprocal of} -4 \text{is} \frac{1}{4} \end{array}$$

The product of the slopes is –1.

$$-4\left(\frac{1}{4}\right) = -1$$

Parallel and Perpendicular Lines

Two lines are parallel lines if they do not intersect. The slopes of the lines are the same.

$$f(x) = m_1 x \quad + \quad b_1 \text{and} g(x) = m_2 x \quad + \quad b_2 \text{are parallel if} m_1 \quad = \quad m_2.$$

If and only if $b_1 = b_2$ and $m_1 = m_2$, we say the lines coincide. Coincident lines are the same line.

Two lines are perpendicular lines if they intersect at right angles.

$$f(x) = m_1 x + b_1 \text{and} g(x) = m_2 x + b_2 \text{are perpendicular if} m_1 m_2 = -1, \text{and so} m_2 = -\frac{1}{m_1}.$$

Example 8

Identifying Parallel and Perpendicular Lines

Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.

$$\begin{array}{ll} f(x) = 2x + 3 & h(x) = -2x + 2 \\ g(x) = \frac{1}{2}x - 4 & j(x) = 2x - 6 \end{array}$$

Answer

Parallel lines have the same slope. Because the functions $f(x) = 2x + 3$ and $j(x) = 2x - 6$ each have a slope of 2, they represent parallel lines. Perpendicular lines have negative reciprocal slopes. Because −2 and $\frac{1}{2}$ are negative reciprocals, the equations, $g(x) = \frac{1}{2}x - 4$ and $h(x) = -2x + 2$ represent perpendicular lines.

Analysis

A graph of the lines is shown in Figure 19.

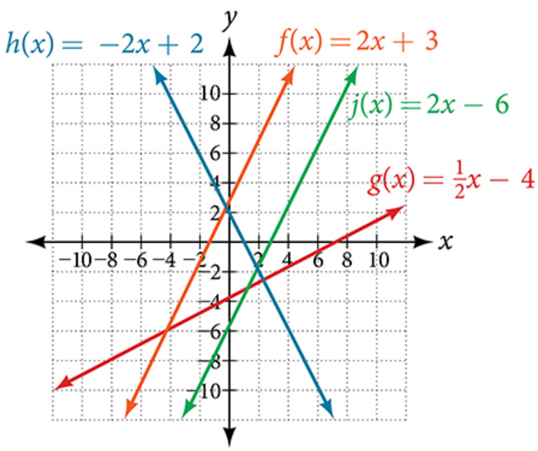


Figure 19

The graph shows that the lines $f(x) = 2x + 3$ and $j(x) = 2x - 6$ are parallel, and the lines $g(x) = \frac{1}{2}x - 4$ and $h(x) = -2x + 2$ are perpendicular.

Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Writing Equations of Parallel Lines

Suppose for example, we are given the following equation.

$$f(x) = 3x + 1$$

We know that the slope of the line formed by the function is 3. We also know that the *y*-intercept is $(0, 1)$. Any other line with a slope of 3 will be parallel to $f(x)$. So the lines formed by all of the following functions will be parallel to $f(x)$.

$$g(x) = 3x + 6$$
$$h(x) = 3x + 1$$
$$p(x) = 3x + \frac{2}{3}$$

Suppose then we want to write the equation of a line that is parallel to f and passes through the point $(1, 7)$. We already know that the slope is 3. We just need to determine which value for b will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 7 &= 3(x - 1) \\ y - 7 &= 3x - 3 \\ y &= 3x + 4 \end{aligned}$$

So $g(x) = 3x + 4$ is parallel to $f(x) = 3x + 1$ and passes through the point $(1, 7)$.

How To

Given the equation of a function and a point through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.
3. Simplify.

Example 9

Finding a Line Parallel to a Given Line

Find a line parallel to the graph of $f(x) = 3x + 6$ that passes through the point $(3, 0)$.

Answer

The slope of the given line is 3. If we choose the slope-intercept form, we can substitute $m = 3$, $x = 3$, and $f(x) = 0$ into the slope-intercept form to find the *y*-intercept.

$$\begin{aligned} g(x) &= 3x + b \\ 0 &= 3(3) + b \\ b &= -9 \end{aligned}$$

The line parallel to $f(x)$ that passes through $(3, 0)$ is $g(x) = 3x - 9$.

Analysis

We can confirm that the two lines are parallel by graphing them. Figure 20 shows that the two lines will never intersect.

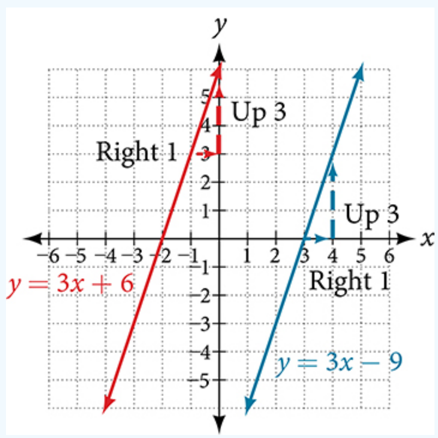


Figure 20

Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the following function:

$$f(x) = 2x + 4$$

The slope of the line is 2, and its negative reciprocal is $-\frac{1}{2}$. Any function with a slope of $-\frac{1}{2}$ will be perpendicular to $f(x)$. So the lines formed by all of the following functions will be perpendicular to $f(x)$.

$$\begin{aligned} g(x) &= -\tfrac{1}{2}x + 4 \\ h(x) &= -\tfrac{1}{2}x + 2 \\ p(x) &= -\tfrac{1}{2}x - \tfrac{1}{2} \end{aligned}$$

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to $f(x)$ and passes through the point $(4, 0)$. We already know that the slope is $-\frac{1}{2}$. Now we can use the point to find the y-intercept by substituting the given values into the slope-intercept form of a line and solving for b.

$$\begin{aligned} g(x) &= mx+b \\ 0 &= -\tfrac{1}{2}(4)+b \\ 0 &= -2+b \\ 2 &= b \\ b &= 2 \end{aligned}$$

The equation for the function with a slope of $-\frac{1}{2}$ and a y-intercept of 2 is

$$g(x) = -\frac{1}{2}x+2.$$

So $g(x) = -\frac{1}{2}x+2$ is perpendicular to $f(x) = 2x+4$ and passes through the point $(4,0)$. Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

Q&A

A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not –1. Doesn't this fact contradict the definition of perpendicular lines?

No. For two perpendicular linear functions, the product of their slopes is –1. However, a vertical line is not a function so the definition is not contradicted.

How To

Given the equation of a function and a point through which its graph passes, write the equation of a line perpendicular to the given line.

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for x and y from the coordinate pair provided into $g(x) = mx+b$.
4. Solve for b.
5. Write the equation for the line.

Example 10

Finding the Equation of a Perpendicular Line

Find the equation of a line perpendicular to $f(x) = 3x+3$ that passes through the point $(3,0)$.

Answer

The original line has slope $m=3$, so the slope of the perpendicular line will be its negative reciprocal, or $-\frac{1}{3}$. Using this slope and the given point, we can find the equation for the line.

$$\begin{aligned} g(x) &= -\tfrac{1}{3}x+b \\ 0 &= -\tfrac{1}{3}(3)+b \\ 1 &= b \\ b &= 1 \end{aligned}$$

The line perpendicular to $f(x)$ that passes through $(3,0)$ is $g(x) = -\frac{1}{3}x+1$.

Analysis

A graph of the two lines is shown in Figure 21 below.

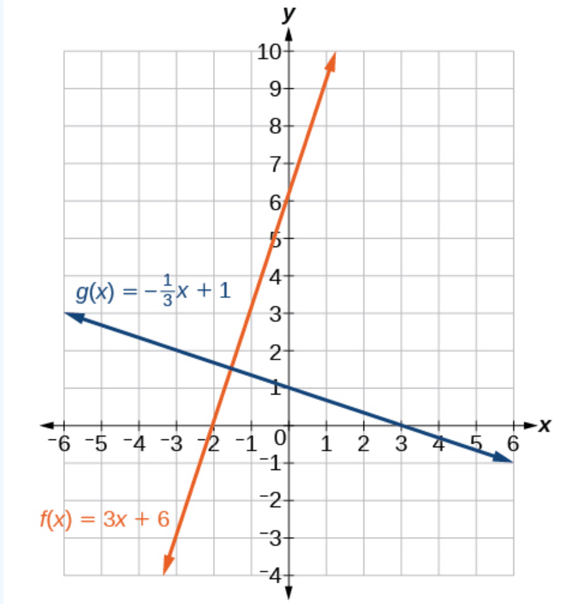


Figure 21

Try It #5

Given the function $h(x) = 2x - 4,$ write an equation for the line passing through $(0, 0)$ that is

1. ⓐ parallel to $h(x)$
2. ⓑ perpendicular to $h(x)$

How To

Given two points on a line and a third point, write the equation of the perpendicular line that passes through the point.

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Use the slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

Example 11

Finding the Equation of a Line Perpendicular to a Given Line Passing through a Point

A line passes through the points $(-2, 6)$ and $(4, 5)$. Find the equation of a perpendicular line that passes through the point $(4, 5)$.

Answer

From the two points of the given line, we can calculate the slope of that line.

$$\begin{aligned} m_1 &= \frac{5-6}{4-(-2)} \\ &= \frac{-1}{6} \\ &= -\frac{1}{6} \end{aligned}$$

Find the negative reciprocal of the slope.

$$\begin{aligned} m_2 &= \frac{-1}{-\frac{1}{6}} \\ &= -1\left(-\frac{6}{1}\right) \\ &= 6 \end{aligned}$$

We can then solve for the *y*-intercept of the line passing through the point $(4,5)$.

$$\begin{aligned} g(x) &= 6x+b \\ 5 &= 6(4)+b \\ 5 &= 24+b \\ -19 &= b \\ b &= -19 \end{aligned}$$

The equation for the line that is perpendicular to the line passing through the two given points and also passes through point $(4,5)$ is

$$y = 6x - 19$$

Try It #6

A line passes through the points, $(-2,-15)$ and $(2,-3)$. Find the equation of a perpendicular line that passes through the point, $(6,4)$.

Solving a System of Linear Equations Using a Graph

A system of linear equations includes two or more linear equations. The graphs of two lines will intersect at a single point if they are not parallel. Two parallel lines can also intersect if they are coincident, which means they are the same line and they intersect at every point. For two lines that are not parallel, the single point of intersection will satisfy both equations and therefore represent the solution to the system.

To find this point when the equations are given as functions, we can solve for an input value so that $f(x)=g(x)$. In other words, we can set the formulas for the lines equal to one another, and solve for the input that satisfies the equation.

Example 12

Finding a Point of Intersection Algebraically

Find the point of intersection of the lines $h(t)=3t-4$ and $j(t)=5-t$.

Answer

Set $h(t)=j(t)$.

$$\begin{aligned} 3t-4 &= 5-t \\ 4t &= 9 \\ t &= \frac{9}{4} \end{aligned}$$

This tells us the lines intersect when the input is $\frac{9}{4}$.

We can then find the output value of the intersection point by evaluating either function at this input.

$$\begin{aligned} j\left(\frac{9}{4}\right) &= 5-\frac{9}{4} \\ &= \frac{11}{4} \end{aligned}$$

These lines intersect at the point $\left(\frac{9}{4},\frac{11}{4}\right)$.

Analysis

Looking at Figure 22, this result seems reasonable.

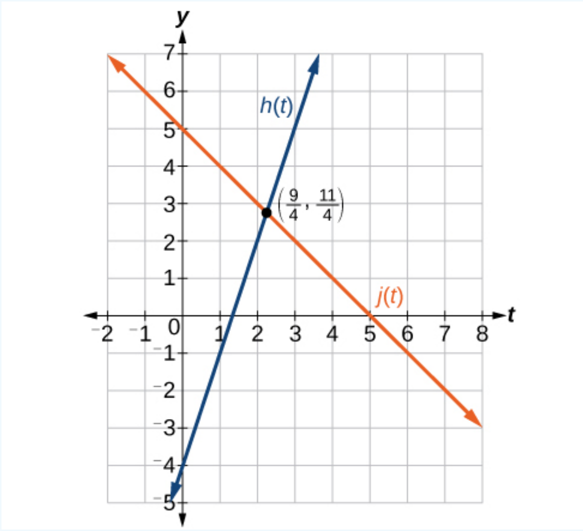


Figure 22

Q&A

If we were asked to find the point of intersection of two distinct parallel lines, should something in the solution process alert us to the fact that there are no solutions?

Yes. After setting the two equations equal to one another, the result would be the contradiction "0 = non-zero real number".

Try It #7

Look at the graph in Figure 22 and identify the following for the function $j(t)$:

1. ⓐ *y*-intercept
2. ⓑ *x*-intercept(s)
3. ⓒ slope
4. ⓓ Is $j(t)$ parallel or perpendicular to $h(t)$ (or neither)?
5. ⓔ Is $j(t)$ an increasing or decreasing function (or neither)?
6. ⓕ Write a transformation description for $j(t)$ from the identity toolkit function $f(x) = x$.

Example 13

Finding a Break-Even Point

A company sells sports helmets. The company incurs a one-time fixed cost for $250,000. Each helmet costs $120 to produce, and sells for $140.

1. ⓐ Find the cost function, C, to produce x helmets, in dollars.
2. ⓑ Find the revenue function, R, from the sales of x helmets, in dollars.
3. ⓒ Find the break-even point, the point of intersection of the two graphs C and R.

Answer

1. ⓐ The cost function in the sum of the fixed cost, $125,000, and the variable cost, $120 per helmet.

$$C(x) = 120x + 250,000$$

2. ⓑ The revenue function is the total revenue from the sale of x helmets, $R(x) = 140x$.

3.

$$\begin{aligned}&C(x)=R(x)\\&250,000+120x=140x\\&250,000=20x\\&12,500=x\\&x=12,500\end{aligned}$$

$$\begin{aligned}R(x)&=140(12,500)\\&=\$1,750,000\end{aligned}$$

The break-even point is $(12{,}500{,}1{,}750{,}000)$.

Analysis

This means if the company sells 12,500 helmets, they break even; both the sales and cost incurred equaled 1.75 million dollars. See Figure 23

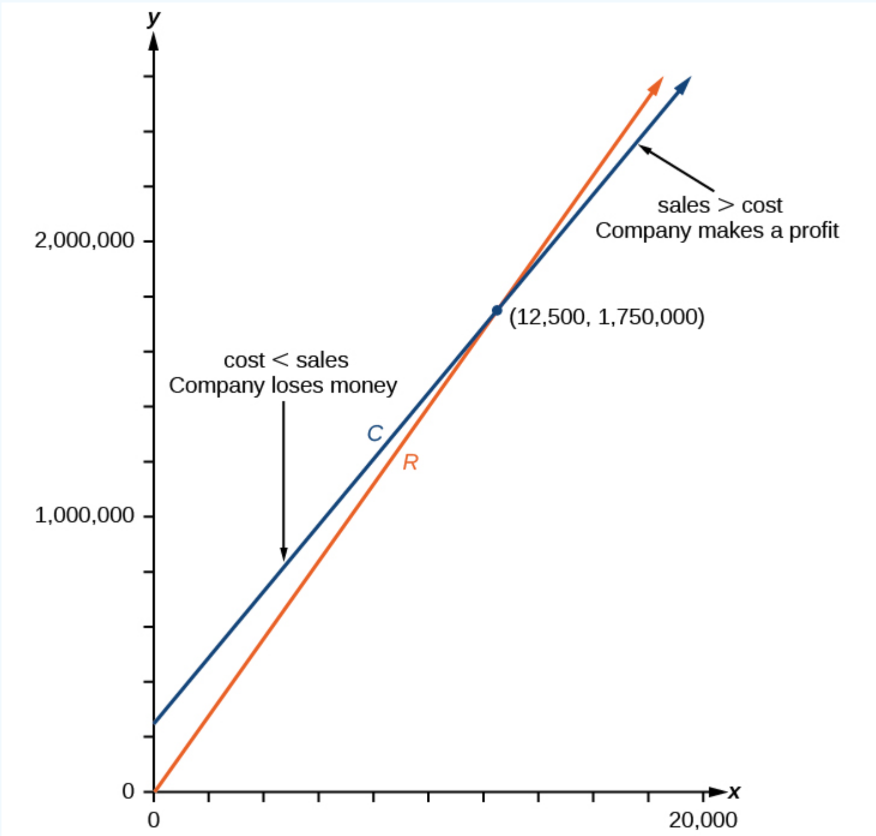


Figure 23

Media

Access these online resources for additional instruction and practice with graphs of linear functions.

- Finding Input of Function from the Output and Graph
- Graphing Functions using Tables

2.2 Section Exercises

Verbal

1.

If the graphs of two linear functions are parallel, describe the relationship between the slopes and the *y*-intercepts.

2.

If the graphs of two linear functions are perpendicular, describe the relationship between the slopes and the *y*-intercepts.

3.

If a horizontal line has the equation $f(x) = a$ and a vertical line has the equation $x = a$, what is the point of intersection? Explain why what you found is the point of intersection.

4.

Explain how to find a line parallel to a linear function that passes through a given point.

5.

Explain how to find a line perpendicular to a linear function that passes through a given point.

Algebraic

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

6.

$4x - 7y = 10$
$7x + 4y = 1$

7.

$3y + x = 12$
$-y = 8x + 1$

8.

$3y + 4x = 12$
$-6y = 8x + 1$

9.

$6x - 9y = 10$
$3x + 2y = 1$

10.

$y = \frac{2}{3}x + 1$
$3x + 2y = 1$

11.

$y = \frac{3}{4}x + 1$
$-3x + 4y = 1$

For the following exercises, find the *x*- and *y*-intercepts of each equation

12.

$f(x) = -x + 2$

13.

$g(x) = 2x + 4$

14.

$h(x) = 3x - 5$

15.

$k(x) = -5x + 1$

16.

$-2x + 5y = 20$

17.

$7x+2y=56$

For the following exercises, use the descriptions of each pair of lines given below to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

18.

- Line 1: Passes through $(0,6)$ and $(3,-24)$
- Line 2: Passes through $(-1,19)$ and $(8,-71)$

19.

- Line 1: Passes through $(-8,-55)$ and $(10, \quad 89)$
- Line 2: Passes through $(9,-44)$ and $(4,-14)$

20.

- Line 1: Passes through $(2,3)$ and $(4,-1)$
- Line 2: Passes through $(6,3)$ and $(8,5)$

21.

- Line 1: Passes through $(1,7)$ and $(5,5)$
- Line 2: Passes through $(-1,-3)$ and $(1,1)$

22.

- Line 1: Passes through $(0,5)$ and $(3,3)$
- Line 2: Passes through $(1,-5)$ and $(3,-2)$

23.

- Line 1: Passes through $(2,5)$ and $(5,-1)$
- Line 2: Passes through $(-3,7)$ and $(3,-5)$

24.

Write an equation for a line parallel to $f(x)=-5x-3$ and passing through the point $(2,-12)$.

25.

Write an equation for a line parallel to $g(x)=3x-1$ and passing through the point $(4,9)$.

26.

Write an equation for a line perpendicular to $h(t)=-2t+4$ and passing through the point (-4, -1).

27.

Write an equation for a line perpendicular to $p(t)=3t+4$ and passing through the point $(3,1)$.

28.

Find the point at which the line $f(x)=-2x-1$ intersects the line $g(x)=-x$.

29.

Find the point at which the line $f(x)=2x+5$ intersects the line $g(x)=-3x-5$.

30.

Use algebra to find the point at which the line $f(x)= \quad -\frac{4}{5}x \quad +\frac{274}{25}$ intersects the line $h(x)=\frac{9}{4}x \quad + \quad \frac{73}{10}$.

31.

Use algebra to find the point at which the line $f(x)=\frac{7}{4}x \quad + \quad \frac{457}{60}$ intersects the line $g(x)=\frac{4}{3}x \quad + \quad \frac{31}{5}$.

Graphical

For the following exercises, match the given linear equation with its graph in Figure 24.

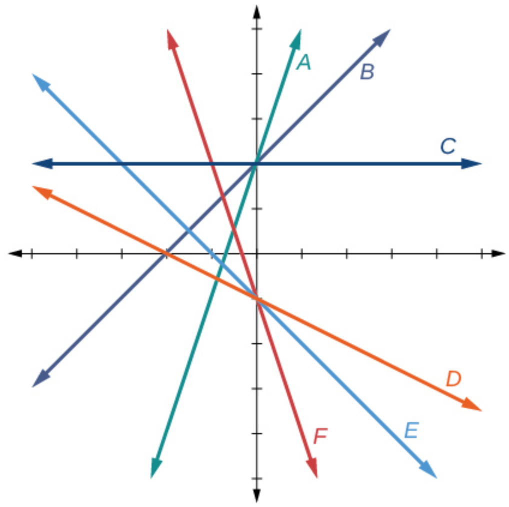


Figure 24

32.

$f\left(x\right)=-x-1$

33.

$f\left(x\right)=-2x-1$

34.

$f\left(x\right)=-\frac{1}{2}x-1$

35.

$f\left(x\right)=2$

36.

$f\left(x\right)=2+x$

37.

$f\left(x\right)=3x+2$

For the following exercises, sketch a line with the given features.

38.

An *x*-intercept of $(-4,0)$ and *y*-intercept of $(0,-2)$

39.

An *x*-intercept of $(-2,0)$ and *y*-intercept of $(0,4)$

40.

A *y*-intercept of $(0,7)$ and slope $-\frac{3}{2}$

41.

A *y*-intercept of $(0,3)$ and slope $\frac{2}{5}$

42.

Passing through the points $(-6,-2)$ and $(6,-6)$

43.

Passing through the points $(-3, -4)$ and $(3, 0)$

For the following exercises, sketch the graph of each equation.

44.

$f(x) = -2x - 1$

45.

$g(x) = -3x + 2$

46.

$h(x) = \frac{1}{3}x + 2$

47.

$k(x) = \frac{2}{3}x - 3$

48.

$f(t) = 3 + 2t$

49.

$p(t) = -2 \quad + \quad 3t$

50.

$x = 3$

51.

$x = -2$

52.

$r(x) = 4$

53.

$q(x) = 3$

54.

$4x = -9y + 36$

55.

$\frac{x}{3} - \frac{y}{4} = 1$

56.

$3x - 5y = 15$

57.

$3x = 15$

58.

$3y = 12$

59.

If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $\frac{3}{4}$, a shift right by 2, and a shift down by 4

1. ⓐ Write an equation for $g(x)$.
2. ⓑ What is the slope of this line?
3. ⓒ Find the *y*-intercept of this line.

60.

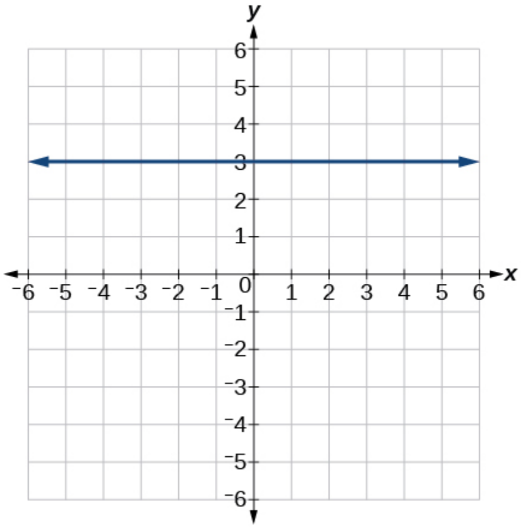
If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $\frac{1}{3}$, a shift left by 1, and a shift up by 3

1. ⓐ Write an equation for $g(x)$.

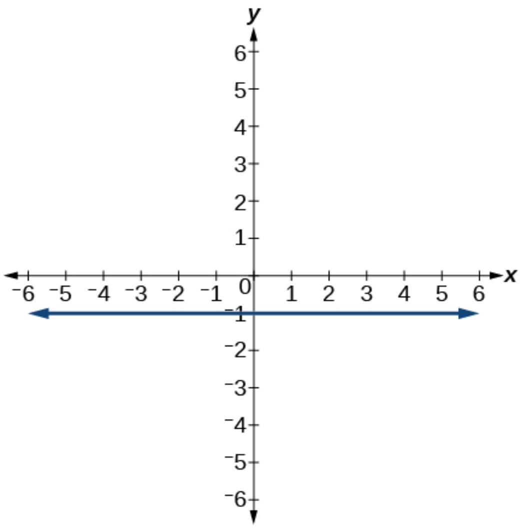
2. ⓑ What is the slope of this line?
3. ⓒ Find the y-intercept of this line.

For the following exercises,, write the equation of the line shown in the graph.

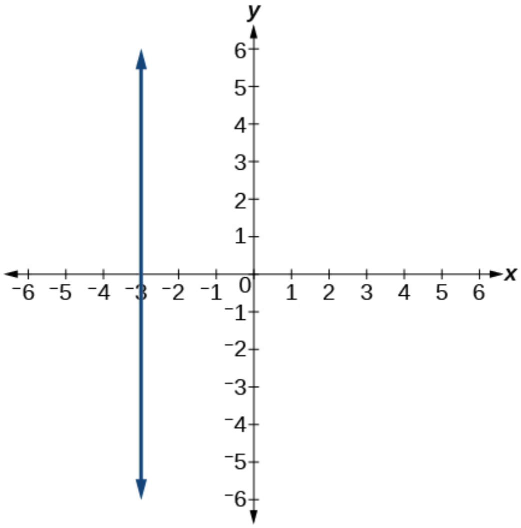
61.



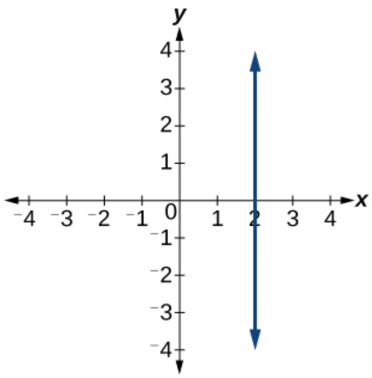
62.



63.



64.



For the following exercises, find the point of intersection of each pair of lines if it exists. If it does not exist, indicate that there is no point of intersection.

65.

$$\begin{aligned} y &= \tfrac{3}{4}x + 1 \\ -3x + 4y &= 12 \end{aligned}$$

66.

$$\begin{aligned} 2x - 3y &= 12 \\ 5y + x &= 30 \end{aligned}$$

67.

$$\begin{aligned} 2x &= y - 3 \\ y + 4x &= 15 \end{aligned}$$

68.

$$\begin{aligned} x - 2y + 2 &= 3 \\ x - y &= 3 \end{aligned}$$

69.

$$\begin{aligned} 5x + 3y &= -65 \\ x - y &= -5 \end{aligned}$$

Extensions

70.

Find the equation of the line parallel to the line $g(x) = -0.01x + 2.01$ through the point $(1, 2)$.

71.

Find the equation of the line perpendicular to the line $g(x) = -0.01x+2.01$ through the point $(1, 2)$.

For the following exercises, use the functions $f(x) = -0.1x+200$ and $g(x) = 20x + 0.1$.

72.

Find the point of intersection of the lines f and g.

73.

Where is $f(x)$ greater than $g(x)$? Where is $g(x)$ greater than $f(x)$?

Real-World Applications

74.

A car rental company offers two plans for renting a car.

- Plan A: $30 per day and $0.18 per mile
- Plan B: $50 per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

75.

A cell phone company offers two plans for minutes.

- Plan A: $20 per month and $1 for every one hundred texts.
- Plan B: $50 per month with free unlimited texts.

How many texts would you need to send per month for plan B to save you money?

76.

A cell phone company offers two plans for minutes.

- Plan A: $15 per month and $2 for every 300 texts.
- Plan B: $25 per month and $0.50 for every 100 texts.

How many texts would you need to send per month for plan B to save you money?

1.5: Modeling with Linear Functions

Learning Objectives

In this section, you will:

- Identify steps for modeling and solving.
- Build linear models from verbal descriptions.
- Build systems of linear models.

Figure 1 (credit: EEK Photography/Flickr)

Emily is a college student who plans to spend a summer in Seattle. She has saved \$3,500 for her trip and anticipates spending \$400 each week on rent, food, and activities. How can we write a linear model to represent her situation? What would be the x-intercept, and what can she learn from it? To answer these and related questions, we can create a model using a linear function. Models such as this one can be extremely useful for analyzing relationships and making predictions based on those relationships. In this section, we will explore examples of linear function models.

Identifying Steps to Model and Solve Problems

When modeling scenarios with linear functions and solving problems involving quantities with a constant rate of change, we typically follow the same problem strategies that we would use for any type of function. Let's briefly review them:

1. Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
3. Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
5. When needed, write a formula for the function.
6. Solve or evaluate the function using the formula.

7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Building Linear Models

Now let's take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.

- Output: M, money remaining, in dollars
- Input: t, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$

We can also identify the initial value and the rate of change.

- Initial Value: She saved \$3,500, so \$3,500 is the initial value for M.
- Rate of Change: She anticipates spending \$400 each week, so –\$400 per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The rate of change is constant, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = \underbrace{m}_{-400}t + \underbrace{b}_{3500}$$

$$M(t) = -400t + 3500$$

To find the x- intercept, we set the output to zero, and solve for the input.

$$\begin{aligned} 0 &= -400t + 3500 \\ t &= \tfrac{3500}{400} \\ &= 8.75 \end{aligned}$$

The x- intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved \$3,500, but the scenario discussed poses the question once she saved \$3,500 because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the x- intercept, unless Emily will use a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \leq t \leq 8.75$.

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Using a Given Intercept to Build a Model

Some real-world problems provide the y- intercept, which is the constant or initial value. Once the y- intercept is known, the x- intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is \$1,000. She plans to pay \$250 per month until her balance is \$0. The y- intercept is the initial amount of her debt, or \$1,000. The rate of change, or slope, is -\$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$\begin{aligned} f(x) &= mx + b \\ &= -250x + 1000 \end{aligned}$$

Now we can set the function equal to 0, and solve for x to find the x- intercept.

$$\begin{aligned} 0 &= -250x + 1000 \\ 1000 &= 250x \\ 4 &= x \\ x &= 4 \end{aligned}$$

The x- intercept is the number of months it takes her to reach a balance of $0. The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

How To

Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given x- value.
6. Use the model to identify an x- value that results in a given y- value.
7. Answer the question posed.

Example 1

Using a Linear Model to Investigate a Town’s Population

A town’s population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. Assume this trend continues.

1. ⓐ Predict the population in 2013.
2. ⓑ Identify the year in which the population will reach 15,000.

Answer

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the y- intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004:

- Input: t, years since 2004
- Output: $P(t)$, the town’s population

To predict the population in 2013 $(t=9)$, we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

$$m = \frac{\text{change in output}}{\text{change in input}}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t=0$, giving the point $(0,6200)$. Notice that through our clever choice of variable definition, we have “given” ourselves the y-intercept of the function. The year 2009 would correspond to $t=5$, giving the point $(5,8100)$.

The two coordinate pairs are $(0, 6200)$and $(5, 8100)$ .Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

$$\begin{aligned} m &= \frac{8100-6200}{5-0} \\ &= \frac{1900}{5} \\ &= 380 \text{people per year} \end{aligned}$$

We already know the *y*-intercept of the line, so we can immediately write the equation:

$$P(t) = 380t + 6200$$

To predict the population in 2013, we evaluate our function at $t = 9$.

$$\begin{aligned} P(9) &= 380(9) + 6,200 \\ &= 9,620 \end{aligned}$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$\begin{aligned} 15000 &= 380t + 6200 \\ 8800 &= 380t \\ t &\approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Try It #1

A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

1. ⓐ Write a linear model to represent the cost C of the company as a function of x, the number of doughnuts produced.
2. ⓑ Find and interpret the *y*-intercept.

Try It #2

A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

1. ⓐ Predict the population in 2014.
2. ⓑ Identify the year in which the population will reach 54,000.

Using a Diagram to Model a Problem

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

Example 2

Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

Answer

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question:

"How long will it take them to be 2 miles apart?"

In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.

- Input: t, time in hours.
- Output: $A(t)$, distance in miles, and $E(t)$, distance in miles

Because it is not obvious how to define our output variable, we'll start by drawing a picture such as Figure 2.

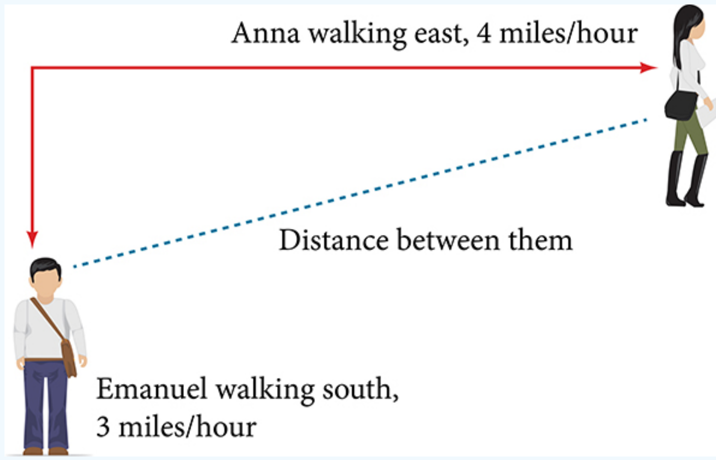


Figure 2

Initial Value: They both start at the same intersection so when $t=0$, the distance traveled by each person should also be 0. Thus the initial value for each is 0.

Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for A is 4 and the slope for E is 3.

Using those values, we can write formulas for the distance each person has walked.

$$A(t)=4t$$
$$E(t)=3t$$

For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the "starting point" at the intersection where they both started. Then we can use the variable, A, which we introduced above, to represent Anna's position, and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.

We can then define a third variable, D, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful, as we can see from Figure 3.

Recall that we need to know how long it takes for D, the distance between them, to equal 2 miles. Notice that for any given input t, the outputs $A(t)$, $E(t)$, and $D(t)$ represent distances.

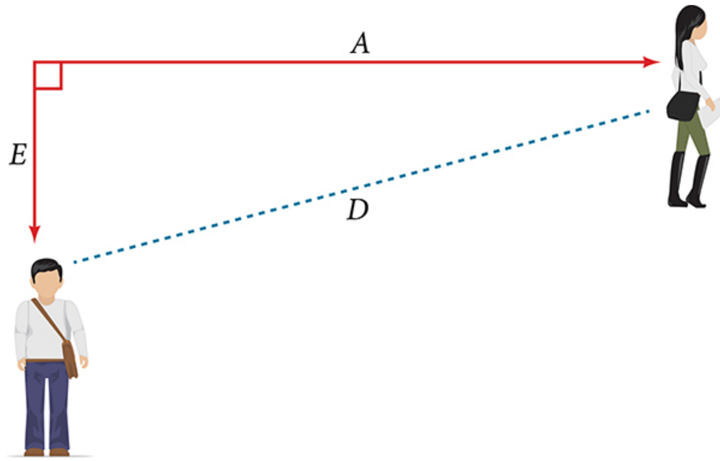


Figure 3

Figure 2 shows us that we can use the Pythagorean Theorem because we have drawn a right angle.

Using the Pythagorean Theorem, we get:

$$\begin{aligned} D(t)^2 &= A(t)^2 + E(t)^2 \\ &= (4t)^2 + (3t)^2 \\ &= 16t^2 + 9t^2 \\ &= 25t^2 \\ D(t) &= \pm\sqrt{25t^2} && \text{Solve for} D(t) \text{using the square root} \\ &= \pm 5|t| \end{aligned}$$

In this scenario we are considering only positive values of t, so our distance $D(t)$ will always be positive. We can simplify this answer to $D(t) = 5t$. This means that the distance between Anna and Emanuel is also a linear function. Because D is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output $D(t) = 2$ and solve for t.

$$\begin{aligned} D(t) &= 2 \\ 5t &= 2 \\ t &= \tfrac{2}{5} = 0.4 \end{aligned}$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

Q&A

Should I draw diagrams when given information based on a geometric shape?

Yes. Sketch the figure and label the quantities and unknowns on the sketch.

Example 3

Using a Diagram to Model Distance between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of

Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Answer

It might help here to draw a picture of the situation. See Figure 4. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates $(30, 10)$,and Eastborough at $(20, 0)$.

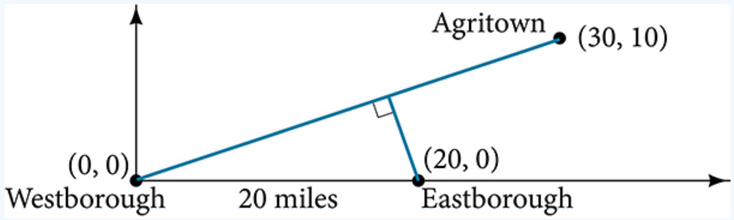


Figure 4

Using this point along with the origin, we can find the slope of the line from Westborough to Agritown:

$$m = \frac{10-0}{30-0} = \frac{1}{3}$$

The equation of the road from Westborough to Agritown would be

$$W(x) = \frac{1}{3}x$$

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Because the town of Eastborough is at the point (20, 0), we can find the equation:

$$\begin{aligned} E(x) &= -3x + b \\ 0 &= -3(20) + b \quad \text{Substitute in } (20, 0) \\ b &= 60 \\ E(x) &= -3x + 60 \end{aligned}$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\begin{aligned} \tfrac{1}{3}x &= -3x + 60 \\ \tfrac{10}{3}x &= 60 \\ 10x &= 180 \\ x &= 18 \qquad \text{Substituting this back into} W(x) \\ y &= W(18) \\ &= \tfrac{1}{3}(18) \\ &= 6 \end{aligned}$$

The roads intersect at the point (18, 6). Using the distance formula, we can now find the distance from Westborough to the junction.

$$\begin{aligned} \text{distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(18-0)^2 + (6-0)^2} \\ &\approx 18.974 \text{miles} \end{aligned}$$

Analysis

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

Try It #3

There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

Building Systems of Linear Models

Real-world situations including two or more linear functions may be modeled with a system of linear equations. Remember, when solving a system of linear equations, we are looking for points the two lines have in common. Typically, there are three types of answers possible, as shown in Figure 5.

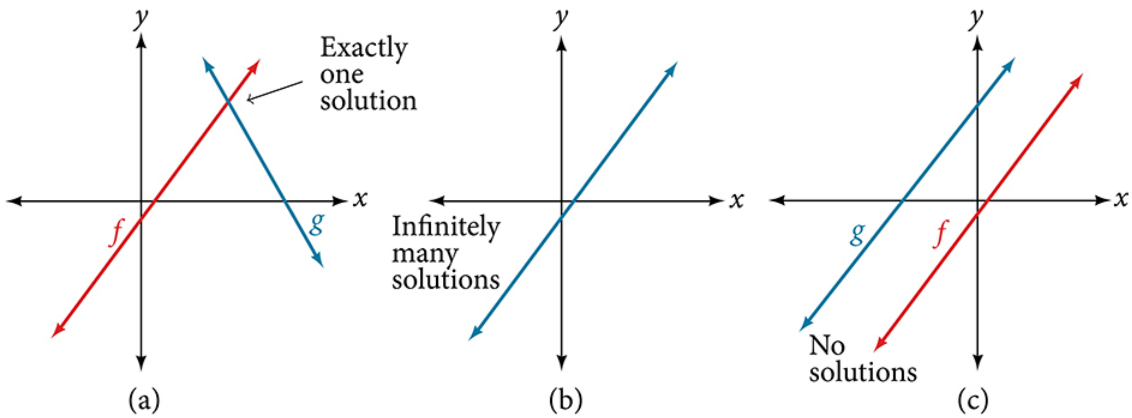


Figure 5

How To

Given a situation that represents a system of linear equations, write the system of equations and identify the solution.

1. Identify the input and output of each linear model.
2. Identify the slope and *y*-intercept of each linear model.
3. Find the solution by setting the two linear functions equal to one another and solving for x, or find the point of intersection on a graph.

Example 4

Building a System of Linear Models to Choose a Truck Rental Company

Jamal is choosing between two truck-rental companies. The first, Keep on Trucking, Inc., charges an up-front fee of $20, then 59 cents a mile. The second, Move It Your Way, charges an up-front fee of $16, then 63 cents a mile[4]. When will Keep on Trucking, Inc. be the better choice for Jamal?

Answer

The two important quantities in this problem are the cost and the number of miles driven. Because we have two companies to consider, we will define two functions.

Input	d, distance driven in miles
Outputs	$K(d)$: cost, in dollars, for renting from Keep on Trucking $M(d)$ cost, in dollars, for renting from Move It Your Way
Initial Value	Up-front fee: $K(0)=20$ and $M(0)=16$
Rate of Change	$K(d)=\$0.59$ /mile and $P(d)=\$0.63$ /mile

Table 1

A linear function is of the form $f(x)=mx+b$. Using the rates of change and initial charges, we can write the equations

$$K(d) = 0.59d + 20$$
$$M(d) = 0.63d + 16$$

Using these equations, we can determine when Keep on Trucking, Inc., will be the better choice. Because all we have to make that decision from is the costs, we are looking for when Move It Your Way, will cost less, or when $K(d) < M(d)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, and then see where the $K(d)$ function is smaller.

These graphs are sketched in Figure 6, with $K(d)$ in blue.

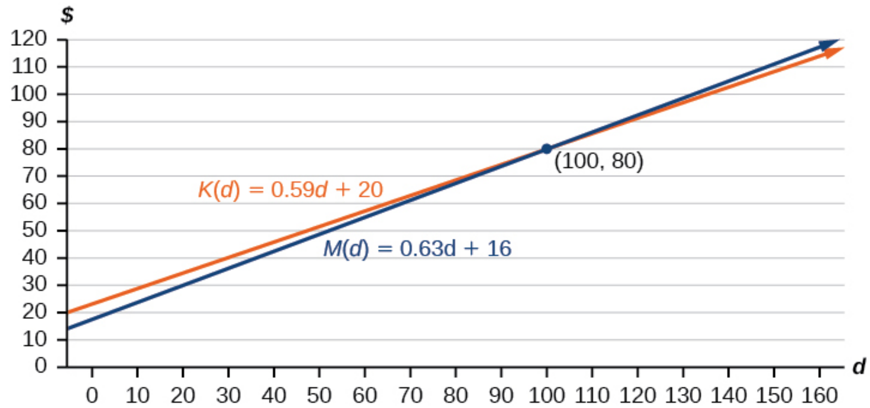


Figure 6

To find the intersection, we set the equations equal and solve:

$$\begin{aligned} K(d) &= M(d) \\ 0.59d + 20 &= 0.63d + 16 \\ 4 &= 0.04d \\ 100 &= d \\ d &= 100 \end{aligned}$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $K(d)$ is growing at a slower rate, we can conclude that Keep on Trucking, Inc. will be the cheaper price when more than 100 miles are driven, that is $d > 100$.

Media

Access this online resource for additional instruction and practice with linear function models.

- Interpreting a Linear Function

2.3 Section Exercises

Verbal

1.

Explain how to find the input variable in a word problem that uses a linear function.

2.

Explain how to find the output variable in a word problem that uses a linear function.

3.

Explain how to interpret the initial value in a word problem that uses a linear function.

4.

Explain how to determine the slope in a word problem that uses a linear function.

Algebraic

5.

Find the area of a parallelogram bounded by the *y*-axis, the line $x=3$, the line $f(x)=1+2x$, and the line parallel to $f(x)$ passing through $(2,7)$.

6.

Find the area of a triangle bounded by the *x*-axis, the line $f(x)=12-\frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.

7.

Find the area of a triangle bounded by the *y*-axis, the line $f(x)=9-\frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.

8.

Find the area of a parallelogram bounded by the *x*-axis, the line $g(x)=2$, the line $f(x)=3x$, and the line parallel to $f(x)$ passing through $(6,1)$.

For the following exercises, consider this scenario: A town's population has been decreasing at a constant rate. In 2010 the population was 5,900. By 2012 the population had dropped 4,700. Assume this trend continues.

9.

Predict the population in 2016.

10.

Identify the year in which the population will reach 0.

For the following exercises, consider this scenario: A town's population has been increased at a constant rate. In 2010 the population was 46,020. By 2012 the population had increased to 52,070. Assume this trend continues.

11.

Predict the population in 2016.

12.

Identify the year in which the population will reach 75,000.

For the following exercises, consider this scenario: A town has an initial population of 75,000. It grows at a constant rate of 2,500 per year for 5 years.

13.

Find the linear function that models the town's population P as a function of the year, t, where t is the number of years since the model began.

14.

Find a reasonable domain and range for the function P.

15.

If the function P is graphed, find and interpret the *x*- and *y*-intercepts.

16.

If the function P is graphed, find and interpret the slope of the function.

17.

When will the output reached 100,000?

18.

What is the output in the year 12 years from the onset of the model?

For the following exercises, consider this scenario: The weight of a newborn is 7.5 pounds. The baby gained one-half pound a month for its first year.

19.

Find the linear function that models the baby's weight W as a function of the age of the baby, in months, t.

20.

Find a reasonable domain and range for the function W.

21.

If the function W is graphed, find and interpret the *x*- and *y*-intercepts.

22.

If the function *W* is graphed, find and interpret the slope of the function.

23.

When did the baby weight 10.4 pounds?

24.

What is the output when the input is 6.2? Interpret your answer.

For the following exercises, consider this scenario: The number of people afflicted with the common cold in the winter months steadily decreased by 205 each year from 2005 until 2010. In 2005, 12,025 people were afflicted.

25.

Find the linear function that models the number of people inflicted with the common cold C as a function of the year, t.

26.

Find a reasonable domain and range for the function C.

27.

If the function C is graphed, find and interpret the *x*- and *y*-intercepts.

28.

If the function C is graphed, find and interpret the slope of the function.

29.

When will the output reach 0?

30.

In what year will the number of people be 9,700?

Graphical

For the following exercises, use the graph in Figure 7, which shows the profit, y, in thousands of dollars, of a company in a given year, t, where t represents the number of years since 1980.

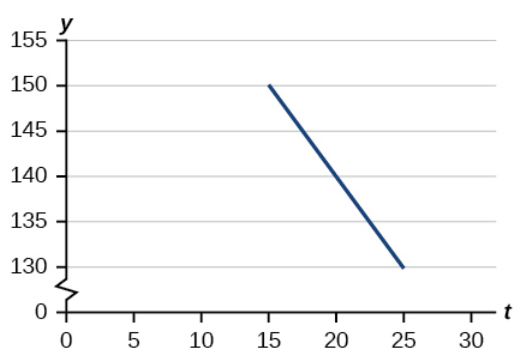


Figure 7

31.

Find the linear function y, where y depends on t, the number of years since 1980.

32.

Find and interpret the y-intercept.

33.

Find and interpret the x-intercept.

34.

Find and interpret the slope.

For the following exercises, use the graph in Figure 8, which shows the profit, y, in thousands of dollars, of a company in a given year, t, where t represents the number of years since 1980.

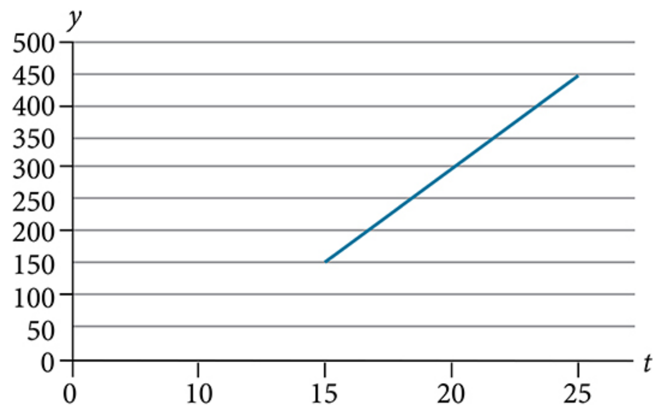


Figure 8

35.

Find the linear function y, where y depends on t, the number of years since 1980.

36.

Find and interpret the y-intercept.

37.

Find and interpret the x-intercept.

38.

Find and interpret the slope.

Numeric

For the following exercises, use the median home values in Mississippi and Hawaii (adjusted for inflation) shown in Table 2. Assume that the house values are changing linearly.

Year	Mississippi	Hawaii
1950	\$25,200	\$74,400
2000	\$71,400	\$272,700

Table 2

39.

In which state have home values increased at a higher rate?

40.

If these trends were to continue, what would be the median home value in Mississippi in 2010?

41.

If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

For the following exercises, use the median home values in Indiana and Alabama (adjusted for inflation) shown in Table 3. Assume that the house values are changing linearly.

Year	Indiana	Alabama
1950	$37,700	$27,100
2000	$94,300	$85,100

Table 3
42.

In which state have home values increased at a higher rate?

43.

If these trends were to continue, what would be the median home value in Indiana in 2010?

44.

If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

Real-World Applications

45.

In 2004, a school population was 1,001. By 2008 the population had grown to 1,697. Assume the population is changing linearly.

1. ⓐ How much did the population grow between the year 2004 and 2008?
2. ⓑ How long did it take the population to grow from 1,001 students to 1,697 students?
3. ⓒ What is the average population growth per year?
4. ⓓ What was the population in the year 2000?
5. ⓔ Find an equation for the population, P, of the school t years after 2000.
6. ⓕ Using your equation, predict the population of the school in 2011.

46.

In 2003, a town's population was 1,431. By 2007 the population had grown to 2,134. Assume the population is changing linearly.

1. ⓐ How much did the population grow between the year 2003 and 2007?
2. ⓑ How long did it take the population to grow from 1,431 people to 2,134 people?
3. ⓒ What is the average population growth per year?
4. ⓓ What was the population in the year 2000?
5. ⓔ Find an equation for the population, P of the town t years after 2000.
6. ⓕ Using your equation, predict the population of the town in 2014.

47.

A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used on the phone. If a customer uses 410 minutes, the monthly cost will be $71.50. If the customer uses 720 minutes, the monthly cost will be $118.

1. ⓐ Find a linear equation for the monthly cost of the cell plan as a function of x, the number of monthly minutes used.
2. ⓑ Interpret the slope and y-intercept of the equation.
3. ⓒ Use your equation to find the total monthly cost if 687 minutes are used.

48.

A phone company has a monthly cellular data plan where a customer pays a flat monthly fee of $10 and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be $11.20. If the customer uses 130 MB, the monthly cost will be $17.80.

1. ⓐ Find a linear equation for the monthly cost of the data plan as a function of x, the number of MB used.
2. ⓑ Interpret the slope and y-intercept of the equation.
3. ⓒ Use your equation to find the total monthly cost if 250 MB are used.

49.

In 1991, the moose population in a park was measured to be 4,360. By 1999, the population was measured again to be 5,880. Assume the population continues to change linearly.

1. ⓐ Find a formula for the moose population, *P* since 1990.
2. ⓑ What does your model predict the moose population to be in 2003?

50.

In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. The population changes linearly. Let the input be years since 1990.

1. ⓐ Find a formula for the owl population, P. Let the input be years since 2003.
2. ⓑ What does your model predict the owl population to be in 2012?

51.

The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010 and is being depleted by about 2.1 billion cubic feet each year.

1. ⓐ Give a linear equation for the remaining federal helium reserves, R, in terms of t, the number of years since 2010.
2. ⓑ In 2015, what will the helium reserves be?
3. ⓒ If the rate of depletion doesn't change, in what year will the Federal Helium Reserve be depleted?

52.

Suppose the world's oil reserves in 2014 are 1,820 billion barrels. If, on average, the total reserves are decreasing by 25 billion barrels of oil each year:

1. ⓐ Give a linear equation for the remaining oil reserves, R, in terms of t, the number of years since now.
2. ⓑ Seven years from now, what will the oil reserves be?
3. ⓒ If the rate at which the reserves are decreasing is constant, when will the world's oil reserves be depleted?

53.

You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of $19.95 *plus* 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

54.

You are choosing between two different window washing companies. The first charges $5 per window. The second charges a base fee of $40 plus $3 per window. How many windows would you need to have for the second company to be preferable?

55.

When hired at a new job selling jewelry, you are given two pay options:

- Option A: Base salary of $17,000 a year with a commission of 12% of your sales
- Option B: Base salary of $20,000 a year with a commission of 5% of your sales

How much jewelry would you need to sell for option A to produce a larger income?

56.

When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $14,000 a year with a commission of 10% of your sales
- Option B: Base salary of $19,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

57.

When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $20,000 a year with a commission of 12% of your sales
- Option B: Base salary of $26,000 a year with a commission of 3% of your sales

How much electronics would you need to sell for option A to produce a larger income?

58.

When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $10,000 a year with a commission of 9% of your sales
- Option B: Base salary of $20,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

Footnotes

- 4Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/

6.1: Applications

Learning Objectives

In this section, you will learn to use linear functions to model real-world applications

Now that we have learned to determine equations of lines, we get to apply these ideas in a variety of real-life situations.

Read the problem carefully. Highlight important information. Keep track of which values correspond to the independent variable (x) and which correspond to the dependent variable (y).

Example 6.1.1

A taxi service charges $0.50 per mile plus a $5 flat fee. What will be the cost of traveling 20 miles? What will be cost of traveling x miles?

Solution

x = distance traveled, in miles and y = cost in dollars

The cost of traveling 20 miles is

$$y=(0.50)(20)+5=10+5=15$$

The cost of traveling x miles is

$$y=(0.50)(x)+5=0.50x+5$$

In this problem, $0.50 per mile is referred to as the **variable cost**, and the flat charge $5 as the **fixed cost**. Now if we look at our cost equation $y=.50x+5$, we can see that the variable cost corresponds to the slope and the fixed cost to the y-intercept.

Example 6.1.2

The variable cost to manufacture a product is $10 per item and the fixed cost $2500. If x represents the number of items manufactured and y represents the total cost, write the cost function.

Solution

- The variable cost of $10 per item tells us that $m=10$.
- The fixed cost represents the y-intercept. So $b=2500$.

Therefore, the cost equation is $y=10x+2500$.

Example 6.1.3

It costs $750 to manufacture 25 items, and $1000 to manufacture 50 items. Assuming a linear relationship holds, find the cost equation, and use this function to predict the cost of 100 items.

Solution

We let x = the number of items manufactured, and let y = the cost.

Solving this problem is equivalent to finding an equation of a line that passes through the points (25, 750) and (50, 1000).

$$m=\frac{1000-750}{50-25}=10$$

Therefore, the partial equation is $y=10x+b$

By substituting one of the points in the equation, we get $b=500$

Therefore, the cost equation is $y=10x+500$

To find the cost of 100 items, substitute $x=100$ in the equation $y=10x+500$

So the cost is

$$y = 10(100) + 500 = 1500$$

It costs \$1500 to manufacture 100 items.

Example 6.1.4

The freezing temperature of water in Celsius is 0 degrees and in Fahrenheit 32 degrees. And the boiling temperatures of water in Celsius, and Fahrenheit are 100 degrees, and 212 degrees, respectively. Write a conversion equation from Celsius to Fahrenheit and use this equation to convert 30 degrees Celsius into Fahrenheit.

Solution

Let us look at what is given.

Celsius	Fahrenheit
0	32
100	212

Again, solving this problem is equivalent to finding an equation of a line that passes through the points (0, 32) and (100, 212).

Since we are finding a linear relationship, we are looking for an equation $y = mx + b$, or in this case $F = mC + b$, where x or C represent the temperature in Celsius, and y or F the temperature in Fahrenheit.

$$\text{slope m } = \frac{312 - 32}{100 - 0} = \frac{9}{5}$$

The equation is $F = \frac{9}{5}C + b$

Substituting the point (0, 32), we get

$$F = \frac{9}{5}C + 32.$$

To convert 30 degrees Celsius into Fahrenheit, substitute $C = 30$ in the equation

$$\begin{aligned} \mathrm{F} &= \frac{9}{5}\mathrm{C} + 32 \\ \mathrm{F} &= \frac{9}{5}(30) + 32 = 86 \end{aligned}$$

Example 6.1.5

The population of Canada in the year 1980 was 24.5 million, and in the year 2010 it was 34 million. The population of Canada over that time period can be approximately modelled by a linear function. Let x represent time as the number of years after 1980 and let y represent the size of the population.

a. Write the linear function that gives a relationship between the time and the population.
b. Assuming the population continues to grow linearly in the future, use this equation to predict the population of Canada in the year 2025.

Solution

The problem can be made easier by using 1980 as the base year, that is, we choose the year 1980 as the year zero. This will mean that the year 2010 will correspond to year 30. Now we look at the information we have:

Year	Population
0 (1980)	24.5 million
30 (2010)	34 million

a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m = \frac{34 - 24.5}{30 - 0} = \frac{9.5}{30} = 0.32$$

The y-intercept occurs when $x = 0$, so $b = 24.5$

$$y = 0.32x + 24.5$$

b. Now to predict the population in the year 2025, we let $x = 2025 - 1980 = 45$

$$\begin{aligned} y &= 0.32x + 24.5 \\ y &= 0.32(45) + 24.5 = 38.9 \end{aligned}$$

In the year 2025, we predict that the population of Canada will be 38.9 million people.

Note that we assumed the population trend will continue to be linear. Therefore if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

1.6.1: Applications (Exercises)

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

1) The variable cost to manufacture a product is $25 per item, and the fixed costs are $1200. If x is the number of items manufactured and y is the cost, write the cost function.	2) It costs $90 to rent a car driven 100 miles and $140 for one driven 200 miles. If x is the number of miles driven and y the total cost of the rental, write the cost function.
3) The variable cost to manufacture an item is $20 per item, and it costs a total of $750 to produce 20 items. If x represents the number of items manufactured and y is the cost, write the cost function.	4) To manufacture 30 items, it costs $2700, and to manufacture 50 items, it costs $3200. If x represents the number of items manufactured and y the cost, write the cost function.
5) To manufacture 100 items, it costs $32,000, and to manufacture 200 items, it costs $40,000. If x is the number of items manufactured and y is the cost, write the cost function.	6) It costs $1900 to manufacture 60 items, and the fixed costs are $700. If x represents the number of items manufactured and y the cost, write the cost function.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

7) A person who weighs 150 pounds has 60 pounds of muscles; a person that weighs 180 pounds has 72 pounds of muscles. If x represents body weight and y is muscle weight, write an equation describing their relationship. Use this relationship to determine the muscle weight of a person that weighs 170 pounds.	8) A spring on a door stretches 6 inches if a force of 30 pounds is applied. It stretches 10 inches if a 50 pound force is applied. If x represents the number of inches stretched, and y is the force, write a linear equation describing the relationship. Use it to determine the amount of force required to stretch the spring 12 inches.
9). A male college student who is 64 inches tall weighs 110 pounds. Another student who is 74 inches tall weighs 180 pounds. Assuming the relationship between male students' heights (x), and weights (y) is linear, write a function to express weights in terms of heights, and use this function to predict the weight of a student who is 68 inches tall.	10) EZ Clean company has determined that if it spends $30,000 on advertising, it can hope to sell 12,000 of its Minivacs a year, but if it spends $50,000, it can sell 16,000. Write an equation that gives a relationship between the number of dollars spent on advertising (x) and the number of minivacs sold(y).
11) The freezing temperatures for water for Celsius and Fahrenheit scales are 0ºC and 32ºF. The boiling temperatures for water are 100 ºC and 212 ºF. Let C denote the temperature in Celsius and F in Fahrenheit. Write the conversion function from Celsius to Fahrenheit. Use the function to convert 25 ºC into ºF.	12) By reversing the coordinates in the previous problem, find a conversion function that converts Fahrenheit into Celsius, and use this conversion function to convert 72 ºF into an equivalent Celsius measure.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

13) California’s population was 29.8 million in the year 1990, and 37.3 million in 2010. Assume that the population trend was and continues to be linear, write the population function. Use this function to predict the population in 2025. *Hint: Use 1990 as the base year (year 0); then 2010 and 2025 are years 20, and 35, respectively.)*	14) Use the population function for California in the previous problem to find the year in which the population will be 40 million people.
15) A college’s enrollment was 13,200 students in the year 2000, and 15,000 students in 2015. Enrollment has followed a linear pattern. Write the function that models enrollment as a function of time. Use the function to find the college’s enrollment in the year 2010. *Hint: Use year 2000 as the base year.*	16) If the college’s enrollment continues to follow this pattern, in what year will the college have 16,000 students enrolled.

17) The cost of electricity in residential homes is a linear function of the amount of energy used. In Grove City, a home using 250 kilowatt hours (kwh) of electricity per month pays $55. A home using 600 kwh per month pays $118. Write the cost of electricity as a function of the amount used. Use the function to find the cost for a home using 400 kwh of electricity per month.	18) Find the level of electricity use that would correspond to a monthly cost of $100.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

19) At ABC Co., sales revenue is $170,000 when it spends $5000 on advertising. Sales revenue is $254,000 when $12,000 is spent on advertising. a) Find a linear function for y = amount of sales revenue as a function of x = amount spent on advertising. b) Find revenue if $10,000 is spent on advertising. c) Find the amount that should be spent on advertising to achieve $200,000 in revenue.	20) For problem 19, explain the following: a. Explain what the slope of the line tells us about the effect on sales revenue of money spent on advertising. Be specific, explaining both the number and the sign of the slope in the context of this problem. b. Explain what the y intercept of the line tells us about the sales revenue in the context of this problem
21) Mugs Café sells 1000 cups of coffee per week if it does not advertise. For every $50 spent in advertising per week, it sells an additional 150 cups of coffee. a) Find a linear function that gives y = number of cups of coffee sold per week x = amount spent on advertising per week. b) How many cups of coffee does Mugs Café expect to sell if $100 per week is spent on advertising?	22) Party Sweets makes baked goods that can be ordered for special occasions. The price is $24 to order one dozen (12 cupcakes) and $9 for each additional 6 cupcakes. a. Find a linear function that gives the total price of a cupcake order as a function of the number of cupcakes ordered b. Find the price for an order of 5 dozen (60) cupcakes

6.2: More Applications

Learning Objectives

In this section, you will learn to:

1. Solve a linear system in two variables.
2. Find the equilibrium point when a demand and a supply equation are given.
3. Find the break-even point when the revenue and the cost functions are given.

Finding the Point of Intersection of Two Lines

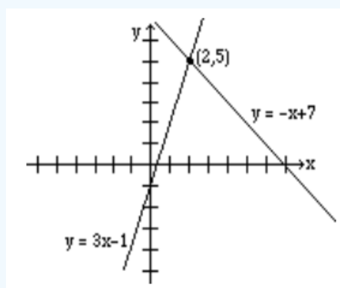
In this section, we will do application problems that involve the intersection of lines. Therefore, before we proceed any further, we will first learn how to find the intersection of two lines.

Example 6.2.1

Find the intersection of the line $y=3x-1$ and the line $y=-x+7$.

Solution

We graph both lines on the same axes, as shown below, and read the solution (2, 5).



Finding an intersection of two lines graphically is not always easy or practical; therefore, we will now learn to solve these problems algebraically.

At the point where two lines intersect, the x and y values for both lines are the same. So in order to find the intersection, we either let the x-values or the y-values equal.

If we were to solve the above example algebraically, it will be easier to let the y-values equal. Since $y=3x-1$ for the first line, and $y=-x+7$ for the second line, by letting the y-values equal, we get

$$\begin{aligned} 3x-1 &= -x+7 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

By substituting $x=2$ in any of the two equations, we obtain $y=5$.

Hence, the solution (2, 5).

A common algebraic method used to solve systems of equations is called the **elimination method**. The object is to eliminate one of the two variables by adding the left and right sides of the equations together. Once one variable is eliminated, we have an equation with only one variable for can be solved. Finally, by substituting the value of the variable that has been found in one of the original equations, we get the value of the other variable.

Example 6.2.2

Find the intersection of the lines $2x + y = 7$ and $3x - y = 3$ by the elimination method.

Solution

We add the left and right sides of the two equations.

$$\begin{array}{r} 2x + y = 7 \\ 3x - y = 3 \\ \hline 5x = 10 \\ x = 2 \end{array}$$

Now we substitute $x = 2$ in any of the two equations and solve for y.

$$\begin{array}{r} 2(2) + y = 7 \\ y = 3 \end{array}$$

Therefore, the solution is (2, 3).

Example 6.2.3

Solve the system of equations $x + 2y = 3$ and $2x + 3y = 4$ by the elimination method.

Solution

If we add the two equations, none of the variables are eliminated. But the variable x can be eliminated by multiplying the first equation by -2, and leaving the second equation unchanged.

$$\begin{array}{r} -2x - 4y = -6 \\ 2x + 3y = 4 \\ \hline -y = -2 \\ y = 2 \end{array}$$

Substituting $y = 2$ in $x + 2y = 3$, we get

$$\begin{aligned} x + 2(2) &= 3 \\ x &= -1 \end{aligned} \tag{6.2.1}$$

Therefore, the solution is (-1, 2).

Example 6.2.4

Solve the system of equations $3x - 4y = 5$ and $4x - 5y = 6$.

Solution

This time, we multiply the first equation by - 4 and the second by 3 before adding. (The choice of numbers is not unique.)

$$\begin{array}{r} -12x + 16y = -20 \\ 12x - 15y = 18 \\ \hline y = -2 \end{array}$$

By substituting y = - 2 in any one of the equations, we get x = -1.
Hence the solution is (-1, -2).

SUPPLY, DEMAND AND THE EQUILIBRIUM MARKET PRICE

In a free market economy the supply curve for a commodity is the number of items of a product that can be made available at different prices, and the demand curve is the number of items the consumer will buy at different prices.

As the price of a product increases, its demand decreases and supply increases. On the other hand, as the price decreases the demand increases and supply decreases. The **equilibrium price** is reached when the demand equals the supply.

✔ Example 6.2.5

The supply curve for a product is $y = 3.5x - 14$ and the demand curve for the same product is $y = -2.5x + 34$, where x is the price and y the number of items produced. Find the following.

a. How many items will be supplied at a price of $10?
b. How many items will be demanded at a price of $10?
c. Determine the equilibrium price.
d. How many items will be produced at the equilibrium price?

Solution

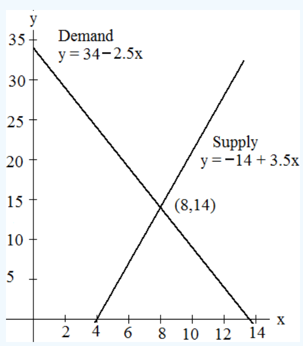
a) We substitute $x = 10$ in the supply equation, $y = 3.5x - 14$; the answer is $y = 3.5(10) - 14 = 21$.

b) We substitute $x = 10$ in the demand equation, $y = -2.5x + 34$; the answer is $y = -2.5(10) + 34 = 9$.

c) By letting the supply equal the demand, we get

$$\begin{aligned} 3.5x - 14 &= -25x + 34 \\ 6x &= 48 \\ x &= \$8 \end{aligned}$$

d) We substitute $x = 8$ in either the supply or the demand equation; we get $y = 14$.



The graph shows the intersection of the supply and the demand functions and their point of intersection, (8,14).

Interpretation: At equilibrium, the price is $8 per item, and 14 items are produced by suppliers and purchased by consumers.

The Break-Even Point

In a business, the profit is generated by selling products.

- If a company sells x number of items at a price P, then the **revenue R** is the price multiplied by number of items sold: $R = P \cdot x$.
- The **production costs C** are the sum of the variable costs and the fixed costs, and are often written as C = mx + b, where x is the number of items manufactured.
-
  - The slope m is the called marginal cost and represents the cost to produce one additional item or unit.
  - The variable cost, mx, depends on how much is being produced
  - The fixed cost b is constant; it does not change no matter how much is produced.
- **Profit** is equal to Revenue minus Cost: Profit = R − C

A company makes a profit if the revenue is greater than the cost. There is a loss if the cost is greater than the revenue. The point on the graph where the revenue equals the cost is called the **break-even point**. At the break-even point, profit is 0.

Example 6.2.6

If the revenue function of a product is $R = 5x$ and the cost function is $y = 3x + 12$, find the following.

a. If 4 items are produced, what will the revenue be?
b. What is the cost of producing 4 items?
c. How many items should be produced to break even?
d. What will be the revenue and the cost at the break-even point?

Solution

a) We substitute $x = 4$ in the revenue equation $R = 5x$, and the answer is $R = 20$.

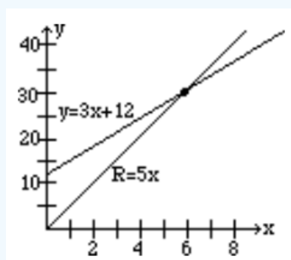
b) We substitute $x = 4$ in the cost equation $C = 3x + 12$, and the answer is $C = 24$.

c) By letting the revenue equal the cost, we get

$$\begin{aligned} 5x &= 3x + 12 \\ x &= 6 \end{aligned}$$

d) Substitute $x = 6$ in either the revenue or the cost equation: we get $R = C = 30$.

The graph below shows the intersection of the revenue and cost functions and their point of intersection, (6, 30).



1.7.1: More Applications (Exercises)

SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

1) Solve for x and y. y = 3x + 4 y = 5x - 2	2) Solve for x and y. 2x - 3y = 4 3x - 4y = 5
3) The supply and demand curves for a product are: Supply y = 2000x - 6500 Demand y = - 1000x + 28000, where x is price and y is the number of items. At what price will supply equal demand and how many items will be produced at that price?	4) The supply and demand curves for a product are Supply y = 300x - 18000 and Demand y = - 100x + 14000, where x is price and y is the number of items. At what price will supply equal demand, and how many items will be produced at that price?
5) A car rental company offers two plans for one way rentals. Plan I charges $36 per day and 17 cents per mile. Plan II charges $24 per day and 25 cents per mile. a. If you were to drive 300 miles in a day, which plan is better? b. For what mileage are both rates equal?	

SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

6) A demand curve for a product is the number of items the consumer will buy at different prices. At a price of $2 a store can sell 2400 of a particular type of toy truck. At a price of $8 the store can sell 600 such trucks. If x represents the price of trucks and y the number of items sold, write an equation for the demand curve.	7) A supply curve for a product is the number of items that can be made available at different prices. A manufacturer of toy trucks can supply 2000 trucks if they are sold for $8 each; it can supply only 400 trucks if they are sold for $4 each. If x is the price and y the number of items, write an equation for the supply curve.
8) The equilibrium price is the price where the supply equals the demand. From the demand and supply curves obtained in the previous two problems, find the equilibrium price, and determine the number of items that can be sold at that price.	9) A break-even point is the intersection of the cost function and the revenue function, that is, where total cost equals revenue, and profit is zero. Mrs. Jones Cookies Store's cost and revenue, in dollars, for x number of cookies is given by C = .05x + 3000 and R = .80x. Find the number of cookies that must be sold to break even.

SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

10) A company's revenue and cost in dollars are given by R = 225x and C = 75x + 6000, where x is the number of items. Find the number of items that must be produced to break-even.	11) A firm producing socks has a fixed cost of $20,000 and variable cost of $2 per pair of socks. Let x = the number of pairs of socks. Find the break-even point if the socks sell for $4.50 per pair.
12) Whackemhard Sports is planning to introduce a new line of tennis rackets. The fixed costs for the new line are $25,000 and the variable cost of producing each racket is $60. x is the number of rackets; y is in dollars. If the racket sells for $80, how many rackets must be sold in order to break even?	13) It costs $1,200 to produce 50 pounds of a chemical and it costs $2,200 to produce 150 pounds. The chemical sells for $15 per pound x is the amount of chemical; y is in dollars. a. Find the cost function. b. What is the fixed cost? c. How many pounds must be sold to break even? d. Find the cost and revenue at the break-even point.

1.8: Fitting Linear Models to Data

Learning Objectives

In this section, you will:

- Draw and interpret scatter plots.
- Find the line of best fit.
- Distinguish between linear and nonlinear relations.
- Use a linear model to make predictions.

A professor is attempting to identify trends among final exam scores. His class has a mixture of students, so he wonders if there is any relationship between age and final exam scores. One way for him to analyze the scores is by creating a diagram that relates the age of each student to the exam score received. In this section, we will examine one such diagram known as a scatter plot.

Drawing and Interpreting Scatter Plots

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a linear model, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. Figure 1 shows a sample scatter plot.

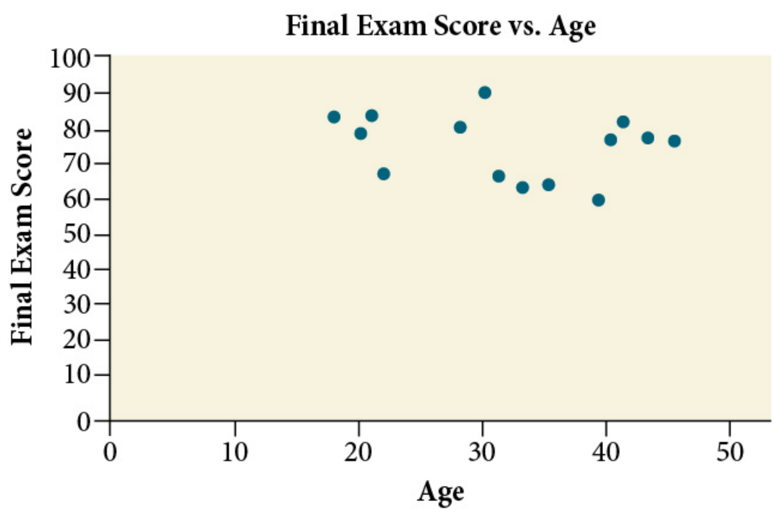


Figure 1 A scatter plot of age and final exam score variables

Notice this scatter plot does *not* indicate a linear relationship. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

Example 1

Using a Scatter Plot to Investigate Cricket Chirps

Table 1 shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit[5]. Plot this data, and determine whether the data appears to be linearly related.

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

Table 1

Answer

Plotting this data, as depicted in Figure 2 suggests that there may be a trend. We can see from the trend in the data that the number of chirps increases as the temperature increases. The trend appears to be roughly linear, though certainly not perfectly so.

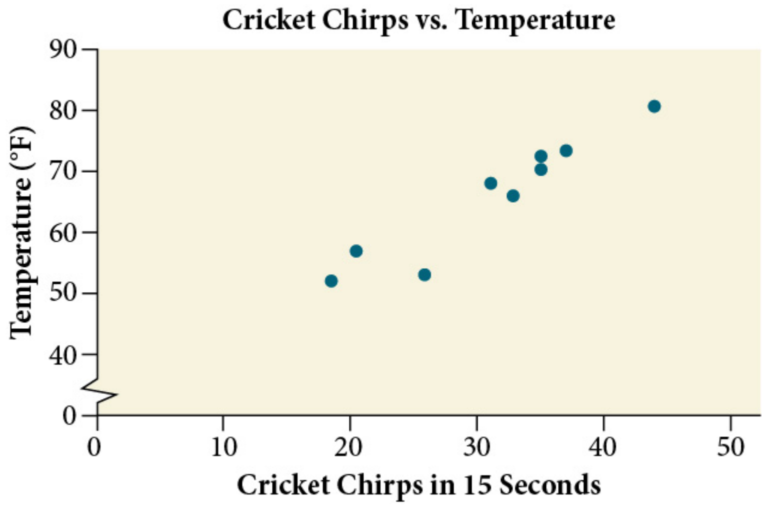


Figure 2

Finding the Line of Best Fit

Once we recognize a need for a linear function to model that data, the natural follow-up question is "what is that linear function?" One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the *y*-intercept. We can approximate the slope of the line by extending it until we can estimate the $\frac{\text{rise}}{\text{run}}$.

Example 2

Finding a Line of Best Fit

Find a linear function that fits the data in Table 1 by "eyeballing" a line that seems to fit.

Answer

On a graph, we could try sketching a line.

Using the starting and ending points of our hand drawn line, points (0, 30) and (50, 90), this graph has a slope of

$$m=\frac{60}{50}=1.2$$

and a *y*-intercept at 30. This gives an equation of

$$T(c)=1.2c+30$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit. The resulting equation is represented in Figure 3.

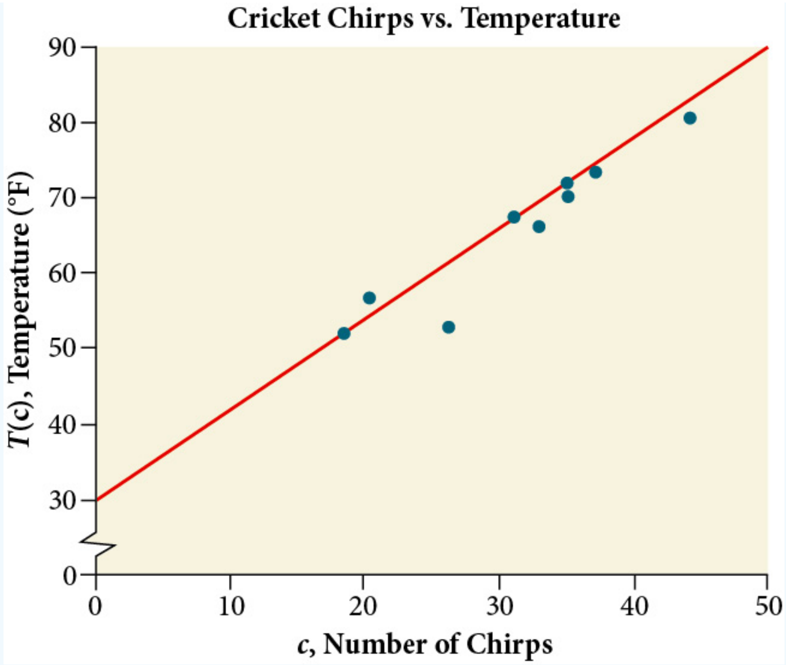


Figure 3

Analysis

This linear equation can then be used to approximate answers to various questions we might ask about the trend.

Recognizing Interpolation or Extrapolation

While the data for most examples does not fall perfectly on the line, the equation is our best guess as to how the relationship will behave outside of the values for which we have data. We use a process known as **interpolation** when we predict a value inside the domain and range of the data. The process of **extrapolation** is used when we predict a value outside the domain and range of the data.

Figure 4 compares the two processes for the cricket-chirp data addressed in Example 2. We can see that interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44. Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

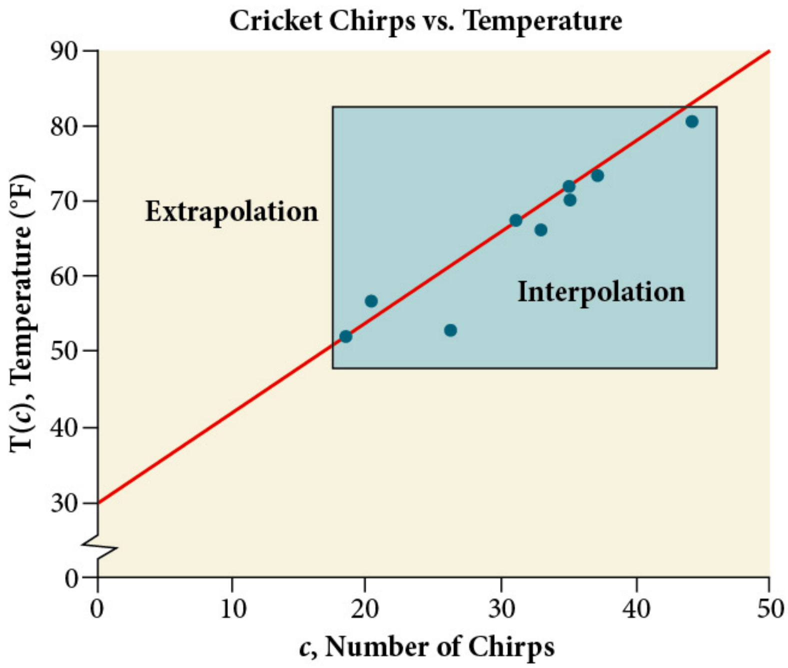


Figure 4 Interpolation occurs within the domain and range of the provided data whereas extrapolation occurs outside.

There is a difference between making predictions inside the domain and range of values for which we have data and outside that domain and range. Predicting a value outside of the domain and range has its limitations. When our model no longer applies after a certain point, it is sometimes called model breakdown. For example, predicting a cost function for a period of two years may involve examining the data where the input is the time in years and the output is the cost. But if we try to extrapolate a cost when $x = 50$, that is in 50 years, the model would not apply because we could not account for factors fifty years in the future.

Interpolation and Extrapolation

Different methods of making predictions are used to analyze data.

- The method of interpolation involves predicting a value inside the domain and/or range of the data.
- The method of extrapolation involves predicting a value outside the domain and/or range of the data.
- **Model breakdown** occurs at the point when the model no longer applies.

Example 3

Understanding Interpolation and Extrapolation

Use the cricket data from Table 1 to answer the following questions:

1. ⓐ Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.
2. ⓑ Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

Answer

1. ⓐ The number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:

$$\begin{aligned} T(30) &= 30 + 1.2(30) \\ &= 66 \quad \text{degrees} \end{aligned}$$

Based on the data we have, this value seems reasonable.

2. ⓑ The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation because 40 is outside the range of our data. Using our model:

$$\begin{aligned} 40 &= 30 + 1.2c \\ 10 &= 1.2c \\ c &\approx 8.33 \end{aligned}$$

We can compare the regions of interpolation and extrapolation using Figure 5.

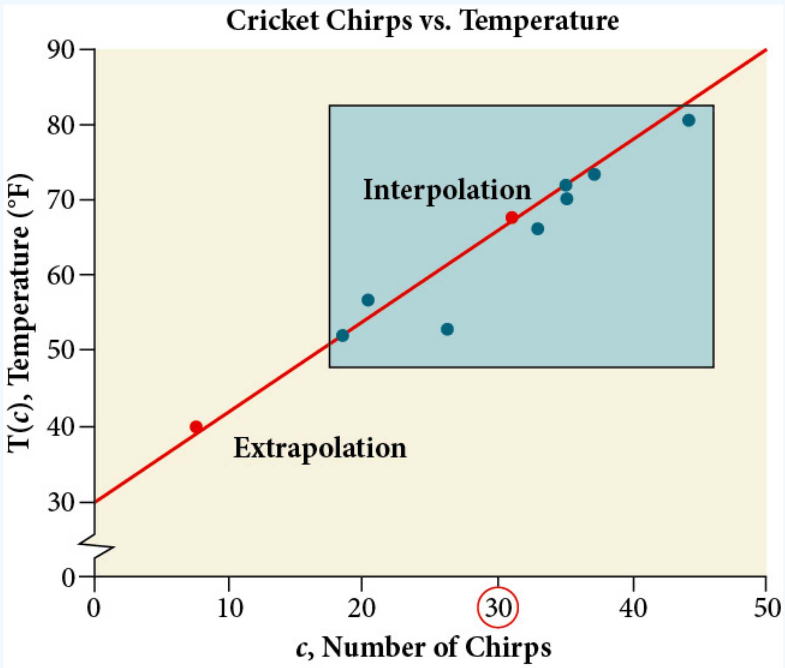


Figure 5

Analysis

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

Try It #1

According to the data from Table 1, what temperature can we predict it is if we counted 20 chirps in 15 seconds?

Finding the Line of Best Fit Using a Graphing Utility

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values[6]. One such technique is called least squares regression and can be computed by many graphing calculators, spreadsheet software, statistical software, and many web-based calculators[7]. Least squares regression is one means to determine the line that best fits the data, and here we will refer to this method as linear regression.

How To

Given data of input and corresponding outputs from a linear function, find the best fit line using linear regression.

1. Enter the input in List 1 (**L1**).

2. Enter the output in List 2 (**L2**).
3. On a graphing utility, select Linear Regression (**LinReg**).

Example 4

Finding a Least Squares Regression Line

Find the least squares regression line using the cricket-chirp data in Table 1.

Answer

1. Enter the input (chirps) in List 1 (L1).
2. Enter the output (temperature) in List 2 (L2). See Table 2.

L1	44	35	20.4	33	31	35	18.5	37	26
L2	80.5	70.5	57	66	68	72	52	73.5	53

Table 2

- On a graphing utility, select Linear Regression (LinReg). Using the cricket chirp data from earlier, with technology we obtain the equation:

$$T(c) = 30.281 + 1.143c$$

Analysis

Notice that this line is quite similar to the equation we “eyeballed” but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$\begin{aligned} T(30) &= 30.281 + 1.143(30) \\ &= 64.571 \\ &\approx 64.6 \text{degrees} \end{aligned}$$

The graph of the scatter plot with the least squares regression line is shown in Figure 6.

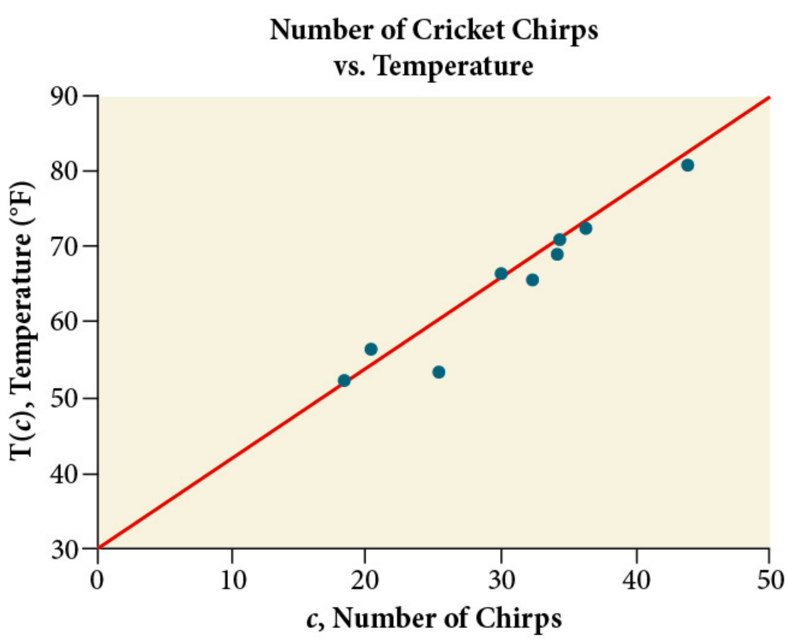


Figure 6

Q&A

Will there ever be a case where two different lines will serve as the best fit for the data?

No. There is only one best fit line.

Distinguishing Between Linear and Non-Linear Models

As we saw above with the cricket-chirp model, some data exhibit strong linear trends, but other data, like the final exam scores plotted by age, are clearly nonlinear. Most calculators and computer software can also provide us with the **correlation coefficient**, which is a measure of how closely the line fits the data. Many graphing calculators require the user to turn a "diagnostic on" selection to find the correlation coefficient, which mathematicians label as r. The correlation coefficient provides an easy way to get an idea of how close to a line the data falls.

We should compute the correlation coefficient only for data that follows a linear pattern or to determine the degree to which a data set is linear. If the data exhibits a nonlinear pattern, the correlation coefficient for a linear regression is meaningless. To get a sense for the relationship between the value of r and the graph of the data, Figure 7 shows some large data sets with their correlation coefficients. Remember, for all plots, the horizontal axis shows the input and the vertical axis shows the output.

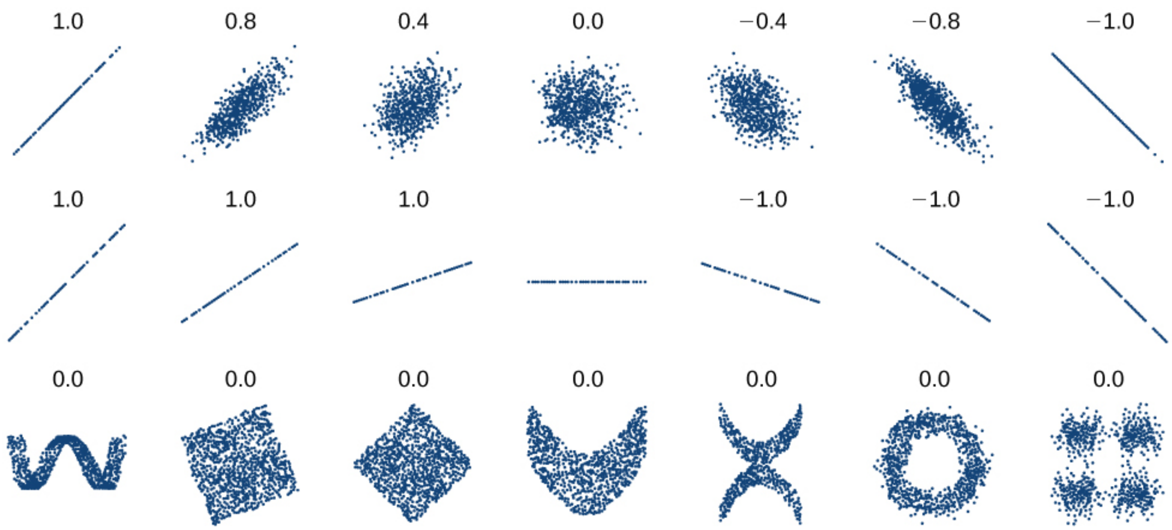


Figure 7 Plotted data and related correlation coefficients. (credit: "DenisBoigelot," Wikimedia Commons)

Correlation Coefficient

The correlation coefficient is a value, r, between –1 and 1.

- $r > 0$ suggests a positive (increasing) relationship
- $r < 0$ suggests a negative (decreasing) relationship
- The closer the value is to 0, the more scattered the data.
- The closer the value is to 1 or –1, the less scattered the data is.

Example 5

Finding a Correlation Coefficient

Calculate the correlation coefficient for cricket-chirp data in Table 1.

Answer

Because the data appear to follow a linear pattern, we can use technology to calculate r. Enter the inputs and corresponding outputs and select the Linear Regression. The calculator will also provide you with the correlation coefficient, $r = 0.9509$. This value is very close to 1, which suggests a strong increasing linear relationship.

Note: For some calculators, the Diagnostics must be turned "on" in order to get the correlation coefficient when linear regression is performed: [2nd]>[0]>[alpha][x –1], then scroll to **DIAGNOSTICSON**.

Predicting with a Regression Line

Once we determine that a set of data is linear using the correlation coefficient, we can use the regression line to make predictions. As we learned above, a regression line is a line that is closest to the data in the scatter plot, which means that only one such line is a best fit for the data.

Example 6

Using a Regression Line to Make Predictions

Gasoline consumption in the United States has been steadily increasing. Consumption data from 1994 to 2004 is shown in Table 3[8]. Determine whether the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133	136

Table 3

The scatter plot of the data, including the least squares regression line, is shown in Figure 8.

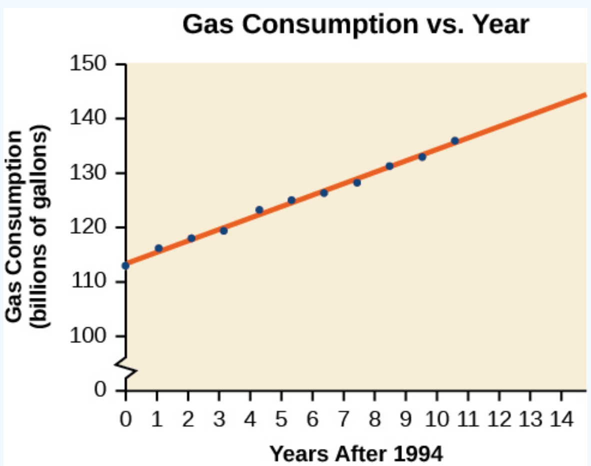


Figure 8

Answer

We can introduce new input variable, t, representing years since 1994.

The least squares regression equation is:

$$C(t) = 113.318 + 2.209t$$

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.

Using this to predict consumption in 2008 $(t = 14)$,

$$\begin{aligned} C(14) &= 113.318 + 2.209(14) \\ &= 144.244 \end{aligned}$$

The model predicts 144.244 billion gallons of gasoline consumption in 2008.

Try It #2

Use the model we created using technology in Example 6 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

Access these online resources for additional instruction and practice with fitting linear models to data.

- Introduction to Regression Analysis
- Linear Regression

2.4 Section Exercises

Verbal

1.

Describe what it means if there is a model breakdown when using a linear model.

2.

What is interpolation when using a linear model?

3.

What is extrapolation when using a linear model?

4.

Explain the difference between a positive and a negative correlation coefficient.

5.

Explain how to interpret the absolute value of a correlation coefficient.

Algebraic

6.

A regression was run to determine whether there is a relationship between hours of TV watched per day (x) and number of sit-ups a person can do (y). The results of the regression are given below. Use this to predict the number of sit-ups a person who watches 11 hours of TV can do.

$$\begin{aligned}&y=ax+b\\&a=-1.341\\&b=32.234\\&r=-0.896\end{aligned}$$

7.

A regression was run to determine whether there is a relationship between the diameter of a tree (x, in inches) and the tree's age (y, in years). The results of the regression are given below. Use this to predict the age of a tree with diameter 10 inches.

$$\begin{aligned}&y=ax+b\\&a=6.301\\&b=-1.044\\&r=-0.970\end{aligned}$$

For the following exercises, draw a scatter plot for the data provided. Does the data appear to be linearly related?

8.

0	2	4	6	8	10
–22	–19	–15	–11	–6	–2

9.

1	2	3	4	5	6

46	50	59	75	100	136

10.

100	250	300	450	600	750
12	12.6	13.1	14	14.5	15.2

11.

1	3	5	7	9	11
1	9	28	65	125	216

12.

For the following data, draw a scatter plot. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation? Eyeball the line, and estimate the answer.

Year	Population
1990	11,500
1995	12,100
2000	12,700
2005	13,000
2010	13,750

13.

For the following data, draw a scatter plot. If we wanted to know when the temperature would reach 28 °F, would the answer involve interpolation or extrapolation? Eyeball the line and estimate the answer.

Temperature, °F	16	18	20	25	30
Time, seconds	46	50	54	55	62

Graphical

For the following exercises, match each scatterplot with one of the four specified correlations in Figure 9 and Figure 10.

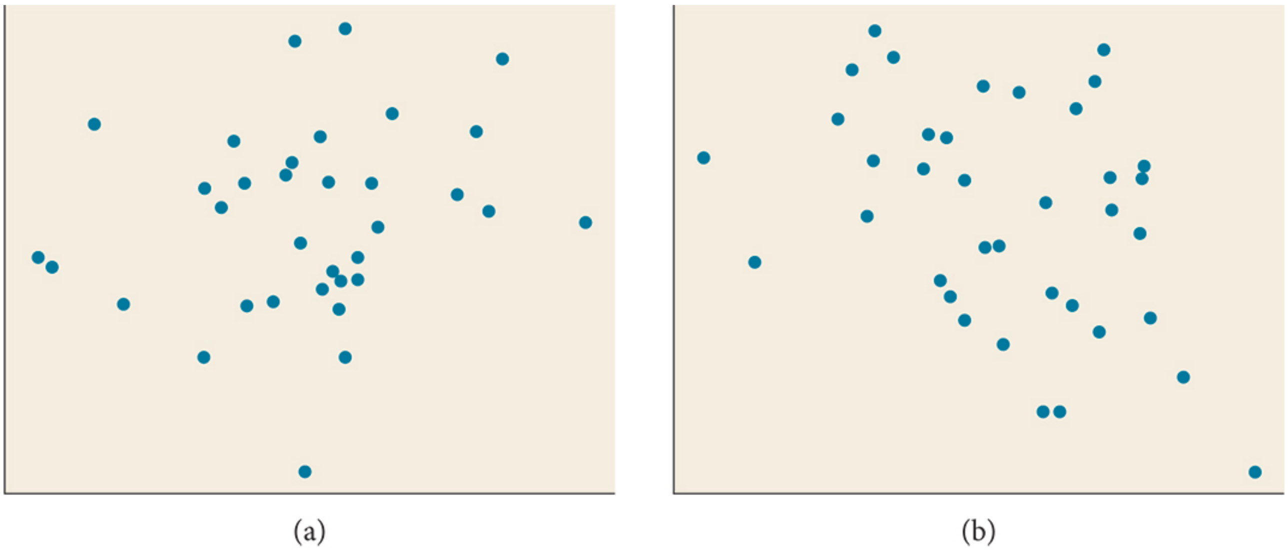


Figure 9

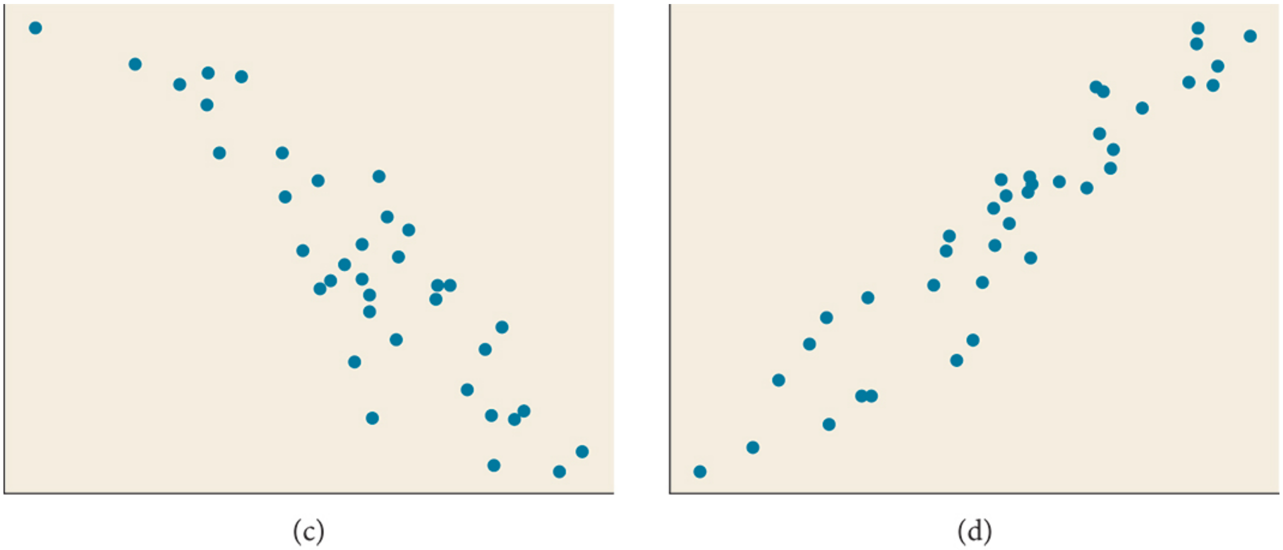


Figure 10

14.

$r = 0.95$

15.

$r = -0.89$

16.

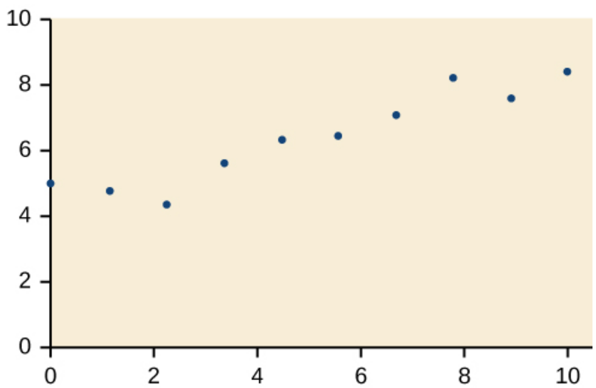
$r = 0.26$

17.

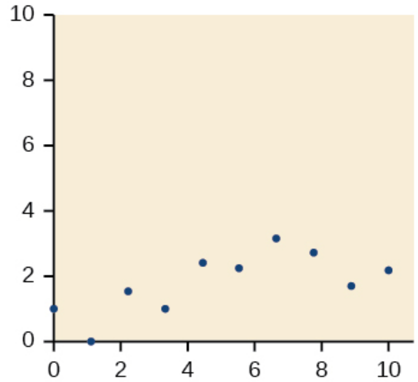
$r = -0.39$

For the following exercises, draw a best-fit line for the plotted data.

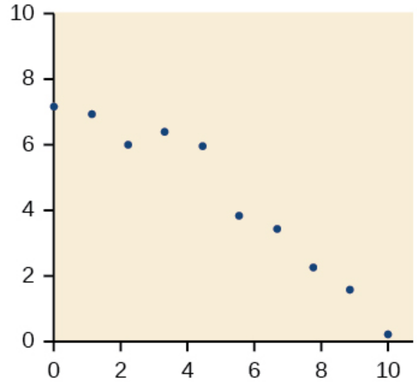
18.



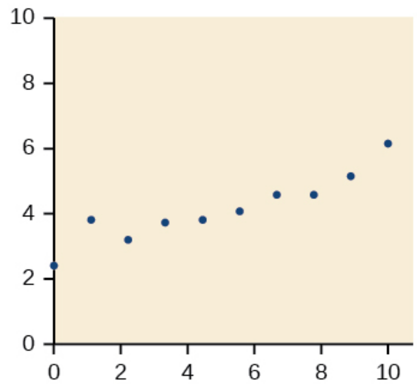
19.



20.



21.



Numeric

22.

The U.S. Census tracks the percentage of persons 25 years or older who are college graduates. That data for several years is given in Table 4[9]. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the percentage exceed 35%?

Year	Percent Graduates
1990	21.3
1992	21.4
1994	22.2
1996	23.6
1998	24.4
2000	25.6

Year	Percent Graduates
2002	26.7
2004	27.7
2006	28
2008	29.4

Table 4
23.

The U.S. import of wine (in hectoliters) for several years is given in Table 5. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will imports exceed 12,000 hectoliters?

Year	Imports
1992	2665
1994	2688
1996	3565
1998	4129
2000	4584
2002	5655
2004	6549
2006	7950
2008	8487
2009	9462

Table 5
24.

Table 6 shows the year and the number of people unemployed in a particular city for several years. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the number of unemployed reach 5?

Year	Number Unemployed
1990	750
1992	670
1994	650
1996	605
1998	550
2000	510
2002	460
2004	420
2006	380
2008	320

Table 6

Technology

For the following exercises, use each set of data to calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to 3 decimal places of accuracy.

25.

x	8	15	26	31	56
y	23	41	53	72	103

26.

x	5	7	10	12	15
y	4	12	17	22	24

27.

x	y	x	y
3	21.9	11	15.76
4	22.22	12	13.68
5	22.74	13	14.1
6	22.26	14	14.02
7	20.78	15	11.94
8	17.6	16	12.76
9	16.52	17	11.28
10	18.54	18	9.1

28.

x	y
4	44.8
5	43.1
6	38.8
7	39
8	38
9	32.7
10	30.1
11	29.3
12	27
13	25.8

29.

x	21	25	30	31	40	50
y	17	11	2	-1	-18	-40

30.

x	100	80	60	55	40	20
y	2000	1798	1589	1580	1390	1202

31.

x	900	988	1000	1010	1200	1205
y	70	80	82	84	105	108

Extensions

32.

Graph $f(x) = 0.5x + 10$. Pick a set of 5 ordered pairs using inputs $x = -2, 1, 5, 6, 9$ and use linear regression to verify that the function is a good fit for the data.

33.

Graph $f(x) = -2x - 10$. Pick a set of 5 ordered pairs using inputs $x = -2, 1, 5, 6, 9$ and use linear regression to verify the function.

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs shows dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span, (number of units sold, profit) for specific recorded years:

$(46, 1, 600), \quad (48, 1, 550), \quad (50, 1, 505), \quad (52, 1, 540), \quad (54, 1, 49.5)$

34.

Use linear regression to determine a function P where the profit in thousands of dollars depends on the number of units sold in hundreds.

35.

Find to the nearest tenth and interpret the *x*-intercept.

36.

Find to the nearest tenth and interpret the *y*-intercept.

Real-World Applications

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population and the year over the ten-year span, (population, year) for specific recorded years:

$(2500, 2000), (2650, 2001), (3000, 2003), (3500, 2006), (4200, 2010)$

37.

Use linear regression to determine a function y, where the year depends on the population. Round to three decimal places of accuracy.

38.

Predict when the population will hit 8,000.

For the following exercises, consider this scenario: The profit of a company increased steadily over a ten-year span. The following ordered pairs show the number of units sold in hundreds and the profit in thousands of over the ten year span, (number of units sold, profit) for specific recorded years:

$(46, 250), \quad (48, 305), \quad (50, 350), \quad (52, 390), \quad (54, 410)$

39.

Use linear regression to determine a function *y*, where the profit in thousands of dollars depends on the number of units sold in hundreds .

40.

Predict when the profit will exceed one million dollars.

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs show dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span (number of units sold, profit) for specific recorded years:

$(46, 250), (48, 225), (50, 205), (52, 180), (54, 165).$

41.

Use linear regression to determine a function y, where the profit in thousands of dollars depends on the number of units sold in hundreds .

42.

Predict when the profit will dip below the $25,000 threshold.

Footnotes

- 5Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010
- 6Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.
- 7For example, http://www.shodor.org/unchem/math/lls/leastsq.html
- 8http://www.bts.gov/publications/nati...ble_04_10.html
- 9http://www.census.gov/hhes/socdemo/e...cal/index.html. Accessed 5/1/2014.

1.9: Chapter Review

1.9.1: Key Terms

Key Terms

correlation coefficient

a value, r, between –1 and 1 that indicates the degree of linear correlation of variables, or how closely a regression line fits a data set.

decreasing linear function

a function with a negative slope: If $f(x) = mx + b, \quad \text{then} \quad m < 0.$

extrapolation

predicting a value outside the domain and range of the data

horizontal line

a line defined by $f(x) = b,$ where b is a real number. The slope of a horizontal line is 0.

increasing linear function

a function with a positive slope: If $f(x) = mx + b, \quad \text{then} \quad m > 0.$

interpolation

predicting a value inside the domain and range of the data

least squares regression

a statistical technique for fitting a line to data in a way that minimizes the differences between the line and data values

linear function

a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

model breakdown

when a model no longer applies after a certain point

parallel lines

two or more lines with the same slope

perpendicular lines

two lines that intersect at right angles and have slopes that are negative reciprocals of each other

point-slope form

the equation for a line that represents a linear function of the form $y - y_1 = m\left(x - x_1\right)$

slope

the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form

the equation for a line that represents a linear function in the form $f(x) = mx + b$

vertical line

a line defined by $x = a,$ where a is a real number. The slope of a vertical line is undefined.

***x*-intercept**

the point on the graph of a linear function when the output value is 0; the point at which the graph crosses the horizontal axis

***y*-intercept**

the value of a function when the input value is zero; also known as initial value

1.9.2: Key Equations

Key Equations

slope-intercept form of a line	$f(x)=mx+b$
slope	$m=\frac{\text{change in output (rise)}}{\text{change in input (run)}}=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$
point-slope form of a line	$y-y_1=m(x-x_1)$

1.9.3: Key Concepts

Key Concepts

2.1 Linear Functions

- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form, and graphical form. See Example 1.
- The rate of change of a linear function is also known as the slope.
- An equation in the slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or *y*-intercept, is the output value when the input of a linear function is zero. It is the *y*-value of the point at which the line crosses the *y*-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant. See Example 2.
- The slope of a linear function can be calculated by dividing the difference between *y*-values by the difference in corresponding *x*-values of any two points on the line. See Example 3 and Example 4.
- The slope and initial value can be determined given a graph or any two points on the line.
- One type of function notation is the slope-intercept form of an equation.
- The point-slope form is useful for finding a linear equation when given the slope of a line and one point. See Example 5.
- The point-slope form is also convenient for finding a linear equation when given two points through which a line passes. See Example 6.
- The equation for a linear function can be written if the slope m and initial value b are known. See Example 7, Example 8, and Example 9.
- A linear function can be used to solve real-world problems. See Example 10 and Example 11.
- A linear function can be written from tabular form. See Example 12.

2.2 Graphs of Linear Functions

- Linear functions may be graphed by plotting points or by using the *y*-intercept and slope. See Example 1 and Example 2.
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections. See Example 3.
- The *y*-intercept and slope of a line may be used to write the equation of a line.
- The *x*-intercept is the point at which the graph of a linear function crosses the *x*-axis. See Example 4 and Example 5.
- Horizontal lines are written in the form, $f(x)=b$. See Example 6.
- Vertical lines are written in the form, $x=b$. See Example 7.
- Parallel lines have the same slope.
- Perpendicular lines have negative reciprocal slopes, assuming neither is vertical. See Example 8.
- A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the *x*- and *y*-values of the given point into the equation, $f(x)=mx+b$, and using the b that results. Similarly, the point-slope form of an equation can also be used. See Example 9.
- A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope. See Example 10 and Example 11.
- A system of linear equations may be solved setting the two equations equal to one another and solving for x. The *y*-value may be found by evaluating either one of the original equations using this *x*-value.
- A system of linear equations may also be solved by finding the point of intersection on a graph. See Example 12 and Example 13.

2.3 Modeling with Linear Functions

- We can use the same problem strategies that we would use for any type of function.
- When modeling and solving a problem, identify the variables and look for key values, including the slope and *y*-intercept. See Example 1.
- Draw a diagram, where appropriate. See Example 2 and Example 3.

- Check for reasonableness of the answer.
- Linear models may be built by identifying or calculating the slope and using the *y*-intercept.
- The *x*-intercept may be found by setting $y = 0$, which is setting the expression $mx + b$ equal to 0.
- The point of intersection of a system of linear equations is the point where the *x*- and *y*-values are the same. See Example 4.
- A graph of the system may be used to identify the points where one line falls below (or above) the other line.

2.4 Fitting Linear Models to Data

- Scatter plots show the relationship between two sets of data. See Example 1.
- Scatter plots may represent linear or non-linear models.
- The line of best fit may be estimated or calculated, using a calculator or statistical software. See Example 2.
- Interpolation can be used to predict values inside the domain and range of the data, whereas extrapolation can be used to predict values outside the domain and range of the data. See Example 3.
- The correlation coefficient, r, indicates the degree of linear relationship between data. See Example 5.
- A regression line best fits the data. See Example 6.
- The least squares regression line is found by minimizing the squares of the distances of points from a line passing through the data and may be used to make predictions regarding either of the variables. See Example 4.

1.10: Exercises

Review Exercises

Linear Functions

1.

Determine whether the algebraic equation is linear. $2x+3y=$

2.

Determine whether the algebraic equation is linear. $6x^2-y=$

3.

Determine whether the function is increasing or decreasing.

$f(x)=7x-2$

4.

Determine whether the function is increasing or decreasing.

$g(x)=-x+2$

5.

Given each set of information, find a linear equation that satisfies the given conditions, if possible.

Passes through $(7,5)$ and $(3,17)$

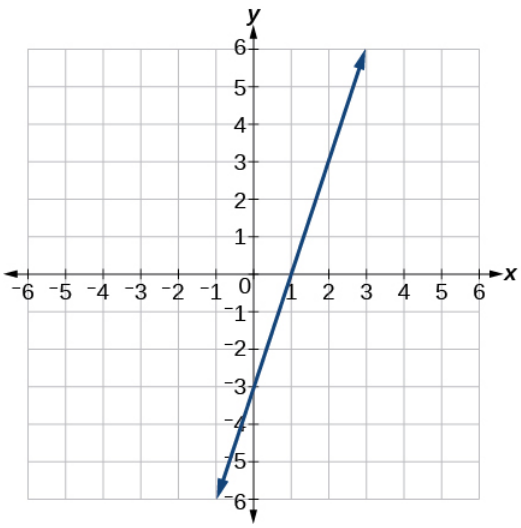
6.

Given each set of information, find a linear equation that satisfies the given conditions, if possible.

x-intercept at $(6,0)$ and *y*-intercept at $(0,10)$

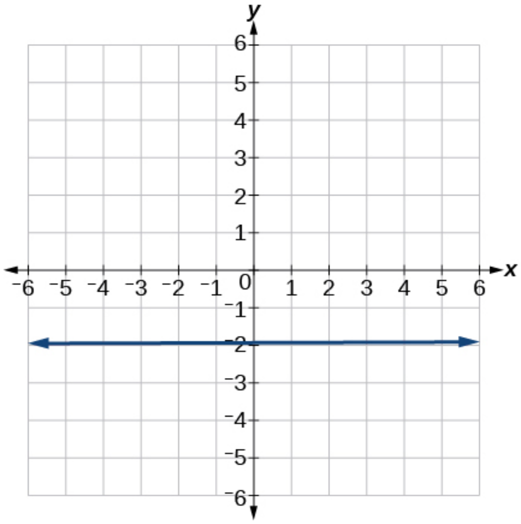
7.

Find the slope of the line shown in the line graph.



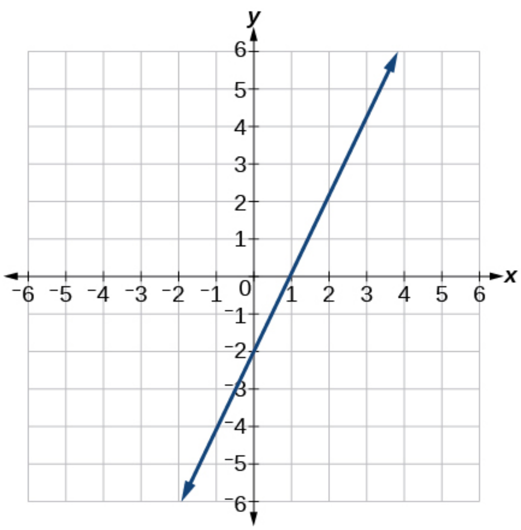
8.

Find the slope of the line graphed.



9.

Write an equation in slope-intercept form for the line shown.



10.

Does the following table represent a linear function? If so, find the linear equation that models the data.

x	–4	0	2	10
$g(x)$	18	–2	–12	–52

11.

Does the following table represent a linear function? If so, find the linear equation that models the data.

x	6	8	12	26
$g(x)$	–8	–12	–18	–46

12.

On June 1st, a company has $4,000,000 profit. If the company then loses 150,000 dollars per day thereafter in the month of June, what is the company's profit n^{th}day after June 1st?

Graphs of Linear Functions

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

13.

$2x - 6y = 12$
$-x + 3y = 1$

14.

$y = \frac{1}{3}x - 2$
$3x + y = -9$

For the following exercises, find the *x*- and *y*- intercepts of the given equation

15.

$7x + 9y = -$

16.

$f(x) = 2x - 1$

For the following exercises, use the descriptions of the pairs of lines to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

17.

- Line 1: Passes through $(5, 11)$ and $(10, 1)$
- Line 2: Passes through $(-1, 3)$ and $(-5, 11)$

18.

- Line 1: Passes through $(8, -10)$ and $(0, -26)$
- Line 2: Passes through $(2, 5)$ and $(4, 4)$

19.

Write an equation for a line perpendicular to $f(x) = 5x - 1$ and passing through the point (5, 20).

20.

Find the equation of a line with a *y*- intercept of $(0, \quad 2)$ and slope $-\frac{1}{2}$.

21.

Sketch a graph of the linear function $f(t) = 2t - 5$.

22.

Find the point of intersection for the 2 linear functions: $\begin{array}{l} x = y + 6 \\ 2x - y = 13 \end{array}$

23.

A car rental company offers two plans for renting a car.

- Plan A: 25 dollars per day and 10 cents per mile
- Plan B: 50 dollars per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

Modeling with Linear Functions

24.

Find the area of a triangle bounded by the *y* axis, the line $f(x) = 10 - 2x$, and the line perpendicular to f that passes through the origin.

25.

A town's population increases at a constant rate. In 2010 the population was 55,000. By 2012 the population had increased to 76,000. If this trend continues, predict the population in 2016.

26.

The number of people afflicted with the common cold in the winter months dropped steadily by 50 each year since 2004 until 2010. In 2004, 875 people were inflicted.

Find the linear function that models the number of people afflicted with the common cold C as a function of the year, t. When will no one be afflicted?

For the following exercises, use the graph in Figure 1 showing the profit, y, in thousands of dollars, of a company in a given year, x, where x represents years since 1980.

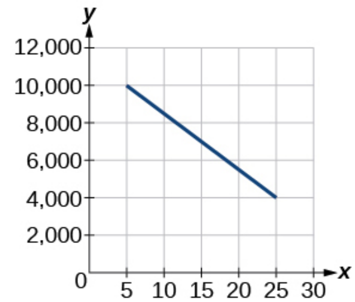


Figure 1
27.

Find the linear function y, where y depends on x, the number of years since 1980.

28.

Find and interpret the y-intercept.

For the following exercise, consider this scenario: In 2004, a school population was 1,700. By 2012 the population had grown to 2,500.

29.

Assume the population is changing linearly.

1. ⓐ How much did the population grow between the year 2004 and 2012?
2. ⓑ What is the average population growth per year?
3. ⓒ Find an equation for the population, P, of the school t years after 2004.

For the following exercises, consider this scenario: In 2000, the moose population in a park was measured to be 6,500. By 2010, the population was measured to be 12,500. Assume the population continues to change linearly.

30.

Find a formula for the moose population, P.

31.

What does your model predict the moose population to be in 2020?

For the following exercises, consider this scenario: The median home values in subdivisions Pima Central and East Valley (adjusted for inflation) are shown in Table 1. Assume that the house values are changing linearly.

Year	Pima Central	East Valley
1970	32,000	120,250
2010	85,000	150,000

Table 1
32.

In which subdivision have home values increased at a higher rate?

33.

If these trends were to continue, what would be the median home value in Pima Central in 2015?

Fitting Linear Models to Data

34.

Draw a scatter plot for the data in Table 2. Then determine whether the data appears to be linearly related.

0	2	4	6	8	10
–105	–50	1	55	105	160

Table 2

35.

Draw a scatter plot for the data in Table 3. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation?

Year	**Population**
1990	5,600
1995	5,950
2000	6,300
2005	6,600
2010	6,900

Table 3

36.

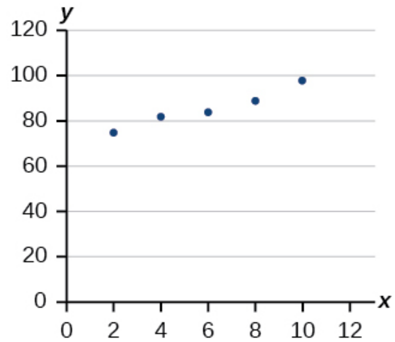
Eight students were asked to estimate their score on a 10-point quiz. Their estimated and actual scores are given in Table 4. Plot the points, then sketch a line that fits the data.

Predicted	**Actual**
6	6
7	7
7	8
8	8
7	9
9	10
10	10
10	9

Table 4

37.

Draw a best-fit line for the plotted data.



For the following exercises, consider the data in Table 5, which shows the percent of unemployed in a city of people 25 years or older who are college graduates is given below, by year.

Year	2000	2002	2005	2007	2010
Percent Graduates	6.5	7.0	7.4	8.2	9.0

Table 5
38.

Determine whether the trend appears to be linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

39.

In what year will the percentage exceed 12%?

40.

Based on the set of data given in Table 6, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	17	20	23	26	29
y	15	25	31	37	40

Table 6
41.

Based on the set of data given in Table 7, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	10	12	15	18	20
y	36	34	30	28	22

Table 7

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs show the population and the year over the ten-year span (population, year) for specific recorded years:

$(3{,}600, 2000)$; $(4{,}000, 2001)$; $(4{,}700, 2003)$; $(6{,}000, 2006)$

42.

Use linear regression to determine a function y, where the year depends on the population, to three decimal places of accuracy.

43.

Predict when the population will hit 12,000.

44.

What is the correlation coefficient for this model to three decimal places of accuracy?

45.

According to the model, what is the population in 2014?

1.10.1: Review Exercises

Review Exercises

Linear Functions

1.

Determine whether the algebraic equation is linear. $2x + 3y =$

2.

Determine whether the algebraic equation is linear. $6x^2 - y =$

3.

Determine whether the function is increasing or decreasing.

$f(x) = 7x - 2$

4.

Determine whether the function is increasing or decreasing.

$g(x) = -x + 2$

5.

Given each set of information, find a linear equation that satisfies the given conditions, if possible.

Passes through $(7, 5)$ and $(3, 17)$

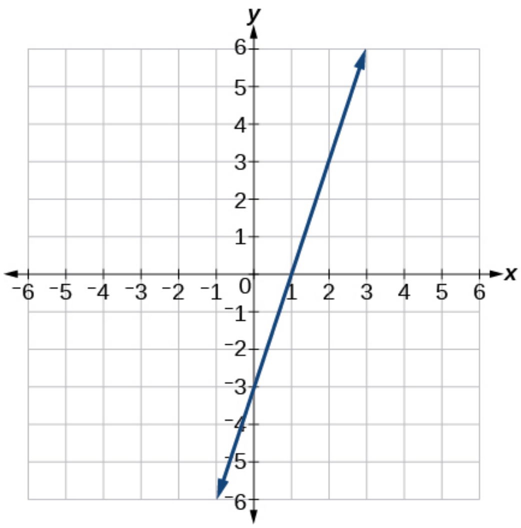
6.

Given each set of information, find a linear equation that satisfies the given conditions, if possible.

x-intercept at $(6, 0)$ and *y*-intercept at $(0, 10)$

7.

Find the slope of the line shown in the line graph.



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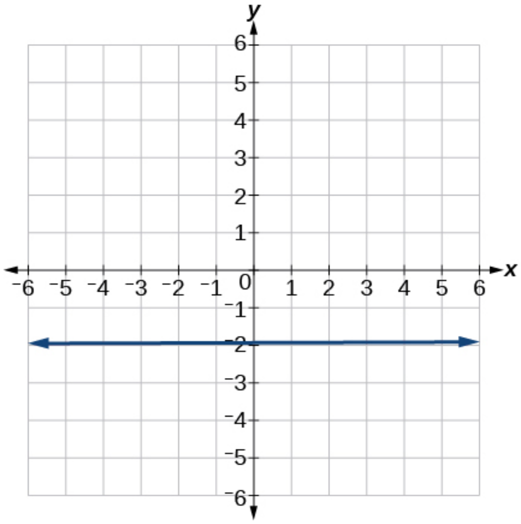

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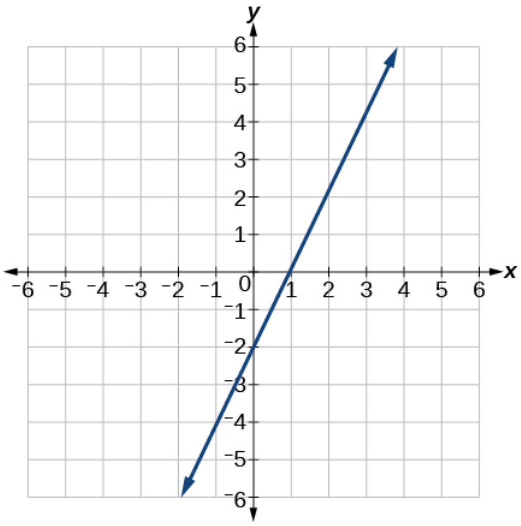

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Does the following table represent a linear function? If so, find the linear equation that models the data.

x	6	8	12	26
$g(x)$	–8	–12	–18	–46

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On June 1^{st}, a company has \$4,000,000 profit. If the company then loses 150,000 dollars per day thereafter in the month of June, what is the company's profit n^{th}day after June 1^{st}?

Graphs of Linear Functions

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

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$2x - 6y = 12$
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Write an equation for a line perpendicular to $f(x) = 5x - 1$ and passing through the point (5, 20).

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Find the equation of a line with a *y*- intercept of $(0, \quad 2)$ and slope $-\frac{1}{2}$.

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Sketch a graph of the linear function $f(t) = 2t - 5$.

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Find the point of intersection for the 2 linear functions: $x = y+$
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A car rental company offers two plans for renting a car.

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How many miles would you need to drive for plan B to save you money?

Modeling with Linear Functions

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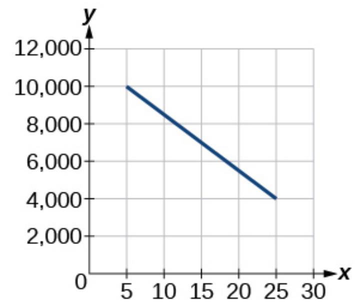

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29.

Assume the population is changing linearly.

1. ⓐ How much did the population grow between the year 2004 and 2012?
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Find a formula for the moose population, P.

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Year	Pima Central	East Valley
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2010	85,000	150,000

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In which subdivision have home values increased at a higher rate?

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0	2	4	6	8	10
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Draw a scatter plot for the data in Table 3. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation?

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2000	6,300
2005	6,600
2010	6,900

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36.

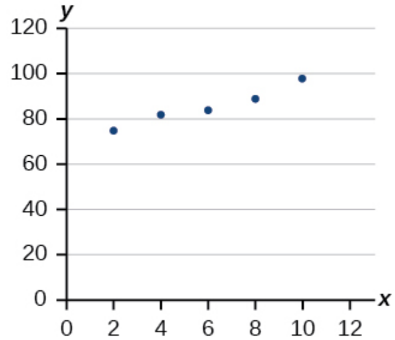
Eight students were asked to estimate their score on a 10-point quiz. Their estimated and actual scores are given in Table 4. Plot the points, then sketch a line that fits the data.

Predicted	**Actual**
6	6
7	7
7	8
8	8
7	9
9	10
10	10
10	9

Table 4

37.

Draw a best-fit line for the plotted data.



For the following exercises, consider the data in Table 5, which shows the percent of unemployed in a city of people 25 years or older who are college graduates is given below, by year.

Year	2000	2002	2005	2007	2010
Percent Graduates	6.5	7.0	7.4	8.2	9.0

Table 5

38.

Determine whether the trend appears to be linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

39.

In what year will the percentage exceed 12%?

40.

Based on the set of data given in Table 6, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	17	20	23	26	29
y	15	25	31	37	40

Table 6

41.

Based on the set of data given in Table 7, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	10	12	15	18	20
y	36	34	30	28	22

Table 7

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs show the population and the year over the ten-year span (population, year) for specific recorded years:

$(3{,}600,\ 2000)$; $(4{,}000,\ 2001)$; $(4{,}700,\ 2003)$; $(6{,}000,\ 2006)$

42.

Use linear regression to determine a function y, where the year depends on the population, to three decimal places of accuracy.

43.

Predict when the population will hit 12,000.

44.

What is the correlation coefficient for this model to three decimal places of accuracy?

45.

According to the model, what is the population in 2014?

1.10.2: Practice Test

Practice Test

1.

Determine whether the following algebraic equation can be written as a linear function. $2x + 3y =$

2.

Determine whether the following function is increasing or decreasing. $f(x) = -2x + 5$

3.

Determine whether the following function is increasing or decreasing. $f(x) = 7x + 9$

4.

Given the following set of information, find a linear equation satisfying the conditions, if possible.

Passes through (5, 1) and (3, –9)

5.

Given the following set of information, find a linear equation satisfying the conditions, if possible.

x intercept at (–4, 0) and *y*-intercept at (0, –6)

6.

Find the slope of the line in Figure 1.

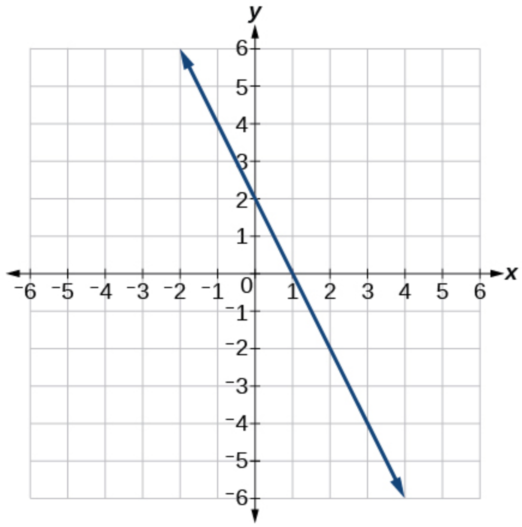


Figure 1

7.

Write an equation for line in Figure 2.

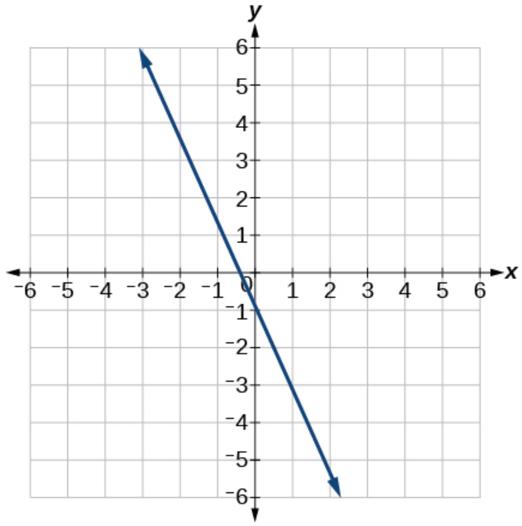


Figure 2
8.

Does Table 1 represent a linear function? If so, find a linear equation that models the data.

x	–6	0	2	4
$g(x)$	14	32	38	44

Table 1
9.

Does Table 2 represent a linear function? If so, find a linear equation that models the data.

x	1	3	7	11
$g(x)$	4	9	19	12

Table 2
10.

At 6 am, an online company has sold 120 items that day. If the company sells an average of 30 items per hour for the remainder of the day, write an expression to represent the number of items that were sold n after 6 am.

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

11.

$y = \frac{3}{4}x - 9$
$-4x - 3y = 8$

12.

$-2x + y = 3$
$3x + \frac{3}{2}y = 5$

13.

Find the *x*- and *y*-intercepts of the equation $2x \quad + \quad 7y = -14$.

14.

Given below are descriptions of two lines. Find the slopes of Line 1 and Line 2. Is the pair of lines parallel, perpendicular, or neither?

Line 1: Passes through $(-2, -6)$ and $(3, 14)$

Line 2: Passes through $(2, 6)$ and $(4, 14)$

15.

Write an equation for a line perpendicular to $f(x) = 4x + 3$ and passing through the point $(8, 10)$.

16.

Sketch a line with a *y*-intercept of $(0, 5)$ and slope $-\frac{5}{2}$.

17.

Graph of the linear function $f(x) = -\mathrm{x} + 6$.

18.

For the two linear functions, find the point of intersection: $x = y+$ 2 $2x$ $-3y = -1$

19.

A car rental company offers two plans for renting a car.

- Plan A: \$25 per day and \$0.10 per mile
- Plan B: \$40 per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

20.

Find the area of a triangle bounded by the *y* axis, the line $f(x) = 12-$, and the line perpendicular to f that passes through the origin.

21.

A town's population increases at a constant rate. In 2010 the population was 65,000. By 2012 the population had increased to 90,000. Assuming this trend continues, predict the population in 2018.

22.

The number of people afflicted with the common cold in the winter months dropped steadily by 25 each year since 2002 until 2012. In 2002, 8,040 people were inflicted. Find the linear function that models the number of people afflicted with the common cold C as a function of the year, t. When will less than 6,000 people be afflicted?

For the following exercises, use the graph in Figure 3, showing the profit, y, in thousands of dollars, of a company in a given year, x, where x represents years since 1980.

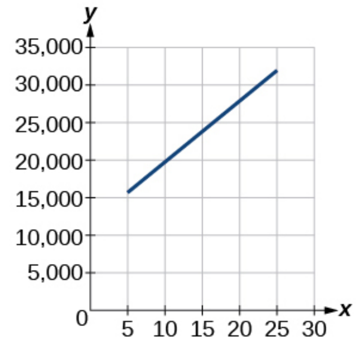


Figure 3
23.

Find the linear function y, where y depends on x, the number of years since 1980.

24.

Find and interpret the *y*-intercept.

25.

In 2004, a school population was 1250. By 2012 the population had dropped to 875. Assume the population is changing linearly.

1. ⓐ How much did the population drop between the year 2004 and 2012?
2. ⓑ What is the average population decline per year?
3. ⓒ Find an equation for the population, *P*, of the school *t* years after 2004.

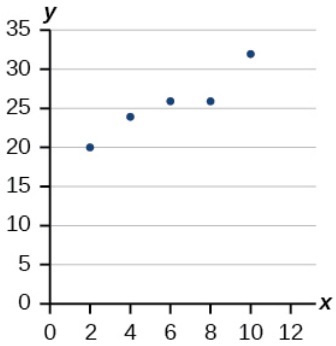
26.

Draw a scatter plot for the data provided in Table 3. Then determine whether the data appears to be linearly related.

0	2	4	6	8	10
–450	–200	10	265	500	755

Table 3
27.

Draw a best-fit line for the plotted data.



For the following exercises, use Table 4, which shows the percent of unemployed persons 25 years or older who are college graduates in a particular city, by year.

Year	2000	2002	2005	2007	2010
Percent Graduates	8.5	8.0	7.2	6.7	6.4

Table 4
28.

Determine whether the trend appears linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

29.

In what year will the percentage drop below 4%?

30.

Based on the set of data given in Table 5, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient. Round to three decimal places of accuracy.

x	16	18	20	24	26
y	106	110	115	120	125

Table 5

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population (in hundreds) and the year over the ten-year span, (population, year) for specific recorded years:

$(4{,}500,\ 2000)$; $(4{,}700,\ 2001)$; $(5{,}200,\ 2003)$; $(5{,}800,\ 2006)$

31.

Use linear regression to determine a function *y*, where the year depends on the population. Round to three decimal places of accuracy.

32.

Predict when the population will hit 20,000.

33.

What is the correlation coefficient for this model?

CHAPTER OVERVIEW

2: Systems of Linear Equations and Matrices

Learning Objectives

In this chapter, you will learn to:

1. Do matrix operations.
2. Solve linear systems using the Gauss-Jordan method.
3. Solve linear systems using the matrix inverse method.
4. Do application problems.

Thumbnail: (via Wikipedia)

2.1: Introduction to Systems of Equations and Inequalities

Enigma machines like this one, once owned by Italian dictator Benito Mussolini, were used by government and military officials for enciphering and deciphering top-secret communications during World War II. (credit: Dave Addey, Flickr)

Chapter Outline

At the start of the Second World War, British military and intelligence officers recognized that defeating Nazi Germany would require the Allies to know what the enemy was planning. This task was complicated by the fact that the German military transmitted all of its communications through a presumably uncrackable code created by a machine called Enigma. The Germans had been encoding their messages with this machine since the early 1930s. Not long after the war started, the British recruited a team of brilliant codebreakers to crack the Enigma code. The codebreakers used what they knew about the Enigma machine to build a mechanical computer that could crack the code. The Germans were so confident that the code could not be cracked, that they felt comfortable transmitting all manner of battlefield intelligence encoded with the machine. But the Allies had cracked it. And that knowledge of what the Germans were planning proved to be a key part of the ultimate Allied victory of Nazi Germany in 1945.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

2.2: Systems of Linear Equations - Two Variables

> **Learning Objectives**
>
> In this section, you will:
>
> - Solve systems of equations by graphing.
> - Solve systems of equations by substitution.
> - Solve systems of equations by addition.
> - Identify inconsistent systems of equations containing two variables.
> - Express the solution of a system of dependent equations containing two variables.

Figure 1 (credit: Thomas Sørenes)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. A system of linear equations consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x+y= & \quad 15 \\ 3x-y= & \quad 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair (4, 7) is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$\begin{gathered} 2(4)+(7)=15 \quad \text{True} \\ 3(4)-(7)=5 \quad \text{True} \end{gathered}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an **independent**

system if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same *y*-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different *y*-intercepts. There are no points common to both lines; hence, there is no solution to the system.

Types of Linear Systems

There are three types of systems of linear equations in two variables, and three types of solutions.

- An independent system has exactly one solution pair (x, y). The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Figure 2 compares graphical representations of each type of system.

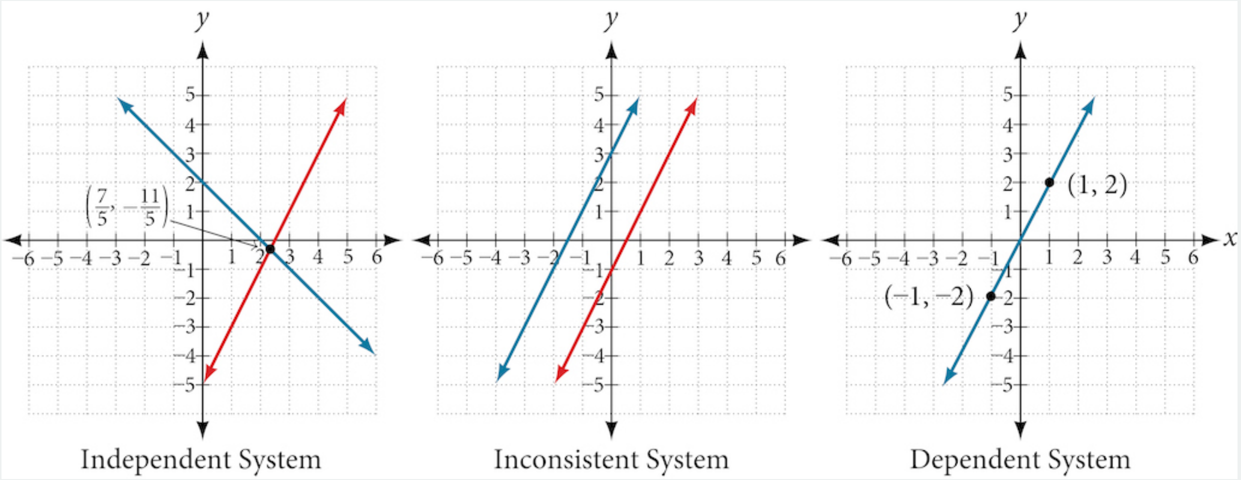


Figure 2

How To

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution.

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example 1

Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$\begin{aligned} x + 3y &= 8 \\ 2x - 9 &= y \end{aligned}$$

Answer

Substitute the ordered pair $(5, 1)$ into both equations.

$$
\begin{aligned}
(5)+3(1)&=8 \\
8&=8 \qquad \text{True} \\
2(5)-9&=(1) \\
1&=1 \qquad \text{True}
\end{aligned}
$$

The ordered pair $(5,1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure 3.

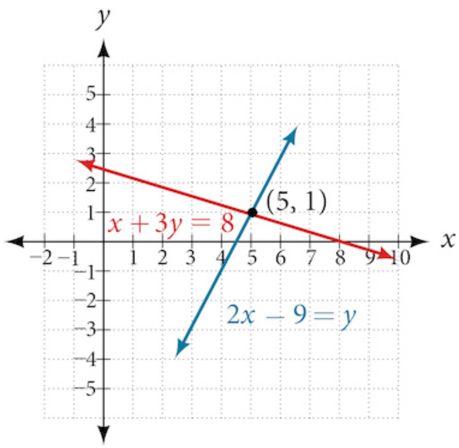


Figure 3

Try It #1

Determine whether the ordered pair $(8,5)$ is a solution to the following system.

$$
\begin{aligned}
5x-4y&=20 \\
2x+1&=3y
\end{aligned}
$$

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

Example 2

Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$
\begin{aligned}
2x+y&=-8 \\
x-y&=-1
\end{aligned}
$$

Answer

Solve the first equation for y.

$$
\begin{aligned}
2x+y&=-8 \\
y&=-2x-8
\end{aligned}
$$

Solve the second equation for y.

$$
\begin{aligned}
x-y&=-1 \\
y&=x+1
\end{aligned}
$$

Graph both equations on the same set of axes as in Figure 4.

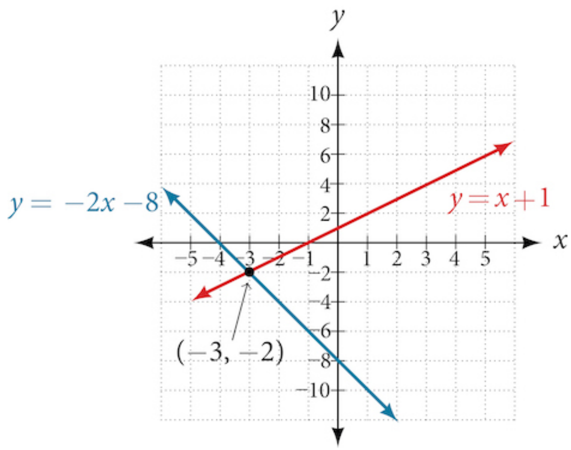


Figure 4

The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3)+(-2) &= -8 \\ -8 &= -8 \qquad \text{True} \\ (-3)-(-2) &= -1 \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$, so the system is independent.

Try It #2

Solve the following system of equations by graphing.

$$\begin{aligned} 2x-5y &= -25 \\ -4x+5y &= 35 \end{aligned}$$

Q&A

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the substitution method, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How To

Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.

3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example 3

Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned} -x+y&=-5 \\ 2x-5y&=1 \end{aligned}$$

Answer

First, we will solve the first equation for y.

$$\begin{aligned} -x+y&=-5 \\ y&=x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned} 2x-5y&=1 \\ 2x-5(x-5)&=1 \\ 2x-5x+25&=1 \\ -3x&=-24 \\ x&=8 \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for y.

$$\begin{aligned} -(8)+y&=-5 \\ y&=3 \end{aligned}$$

Our solution is $(8,3)$.

Check the solution by substituting $(8,3)$ into both equations.

$$\begin{aligned} -x+y&=-5 \\ -(8)+(3)&=-5 && \text{True} \\ 2x-5y&=1 \\ 2(8)-5(3)&=1 && \text{True} \end{aligned}$$

Try It #3

Solve the following system of equations by substitution.

$$\begin{aligned} x&=y+3 \\ 4&=3x-2y \end{aligned}$$

Q&A

Can the substitution method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of 1 or –1 so that we do not have to deal with fractions.

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How To

Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example 4

Solving a System by the Addition Method

Solve the given system of equations by addition.

$$\begin{aligned} x+2y&=-1 \\ -x+y&=3 \end{aligned}$$

Answer

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, –1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{array}{r} x+2y=-1 \\ \underline{-x+y=3} \\ 3y=2 \end{array}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$\begin{aligned} 3y&=2 \\ y&=\tfrac{2}{3} \end{aligned}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x+y&=3 \\ -x+\tfrac{2}{3}&=3 \\ -x&=3-\tfrac{2}{3} \\ -x&=\tfrac{7}{3} \\ x&=-\tfrac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3},\frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned}&x+2y=-1\\&\left(-\tfrac{7}{3}\right)+2\left(\tfrac{2}{3}\right)=\\&-\tfrac{7}{3}+\tfrac{4}{3}=\\&-\tfrac{3}{3}=\\&-1=-1 &&\text{True}\end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 5 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

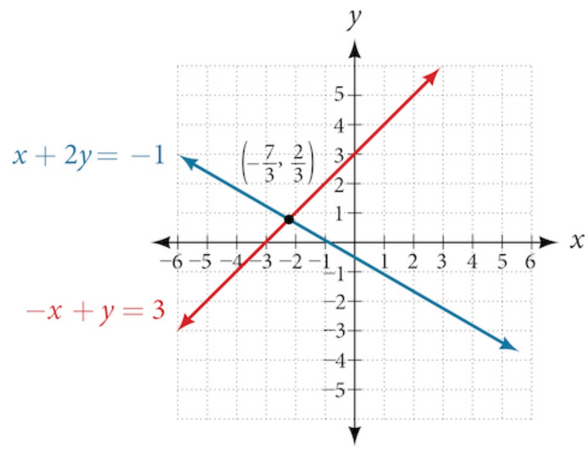


Figure 5

Example 5

Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$3x+5y=-11$$

$$x-2y=11$$

Answer

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by $-3,$ the *x*-terms will add to zero.

$$\begin{aligned}&x-2y=11\\&-3(x-2y)=-3(11) &&\text{Multiply both sides by}-3.\\&-3x+6y=-33 &&\text{Use the distributive property.}\end{aligned}$$

Now, let's add them.

$$\begin{aligned}3x+5y&=-11\\-3x+6y&=-33\\\hline 11y&=-44\\y&=-4\end{aligned}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.

$$\begin{aligned}3x+5y&=-11\\3x+5(-4)&=-11\\3x-20&=-11\\3x&=9\\x&=3\end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See Figure 6. Check the solution in the original second equation.

$$\begin{aligned}x-2y&=11\\(3)-2(-4)&=3+8\\11&=11 \qquad\qquad \text{True}\end{aligned}$$

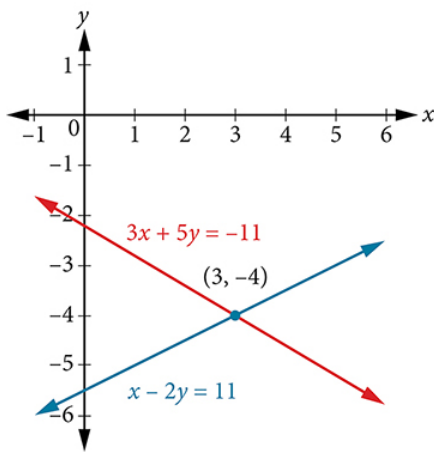


Figure 6

Try It #4

Solve the system of equations by addition.

$$\begin{aligned}2x-7y&=2\\3x+y&=-20\end{aligned}$$

Example 6

Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$\begin{aligned}2x+3y&=-16\\5x-10y&=30\end{aligned}$$

Answer

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned}-5(2x+3y)&=-5(-16)\\-10x-15y&=80\\2(5x-10y)&=2(30)\\10x-20y&=60\end{aligned}$$

Then, we add the two equations together.

$$
\begin{aligned}
-10x - 15y &= 80 \\
\underline{10x - 20y = 60} \\
-35y &= 140 \\
y &= -4
\end{aligned}
$$

Substitute $y = -4$ into the original first equation.

$$
\begin{aligned}
2x + 3(-4) &= -16 \\
2x - 12 &= -16 \\
2x &= -4 \\
x &= -2
\end{aligned}
$$

The solution is $(-2, -4)$. Check it in the other equation.

$$
\begin{aligned}
5x - 10y &= 30 \\
5(-2) - 10(-4) &= 30 \\
-10 + 40 &= 30 \\
30 &= 30
\end{aligned}
$$

See Figure 7.

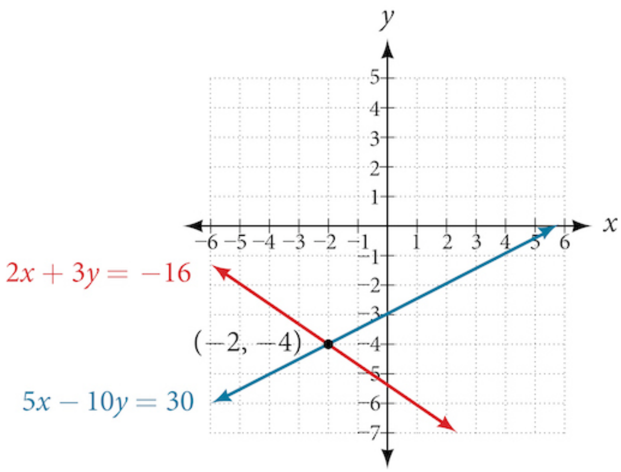


Figure 7

Example 7

Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$
\begin{aligned}
\frac{x}{3} + \frac{y}{6} &= 3 \\
\frac{x}{2} - \frac{y}{4} &= 1
\end{aligned}
$$

Answer

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$
\begin{aligned}
6\left(\frac{x}{3} + \frac{y}{6}\right) &= 6(3) \\
2x + y &= 18 \\
4\left(\frac{x}{2} - \frac{y}{4}\right) &= 4(1) \\
2x - y &= 4
\end{aligned}
$$

Now multiply the second equation by -1 so that we can eliminate the *x*-variable.

$$\begin{aligned} -1(2x-y) &= -1(4) \\ -2x+y &= -4 \end{aligned}$$

Add the two equations to eliminate the *x*-variable and solve the resulting equation.

$$\begin{aligned} &\;\;2x+y=18 \\ &\underline{-2x+y=-4} \\ &2y=14 \\ &y=7 \end{aligned}$$

Substitute $y=7$ into the first equation.

$$\begin{aligned} 2x+(7) &= 18 \\ 2x &= 11 \\ x &= \frac{11}{2} \\ &= 5.5 \end{aligned}$$

The solution is $\left(\frac{11}{2}, 7\right)$. Check it in the other equation.

$$\begin{aligned} \frac{x}{2}-\frac{y}{4} &= 1 \\ \frac{\frac{11}{2}}{2}-\frac{7}{4} &= 1 \\ \frac{11}{4}-\frac{7}{4} &= 1 \\ \frac{4}{4} &= 1 \end{aligned}$$

Try It #5

Solve the system of equations by addition.

$$\begin{aligned} 2x+3y &= 8 \\ 3x+5y &= 10 \end{aligned}$$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y -intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12=0$.

Example 8

Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{aligned} x &= 9-2y \\ x+2y &= 13 \end{aligned}$$

Answer

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} &x+2y=13 \\ &(9-2y)+2y=13 \\ &9+0y=13 \\ &9=13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9 - 2y \\ 2y &= -x + 9 \\ y &= -\tfrac{1}{2}x + \tfrac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x + 2y &= 13 \\ 2y &= -x + 13 \\ y &= -\tfrac{1}{2}x + \tfrac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different *y*-intercepts. Therefore, the lines are parallel and do not intersect.

$$\begin{aligned} y &= -\tfrac{1}{2}x + \tfrac{9}{2} \\ y &= -\tfrac{1}{2}x + \tfrac{13}{2} \end{aligned}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure 8.

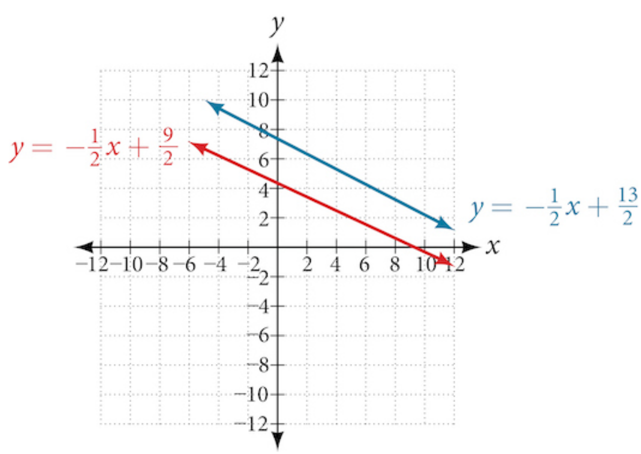


Figure 8

Try It #6

Solve the following system of equations in two variables.

$$\begin{aligned} 2y - 2x &= 2 \\ 2y - 2x &= 6 \end{aligned}$$

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0 = 0$.

Example 9

Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the addition method.

$$\begin{aligned} x+3y&=2 \\ 3x+9y&=6 \end{aligned}$$

Answer

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x -variable.

$$\begin{aligned} x+3y&=2 \\ (-3)(x+3y)&=(-3)(2) \\ -3x-9y&=-6 \end{aligned}$$

Now add the equations.

$$\begin{array}{rl} -3x-9y & =-6 \\ +\quad 3x+9y & =6 \\ \hline 0 & =0 \end{array}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x+3y&=2 \\ 3y&=-x+2 \\ y&=-\tfrac{1}{3}x+\tfrac{2}{3} \\ 3x+9y&=6 \\ 9y&=-3x+6 \\ y&=-\tfrac{3}{9}x+\tfrac{6}{9} \\ y&=-\tfrac{1}{3}x+\tfrac{2}{3} \end{aligned}$$

See Figure 9. Notice the results are the same. The general solution to the system is $\left(x\,,\,-\frac{1}{3}x+\frac{2}{3}\right)$.

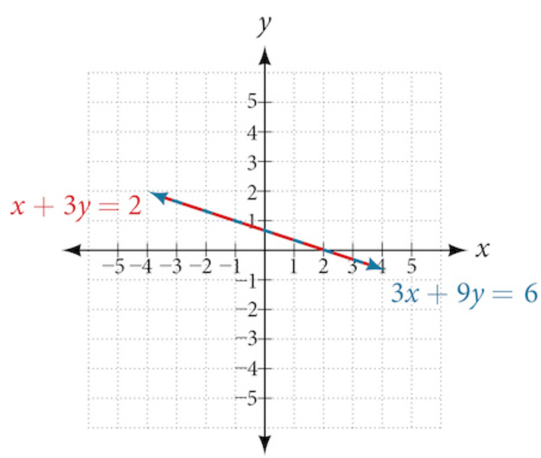


Figure 9

Try It #7

Solve the following system of equations in two variables.

$$y - 2x = 5$$
$$-3y + 6x = -15$$

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where $x =$ quantity and $p =$ price. The revenue function is shown in orange in Figure 10.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 10. The x -axis represents quantity in hundreds of units. The *y*-axis represents either cost or revenue in hundreds of dollars.

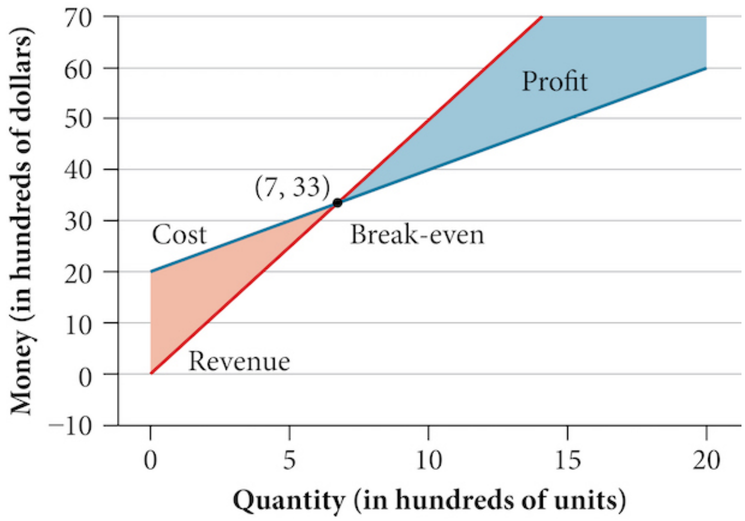


Figure 10

The point at which the two lines intersect is called the break-even point. We can see from the graph that if 700 units are produced, the cost is \$3,300 and the revenue is also \$3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example 10

Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35{,}000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Answer

Write the system of equations using y to replace function notation.

$$y = 0.85x + 35{,}000$$
$$y = 1.55x$$

Substitute the expression $0.85x + 35{,}000$ from the first equation into the second equation and solve for x.

$$0.85x + 35{,}000 = 1.55x$$
$$35{,}000 = 0.7x$$
$$50{,}000 = x$$

Then, we substitute $x = 50{,}000$ into either the cost function or the revenue function.

$$1.55\,(50{,}000) = 77{,}500$$

The break-even point is $(50{,}000, 77{,}500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$P(x) = 1.55x - (0.85x + 35,000)$$
$$= 0.7x - 35,000$$

The profit function is $P(x) = 0.7x - 35{,}000$.

Analysis

The cost to produce 50,000 units is $77,500, and the revenue from the sales of 50,000 units is also $77,500. To make a profit, the business must produce and sell more than 50,000 units. See Figure 11.

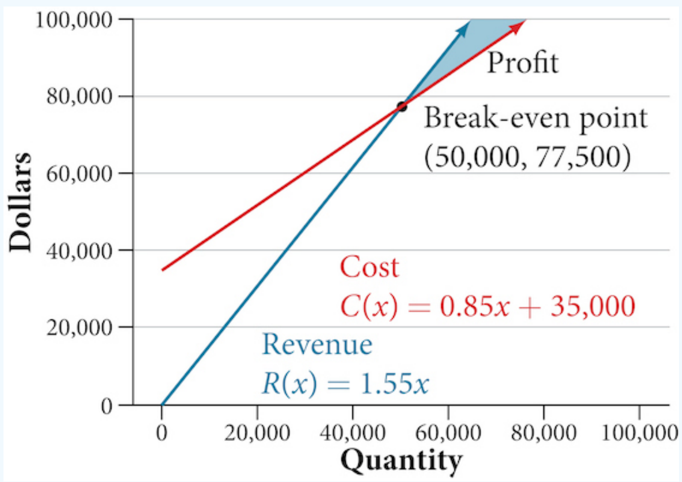


Figure 11

We see from the graph in Figure 12 that the profit function has a negative value until $x = 50{,}000$, when the graph crosses the *x*-axis. Then, the graph emerges into positive *y*-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

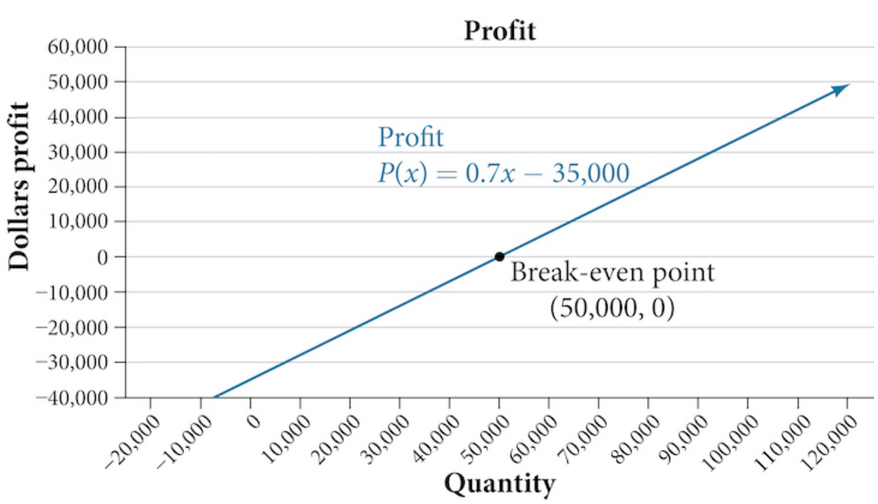


Figure 12

Example 11

Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000$. How many children and how many adults bought tickets?

Answer

Let c = the number of children and a = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c+a=2,000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70,000$. We can use this to write an equation for the revenue.

$$25c+50a=70,000$$

We now have a system of linear equations in two variables.

$$\begin{aligned} c+a&=2,000 \\ 25c+50a&=70,000 \end{aligned}$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$\begin{aligned} c+a&=2,000 \\ a&=2,000-c \end{aligned}$$

Substitute the expression $2,000-c$ in the second equation for a and solve for c.

$$\begin{aligned} 25c+50(2,000-c)&=70,000 \\ 25c+100,000-50c&=70,000 \\ -25c&=-30,000 \\ c&=1,200 \end{aligned}$$

Substitute $c=1,200$ into the first equation to solve for a.

$$\begin{aligned} 1,200+a&=2,000 \\ a&=800 \end{aligned}$$

We find that $1,200$ children and 800 adults bought tickets to the circus that day.

Try It #8

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1,650$ meal tickets were bought for a total of $\$14,200$, how many children and how many adults bought meal tickets?

Media

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations

9.1 Section Exercises

Verbal

1.

Can a system of linear equations have exactly two solutions? Explain why or why not.

2.

If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.

3.

If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

4.

If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

5.

Given a system of equations, explain at least two different methods of solving that system.

Algebraic

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

6.

$$\begin{aligned} 5x-y&=4\\ x+6y&=2 \end{aligned} \quad \text{and } (4,0)$$

7.

$$\begin{aligned} -3x-5y&=13\\ -x+4y&=10 \end{aligned} \quad \text{and } (-6,1)$$

8.

$$\begin{aligned} 3x+7y&=1\\ 2x+4y&=0 \end{aligned} \quad \text{and } (2,3)$$

9.

$$\begin{aligned} -2x+5y&=7\\ 2x+9y&=7 \end{aligned} \quad \text{and } (-1,1)$$

10.

$$\begin{aligned} x+8y&=43\\ 3x-2y&=-1 \end{aligned} \quad \text{and } (3,5)$$

For the following exercises, solve each system by substitution.

11.

$$\begin{aligned} x+3y&=5\\ 2x+3y&=4 \end{aligned}$$

12.

$$\begin{aligned} 3x-2y&=18\\ 5x+10y&=-10 \end{aligned}$$

13.

$$\begin{aligned} 4x+2y&=-10\\ 3x+9y&=0 \end{aligned}$$

14.

$2x+4y=-3.8$
$9x-5y=1.3$

15.

$-2x+3y=1.2$
$-3x-6y=1.8$

16.

$x-0.2y=1$
$-10x+2y=5$

17.

$3x+5y=9$
$30x+50y=-90$

18.

$-3x+y=2$
$12x-4y=-8$

19.

$\frac{1}{2}x+\frac{1}{3}y=16$
$\frac{1}{6}x+\frac{1}{4}y=9$

20.

$-\frac{1}{4}x+\frac{3}{2}y=11$
$-\frac{1}{8}x+\frac{1}{3}y=3$

For the following exercises, solve each system by addition.

21.

$-2x+5y=-42$
$7x+2y=30$

22.

$6x-5y=-34$
$2x+6y=4$

23.

$5x-y=-2.6$
$-4x-6y=1.4$

24.

$7x-2y=3$
$4x+5y=3.25$

25.

$-\mathrm{x}+2y=-1$
$5x-10y=6$

26.

$7x+6y=2$
$-28x-24y=-8$

27.

$\frac{5}{6}x+\frac{1}{4}y=0$
$\frac{1}{8}x-\frac{1}{2}y=-\frac{43}{120}$

28.

$$\frac{1}{3}x + \frac{1}{9}y = \frac{2}{9}$$
$$-\frac{1}{2}x + \frac{4}{5}y = -\frac{1}{3}$$

29.

$$-0.2x + 0.4y = 0.6$$
$$x - 2y = -3$$

30.

$$-0.1x + 0.2y = 0.6$$
$$5x - 10y = 1$$

For the following exercises, solve each system by any method.

31.

$$5x + 9y = 16$$
$$x + 2y = 4$$

32.

$$6x - 8y = -0.6$$
$$3x + 2y = 0.9$$

33.

$$5x - 2y = 2.25$$
$$7x - 4y = 3$$

34.

$$x - \frac{5}{12}y = -\frac{55}{12}$$
$$-6x + \frac{5}{2}y = \frac{55}{2}$$

35.

$$7x - 4y = \frac{7}{6}$$
$$2x + 4y = \frac{1}{3}$$

36.

$$3x + 6y = 11$$
$$2x + 4y = 9$$

37.

$$\frac{7}{3}x - \frac{1}{6}y = 2$$
$$-\frac{21}{6}x + \frac{3}{12}y = -3$$

38.

$$\frac{1}{2}x + \frac{1}{3}y = \frac{1}{3}$$
$$\frac{3}{2}x + \frac{1}{4}y = -\frac{1}{8}$$

39.

$$2.2x + 1.3y = -0.1$$
$$4.2x + 4.2y = 2.1$$

40.

$$0.1x + 0.2y = 2$$
$$0.35x - 0.3y = 0$$

Graphical

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

41.

$$\begin{aligned} 3x - y &= 0.6 \\ x - 2y &= 1.3 \end{aligned}$$

42.

$$\begin{aligned} -x + 2y &= 4 \\ 2x - 4y &= 1 \end{aligned}$$

43.

$$\begin{aligned} x + 2y &= 7 \\ 2x + 6y &= 12 \end{aligned}$$

44.

$$\begin{aligned} 3x - 5y &= 7 \\ x - 2y &= 3 \end{aligned}$$

45.

$$\begin{aligned} 3x - 2y &= 5 \\ -9x + 6y &= -15 \end{aligned}$$

Technology

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

46.

$$\begin{aligned} 0.1x + 0.2y &= 0.3 \\ -0.3x + 0.5y &= 1 \end{aligned}$$

47.

$$\begin{aligned} -0.01x + 0.12y &= 0.62 \\ 0.15x + 0.20y &= 0.52 \end{aligned}$$

48.

$$\begin{aligned} 0.5x + 0.3y &= 4 \\ 0.25x - 0.9y &= 0.46 \end{aligned}$$

49.

$$\begin{aligned} 0.15x + 0.27y &= 0.39 \\ -0.34x + 0.56y &= 1.8 \end{aligned}$$

50.

$$\begin{aligned} -0.71x + 0.92y &= 0.13 \\ 0.83x + 0.05y &= 2.1 \end{aligned}$$

Extensions

For the following exercises, solve each system in terms of $A, B, C, D, E,$ and F where $A – F$ are nonzero numbers. Note that $A \neq B$ and $AE \neq BD$.

51.

$x+y=A$

$x-y=B$

52.

$x+Ay=1$

$x+By=1$

53.

$Ax+y=0$

$Bx+y=1$

54.

$Ax+By=C$

$x+y=1$

55.

$Ax+By=C$

$Dx+Ey=F$

Real-World Applications

For the following exercises, solve for the desired quantity.

56.

A stuffed animal business has a total cost of production $C=12x+30$ and a revenue function $R=20x$. Find the break-even point.

57.

A fast-food restaurant has a cost of production $C(x)=11x+120$ and a revenue function $R(x)=5x$. When does the company start to turn a profit?

58.

A cell phone factory has a cost of production $C(x)=150x+10,000$ and a revenue function $R(x)=200x$. What is the break-even point?

59.

A musician charges $C(x)=64x+20{,}000$ where x is the total number of attendees at the concert. The venue charges \$80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

60.

A guitar factory has a cost of production $C(x)=75x+50{,}000$. If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

61.

Find two numbers whose sum is 28 and difference is 13.

62.

A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.

63.

The startup cost for a restaurant is \$120,000, and each meal costs \$10 for the restaurant to make. If each meal is then sold for \$15, after how many meals does the restaurant break even?

64.

A moving company charges a flat rate of \$150, and an additional \$5 for each box. If a taxi service would charge \$20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

65.

A total of 1,595 first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?

66.

276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

67.

There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?

68.

A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.

69.

If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?

70.

An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?

71.

An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

72.

If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays $2\frac{7}{8}\%$ interest, and the investor earns $737.50 annual interest, how much was invested in each account?

73.

If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?

74.

CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?

75.

A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?

76.

A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?

77.

Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child's ticket and the price of the adult ticket?

2.3: Introduction to Matrices

Learning Objectives

In this section, you will learn to:

1. Add and subtract matrices.
2. Multiply a matrix by a scalar.
3. Multiply two matrices.

A matrix is a 2 dimensional array of numbers arranged in rows and columns. Matrices provide a method of organizing, storing, and working with mathematical information. Matrices have an abundance of applications and use in the real world. Matrices provide a useful tool for working with models based on systems of linear equations. We'll use matrices in sections 2.2, 2.3, and 2.4 to solve systems of linear equations with several variables in this chapter.

Matrices are used in encryption, which we will explore in section 2.5 and in economic modelling, explored in section 2.6. We use matrices again in chapter 4, in optimization problems such as maximizing profit or revenue, or minimizing cost. Matrices are used in business for scheduling, routing transportation and shipments, and managing inventory.

Just about any application that collects and manages data can apply matrices. Use of matrices has grown as the availability of data in many areas of life and business has increased. They are important tools for organizing data and solving problems in all fields of science, from physics and chemistry, to biology and genetics, to meteorology, and economics. In computer science, matrix mathematics lies behind animation of images in movies and video games.

Computer science analyzes diagrams of networks to understand how things are connected to each other, such as relationships between people on a social website, and relationships between results in line search and how people link from one website to another. The mathematics to work with network diagrams comprise the field of "graph theory"; it relies on matrices to organize the information in the graphs that diagram connections and associations in a network. For example, if you use Facebook or Linked-In, or other social media sites, these sites use network graphs and matrices to organize your relationships with other users.

Introduction to Matrices

A matrix is a rectangular array of numbers. Matrices are useful in organizing and manipulating large amounts of data. In order to get some idea of what matrices are all about, we will look at the following example.

Example 2.3.1

Fine Furniture Company makes chairs and tables at its San Jose, Hayward, and Oakland factories. The total production, in hundreds, from the three factories for the years 2014 and 2015 is listed in the table below.

	2014		2015	
	CHAIRS	TABLES	CHAIRS	TABLES
SAN JOSE	30	18	36	20
HAYWARD	20	12	24	18
OAKLAND	16	10	20	12

a. Represent the production for the years 2014 and 2015 as the matrices A and B.
b. Find the difference in sales between the years 2014 and 2015.
c. The company predicts that in the year 2020 the production at these factories will be double that of the year 2014. What will the production be for the year 2020?

Solution

a) The matrices are as follows:

$$A=\begin{bmatrix}30&18\\20&12\\16&10\end{bmatrix}$$

$$\mathrm{B}=\begin{bmatrix}36&20\\24&18\\20&12\end{bmatrix}$$

b) We are looking for the matrix $B-A$. When two matrices have the same number of rows and columns, the matrices can be added or subtracted entry by entry. Therefore, we get

$$\mathrm{B}-\mathrm{A}=\begin{bmatrix}36-30&20-18\\24-20&18-12\\20-16&12-10\end{bmatrix}=\begin{bmatrix}6&2\\4&6\\4&2\end{bmatrix}$$

c) We would like a matrix that is twice the matrix of 2014, i.e., $2A$.

Whenever a matrix is multiplied by a number, each entry is multiplied by the number.

$$2\mathrm{A}=2\begin{bmatrix}30&18\\20&12\\16&10\end{bmatrix}=\begin{bmatrix}60&36\\40&24\\32&20\end{bmatrix}$$

Before we go any further, we need to familiarize ourselves with some terms that are associated with matrices. The numbers in a matrix are called the **entries** or the **elements** of a matrix.

Whenever we talk about a matrix, we need to know the **size** or the **dimension** of the matrix. The dimension of a matrix is the number of rows and columns it has. When we say a matrix is a “3 by 4 matrix”, we are saying that it has 3 rows and 4 columns. The rows are always mentioned first and the columns second. This means that a 3×4 matrix does not have the same dimension as a 4×3 matrix.

$$A=\begin{bmatrix}1&4&-2&0\\3&-1&7&9\\6&2&0&5\end{bmatrix}$$

$$B=\begin{bmatrix}2&9&8\\-3&0&1\\6&5&-2\\-4&7&8\end{bmatrix}$$

Matrix A has dimensions 3×4 and matrix B has dimensions 4×3.

A matrix that has the same number of rows as columns is called a **square matrix**. A matrix with all entries zero is called a **zero matrix.** A square matrix with 1's along the main diagonal and zeros everywhere else, is called an **identity matrix**. When a square matrix is multiplied by an identity matrix of same size, the matrix remains the same.

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Matrix I is a 3×3 identity matrix

A matrix with only one row is called a row matrix or a **row vector,** and a matrix with only one column is called a column matrix or a **column vector**. Two matrices are **equal** if they have the same size and the corresponding entries are equal.

We can perform arithmetic operations with matrices. Next we will define and give examples illustrating the operations of matrix addition and subtraction, scalar multiplication, and matrix multiplication. Note that matrix multiplication is quite different from what you would intuitively expect, so pay careful attention to the explanation. Note also that the ability to perform matrix operations depends on the matrices involved being compatible in size, or dimensions, for that operation. The definition of compatible dimensions is different for different operations, so note the requirements carefully for each.

Matrix Addition and Subtraction

If two matrices have the same size, they can be added or subtracted. The operations are performed on corresponding entries.

Example 2.3.2

Given the matrices A, B, C and D, below

$$A=\begin{bmatrix}1&2&4\\2&3&1\\5&0&3\end{bmatrix}\quad B=\begin{bmatrix}2&-1&3\\2&4&2\\3&6&1\end{bmatrix}\quad C=\begin{bmatrix}4\\2\\3\end{bmatrix}\quad D=\begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

Find, if possible.

a. $A+B$
b. $C-D$
c. $A+D$.

Solution

As we mentioned earlier, matrix addition and subtraction involves performing these operations entry by entry.

a) We add each element of A to the corresponding entry of B.

$$A+B=\begin{bmatrix}3&1&7\\4&7&3\\8&6&4\end{bmatrix}$$

b) Just like the problem above, we perform the subtraction entry by entry.

$$\mathrm{C}-\mathrm{D}=\begin{bmatrix}6\\5\\-1\end{bmatrix}$$

c) The sum $A+D$ cannot be found because the two matrices have different sizes.

Note: Two matrices can *only* be added or subtracted if they have the same dimension.

Multiplying a Matrix by a Scalar

If a matrix is multiplied by a scalar, each entry is multiplied by that scalar. We can consider scalar multiplication as multiplying a number and a matrix to obtain a new matrix as the product.

Example 2.3.3

Given the matrix A and C in the example above, find $2A$ and $-3C$.

Solution

To find $2A$, we multiply each entry of matrix A by 2, and to find $-3C$, we multiply each entry of C by -3. The results are given below.

a) We multiply each entry of A by 2.

$$2\mathrm{A}=\begin{bmatrix}2&4&8\\4&6&2\\10&0&6\end{bmatrix}$$

b) We multiply each entry of C by -3.

$$-3C = \begin{bmatrix} -12 \\ -6 \\ -9 \end{bmatrix}$$

Multiplication of Two Matrices

To multiply a matrix by another is not as easy as the addition, subtraction, or scalar multiplication of matrices. Because of its wide use in application problems, it is important that we learn it well. Therefore, we will try to learn the process in a step by step manner. We first begin by finding a product of a row matrix and a column matrix.

Example 2.3.4

Find the product AB, given

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Solution

The product is a 1×1 matrix whose entry is obtained by multiplying the corresponding entries and then forming the sum.

$$\begin{aligned} \mathrm{AB} &= \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix} \\ &= [2(\mathrm{a} + 3\mathrm{b} + 4\mathrm{c})] \end{aligned}$$

Note that AB is a 1×1 matrix, and its only entry is $2a + 3b + 4c$.

Example 2.3.5

Find the product AB, given

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Solution

Again, we multiply the corresponding entries and add.

$$\begin{aligned} \mathrm{AB} &= \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \\ &= [2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7] \\ &= [10 + 18 + 28] \\ &= [56] \end{aligned}$$

Note: In order for a product of a row matrix and a column matrix to exist, the number of entries in the row matrix must be the same as the number of entries in the column matrix.

Example 2.3.6

Find the product AB, given

$$A=\begin{bmatrix}2 & 3 & 4\end{bmatrix}$$

and

$$B=\begin{bmatrix}5 & 3\\ 6 & 4\\ 7 & 5\end{bmatrix}.$$

Solution

We know how to multiply a row matrix by a column matrix. To find the product AB, in this example, we will multiply the row matrix A to both the first and second columns of matrix B, resulting in a 1×2 matrix.

$$\mathrm{AB}=\begin{bmatrix}2\cdot 5+3\cdot 6+4\cdot 7 & 2\cdot 3+3\cdot 4+4\cdot 5\end{bmatrix}=\begin{bmatrix}56 & 38\end{bmatrix}$$

We multiplied a 1×3 matrix by a matrix whose size is 3×2. So unlike addition and subtraction, it is possible to multiply two matrices with different dimensions, if the number of entries in the rows of the first matrix is the same as the number of entries in the columns of the second matrix.

Example 2.3.7

Find the product AB, given:

$$A=\begin{bmatrix}2 & 3 & 4\\ 1 & 2 & 3\end{bmatrix}$$

and

$$\mathrm{B}=\begin{bmatrix}5 & 3\\ 6 & 4\\ 7 & 5\end{bmatrix}$$

Solution

This time we are multiplying two rows of the matrix A with two columns of the matrix B. Since the number of entries in each row of A is the same as the number of entries in each column of B, the product is possible. We do exactly what we did in the last example. The only difference is that the matrix A has one more row.

We multiply the first row of the matrix A with the two columns of B, one at a time, and then repeat the process with the second row of A. We get

$$\mathrm{AB}=\begin{bmatrix}2 & 3 & 4\\ 1 & 2 & 3\end{bmatrix}\begin{bmatrix}5 & 3\\ 6 & 4\\ 7 & 5\end{bmatrix}=\begin{bmatrix}2\cdot 5+3\cdot 6+4\cdot 7 & 2\cdot 3+3\cdot 4+4\cdot 5\\ 1\cdot 5+2\cdot 6+3\cdot 7 & 1\cdot 3+2\cdot 4+3\cdot 5\end{bmatrix}$$

$$\mathrm{AB}=\begin{bmatrix}56 & 38\\ 38 & 26\end{bmatrix}$$

Example 2.3.8

Find, if possible:

a. EF
b. FE
c. FH
d. GH
e. HG

$$E=\begin{bmatrix}1&2\\4&2\\3&1\end{bmatrix} \quad F=\begin{bmatrix}2&-1\\3&2\end{bmatrix} \quad G=\begin{bmatrix}4&1\end{bmatrix} \quad H=\begin{bmatrix}-3\\-1\end{bmatrix}$$

Solution

a) To find EF, we multiply the first row $\begin{bmatrix}1&2\end{bmatrix}$

of E with the columns $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}1\\-2\end{bmatrix}$ of the matrix F, and then repeat the process by multiplying the other two rows of E with these columns of F. The result is as follows:

$$EF=\begin{bmatrix}1&2\\4&2\\3&1\end{bmatrix}\begin{bmatrix}2&-1\\3&2\end{bmatrix}=\begin{bmatrix}1\cdot2+2\cdot3&1\cdot-1+2\cdot2\\4\cdot2+2\cdot3&4\cdot-1+2\cdot2\\3\cdot2+1\cdot3&3\cdot-1+1\cdot2\end{bmatrix}=\begin{bmatrix}8&3\\14&0\\9&-1\end{bmatrix}$$

b) Product FE is not possible because the matrix F has two entries in each row, while the matrix E has three entries in each column. In other words, the matrix F has two columns, while the matrix E has three rows.

c)

$$FH=\begin{bmatrix}2&-1\\3&2\end{bmatrix}\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}2\cdot-3+-1\cdot-1\\3\cdot-3+2\cdot-1\end{bmatrix}=\begin{bmatrix}-5\\-11\end{bmatrix}$$

d)

$$GH=\begin{bmatrix}4&1\end{bmatrix}\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}4\cdot-3+1\cdot-1\\-1\end{bmatrix}=\begin{bmatrix}-13\end{bmatrix}$$

e)

$$HG=\begin{bmatrix}-3\\-1\end{bmatrix}\begin{bmatrix}4&1\end{bmatrix}=\begin{bmatrix}-3\cdot4&-3\cdot1\\-1\cdot4&-1\cdot1\end{bmatrix}=\begin{bmatrix}-12&-3\\-4&-1\end{bmatrix}$$

We summarize some important properties of matrix multiplication that we observed in the previous examples.

In order for product $\mathbf{AB}$ to exist:

- **the number of columns of $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$**
- **if matrix $\mathbf{A}$ has dimension $\mathbf{m}\times\mathbf{n}$ and matrix $\mathbf{B}$ has dimension $\mathbf{n}\times\mathbf{p}$, then the product $\mathbf{AB}$ will be a matrix with dimension $\mathbf{m}\times\mathbf{p}$.**

Matrix multiplication is not commutative: if both matrix products $\mathbf{AB}$ and $\mathbf{BA}$ exist, most of the time $\mathbf{AB}$ will not equal $\mathbf{BA}$.

✓ Example 2.3.9

Given matrices R, S, and T below, find $2RS-3ST$.

$$R=\begin{bmatrix}1&0&2\\2&1&5\\2&3&1\end{bmatrix} \quad S=\begin{bmatrix}0&-1&2\\3&1&0\\4&2&1\end{bmatrix} \quad T=\begin{bmatrix}-2&3&0\\-3&2&2\\-1&1&0\end{bmatrix}$$

Solution

We multiply the matrices R and S.

$$\mathrm{RS}=\begin{bmatrix}8&3&4\\23&9&9\\13&3&5\end{bmatrix}$$

$$2\mathrm{RS}=2\begin{bmatrix}8&3&4\\23&9&9\\13&3&5\end{bmatrix}=\begin{bmatrix}16&6&8\\46&18&18\\26&6&10\end{bmatrix}$$

$$\mathrm{ST}=\begin{bmatrix}1&0&-2\\-9&11&2\\-15&17&4\end{bmatrix}$$

$$3\mathrm{ST}=3\begin{bmatrix}1&0&-2\\-9&11&2\\-15&17&4\end{bmatrix}=\begin{bmatrix}3&0&-6\\-27&33&6\\-45&51&12\end{bmatrix}$$

Thus

$$2\mathrm{RS}-3\mathrm{ST}=\begin{bmatrix}16&6&8\\46&18&18\\26&6&10\end{bmatrix}-\begin{bmatrix}3&0&-6\\-27&33&6\\-45&51&12\end{bmatrix}=\begin{bmatrix}13&6&14\\73&-15&12\\71&-45&-2\end{bmatrix}$$

Example 2.3.10

Find F^2 given matrix

$$\mathrm{F}=\begin{bmatrix}2&-1\\3&2\end{bmatrix}$$

Solution

F^2 is found by multiplying matrix F by itself, using matrix multiplication.

$$\mathrm{F}^2=\begin{bmatrix}2&-1\\3&2\end{bmatrix}\begin{bmatrix}2&-1\\3&2\end{bmatrix}=\begin{bmatrix}2\cdot2+(-1)\cdot3&2\cdot(-1)+(-1)\cdot2\\3\cdot2+2\cdot3&3\cdot(-1)+2\cdot2\end{bmatrix}=\begin{bmatrix}1&-4\\12&1\end{bmatrix}$$

Note that F^2 is not found by squaring each entry of matrix F. The process of raising a matrix to a power, such as finding F^2, is only possible if the matrix is a square matrix.

USING MATRICES TO REPRESENT A SYSTEM OF LINEAR EQUATIONS

In this chapter, we will be using matrices to solve linear systems. In section 2.4, we will be asked to express linear systems as the **matrix equation $\mathbf{AX}=\mathbf{B}$**, where A, X, and B are matrices.

- Matrix A is called the **coefficient matrix.**
- Matrix X is a matrix with 1 column that contains the variables.
- Matrix B is a matrix with 1 column that contains the constants.

Example 2.3.11

Verify that the system of two linear equations with two unknowns:

$$\begin{aligned}ax+by&=h\\cx+dy&=k\end{aligned}\tag{2.3.1}$$

can be written as $AX=B$, where

$$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\quad X=\begin{bmatrix}x\\y\end{bmatrix}\text{ and }B=\begin{bmatrix}h\\k\end{bmatrix}$$

Solution

If we multiply the matrices A and X, we get

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

If $AX = B$ then

$$\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

If two matrices are equal, then their corresponding entries are equal. It follows that

$$\begin{aligned} ax+by &= h \\ cx+dy &= k \end{aligned} \tag{2.3.2}$$

Example 2.3.12

Express the following system as a matrix equation in the form $AX = B$.

$$\begin{aligned} 2x+3y-4z &= 5 \\ 3x+4y-5z &= 6 \\ 5x \quad\ -6z &= 7 \end{aligned} \tag{2.3.3}$$

Solution

This system of equations can be expressed in the form $AX = B$ as shown below.

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

2.3.1: Introduction to Matrices (Exercises)

A vendor sells hot dogs and corn dogs at three different locations. His total sales(in hundreds) for January and February from the three locations are given in the table below.

	JANUARY			FEBRUARY	
	HOT DOGS	CORN DOGS		HOT DOGS	CORN DOGS
PLACE I	10	8		8	7
PLACE II	8	6		6	7
PLACE III	6	4		6	5

Represent these tables as 3×2 matrices J and F, and answer problems 1 - 5.

1) Determine total sales for the two months, that is, find $J+F$.

2) Find the difference in sales, $J-F$.

3) If hot dogs sell for $3 and corn dogs for $2, find the revenue from the sale of hot dogs and corn dogs. *Hint: Let P be a 2×1 matrix. Find $(J+F)P$.*

4) If March sales will be up from February by 10%, 15%, and 20% at Place I, Place II, and Place III, respectively, find the expected number of hot dogs and corn dogs to be sold in March. *Hint: Let R be a 1×3 matrix with entries 1.10, 1.15, and 1.20. Find $M = RF$.*

5) Hots dogs sell for $3 and corn dogs sell for $2. Using matrix M that predicts the number of hot dogs and corn dogs expected to be sold in March from problem (4), find the 1×1 matrix that predicts total revenue in March. *Hint: Use 2×1 price matrix P from problem (3) and find MP.*

Determine the sums and products in problems 6-13. Given the matrices A, B, C, and D as follows:

$$\mathrm{A}=\begin{bmatrix}3 & 6 & 1\\ 0 & 1 & 3\\ 2 & 4 & 1\end{bmatrix} \quad \mathrm{B}=\begin{bmatrix}1 & -1 & 2\\ 1 & 4 & 2\\ 3 & 1 & 1\end{bmatrix} \quad \mathrm{C}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix} \quad \mathrm{D}=\begin{bmatrix}2 & 3 & 2\end{bmatrix}$$

6) $3A-2B$

7) AB

8) BA

9) $AB+BA$

10) A^2

11) $2BC$

12) $2CD+3AB$

13) A^2B

14) Let $E=\begin{bmatrix}m & n\\ p & q\end{bmatrix}$ and F $=\begin{bmatrix}\mathrm{a} & \mathrm{b}\\ \mathrm{c} & \mathrm{d}\end{bmatrix}$, find EF.

15) Let $E=\begin{bmatrix}m & n\\ p & q\end{bmatrix}$ and F $=\begin{bmatrix}\mathrm{a} & \mathrm{b}\\ \mathrm{c} & \mathrm{d}\end{bmatrix}$, find FE.

16) Let $G=\begin{bmatrix}3 & 6 & 1\\ 0 & 1 & 3\\ 2 & 4 & 1\end{bmatrix}$ and $H=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$, find GH.

17) Let $G=\begin{bmatrix}3 & 6 & 1\\ 0 & 1 & 3\\ 2 & 4 & 1\end{bmatrix}$ and $H=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$. Explain why the product HG does not exist.

Express the following systems as $AX=B$, where A, X, and B are matrices.

18)

$$\begin{aligned}4x-5y&=6\\ 5x-6y&=7\end{aligned} \tag{2.3.1.1}$$

19)

$$\begin{aligned}x-2y+2z&=3\\ x-3y+4z&=7\\ x-2y-3z&=-12\end{aligned} \tag{2.3.1.2}$$

20) $$\begin{array}{l} 2x+3z=17 \\ 3x-2y=10 \\ 5y+2z=11 \end{array} \tag{2.3.1.3}$$	21) $$\begin{array}{rrrrrrrr} x & + & 2y & + & 3z & + & 2w & =14 \\ x & - & 2y & - & z & & & =-5 \\ & & y & - & 2z & + & 4w & =9 \\ x & & & + & 3z & + & 3w & =15 \end{array} \tag{2.3.1.4}$$

2.4: Systems of Linear Equations and the Gauss-Jordan Method

Learning Objectives

In this section you will learn to

1. Represent a system of linear equations as an augmented matrix
2. Solve the system using elementary row operations.

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the **augmented matrix**, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a **row operation**.

Example 2.4.1

Write the following system as an augmented matrix.

$$\begin{aligned}2x+3y-4z&=5\\3x+4y-5z&=-6\\4x+5y-6z&=7\end{aligned}$$

Solution

We express the above information in matrix form. Since a system is entirely determined by its coefficient matrix and by its matrix of constant terms, the augmented matrix will include only the coefficient matrix and the constant matrix. So the augmented matrix we get is as follows:

$$\left[\begin{array}{ccc|c}2&3&-4&5\\3&4&-5&-6\\4&5&-6&7\end{array}\right]$$

In the last section, we expressed the system of equations as $AX=B$, where A represented the coefficient matrix, and B the matrix of constant terms. As an augmented matrix, we write the matrix as $\left[\begin{array}{c|c}A&B\end{array}\right]$. It is clear that all of the information is maintained in this matrix form, and only the letters x, y and z are missing. A student may choose to write x, y and z on top of the first three columns to help ease the transition.

Example 2.4.2

For the following augmented matrix, write the system of equations it represents.

$$\left[\begin{array}{cccccc}1&3&-5&|&2\\2&0&-3&|&-5\\3&2&-3&|&-1\end{array}\right]$$

Solution

The system is readily obtained as below.

$$\begin{aligned}x+3y-5z&=2\\2x-3z&=-5\\3x+2y-3z&=-1\end{aligned}$$

Once a system is expressed as an augmented matrix, the Gauss-Jordan method reduces the system into a series of equivalent systems by using the row operations. This row reduction continues until the system is expressed in what is called the **reduced row**

echelon form. The reduced row echelon form of the coefficient matrix has 1's along the main diagonal and zeros elsewhere. The solution is readily obtained from this form.

The method is not much different form the algebraic operations we employed in the elimination method in the first chapter. The basic difference is that it is algorithmic in nature, and, therefore, can easily be programmed on a computer.

We will next solve a system of two equations with two unknowns, using the elimination method, and then show that the method is analogous to the Gauss-Jordan method.

Example 2.4.3

Solve the following system by the elimination method.

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

Solution

We multiply the first equation by – 3, and add it to the second equation.

$$\begin{array}{r} -3x-9y=-21 \\ \underline{3x+4y=11} \\ -5y=-10 \end{array}$$

By doing this we transformed our original system into an equivalent system:

$$\begin{aligned} x+3y &= 7 \\ -5y &= -10 \end{aligned}$$

We divide the second equation by – 5, and we get the next equivalent system.

$$\begin{aligned} x+3y &= 7 \\ y &= 2 \end{aligned}$$

Now we multiply the second equation by – 3 and add to the first, we get

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

Example 2.4.4

Solve the following system from Example 3 by the Gauss-Jordan method, and show the similarities in both methods by writing the equations next to the matrices.

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned} \tag{2.4.1}$$

Solution

The augmented matrix for the system is as follows.

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 3 & 4 & | & 11 \end{bmatrix} \quad \begin{bmatrix} x+3y=7 \\ 3x+4y=11 \end{bmatrix}$$

We multiply the first row by – 3, and add to the second row.

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 0 & -5 & | & -10 \end{bmatrix} \quad \begin{bmatrix} x+3y & =7 \\ -5y & =-10 \end{bmatrix}$$

We divide the second row by – 5, we get,

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 0 & 1 & | & 2 \end{bmatrix} \quad \begin{bmatrix} x+3y & =7 \\ y & =2 \end{bmatrix}$$

Finally, we multiply the second row by – 3 and add to the first row, and we get,

$$\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \end{bmatrix} \quad \begin{bmatrix} x=1 \\ y=2 \end{bmatrix}$$

Now we list the three row operations the Gauss-Jordan method employs.

Row Operations

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

One can easily see that these three row operation may make the system look different, but they do not change the solution of the system.

The first row operation states that if any two rows of a system are interchanged, the new system obtained has the same solution as the old one. Let us look at an example in two equations with two unknowns. Consider the system

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

We interchange the rows, and we get,

$$\begin{aligned} 3x+4y &= 11 \\ x+3y &= 7 \end{aligned}$$

Clearly, this system has the same solution as the one above.

The second operation states that if a row is multiplied by any non-zero constant, the new system obtained has the same solution as the old one. Consider the above system again,

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

We multiply the first row by –3, we get,

$$\begin{aligned} -3x-9y &= -21 \\ 3x+4y &= 11 \end{aligned}$$

Again, it is obvious that this new system has the same solution as the original.

The third row operation states that any constant multiple of one row added to another preserves the solution. Consider our system,

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

If we multiply the first row by –3, and add it to the second row, we get,

$$\begin{aligned} x+3y &= 7 \\ -5y &= -10 \end{aligned}$$

And once again, the same solution is maintained.

Now that we understand how the three row operations work, it is time to introduce the Gauss-Jordan method to solve systems of linear equations. As mentioned earlier, the Gauss-Jordan method starts out with an augmented matrix, and by a series of row operations ends up with a matrix that is in the **reduced row echelon form**.

A matrix is in the **reduced row echelon form** if the first nonzero entry in each row is a 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom. We state the Gauss-Jordan method as follows.

Gauss-Jordan Method

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.
6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

The final matrix is called the reduced row-echelon form.

Example 2.4.5

Solve the following system by the Gauss-Jordan method.

$$\begin{aligned} 2x+y+2z &= 10 \\ x+2y+z &= 8 \\ 3x+y-z &= 2 \end{aligned}$$

Solution

We write the augmented matrix.

$$\left[\begin{array}{ccccc} 2 & 1 & 2 & | & 10 \\ 1 & 2 & 1 & | & 8 \\ 3 & 1 & -1 & | & 2 \end{array}\right]$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2, or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\left[\begin{array}{ccccc} 1 & 2 & 1 & | & 8 \\ 2 & 1 & 2 & | & 10 \\ 3 & 1 & -1 & | & 2 \end{array}\right] \quad \text{we interchanged row 1 (R1) and row 2 (R2)}$$

We need to make all other entries zeros in column 1. To make the entry (2) a zero in row 2, column 1, we multiply row 1 by - 2 and add it to the second row. We get,

$$\left[\begin{array}{ccccc} 1 & 2 & 1 & | & 8 \\ 0 & -3 & 0 & | & -6 \\ 3 & 1 & -1 & | & 2 \end{array}\right] \quad -2R1+R2$$

To make the entry (3) a zero in row 3, column 1, we multiply row 1 by - 3 and add it to the third row. We get,

$$\left[\begin{array}{ccccc} 1 & 2 & 1 & | & 8 \\ 0 & -3 & 0 & | & -6 \\ 0 & -5 & -4 & | & -22 \end{array}\right] \quad -3R1+R3$$

So far we have made a 1 in the left corner and all other entries zeros in that column. Now we move to the next diagonal entry, row 2, column 2. We need to make this entry(–3) a 1 and make all other entries in this column zeros. To make row 2, column 2 entry a 1, we divide the entire second row by –3.

$$\left[\begin{array}{ccccc} 1 & 2 & 1 & | & 8 \\ 0 & 1 & 0 & | & 2 \\ 0 & -5 & -4 & | & -22 \end{array}\right] \quad \text{R2} \div (-3)$$

Next, we make all other entries zeros in the second column.

$$\begin{bmatrix} 1 & 0 & 1 & | & 4 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & -4 & | & -12 \end{bmatrix} \quad -2R2+R1 \text{ and } 5R2+R3$$

We make the last diagonal entry a 1, by dividing row 3 by – 4.

$$\begin{bmatrix} 1 & 0 & 1 & | & 4 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \qquad R3 \div (-4)$$

Finally, we make all other entries zeros in column 3.

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad -\mathrm{R3}+\mathrm{R1}$$

Clearly, the solution reads $x=1$, $y=2$, and $z=3$.

Before we leave this section, we mention some terms we may need in the fourth chapter.

The process of obtaining a 1 in a location, and then making all other entries zeros in that column, is called **pivoting**.

The number that is made a 1 is called the **pivot element,** and the row that contains the pivot element is called the **pivot row**.

We often multiply the pivot row by a number and add it to another row to obtain a zero in the latter. The row to which a multiple of pivot row is added is called the **target row**.

2.4.1: Systems of Linear Equations and the Gauss-Jordan Method (Exercises)

SECTION 2.2 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS

Solve the following by the Gauss-Jordan Method. Show all work.

1) $$\begin{gathered} x+3y=1 \\ 2x-5y=13 \end{gathered}$$	2) $$\begin{gathered} x-y-z=-1 \\ x-3y+2z=7 \\ 2x-y+z=3 \end{gathered}$$
3) $$\begin{gathered} x+2y+3z=9 \\ 3x+4y+z=5 \\ 2x-y+2z=11 \end{gathered}$$	4) $$\begin{gathered} x+2y=0 \\ y+z=3 \\ x+3z=14 \end{gathered}$$

SECTION 2.2 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS

Solve the following by the Gauss-Jordan Method. Show all work.

5) Two apples and four bananas cost \$2.00 and three apples and five bananas cost \$2.70. Find the price of each.	6) A bowl of corn flakes, a cup of milk, and an egg provide 16 grams of protein. A cup of milk and two eggs provide 21 grams of protein. Two bowls of corn flakes with two cups of milk provide 16 grams of protein. How much protein is provided by one unit of food?
7) $$\begin{gathered} x+2y=10 \\ y+z=5 \\ z+w=3 \\ x+w=5 \end{gathered}$$	8) $$\begin{gathered} x+w=6 \\ 2x+y+w=16 \\ x-2z=0 \\ z+w=5 \end{gathered}$$

2.5: Systems of Linear Equations – Special Cases

Learning Objectives

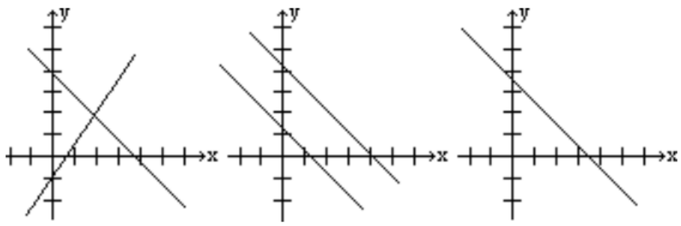
In this section, you will learn to:

1. Determine the linear systems that have no solution.
2. Solve the linear systems that have infinitely many solutions.

If we consider the intersection of two lines in a plane, three things can happen.

1. The lines intersect in exactly one point. This is called an **independent system**.
2. The lines are parallel, so they do not intersect. This is called an **inconsistent system**.
3. The lines coincide; they intersect at infinitely many points. This is a **dependent system**.

The figures below show all three cases.



Every system of equations has either one solution, no solution, or infinitely many solutions.

In the last section, we used the Gauss-Jordan method to solve systems that had exactly one solution. In this section, we will determine the systems that have no solution, and solve the systems that have infinitely many solutions.

Example 2.5.1

Solve the following system of equations:

$$\begin{array}{l} x+y=7 \\ x+y=9 \end{array}$$

Solution

Let us use the Gauss-Jordan method to solve this system. The augmented matrix is

$$\left[\begin{array}{rrrr} 1 & 1 & \mid & 7 \\ 1 & 1 & \mid & 9 \end{array}\right] \quad \left[\begin{array}{l} x+y=7 \\ x+y=9 \end{array}\right]$$

If we multiply the first row by - 1 and add to the second row, we get

$$\left[\begin{array}{rrrr} 1 & 1 & \mid & 7 \\ 0 & 0 & \mid & 2 \end{array}\right] \quad \left[\begin{array}{l} x+y=7 \\ 0x+0y=2 \end{array}\right]$$

Since 0 cannot equal 2, the last equation cannot be true for any choices of x and y.

Alternatively, it is clear that the two lines are parallel; therefore, they do not intersect.

In the examples that follow, we are going to start using a calculator to row reduce the augmented matrix, in order to focus on understanding the answer rather than focusing on the process of carrying out the row operations.

Example 2.5.2

Solve the following system of equations.

$$\begin{aligned} 2x+3y-4z&=7\\ 3x+4y-2z&=9\\ 5x+7y-6z&=20 \end{aligned}$$

Solution

We enter the following augmented matrix in the calculator.

$$\left[\begin{array}{ccccc} 2 & 3 & -4 & | & 7 \\ 3 & 4 & -2 & | & 9 \\ 5 & 7 & -6 & | & 20 \end{array}\right]$$

Now by pressing the key to obtain the reduced row-echelon form, we get

$$\left[\begin{array}{ccccc} 1 & 0 & 10 & | & 0 \\ 0 & 1 & -8 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{array}\right]$$

The last row states that $0x+0y+0z=1$. But the left side of the equation is equal to 0. So this last row states 0 = 1, which is a contradiction, a false statement.

This bottom row indicates that the system is inconsistent; therefore, there is no solution.

Example 2.5.3

Solve the following system of equations.

$$\begin{aligned} x+y&=7\\ x+y&=7 \end{aligned}$$

Solution

The problem clearly asks for the intersection of two lines that are the same; that is, the lines coincide. This means the lines intersect at an infinite number of points.

A few intersection points are listed as follows: (3, 4), (5, 2), (-1, 8), (-6, 13) etc. However, when a system has an infinite number of solutions, the solution is often expressed in the parametric form. This can be accomplished by assigning an arbitrary constant, t, to one of the variables, and then solving for the remaining variables. Therefore, if we let $y=t$, then $x=7-t$. Or we can say all ordered pairs of the form $(7-t, t)$ satisfy the given system of equations.

Alternatively, while solving the Gauss-Jordan method, we will get the reduced row-echelon form given below.

$$\left[\begin{array}{cccc} 1 & 1 & | & 7 \\ 0 & 0 & | & 0 \end{array}\right]$$

The row of all zeros, can simply be ignored. This row says $0x+0y=0$; it provides no further information about the values of x and y that solve this system.

This leaves us with only one equation but two variables. And whenever there are more variables than the equations, the solution must be expressed as a parametric solution in terms of an arbitrary constant, as above.

Parametric Solution: $\mathbf{x=7-t, y=t}$.

Example 2.5.4

Solve the following system of equations.

$$\begin{aligned} x+y+z&=2\\ 2x+y-z&=3\\ 3x+2y&=5 \end{aligned}$$

Solution

The augmented matrix and the reduced row-echelon form are given below.

$$\begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 2 & 1 & -1 & | & 3 \\ 3 & 2 & 0 & | & 5 \end{bmatrix} \text{ Augmented Matrix for this system}$$

$$\begin{bmatrix} 1 & 0 & -2 & | & 1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \text{ Reduced Row Echelon Form}$$

Since the last equation dropped out, we are left with two equations and three variables. This means the system has infinite number of solutions. We express those solutions in the parametric form by letting the last variable z equal the parameter t.

The first equation reads $x - 2z = 1$, therefore, $x = 1 + 2z$.

The second equation reads $y + 3z = 1$, therefore, $y = 1 - 3z$.

And now if we let $z = t$, the parametric solution is expressed as follows:

Parametric Solution: $\mathbf{x = 1 + 2t, \quad y = 1 - 3t, \quad z = t}$ **.**

The reader should note that particular solutions, or specific solutions, to the system can be obtained by assigning values to the parameter t. For example:

- if we let $t = 2$, we have the solution $x = 5, y = -5, z = 2$: $(5, -5, 2)$
- if we let $t = 0$, we have the solution $x = 1, y = 1, z = 0$: $(1, 1, 0)$.

Example 2.5.5

Solve the following system of equations.

$$\begin{aligned} x + 2y - 3z &= 5 \\ 2x + 4y - 6z &= 10 \\ 3x + 6y - 9z &= 15 \end{aligned}$$

Solution

The reduced row-echelon form is given below.

$$\begin{bmatrix} 1 & 2 & -3 & | & 5 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This time the last two equations drop out. We are left with one equation and three variables. Again, there are an infinite number of solutions. But this time the answer must be expressed in terms of two arbitrary constants.
If we let $z = t$ and let $y = s$, the first equation $x + 2y - 3z = 5$ results in $x = 5 - 2s + 3t$.

We rewrite the **parametric solution:** $\mathbf{x = 5 - 2s + 3t, \quad y = s, \quad z = t}$ **.**

We summarize our discussion in the following table.

1. If any row of the reduced row-echelon form of the matrix gives a false statement such as 0 = 1, the system is inconsistent and has no solution.
2. If the reduced row echelon form has fewer equations than the variables and the system is consistent, then the system has an infinite number of solutions. Remember the rows that contain all zeros are dropped.
    a. If a system has an infinite number of solutions, the solution must be expressed in the parametric form.
    b. The number of arbitrary parameters equals the number of variables minus the number of equations.

2.5.1: Systems of Linear Equations – Special Cases (Exercises)

SECTION 2.3 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS - SPECIAL CASES

Solve the following inconsistent or dependent systems by using the Gauss-Jordan method.

1. $$\begin{aligned} 2x+6y&=8 \\ x+3y&=4 \end{aligned}$$	2. The sum of digits of a two digit number is 9. The sum of the number and the number obtained by interchanging the digits is 99. Find the number.
3. $$\begin{aligned} 2x-y&=10 \\ -4x+2y&=15 \end{aligned}$$	4. $$\begin{aligned} x+y+z&=6 \\ 3x+2y+z&=14 \\ 4x+3y+2z&=20 \end{aligned}$$
5. $$\begin{aligned} x+2y-4z&=1 \\ 2x-3y+8z&=9 \end{aligned}$$	6. Jessica has a collection of 15 coins consisting of nickels, dimes and quarters. If the total worth of the coins is \$1.80, how many are there of each? Find all three solutions.

SECTION 2.3 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS - SPECIAL CASES

Solve the following inconsistent or dependent systems by using the Gauss-Jordan method.

7. A company is analyzing sales reports for three products: products X, Y, Z. One report shows that a combined total of 20,000 of items X, Y, and Z were sold. Another report shows that the sum of the number of item Z sold and twice the number of item X sold equals 10,000. Also item X has 5,000 more items sold than item Y. Are these reports consistent?	8. $$\begin{aligned} x+y+2z&=0 \\ x+2y+z&=0 \\ 2x+3y+3z&=0 \end{aligned}$$
9. Find three solutions to the following system of equations. $$\begin{aligned} x+2y+z&=12 \\ y&=3 \end{aligned}$$	10. $$\begin{aligned} x+2y&=5 \\ 2x+4y&=k \end{aligned}$$ For what values of k does this system of equations have a. No solution? b. Infinitely many solutions?
11. $x+3y-z=5$	12. Why is it not possible for a linear system to have exactly two solutions? Explain geometrically.

2.6: Inverse Matrices

Learning Objectives

In this section, you will learn to:

1. Find the inverse of a matrix, if it exists.
2. Use inverses to solve linear systems.

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

Definition of an Inverse: An $n \times n$ matrix has an inverse if there exists a matrix B such that $AB = BA = I_n$, where I_n is an $n \times n$ identity matrix. The inverse of a matrix A, if it exists, is denoted by the symbol A^{-1}.

Example 2.6.1

Given matrices A and B below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution

The matrices are inverses if the product AB and BA both equal the identity matrix of dimension 2×2: I_2,

$$\mathrm{AB} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2$$

and

$$\mathrm{BA} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2$$

Clearly that is the case; therefore, the matrices A and B are inverses of each other.

Example 2.6.2

Find the inverse of the matrix $\mathrm{A} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

Solution

Suppose A has an inverse, and it is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $AB = I_2$: $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

After multiplying the two matrices on the left side, we get

$$\begin{bmatrix} 3a + c & 3b + d \\ 5a + 2c & 5b + 2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries, we get four equations with four unknowns:

$$\begin{array}{ll} 3a + c = 1 & 3b + d = 0 \\ 5a + 2c = 0 & 5b + 2d = 1 \end{array}$$

Solving this system, we get: $a=2 \quad b=-1 \quad c=-5 \quad d=3$

Therefore, the inverse of the matrix A is $B=\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

In this problem, finding the inverse of matrix A amounted to solving the system of equations:

$$\begin{array}{ll} 3a+c=1 & 3b+d=0 \\ 5a+2c=0 & 5b+2d=1 \end{array}$$

Actually, it can be written as two systems, one with variables a and c, and the other with b and d. The augmented matrices for both are given below.

$$\begin{bmatrix} 3 & 1 & | & 1 \\ 5 & 2 & | & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 1 & | & 0 \\ 5 & 2 & | & 1 \end{bmatrix}$$

As we look at the two augmented matrices, we notice that the coefficient matrix for both the matrices is the same. This implies the row operations of the Gauss-Jordan method will also be the same. A great deal of work can be saved if the two right hand columns are grouped together to form one augmented matrix as below.

$$\begin{bmatrix} 3 & 1 & | & 1 & 0 \\ 5 & 2 & | & 0 & 1 \end{bmatrix}$$

And solving this system, we get

The matrix on the right side of the vertical line is the A^{-1} matrix.

What you just witnessed is no coincidence. This is the method that is often employed in finding the inverse of a matrix. We list the steps, as follows:

The Method for Finding the Inverse of a Matrix

1. Write the augmented matrix $[A|I_n]$.

2. Write the augmented matrix in step 1 in reduced row echelon form.

3. If the reduced row echelon form in 2 is $[I_n|B]$, then B is the inverse of A.

4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

Example 2.6.3

Given the matrix A below, find its inverse.

$$A=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution

We write the augmented matrix as follows.

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 2 & 3 & 0 & | & 0 & 1 & 0 \\ 0 & -2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

We will reduce this matrix using the Gauss-Jordan method.

Multiplying the first row by -2 and adding it to the second row, we get

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & 5 & -2 & | & -2 & 1 & 0 \\ 0 & -2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

If we swap the second and third rows, we get

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & -2 & 1 & | & 0 & 0 & 1 \\ 0 & 5 & -2 & | & -2 & 1 & 0 \end{bmatrix}$$

Divide the second row by -2. The result is

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -1/2 & | & 0 & 0 & -1/2 \\ 0 & 5 & -2 & | & -2 & 1 & 0 \end{bmatrix}$$

Let us do two operations here. 1) Add the second row to first, 2) Add -5 times the second row to the third. And we get

$$\begin{bmatrix} 1 & 0 & 1/2 & | & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & | & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & | & -2 & 1 & 5/2 \end{bmatrix}$$

Multiplying the third row by 2 results in

$$\begin{bmatrix} 1 & 0 & 1/2 & | & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & | & 0 & 0 & -1/2 \\ 0 & 0 & 1 & | & -4 & 2 & 5 \end{bmatrix}$$

Multiply the third row by 1/2 and add it to the second.
Also, multiply the third row by -1/2 and add it to the first.

$$\begin{bmatrix} 1 & 0 & 0 & | & 3 & -1 & -3 \\ 0 & 1 & 0 & | & -2 & 1 & 2 \\ 0 & 0 & 1 & | & -4 & 2 & 5 \end{bmatrix}$$

Therefore, the inverse of matrix A is $\mathrm{A}^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

One should verify the result by multiplying the two matrices to see if the product does, indeed, equal the identity matrix.

Now that we know how to find the inverse of a matrix, we will use inverses to solve systems of equations. The method is analogous to solving a simple equation like the one below.

$$\frac{2}{3}x = 4$$

Example 2.6.4

Solve the following equation: $\frac{2}{3}x = 4$

Solution

To solve the above equation, we multiply both sides of the equation by the multiplicative inverse of $\frac{2}{3}$ which happens to be $\frac{3}{2}$. We get

$$\begin{aligned} &\tfrac{3}{2} \cdot \tfrac{2}{3}x = 4 \cdot \tfrac{3}{2} \\ &x = 6 \end{aligned}$$

We use the Example 2.6.4 as an analogy to show how linear systems of the form $AX = B$ are solved.

To solve a linear system, we first write the system in the matrix equation $AX = B$, where A is the coefficient matrix, X the matrix of variables, and B the matrix of constant terms.
We then multiply both sides of this equation by the multiplicative inverse of the matrix A.

Consider the following example.

Example 2.6.5

Solve the following system

$$\begin{aligned} 3x + y &= 3 \\ 5x + 2y &= 4 \end{aligned}$$

Solution

To solve the above equation, first we express the system as

$$AX = B$$

where A is the coefficient matrix, and B is the matrix of constant terms. We get

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To solve this system, we multiply both sides of the matrix equation $AX = B$ by A^{-1}. Matrix multiplication is not commutative, so we need to multiply by A^{-1} on the left on both sides of the equation.

Matrix A is the same matrix A whose inverse we found in Example $2.6.2$, so $\mathrm{A}^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Multiplying both sides by A^{-1}, we get

$$\begin{gathered} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \end{gathered}$$

Therefore, $x = 2$, and $y = -3$.

Example 2.6.6

Solve the following system:

$$\begin{aligned} x - y + z &= 6 \\ 2x + 3y &= 1 \\ -2y + z &= 5 \end{aligned}$$

Solution

To solve the above equation, we write the system in matrix form $AX = B$ as follows:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

To solve this system, we need inverse of A. From Example $2.6.3$, $\mathrm{A}^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

Multiplying both sides of the matrix equation $AX = B$ on the left by A^{-1}, we get

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

After multiplying the matrices, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

We remind the reader that not every system of equations can be solved by the matrix inverse method. Although the Gauss-Jordan method works for every situation, the matrix inverse method works only in cases where the inverse of the square matrix exists. In such cases the system has a unique solution.

The Method for Finding the Inverse of a Matrix

1. Write the augmented matrix $[\mathrm{A}|\mathrm{I_n}]$.
2. Write the augmented matrix in step 1 in reduced row echelon form.
3. If the reduced row echelon form in 2 is $[\mathrm{I_n}|\mathrm{B}]$, then B is the inverse of A.
4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

The Method for Solving a System of Equations When a Unique Solution Exists

1. Express the system in the matrix equation $AX = B$.

2. To solve the equation $AX = B$, we multiply on both sides by A^{-1}.

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B \text{ where } I \text{ is the identity matrix}$$

2.6.1: Inverse Matrices (Exercises)

SECTION 2.4 PROBLEM SET: INVERSE MATRICES

In problems 1- 2, verify that the given matrices are inverses of each other.

1. $\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$	2. $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} 3 & -4 & 1 \\ 2 & -4 & 1 \\ 3 & -5 & 1 \end{bmatrix}$

In problems 3- 6, find the inverse of each matrix by the row-reduction method.

3. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$	4. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

SECTION 2.4 PROBLEM SET: INVERSE MATRICES

In problems 5 - 6, find the inverse of each matrix by the row-reduction method.

5. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$	6. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Problems 7 -10: Express the system as $AX = B$; then solve using matrix inverses found in problems 3 - 6.

7. $\begin{array}{l} 3x-5y=2 \\ -x+2y=0 \end{array}$	8. $\begin{array}{r} x+\quad 2z=8 \\ y+4z=8 \\ z=3 \end{array}$

SECTION 2.4 PROBLEM SET: INVERSE MATRICES

Problems 9 -10: Express the system as $AX = B$; then solve using matrix inverses found in problems 3 - 6.

9. $\begin{array}{r} x+y-z=2 \\ x+z=7 \\ 2x+y+z=13 \end{array}$	10. $\begin{array}{l} x+y+z=2 \\ 3x+y=7 \\ x+y+2z=3 \end{array}$
11. Why is it necessary that a matrix be a square matrix for its inverse to exist? Explain by relating the matrix to a system of equations.	12. Suppose we are solving a system $AX = B$ by the matrix inverse method, but discover A has no inverse. How else can we solve this system? What can be said about the solutions of this system?

2.7: Application of Matrices in Cryptography

Learning Objectives

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems. In this section you will learn to

1. encode a message using matrix multiplication.
2. decode a coded message using the matrix inverse and matrix multiplication

Encryption dates back approximately 4000 years. Historical accounts indicate that the Chinese, Egyptians, Indian, and Greek encrypted messages in some way for various purposes. One famous encryption scheme is called the Caesar cipher, also called a substitution cipher, used by Julius Caesar, involved shifting letters in the alphabet, such as replacing A by C, B by D, C by E, etc, to encode a message. Substitution ciphers are too simple in design to be considered secure today.

In the middle ages, European nations began to use encryption. A variety of encryption methods were used in the US from the Revolutionary War, through the Civil War, and on into to modern times.

Applications of mathematical theory and methods to encryption became widespread in military usage in the 20$^{\text{th}}$ century. The military would encode messages before sending and the recipient would decode the message, in order to send information about military operations in a manner that kept the information safe if the message was intercepted. In World War II, encryption played an important role, as both Allied and Axis powers sent encrypted messages and devoted significant resources to strengthening their own encryption while also trying to break the opposition's encryption.

In this section we will examine a method of encryption that uses matrix multiplication and matrix inverses. This method, known as the Hill Algorithm, was created by Lester Hill, a mathematics professor who taught at several US colleges and also was involved with military encryption. The Hill algorithm marks the introduction of modern mathematical theory and methods to the field of cryptography.

These days, the Hill Algorithm is not considered a secure encryption method; it is relatively easy to break with modern technology. However, in 1929 when it was developed, modern computing technology did not exist. This method, which we can handle easily with today's technology, was too cumbersome to use with hand calculations. Hill devised a mechanical encryption machine to help with the mathematics; his machine relied on gears and levers, but never gained widespread use. Hill's method was considered sophisticated and powerful in its time and is one of many methods influencing techniques in use today. Other encryption methods at that time also utilized special coding machines. Alan Turing, a computer scientist pioneer in the field of artificial intelligence, invented a machine that was able to decrypt messages encrypted by the German Enigma machine, helping to turn the tide of World War II.

With the advent of the computer age and internet communication, the use of encryption has become widespread in communication and in keeping private data secure; it is no longer limited to military uses. Modern encryption methods are more complicated, often combining several steps or methods to encrypt data to keep it more secure and harder to break. Some modern methods make use of matrices as part of the encryption and decryption process; other fields of mathematics such as number theory play a large role in modern cryptography.

To use matrices in encoding and decoding secret messages, our procedure is as follows.

We first convert the secret message into a string of numbers by arbitrarily assigning a number to each letter of the message. Next we convert this string of numbers into a new set of numbers by multiplying the string by a square matrix of our choice that has an inverse. This new set of numbers represents the coded message.

To decode the message, we take the string of coded numbers and multiply it by the inverse of the matrix to get the original string of numbers. Finally, by associating the numbers with their corresponding letters, we obtain the original message.

In this section, we will use the correspondence shown below where letters A to Z correspond to the numbers 1 to 26, a space is represented by the number 27, and punctuation is ignored.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Example 2.7.1

Use matrix $A=\begin{bmatrix}1&2\\1&3\end{bmatrix}$ to encode the message: ATTACK NOW!

Solution

We divide the letters of the message into groups of two.

$$\text{AT}\quad \text{TA}\quad \text{CK}\quad \text{-N}\quad \text{OW}$$

We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix}\mathrm{A}\\\mathrm{T}\end{bmatrix}=\begin{bmatrix}1\\20\end{bmatrix}\quad\begin{bmatrix}\mathrm{T}\\\mathrm{A}\end{bmatrix}=\begin{bmatrix}20\\1\end{bmatrix}\quad\begin{bmatrix}\mathrm{C}\\\mathrm{K}\end{bmatrix}=\begin{bmatrix}3\\11\end{bmatrix}$$
$$\begin{bmatrix}-\\\mathrm{N}\end{bmatrix}=\begin{bmatrix}27\\14\end{bmatrix}\quad\begin{bmatrix}\mathrm{O}\\\mathrm{W}\end{bmatrix}=\begin{bmatrix}15\\23\end{bmatrix}$$

So at this stage, our message expressed as 2×1 matrices is as follows.

$$\begin{bmatrix}1\\20\end{bmatrix}\begin{bmatrix}20\\1\end{bmatrix}\begin{bmatrix}3\\11\end{bmatrix}\begin{bmatrix}27\\14\end{bmatrix}\begin{bmatrix}15\\23\end{bmatrix}\quad\mathbf{(I)}$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A. For example, the product of A with our first matrix is: $\begin{bmatrix}1&2\\1&3\end{bmatrix}\begin{bmatrix}1\\20\end{bmatrix}=\begin{bmatrix}41\\61\end{bmatrix}$

And the product of A with our second matrix is: $\begin{bmatrix}1&2\\1&3\end{bmatrix}\begin{bmatrix}20\\1\end{bmatrix}=\begin{bmatrix}22\\23\end{bmatrix}$

Multiplying each matrix in $\mathbf{(I)}$ by matrix A, in turn, gives the desired coded message:

$$\begin{bmatrix}41\\61\end{bmatrix}\begin{bmatrix}22\\23\end{bmatrix}\begin{bmatrix}25\\36\end{bmatrix}\begin{bmatrix}55\\69\end{bmatrix}\begin{bmatrix}61\\84\end{bmatrix}$$

Example 2.7.2

Decode the following message that was encoded using matrix $A=\begin{bmatrix}1&2\\1&3\end{bmatrix}$.

$$\begin{bmatrix}21\\26\end{bmatrix}\begin{bmatrix}37\\53\end{bmatrix}\begin{bmatrix}45\\54\end{bmatrix}\begin{bmatrix}74\\101\end{bmatrix}\begin{bmatrix}53\\69\end{bmatrix}\quad\mathbf{(II)}$$

Solution

Since this message was encoded by multiplying by the matrix A in Example $2.7.1$, we decode this message by first multiplying each matrix, on the left, by the inverse of matrix A given below.

$$A^{-1}=\begin{bmatrix}3&-2\\-1&1\end{bmatrix}$$

For example: $\begin{bmatrix}3&-2\\-1&1\end{bmatrix}\begin{bmatrix}21\\26\end{bmatrix}=\begin{bmatrix}11\\5\end{bmatrix}$

By multiplying each of the matrices in $\mathbf{(II)}$ by the matrix A^{-1}, we get the following.

$$\begin{bmatrix}11\\5\end{bmatrix}\begin{bmatrix}5\\16\end{bmatrix}\begin{bmatrix}27\\9\end{bmatrix}\begin{bmatrix}20\\27\end{bmatrix}\begin{bmatrix}21\\16\end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix}\mathrm{K}\\\mathrm{E}\end{bmatrix}\begin{bmatrix}\mathrm{E}\\\mathrm{P}\end{bmatrix}\begin{bmatrix}-\\\mathrm{I}\end{bmatrix}\begin{bmatrix}\mathrm{T}\\-\end{bmatrix}\begin{bmatrix}\mathrm{U}\\\mathrm{P}\end{bmatrix}$$

And the message reads: KEEP IT UP.

Now suppose we wanted to use a 3×3 matrix to encode a message, then instead of dividing the letters into groups of two, we would divide them into groups of three.

Example 2.7.3

Using the matrix $B = \begin{bmatrix}1 & 1 & -1\\1 & 0 & 1\\2 & 1 & 1\end{bmatrix}$, encode the message: ATTACK NOW!

Solution

We divide the letters of the message into groups of three.

ATT ACK -NO W- -

Note that since the single letter "W" was left over on the end, we added two spaces to make it into a triplet.

Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into 3×1 matrices. We get

$$\begin{bmatrix}\mathrm{A}\\\mathrm{T}\\\mathrm{T}\end{bmatrix}=\begin{bmatrix}1\\20\\20\end{bmatrix}\begin{bmatrix}\mathrm{A}\\\mathrm{C}\\\mathrm{K}\end{bmatrix}=\begin{bmatrix}1\\3\\11\end{bmatrix}\begin{bmatrix}-\\\mathrm{N}\\\mathrm{O}\end{bmatrix}=\begin{bmatrix}27\\14\\15\end{bmatrix}\begin{bmatrix}\mathrm{W}\\-\\-\end{bmatrix}=\begin{bmatrix}23\\27\\27\end{bmatrix}$$

So far we have,

$$\begin{bmatrix}1\\20\\20\end{bmatrix}\begin{bmatrix}1\\3\\11\end{bmatrix}\begin{bmatrix}27\\14\\15\end{bmatrix}\begin{bmatrix}23\\27\\27\end{bmatrix} \quad \textbf{(III)}$$

We multiply, on the left, each matrix of our message by the matrix B. For example,

$$\begin{bmatrix}1 & 1 & -1\\1 & 0 & 1\\2 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\20\\20\end{bmatrix}=\begin{bmatrix}1\\21\\42\end{bmatrix}$$

By multiplying each of the matrices in $\textbf{(III)}$ by the matrix B, we get the desired coded message as follows:

$$\begin{bmatrix}1\\21\\42\end{bmatrix}\begin{bmatrix}-7\\12\\16\end{bmatrix}\begin{bmatrix}26\\42\\83\end{bmatrix}\begin{bmatrix}23\\50\\100\end{bmatrix}$$

If we need to decode this message, we simply multiply the coded message by B^{-1}, and associate the numbers with the corresponding letters of the alphabet.

In Example $2.7.4$ we will demonstrate how to use matrix B^{-1} to decode an encrypted message.

Example 2.7.4

Decode the following message that was encoded using matrix $B=\begin{bmatrix}1&1&-1\\1&0&1\\2&1&1\end{bmatrix}$.

$$\begin{bmatrix}11\\20\\43\end{bmatrix}\begin{bmatrix}25\\10\\41\end{bmatrix}\begin{bmatrix}22\\14\\41\end{bmatrix}\quad\textbf{(IV)}$$

Solution

Since this message was encoded by multiplying by the matrix B. We first determine inverse of B.

$$\mathrm{B}^{-1}=\begin{bmatrix}1&2&-1\\-1&-3&2\\-1&-1&1\end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by B^{-1}. For example,

$$\begin{bmatrix}1&2&-1\\-1&-3&2\\-1&-1&1\end{bmatrix}\begin{bmatrix}11\\20\\43\end{bmatrix}=\begin{bmatrix}8\\15\\12\end{bmatrix}$$

Multiplying each of the matrices in $\textbf{(IV)}$ by the matrix B^{-1} gives the following.

$$\begin{bmatrix}8\\15\\12\end{bmatrix}\begin{bmatrix}4\\27\\6\end{bmatrix}\begin{bmatrix}9\\18\\5\end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain

$$\begin{bmatrix}\mathrm{H}\\\mathrm{O}\\\mathrm{L}\end{bmatrix}\begin{bmatrix}\mathrm{D}\\-\\\mathrm{F}\end{bmatrix}\begin{bmatrix}\mathrm{I}\\\mathrm{R}\\\mathrm{E}\end{bmatrix}\quad\text{The message reads: HOLD FIRE}$$

The message reads: HOLD FIRE.

We summarize:

TO ENCODE A MESSAGE

1. Divide the letters of the message into groups of two or three.

2. Convert each group into a string of numbers by assigning a number to each letter of the message. Remember to assign letters to blank spaces.

3. Convert each group of numbers into column matrices.

3. Convert these column matrices into a new set of column matrices by multiplying them with a compatible square matrix of your choice that has an inverse. This new set of numbers or matrices represents the coded message.

TO DECODE A MESSAGE

1. Take the string of coded numbers and multiply it by the inverse of the matrix that was used to encode the message.

2. Associate the numbers with their corresponding letters.

2.7.1: Application of Matrices in Cryptography (Exercises)

SECTION 2.5 PROBLEM SET: APPLICATION OF MATRICES IN CRYPTOGRAPHY

In problems 1 - 8, the letters A to Z correspond to the numbers 1 to 26, as shown below, and a space is represented by the number 27.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

In problems 1 - 2, use the matrix A, given below, to encode the given messages.

$$A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$$

In problems 3 - 4, decode the messages that were encoded using matrix A.
Make sure to consider the spaces between words, but ignore all punctuation. Add a final space if necessary.

1. Encode the message: WATCH OUT!	2. Encode the message: HELP IS ON THE WAY.
3. Decode the following message: 64 23 102 41 82 32 97 35 71 28 69 32	4. Decode the following message: 105 40 117 48 39 19 69 32 72 27 37 15 114 47

SECTION 2.5 PROBLEM SET: APPLICATION OF MATRICES IN CRYPTOGRAPHY

In problems 5 - 6, use the matrix B, given below, to encode the given messages.

$$\mathrm{B}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 2\\ 1 & 0 & -1\end{bmatrix}$$

In problems 7 - 8, decode the messages that were encoded using matrix B.

Make sure to consider the spaces between words, but ignore all punctuation. Add a final space if necessary.

5. Encode the message using matrix B: LUCK IS ON YOUR SIDE.	6. Encode the message using matrix B: MAY THE FORCE BE WITH YOU.
7. Decode the following message that was encoded using matrix B: 8 23 7 4 47 - 2 15 102 -12 20 58 15 27 80 18 12 74 -7	8. Decode the following message that was encoded using matrix B: 12 69 - 3 11 53 9 5 46 -10 18 95 - 9 25 107 4 27 76 22 1 72 -26

2.8: Applications – Leontief Models

Learning Objectives

In this section we will examine an application of matrices to model economic systems.

In the 1930's, Wassily Leontief used matrices to model economic systems. His models, often referred to as the input-output models, divide the economy into sectors where each sector produces goods and services not only for itself but also for other sectors. These sectors are dependent on each other and the total input always equals the total output. In 1973, he won the Nobel Prize in Economics for his work in this field. In this section we look at both the closed and the open models that he developed.

The Closed Model

As an example of the closed model, we look at a very simple economy, where there are only three sectors: food, shelter, and clothing.

Example 2.8.1

We assume that in a village there is a farmer, carpenter, and a tailor, who provide the three essential goods: food, shelter, and clothing. Suppose the farmer himself consumes 40% of the food he produces, and gives 40% to the carpenter, and 20% to the tailor. Thirty percent of the carpenter's production is consumed by himself, 40% by the farmer, and 30% by the carpenter. Fifty percent of the tailor's production is used by himself, 30% by the farmer, and 20% by the tailor. Write the matrix that describes this closed model.

Solution

The table below describes the above information.

	Proportion produced by the farmer	Proportion produced by the carpenter	Proportion produced by the tailor
The proportion used by the farmer	.40	.40	.30
The proportion used by the carpenter	.40	.30	.20
The proportion used by the tailor	.20	.30	.50

In a matrix form it can be written as follows.

$$A=\begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix}$$

This matrix is called the **input-output matrix.** It is important that we read the matrix correctly. For example the entry A_{23}, the entry in row 2 and column 3, represents the following.

A_{23} = 20% of the tailor's production is used by the carpenter.

A_{33} = 50% of the tailor's production is used by the tailor.

Example 2.8.2

In Example $2.8.1$ above, how much should each person get for his efforts?

Solution

We choose the following variables.

x = Farmer's pay y = Carpenter's pay z = Tailor's pay

As we said earlier, in this model input must equal output. That is, the amount paid by each equals the amount received by each.

Let us say the farmer gets paid x dollars. Let us now look at the farmer's expenses. The farmer uses up 40% of his own production, that is, of the x dollars he gets paid, he pays himself .40x dollars, he pays .40y dollars to the carpenter, and .30z to the tailor. Since the expenses equal the wages, we get the following equation.

$$x = .40x + .40y + .30z$$

In the same manner, we get

$$\begin{aligned} y &= .40x + .30y + .20z \\ z &= .20x + .30y + .50z \end{aligned}$$

The above system can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .40 & .40 & 30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This system is often referred to as $X = AX$

Simplification results in the system of equations $(I - A)X = 0$

$$\begin{aligned} .60x - .40y - .30z &= 0 \\ -.40x + .70y - .20z &= 0 \\ -.20x - .30y + .50z &= 0 \end{aligned}$$

Solving for x, y, and z using the Gauss-Jordan method, we get

$$x = \frac{29}{26}t \quad y = \frac{12}{13}t \quad \text{ and } z = t$$

Since we are only trying to determine the proportions of the pay, we can choose t to be any value. Suppose we let \(t\(= $2600, then we get

$$x = \$2900 \quad y = \$2400 \quad \text{ and } z = \$2600$$

Note: The use of a graphing calculator or computer application in solving the systems of linear matrix equations in these problems is strongly recommended.

The Open Model

The open model is more realistic, as it deals with the economy where sectors of the economy not only satisfy each other's' needs, but they also satisfy some outside demands. In this case, the outside demands are put on by the consumer. But the basic assumption is still the same; that is, whatever is produced is consumed.

Let us again look at a very simple scenario. Suppose the economy consists of three people, the farmer F, the carpenter C, and the tailor T. A part of the farmer's production is used by all three, and the rest is used by the consumer. In the same manner, a part of the carpenter's and the tailor's production is used by all three, and rest is used by the consumer.

Let us assume that whatever the farmer produces, 20% is used by him, 15% by the carpenter, 10% by the tailor, and the consumer uses the other 40 billion dollars worth of the food. Ten percent of the carpenter's production is used by him, 25% by the farmer, 5% by the tailor, and 50 billion dollars worth by the consumer. Fifteen percent of the clothing is used by the tailor, 10% by the farmer, 5% by the carpenter, and the remaining 60 billion dollars worth by the consumer. We write the internal consumption in the following table, and express the demand as the matrix D.

	F produces	C produces	T produces
F uses	.20	.25	.10
C uses	.15	.10	.05
T uses	.10	.05	.15

The consumer demand for each industry in billions of dollars is given below.

$$\mathrm{D}=\begin{bmatrix}40\\50\\60\end{bmatrix}$$

Example 2.8.3

In the example above, what should be, in billions of dollars, the required output by each industry to meet the demand given by the matrix D?

Solution

We choose the following variables.

x = Farmer's output

y = Carpenter's output

z = Tailor's output

In the closed model, our equation was $X=AX$, that is, the total input equals the total output. This time our equation is similar with the exception of the demand by the consumer.

So our equation for the open model should be $X=AX+D$, where D represents the demand matrix.

We express it as follows:

$$X=AX+D$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}.20&.25&.10\\.15&.10&.05\\.10&.05&.15\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}40\\50\\60\end{bmatrix}$$

To solve this system, we write it as

$$\begin{aligned}&X=AX+D\\&(I-A)X=D\quad\text{where I is a 3 by 3 identity matrix}\\&X=(I-A)^{-1}D\end{aligned}$$

$$\mathrm{I}-\mathrm{A}=\begin{bmatrix}.80&-.25&-.10\\-.15&.90&-.05\\-.10&-.05&.85\end{bmatrix}$$

$$(\mathrm{I}-\mathrm{A})^{-1}=\begin{bmatrix}1.3445&.3835&.1807\\.2336&1.1814&.097\\.1719&.1146&1.2034\end{bmatrix}$$

$$X=\begin{bmatrix}1.3445&.3835&.1807\\.2336&1.1814&.097\\.1719&.1146&1.2034\end{bmatrix}\begin{bmatrix}40\\50\\60\end{bmatrix}$$

$$X=\begin{bmatrix}83.7999\\74.2341\\84.8138\end{bmatrix}$$

The three industries must produce the following amount of goods in billions of dollars.

Farmer = \$83.7999 Carpenter = \$74.2341 Tailor = \$84.813

We will do one more problem like the one above, except this time we give the amount of internal and external consumption in dollars and ask for the proportion of the amounts consumed by each of the industries. In other words, we ask for the matrix A.

Example 2.8.4

Suppose an economy consists of three industries F, C, and T. Each of the industries produces for internal consumption among themselves, as well as for external demand by the consumer. The table shows the use of each industry's production ,in dollars.

	F	C	T	Demand	Total
F	40	50	60	100	250
C	30	40	40	110	220
T	20	30	30	120	200

The first row says that of the $250 dollars worth of production by the industry F, $40 is used by F, $50 is used by C, $60 is used by T, and the remainder of $100 is used by the consumer. The other rows are described in a similar manner.

Once again, the total input equals the total output. Find the proportion of the amounts consumed by each of the industries. In other words, find the matrix A.

Solution

We are being asked to determine the following:

How much of the production of each of the three industries, F, C, and T is required to produce one unit of F? In the same way, how much of the production of each of the three industries, F, C, and T is required to produce one unit of C? And finally, how much of the production of each of the three industries, F, C, and T is required to produce one unit of T?

Since we are looking for proportions, we need to divide the production of each industry by the total production for each industry.

We analyze as follows:

To produce 250 units of F, we need to use 40 units of F, 30 units of C, and 20 units of T.

Therefore, to produce 1 unit of F, we need to use 40/250 units of F, 30/250 units of C, and 20/250 units of T.

To produce 220 units of C, we need to use 50 units of F, 40 units of C, and 30 units of T.

Therefore, to produce 1 unit of C, we need to use 50/220 units of F, 40/220 units of C, and 30/220 units of T.

To produce 200 units of T, we need to use 60 units of F, 40 units of C, and 30 units of T.

Therefore, to produce 1 unit of T, we need to use 60/200 units of F, 40/200 units of C, and 30/200 units of T.

We obtain the following matrix.

$$\mathrm{A}=\begin{bmatrix} 40/250 & 50/220 & 60/200 \\ 30/250 & 40/220 & 40/200 \\ 20/250 & 30/220 & 30/200 \end{bmatrix}=\begin{bmatrix} .1600 & .2273 & .3000 \\ .1200 & .1818 & .2000 \\ .0800 & .1364 & .1500 \end{bmatrix}$$

Clearly $AX+D=X$

$$\begin{bmatrix} 40/250 & 50/220 & 60/200 \\ 30/250 & 40/220 & 40/200 \\ 20/250 & 30/220 & 30/200 \end{bmatrix}\begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix}+\begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix}=\begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix}$$

We summarize as follows:

LEONTIEF'S CLOSED MODEL

1. All consumption is within the industries. There is no external demand.
2. Input = Output
3. $X=AX$ or $(I-A)X=0$

LEONTIEF'S OPEN MODEL

1. In addition to internal consumption, there is an outside demand by the consumer.
2. Input = Output
3. $X = AX + D$ or $X = (I - A)^{-1} D$

2.8.1: Applications – Leontief Models (Exercises)

SECTION 2.6 PROBLEM SET: APPLICATIONS - LEONTIEF MODELS

1) Solve the following homogeneous system.

$$\begin{aligned} x+y+z &= 0 \\ 3x+2y+z &= 0 \\ 4x+3y+2z &= 0 \end{aligned}$$

2) Solve the following homogeneous system.

$$\begin{aligned} x-y-z &= 0 \\ x-3y+2z &= 0 \\ 2x-4y+z &= 0 \end{aligned}$$

3) Chris and Ed decide to help each other by doing repairs on each others houses. Chris is a carpenter, and Ed is an electrician. Chris does carpentry work on his house as well as on Ed's house. Similarly, Ed does electrical repairs on his house and on Chris' house. When they are all finished they realize that Chris spent 60% of his time on his own house, and 40% of his time on Ed's house. On the other hand Ed spent half of his time on his house and half on Chris's house. If they originally agreed that each should get about a $1000 for their work, how much money should each get for their work?

4) Chris, Ed, and Paul decide to help each other by doing repairs on each others houses. Chris is a carpenter, Ed is an electrician, and Paul is a plumber. Each does work on his own house as well as on the others houses. When they are all finished they realize that Chris spent 30% of his time on his own house, 40% of his time on Ed's house, and 30% on Paul's house. Ed spent half of his time on his own house, 30% on Chris' house, and remaining on Paul's house. Paul spent 40% of the time on his own house, 40% on Chris' house, and 20% on Ed's house. If they originally agreed that each should get about a $1000 for their work, how much money should each get for their work?

SECTION 2.6 PROBLEM SET: APPLICATIONS - LEONTIEF MODELS

5) Given the internal consumption matrix A, and the external demand matrix D as follows.

$$A=\begin{bmatrix} .30 & .20 & .10 \\ .20 & .10 & .30 \\ .10 & .20 & .30 \end{bmatrix} \quad D=\begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix}$$

Solve the system using the open model: $X=AX+D$ or $X=(I-A)^{-1}D$

6) Given the internal consumption matrix A, and the external demand matrix D as follows.

$$A=\begin{bmatrix} .05 & .10 & .10 \\ .10 & .15 & .05 \\ .05 & .20 & .20 \end{bmatrix} \quad D=\begin{bmatrix} 50 \\ 100 \\ 80 \end{bmatrix}$$

Solve the system using the open model: $X=AX+D$ or $X=(I-A)^{-1}D$

7) An economy has two industries, farming and building. For every $1 of food produced, the farmer uses $.20 and the builder uses $.15. For every $1 worth of building, the builder uses $.25 and the farmer uses $.20. If the external demand for food is $100,000, and for building $200,000, what should be the total production for each industry in dollars?

SECTION 2.6 PROBLEM SET: APPLICATIONS - LEONTIEF MODELS

8) An economy has three industries, farming, building, and clothing. For every $1 of food produced, the farmer uses $.20, the builder uses $.15, and the tailor $.05. For every $1 worth of building, the builder uses $.25, the farmer uses $.20, and the tailor $.10. For every $1 worth of clothing, the tailor uses $.10, the builder uses $.20, the farmer uses $.15. If the external demand for food is $100 million, for building $200 million, and for clothing $300 million, what should be the total production for each in dollars?

9) Suppose an economy consists of three industries F, C, and T. The following table gives information about the internal use of each industry's production and external demand in dollars.

	F	C	T	Demand	Total
F	30	10	20	40	100
C	20	30	20	50	120
T	10	10	30	60	110

Find the proportion of the amounts consumed by each of the industries; that is, find the matrix A.

10) If in problem 9, the consumer demand for F, C, and T becomes 60, 80, and 100, respectively, find the total output and the internal use by each industry to meet that demand.

2.9: Chapter Review

SECTION 2.7 PROBLEM SET: CHAPTER REVIEW

1. To reinforce her diet, Mrs. Tam bought a bottle containing 30 tablets of Supplement A and a bottle containing 50 tablets of Supplement B. Each tablet of supplement A contains 1000 mg of calcium, 400 mg of magnesium, and 15 mg of zinc, and each tablet of supplement B contains 800 mg of calcium, 500 mg of magnesium, and 20 mg of zinc.}
   a. Represent the amount of calcium, magnesium and zinc in each tablet as a 2×3 matrix.
   b. Represent the number of tablets in each bottle as a row matrix.
   c. Use matrix multiplication to find the total amount of calcium, magnesium, and zinc in both bottles.
2. Let matrix $A=\begin{bmatrix}1 & -1 & 3\\ 3 & -2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}3 & 3 & -1\\ 1 & 4 & -3\end{bmatrix}$. Find the following.
   a. $\frac{1}{2}(A+B)$
   b. $3A=2B$
3. Let matrix $\mathrm{C}=\begin{bmatrix}1 & 1 & -1\\ 2 & 1 & 1\\ 1 & 0 & 1\end{bmatrix}$ and $D=\begin{bmatrix}2 & -3 & -1\\ 3 & -1 & -2\\ 3 & -3 & -2\end{bmatrix}$. Find the following.
   a. $2(C-D)$
   b. $C-3D$
4. Let matrix $E=\begin{bmatrix}1 & -1\\ 2 & 3\\ 1 & 2\end{bmatrix}$ and $\mathrm{F}=\begin{bmatrix}2 & 1 & -1\\ 1 & 2 & -3\end{bmatrix}$. Find the following.
   a. $2EF$
   b. $3FE$
5. Let matrix $G=\begin{bmatrix}1 & -1 & 3\\ 3 & 2 & 1\end{bmatrix}$ and $H=\begin{bmatrix}a & b\\ c & d\\ e & f\end{bmatrix}$. Find the following.
   a. $2GH$
   b. HG
6. Solve the following systems using the Gauss-Jordan Method.
   a. $$\begin{aligned} x+3y-2z&=7\\ 2x+7y-5z&=16\\ x+5y-3z&=10\end{aligned}$$
   b. $$\begin{aligned} 2x-4y+4z&=2\\ 2x+y+9z&=17\\ 3x-2y+2z&=7\end{aligned}$$
7. An apple, a banana and three oranges or two apples, two bananas, and an orange, or four bananas and two oranges cost $2. Find the price of each.
8. Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then determine one particular solution.
   a. $$\begin{aligned} x+y+z&=6\\ 2x-3y+2z&=12\\ 3x-2y+3z&=18\end{aligned}$$
   b. $$\begin{aligned} x+y+3z&=4\\ x+z&=1\\ 2x-y&=2\end{aligned}$$

9. Elise has a collection of 12 coins consisting of nickels, dimes and quarters. If the total worth of the coins is $1.80, how many are there of each? Find all possible solutions.

SECTION 2.7 PROBLEM SET: CHAPTER REVIEW

10. Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then find a particular solution.

    a.
$$\begin{aligned} 2x+y-2z&=0\\ 2x+2y-3z&=0\\ 6x+4y-7z&=0 \end{aligned}$$

    b.
$$\begin{aligned} 3x+4y-3z&=5\\ 2x+3y-z&=4\\ x+2y+z&=1 \end{aligned}$$

11. Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then provide one particular solution.

    a.
$$\begin{aligned} 2x+y-2z&=0\\ 2x+2y-3z&=0\\ 6x+4y-7z&=0 \end{aligned}$$

    b.
$$\begin{aligned} 3x+4y-3z&=5\\ 2x+3y-z&=4\\ x+2y+z&=1 \end{aligned}$$

12. Find the inverse of the following matrices:

    a.
$$\begin{bmatrix} 2 & 3\\ 3 & 5 \end{bmatrix}$$

    b.
$$\begin{bmatrix} 1 & 1 & 1\\ 1 & 2 & 1\\ 2 & 3 & 1 \end{bmatrix}$$

13. Solve the following systems using the matrix inverse method.

    a.
$$\begin{aligned} 2x+3y+z&=12\\ x+2y+z&=9\\ x+y+z&=5 \end{aligned}$$

    b.
$$\begin{aligned} x+2y-3z+w&=\\ x-z&=4\\ x-2y+z&=0\\ y-2z+w&=-11=0 \end{aligned}$$

14. Use matrix A to encode the following messages. The space between the letters is represented by the number 27, and all punctuation is ignored.

$$A=\begin{bmatrix} 1 & 2 & 0\\ 1 & 2 & 1\\ 0 & 1 & 0 \end{bmatrix}$$

    a. TAKE IT AND RUN
    b. GET OUT QUICK

15. Decode the following messages that were encoded using matrix A in the above problem.

    a. 44, 71, 15, 18, 27, 1, 68, 82, 27, 69, 76, 27, 19, 33, 9
    b. 37, 64, 15, 36, 54, 15, 67, 75, 20, 59, 66, 27, 39, 43, 12

16. Chris, Bob, and Matt decide to help each other study during the final exams. Chris's favorite subject is chemistry, Bob loves biology, and Matt knows his math. Each studies his own subject as well as helps the others learn their subjects. After the finals, they realize that Chris spent 40% of his time studying his own subject chemistry, 30% of his time helping Bob learn chemistry, and 30% of the time helping Matt learn chemistry. Bob spent 30% of his time studying his own subject biology, 30% of his time helping Chris learn biology, and 40% of the time helping Matt learn biology. Matt spent 20% of his time studying his own subject math, 40% of his time helping Chris learn math, and 40% of the time helping Bob learn math. If they originally agreed that each should work about 33 hours, how long did each work?
17. As in the previous problem, Chris, Bob, and Matt decide to not only help each other study during the final exams, but also tutor others to make a little money. Chris spends 30% of his time studying chemistry, 15% of his time helping Bob with chemistry, and 25% helping Matt with chemistry. Bob spends 25% of his time studying biology, 15% helping Chris with biology, and 30% helping Matt. Similarly, Matt spends 20% of his time on his own math, 20% helping Chris, and 20% helping Bob. If they spend respectively, 12, 12, and 10 hours tutoring others, how many total hours are they going to end up working?

CHAPTER OVERVIEW

3: Linear Programming- A Geometric Approach

3.1: Graphing Systems of Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of linear inequalities
- Solve a system of linear inequalities by graphing
- Solve applications of systems of inequalities

Before you get started, take this readiness quiz.

1. Solve the inequality $2a < 5a + 12$.
   If you missed this problem, review [link].
2. Determine whether the ordered pair $(3, 12)$ is a solution to the system $y > 2x + 3$.
   If you missed this problem, review [link].

Determine whether an ordered pair is a solution of a system of linear inequalities

The definition of a **system of linear inequalities** is very similar to the definition of a system of linear equations.

Definition: System of Linear Inequalities

Two or more linear inequalities grouped together form a system of linear inequalities.

A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\left\{\begin{array}{l} x+4y \geq 10 \\ 3x-2y<12 \end{array}\right.$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true.

Solutions of a System of Linear Inequalities

Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true.

The solution of a system of linear inequalities is shown as a shaded region in the xy-coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, we substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

Example 3.1.1

Determine whether the ordered pair is a solution to the system $\left\{\begin{array}{l} x+4y \geq 10 \\ 3x-2y<12 \end{array}\right.$

a. $(-2, 4)$ b. $(3, 1)$

Solution:

a. Is the ordered pair $(-2, 4)$ a solution?

We substitute $x = -2$ and $y = 4$ into both inequalities.

$$x + 4y \geq 10 \qquad\qquad 3x - 2y < 12$$

$$-2 + 4(4) \overset{?}{\geq} 10 \qquad\qquad 3(-2) - 2(4) \overset{?}{<} 12$$

$$14 \geq 10 \text{ true} \qquad\qquad -14 < 12 \text{ true}$$

The ordered pair $(-2, 4)$ made both inequalities true. Therefore $(-2, 4)$ is a solution to this system.

b. Is the ordered pair $(3, 1)$ a solution?

We substitute $x = 3$ and $y = 1$ into both inequalities.

$$x + 4y \geq 10 \qquad\qquad 3x - 2y < 12$$

$$3 + 4(1) \overset{?}{\geq} 10 \qquad\qquad 3(3) - 2(1) \overset{?}{<} 12$$

$$7 \geq 10 \text{ false} \qquad\qquad 7 < 12 \text{ true}$$

The ordered pair $(3, 1)$ made one inequality true, but the other one false. Therefore $(3, 1)$ is not a solution to this system.

Try It! 3.1.1

Determine whether the ordered pair is a solution to the system: $\begin{cases} x - 5y > 10 \\ 2x + 3y > -2 \end{cases}$

a. $(3, -1)$ b. $(6, -3)$

Answer

a. no
b. yes

Try It! 3.1.2

Determine whether the ordered pair is a solution to the system: $\begin{cases} y > 4x - 2 \\ 4x - y < 20 \end{cases}$

a. $(-2, 1)$ b. $(4, -1)$

Answer

a. yes
b. no

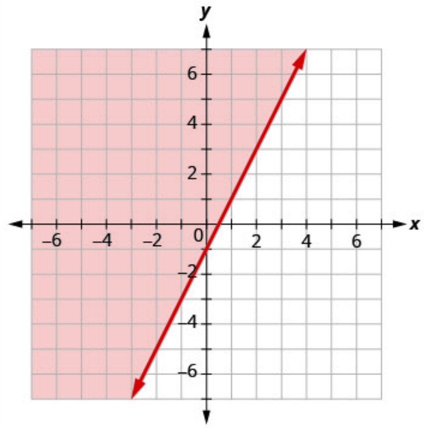
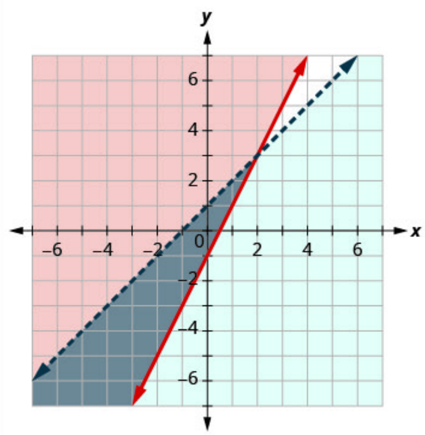
Solve a System of Linear Inequalities by Graphing

The solution to a single linear inequality is the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

Example 3.1.2: How to Solve a System of Linear Inequalities by Graphing

Solve the system by graphing: $\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$

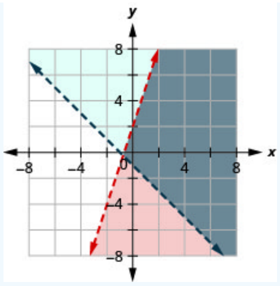
Solution:

Step 1. Graph the first inequality. • Graph the boundary line. • Shade in the side of the boundary line where the inequality is true.	We will graph $y \geq 2x - 1$. We graph the line $y = 2x - 1$. It is a solid line because the inequality sign is $\geq$. We choose (0, 0) as a test point. It is a solution to $y \geq 2x - 1$, so we shade in above the boundary line.	$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$ 
Step 2. On the same grid, graph the second inequality. • Graph the boundary line. • Shade in the side of that boundary line where the inequality is true.	We will graph $y < x + 1$ on the same grid. We graph the line $y = x + 1$. It is a dashed line because the inequality sign is $<$. Again, we use (0, 0) as a test point. It is a solution so we shade in that side of the line $y = x + 1$.	
Step 3. The solution is the region where the shading overlaps.	The point where the boundary lines intersect is not a solution because it is not a solution to $y < x + 1$.	The solution is all points in the area shaded twice—which appears as the darkest shaded region.
Step 4. Check by choosing a test point.	We'll use $(-1, -1)$ as a test point.	Is $(-1, -1)$ a solution to $y \geq 2x - 1$? $-1 \overset{?}{\geq} 2(-1) - 1$ $-1 \geq -3$ true Is $(-1, -1)$ a solution to $y < x + 1$? $-1 \overset{?}{<} -1 + 1$ $-1 < 0$ true The region containing $(-1, -1)$ is the solution to this system.

? Try It! 3.1.3

Solve the system by graphing: $\begin{cases} y < 3x + 2 \\ y > -x - 1 \end{cases}$

Answer

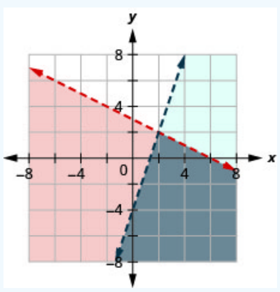


The solution is the grey region.

Try It! 3.1.4

Solve the system by graphing: $\begin{cases} y < -12x + 3 \\ y < 3x - 4 \end{cases}$

Answer



The solution is the grey region.

Solve a System of Linear Inequalities by Graphing

1. Graph the first inequality.
   - Graph the boundary line.
   - Shade in the side of the boundary line where the inequality is true.
2. On the same grid, graph the second inequality.
   - Graph the boundary line.
   - Shade in the side of that boundary line where the inequality is true.
3. The solution is the region where the shading overlaps.
4. Check by choosing a test point.

Example 3.1.3

Solve the system by graphing: $\begin{cases} x - y > 3 \\ y < -15x + 4 \end{cases}$

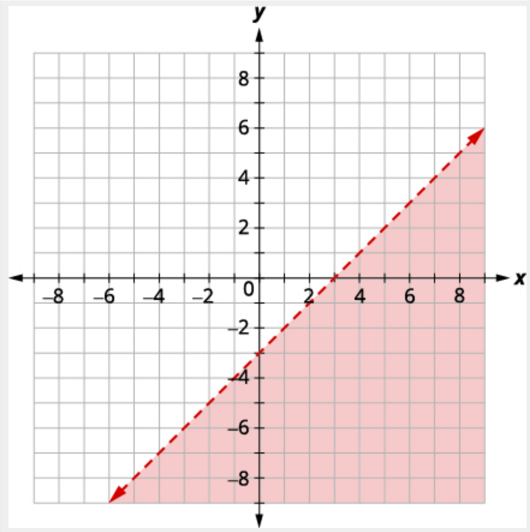
Solution:

	$\begin{cases} x - y > 3 \\ y < -15x + 4 \end{cases}$

Graph $x - y > 3$, by graphing $x - y = 3$ and testing a point.

The intercepts are $x = 3$ and $y = -3$ and the boundary line will be dashed.

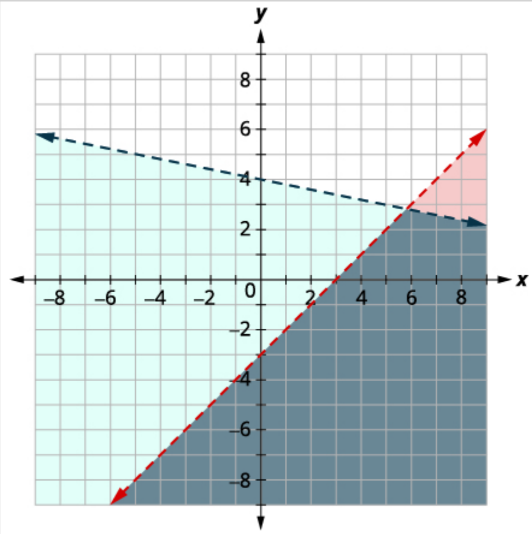
Test $(0,0)$ which makes the inequality false so shade (red) the side that does not contain $(0,0)$.



Graph $y < -15x + 4$ by graphing $y = -15x + 4$ using the slope $m = -15$ and y-intercept $b = 4$. The boundary line will be dashed

Test $(0,0)$ which makes the inequality true, so shade (blue) the side that contains $(0,0)$.

Choose a test point in the solution and verify that it is a solution to both inequalties.

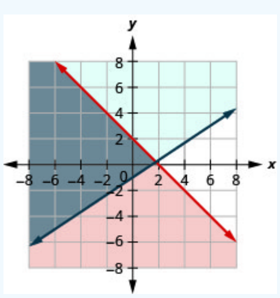


The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice—which appears as the darkest shaded region.

? Try It! 3.1.5

Solve the system by graphing: $\begin{cases} x + y \leq 2 \\ y \geq \frac{2}{3}x - 1 \end{cases}$

Answer

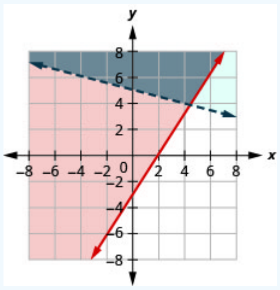


The solution is the grey region.

Try It! 3.1.6

Solve the system by graphing: $\begin{cases} 3x-2y \leq 6 \\ y > -\frac{1}{4}x+5 \end{cases}$

Answer

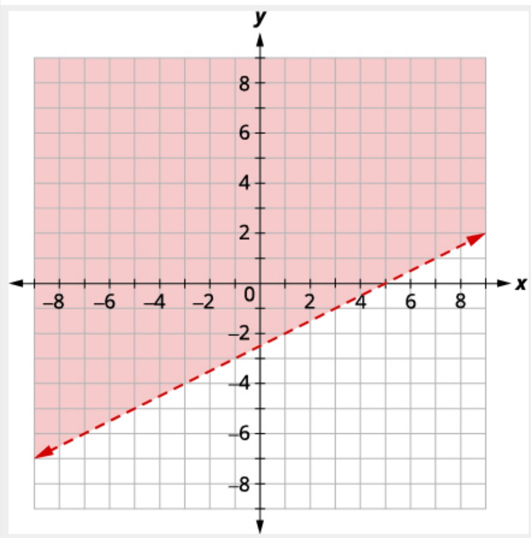


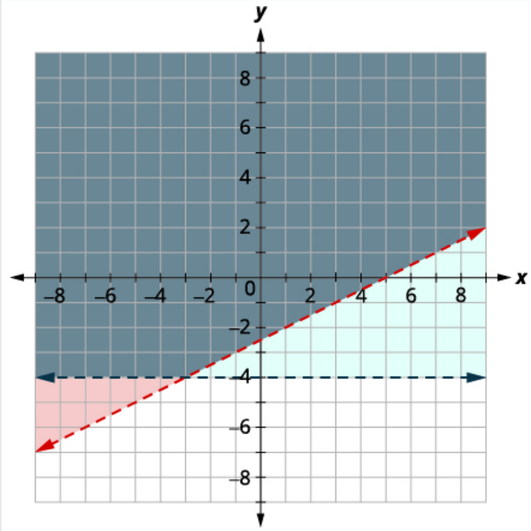
The solution is the grey region.

Example 3.1.4

Solve the system by graphing: $\begin{cases} x-2y<5 \\ y>-4 \end{cases}$

Solution:

	$\begin{cases} x-2y<5 \\ y>-4 \end{cases}$
Graph $x-2y<5$, by graphing $x-2y=5$ and testing a point. The intercepts are $x=5$ and $y=-2.5$ and the boundary line will be dashed. Test $(0,0)$ which makes the inequality true, so shade (red) the side that contains $(0,0)$.	

Graph $y > -4$, by graphing $y = -4$ and recognizing that it is a horizontal line through $y = -4$. The boundary line will be dashed. Test $(0,0)$ which makes the inequality true so shade (blue) the side that contains $(0,0)$.	

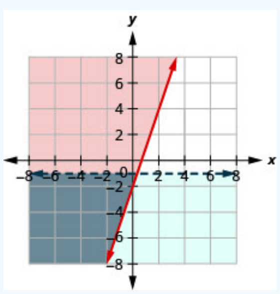
The point $(0, 0)$ is in the solution and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed.

The solution is the area shaded twice—which appears as the darkest shaded region.

Try It! 3.1.7

Solve the system by graphing: $\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$

Answer

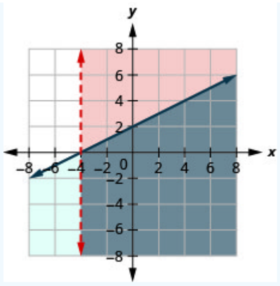


The solution is the grey region.

Try It! 3.1.8

Solve the system by graphing: $\begin{cases} x > -4x - 2 \\ y \geq -4 \end{cases}$

Answer



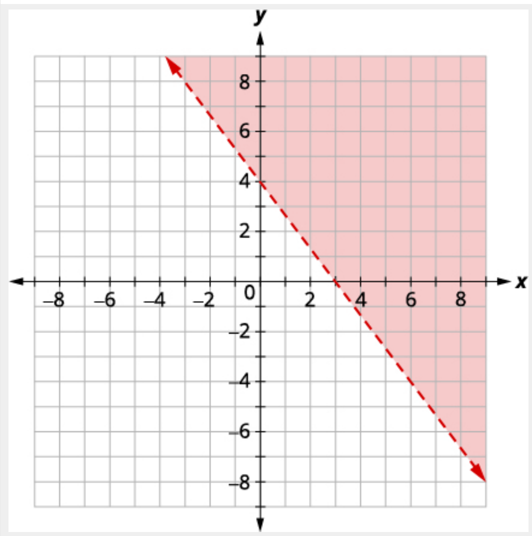
The solution is the grey region.

Systems of linear inequalities where the boundary lines are parallel might have no solution. We'll see this in the next example.

Example 3.1.5

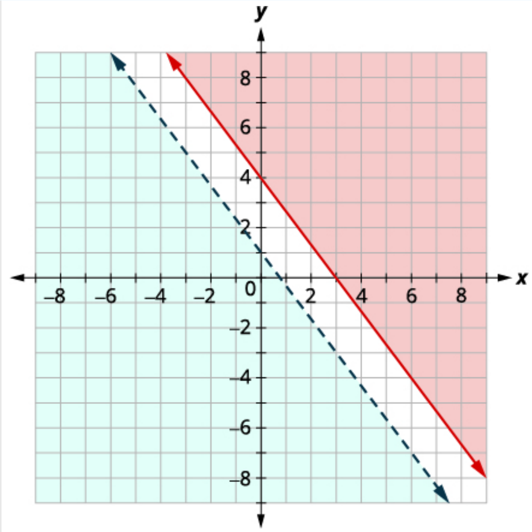
Solve the system by graphing: $\begin{cases} 4x+3y \geq 12 \\ y < -\frac{4}{3}x+1 \end{cases}$

Solution:

	$\begin{cases} 4x+3y \geq 12 \\ y < -\frac{4}{3}x+1 \end{cases}$
Graph $4x+3y \geq 12$, by graphing $4x+3y=12$ and testing a point. The intercepts are $x=3$ and $y=4$ and the boundary line will be solid. Test $(0,0)$ which makes the inequality false, so shade (red) the side that does not contain $(0,0)$.	

Graph $y < -\frac{4}{3}x + 1$ by graphing $y = -\frac{4}{3}x + 1$ using the slope $m = -\frac{4}{3}$ and y-intercept $b = 1$. The boundary line will be dashed.

Test $(0,0)$ which makes the inequality true, so shade (blue) the side that contains $(0,0)$.

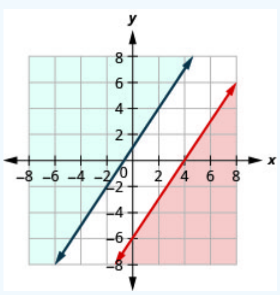


There is no point in both shaded regions, so the system has no solution.

Try It! 3.1.9

Solve the system by graphing: $\begin{cases} 3x - 2y \geq 12 \\ y \geq \frac{3}{2}x + 1 \end{cases}$

Answer

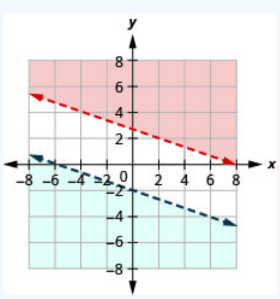


No solution.

Try It! 3.1.10

Solve the system by graphing: $\begin{cases} x + 3y > 8 \\ y < -\frac{1}{3}x - 2 \end{cases}$

Answer



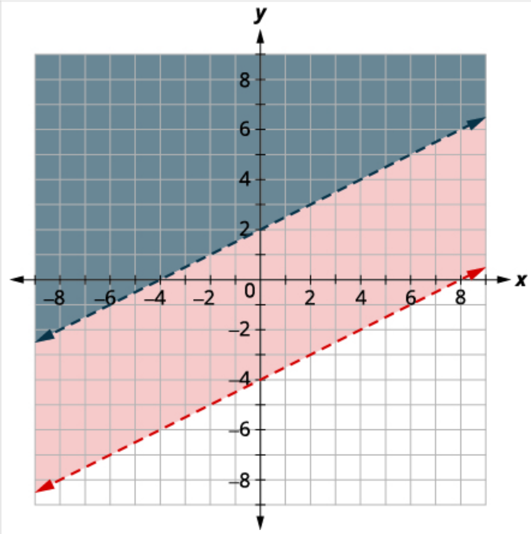
No solution.

Some systems of linear inequalities where the boundary lines are parallel will have a solution. We'll see this in the next example.

✔ Example 3.1.6

Solve the system by graphing: $\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$

Solution:

	$\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$
Graph $y > \frac{1}{2}x - 4$ by graphing $y = \frac{1}{2}x - 4$ using the slope $m = \frac{1}{2}$ and the intercept $b = -4$. The boundary line will be dashed. Test $(0,0)$ which makes the inequality true, so shade (red) the side that contains $(0,0)$.	
Graph $x - 2y < -4$ by graphing $x - 2y = -4$ and testing a point. The intercepts are $x = -4$ and $y = 2$ and the boundary line will be dashed. Choose a test point in the solution and verify that it is a solution to both inequalties. Test $(0,0)$ which makes the inequality false, so shade (blue) the side that does not contain $(0,0)$.	

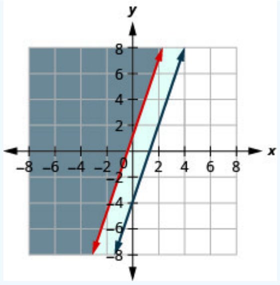
No point on the boundary lines is included in the solution as both lines are dashed.

The solution is the region that is shaded twice which is also the solution to $x - 2y < -4$.

? Try It! 3.1.11

Solve the system by graphing: $\begin{cases} y \geq 3x + 1 \\ -3x + y \geq -4 \end{cases}$

Answer

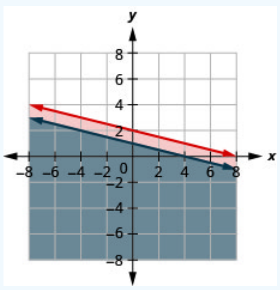


The solution is the grey region.

Try It! 3.1.12

Solve the system by graphing: $\begin{cases} y \leq -\frac{1}{4}x+2 \\ x+4y \leq 4 \end{cases}$

Answer



The solution is the grey region.

Solve Applications of Systems of Inequalities

The first thing we'll need to do to solve applications of systems of inequalities is to translate each condition into an inequality. Then we graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so we add inequalities to the system as additional requirements.

Example 3.1.7

Christy sells her photographs at a booth at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her $4 and each large photo costs her $10. She doesn't want to spend more than $200 on photos to display.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could she display 10 small and 20 large photos?
d. Could she display 20 large and 10 small photos?

Solution:

a.
$$\begin{array}{ll} \text{Let} & x = \text{the number of small photos.} \\ & y = \text{the number of large photos} \end{array}$$

To find the system of equations translate the information.

She wants to have at least 25 photos.
The number of small plus the number of large should be at least 25.

$$x+y \geq 25$$

\$4 for each small and \$10 for each large must be no more than \$200

$$4x+10y \leq 200$$

The number of small photos must be greater than or equal to 0.

$$x \geq 0$$

The number of large photos must be greater than or equal to 0.

$$y \geq 0$$

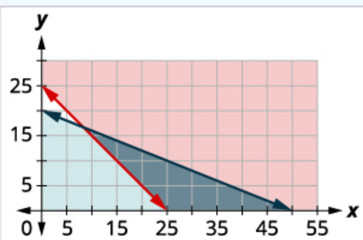
We have our system of equations.

$$\begin{cases} x+y \geq 25 \\ 4x+10y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

b.
Since $x \geq 0$ and $y \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $x+y \geq 25$, graph $x+y=25$ as a solid line.
Choose $(0,0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0,0)$.

To graph $4x+10y \leq 200$, graph $4x+10y=200$ as a solid line.
Choose $(0,0)$ as a test point. Since it does make the inequality true, shade (blue) the side that include the point $(0,0)$.



The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly-shaded section are included in the solution as are the points on the x-axis from $(25,0)$ to $(55,0)$.

c. To determine if 10 small and 20 large photos would work, we look at the graph to see if the point $(10,20)$ is in the solution region. We could also test the point to see if it is a solution of both equations.

It is not, Christy would not display 10 small and 20 large photos.

d. To determine if 20 small and 10 large photos would work, we look at the graph to see if the point $(20,10)$ is in the solution region. We could also test the point to see if it is a solution of both equations.

It is, so Christy could choose to display 20 small and 10 large photos.

Notice that we could also test the possible solutions by substituting the values into each inequality.

Try It! 3.1.13

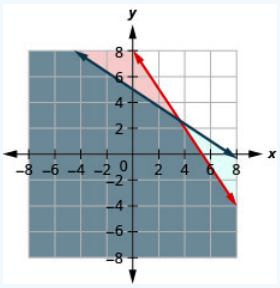
A trailer can carry a maximum weight of 160 pounds and a maximum volume of 15 cubic feet. A microwave oven weighs 30 pounds and has 2 cubic feet of volume, while a printer weighs 20 pounds and has 3 cubic feet of space.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could 4 microwaves and 2 printers be carried on this trailer?
d. Could 7 microwaves and 3 printers be carried on this trailer?

Answer

a. $\begin{cases} 30m+20p \leq 160 \\ 2m+3p \leq 15 \end{cases}$

b.



c. yes
d. no

Try It! 3.1.14

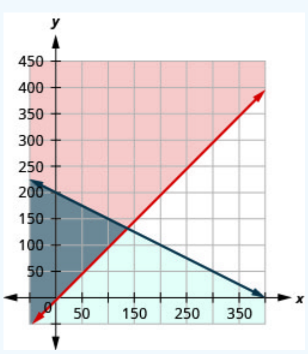
Mary needs to purchase supplies of answer sheets and pencils for a standardized test to be given to the juniors at her high school. The number of the answer sheets needed is at least 5 more than the number of pencils. The pencils cost \$2 and the answer sheets cost \$1. Mary's budget for these supplies allows for a maximum cost of \$400.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could Mary purchase 100 pencils and 100 answer sheets?
d. Could Mary purchase 150 pencils and 150 answer sheets?

Answer

a. $\begin{cases} a \geq p+5 \\ a+2p \leq 400 \end{cases}$

b.



c. no
d. no

When we use variables other than x and y to define an unknown quantity, we must change the names of the axes of the graph as well.

Example 3.1.8

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than \$5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs \$1.40. Each cookie has 160 calories and costs \$0.50.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could he eat 3 hamburgers and 1 cookie?
d. Could he eat 2 hamburgers and 4 cookies?

Solution:

a.

$$\begin{aligned} \text{Let} \quad & h = \text{the number of hamburgers.} \\ & c = \text{the number of cookies} \end{aligned}$$

To find the system of equations translate the information.

The calories from hamburgers at 240 calories each, plus the calories from cookies at 160 calories each must be more that 800.

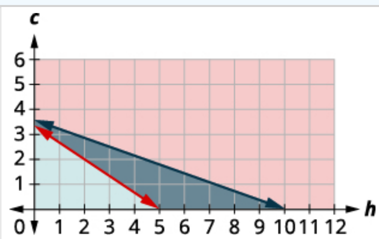
$$240h + 160c \geq 800$$

$$\text{The amount spent on hamburgers at \$1.40 each, plus the amount spent on cookies}$$
$$\text{at \$0.50 each must be no more than \$5.00.}$$
$$1.40h + 0.50c \leq 5$$

$$\text{The number of hamburgers must be greater than or equal to 0.}$$
$$h \geq 0$$
$$\text{The number of cookies must be greater than or equal to 0.}$$
$$c \geq 0$$

$$\text{We have our system of equations.} \qquad \begin{cases} 240h + 160c \geq 800 \\ 1.40h + 0.50c \leq 5 \\ h \geq 0 \\ c \geq 0 \end{cases}$$

b.
Since $h \geq 0$ and $c \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $240h + 160c \geq 800$, graph $240h + 160c = 800$ as a solid line. Choose $(0,0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0,0)$.	

Graph $1.40h+0.50c \leq 5$. The boundary line is $1.40h+0.50c=5$. We test $(0,0)$ and it makes the inequality true. We shade the side of the line that includes $(0,0)$.

The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the x-axis from $(5,0)$ to $(10,0)$.

c. To determine if 3 hamburgers and 2 cookies would meet Omar’s criteria, we see if the point $(3,2)$ is in the solution region. It is, so Omar might choose to eat 3 hamburgers and 2 cookies.

d. To determine if 2 hamburgers and 4 cookies would meet Omar’s criteria, we see if the point $(2,4)$ is in the solution region. It is, Omar might choose to eat 2 hamburgers and 4 cookies.

We could also test the possible solutions by substituting the values into each inequality.

? Try It! 3.1.15

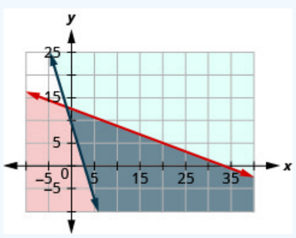
Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only $25 to spend on the extra food he needs and will spend it on $0.75 donuts which have 360 calories each and $2 energy drinks which have 110 calories.

a. Write a system of inequalities that models this situation.
b. Graph the system.
c. Can he buy 8 donuts and 4 energy drinks and satisfy his caloric needs?
d. Can he buy 1 donut and 3 energy drinks and satisfy his caloric needs?

Answer

a. $\begin{cases} 0.75d+2e \leq 25 \\ 360d+110e \geq 1000 \end{cases}$

b.



c. yes
d. no

? Try It! 3.1.16

Philip’s doctor tells him he should add at least 1,000 more calories per day to his usual diet. Philip wants to buy protein bars that cost $1.80 each and have 140 calories and juice that costs $1.25 per bottle and have 125 calories. He doesn’t want to spend more than $12.

a. Write a system of inequalities that models this situation.
b. Graph the system.
c. Can he buy 3 protein bars and 5 bottles of juice?
d. Can he buy 5 protein bars and 3 bottles of juice?

Answer

a. $\begin{cases} 140p+125j \geq 1000 \\ 1.80p+1.25j \leq 12 \end{cases}$

b.

c. yes
d. no

Access these online resources for additional instruction and practice with solving systems of linear inequalities by graphing.

- Solving Systems of Linear Inequalities by Graphing
- Systems of Linear Inequalities

Key Concepts

- **Solutions of a System of Linear Inequalities:** Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true. The solution of a system of linear inequalities is shown as a shaded region in the xy-coordinate system that includes all the points whose ordered pairs make the inequalities true.
- **How to solve a system of linear inequalities by graphing.**
  1. Graph the first inequality.
     Graph the boundary line.
     Shade in the side of the boundary line where the inequality is true.
  2. On the same grid, graph the second inequality.
     Graph the boundary line.
     Shade in the side of that boundary line where the inequality is true.
  3. The solution is the region where the shading overlaps.
  4. Check by choosing a test point.

Glossary

system of linear inequalities

Two or more linear inequalities grouped together form a system of linear inequalities.

3.1.1: Exercises

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.

1. $\begin{cases} 3x+y>5 \\ 2x-y\leq 10 \end{cases}$

ⓐ $(3,-3)$
ⓑ $(7,1)$

2. $\begin{cases} 4x-y<10 \\ -2x+2y>-8 \end{cases}$

ⓐ $(5,-2)$
ⓑ $(-1,3)$

Answer

ⓐ false ⓑ true

3. $\begin{cases} y>\frac{2}{3}x-5 \\ x+\frac{1}{2}y\leq 4 \end{cases}$

ⓐ $(6,-4)$
ⓑ $(3,0)$

4. $\begin{cases} y<\frac{3}{2}x+3 \\ \frac{3}{4}x-2y<5 \end{cases}$

ⓐ $(-4,-1)$
ⓑ $(8,3)$

Answer

ⓐ false ⓑ true

5. $\begin{cases} 7x+2y>14 \\ 5x-y\leq 8 \end{cases}$

ⓐ $(2,3)$
ⓑ $(7,-1)$

6. $\begin{cases} 6x-5y<20 \\ -2x+7y>-8 \end{cases}$

ⓐ $(1,-3)$
ⓑ $(-4,4)$

Answer

ⓐ false ⓑ true

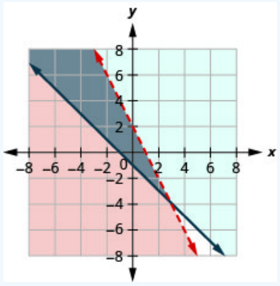
Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

7. $\begin{cases} y \leq 3x+2 \\ y > x-1 \end{cases}$

8. $\begin{cases} y < -2x+2 \\ y \geq -x-1 \end{cases}$

Answer

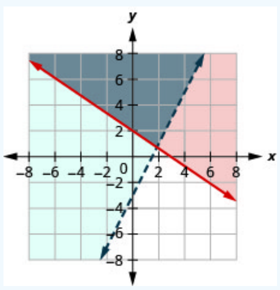


The solution is the grey region.

9. $\begin{cases} y < 2x-1 \\ y \leq -\frac{1}{2}x+4 \end{cases}$

10. $\begin{cases} y \geq -\frac{2}{3}x+2 \\ y > 2x-3 \end{cases}$

Answer

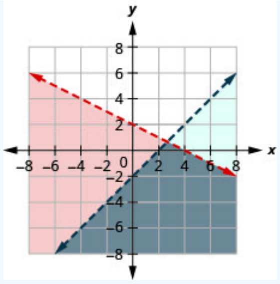


The solution is the grey region.

11. $\begin{cases} x-y > 1 \\ y < -\frac{1}{4}x+3 \end{cases}$

12. $\begin{cases} x+2y < 4 \\ y < x-2 \end{cases}$

Answer

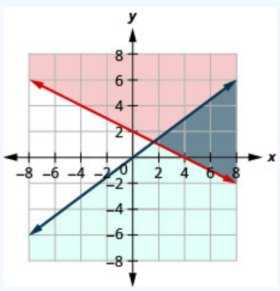


The solution is the grey region.

13. $\begin{cases} 3x - y \geq 6 \\ y \geq -\frac{1}{2}x \end{cases}$

14. $\begin{cases} x + 4y \geq 8 \\ y \leq \frac{3}{4}x \end{cases}$

Answer

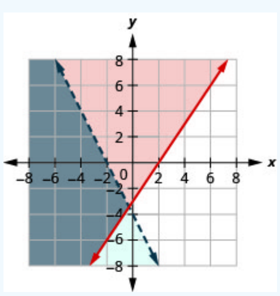


The solution is the grey region.

15. $\begin{cases} 2x - 5y < 10 \\ 3x + 4y \geq 12 \end{cases}$

16. $\begin{cases} 3x - 2y \leq 6 \\ -4x - 2y > 8 \end{cases}$

Answer

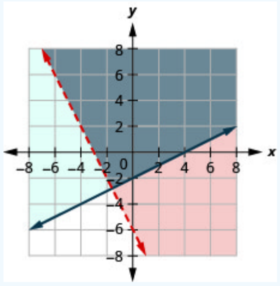


The solution is the grey region.

17. $\begin{cases} 2x + 2y > -4 \\ -x + 3y \geq 9 \end{cases}$

18. $\begin{cases} 2x+y>-6 \\ -x+2y\geq -4 \end{cases}$

Answer

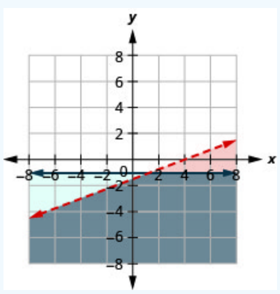


The solution is the grey region.

19. $\begin{cases} x-2y<3 \\ y\leq 1 \end{cases}$

20. $\begin{cases} x-3y>4 \\ y\leq -1 \end{cases}$

Answer

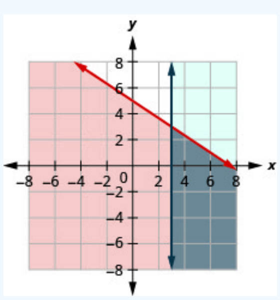


The solution is the grey region.

21. $\begin{cases} y\geq -\frac{1}{2}x-3 \\ x\leq 2 \end{cases}$

22. $\begin{cases} y\leq -\frac{2}{3}x+5 \\ x\geq 3 \end{cases}$

Answer

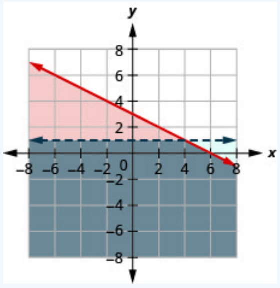


The solution is the grey region.

23. $\begin{cases} y \geq \frac{3}{4}x - 2 \\ y < 2 \end{cases}$

24. $\begin{cases} y \leq -\frac{1}{2}x + 3 \\ y < 1 \end{cases}$

Answer

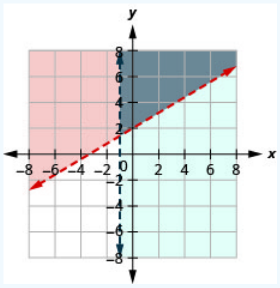


The solution is the grey region.

25. $\begin{cases} 3x - 4y < 8 \\ x < 1 \end{cases}$

26. $\begin{cases} -3x + 5y > 10 \\ x > -1 \end{cases}$

Answer

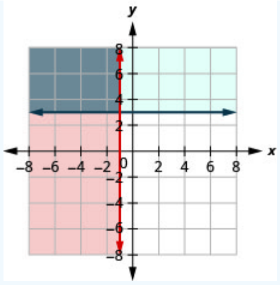


The solution is the grey region.

27. $\begin{cases} x \geq 3 \\ y \leq 2 \end{cases}$

28. $\begin{cases} x \leq -1 \\ y \geq 3 \end{cases}$

Answer

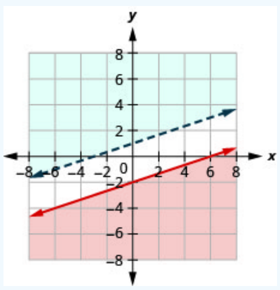


The solution is the grey region.

29. $\begin{cases} 2x+4y>4 \\ y \leq -\frac{1}{2}x-2 \end{cases}$

30. $\begin{cases} x-3y \geq 6 \\ y > \frac{1}{3}x+1 \end{cases}$

Answer

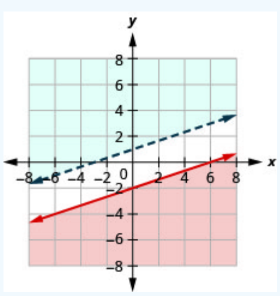


No solution.

31. $\begin{cases} -2x+6y<0 \\ 6y>2x+4 \end{cases}$

32. $\begin{cases} -3x+6y>12 \\ 4y \leq 2x-4 \end{cases}$

Answer

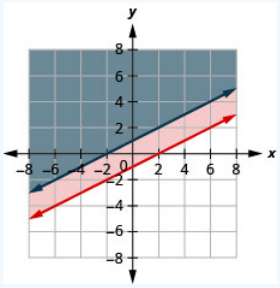


No solution.

33. $\begin{cases} y \geq -3x+2 \\ 3x+y>5 \end{cases}$

34. $\begin{cases} y \geq \frac{1}{2}x - 1 \\ -2x + 4y \geq 4 \end{cases}$

Answer

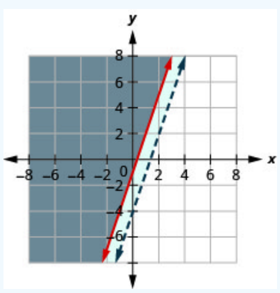


The solution is the grey region.

35. $\begin{cases} y \leq -\frac{1}{4}x - 2 \\ x + 4y < 6 \end{cases}$

36. $\begin{cases} y \geq 3x - 1 \\ -3x + y > -4 \end{cases}$

Answer

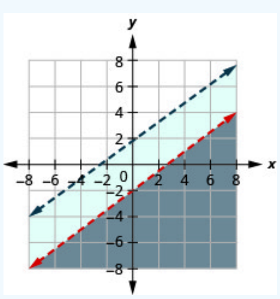


The solution is the grey region.

37. $\begin{cases} 3y > x + 2 \\ -2x + 6y > 8 \end{cases}$

38. $\begin{cases} y < \frac{3}{4}x - 2 \\ -3x + 4y < 7 \end{cases}$

Answer



The solution is the grey region.

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.

39. Caitlyn sells her drawings at the county fair. She wants to sell at least 60 drawings and has portraits and landscapes. She sells the portraits for $15 and the landscapes for $10. She needs to sell at least $800 worth of drawings in order to earn a profit.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Will she make a profit if she sells 20 portraits and 35 landscapes?
ⓓ Will she make a profit if she sells 50 portraits and 20 landscapes?

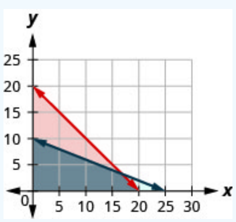
40. Jake does not want to spend more than $50 on bags of fertilizer and peat moss for his garden. Fertilizer costs $2 a bag and peat moss costs $5 a bag. Jake's van can hold at most 20 bags.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can he buy 15 bags of fertilizer and 4 bags of peat moss?
ⓓ Can he buy 10 bags of fertilizer and 10 bags of peat moss?

Answer

ⓐ $\begin{cases} f \geq 0 \\ p \geq 0 \\ f + p \leq 202 \\ f + 5p \leq 50 \end{cases}$

ⓑ



ⓒ yes
ⓓ no

41. Reiko needs to mail her Christmas cards and packages and wants to keep her mailing costs to no more than $500. The number of cards is at least 4 more than twice the number of packages. The cost of mailing a card (with pictures enclosed) is $3 and for a package the cost is $7.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she mail 60 cards and 26 packages?
ⓓ Can she mail 90 cards and 40 packages?

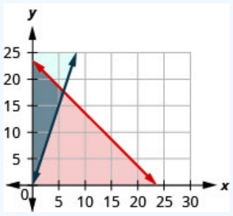
42. Juan is studying for his final exams in chemistry and algebra. he knows he only has 24 hours to study, and it will take him at least three times as long to study for algebra than chemistry.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can he spend 4 hours on chemistry and 20 hours on algebra?
ⓓ Can he spend 6 hours on chemistry and 18 hours on algebra?

Answer

ⓐ $\begin{cases} c \geq 0 \\ a \geq 0 \\ c+a \leq 24 \\ a \geq 3c \end{cases}$

ⓑ



ⓒ yes
ⓓ no

43. Jocelyn is pregnant and so she needs to eat at least 500 more calories a day than usual. When buying groceries one day with a budget of $15 for the extra food, she buys bananas that have 90 calories each and chocolate granola bars that have 150 calories each. The bananas cost $0.35 each and the granola bars cost $2.50 each.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could she buy 5 bananas and 6 granola bars?
ⓓ Could she buy 3 bananas and 4 granola bars?

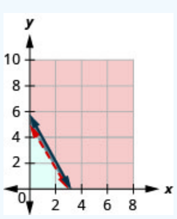
44. Mark is attempting to build muscle mass and so he needs to eat at least an additional 80 grams of protein a day. A bottle of protein water costs $3.20 and a protein bar costs $1.75. The protein water supplies 27 grams of protein and the bar supplies 16 gram. If he has $10 dollars to spend

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could he buy 3 bottles of protein water and 1 protein bar?
ⓓ Could he buy no bottles of protein water and 5 protein bars?

Answer

ⓐ $\begin{cases} w \geq 0 \\ b \geq 0 \\ 27w+16b > 80 \\ 3.20w+1.75b \leq 10 \end{cases}$

ⓑ



ⓒ no
ⓓ yes

45. Jocelyn desires to increase both her protein consumption and caloric intake. She desires to have at least 35 more grams of protein each day and no more than an additional 200 calories daily. An ounce of cheddar cheese has 7 grams of protein and 110 calories. An ounce of parmesan cheese has 11 grams of protein and 22 calories.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could she eat 1 ounce of cheddar cheese and 3 ounces of parmesan cheese?
ⓓ Could she eat 2 ounces of cheddar cheese and 1 ounce of parmesan cheese?

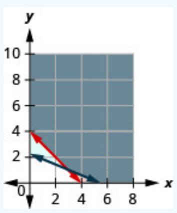
46. Mark is increasing his exercise routine by running and walking at least 4 miles each day. His goal is to burn a minimum of 1500 calories from this exercise. Walking burns 270 calories/mile and running burns 650 calories.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could he meet his goal by walking 3 miles and running 1 mile?
ⓓ Could he his goal by walking 2 miles and running 2 mile

Answer

ⓐ $\begin{cases} w \geq 0 \\ r \geq 0 \\ w + r \geq 4 \\ 270w + 650r \geq 1500 \end{cases}$

ⓑ



ⓒ no
ⓓ yes

Writing Exercises

47. Graph the inequality $x - y \geq 3$. How do you know which side of the line $x - y = 3$ should be shaded?

48. Graph the system $\begin{cases} x + 2y \leq 6 \\ y \geq -\frac{1}{2}x - 4 \end{cases}$. What does the solution mean?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered pair is a solution of a system of linear inequalities.			
solve a system of linear inequalities by graphing.			
solve applications of systems of inequalities.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

3.5: Linear Programming - Maximization Applications

Learning Objectives

In this section, you will learn to:

- Recognize the typical form of a linear programming problem.
- Formulate maximization linear programming problems.
- Graph feasible regions for maximization linear programming problems.
- Determine optimal solutions for maximization linear programming problems.

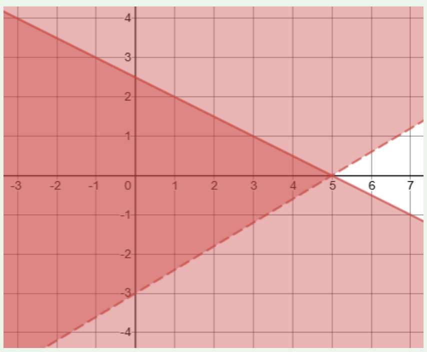
Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Graph this system of inequalities:

$$\begin{cases} 2x + 4y \leq 10 \\ 3x - 5y < 15 \end{cases}$$

Click here to check your answer

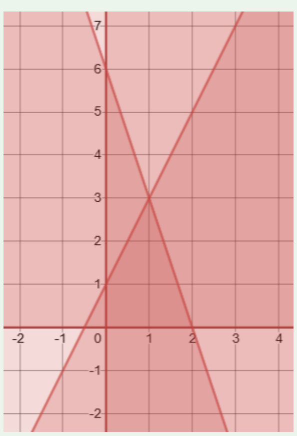


If you missed this problem, review Section 2.3. (Note that this will open in a new window.)

2. Graph this system of inequalities:

$$\begin{cases} y \leq 2x + 1 \\ y \leq -3x + 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Click here to check your answer



(The solution is in the center region.)

If you missed this problem, review Section 2.3. (Note that this will open in a new window.)

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. The conditions or constraints often take the form of inequalities. In this section, we will begin to formulate, analyze, and solve such problems, at a simple level, to understand the many components of such a problem.

A typical **linear programming** problem consists of finding an extreme value of a linear equation subject to certain constraints. We are either trying to maximize or minimize the value of this linear equation, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as **maximization** or **minimization problems**, or just **optimization problems**. The value we are trying to optimize is called an **objective function**, and the conditions that must be satisfied are called **constraints**.

When we graph all constraints, the area of the graph that satisfies all constraints is called the **feasible region**. The **Fundamental Theorem of Linear Programming** states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasible region. We call these vertices **critical points.** These are found using any methods from Section 1.4 as we are looking for the points where any two of the boundary lines intersect.

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize cost of) these activities, subject to limitations on time and other resources available for each activity.

In this chapter, we will work with problems that involve only two variables, and therefore, can be solved by graphing. Here are the steps we'll follow:

The Maximization Linear Programming Problems

1. Define the unknowns.
2. Write the objective function that needs to be maximized.
3. Write the constraints.
   a. For the standard maximization linear programming problems, constraints are of the form: $ax + by \leq c$
   b. When the variables represent quantities that cannot be negative, we include the constraints: $x \geq 0$ and $y \geq 0$.
4. Graph the constraints.
5. Determine the feasible region.
6. Find the corner points.
7. Find the value of the objective function at each corner point to determine the corner point that gives the maximum value.
8. State the answer to the original question.

Example 3.5.1

Niki holds two part-time jobs, Job I and Job II. She never wants to be at work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation.

If Niki makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

1. We start by defining our unknowns.
   - Let the number of hours per week Niki will work at Job I = x.
   - Let the number of hours per week Niki will work at Job II = y.
2. Now we write the objective function. We're trying to maximize her income, so the equation will represent the calculation for finding her income. Since Niki gets paid $40 an hour at Job I, and $30 an hour at Job II, her total income I is given by the equation

$$I = 40x + 30y$$

3. Our next task is to find the constraints. Each limitation in the problem will give a constraint.
   - The second sentence in the problem states, "She never wants to work more than a total of 12 hours a week." This translates into the following constraint:

   $$x + y \leq 12$$

   - The third sentence states, "For every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation." Therefore the next constraint is

   $$2x + y \leq 16$$

   - The fact that x and y can never be negative is represented by the following two constraints:

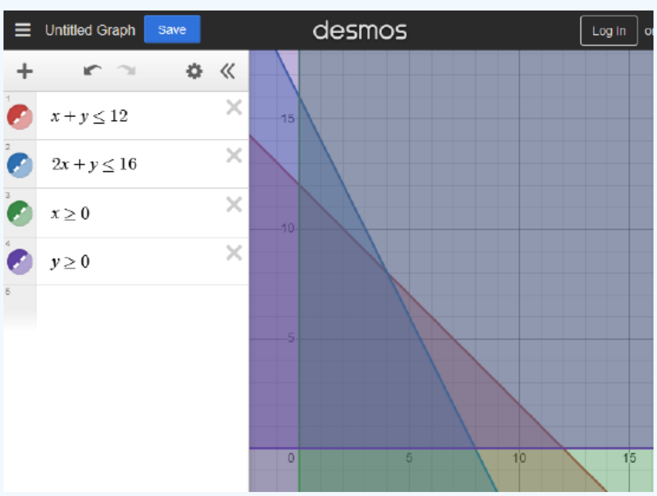
   $$x \geq 0, \text{ and } y \geq 0.$$

   - In summary, our entire problem can be written as:

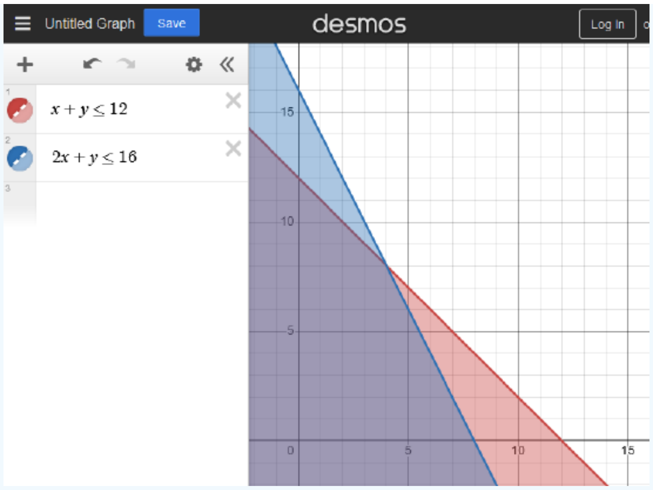
   $$\begin{array}{ll} \textbf{Maximize} & \mathrm{I = 40x + 30y} \\ \textbf{Subject to:} & \mathrm{x + y \leq 12} \\ & \mathrm{2x + y \leq 16} \\ & \mathrm{x \geq 0; y \geq 0} \end{array}$$

4. In order to solve the problem, we graph the constraints and shade the region that satisfies **all** the inequality constraints. Any appropriate method can be used to graph the lines for the constraints.
   i. Graph each inequality.
      - When doing this by hand, refer to the strategies in Section 2.2.
      - This may also be done in the Desmos app or on https://www.desmos.com/calculator. On a computer, the $\leq$ symbol can be written by typing the $<$ symbol followed by the $=$ symbol.

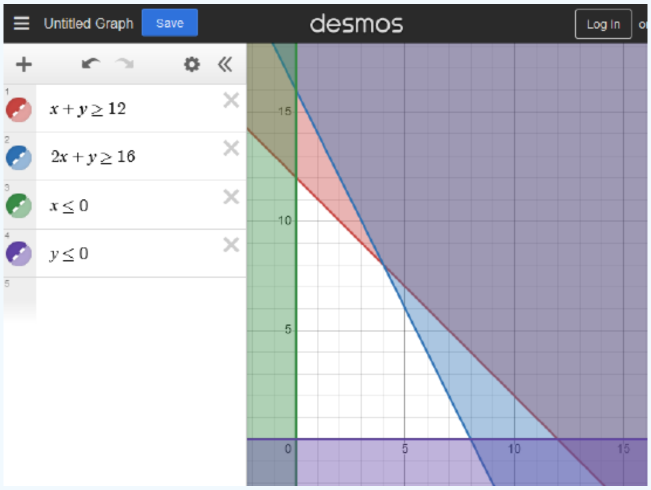
   ii. Determine which direction to shade for each line. There are three main strategies to decide how much shading is put on the graph:
      - Strategy A: Graph all inequalities exactly as written. The negative of this method is that it is often difficult to see exactly which region is covered by all colors. In the graphic below, the feasible region is the darkest gray four-sided figure in the lower left portion of the graph.

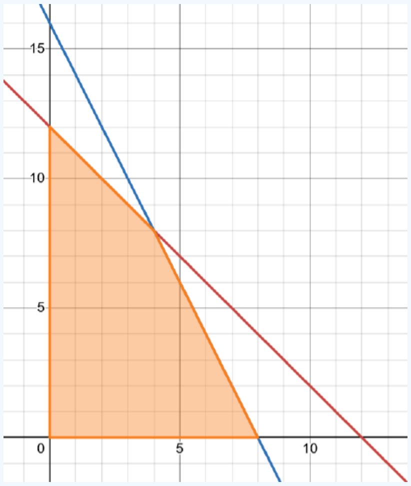
      

      - Strategy B: Graph ALMOST all inequalities, omitting $x \geq 0$ and $y \geq 0$. The feasible region will be slightly easier to identify, but the negative of this method is that students must remember that the feasible region actually stops at the axes.



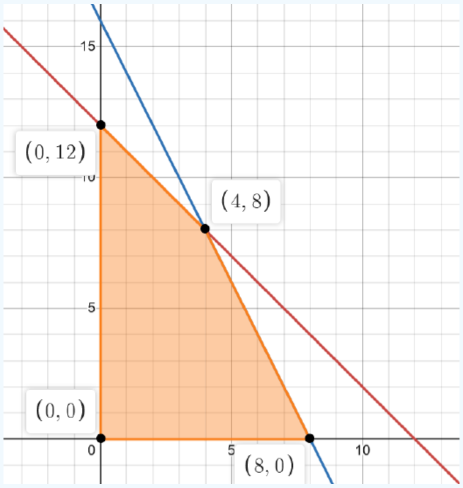
- Strategy C: Graph the OPPOSITE of each inequality. The feasible region will be clearly visible as the void or the white space between the graphs. The negative of this strategy is that it can be easy to make a mistake when typing in the inequalities, as they will be different than what is in the problem. Often students switch some inequalities but accidentally don't switch ALL inequalities.



5. Determine the feasible region, which is the region that makes all inequalities true. No matter which strategy is used, the inequalities above should yield the feasible region shown below.
    - 

6. Determine the corner points of the feasible region. These are known as the critical points or critical values.

- When graphing by hand, this can be done using any of the methods discussed in Section 1.3.
- When graphing in Desmos, this can be accomplished by clicking on the point of intersection.



7. The Fundamental Theorem of Linear Programming states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasible region. Therefore, we will substitute these points in the objective function to see which point gives us the maximum value. In this case, we are looking for the highest income per week. We list the results below.

Critical Points	Income = $40x + 30y$
(0, 0)	I = 40(0) + 30(0) = $0
(0, 12)	I = 40(0) + 30(12) = $360
(4, 8)	I = 40(4) + 30(8) = $400
(8, 0)	I = 40(8) + 30(0) = $320

8. The point (4, 8) gives the most profit: $400. Therefore, we conclude that Niki should work 4 hours at Job I, and 8 hours at Job II.

Example 3.5.2

A factory manufactures two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of $20 is realized for each regular gadget and $30 for a premium gadget, how many of each should be manufactured to maximize profit?

Solution

1. We start by defining our unknowns:

- Let the number of regular gadgets manufactured each day = x.
- and the number of premium gadgets manufactured each day = y.

2. Now we write the objective function. We are trying to maximize profit, which will be $20 for however many regular gadgets we manufacture (x and $30 for however many premium gadgets we manufacture (y. Therefore profit is is given by the equation

$$P = 20x + 30y$$

3. We now write the constraints.

- The fourth sentence states that the company can make at most 7 total gadgets a day. This translates as

$$x + y \leq 7$$

- One limitation given is the hours for the assembly operation. Since the regular gadget requires one hour of assembly and the premium gadget requires two hours of assembly, and there are at most 12 hours available for this operation, we get

$$x + 2y \leq 12$$

- Similarly, we limited by the finishing operation. The regular gadget requires two hours of finishing and the premium gadget one hour. Again, there are at most 12 hours available for finishing. This gives us the following constraint.

$$2x + y \leq 12$$

- We are also limited by reality. The fact that x and y can never be negative is represented by the following two constraints:

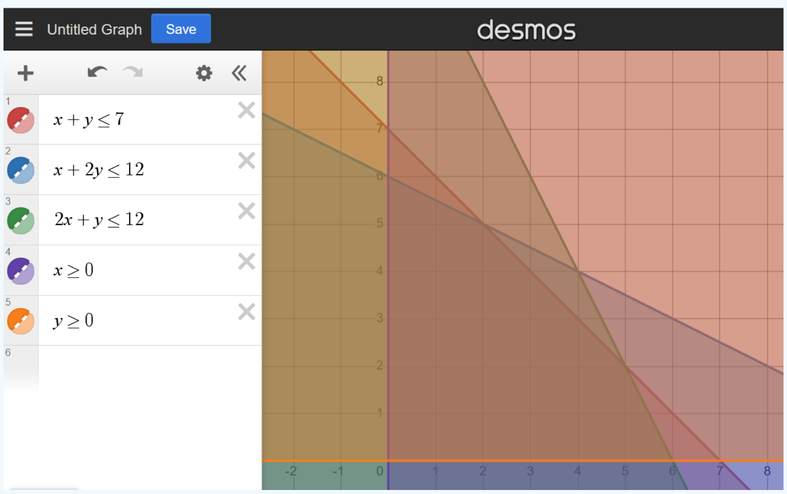
$$x \geq 0, \text{ and } y \geq 0.$$

- In summary, our entire problem can be written as follows:

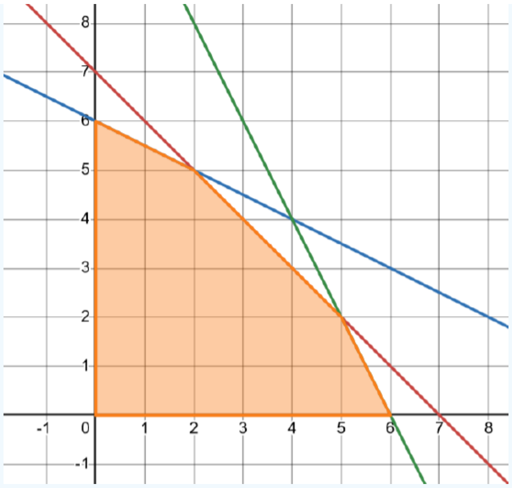
$$\begin{array}{ll} \textbf{Maximize} & \mathrm{P = 20x + 30y} \\ \textbf{Subject to:} & \mathrm{x + y \leq 7} \\ & \mathrm{x + 2y \leq 12} \\ & \mathrm{2x + y \leq 12} \\ & \mathrm{x \geq 0; y \geq 0} \end{array}$$

4. In order to solve the problem, we next graph each of the constraints.

By using Strategy A shown in the example above, we get a graph like this:

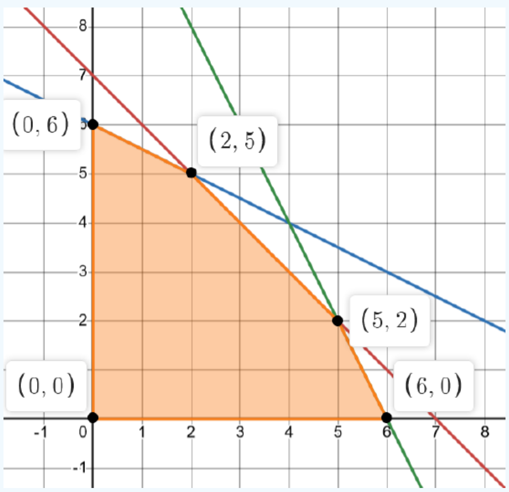


5. Determine the feasible region, which is the region that makes all inequalities true.



6. Determine the corner points of the feasible region.

- When graphing by hand, this can be done using any of the methods discussed in Section 1.3.
- When graphing in Desmos, this can be accomplished by clicking on the point of intersection.



7. We substitute these points in the objective function to see which point gives us the maximum profit each day. The results are listed below.

Critical Point	Profit = $20x + 30y$
(0, 0)	P = 20(0) + 30(0) = $0
(0, 6)	P = 20(0) + 30(6) = $180
(2, 5)	P = 20(2) + 30(5) = $190
(5, 2)	P = 20(5) + 30(2) = $160
(6,0)	P = 20(6) + 30(0) = $120

8. The point (2, 5) gives the most profit, and that profit is $190. Therefore, we conclude that we should manufacture 2 regular gadgets and 5 premium gadgets daily to obtain the maximum profit of $190.

So far we have focused on "**standard maximization problems**" in which

1. The objective function is to be maximized
2. All constraints are of the form $ax + by \leq c$
3. All variables are constrained to by non-negative ($x \geq 0, y \geq 0$)

We will next consider an example where that is not the case. Our next problem is said to have “**mixed constraints**”, since some of the inequality constraints are of the form $ax+by \leq c$ and some are of the form $ax+by \geq c$. The non-negativity constraints are still an important requirement in any linear program.

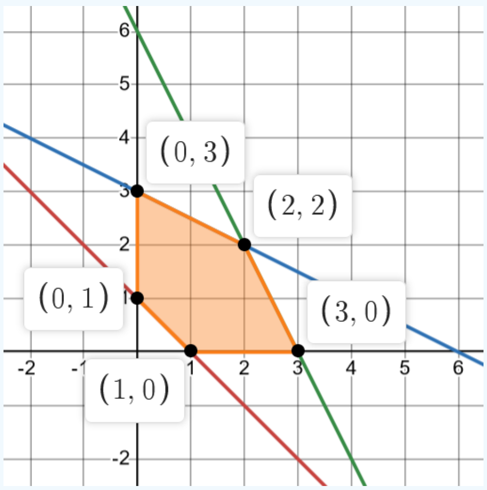
Example 3.5.3

Solve the following maximization problem graphically.

$$\begin{array}{ll}\textbf{Maximize} & \mathrm{Z}=10\mathrm{x}+15\mathrm{y} \\ \textbf{Subject to:} & \mathrm{x}+\mathrm{y} \geq 1 \\ & \mathrm{x}+2\mathrm{y} \leq 6 \\ & 2\mathrm{x}+\mathrm{y} \leq 6 \\ & \mathrm{x} \geq 0 ; \mathrm{y} \geq 0\end{array}$$

Solution

The graph is shown below.



The five critical points are listed in the above figure.

Critical point	Z = $10x+15y$
(1, 0)	Z = 10(1) + 15(0) = 10
(3, 0)	Z = 10(3) + 15(0) = 30
(2, 2)	Z = 10(2) + 15(2) = 50
(0, 3)	Z = 10(0) + 15(3) = 45
(0,1)	Z = 10(0) + 15(1) = 15

The point (2, 2) maximizes the objective function to a maximum z-value of 50.

We summarize:

The Maximization Linear Programming Problems

1. Define the unknowns.
2. Write the objective function that needs to be maximized.
3. Write the constraints.
    a. For the standard maximization linear programming problems, constraints are of the form: $ax+by \leq c$
    b. Since the variables are non-negative, we include the constraints: $x \geq 0$; $y \geq 0$.
4. Graph the constraints.

5. Determine the feasible region.
6. Find the corner points.
7. Find the value of the objective function at each corner point to determine the corner point that gives the maximum value.
8. State the answer to the original question.

3.5E: Exercises - Linear Programming Maximization Applications

For the following maximization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

2) Mr. Tran has $24,000 to invest, some in bonds and the rest in stocks. He has decided that the money invested in bonds must be at least twice as much as that in stocks. But the money invested in bonds must not be greater than $18,000. If the bonds earn 6%, and the stocks earn 8%, how much money should he invest in each to maximize profit?

3) A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair and $30 for a table, how many units of each should be manufactured to maximize profit?

4) The Silly Nut Company makes two mixtures of nuts: Mixture A and Mixture B. A pound of Mixture A contains 12 oz of peanuts, 3 oz of almonds and 1 oz of cashews and sells for $4. A pound of Mixture B contains 12 oz of peanuts, 2 oz of almonds and 2 oz of cashews and sells for $5. The company has 1080 lb. of peanuts, 240 lb. of almonds, 160 lb. of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit?
(Hint: Use consistent units. Work the entire problem in pounds by converting all values given in ounces into fractions of pounds).

5)

$$\begin{array}{ll}\text{Maximize:} & Z=4x+10y\\ \text{Subject to:} & x+y\leq 5\\ & 2x+y\leq 8\\ & x+2y\leq 8\\ & x\geq 0, y\geq 0\end{array}$$

6) This maximization linear programming problem is not in “standard” form. It has mixed constraints, some involving ≤ inequalities and some involving ≥ inequalities. However with careful graphing, we can solve this using the techniques we have learned in this section.

$$\begin{array}{ll}\text{Maximize:} & Z=5x+7y\\ \text{Subject to:} & x+y\leq 30\\ & 2x+y\leq 50\\ & 4x+3y\geq 60\\ & 2x\geq y\\ & x\geq 0, y\geq 0\end{array}$$

3.6: Linear Programming - Minimization Applications

Learning Objectives

In this section, you will learn to:

- Formulate minimization linear programming problems.
- Graph feasible regions for minimization linear programming problems.
- Determine optimal solutions for minimization linear programming problems.

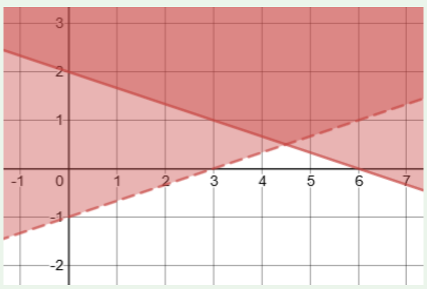
Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Graph this system of inequalities:

$$\begin{cases} x+3y \geq 6 \\ y > \dfrac{1}{3}x-1 \end{cases}$$

Click here to check your answer

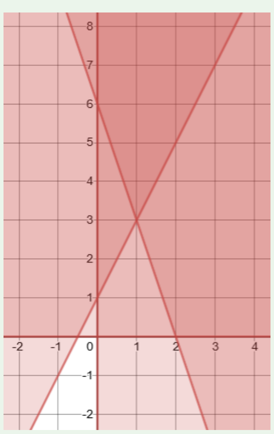


If you missed this problem, review Section 2.3. (Note that this will open a different textbook in a new window.)

2. Graph this system of inequalities:

$$\begin{cases} y \geq 2x+1 \\ y \geq -3x+6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Click here to check your answer



(The solution is in the upper center region.)

If you missed this problem, review Section 2.3. (Note that this will open a different textbook in a new window.)

Minimization linear programming problems are solved in much the same way as the maximization problems.

For the **standard minimization linear program**, the constraints are of the form $ax + by \geq c$, as opposed to the form $ax + by \leq c$ for the standard maximization problem. As a result, the feasible solution extends indefinitely to the upper right of the first quadrant, and is unbounded. But that is not a concern, since in order to minimize the objective function, the line associated with the objective function is moved towards the origin, and the critical point that minimizes the function is closest to the origin.

However, one should be aware that in the case of an unbounded feasible region, the possibility of no optimal solution exists.

Minimization Linear Programming Problems

1. Define the unknowns.
2. Write the objective function that needs to be minimized.
3. Write the constraints.
   a. For standard minimization linear programming problems, constraints are of the form: $ax + by \geq c$
   b. Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
4. Graph the constraints.
5. Shade the feasible region.
6. Find the corner points.
7. Find the value of the objective function at each corner point to determine the corner point that gives the minimum value.
8. State the answer to the original question.

Example 3.6.1

At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns $15 per hour for grading papers. Mary is a post-doctoral associate and can grade 30 papers per hour; Mary earns $25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment.

If Prof. Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?

Solution

1. We define the unknowns:

- Let the number of hours per week John is employed = x.
- and the number of hours per week Mary is employed = y.

2. Write the objective function. We are trying to minimize cost, so the function should represent the total cost. In this case, cost is given by:

$$C = 15x + 25y$$

3. Find the constraints.

- The fact that each must work at least one hour each week results in the following two constraints:

$$x \geq 1$$
$$y \geq 1$$

  Note that this already includes or covers the usual constraints that x and y are non-negative, so we don't need to include additional inequalities to state this.

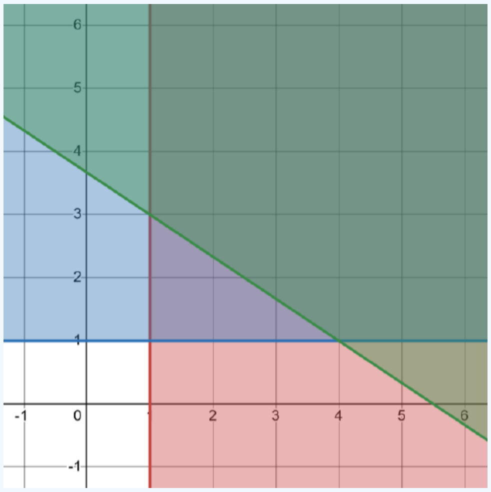
- We are limited by the minimum number of papers that must be graded. Since John can grade 20 papers per hour and Mary 30 papers per hour, and there are at least 110 papers to be graded per week, we get

$$20x + 30y \geq 110$$

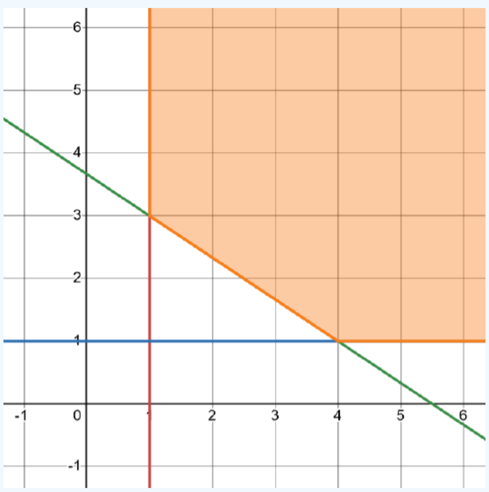
- The problem has been formulated as follows.

$$\begin{array}{ll} \textbf{Minimize} & C = 15x + 25y \\ \textbf{Subject to:} & x \geq 1 \\ & y \geq 1 \\ & 20x + 30y \geq 110 \end{array}$$

4. Next, we graph all constraints:

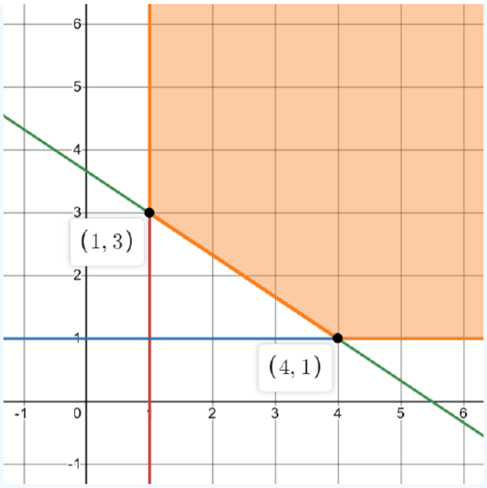


5. Determine the feasible region.



Note that this feasible region extends indefinitely up and to the right. This region is called **unbounded** as it extends without an upper bound. (In practicality, a maximum number of hours for each employee would prevent this, if we had been given this information in this problem.)

6. Determine the corner points of the feasible region.



7. Substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below.

Critical points	Cost = $15x + 25y$
(1, 3)	C = 15(1) + 25(3) = $90
(4, 1)	C = 15(4) + 25(1) = $85

8. The point (4, 1) gives the least cost, and that cost is $85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John 4 hours a week, and Mary 1 hour a week at a cost of $85 per week.

Example 3.6.2

Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals: pasta or tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamin C for lunch each month. He plans to eat lunch in the cafeteria at least 20 times. How many days should he have the pasta meal, and how many days the tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?

	PASTA	**TOFU**
PROTEIN	8g	16g
CARBOHYDRATES	60g	40g
VITAMIN C	2g	2g
CHOLESTEROL	60mg	50mg

Solution

1. We define the unknowns as follows.

- Let the number of days Mr. Hamer eats pasta = x.
- Let the number of days Mr. Hamer eats tofu = y.

2. The objective function represents the quantity to be minimized. Since he is trying to minimize his cholesterol intake, our objective function represents the total amount of cholesterol C provided by both meals.

$$C = 60x + 50y$$

3. Write all constraints.

- He needs to include a minimum amount of protein. The total amount of protein provided by both meals is

$$8x + 16y \geq 200$$

- Additionally, he needs to include a minimum amount of carbohydrates . The total amount of carbohydrates is

$$60x + 40y \geq 960$$

- Similarly, he needs to include a minimum amount of vitamins. The total amount of vitamin C is

$$2x + 2y \geq 40$$

- Finally, he needs to eat at least 20 meals in the cafeteria. Therefore

$$x + y \geq 20$$

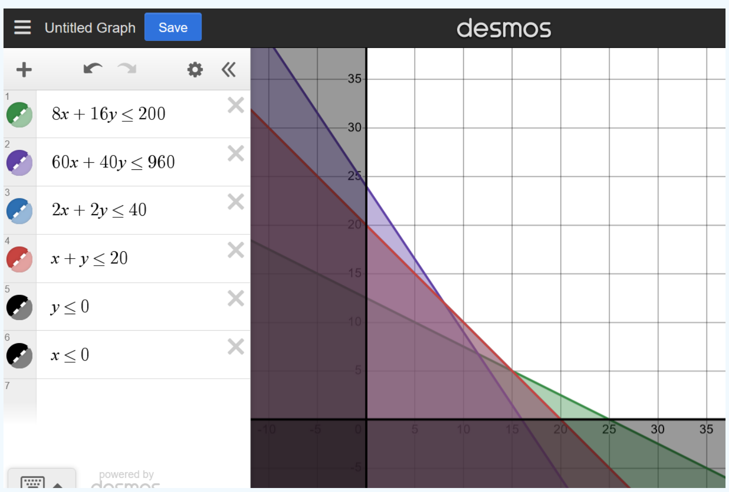
- He cannot eat a negative number of meals, so x and y are both non-negative:

$$x \geq 0, \text{ and } y \geq 0$$

- We summarize all information as follows:

$$\begin{array}{ll} \textbf{Minimize} & \mathrm{C} = 60\mathrm{x} + 50\mathrm{y} \\ \textbf{Subject to:} & 8\mathrm{x} + 16\mathrm{y} \geq 200 \\ & 60\mathrm{x} + 40\mathrm{y} \geq 960 \\ & 2\mathrm{x} + 2\mathrm{y} \geq 40 \\ & \mathrm{x} + \mathrm{y} \geq 20 \\ & \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{array}$$

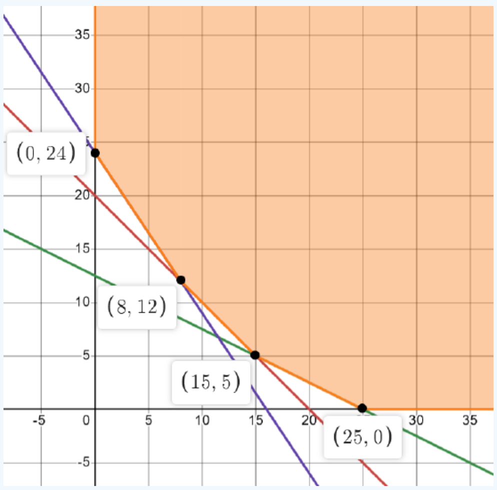
4. We graph the constraints. As this particular graph has many inequalities, graphing all of them gets quite cluttered. The graph below shows the OPPOSITE of each inequality, leaving the feasible region as the void or unshaded portion of the graph.



Also note that the line $2x + 2y = 40$ appears to be missing. This is because it is mathematically equivalent to the line $x + y = 20$. In essence, the blue line above is "hidden" underneath the red line.

5. Determine the feasible region.

6. Determine the corner points of the feasible region.



7. Substitute these points in the objective function to see which point gives us the smallest value. The results are listed below.

Critical points	Carbohydrates = $60x + 50y$
(0, 24)	C = 60(0) + 50(24) = 1200
(8, 12)	C = 60(8) + 50(12) = 1080
(15, 5)	C = 60(15) + 50(5) = 1150
(25, 0)	C = 60(25) + 50(0) = 1500

8. The point (8, 12) gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat the pasta meal 8 days and the tofu meal 12 days.

We must be aware that in some cases, a linear program may not have an optimal solution.

- A linear program can fail to have an optimal solution is if there is not a feasible region. If the inequality constraints are not compatible, there may not be a region in the graph that satisfies **all** the constraints. If the linear program does not have a feasible solution satisfying all constraints, then it can not have an optimal solution.
- A linear program can fail to have an optimal solution if the feasible region is unbounded.
  - The two minimization linear programs we examined had unbounded feasible regions. The feasible region was bounded by constraints on some sides but was not entirely enclosed by the constraints. Both of the minimization problems had optimal solutions.

- However, if we were to consider a maximization problem with a similar unbounded feasible region, the linear program would have no optimal solution. No matter what values of x and y were selected, we could always find other values of x and y that would produce a higher value for the objective function. In other words, if the value of the objective function can be increased without bound in a linear program with an unbounded feasible region, there is no optimal maximum solution.

Although the method of solving minimization problems is similar to that of the maximization problems, we still feel that we should summarize the steps involved.

Minimization Linear Programming Problems

1. Define the unknowns.
2. Write the objective function that needs to be minimized.
3. Write the constraints.
   a. For standard minimization linear programming problems, constraints are of the form: $ax + by \geq c$
   b. Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
4. Graph the constraints.
5. Shade the feasible region.
6. Find the corner points.
7. Find the value of the objective function at each corner point to determine the corner point that gives the minimum value.
8. State the answer to the original question.

3.6.1: Exercises - Linear Programming Minimization Applications

For each of the following minimization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. Food A costs \$2 per unit and Food B cost \$1 per unit. How many units of each food should be purchased to keep costs at a minimum?

2) A computer store sells two types of computers, laptops and desktops. The supplier demands that at least 150 computers be sold a month. Experience shows that most consumers prefer laptops, but some business customers require desktops. The result is that the number of laptops sold is at least twice of the number of desktops. The store pays its sales staff a \$60 commission for each laptop, and a \$40 commission for each desktop. Let x = the number of laptops and y = the number of desktop computers. How many of each type must be sold to minimize commission to its sales people?
What is the minimum commission?

3) An oil company has two refineries. Each day, Refinery A produces 200 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs \$12,000 to operate. Each day, Refinery B produces 100 barrels of high-grade oil, 100 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs \$10,000 to operate. The company must produce at least 800 barrels of high-grade oil, 900 barrels of medium-grade oil, and 1,000 barrels of low-grade oil.
How many days should each refinery be operated to meet the goals at a minimum cost?

4) A print shop at a community college in Cupertino, California, employs two different contractors to maintain its copying machines. The print shop needs to have 12 IBM, 18 Xerox, and 20 Canon copying machines serviced. Contractor A can repair 2 IBM, 1 Xerox, and 2 Canon machines at a cost of \$800 per month, while Contractor B can repair 1 IBM, 3 Xerox, and 2 Canon machines at a cost of \$1000 per month. How many months should each of the two contractors be employed to minimize the cost?

CHAPTER OVERVIEW

4: Linear Programming - An Algebraic Approach

Learning Objectives

In this chapter, you will:

1. Investigate real world applications of linear programming and related methods.
2. Solve linear programming maximization problems using the simplex method.
3. Solve linear programming minimization problems using the simplex method.

Thumbnail: Polyhedron of simplex algorithm in 3D. (CC BY-SA 3.0; Sdo via Wikipedia)

4.1: Introduction to Linear Programming Applications in Business, Finance, Medicine, and Social Science

Learning Objectives

In this section, you will learn about real world applications of linear programming and related methods.

The linear programs we solved in Chapter 3 contain only two variables, x and y, so that we could solve them graphically. In practice, linear programs can contain thousands of variables and constraints. Later in this chapter we'll learn to solve linear programs with more than two variables using the simplex algorithm, which is a numerical solution method that uses matrices and row operations. However, in order to make the problems practical for learning purposes, our problems will still have only several variables. Now that we understand the main concepts behind linear programming, we can also consider how linear programming is currently used in large scale real-world applications.

Linear programming is used in business and industry in production planning, transportation and routing, and various types of scheduling. Airlines use linear programs to schedule their flights, taking into account both scheduling aircraft and scheduling staff. Delivery services use linear programs to schedule and route shipments to minimize shipment time or minimize cost. Retailers use linear programs to determine how to order products from manufacturers and organize deliveries with their stores. Manufacturing companies use linear programming to plan and schedule production. Financial institutions use linear programming to determine the mix of financial products they offer, or to schedule payments transferring funds between institutions. Health care institutions use linear programming to ensure the proper supplies are available when needed. And as we'll see below, linear programming has also been used to organize and coordinate life saving health care procedures.

In some of the applications, the techniques used are related to linear programming but are more sophisticated than the methods we study in this class. One such technique is called integer programming. In these situations, answers must be integers to make sense, and can not be fractions. Problems where solutions must be integers are more difficult to solve than the linear programs we've worked with. In fact, many of our problems have been very carefully constructed for learning purposes so that the answers just happen to turn out to be integers, but in the real world unless we specify that as a restriction, there is no guarantee that a linear program will produce integer solutions. There are also related techniques that are called non-linear programs, where the functions defining the objective function and/or some or all of the constraints may be non-linear rather than straight lines.

Many large businesses that use linear programming and related methods have analysts on their staff who can perform the analyses needed, including linear programming and other mathematical techniques. Consulting firms specializing in use of such techniques also aid businesses who need to apply these methods to their planning and scheduling processes.

When used in business, many different terms may be used to describe the use of techniques such as linear programming as part of mathematical business models. Optimization, operations research, business analytics, data science, industrial engineering hand management science are among the terms used to describe mathematical modelling techniques that may include linear programming and related met

In the rest of this section we'll explore six real world applications, and investigate what they are trying to accomplish using optimization, as well as what their constraints might represent.

Airline Scheduling

Airlines use techniques that include and are related to linear programming to schedule their aircrafts to flights on various routes, and to schedule crews to the flights. In addition, airlines also use linear programming to determine ticket pricing for various types of seats and levels of service or amenities, as well as the timing at which ticket prices change.

The process of scheduling aircraft and departure times on flight routes can be expressed as a model that minimizes cost, of which the largest component is generally fuel costs.

Constraints involve considerations such as:

- Each aircraft needs to complete a daily or weekly tour to return back to its point of origin.
- Scheduling sufficient flights to meet demand on each route.
- Scheduling the right type and size of aircraft on each route to be appropriate for the route and for the demand for number of passengers.

- Aircraft must be compatible with the airports it departs from and arrives at - not all airports can handle all types of planes.

A model to accomplish this could contain thousands of variables and constraints. Highly trained analysts determine ways to translate all the constraints into mathematical inequalities or equations to put into the model.

After aircraft are scheduled, crews need to be assigned to flights. Each flight needs a pilot, a co-pilot, and flight attendants. Each crew member needs to complete a daily or weekly tour to return back to his or her home base. Additional constraints on flight crew assignments take into account factors such as:

- Pilot and co-pilot qualifications to fly the particular type of aircraft they are assigned to
- Flight crew have restrictions on the maximum amount of flying time per day and the length of mandatory rest periods between flights or per day that must meet certain minimum rest time regulations.
- Numbers of crew members required for a particular type or size of aircraft.

When scheduling crews to flights, the objective function would seek to minimize total flight crew costs, determined by the number of people on the crew and pay rates of the crew members. However the cost for any particular route might not end up being the lowest possible for that route, depending on tradeoffs to the total cost of shifting different crews to different routes.

An airline can also use linear programming to revise schedules on short notice on an emergency basis when there is a schedule disruption, such as due to weather. In this case the considerations to be managed involve:

- Getting aircrafts and crews back on schedule as quickly as possible
- Moving aircraft from storm areas to areas with calm weather to keep the aircraft safe from damage and ready to come back into service as quickly and conveniently as possible
- Ensuring crews are available to operate the aircraft and that crews continue to meet mandatory rest period requirements and regulations.

Kidney Donation Chain

For patients who have kidney disease, a transplant of a healthy kidney from a living donor can often be a lifesaving procedure. Criteria for a kidney donation procedure include the availability of a donor who is healthy enough to donate a kidney, as well as a compatible match between the patient and donor for blood type and several other characteristics. Ideally, if a patient needs a kidney donation, a close relative may be a match and can be the kidney donor. However often there is not a relative who is a close enough match to be the donor. Considering donations from unrelated donor allows for a larger pool of potential donors. Kidney donations involving unrelated donors can sometimes be arranged through a chain of donations that pair patients with donors. For example a kidney donation chain with three donors might operate as follows:

- Donor A donates a kidney to Patient B.
- Donor B, who is related to Patient B, donates a kidney to Patient C.
- Donor C, who is related to Patient C, donates a kidney to Patient A, who is related to Donor A.

Linear programming is one of several mathematical tools that have been used to help efficiently identify a kidney donation chain. In this type of model, patient/donor pairs are assigned compatibility scores based on characteristics of patients and potential donors.

The objective is to maximize the total compatibility scores. Constraints ensure that donors and patients are paired only if compatibility scores are sufficiently high to indicate an acceptable match.

Advertisements in Online Marketing

Did you ever make a purchase online and then notice that as you browse websites, search, or use social media, you now see more ads related the item you purchased?

Marketing organizations use a variety of mathematical techniques, including linear programming, to determine individualized advertising placement purchases.

Instead of advertising randomly, online advertisers want to sell bundles of advertisements related to a particular product to batches of users who are more likely to purchase that product. Based on an individual's previous browsing and purchase selections, he or she is assigned a "propensity score" for making a purchase if shown an ad for a certain product. The company placing the ad generally does not know individual personal information based on the history of items viewed and purchased, but instead has aggregated information for groups of individuals based on what they view or purchase. However, the company may know more about an individual’s history if he or she logged into a website making that information identifiable, within the privacy provisions and terms of use of the site.

The company’s goal is to buy ads to present to specified size batches of people who are browsing. The linear program would assign ads and batches of people to view the ads using an objective function that seeks to maximize advertising response modelled using the propensity scores. The constraints are to stay within the restrictions of the advertising budget.

Loans

A car manufacturer sells its cars though dealers. Dealers can offer loan financing to customers who need to take out loans to purchase a car. Here we will consider how car manufacturers can use linear programming to determine the specific characteristics of the loan they offer to a customer who purchases a car. In a future chapter we will learn how to do the financial calculations related to loans.

A customer who applies for a car loan fills out an application. This provides the car dealer with information about that customer. In addition, the car dealer can access a credit bureau to obtain information about a customer’s credit score.

Based on this information obtained about the customer, the car dealer offers a loan with certain characteristics, such as interest rate, loan amount, and length of loan repayment period.

Linear programming can be used as part of the process to determine the characteristics of the loan offer. The linear program seeks to maximize the profitability of its portfolio of loans. The constraints limit the risk that the customer will default and will not repay the loan. The constraints also seek to minimize the risk of losing the loan customer if the conditions of the loan are not favorable enough; otherwise the customer may find another lender, such as a bank, which can offer a more favorable loan.

Production Planning and Scheduling in Manufacturing

Consider the example of a company that produces yogurt. There are different varieties of yogurt products in a variety of flavors. Yogurt products have a short shelf life; it must be produced on a timely basis to meet demand, rather than drawing upon a stockpile of inventory as can be done with a product that is not perishable. Most ingredients in yogurt also have a short shelf life, so can not be ordered and stored for long periods of time before use; ingredients must be obtained in a timely manner to be available when needed but still be fresh. Linear programming can be used in both production planning and scheduling.

To start the process, sales forecasts are developed to determine demand to know how much of each type of product to make.

There are often various manufacturing plants at which the products may be produced. The appropriate ingredients need to be at the production facility to produce the products assigned to that facility. Transportation costs must be considered, both for obtaining and delivering ingredients to the correct facilities, and for transport of finished product to the sellers.

The linear program that monitors production planning and scheduling must be updated frequently - daily or even twice each day - to take into account variations from a master plan.

Bike Share Programs

Over 600 cities worldwide have bikeshare programs. Although bikeshare programs have been around for a long time, they have proliferated in the past decade as technology has developed new methods for tracking the bicycles.

Bikeshare programs vary in the details of how they work, but most typically people pay a fee to join and then can borrow a bicycle from a bike share station and return the bike to the same or a different bike share station. Over time the bikes tend to migrate; there may be more people who want to pick up a bike at station A and return it at station B than there are people who want to do the opposite. In chapter 9, we’ll investigate a technique that can be used to predict the distribution of bikes among the stations.

Once other methods are used to predict the actual and desired distributions of bikes among the stations, bikes may need to be transported between stations to even out the distribution. Bikeshare programs in large cities have used methods related to linear programming to help determine the best routes and methods for redistributing bicycles to the desired stations once the desire distributions have been determined. The optimization model would seek to minimize transport costs and/or time subject to constraints of having sufficient bicycles at the various stations to meet demand.

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4.2: Maximization By The Simplex Method

Learning Objectives

In this section, you will learn to solve linear programming maximization problems using the Simplex Method:

1. Identify and set up a linear program in standard maximization form
2. Convert inequality constraints to equations using slack variables
3. Set up the initial simplex tableau using the objective function and slack equations
4. Find the optimal simplex tableau by performing pivoting operations.
5. Identify the optimal solution from the optimal simplex tableau.

In the last chapter, we used the geometrical method to solve linear programming problems, but the geometrical approach will not work for problems that have more than two variables. In real life situations, linear programming problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, 5 variables and 10 constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than 250 corner points, and testing each point will be very tedious. So we need a method that has a systematic algorithm and can be programmed for a computer. The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the **simplex method**.

The simplex method was developed during the Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems. In 1979, a Soviet scientist named Leonid Khachian developed a method called the ellipsoid algorithm which was supposed to be revolutionary, but as it turned out it is not any better than the simplex method. In 1984, Narendra Karmarkar, a research scientist at AT&T Bell Laboratories developed Karmarkar's algorithm which has been proven to be four times faster than the simplex method for certain problems. But the simplex method still works the best for most problems.

The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

To learn the simplex method, we try a rather unconventional approach. We first list the algorithm, and then work a problem. We justify the reasoning behind each step during the process. A thorough justification is beyond the scope of this course.

We start out with an example we solved in the last chapter by the graphical method. This will provide us with some insight into the simplex method and at the same time give us the chance to compare a few of the feasible solutions we obtained previously by the graphical method. But first, we list the algorithm for the simplex method.

THE SIMPLEX METHOD

1. **Set up the problem.** That is, write the objective function and the inequality constraints.
2. **Convert the inequalities into equations.** This is done by adding one slack variable for each inequality.
3. **Construct the initial simplex tableau.** Write the objective function as the bottom row.
4. **The most negative entry in the bottom row identifies the pivot column.**
5. **Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.** The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored.
6. **Perform pivoting to make all other entries in this column zero.** This is done the same way as we did with the Gauss-Jordan method.
7. **When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.**
8. **Read off your answers.** Get the variables using the columns with 1 and 0s. All other variables are zero. The maximum value you are looking for appears in the bottom right hand corner.

Now, we use the simplex method to solve Example 3.1.1 solved geometrically in section 3.1.

Example 4.2.1

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

In solving this problem, we will follow the algorithm listed above.

STEP 1. Set up the problem. Write the objective function and the constraints.

Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x, y, z etc. We use symbols x_1, x_2, x_3, and so on.

Let

- x_1 = The number of hours per week Niki will work at Job I and
- x_2 = The number of hours per week Niki will work at Job II.

It is customary to choose the variable that is to be maximized as Z.

The problem is formulated the same way as we did in the last chapter.

$$\begin{array}{ll}\textbf{Maximize} & \mathrm{Z=40x_1+30x_2} \\ \textbf{Subject to:} & \mathrm{x_1+x_2 \leq 12} \\ & \mathrm{2x_1+x_2 \leq 16} \\ & \mathrm{x_1 \geq 0; x_2 \geq 0}\end{array}$$

STEP 2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

For example to convert the inequality $x_1+x_2 \leq 12$ into an equation, we add a non-negative variable y_1, and we get

$$x_1+x_2+y_1=12$$

Here the variable y_1 picks up the slack, and it represents the amount by which x_1+x_2 falls short of 12. In this problem, if Niki works fewer than 12 hours, say 10, then y_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

We rewrite the objective function $Z=40x_1+30x_2$ as $-40x_1-30x_2+Z=0$.

After adding the slack variables, our problem reads

$$\begin{array}{ll}\text{Objective function} & -40x_1-30x_2+Z=0 \\ \text{Subject to constraints:} & x_1+x_2+y_1=12 \\ & 2x_1+x_2+y_2=16 \\ & x1 \geq 0; x2 \geq 0\end{array}$$

STEP 3. Construct the initial simplex tableau. Each inequality constraint appears in its own row. (The non-negativity constraints do *not* appear as rows in the simplex tableau.) Write the objective function as the bottom row.

Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

x1	x2	y1	y2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
–40	–30	0	0	1	0

Here the vertical line separates the left hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

The reader needs to observe that the last four columns of this matrix look like the final matrix for the solution of a system of equations. If we arbitrarily choose $x_1 = 0$ and $x_2 = 0$, we get

$$\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & Z & C \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

which reads

$$y_1 = 12 \quad y_2 = 16 \quad Z = 0$$

The solution obtained by arbitrarily assigning values to some variables and then solving for the remaining variables is called the **basic solution** associated with the tableau. So the above solution is the basic solution associated with the initial simplex tableau. We can label the basic solution variable in the right of the last column as shown in the table below.

x1	x2	y1	y2	Z		
1	1	1	0	0	12	y1
2	1	0	1	0	16	y2
–40	–30	0	0	1	0	Z

STEP 4. The most negative entry in the bottom row identifies the pivot column.

The most negative entry in the bottom row is -40; therefore the column 1 is identified.

x1	x2	y1	y2	Z		
1	1	1	0	0	12	y1
2	1	0	1	0	16	y2
–40	–30	0	0	1	0	Z
↑						

Question Why do we choose the most negative entry in the bottom row?

Answer The most negative entry in the bottom row represents the largest coefficient in the objective function - the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3 etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function. In the case of the objective function $Z = 40x_1 + 30x2$, it will make more sense to increase the value of x_1 rather than x_2. The variable x_1 represents the number of hours per week Niki works at Job I. Since Job I pays $40 per hour as opposed to Job II which pays only $30, the variable x_1 will increase the objective function by $40 for a unit of increase in the variable x_1.

STEP 5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.

Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x1	x2	y1	y2	Z			
1	1	1	0	0	12	y1	12 ÷ 1 = 12
$\boxed{2}$	1	0	1	0	16	y2	← 16 ÷ 2 = 8
–40	–30	0	0	1	0	Z	
↑							

The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

Question Why do we find quotients, and why does the smallest quotient identify a row?

Answer When we choose the most negative entry in the bottom row, we are trying to increase the value of the objective function by bringing in the variable x_1. But we cannot choose any value for x_1. Can we let $x_1 = 100$? Definitely not! That is because Niki never wants to work for more than 12 hours at both jobs combined: $x_1 + x_2 \leq 12$. Can we let $x_1 = 12$? Again, the answer is no because the preparation time for Job I is two times the time spent on the job. Since Niki never wants to spend more than 16 hours for preparation, the maximum time she can work is 16 ÷ 2 = 8.

Now you see the purpose of computing the quotients; using the quotients to identify the pivot element guarantees that we do not violate the constraints.

Question Why do we identify the pivot element?

Answer As we have mentioned earlier, the simplex method begins with a corner point and then moves to the next corner point always improving the value of the objective function. The value of the objective function is improved by changing the number of units of the variables. We may add the number of units of one variable, while throwing away the units of another. Pivoting allows us to do just that.

The variable whose units are being added is called the **entering variable,** and the variable whose units are being replaced is called the **departing variable**. The entering variable in the above table is x_1, and it was identified by the most negative entry in the bottom row. The departing variable y_2 was identified by the lowest of all quotients.

STEP 6. Perform pivoting to make all other entries in this column zero.

In chapter 2, we used pivoting to obtain the row echelon form of an augmented matrix. Pivoting is a process of obtaining a 1 in the location of the pivot element, and then making all other entries zeros in that column. So now our job is to make our pivot element a 1 by dividing the entire second row by 2. The result follows.

x1	x2	y1	y2	Z	
1	1	1	0	0	12
$\boxed{1}$	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the entry first above the pivot element, we multiply the second row by -1 and add it to row 1. We get

x1	x2	y1	y2	Z	
0	1/2	1	–1/2	0	4
$\boxed{1}$	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the element below the pivot, we multiply the second row by 40 and add it to the last row.

x1	x2	y1	y2	Z		
0	1/2	1	–1/2	0	4	y1
$\boxed{1}$	1/2	0	1/2	0	8	x1
0	–10	0	20	1	320	Z

We now determine the basic solution associated with this tableau. By arbitrarily choosing $x_2 = 0$ and $y_2 = 0$, we obtain $x_1 = 8$, $y_1 = 4$, and $z = 320$. If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix stating the same thing.

$$\begin{bmatrix} \mathrm{x}_1 & \mathrm{y}_1 & \mathrm{Z} & | & \mathrm{C} \\ 0 & 1 & 0 & | & 4 \\ 1 & 0 & 0 & | & 8 \\ 0 & 0 & 1 & | & 320 \end{bmatrix}$$

We can restate the solution associated with this matrix as $x_1 = 8$, $x_2 = 0$, $y_1 = 4$, $y_2 = 0$ and $z = 320$. At this stage of the game, it reads that if Niki works 8 hours at Job I, and no hours at Job II, her profit Z will be \$320. Recall from Example 3.1.1 in section 3.1 that (8, 0) was one of our corner points. Here $y_1 = 4$ and $y_2 = 0$ mean that she will be left with 4 hours of working time and no preparation time.

STEP 7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.

Since there is still a negative entry, -10, in the bottom row, we need to begin, again, from step 4. This time we will not repeat the details of every step, instead, we will identify the column and row that give us the pivot element, and highlight the pivot element. The result is as follows.

x1	x2	y1	y2	Z			
0	$\boxed{1/2}$	1	−1/2	0	4	y1	← 4 ÷ 1/2 = 8
1	1/2	0	1/2	0	8	x1	8 ÷ 1/2 = 16
0	−10	0	20	1	320	Z	
	↑						

We make the pivot element 1 by multiplying row 1 by 2, and we get

x1	x2	y1	y2	Z	
0	$\boxed{1}$	2	−1	0	8
1	1/2	0	1/2	0	8
0	−10	0	20	1	320

Now to make all other entries as zeros in this column, we first multiply row 1 by -1/2 and add it to row 2, and then multiply row 1 by 10 and add it to the bottom row.

x1	x2	y1	y2	Z		
0	1	2	−1	0	8	x2
1	0	−1	1	0	4	x1
0	0	20	10	1	400	Z

We no longer have negative entries in the bottom row, therefore we are finished.

Question Why are we finished when there are no negative entries in the bottom row?

Answer The answer lies in the bottom row. The bottom row corresponds to the equation:

$$0x_1 + 0x_2 + 20y_1 + 10y_2 + Z = 400 \quad \text{or}$$
$$z = 400 - 20y1 - 10y2$$

Since all variables are non-negative, the highest value Z can ever achieve is 400, and that will happen only when y_1 and y_2 are zero.

STEP 8. Read off your answers.

We now read off our answers, that is, we determine the basic solution associated with the final simplex tableau. Again, we look at the columns that have a 1 and all other entries zeros. Since the columns labeled y_1 and y_2 are not such columns, we arbitrarily choose $y_1 = 0$, and $y_2 = 0$, and we get

$$\begin{bmatrix} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{Z} & | & \mathrm{C} \\ 0 & 1 & 0 & | & 8 \\ 1 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & | & 400 \end{bmatrix}$$

The matrix reads $x_1 = 4$, $x_2 = 8$ and $z = 400$.

The final solution says that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to \$400. Since both slack variables are zero, it means that she would have used up all the working time, as well as the preparation time, and none will be left.

4.2.1: Maximization By The Simplex Method (Exercises)

SECTION 4.2 PROBLEM SET: MAXIMIZATION BY THE SIMPLEX METHOD

Solve the following linear programming problems using the simplex method.

1)

$$\begin{array}{ll} \text{Maximize} & \mathrm{z = x_1 + 2x_2 + 3x_3} \\ \text{subject to} & \mathrm{x_1 + x_2 + x_3 \leq 12} \\ & \mathrm{2x_1 + x_2 + 3x_3 \leq 18} \\ & \mathrm{x_1, x_2, x_3 \geq 0} \end{array}$$

2)

$$\begin{array}{lll} \text{Maximize} & z = & x_1 + 2x_2 + x_3 \\ \text{subject to} & & x_1 + x_2 \leq 3 \\ & & x_2 + x_3 \leq 4 \\ & & x_1 + x_3 \leq 5 \\ & & x_1, x_2, x_3 \geq 0 \end{array}$$

3) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and \$20 of capital, and each acre of corn requires 16 hours of labor and \$40 of capital. The farmer has at most 800 hours of labor and \$2400 of capital available. If the profit from an acre of wheat is \$80 and from an acre of corn is \$100, how many acres of each crop should she plant to maximize her profit?

SECTION 4.2 PROBLEM SET: MAXIMIZATION BY THE SIMPLEX METHOD

Solve the following linear programming problems using the simplex method.

4) A factory manufactures chairs, tables and bookcases each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 600 hours; the second at most 500 hours; and the third at most 300 hours. A chair requires 1 hour of cutting, 1 hour of assembly, and 1 hour of finishing; a table needs 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; and a bookcase requires 3 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is \$20 per unit for a chair, \$30 for a table, and \$25 for a bookcase, how many units of each should be manufactured to maximize profit?

5). The Acme Apple company sells its Pippin, Macintosh, and Fuji apples in mixes. Box I contains 4 apples of each kind; Box II contains 6 Pippin, 3 Macintosh, and 3 Fuji; and Box III contains no Pippin, 8 Macintosh and 4 Fuji apples. At the end of the season, the company has altogether 2800 Pippin, 2200 Macintosh, and 2300 Fuji apples left. Determine the maximum number of boxes that the company can make.

4.3: Minimization By The Simplex Method

Learning Objectives

In this section, you will learn to solve linear programming minimization problems using the simplex method.

1. Identify and set up a linear program in standard minimization form
2. Formulate a dual problem in standard maximization form
3. Use the simplex method to solve the dual maximization problem
4. Identify the optimal solution to the original minimization problem from the optimal simplex tableau.

In this section, we will solve the standard linear programming minimization problems using the simplex method. Once again, we remind the reader that in the standard minimization problems all constraints are of the form $ax + by \geq c$.

The procedure to solve these problems was developed by Dr. John Von Neuman. It involves solving an associated problem called the **dual problem.** To every minimization problem there corresponds a dual problem. The solution of the dual problem is used to find the solution of the original problem. The dual problem is a maximization problem, which we learned to solve in the last section. We first solve the dual problem by the simplex method.

From the final simplex tableau, we then extract the solution to the original minimization problem.

Before we go any further, however, we first learn to convert a minimization problem into its corresponding maximization problem called its **dual**.

Example 4.3.1

Convert the following minimization problem into its dual.

$$\begin{array}{ll}\textbf{Minimize} & \mathrm{Z}=12 \mathrm{x}_1+16 \mathrm{x}_2 \\ \textbf{Subject to:} & \mathrm{x}_1+2 \mathrm{x}_2 \geq 40 \\ & \mathrm{x}_1+\mathrm{x}_2 \geq 30 \\ & \mathrm{x}_1 \geq 0 ; \mathrm{x}_2 \geq 0\end{array}$$

Solution

To achieve our goal, we first express our problem as the following matrix.

$$\begin{array}{cc|c} 1 & 2 & 40 \\ 1 & 1 & 30 \\ \hline 12 & 16 & 0 \end{array}$$

Observe that this table looks like an initial simplex tableau without the slack variables. Next, we write a matrix whose columns are the rows of this matrix, and the rows are the columns. Such a matrix is called a **transpose** of the original matrix. We get:

$$\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 1 & 16 \\ \hline 40 & 30 & 0 \end{array}$$

The following maximization problem associated with the above matrix is called its dual.

$$\begin{array}{ll}\textbf{Maximize} & \mathrm{Z}=40 \mathrm{y}_1+30 \mathrm{y}_2 \\ \textbf{Subject to:} & \mathrm{y}_1+\mathrm{y}_2 \leq 12 \\ & 2 \mathrm{y}_1+\mathrm{y}_2 \leq 16 \\ & \mathrm{y}_1 \geq 0 ; \mathrm{y}_2 \geq 0\end{array}$$

Note that we have chosen the variables as y's, instead of x's, to distinguish the two problems.

Example 4.3.2

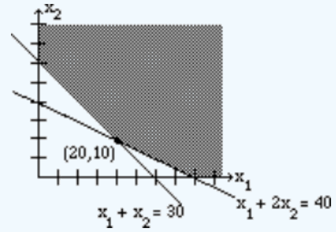
Solve graphically both the minimization problem and its dual maximization problem.

Solution

Our minimization problem is as follows.

$$\begin{array}{ll} \textbf{Minimize} & Z = 12x_1 + 16x_2 \\ \textbf{Subject to:} & x_1 + 2x_2 \geq 40 \\ & x_1 + x_2 \geq 30 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

We now graph the inequalities:

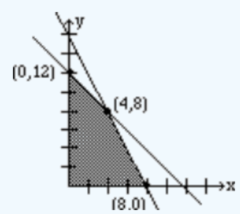


We have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (20, 10) gives the lowest value for the objective function and that value is 400.

Now its dual is:

$$\begin{array}{ll} \textbf{Maximize} & Z = 40y_1 + 30y_2 \\ \textbf{Subject to:} & y_1 + y_2 \leq 12 \\ & 2y_1 + y2 \leq 16 \\ & y_1 \geq 0; y_2 \geq 0 \end{array}$$

We graph the inequalities:



Again, we have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (4, 8) gives the highest value for the objective function, with a value of 400.

The reader may recognize that Example $4.3.2$ above is the same as Example 3.1.1, in section 3.1. It is also the same problem as Example 4.1.1 in section 4.1, where we solved it by the simplex method.

We observe that the minimum value of the minimization problem is the same as the maximum value of the maximization problem; in Example $4.3.2$ the minimum and maximum are both 400. This is not a coincident. We state the duality principle.

The Duality Principle

The Duality Principle

The objective function of the minimization problem reaches its minimum if and only if the objective function of its dual reaches its maximum. And when they do, they are equal.

Our next goal is to extract the solution for our minimization problem from the corresponding dual. To do this, we solve the dual by the simplex method.

✓ Example 4.3.3

Find the solution to the minimization problem in Example 4.3.1 by solving its dual using the simplex method. We rewrite our problem.

$$\begin{array}{ll} \textbf{Minimize} & Z = 12x_1 + 16x_2 \\ \textbf{Subject to:} & x_1 + 2x_2 \ge 40 \\ & x_1 + x_2 \ge 30 \\ & x_1 \ge 0; x_2 \ge 0 \end{array}$$

Solution

$$\begin{array}{ll} \textbf{Maximize} & Z = 40y_1 + 30y_2 \\ \textbf{Subject to:} & y_1 + y_2 \le 12 \\ & 2y_1 + y_2 \le 16 \\ & y_1 \ge 0; y_2 \ge 0 \end{array}$$

Recall that we solved the above problem by the simplex method in Example 4.1.1, section 4.1. Therefore, we only show the initial and final simplex tableau.

The initial simplex tableau is

$$\begin{array}{ccccc|c} y_1 & y_2 & x_1 & x_2 & Z & C \\ 1 & 1 & 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 1 & 0 & 16 \\ \hline -40 & -30 & 0 & 0 & 1 & 0 \end{array}$$

Observe an important change. Here our main variables are y_1 and y_2 and the slack variables are $x_1 and x_2$.

The final simplex tableau reads as follows:

$$\begin{array}{ccccc|c} y_1 & y_2 & x_1 & x_2 & Z & \\ 0 & 1 & 2 & -1 & 0 & 8 \\ 1 & 0 & -1 & 1 & 0 & 4 \\ \hline 0 & 0 & 20 & 10 & 1 & 400 \end{array}$$

A closer look at this table reveals that the x_1 and x_2 values along with the minimum value for the minimization problem can be obtained from the last row of the final tableau. We have highlighted these values by the arrows.

$$\begin{array}{ccccc|c} y_1 & y_2 & x_1 & x_2 & Z & \\ 0 & 1 & 2 & -1 & 0 & 8 \\ 1 & 0 & -1 & 1 & 0 & 4 \\ \hline 0 & 0 & 20 & 10 & 1 & 400 \\ & & \uparrow & \uparrow & & \uparrow \end{array}$$

We restate the solution as follows:

The minimization problem has a minimum value of 400 at the corner point (20, 10)

We now summarize our discussion.

Minimization by the Simplex Method

1. Set up the problem.
2. Write a matrix whose rows represent each constraint with the objective function as its bottom row.
3. Write the transpose of this matrix by interchanging the rows and columns.
4. Now write the dual problem associated with the transpose.
5. Solve the dual problem by the simplex method learned in section 4.1.

6. The optimal solution is found in the bottom row of the final matrix in the columns corresponding to the slack variables, and the minimum value of the objective function is the same as the maximum value of the dual.

4.3.1: Minimization By The Simplex Method (Exercises)

SECTION 4.3 PROBLEM SET: MINIMIZATION BY THE SIMPLEX METHOD

In problems 1-2, convert each minimization problem into a maximization problem, the dual, and then solve by the simplex method.

1)

$$\begin{array}{ll} \text{Minimize} & z=6x_1+8x_2 \\ \text{subject to} & 2x_1+3x_2 \geq 7 \\ & 4x_1+5x_2 \geq 9 \\ & x_1, x_2 \geq 0 \end{array}$$

2)

$$\begin{array}{ll} \text{Minimize} & z=5x_1+6x_2+7x_3 \\ \text{subject to} & 3x_1+2x_2+3x_3 \geq 10 \\ & 4x_1+3x_2+5x_3 \geq 12 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

SECTION 4.3 PROBLEM SET: MINIMIZATION BY THE SIMPLEX METHOD

In problems 3-4, convert each minimization problem into a maximization problem, the dual, and then solve by the simplex method.

3)

$$\begin{array}{ll} \text{Minimize} & z=4x_1+3x_2 \\ \text{subject to} & x_1+x_2 \geq 10 \\ & 3x_1+2x_2 \geq 24 \\ & x_1, x_2 \geq 0 \end{array}$$

4) A diet is to contain at least 8 units of vitamins, 9 units of minerals, and 10 calories. Three foods, Food A, Food B, and Food C are to be purchased. Each unit of Food A provides 1 unit of vitamins, 1 unit of minerals, and 2 calories. Each unit of Food B provides 2 units of vitamins, 1 unit of minerals, and 1 calorie. Each unit of Food C provides 2 units of vitamins, 1 unit of minerals, and 2 calories. If Food A costs \$3 per unit, Food B costs \$2 per unit and Food C costs \$3 per unit, how many units of each food should be purchased to keep costs at a minimum?

4.4: Chapter Review

SECTION 4.4 PROBLEM SET: CHAPTER REVIEW

Solve the following linear programming problems using the simplex method.

1)

$$\begin{array}{ll}\text{Maximize} & z=5x_1+3x_2\\ \text{subject to} & x_1+x_2\le 12\\ & 2x_1+x_2\le 16\\ & x_1\ge 0;x_2\ge 0\end{array}$$

2)

$$\begin{array}{ll}\text{Maximize} & z=5x_1+8x_2\\ \text{subject to} & x1+2x_2\le 30\\ & 3x_1+x_2\le 30\\ & x_1\ge 0;x_2\ge 0\end{array}$$

3)

$$\begin{array}{ll}\text{Maximize} & z=2x_1+3x_2+x_3\\ \text{subject to} & 4x_1+2x_2+5x_3\le 32\\ & 2x_1+4x_2+3x_3\le 28\\ & x_1,x_2,x_3\ge 0\end{array}$$

4)

$$\begin{array}{ll}\text{Maximize} & z=x_1+6x_2+8x_3\\ \text{subject to} & x_1+2x_2\le 1200\\ & 2x_2+x_3\le 1800\\ & 4x_1+x_3\le 3600\\ & x_1,x_2,x_3\ge 0\end{array}$$

5)

$$\begin{array}{ll}\text{Maximize} & z=6x_1+8x_2+5x_3\\ \text{subject to} & 4x_1+x_2+x_3\le 1800\\ & 2x_1+2x_2+x_3\le 2000\\ & 4x_1+2x_2+x_3\le 3200\\ & x_1,x_2,x_3\ge 0\end{array}$$

6)

$$\begin{array}{ll}\text{Minimize} & z=12x_1+10x_2\\ \text{subject to} & x_1+x_2\ge 6\\ & 2x_1+x_2\ge 8\\ & x_1\ge 0;x_2\ge 0\end{array}$$

7)

$$\begin{array}{ll}\text{Minimize} & z=4x_1+6x_2+7x_3\\ \text{subject to} & x_1+x_2+2x_3\\ & x_1+2x_2+x_3\ge 30\\ & x_1,x_2,x_3\ge 0\end{array}$$

8)

$$\begin{array}{ll}\text{Minimize} & z=40x_1+48x_2+30x_3\\ \text{subject to} & 2x_1+2x_2+x_3\ge 25\\ & x_1+3x_2+2x_3\ge 30\\ & x_1,x_2,x_3\ge 0\end{array}$$

9) An appliance store sells three different types of ovens: small, medium, and large. The small, medium, and large ovens require, respectively, 3, 5, and 6 cubic feet of storage space; a maximum of 1,000 cubic feet of storage space is available. Each oven takes 1hour of sales time; there is a maximum of 200 hours of sales labor time available for ovens. The small, medium, and large ovens require, respectively, 1, 1, and 2 hours of installation time; a maximum of 280 hours of installer labor for ovens is available monthly.
If the profit made from sales of small, medium and large ovens is \$50, \$100, and \$150, respectively, how many of each type of oven should be sold to maximize profit, and what is the maximum profit?

SECTION 4.4 PROBLEM SET: CHAPTER REVIEW

10) A factory manufactures three products, A, B, and C. Each product requires the use of two machines, Machine I and Machine II. The total hours available, respectively, on Machine I and Machine II per month are 180 and 300. The time requirements and profit per unit for each product are listed below.

	A	B	C
Machine I	1	2	2
Machine II	2	2	4
Profit	20	30	40

How many units of each product should be manufactured to maximize profit, and what is the maximum profit?

11) A company produces three products, A, B, and C, at its two factories, Factory I and Factory II. Daily production of each factory for each product is listed below.

	Factory I	Factory II
Product A	10	20
Product B	20	20
Product C	20	10

The company must produce at least 1000 units of product A, 1600 units of B, and 700 units of C. If the cost of operating Factory I is $4,000 per day and the cost of operating Factory II is $5000, how many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost?

12) For his classes, Professor Wright gives three types of quizzes, objective, recall, and recall-plus.
To keep his students on their toes, he has decided to give at least 20 quizzes next quarter.

The three types, objective, recall, and recall-plus quizzes, require the students to spend, respectively, 10 minutes, 30 minutes, and 60 minutes for preparation, and Professor Wright would like them to spend at least 12 hours(720 minutes) preparing for these quizzes above and beyond the normal study time.

An average score on an objective quiz is 5, on a recall type 6, and on a recall-plus 7, and Dr. Wright would like the students to score at least 130 points on all quizzes.

It takes the professor one minute to grade an objective quiz, 2 minutes to grade a recall type quiz, and 3 minutes to grade a recall-plus quiz.

How many of each type should he give in order to minimize his grading time?

CHAPTER OVERVIEW

5: Mathematics of Finance

Learning Objectives

In this chapter, you will learn to:

1. Solve financial problems that involve simple interest.
2. Solve problems involving compound interest.
3. Find the future value of an annuity, and the amount of payments to a sinking fund.
4. Find the future value of an annuity, and an installment payment on a loan.

6.1: Simple Interest

Learning Objectives

In this section, you will learn to:

- Use the simple interest formula to find an account balance, principal, rate, or time.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Solve $5 = 2(x + 9)$.

Click here to check your answer

$x = -\frac{13}{2}$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

2. Solve $x = -3(7 - 10)$.

Click here to check your answer

$x = 9$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

3. Solve $-12 = x(3 + 8)$.

Click here to check your answer

$x = -\frac{12}{11}$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

4. Solve $12 = 4(2 - x)$.

Click here to check your answer

$x = -1$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

Simple Interest

It costs to borrow money. The rent one pays for the use of money is called the **interest**. The amount of money that is being borrowed or loaned is called the **principal**, also called the present value. Simple interest is paid only on the original amount borrowed. When the money is loaned out, the person who borrows the money generally pays a fixed rate of interest on the principal for the time period he keeps the money. Although the interest rate is most commonly specified for a year, it may be specified for a week, a month, or a quarter, etc. Credit card companies often list their charges as monthly rates, sometimes it is as high as 1.5% a month.

Definition: Simple Interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$I = P \cdot r \cdot t \tag{6.1.1}$$

Definition: Accumulated Value

The total amount A, also called the **accumulated value** or the future value, is given by

$$\begin{aligned} A &= P + I \\ &= P + Prt \end{aligned}$$

or

$$A = P(1+rt) \tag{6.1.2}$$

where interest rate r is expressed in decimals.

Example 6.1.1

Ursula borrows $600 for 5 months at a simple interest rate of 15% per year. Find the interest, and the total amount she is obligated to pay?

Solution

The interest is computed by multiplying the principal with the interest rate and the time.

$$\begin{aligned} \mathrm{I} &= \mathrm{Prt} \\ &= \$600(0.15)\frac{5}{12} \\ &= \$37.50 \end{aligned}$$

The total amount is

$$\begin{aligned} \mathrm{A} &= \mathrm{P} + \mathrm{I} \\ &= \$600 + \$37.50 \\ &= \$637.50 \end{aligned}$$

Incidentally, the total amount can be computed directly via Equation 6.1.2 as

$$\begin{aligned} A &= P(1+rt) \\ &= \$600[1+(0.15)(5/12)] \\ &= \$600(1+0.0625) \\ &= \$637.50 \end{aligned}$$

Example 6.1.2

Jose deposited $2500 in an account that pays 6% simple interest. How much money will he have at the end of 3 years?

Solution

The total amount or the future value is given by Equation 6.1.2.

$$\begin{aligned} A &= P(1+rt) \\ &= \$2500[1+(.06)(3)] \\ \mathrm{A} &= \$2950 \end{aligned}$$

Example 6.1.3

Darnel owes a total of $3060 which includes 12% simple interest for the three years he borrowed the money. How much did he originally borrow?

Solution

This time we are asked to compute the principal P via Equation 6.1.2.

$$\begin{aligned} \$3060 &= \mathrm{P}[1+(0.12)(3)] \\ \$3060 &= \mathrm{P}(1.36) \\ \frac{\$3060}{1.36} &= \mathrm{P} \\ \$2250 &= \mathrm{P} \quad \text{Darnel originally borrowed } \$2250. \end{aligned}$$

Example 6.1.4

A Visa credit card company charges a 1.5% simple interest finance charge each month on the unpaid balance. If Martha owed \$2350 and has not paid her bill for three months, how much does she owe now?

Solution

Before we attempt the problem, the reader should note that in this problem the rate of finance charge is given per month and not per year.

The total amount Martha owes is the previous unpaid balance plus the finance charge.

$$A = \$2350 + \$2350(.015)(3) = \$2350 + \$105.75 = \$2455.75$$

Alternatively, again, we can compute the amount directly by using formula $A = P(1+rt)$

$$A = \$2350[1+(.015)(3)] = \$2350(1.045) = \$2455.75$$

Summary

Below is a summary of the formulas we developed for calculations involving simple interest:

Simple interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$I = P \cdot r \cdot t$$

The total amount A, also called the accumulated value or the future value, is given by

$$A = P + I = P + Prt$$

or

$$A = P(1+rt)$$

where interest rate r is expressed in decimals.

6.1E: Exercises - Simple Interest

PROBLEM SET: SIMPLE INTEREST

Do the following simple interest problems.

1) If an amount of $2,000 is borrowed at a simple interest rate of 10% for 3 years, how much is the interest?	2) You borrow $4,500 for six months at a simple interest rate of 8%. How much is the interest?
3) John borrows $2400 for 3 years at 9% simple interest. How much will he owe at the end of 3 years?	4) Jessica takes a loan of $800 for 4 months at 12% simple interest. How much does she owe at the end of the 4-month period?
5) If an amount of $2,160, which includes a 10% simple interest for 2 years, is paid back, how much was borrowed 2 years earlier?	6) Jamie just paid off a loan of $2,544, the principal and simple interest. If he took out the loan six months ago at 12% simple interest, what was the amount borrowed?
7) Shanti charged $800 on her charge card and did not make a payment for six months. If there is a monthly charge of 1.5%, how much does she owe?	8) A credit card company charges 18% interest on the unpaid balance. If you owed $2000 three months ago and have been delinquent since, how much do you owe?

9) An amount of $2000 is borrowed for 3 years. At the end of the three years, $2660 is paid back. What was the simple interest rate?	10) Nancy borrowed $1,800 and paid back $1,920, four months later. What was the simple interest rate?

6.2: Compound Interest

Learning Objectives

In this section, you will learn to:

- Find the future value of a lump-sum.
- Find the present value of a lump-sum.
- Find the effective interest rate.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Simplify each expression.

a. $100(3+2^2)$

b. $100(3+2)^2$

Click here to check your answer

a. 700

b. 2500

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

2. If an amount of $2,000 is borrowed at a simple interest rate of 10% for 3 years, how much is the interest?

Click here to check your answer

$\$600$

If you missed this problem, review Section 6.1. (Note that this will open in a new window.)

3. You borrow $4,500 for six months at a simple interest rate of 8%. How much is the interest?

Click here to check your answer

$\$180$

If you missed this problem, review Section 6.1. (Note that this will open in a new window.)

4. John borrows $2400 for 3 years at 9% simple interest. How much will he owe at the end of 3 years?

Click here to check your answer

$\$3048$

If you missed this problem, review Section 6.1. (Note that this will open in a new window.)

Compound Interest

In the last section, we examined problems involving simple interest. Simple interest is generally charged when the lending period is short and often less than a year. When the money is loaned or borrowed for a longer time period, if the interest is paid (or charged) not only on the principal, but also on the past interest, then we say the interest is **compounded.**

Suppose we deposit $200 in an account that pays 8% interest each year. At the end of one year, we will have $200 + $200(.08) = $200(1 + .08) = $216.

Now suppose we put this amount, \$216, in the same account. After another year, we will have \$216 + \$216(.08) = \$216(1 + .08) = \$233.28.

So an initial deposit of \$200 has accumulated to \$233.28 in two years. Further note that had it been simple interest, this amount would have accumulated to only \$232. The reason the amount is slightly higher is because the interest (\$16) we earned the first year, was put back into the account. And this \$16 amount itself earned for one year an interest of \$16(.08) = \$1.28, thus resulting in the increase. So we have earned interest on the principal as well as on the past interest, and that is why we call it compound interest.

Now suppose we leave this amount, \$233.28, in the bank for another year, the final amount will be \$233.28 + \$233.28(.08) = \$233.28(1 + .08) = \$251.94.

Now let us look at the mathematical part of this problem so that we can devise an easier way to solve these problems.

After one year, we had \$200(1 + .08) = \$216

After two years, we had \$216(1 + .08)

But \$216 = \$200(1 + .08), therefore, the above expression becomes

$$\$200(1+.08)(1+.08)=\$200(1+.08)^2=\$233.28$$

After three years, we get

$$\$233.28(1+.08)=\$200(1+.08)(1+.08)(1+.08)$$

which can be written as

$$\$200(1+.08)^3=\$251.94$$

Suppose we are asked to find the total amount at the end of 5 years, we will get

$$200(1+.08)^5=\$293.87$$

We summarize as follows:

The original amount	\$200	= \$200
The amount after one year	\$200(1 + .08)	= \$216
The amount after two years	$\$200(1 + .08)^2$	= \$233.28
The amount after three years	$\$200(1 + .08)^3$	= \$251.94
The amount after five years	$\$200(1 + .08)^5$	= \$293.87
The amount after t years	$\$200(1 + .08)^t$	

Periodically Compounded Interest

Banks often compound interest more than one time a year. Consider a bank that pays 8% interest but compounds it four times a year, or quarterly. This means that every quarter the bank will pay an interest equal to one-fourth of 8%, or 2%.

Now if we deposit \$200 in the bank, after one quarter we will have $\$200\left(1+\frac{.08}{4}\right)$ or \$204.

After two quarters, we will have $\$200\left(1+\frac{.08}{4}\right)^2$ or \$208.08.

After one year, we will have $\$200\left(1+\frac{.08}{4}\right)^4$ or \$216.49.

After three years, we will have $\$200\left(1+\frac{.08}{4}\right)^{12}$ or \$253.65, etc.

The original amount	\$200	= \$200
The amount after one quarter	$\$200\left(1+\frac{.08}{4}\right)$	= \$204
The amount after two quarters	$\$200\left(1+\frac{.08}{4}\right)^2$	= \$208.08

The amount after one year	$\$200\left(1+\frac{.08}{4}\right)^{4}$	= $216.49
The amount after two years	$\$200\left(1+\frac{.08}{4}\right)^{8}$	= $234.31
The amount after three years	$\$200\left(1+\frac{.08}{4}\right)^{12}$	= $253.65
The amount after five years	$\$200\left(1+\frac{.08}{4}\right)^{20}$	= $297.19
The amount after t years	$\$200\left(1+\frac{.08}{4}\right)^{4t}$	

We can see the formula for compound interest emerge.

Definition: Compound Interest, n times per year

If a lump-sum amount of P dollars is invested at an interest rate r, compounded n times a year, then after t years the final amount is given by

$$A=P\left(1+\frac{r}{n}\right)^{nt} \tag{6.2.1}$$

$\mathbf{P}$ is called the **principal** and is also called **the present value**.

Example 6.2.1

If $3500 is invested at 9% compounded monthly, what will the future value be in four years?

Solution

Clearly an interest of .09/12 is paid every month for four years. The interest is compounded $4\times 12=48$ times over the four-year period. We get

$$\mathrm{A}=\$3500\left(1+\frac{.09}{12}\right)^{48}=\$3500(1.0075)^{48}=\$5009.92$$

$3500 invested at 9% compounded monthly will accumulate to $5009.92 in four years.

Example 6.2.2

How much should be invested in an account paying 9% compounded daily for it to accumulate to $5,000 in five years?

Solution

We know the future value, but need to find the principal.

$$\begin{aligned} \$5000&=P\left(1+\tfrac{.09}{365}\right)^{365\times 5}\\ \$5000&=P(1.568225)\\ \$3188.32&=P \end{aligned}$$

$3188,32 invested into an account paying 9% compounded daily will accumulate to $5,000 in five years.

Example 6.2.3

If $4,000 is invested at 4% compounded annually, how long will it take to accumulate to $6,000?

Solution

$n=1$ because annual compounding means compounding only once per year. The formula simplifies to $A=(1+r)^t$ when $n=1$.

$$\begin{aligned}\$6000 &= 4000(1+.04)^t\\ \frac{6000}{4000} &= 1.04^t\\ 1.5 &= 1.04^t\end{aligned}$$

We use logarithms to solve for the value of t because the variable t is in the exponent.

$$t=\log_{1.04}(1.5)$$

Using the change of base formula we can solve for t:

$$t=\frac{\ln(1.5)}{\ln(1.04)}=10.33 \text{ years}$$

It takes 10.33 years for \$4000 to accumulate to \$6000 if invested at 4% interest, compounded annually

Example 6.2.4

If \$5,000 is invested now for 6 years what interest rate compounded quarterly is needed to obtain an accumulated value of \$8000.

Solution

We have $n=4$ for quarterly compounding.

$$\begin{aligned}\$8000 &= \$5000\left(1+\frac{r}{4}\right)^{4\times 6}\\ \frac{\$8000}{\$5000} &= \left(1+\frac{r}{4}\right)^{24}\\ 1.6 &= \left(1+\frac{r}{4}\right)^{24}\end{aligned}$$

We use roots to solve for t because the variable r is in the base, whereas the exponent is a known number.

$$\sqrt[24]{1.6}=1+\frac{\mathrm{r}}{4}$$

Many calculators have a built in “nth root” key or function. In the TI-84 calculator, this is found in the Math menu. Roots can also be calculated as fractional exponents; if necessary, the previous step can be rewritten as

$$1.6^{1/24}=1+\frac{\mathrm{r}}{4}$$

Evaluating the left side of the equation gives

$$\begin{aligned}&1.0197765=1+\tfrac{\mathrm{r}}{4}\\ &0.0197765=\tfrac{\mathrm{r}}{4}\\ &\mathrm{r}=4(0.0197765)=0.0791\end{aligned}$$

An interest rate of 7.91% is needed in order for \$5000 invested now to accumulate to \$8000 at the end of 6 years, with interest compounded quarterly.

Effective Interest Rate

Banks are required to state their interest rate in terms of an **“effective yield”** or **“effective interest rate”**, for comparison purposes. The effective rate is also called the Annual Percentage Yield (APY) or Annual Percentage Rate (APR).

The effective rate is the interest rate compounded annually would be equivalent to the stated rate and compounding periods. We often suppose we invest \$1 over the course of one year to determine the effective rate, as is shown in the next example.

To examine several investments to see which has the best rate, we find and compare the effective rate for each investment.

Example 6.2.5

If Bank A pays 7.2% interest compounded monthly, what is the effective interest rate?
If Bank B pays 7.2% interest compounded semiannually, what is the effective interest rate?

Which bank pays more interest?

Solution

Bank A: Suppose we deposit $1 in this bank and leave it for a year, we will get

$$1\left(1+\frac{0.072}{12}\right)^{12}=1.0744$$
$$\mathbf{r}_{\mathrm{EFF}}=1.0744-1=0.0744$$

We earned interest of $1.0744 - $1.00 = $.0744 on an investment of $1.

The effective interest rate is 7.44%, often referred to as the APY.

Bank B: The effective rate is calculated as

$$\mathbf{r}_{\mathrm{EFF}}=1\left(1+\frac{0.072}{2}\right)^{2}-1=.0733$$

The effective interest rate is 7.33%.

Bank A pays slightly higher interest, with an effective rate of 7.44%, compared to Bank B with effective rate 7.33%.

Definition: Effective Interest Rate, compounded n times per year

If a bank pays an interest rate r per year, compounded n times a year, then **the effective interest rate** is given by

$$\mathbf{r}_{\mathrm{EFF}}=\left(1+\frac{r}{n}\right)^{n}-1 \tag{6.2.2}$$

This is also reffered to as the **annual percentage yield**, or **APY**.

Continuously Compounded Interest

Interest can be compounded yearly, semiannually, quarterly, monthly, and daily. Using the same calculation methods, we could compound every hour, every minute, and even every second. As the compounding period gets shorter and shorter, we move toward the concept of continuous compounding.

But what do we mean when we say the interest is compounded continuously, and how do we compute such amounts? When interest is compounded "infinitely many times", we say that the interest is **compounded continuously**. Our next objective is to derive a formula to model continuous compounding.

Suppose we put $1 in an account that pays 100% interest. If the interest is compounded once a year, the total amount after one year will be $\$1(1+1)=\$2$.

- If the interest is compounded semiannually, in one year we will have $\$1(1+1/2)^2=\$2.25$
- If the interest is compounded quarterly, in one year we will have $\$1(1+1/4)^4=\$2.44$
- If the interest is compounded monthly, in one year we will have $\$1(1+1/12)^{12}=\$2.61$
- If the interest is compounded daily, in one year we will have $\$1(1+1/365)^{365}=\$2.71$

We show the results as follows:

Frequency of compounding	Formula	Total amount
Annually	$\$1(1+1)$	$2
Semiannually	$\$1(1+1/2)^2$	$2.25
Quarterly	$\$1(1+1/4)^4=\$2.44$	$2.44140625
Monthly	$\$1(1+1/12)^{12}$	$2.61303529

Frequency of compounding	Formula	Total amount
Daily	$\$1(1+1/365)^{365}$	$2.71456748
Hourly	$\$1(1+1/8760)^{8760}$	$2.71812699
Every minute	$\$1(1+1/525600)^{525600}$	$2.71827922
Every Second	$\$1(1+1/31536000)^{31536000}$	$2.71828247
Continuously	$\$1(2.718281828\ldots)$	$2.718281828...

We have noticed that the $1 we invested does not grow without bound. It starts to stabilize to an irrational number 2.718281828... given the name "*e*" after the great mathematician Euler.

In mathematics, we say that as n becomes infinitely large the expression equals $\left(1+\frac{1}{n}\right)^{n} = e$.

Therefore, it is natural that the number *e* play a part in continuous compounding.
It can be shown that as n becomes infinitely large the expression $\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$

Therefore, it follows that if we invest $\$P$ at an interest rate r per year, compounded continuously, after t years the final amount will be given by

$$A = P \cdot e^{rt}$$

Definition: Continuously Compounded Interest

If an amount P is invested for t years at an interest rate r per year, **compounded continuously**, then the future value is given by

$$\mathrm{A} = \mathrm{P}e^{rt} \tag{6.2.3}$$

Example 6.2.6

$3500 is invested at 9% compounded continuously. Find the future value in 4 years.

Solution

Using the formula for the continuous compounding, we get $A = Pe^{rt}$.

$$\begin{aligned} A &= \$3500e^{0.09\times 4} \\ A &= \$3500e^{0.36} \\ A &= \$5016.65 \end{aligned}$$

Example 6.2.7

If an amount is invested at 7.2% compounded continuously, what is the effective interest rate?

Solution

If we deposit $1 in the bank at 7.2% compounded continuously for one year, and subtract that $1 from the final amount, we get the effective interest rate in decimals.

$$\begin{aligned} \mathrm{r_{EFF}} &= 1\mathrm{e}^{0.072} - 1 \\ \mathrm{r_{EFF}} &= 1.07466 - 1 \\ \mathrm{r_{EFF}} &= .07466 \text{ or } 7.466\% \end{aligned}$$

Definition: Effective Interest Rate, compounded continously

If a bank pays an interest rate r per year, compounded continuously, then the effective interest rate is given by

$$\mathrm{r_{EFF}} = e^{\mathrm{r}} - 1 \tag{6.2.4}$$

Example 6.2.8

If an amount is invested at 7% compounded continuously, how long will it take to double?

Solution

We don't know the initial value of the principal but we do know that the accumulated value is double (twice) the principal.

$$\mathrm{P} \cdot e^{0.07t} = 2\mathrm{P}$$

We divide both sides by P

$$e^{.07t} = 2$$

Using natural logarithm:

$$\begin{aligned} .07\mathrm{t} &= \ln(2) \\ \mathrm{t} &= \ln(2)/.07 = 9.9 \text{ years} \end{aligned}$$

It takes 9.9 years for money to double if invested at 7% continuous interest.

Example 6.2.9

a. At the peak growth rate in the 1960's the world's population had a doubling time of 35 years. At that time, approximately what was the growth rate?

b. As of 2015, the world population's annual growth rate was approximately 1.14%. Based on that rate, find the approximate doubling time.

Solution

We expect the world's population to grow continuously, not in discrete intervals such as years or months. Therefore, we will use the formula $A = Pe^{rt}$.

a. Substituting $2\mathrm{P}$ for A and 35 for t gives us the equation

$$2\mathrm{P} = \mathrm{P} \cdot e^{r(35)}$$

We divide both sides by P:

$$2 = e^{r(35)}$$

Using natural logarithm:

$$\ln(2) = r(35)$$

Dividing both sides by 35:

$$\begin{gathered} \frac{\ln(2)}{35} = r \\ 0.0198 = r \end{gathered}$$

The growth rate was approximately 1.98%.

b. Substituting $2\mathrm{P}$ for A and 0.0114 for r gives us the equation

$$2\mathrm{P} = \mathrm{P} \cdot e^{0.0114t}$$

We divide both sides by P:

$$2 = e^{0.0114t}$$

Using natural logarithm:

$$\ln(2) = 0.0114t$$

Dividing both sides by 0.014:

$$\frac{\ln(2)}{0.0114} = t$$

$$60.8 = t$$

If the world population were to continue to grow at the annual growth rate of 1.14% , it would take approximately 60.8 years for the population to double.

SECTION 8.2 SUMMARY

Below is a summary of the formulas we developed for calculations involving compound interest:

COMPOUND INTEREST n times per year

1. If an amount P is invested for t years at an interest rate r per year, compounded n times a year, then the future value is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

   $\mathbf{P}$ is called the principal and is also called the present value.
2. If a bank pays an interest rate r per year, compounded n times a year, then the effective interest rate is given by

$$\mathbf{r}_{\mathrm{EFF}} = \left(1 + \frac{r}{n}\right)^{n} - 1$$

CONTINUOUSLY COMPOUNDED INTEREST

3. If an amount P is invested for t years at an interest rate r per year, compounded continuously, then the future value is given by

$$\mathrm{A} = \mathrm{P}e^{rt}$$

4. If a bank pays an interest rate r per year, compounded continuously, then the effective interest rate is given by

$$\mathrm{r}_{\mathrm{EFF}} = e^{\mathbf{r}} - 1$$

6.2E: Exercises - Compound Interest

PROBLEM SET: COMPOUND INTEREST

Do the following compound interest problems involving a lump-sum amount.

1) What will the final amount be in 4 years if $8,000 is invested at 9.2% compounded monthly?	2) How much should be invested at 10.3% compounded quarterly for it to amount to $10,000 in 6 years?
3) Lydia's aunt Rose left her $5,000. Lydia spent $1,000 on her wardrobe and deposited the rest in an account that pays 6.9% compounded daily. How much money will she have in 5 years?	4) Thuy needs $1,850 in eight months for her college tuition. How much money should she deposit lump sum in an account paying 8.2% compounded monthly to achieve that goal?
5) Bank A pays 5% compounded daily, while Bank B pays 5.12% compounded monthly. Which bank pays more? Explain.	6) EZ Photo Company needs five copying machines in 2 1/2 years for a total cost of $15,000. How much money should be deposited now to pay for these machines, if the interest rate is 8% compounded semiannually?
7) Jon's grandfather was planning to give him $12,000 in 10 years. Jon has convinced his grandfather to pay him $6,000 now, instead. If Jon invests this $6,000 at 7.5% compounded continuously, how much money will he have in 10 years?	8) What will be the price of a $20,000 car in 5 years if the inflation rate is 6%?

COMPOUND INTEREST

Do the following compound interest problems.

9) At an interest rate of 8% compounded continuously, how many years will it take to double your money?	10) If an investment earns 10% compounded continuously, in how many years will it triple? .
11) The City Library ordered a new computer system costing $158,000; it will be delivered in 6 months, and the full amount will be due 30 days after delivery. How much must be deposited today into an account paying 7.5% compounded monthly to have $158,000 in 7 months?	12) Mr. and Mrs. Tran are expecting a baby girl in a few days. They want to put away money for her college education now. How much money should they deposit in an account paying 10.2% so they will have $100,000 in 18 years to pay for their daughter's educational expenses?
13) Find the effective interest rate for an account paying 7.2% compounded quarterly.	14) If a bank pays 5.75% compounded monthly, what is the effective interest rate?
15) The population of the African nation of Cameroon was 12 million people in the year 2015; it has been growing at the rate of 2.5% per year. If the population continues to grow that rate,what will the population be in 2030? (http://databank.worldbank.org/data on 4/26/2016)	16) According to the Law of 70, if an amount grows at an annual rate of 1%, then it doubles every seventy years. Suppose a bank pays 5% interest, how long will it take for you to double your money? How about at 15%?

6.3: Future Value of Annuities and Sinking Funds

Learning Objectives

In this section, you will learn to:

- Find the future value of an annuity.
- Find the amount of payments to a sinking fund.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Solve $24 = 6 \cdot x^3$.

Click here to check your answer

$x = \sqrt[3]{4} \approx 1.5874$

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

2. Solve $24 = 6 \cdot 3^x$.

Click here to check your answer

$x = \log_3(4) \approx 1.2619$

If you missed this problem, review Section 5.3. (Note that this will open in a new window.)

3. Simplify $\dfrac{500(2 - \frac{5}{6})^{5 \cdot 4} - 1}{\frac{2}{3}}$ in your calculator.

Click here to check your answer

$16,366.55426$

If you missed this problem, review here. (Note that this will open a YouTube video in a new window.)

Ordinary Annuities

In the first two sections of this chapter, we examined problems where an amount of money was deposited as a lump sum in an account and was left there for the entire time period. Now we will do problems where timely payments are made in an account. When a sequence of payments of some fixed amount are made in an account at equal intervals of time, we call that an **annuity**. And this is the subject of this section.

Example 6.3.1

If at the end of each month a deposit of \$500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution

There are 60 deposits made in this account. The first payment stays in the account for 59 months, the second payment for 58 months, the third for 57 months, and so on. We can use the formula for compound interest to examine each payment.

- The first payment of \$500 will accumulate to an amount of $\$500(1 + 0.08/12)^{59}$.

- The second payment of \$500 will accumulate to an amount of \$500(1 + 0.08/12)^{58}.
- The third payment will accumulate to \$500(1 + 0.08/12)^{57}.
- The fourth payment will accumulate to \$500(1 + 0.08/12)^{56}.

And so on . . .

Finally the next to last (59$^{\text{th}}$) payment will accumulate to $\$500(1+0.08/12)^1$.

The last payment is taken out the same time it is made, and will not earn any interest.

To find the total amount in five years, we need to add the accumulated value of these sixty payments.

In other words, we need to find the sum of the following series.

$$\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\$500(1+0.08/12)^{57}+\ldots+\$500$$

Written backwards, we have

$$\$500+\$500(1+0.08/12)+\$500(1+0.08/12)^2+\ldots+\$500(1+0.08/12)^{59}$$

This is a geometric series with $a=\$500$, $r=(1+0.08/12)$, and $n=59$. The sum is

$$\begin{aligned}\text{sum} &= \frac{\$500\left[(1+0.08/12)^{60}-1\right]}{0.08/12} \\ &= \$500(73.47686) \\ &= \$36,738.43\end{aligned}$$

When the payments are made at the end of each period rather than at the beginning, we call it an **ordinary annuity.**

Future Value of an Ordinary Annuity

If a payment of m dollars is made in an account n times a year at an interest r, then the final amount A after t years is

$$\mathbf{A}=\frac{\mathbf{m}\left[(\mathbf{1}+\mathbf{r}/\mathbf{n})^{\mathbf{nt}}-\mathbf{1}\right]}{\mathbf{r}/\mathbf{n}} \tag{6.3.1}$$

The final amount is also called the future value or the accumulated value.

Example 6.3.2

Tanya deposits \$300 at the end of each quarter in her savings account. If the account earns 5.75% compounded quarterly, how much money will she have in 4 years?

Solution

The future value of this annuity can be found using the above formula.

$$\begin{aligned}A &= \frac{\$300\left[(1+.0575/4)^{16}-1\right]}{0.0575/4} \\ &= \$300(17.8463) \\ &= \$5353.89\end{aligned}$$

If Tanya deposits \$300 into a savings account earning 5.75% compounded quarterly for 4 years, then at the end of 4 years she will have \$5,353.89

Example 6.3.3

Robert needs \$5,000 in three years. How much should he deposit each month in an account that pays 8% compounded monthly in order to achieve his goal?

Solution

If Robert saves m dollars per month, after three years he will have

$$\frac{m\left[(1+.08/12)^{36}-1\right]}{0.08/12}$$

But we'd like this amount to be \$5,000. Therefore,

$$\begin{aligned}\frac{\mathrm{m}\left[(1+.08/12)^{36}-1\right]}{.08/12} &= \$5000\\ \mathrm{m}(40.5356) &= \$5000\\ \mathrm{m} &= \frac{5000}{40.5356}\\ &= \$123.35\end{aligned}$$

Robert needs to deposit \$123.35 at the end of each month for 3 years into an account paying 8% compounded monthly in order to have \$5,000 at the end of 5 years.

Example 6.3.4

Suppose you contribute \$500 each month into an ordinary annuity earning 6.25% compounded monthly. How long do you need to continue making contributions so that you will have at least \$1,000,000?

Solution

Contributing \$500 each month, $m=500$. Interest is 6.25% compounded monthly, so $r=0.0625$ and $n=12$. You want a future value of \$1,000,000 so $A=1,000,000$.

$$1,000,000=\frac{500\left[(\mathbf{1}+0.0625/12)^{\mathbf{12t}}-\mathbf{1}\right]}{0.0625/12}$$

Multiply both sides by $(0.0625/12)$to clear the denominator:

$$1,000,000\cdot(0.0625/12)=500\left[(\mathbf{1}+0.0625/12)^{\mathbf{12t}}-\mathbf{1}\right]$$

Divide both sides by 500:

$$\frac{1,000,000\cdot(0.0625/12)}{500}=(\mathbf{1}+0.0625/12)^{\mathbf{12t}}-\mathbf{1}$$

Add 1 to both sides:

$$\frac{1,000,000\cdot(0.0625/12)}{500}+1=(\mathbf{1}+0.0625/12)^{\mathbf{12t}}$$

Convert to logarithmic form to isolate the exponent:

$$\log_{1+0.0625/12}\left(\frac{1,000,000\cdot(0.0625/12)}{500}+1\right)=12\mathbf{t}$$

Use change of base formula:

$$\frac{\log\left(\frac{1,000,000\cdot(0.0625/12)}{500}+1\right)}{\log(1+0.0625/12)}=12\mathbf{t}$$

Enter into calculator:

$$468.75=t$$

Divide both sides by 12:

$$39.06=t$$

It will take approximately 39 years for you to have at least \$1,000,000.

Sinking Fund

When a business deposits money at regular intervals into an account in order to save for a future purchase of equipment, the savings fund is referred to as a "**sinking fund**". Calculating the sinking fund deposit uses the same method as the previous problem.

Example 6.3.5

A business needs $450,000 in five years. How much should be deposited each quarter in a sinking fund that earns 9% compounded quarterly to have this amount in five years?

Solution

Again, suppose that m dollars are deposited each quarter in the sinking fund. After five years, the future value of the fund should be $450,000. This suggests the following relationship:

$$\begin{aligned} \frac{\mathrm{m}\left[(1+0.09/4)^{20}-1\right]}{0.09/4} &= \$450,000 \\ \mathrm{m}(24.9115) &= 450,000 \\ \mathrm{m} &= \frac{450000}{24.9115} \\ &= \$18,063.93 \end{aligned}$$

The business needs to deposit $18,063.93 at the end of each quarter for 5 years into an sinking fund earning interest of 9% compounded quarterly in order to have $450,000 at the end of 5 years.

Annuity Due

If the payment is made at the beginning of each period, rather than at the end, we call it an **annuity due**. The formula for the annuity due can be derived in a similar manner. Reconsider Example 1, with the change that the deposits are made at the beginning of each month.

Example 6.3.6

If at the beginning of each month a deposit of $500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution

There are 60 deposits made in this account. The first payment stays in the account for 60 months, the second payment for 59 months, the third for 58 months, and so on.

- The first payment of $500 will accumulate to an amount of $\$500(1+0.08/12)^{60}$.
- The second payment of $500 will accumulate to an amount of $\$500(1+.08/12)^{59}$.
- The third payment will accumulate to $\$500(1+0.08/12)^{58}$.

And so on . . .

The last payment is in the account for a month and accumulates to $\$500(1+0.08/12)$

To find the total amount in five years, we need to find the sum of the series:

$$\$500(1+0.08/12)^{60}+\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\ldots+\$500(1+0.08/12)$$

Written backwards, we have

$$\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}$$

If we add $500 to this series, and later subtract that $500, the value will not change. We get

$$\mathbf{\$500}+\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}-\mathbf{\$500}$$

Except for the last term, we have a geometric series with a = $500, r = (1 + .08/12), and n = 60. Therefore the sum is

$$\begin{aligned} \mathrm{A} &= \frac{\$500\left[(1+0.08/12)^{61}-1\right]}{0.08/12}-\$500 \\ &=\$500(74.9667)-\$500 \\ &=\$37483.35-\$500 \\ &=\$36983.35 \end{aligned}$$

So, in the case of an annuity due, to find the future value, we increase the number of periods n by 1, and subtract one payment.

Future Value of an "Annuity Due"

$$\mathrm{A}=\frac{\mathrm{m}\left[(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}+1}-1\right]}{\mathrm{r}/\mathrm{n}}-\mathrm{m}$$

Most of the problems we are going to do in this chapter involve ordinary annuities, therefore, we will down play the significance of the last formula for the annuity due. We mentioned the formula for the annuity due only for completeness.

Summary

Finally, it is the author's wish that the student learn the concepts in a way that he or she will not have to memorize every formula. It is for this reason formulas are kept at a minimum. But before we conclude this section we will once again mention one single equation that will help us find the future value, as well as the sinking fund payment.

If a payment of m dollars is made in an account n times a year at an interest r, then the future value A after t years is

$$\mathbf{A}=\frac{\mathbf{m}\left[(\mathbf{1}+\mathbf{r}/\mathbf{n})^{\mathbf{nt}}-\mathbf{1}\right]}{\mathbf{r}/\mathbf{n}}$$

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

6.3E: Exercises - Annuities and Sinking Funds

PROBLEM SET:ANNUITIES AND SINKING FUNDS

Each of the following problems involve an annuity - a sequence of payments.

1) Find the future value of an annuity of $200 per month for 5 years at 6% compounded monthly.	2) How much money should be deposited at the end of each month in an account paying 7.5% for it to amount to $10,000 in 5 years?
3) At the end of each month Rita deposits $300 in an account that pays 5%. What will the final amount be in 4 years?	4) Mr. Chang wants to retire in 10 years and can save $650 every three months. If the interest rate is 7.8%, how much will he have (a) at the end of 5 years? (b) at the end of 10 years?
5) A firm needs to replace most of its machinery in five years at a cost of $500,000. The company wishes to create a sinking fund to have this money available in five years. How much should the quarterly deposits be if the fund earns 8%?	6) Mrs. Brown needs $5,000 in three years. If the interest rate is 9%, how much should she save at the end of each month to have that amount in three years?
7) A company has a $120,000 note due in 4 years. How much should be deposited at the end of each quarter in a sinking fund to payoff the note in four years if the interest rate is 8%?	8) You are now 20 years of age and decide to save $100 at the end of each month until you are 65. If the interest rate is 9.2%, how much money will you have when you are 65?
9) Is it better to receive $400 at the beginning of each month for six years, or a lump sum of $25,000 today if the interest rate is 7%? Explain.	10) To save money for a vacation, Jill decided to save $125 at the beginning of each month for the next 8 months. If the interest rate is 7%, how much money will she have at the end of 8 months?
11) Mrs. Gill puts $2200 at the end of each year in her IRA account that earns 9% per year. How much total money will she have in this account after 20 years?	12) If the inflation rate stays at 6% per year for the next five years, how much will the price be of a $15,000 car in five years? How much must you save at the end of each month at an interest rate of 7.3% to buy that car in 5 years?

6.4: Present Value of Annuities and Installment Payment

Learning Objectives

In this section, you will learn to:

- Find the present value of an annuity.
- Find the amount of installment payment on a loan.
- Find the outstanding balance on a loan.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. What will the final amount be in 4 years if $8,000 is invested at 9.2% compounded monthly?

Click here to check your answer

$\$11,542.53$

If you missed this problem, review Section 6.2. (Note that this will open in a new window.)

2. How much should be invested at 10.3% compounded quarterly for it to amount to $10,000 in 6 years?

Click here to check your answer

$\$5,432.55$

If you missed this problem, review Section 6.2. (Note that this will open in a new window.)

3. Find the future value of an annuity of $200 per month for 5 years at 6% compounded monthly.

Click here to check your answer

$\$13,954.01$

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

4. How much money should be deposited at the end of each month in an account paying 7.5% for it to amount to $10,000 in 5 years?

Click here to check your answer

$\$137.88$

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

PRESENT VALUE OF AN ANNUITY

In the previous two sections, we learned to find the future value of a lump sum and the future value of an annuity. With these two concepts in hand, we will now learn to amortize a loan, and to find the present value of an annuity.

The **present value** of an annuity is the amount of money we would need now in order to be able to make the payments in the annuity in the future. In other words, the present value is the value now of a future stream of payments.

We start by breaking this down step by step to understand the concept of the present value of an annuity. After that, the examples provide a more efficient way to do the calculations by working with concepts and calculations we have already explored in the last two sections.

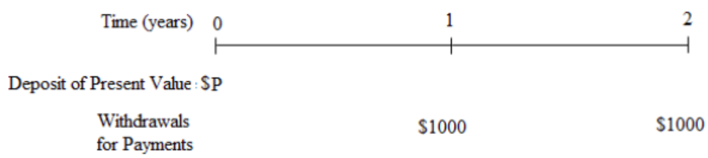
Suppose Carlos owns a small business and employs an assistant manager to help him run the business. Assume it is January 1 now. Carlos plans to pay his assistant manager a $1000 bonus at the end of this year and another $1000 bonus at the end of the following year. Carlos' business had good profits this year so he wants to put the money for his assistant's future bonuses into a savings account now. The money he puts in now will earn interest at the rate of 4% per year compounded annually while in the savings account.

How much money should Carlos put into the savings account now so that he will be able to withdraw $1000 one year from now and another $1000 two years from now?

At first, this sounds like a sinking fund. But it is different. In a sinking fund, we put money into the fund with periodic payments to save to accumulate to a specified lump sum that is the future value at the end of a specified time period.

In this case we want to put a lump sum into the savings account now, so that lump sum is our principal, P. Then we want to withdraw that amount as a series of period payments; in this case the withdrawals are an annuity with $1000 payments at the end of each of two years.

We need to determine the amount we need in the account now, the present value, to be able to make withdraw the periodic payments later.



We use the compound interest formula from Section 6.2 with r = 0.04 and n = 1 for annual compounding to determine the present value of each payment of $1000.

Consider the first payment of $1000 at the end of year 1. Let P_1 be its present value

$$\$1000 = P_1(1.04)^1 \text{ so } P_1 = \$961.54$$

Now consider the second payment of $1000 at the end of year 2. Let P_2 is its present value

$$\$1000 = P_2(1.04)^2 \text{ so } P_2 = \$924.56$$

To make the $1000 payments at the specified times in the future, the amount that Carlos needs to deposit now is the present value $P = P_1 + P_2 = \$961.54 + \$924.56 = \$1886.10$

The calculation above was useful to illustrate the meaning of the present value of an annuity.
But it is not an efficient way to calculate the present value. If we were to have a large number of annuity payments, the step by step calculation would be long and tedious.

Example 6.4.1 investigates and develops an efficient way to calculate the present value of an annuity, by relating the future (accumulated) value of an annuity and its present value.

✔ Example 6.4.1

Suppose you have won a lottery that pays $1,000 per month for the next 20 years. But, you prefer to have the entire amount now. If the interest rate is 8%, how much will you accept?

Solution

This classic present value problem needs our complete attention because the rationalization we use to solve this problem will be used again in the problems to follow.

Consider, for argument purposes, that two people Mr. Cash, and Mr. Credit have won the same lottery of $1,000 per month for the next 20 years. Mr. Credit is happy with his $1,000 monthly payment, but Mr. Cash wants to have the entire amount now.

Our job is to determine how much Mr. Cash should get. We reason as follows:

If Mr. Cash accepts P dollars, then the P dollars deposited at 8% for 20 years should yield the same amount as the $1,000 monthly payments for 20 years. In other words, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like the future values to equal.

Since Mr. Cash is receiving a lump sum of x dollars, its future value is given by the lump sum formula, and it is

$$A = P(1+.08/12)^{240}$$

Since Mr. Credit is receiving a sequence of payments, or an annuity, of $1,000 per month, its future value is given by the annuity formula This value is

$$A = \frac{\$1000\left[(1+.08/12)^{240}-1\right]}{.08/12}$$

The only way Mr. Cash will agree to the amount he receives is if these two future values are equal. So we set them equal and solve for the unknown.

$$\begin{aligned} P(1+.08/12)^{240} &= \frac{\$1000\left[(1+.08/12)^{240}-1\right]}{.08/12} \\ P &= \frac{\frac{\$1000\left[(1+.08/12)^{240}-1\right]}{.08/12}}{(1+.08/12)^{240}} \\ P &= \$119,554.36 \end{aligned}$$

The present value of an ordinary annuity of $1,000 each month for 20 years at 8% is $119,554.36

The reader should also note that if Mr. Cash takes his lump sum of P = $119,554.36 and invests it at 8% compounded monthly, he will have an accumulated value of A=$589,020.41 in 20 years.

If a person or business needs to buy or pay for something now (a car, a home, college tuition, equipment for a business) but does not have the money, they can borrow the money as a loan.

They receive the loan amount called the principal (or present value) now and are obligated to pay back the principal in the future over a stated amount of time (term of the loan), as regular periodic payments with interest.

We can see the formula begin to develop in the example above.

Definition: Present Value of an Annuity

If a payment of m dollars is made in an account n times a year at an interest r, then the **present value P of the annuity** after t years is

$$\mathbf{P(1+r/n)^{nt} = \frac{m\left[(1+r/n)^{nt}-1\right]}{r/n}} \tag{6.4.1}$$

When used for a loan, the amount P is the **loan amount**, and m is the periodic payment needed to repay the loan over a term of t years with n payments per year.

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

Example 6.4.2

You determine that you can afford $550 per month for a car. At the dealership, the finance department offers you a 3-year loan with a 7% interest rate, compounded monthly.

a. What is the highest priced car you can afford in this scenario?
b. How much will you pay over the 3 years?
c. What is the difference between the answers for part a and part b, and what does it represent?

Solution

a. Using the formula above, we are looking for the total amount of the loan, or P. We know that $m=550$, $r=0.07$, $n=12$, and $t=3$.

$$\mathbf{P(1+0.07/12)^{(12\cdot 3)} = \frac{550\left[(1+0.07/12)^{(12\cdot 3)}-1\right]}{0.07/12}}$$

$$\mathbf{P} = \frac{\frac{550\left[(1+0.07/12)^{(12\cdot 3)}-1\right]}{0.07/12}}{(1+0.07/12)^{(12\cdot 3)}}$$

$$P = 17812.55545$$

You can take a loan of $17,812.55 to pay off within 3 years.

b. Over the course of 3 years, you would pay $550 every month, 12 months each year, for 3 years.

Total paid $= \$550 \cdot 12 \cdot 3 = \$19,800$.

c. The difference between $17,812.55 as the purchase price and $19,800 as the repayment price is $\$19800 - \$17812.55 = \$1987.45$ This means that you'll pay $1,987.45 in interest over the 3 years.

Alternate Method to find Present Value of an Annuity

We should note that many finite mathematics and finance books develop the formula for the present value of an annuity differently.

Instead of using the formula:

$$\mathrm{P}(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}} = \frac{\mathrm{m}\left[(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}-1\right]}{\mathrm{r}/\mathrm{n}} \tag{6.4.2}$$

and solving for the present value P after substituting the numerical values for the other items in the formula, many textbooks first solve the formula for P in order to develop a new formula for the present value. Then the numerical information can be substituted into the present value formula and evaluated, without needing to solve algebraically for P.

Starting with formula 6.4.2: $\mathrm{P}(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}} = \frac{\mathrm{m}\left[(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}-1\right]}{\mathrm{r}/\mathrm{n}}$

Divide both sides by $(1+r/n)^{nt}$ to isolate P, and simplify

$$P = \frac{m\left[(1+r/n)^{nt}-1\right]}{r/n} \cdot \frac{1}{(1+r/n)^{nt}}$$

$$P = \frac{m\left[1-(1+r/n)^{-nt}\right]}{r/n} \tag{6.4.3}$$

The authors of this book believe that it is easier to use formula 6.4.2 at the top of this page and solve for P or m as needed. In this approach there are fewer formulas to understand, and many students find it easier to learn. In the problems the rest of this chapter, when a problem requires the calculation of the present value of an annuity, formula 6.4.2 will be used.

However, formula 6.4.3 is ideal when used to solve for t, and some people prefer to use this formula to find present value. It is a mathematically correct option to do so. Note that if you choose to use formula 6.4.3, you need to be careful with the negative exponents in the formula.

INSTALLMENT PAYMENTS ON A LOAN

Example 6.4.3 examines how to calculate the loan payment, using reasoning similar to Example 6.4.1.

✓ Example 6.4.3

Daria purchased a truck costing $45,000 and accepted a 60-month loan at an interest rate of 9%, compounded monthly.

a. Find the monthly installment payments.
b. How much will she pay over the course of the loan?
c. How much interest will she pay over the course of the loan?

Solution

a. The monthly installment payments can be found in a similar process to finding the principal in Example 6.4.2. Note that 60 months is the same as 5 years. We can use $t = 5$ or we can use $nt = 60$.

$$\$45,000(1+.09/12)^{60}=\frac{m\left[(1+.09/12)^{60}-1\right]}{.09/12}$$

We then multiply both sides by the denominator:

$$(\$45,000(1+.09/12)^{60})(\tfrac{.09}{12})=m\left[(1+.09/12)^{60}-1\right]$$

Finally dividing by the entire quantity that is multiplied to the m leaves:

$$\frac{(\$45,000(1+.09/12)^{60})(\frac{.09}{12})}{\left[(1+.09/12)^{60}-1\right]}=m$$
$$\$934.13=m$$

Therefore, the monthly payment needed to repay the loan is $934.13 for five years.

b. Over the course of the loan, she would pay $934.13 every month for 60 months.

Total paid $=\$934.13\cdot 60=\$56047.80$

c. To find interest, we find the difference between purchase price and repayment price. $\$56047.80-\$45000=\$16047.80$ so she will pay $\$16047.80$in interest over the course of the loan.

OUTSTANDING BALANCE ON A LOAN

One of the most common problems deals with finding the balance owed at a given time during the life of a loan. Suppose a person buys a house and amortizes the loan over 30 years, but decides to sell the house a few years later. At the time of the sale, he is obligated to pay off his lender, therefore, he needs to know the balance he owes. Since most long term loans are paid off prematurely, we are often confronted with this problem.

To find the outstanding balance of a loan at a specified time, we need to find the present value P of all future payments that have not yet been paid. In this case t does not represent the entire term of the loan. Instead:

- t represents the time that still remains on the loan
- nt represents the total number of future payments.

If the problem does not directly state the amount of time still remaining in the term of the loan, then it must be calculated FIRST using t = [original term of loan] - [time already passed since the start date of the loan].

✓ Example 6.4.4

Mr. Jackson bought his house in 1995, and financed the loan for 30 years at an interest rate of 7.8%. His monthly payment was $1260. In 2015, Mr. Jackson decides to pay off the loan. Find the balance of the loan he still owes.

Solution

The reader should note that the original amount of the loan is not mentioned in the problem. That is because we don't need to know that to find the balance.

The original loan was for 30 years. 20 years have passed so there are 10 years still remaining. 12(10) = 120 payments still remain to be paid on this loan.

As for the bank or lender is concerned, Mr. Jackson is obligated to pay $1260 each month for 10 more years; he still owes a total of 120 payments. But since Mr. Jackson wants to pay it all off now, we need to find the present value P at the time of repayment of the remaining 10 years of payments of $1260 each month. Using the formula we get for the present value of an annuity, we get

$$\begin{aligned}\mathrm{P}(1+.078/12)^{120}&=\frac{\$1260\left[(1+.078/12)^{120}-1)\right]}{(.078/12)}\\ \mathrm{P}(2.17597)&=\$227957.85\\ \mathrm{P}&=\$104761.48\end{aligned}$$

To find the outstanding balance of a loan

If a loan has a payment of m dollars made n times a year at an interest r, then the outstanding value of the loan when there are t years still remaining on the loan is given by P:

$$\mathbf{P(1+r/n)^{nt}=\frac{m\left[(1+r/n)^{nt}-1\right|}{r/n}} \tag{6.4.4}$$

IMPORTANT: Note that t is not the original term of the loan but instead t is the amount of time still remaining in the future; nt is the number of payments still remaining in the future

If the problem does not directly state the amount of time still remaining in the term of the loan, then it must be calculated BEFORE using the above formula as t = [original term of loan] - [time already passed since the start date of the loan].

Note that there are other methods to find the outstanding balance on a loan, but the method illustrated above is generally considered the easiest.

6.4E: Exercises - Present Value of an Annuity and Installment Payment

PROBLEM SET: PRESENT VALUE OF AN ANNUITY AND INSTALLMENT PAYMENT

For the following problems, show all work.

1) Shawn has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 8.2%, how much money can he hope to get?	2) Sonya bought a car for $15,000. Find the monthly payment if the loan is to be amortized over 5 years at a rate of 10.1%.
3) You determine that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for a car if the interest rate is 9% and you want to repay the loan in 5 years?	4) Compute the monthly payment for a house loan of $200,000 to be financed over 30 years at an interest rate of 10%.
5) If the $200,000 loan in the previous problem is financed over 15 years rather than 30 years at 10%, what will the monthly payment be?	6) Friendly Auto offers Jennifer a car for $2000 down and $300 per month for 5 years. Jason wants to buy the same car but wants to pay cash. How much must Jason pay if the interest rate is 9.4%?
7) The Gomez family bought a house for $450,000. They paid 20% down and amortized the rest at 5.2% over a 30-year period. Find their monthly payment.	8) Mr. and Mrs. Wong purchased their new house for $350,000. They made a down payment of 15%, and amortized the rest over 30 years. If the interest rate is 5.8%, find their monthly payment.
9) A firm needs a piece of machinery that has a useful life of 5 years. It has an option of leasing it for $10,000 a year, or buying it for $40,000 cash. If the interest rate is 10%, which choice is better?	10) Jackie wants to buy a $19,000 car, but she can afford to pay only $300 per month for 5 years. If the interest rate is 6%, how much does she need to put down?
11) Vijay's tuition at college for the next year is $32,000. His parents have decided to pay the tuition by making nine monthly payments. If the interest rate is 6%, what is the monthly payment?	12) Glen borrowed $10,000 for his college education at 8% compounded quarterly. Three years later, after graduating and finding a job, he decided to start paying off his loan. If the loan is amortized over five years at 9%, find his monthly payment for the next five years.

6.5: Classification of Finance Problems

Learning Objectives

In this section, you will learn to:

- Re-examine the types of financial problems and classify them.
- Re-examine the vocabulary words used in describing financial calculations

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Describe the differences between solving $50 = x^9$ vs solving $50 = 9^x$.

Click here to check your answer

If the exponent is known, you'll need to take that root of each side of the equation. If the exponent is unknown, you'll need to take the logarithm of each side of the equation.

If you missed this problem, review Section 5.1. (Note that this will open in a new window.)

2. Describe the differences between simple interest vs compound interest.

Click here to check your answer

Simple interest is calculated once, based on the agreed timeframe of the loan. Compound interest is calculated multiple times and added to the account balance before it is calculated again.

If you missed this problem, review Section 6.2. (Note that this will open in a new window.)

3. Describe the differences between periodically compounded interest vs continuously compounded interest.

Click here to check your answer

Periodically compounded interest is calculated at regular time intervals, such as every quarter, month, day, etc. The n-value tells how many times that time interval occurs in a year. Continuously compounded interest is calculated at an infinitely small time interval, using e instead of an n-value.

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

4. Describe the differences between a lump-sum vs an annuity.

Click here to check your answer

In a lump-sum, money is added to the account once. The only thing that changes the account balance after that is the interest. In an annuity, money is added (and\or removed) from the account at regular time intervals.

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

5. Describe the differences between the future value of an annuity vs the present value of an annuity. (Be careful because either formula can be used to find what should be done now (in the present) and what would be the result in the future!)

Click here to check your answer

The future value formula is used when interest and regular payments both add to the account balance. The present value formula is used when interest and regular payments work in opposite directions, one increasing the balance and one decreasing the balance.

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

We'd like to remind the reader that the hardest part of solving a finance problem is determining the category it falls into. So in this section, we will emphasize the classification of problems rather than finding the actual solution. We suggest that the student read each problem carefully and look for the word or words that may give clues to the kind of problem that is presented.

For instance, students often fail to distinguish a lump-sum problem from an annuity. Since the payments are made each period, an annuity problem contains words such as each, every, per etc. One should also be aware that in the case of a lump-sum, only a single deposit is made, while in an annuity numerous deposits are made at equal spaced time intervals. To help interpret the vocabulary used in the problems, we include a glossary at the end of this section.

Students often confuse the present value of an annuity with the future value of an annuity. Don't get too caught up on the words, as either formula can be used to find what should be done now (in the present) and what would be the result in the future.

- The future value formula will be used in the context of investments, where the regular deposits and interest both work together to add to the account balance.
- The present value formula will be used in the context of loans or bank withdraws, where the regular payments and interest work against each other. In a loan, the monthly payments lower the account balance while the interest payments raise the account balance.

CLASSIFICATION OF PROBLEMS AND EQUATIONS

We now list eight problems that form a basis for all finance problems. Read each question and determine which formula would be used to find the missing information.

Example 6.5.1

Dillon takes out a simple interest loan for $5,000 at 10.5% interest with an agreement to pay it back in 10 months. What is the total amount he will repay at the end of the 10 months?

Classification: Since the loan is a simple interest loan and we are looking for the total amount he will repay, we will use the **Accumulated Value of Simple Interest Formula**.

Equation:

$$\mathrm{A}=5,000(1+0.105(\frac{10}{12})) \tag{6.5.1}$$

Example 6.5.2

If $2,000 is invested at 7% compounded quarterly, what will the final amount be in 5 years?

Classification: Since the $2000 is invested once, this is a lump-sum. The interest is compounded each quarter, so this will use the **Accumulated Value of Periodic Compounding Formula**.

Equation:

$$\mathrm{A}=\$2000(1+.07/4)^{20} \tag{6.5.2}$$

Example 6.5.3

If $2,000 is invested at 7% compounded continuously, what will the final amount be in 5 years?

Classification: Since the $2000 is invested once, this is a lump-sum. The interest is compounded continuously, so this will use the **Accumulated Value of Continuous Compounding Formula**.

Equation:

$$\mathrm{A}=\$2000e^{0.07*5} \tag{6.5.3}$$

Example 6.5.4

How much should be invested at 8% compounded yearly, for the final amount to be \$5,000 in five years?

Classification: Since this question is asking for a one-time investment, this is a lump-sum. The interest is compounded each year, so this will use the **Accumulated Value of Periodic Compounding Formula**. Note that we are given the accumulated value at the end of the 5 years and are looking for the principal.

Equation:

$$\$5,000 = \mathrm{P}(1+.08)^5 \tag{6.5.4}$$

Example 6.5.5

If \$200 is invested *each* month at 8.5% compounded monthly, what will the final amount be in 4 years?

Classification: Since the money is invested each month, this is an example of an annuity. This is an investment where the monthly payments and the interest both add to the value, so this will use the **Future Value of an Annuity Formula.**

Equation:

$$\mathrm{A} = \frac{\$200\left[(1+.085/12)^{12*4}-1\right]}{.085/12} \tag{6.5.5}$$

Example 6.5.6

How much should be invested *each* month at 9% for it to accumulate to \$8,000 in three years?

Classification: Since the money is invested each month, this is an example of an annuity. This is an investment where the monthly payments and the interest both add to the value, so this will use the **Future Value of an Annuity Formula.** Note that we are given the accumulated value at the end of the 3 years and are looking for the monthly payment.

Equation:

$$\$8,000 = \frac{m\left[(1+.09/12)^{36}-1\right]}{.09/12} \tag{6.5.6}$$

Example 6.5.7

Keith wants to buy a new car worth \$30,000 and needs to apply for an auto loan. The dealership is offering him financing at 9.25% compounded monthly over the next 5 years. How much will each monthly payment be?

Classification: Since the money will be paid each month, this is an example of an annuity. This is a loan where the monthly payments lower the account balance and the interest raises the account balance, so this will use the **Present Value of an Annuity Formula.**

Equation:

$$\$30,000(1+.0925/12)^{12*5} = \frac{m\left[(1+.0925/12)^{12*5}-1\right]}{.0925/12} \tag{6.5.7}$$

Example 6.5.8

Maria is saving money for her retirement. The retirement fund earns 8% interest compounded monthly, and she plans to withdraw \$1,500 each month for 25 years after her retirement. How much does she need in this fund before she retires?

Classification: Since the money will be withdrawn each month, this is an example of an annuity. This is a fund where the monthly withdraws lower the account balance and the interest raises the account balance, so this will use the **Present Value of an Annuity Formula.**

Equation:

$$P(1+.08/12)^{12*25} = \frac{\$1,500\left[(1+.08/12)^{12*25}-1\right]}{.08/12} \tag{6.5.8}$$

GLOSSARY: VOCABULARY AND SYMBOLS USED IN FINANCIAL CALCULATIONS

As we've seen in these examples, it's important to read the problems carefully to correctly identify the situation. It is essential to understand to vocabulary for financial problems. Many of the vocabulary words used are listed in the glossary below for easy reference.

t	Term	Time period for a loan or investment. In this book t is represented in years and should be converted into years when it is stated in months or other units.
P	Principal	Principal is the amount of money borrowed in a loan. If a sum of money is invested for a period of time, the sum invested at the start is the Principal.
P	Present Value	Value of money at the beginning of the time period.
A	Accumulated Value Future Value	Value of money at the end of the time period
D	Discount	In loans involving simple interest, a discount occurs if the interest is deducted from the loan amount at the beginning of the loan period, rather than being repaid at the end of the loan period.
m	Periodic Payment	The amount of a constant periodic payment that occurs at regular intervals during the time period under consideration (examples: periodic payments made to repay a loan, regular periodic payments into a bank account as savings, regular periodic payment to a retired person as an annuity,)
$\boldsymbol{n}$	Number of payment periods and compounding periods per year	In this book, when we consider periodic payments, we will always have the compounding period be the same as the payment period. In general the compounding and payment periods do not have to be the same, but the calculations are more complicated if they are different. If the periods differ, formulas for the calculations can be found in finance textbooks or various online resources. Calculations can easily be done using technology such as an online financial calculator, or financial functions in a spreadsheet, or a financial pocket calculator.
nt	Number of periods	nt = (number of periods per year)×(number of years) nt gives the total number of payment and compounding periods In some situations we will calculate nt as the multiplication shown above. In other situations the problem may state nt, such as a problem describing an investment of 18 months duration compounded monthly. In this example: nt = 18 months and n = 12; then t = 1.5 years but t is not stated explicitly in the problem. The TI-84+ calculators built in TVM solver uses $N=nt$.
r	Annual interest rate Nominal rate	The stated annual interest rate. This is stated as a percent but converted to decimal form when using financial calculation formulas. If a bank account pays 3% interest compounded quarterly, then 3% is the nominal rate, and it is included in the financial formulas as r = 0.03
r/n	Interest rate per compounding period	If a bank account pays 3% interest compounded quarterly, then r/n = 0.03/4 = 0. 075, corresponding to a rate of 0.75% per quarter. Some Finite Math books use the symbol i to represent r/n

r_{EFF}	Effective Rate Effective Annual Interest Rate APY Annual Percentage Yield APR Annual Percentage Rate	The effective rate is the interest rate compounded annually that would give the same interest rate as the compounded rate stated for the investment. The effective rate provides a uniform way for investors or borrowers to compare different interest rates with different compounding periods.
I	Interest	Money paid by a borrower for the use of money borrowed as a loan. Money earned over time when depositing money into a savings account, certificate of deposit, or money market account. When a person deposits money in a bank account, the person depositing the funds is essentially temporarily lending the money to the bank and the bank pays interest to the depositor.

	Sinking Fund	A fund set up by making payments over a period of time into a savings or investment account in order to save to fund a future purchase. Businesses use sinking funds to save for a future purchase of equipment at the end of the savings period by making periodic installment payments into a sinking fund.
	Annuity	An annuity is a stream of periodic payments. In this book it refers to a stream of constant periodic payments made at the end of each compounding period for a specific amount of time. In common use the term annuity generally refers to a constant stream of periodic payments received by a person as retirement income, such as from a pension. Annuity payments in general may be made at the end of each payment period (ordinary annuity) or at the start of each period (annuity due). The compounding periods and payment periods do not need to be equal, but in this textbook we only consider situations when these periods are equal.
	Lump Sum	A single sum of money paid or deposited at one time, rather than being spread out over time. An example is lottery winnings if the recipient chooses to receive a single “lump sum” one-time payment, instead of periodic payments over a period of time or as. Use of the word lump sum indicates that this is a one time transaction and is not a stream of periodic payments.
	Loan	An amount of money that is borrowed with the understanding that the borrower needs to repay the loan to the lender in the future by the end of a period of time that is called the term of the loan. The repayment is most often accomplished through periodic payments until the loan has been completely repaid over the term of the loan. However there are also loans that can be repaid as a single sum at the end of the term of the loan, with interest paid either periodically over the term or in a lump sum at the end of the loan or as a discount at the start of the loan.

6.5E: Exercises - Classification of Finance Problems

PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS

Let the letters A, B, C, D, E and F be represented as follows:

$$\begin{array}{lll} A = FV \text{ of a lump-sum} & C = FV \text{ of an annuity} & E = \text{ Installment payment} \\ B = PV \text{ of a lump-sum} & D = \text{ sinking fund payment} & F = PV \text{ of an annuity} \end{array}$$

Classify each by writing the appropriate letter in the box, and write an equation for solution.

1) What monthly deposits made to an account paying 9% will grow to $10,000 in 4 years?

2) An amount of $4000 is invested at 6% compounded daily. What will the final amount be in 5 years?

3) David has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum now. If the current interest rate is 7%, how much money can he hope to get?

4) Each month Linda deposits $250 in an account that pays 9%. How much money will she have in 4 years?

5) Find the monthly payment for a $15,000 car if the loan is amortized over 4 years at a rate of 10%.

6) What lump-sum deposited in an account paying 7% compounded daily will grow to $10,000 in 5 years?

7) What amount of quarterly payments will amount to $250,000 in 5 years at a rate of 8%?

8) The Chang family bought their house 25 years ago. They had their loan financed for 30 years at an interest rate of 11% resulting in a payment of $1350 a month. Find the balance of the loan.

A 10-year $1000 bond pays $35 every six months. If the current interest rate is 8%, in order to find the fair market value of the bond, we need to find the following.

9) The present value of $1000.

10) The present value of the $35 per six month payments.

11) What lump-sum deposit made today is equal to 33 monthly deposits of $500 if the interest rate is 8%?

12) What monthly deposits made to an account paying 10% will accumulated to $10,000 in six years?

13) A department store charges a finance charge of 1.5% per month on the outstanding balance.
If Ned charged $400 three months ago and has not paid his bill, how much does he owe?

14) What will the value of $300 monthly deposits be in 10 years if the account pays 12% compounded monthly?

15) What lump-sum deposited at 6% compounded daily will grow to $2000 in three years?

16) A company buys an apartment complex for $5,000,000 and amortizes the loan over 10 years.
What is the yearly payment if the interest rate is 14%?

17) In 2002, a house in Rock City cost $300,000. Real estate in Rock City has been increasing in value at the annual rate of 5.3%.. Find the price of that house in 2016.

18) You determine that you can afford to pay $400 per month for a car. What is the maximum price you can pay for a car if the interest rate is 11% and you want to repay the loan in 4 years?

19) A business needs $350,000 in 5 years. How much lump-sum should be put aside in an account that pays 9% so that five years from now the company will have $350,000?

20) A person wishes to have $500,000 in a pension fund 20 years from now. How much should he deposit each month in an account paying 9% compounded monthly?

6.6: Additional Application Problems

Learning Objectives

In this section, you will learn to:

- Perform financial calculations in situations involving several stages of savings and/or annuities.
- Find the fair market value of a bond.
- Construct an amortization schedule for a loan.

We have already developed the tools to solve most finance problems. Now we use these tools to solve some additional applications.

PROBLEMS INVOLVING MULTIPLE STAGES OF SAVINGS AND/OR ANNUITIES

Consider the following situations:

a. Suppose a baby, Aisha, is born and her grandparents invest $5000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to need to finish college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?
b. Aisha graduates college and starts a job. She saves $1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

These problems appear complicated. But each can be broken down into two smaller problems involving compound interest on savings or involving annuities. Often the problem involves a savings period followed by an annuity period. ; the accumulated value from first part of the problem may become a present value in the second part. Read each problem carefully to determine what is needed.

Example 6.6.2

Suppose a baby, Aisha, is born and her grandparents invest $8000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to attend college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?

Solution

Part 1: Accumulation of College Savings: Find the accumulated value at the end of 18 years of a sum of $8000 invested at 5% compounded semiannually.

$$\begin{aligned} A &= \$8000(1+.05/2)^{(2\times18)} = \$8000(1.025)^{36} = \$8000(2.432535) \\ A &= \$19460.28 \end{aligned}$$

Part 2: Seminannual annuity payout from savings to put toward college expenses. Find the amount of the semiannual payout for four years using the accumulated savings from part 1 of the problem with an interest rate of 5% compounded semiannually.

A= $19460.28 in Part 1 is the accumulated value at the end of the savings period. This becomes the present value P=$19460.28 when calculating the semiannual payments in Part 2.

$$\begin{aligned} \$19460.28\left(1+\tfrac{.05}{2}\right)^{2\times4} &= \frac{m\left[\left(1+\frac{.05}{2}\right)^{2\times4}-1\right]}{(.05/2)} \\ \$23710.46 &= m(8.73612) \\ m &= \$2714.07 \end{aligned}$$

Aisha will be able to withdraw $2714.07 semiannually for her college expenses.

Example 6.6.3

Aisha graduates college and starts a job. She saves $1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

Solution

Part 1: Accumulation of Retirement Savings: Find the accumulated value at the end of 30 years of $1000 deposited at the end of each quarter into a retirement savings account earning 6% interest compounded quarterly.

$$\begin{aligned} A &= \frac{\$1000\left[(1+.06/4)^{4\times 30}-1\right]}{(.06/4)} \\ A &= \$331288.19 \end{aligned}$$

Part 2: Monthly retirement annuity payout: Find the amount of the monthly annuity payments for 25 years using the accumulated savings from part 1 of the problem with an interest rate of 4.8% compounded monthly.

A= $331288.19 in Part 1 is the accumulated value at the end of the savings period. This amount will become the present value P =$331288.19 when calculating the monthly retirement annuity payments in Part 2.

$$\begin{aligned} &\$331288.19(1+.048/12)^{12\times 25} = \frac{m\left[(1+.048/12)^{12\times 25}-1\right]}{(.048/12)} \\ &\$1097285.90 = m(578.04483) \\ &m = \$1898.27 \end{aligned}$$

Aisha will have a monthly retirement annuity income of $1898.27 when she retires.

FAIR MARKET VALUE OF A BOND

Whenever a business, and for that matter the U. S. government, needs to raise money it does it by selling bonds. A **bond** is a certificate of promise that states the terms of the agreement. Usually the business sells bonds for the **face amount** of $1,000 each for a stated **term**, a period of time ending at a specified **maturity** date.

The person who buys the bond, the **bondholder**, pays $1,000 to buy the bond.

The bondholder is promised two things: First that he will get his $1,000 back at the maturity date, and second that he will receive a fixed amount of interest every six months.

As the market interest rates change, the price of the bond starts to fluctuate. The bonds are bought and sold in the market at their **fair market value**.

The interest rate a bond pays is fixed, but if the market interest rate goes up, the value of the bond drops since the money invested in the bond could earn more if invested elsewhere. When the value of the bond drops, we say it is trading at a **discount**.

On the other hand, if the market interest rate drops, the value of the bond goes up since the bond now yields a higher return than the market interest rate, and we say it is trading at a **premium**.

Example 6.6.4

The Orange Computer Company needs to raise money to expand. It issues a 10-year $1,000 bond that pays $30 every six months. If the current market interest rate is 7%, what is the fair market value of the bond?

Solution

The bond certificate promises us two things - An amount of $1,000 to be paid in 10 years, and a semi-annual payment of $30 for ten years. Therefore, to find the fair market value of the bond, we need to find the present value of the lump sum of $1,000 we are to receive in 10 years, as well as, the present value of the $30 semi-annual payments for the 10 years.

We will let P_1 = the present value of the face amount of $1,000

$$P_1(1+.07/2)^{20} = \$1,000$$

Since the interest is paid twice a year, the interest is compounded twice a year and nt = 2(10)=20

$$\begin{aligned} P_1(1.9898) &= \$1,000 \\ P_1 &= \$502.56 \end{aligned}$$

We will let P_2 = the present value of the \$30 semi-annual payments is

$$\begin{aligned} \mathrm{P}_2(1+.07/2)^{20} &= \frac{\$30\left[(1+.07/2)^{20}-1\right]}{(.07/2)} \\ \mathrm{P}_2(1.9898) &= 848.39 \\ \mathrm{P}_2 &= \$426.37 \end{aligned}$$

The present value of the lump-sum \$1,000 = \$502.56

The present value of the \$30 semi-annual payments = \$426.37

The fair market value of the bond is P = P_{1+} P_2 = \$502.56 + \$426.37 = \$928.93

Note that because the market interest rate of 7% is higher than the bond's implied interest rate of 6% implied by the semiannual payments, the bond is selling at a discount; its fair market value of \$928.93 is less than its face value of \$1000.

Example 6.6.5

A state issues a 15 year \$1000 bond that pays \$25 every six months. If the current market interest rate is 4%, what is the fair market value of the bond?

Solution

The bond certificate promises two things - an amount of \$1,000 to be paid in 15 years, and semi-annual payments of \$25 for 15 years. To find the fair market value of the bond, we find the present value of the \$1,000 face value we are to receive in 15 years and add it to the present value of the \$25 semi-annual payments for the 15 years. In this example, $nt = 2(15) = 30$.

We will let P_1 = the present value of the lump-sum \$1,000

$$\begin{aligned} \mathrm{P}_1(1+.04/2)^{30} &= \$1,000 \\ \mathrm{P}_1 &= \$552.07 \end{aligned}$$

We will let P_2 = the present value of the \$25 semi-annual payments is

$$\begin{aligned} \mathrm{P}_2(1+.04/2)^{30} &= \frac{\$25\left[(1+.04/2)^{30}-1\right]}{(.04/2)} \\ \mathrm{P}_2(1.18114) &= \$1014.20 \\ \mathrm{P}_2 &= \$559.90 \end{aligned}$$

The present value of the lump-sum \$1,000 = \$552.07

The present value of the \$30 semi-annual payments = \$559.90

Therefore, the fair market value of the bond is

$$P = P_1 + P_2 = \$552.07 + \$559.90 = \$1111.97$$

Because the market interest rate of 4% is lower than the interest rate of 5% implied by the semiannual payments, the bond is selling at a premium: the fair market value of \$1,111.97 is more than the face value of \$1,000.

To summarize:

To find the Fair Market Value of a Bond

Find the present value of the face amount A that is payable at the maturity date:

$$\mathbf{A}=\mathbf{P}_{1}(\mathbf{1}+\mathbf{r}/\mathbf{n})^{\mathbf{nt}}; \text{ solve to find } \mathrm{P}_{1} \tag{6.6.1}$$

Find the present value of the semiannually payments of $\$m$ over the term of the bond:

$$\mathbf{P}_{2}(\mathbf{1}+\mathbf{r}/\mathbf{n})^{\mathbf{nt}}=\frac{\mathbf{m}\left[(\mathbf{1}+\mathbf{r}/\mathbf{n})^{\mathbf{nt}}-\mathbf{1}\right]}{\mathbf{r}/\mathbf{n}} \quad ; \text{ solve to find } \mathbf{P}_{2} \tag{6.6.2}$$

The fair market value (or present value or price or current value) of the bond is the sum of the present values calculated above:

$$\mathrm{P}=\mathbf{P}_{1}+\mathbf{P}_{2} \tag{6.6.3}$$

AMORTIZATION SCHEDULE FOR A LOAN

An amortization schedule is a table that lists all payments on a loan, splits them into the portion devoted to interest and the portion that is applied to repay principal, and calculates the outstanding balance on the loan after each payment is made.

Example $6.6.6$

An amount of \$500 is borrowed for 6 months at a rate of 12%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.

Solution

The reader can verify that the monthly payment is \$86.27.

The first month, the outstanding balance is \$500, and therefore, the monthly interest on the outstanding balance is

(outstanding balance)(the monthly interest rate) = (\$500)(.12/12) = \$5

This means, the first month, out of the \$86.27 payment, \$5 goes toward the interest and the remaining \$81.27 toward the balance leaving a new balance of \$500 - \$81.27 = \$418.73.

Similarly, the second month, the outstanding balance is \$418.73, and the monthly interest on the outstanding balance is (\$418.73)(.12/12) = \$4.19. Again, out of the \$86.27 payment, \$4.19 goes toward the interest and the remaining \$82.08 toward the balance leaving a new balance of \$418.73 - \$82.08 = \$336.65. The process continues in the table below.

Payment #	Payment	Interest	Debt Payment	Balance
1	\$86.27	\$5	\$81.27	\$418.73
2	\$86.27	\$4.19	\$82.08	\$336.65
3	\$86.27	\$3.37	\$82.90	\$253.75
4	\$86.27	\$2.54	\$83.73	\$170.02
5	\$86.27	\$1.70	\$84.57	\$85.45
6	\$86.27	\$0.85	\$85.42	\$0.03

Note that the last balance of 3 cents is due to error in rounding off.

An amortization schedule is usually lengthy and tedious to calculate by hand. For example, an amortization schedule for a 30 year mortgage loan with monthly payments would have (12)(30)=360 rows of calculations in the amortization schedule table. A car loan with 5 years of monthly payments would have 12(5)=60 rows of calculations in the amortization schedule table. However it would be straightforward to use a spreadsheet application on a computer to do these repetitive calculations by inputting and copying formulas for the calculations into the cells.

Most of the other applications in this section's problem set are reasonably straightforward, and can be solved by taking a little extra care in interpreting them. And remember, there is often more than one way to solve a problem.

6.6E: Exercises - Miscellaneous Application Problems

PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

For problems 1 - 4, assume a $200,000 house loan is amortized over 30 years at an interest rate of 5.4%.

1) Find the monthly payment.	2) Find the balance owed after 20 years.
3) Find the balance of the loan after 100 payments.	4) Find the monthly payment if the original loan were amortized over 15 years.

MISCELLANEOUS APPLICATION PROBLEMS

5) Mr. Patel wants to pay off his car loan. The monthly payment for his car is $365, and he has 16 payments left. If the loan was financed at 6.5%, how much does he owe?	6) An amount of $2000 is borrowed for a year at a rate of 7%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment going toward reducing the debt, and the balance.
7) Fourteen months after Dan bought his new car he lost his job. His car was repossessed by his lender after he made only 14 monthly payments of $376 each. If the loan was financed over a 4-year period at an interest rate of 6.3%, how much did the car cost the lender? In other words, how much did Dan still owe on the car?	8) You have a choice of either receiving $5,000 at the end of each year for the next 5 years or receiving $3000 per year for the next 10 years. If the current interest rate is 9%, which is better?
9) Mr. Smith is planning to retire in 25 years and would like to have $250,000 then. What monthly payment made at the end of each month to an account that pays 6.5% will achieve his objective?	10) Assume Mr. Smith has reached retirement and has $250,000 in an account which is earning 6.5%. He would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment.
11) Mrs. Garcia is planning to retire in 20 years. She starts to save for retirement by depositing $2000 each quarter into a retirement investment account that earns 6% interest compounded quarterly. Find the accumulated value of her retirement savings at the end of 20 years.	12) Assume Mrs. Garcia has reached retirement and has accumulated the amount found in question 13 in a retirement savings account. She would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment. Assume the account now pays 5.4% compounded monthly.

13) A ten-year $1,000 bond pays $35 every six months. If the current interest rate is 8.2%, find the fair market value of the bond. Hint: You must do the following. a) Find the present value of $1000. b) Find the present value of the $35 payments. c) The fair market value of the bond = a + b	14) Find the fair market value of the ten-year $1,000 bond that pays $35 every six months, if the current interest rate has dropped to 6%. Hint: You must do the following. a) Find the present value of $1000. b) Find the present value of the $35 payments. c) The fair market value of the bond = a + b
15) A twenty-year $1,000 bond pays $30 every six months. If the current interest rate is 4.2%, find the fair market value of the bond. Hint: You must do the following. a) Find the present value of $1000. b) Find the present value of the $30 payments. c) The fair market value of the bond = a + b	16) Find the fair market value of the twenty-year $1,000 bond that pays $30 every six months, if the current interest rate has increased to 7.5%.

17) Mr. and Mrs. Nguyen deposit $10,000 into a college investment account when their new baby grandchild is born. The account earns 6.25% interest compounded quarterly. a) When their grandchild reaches the age of 18, what is the accumulated value of the college investment account? b) The Nguyen's grandchild has just reached the age of 18 and started college. If she is to withdraw the money in the college savings account n equal monthly payments over the next 4 years, how much money will be withdrawn each month?	18) Mr. Singh is 38 and plans to retire at age 65. He opens a retirement savings account. a) Mr. Singh wants to save enough money to accumulate $500,000 by the time he retires. The retirement investment account pays 7% interest compounded monthly. How much does he need to deposit each month to achieve this goal? b) Mr. Singh has now reached at 65 and retires. How much money can he withdraw each month for 25 years if the retirement investment account now pays 5.2% interest, compounded monthly?

6.7: Chapter 6 Review

PROBLEM SET: CHAPTER REVIEW

Simple Interest (6.1)

1. Manuel borrows $800 for 6 months at 18% simple interest. How much does he owe at the end of 6 months?
2. An amount of $2300 is borrowed for 7 months at a simple interest rate of 16%. Find the total that will need to be paid back.

Compound Interest (6.2)

3. In the year 2000, an average house in Star City cost $250,000. If the average annual inflation rate for the past years has been about 4.7%, what was the price of that house in 2015?
4. A Visa credit card company has a finance charge of 1.5% per month (18% per year) on the outstanding balance. John owed $3200 and has been delinquent for 5 months. How much total does he owe, now?
5. A sum of $5000 is deposited in a bank today. What will the final amount be in 20 months if the bank pays 9% and the interest is compounded monthly?
6. City Bank pays an interest rate of 6%, while Western Bank pays 5.8% compounded continuously. Which one is a better deal?
7. If a bank pays 6.8% compounded continuously, how long will it take to double your money?

Future Value of Annuities and Sinking Funds (6.3)

8. A corporation estimates it will need $300,000 in 8 years to replace its existing machinery. How much should it deposit each quarter in a sinking fund earning 8.4% compounded quarterly to meet this obligation?
9. A business must raise $400,000 in 10 years. What should be the size of the owners' monthly payments to a sinking fund paying 6.5% compounded monthly?
10. A mutual fund claims a growth rate of 8.3% per year. If $500 per month is invested, what will the final amount be in 15 years?

Present Value of Annuities and Installment Payments (6.4)

11. You look at your budget and decide that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for the car if the interest rate is 8.6% and you want to finance the loan over 5 years?
12. Lisa buys a car for $16,500, and receives $2400 for her old car as a trade-in value. Find the monthly payment for the balance if the loan is amortized over 5 years at 8.5%.
13. You want to purchase a home for $200,000 with a 30-year mortgage at 9.24% interest. Find
    a. the monthly payment
    b. the balance owed after 20 years.
14. Ali has inherited $20,000 and is planning to invest this amount at 7.9% interest. At the same time she wishes to make equal monthly withdrawals to use up the entire sum in 5 years. How much can she withdraw each month?
15. Mr. Albers borrowed $425,000 from the bank for his new house at an interest rate of 4.7%. He will make equal monthly payments for the next 30 years. How much money will he end up paying the bank over the life of the loan, and how much is the interest?

Classification of Finance Problems (6.5)

16. Determine which formula would be used to answer the question, then find the answer.
    a. The United States paid about 4 cents an acre for the Louisiana Purchase in 1803. Suppose the value of this property grew at a rate of 5.5% annually. What would an acre be worth in the year 2000?
    b. When Jose bought his car, he amortized his loan over 6 years at a rate of 9.2%, and his monthly payment came out to be $350 per month. He has been making these payments for the past 40 months and now wants to pay off the remaining balance. How much does he owe?
    c. What amount should be invested per month at 9.1% compounded monthly so that it will become $5000 in 17 months?
    d. How much should Mr. Shackley deposit in a trust account so that his daughter can withdraw $400 per month for 4 years if the interest rate is 8%?

Additional Application Problems (6.6)

17. Mr. Nakahama bought his house in the year 1998. He had his loan financed for 30 years at an interest rate of 6.2% resulting in a monthly payment of $1500. In 2015, 17 years later, he paid off the balance of the loan. How much did he pay?

18. Find the 'fair market' value of a ten-year $1000 bond which pays $30 every six months if the current interest rate is 7%. What if the current interest rate is 5%?
19. Mrs. Tong puts away $500 per month for 10 years in an account that earns 9.3%. After 10 years, she decides to withdraw $1,000 per month. If the interest rate stays the same, how long will it take Mrs. Tong to deplete the account?
20. An amount of $5000 is borrowed for 15 months at an interest rate of 9%. Find the monthly payment and construct an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the amount of payment contributing towards debt, and the outstanding debt.

CHAPTER OVERVIEW

6: Sets and Counting

Learning Objectives

In this chapter, you will learn to:

- Use set theory and Venn diagrams to solve counting problems.
- Use the Multiplication Axiom to solve counting problems.
- Use Permutations to solve counting problems.
- Use Combinations to solve counting problems.
- Use the Binomial Theorem to expand $(x+y)^n$

6.1: Sets and Counting

Learning Objectives

In this section, you will learn to:

1. Use set notation to represent unions, intersections, and complements of sets
2. Use Venn diagrams to solve counting problems.

Introduction to Sets

In this section, we will familiarize ourselves with set operations and notations, so that we can apply these concepts to both counting and probability problems. We begin by defining some terms.

Definition: Set and Elements

A **set** is a collection of objects, and its members are called the **elements** of the set.

We name the set by using capital letters, and enclose its members in braces. Suppose we need to list the members of the chess club. We use the following set notation.

C = { Ken, Bob, Tran, Shanti, Eric }

Definition: Empty Set

A set that has no members is called an **empty set**. The empty set is denoted by the symbol $\varnothing$.

Definition: Set Equality

Two sets are **equal** if they have the same elements.

Definition: Subset

Set A is a **subset** of a set B if every member of A is also a member of B.

Suppose C = { Al, Bob, Chris, David, Ed } and A = { Bob, David }. Then A is a subset of C, written as A $\subseteq$ C.

Every set is a subset of itself, and the empty set is a subset of every set.

Definition: Union of Two Sets

Let A and B be two sets, then the **union** of A and B, written as $A \cup B$, is the set of all elements that are either in A or in B, or in both A and B.

Definition: Intersection of Two Sets

Let A and \(B\0 be two sets, then the **intersection** of A and B, written as $A \cap B$, is the set of all elements that are common to both sets A and B.

Definition: Universal Set

A **universal set** U is the set consisting of all elements under consideration.

Definition: Complement of a Set and Disjoint Sets

Let A be any set, then the **complement** of set A, written as $\bar{\mathrm{A}}$, is the set consisting of elements in the universal set U that are not in A.

Two sets A and B are called **disjoint sets** if their intersection is an empty set. Clearly, a set and its complement are disjoint; however two sets can be disjoint and not be complements.

Example 6.1.1

List all the subsets of the set of primary colors { red, yellow, blue }.

Solution

The subsets are $\varnothing$, {red}, {yellow}, {blue}, {red, yellow}, {red, blue}, {yellow, blue}, {red, yellow, blue}

Note that the empty set is a subset of every set, and a set is a subset of itself.

Example 6.1.2

Let F = { Aikman, Jackson, Rice, Sanders, Young }, and let B = { Griffey, Jackson, Sanders, Thomas }.
Find the intersection of the sets F and B.

Solution

The intersection of the two sets is the set whose elements belong to both sets. Therefore, F ∩ B = { Jackson, Sanders }

Example 6.1.3

Find the union of the sets F and B given as follows.

- F = { Aikman, Jackson, Rice, Sanders, Young }
- B = { Griffey, Jackson, Sanders, Thomas }

Solution

The union of two sets is the set whose elements are either in A or in B or in both A and B. Observe that when writing the union of two sets, the repetitions are avoided.

F ∪ B = { Aikman, Griffey, Jackson, Rice, Sanders, Thomas, Young }

Example 6.1.4

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue }. Find the complement of P.

Solution

The complement of a set P is the set consisting of elements in the universal set U that are not in P. Therefore,

$\bar{P}$ = { orange, green, indigo, violet }

To achieve a better understanding, let us suppose that the universal set U represents the colors of the spectrum, and P the primary colors, then $\bar{P}$ represents those colors of the spectrum that are not primary colors.

Example 6.1.5

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue } . Find a set R so that R is not the complement of P but R and P are disjoint.

Solution

R = { orange, green } and P = { red, yellow, blue } are disjoint because the intersection of the two sets is the empty set. The sets have no elements in common. However they are not complements because their union P ∪ R = { red, yellow, blue, orange, green } is not equal to the universal set U.

✓ Example 6.1.6

Let U = { red, orange, yellow, green, blue, indigo, violet }, P = { red, yellow, blue }, Q = { red, green }, and R = { orange, green, indigo }. Find $\overline{P \cup Q} \cap \overline{R}$.

Solution

We do the problems in steps:

P ∪ Q = { red, yellow, blue, green }
$\overline{P \cup Q}$ = { orange, indigo, violet }
$\overline{R}$ = { red, yellow, blue, violet }
$\overline{P \cup Q} \cap \overline{R}$ = {violet}

Venn Diagrams

We now use **Venn diagrams** to illustrate the relations between sets. In the late 1800s, an English logician named John Venn developed a method to represent relationship between sets. He represented these relationships using diagrams, which are now known as Venn diagrams.

A Venn diagram represents a set as the interior of a circle. Often two or more circles are enclosed in a rectangle where the rectangle represents the universal set. To visualize an intersection or union of a set is easy. In this section, we will mainly use Venn diagrams to sort various populations and count objects.

✓ Example 6.1.7

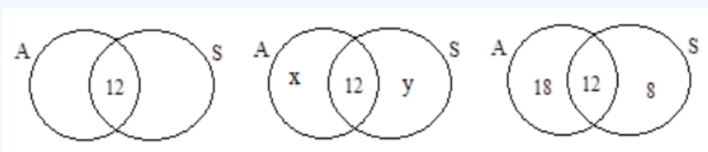
Suppose a survey of car enthusiasts showed that over a certain time period, 30 drove cars with automatic transmissions, 20 drove cars with standard transmissions, and 12 drove cars of both types. If everyone in the survey drove cars with one of these transmissions, how many people participated in the survey?

Solution

We will use Venn diagrams to solve this problem.

Let the set A represent those car enthusiasts who drove cars with automatic transmissions, and set S represent the car enthusiasts who drove the cars with standard transmissions. Now we use Venn diagrams to sort out the information given in this problem.

Since 12 people drove both cars, we place the number 12 in the region common to both sets.



Because 30 people drove cars with automatic transmissions, the circle A must contain 30 elements. This means that

x + 12 = 30, or x = 18.

Similarly, since 20 people drove cars with standard transmissions, the circle B must contain 20 elements.

Thus, y + 12 = 20 which in turn makes y = 8.

Now that all the information is sorted out, it is easy to read from the diagram that 18 people drove cars with automatic transmissions only, 12 people drove both types of cars, and 8 drove cars with standard transmissions only.

Therefore, 18 + 12 + 8 = 38 people took part in the survey.

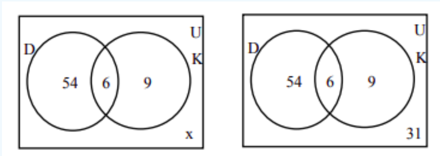
Example 6.1.8

A survey of 100 people in California indicates that 60 people have visited Disneyland, 15 have visited Knott's Berry Farm, and 6 have visited both. How many people have visited neither place?

Solution

The problem is similar to the one in the previous example.

Let the set D represent the people who have visited Disneyland, and K the set of people who have visited Knott's Berry Farm.



We fill the three regions associated with the sets D and K in the same manner as before. Since 100 people participated in the survey, the rectangle representing the universal set U must contain 100 objects. Let x represent those people in the universal set that are neither in the set D nor in K. This means $54 + 6 + 9 + x = 100$, or $x = 31$.

Therefore, there are 31 people in the survey who have visited neither place.

Example 6.1.9

A survey of 100 exercise conscious people resulted in the following information:

- 50 jog, 30 swim, and 35 cycle
- 14 jog and swim
- 7 swim and cycle
- 9 jog and cycle
- 3 people take part in all three activities

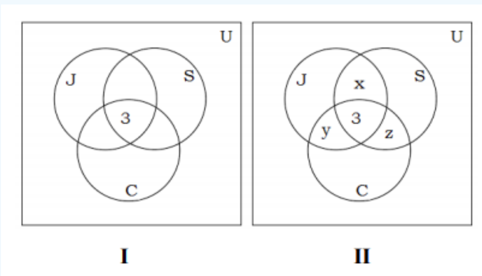
a. How many jog but do not swim or cycle?
b. How many take part in only one of the activities?
c. How many do not take part in any of these activities?

Solution

Let J represent the set of people who jog, S the set of people who swim, and C who cycle.

In using Venn diagrams, our ultimate aim is to assign a number to each region. We always begin by first assigning the number to the innermost region and then working our way out.

We'll show the solution step by step. As you practice working out such problems, you will find that with practice you will not need to draw multiple copies of the diagram.



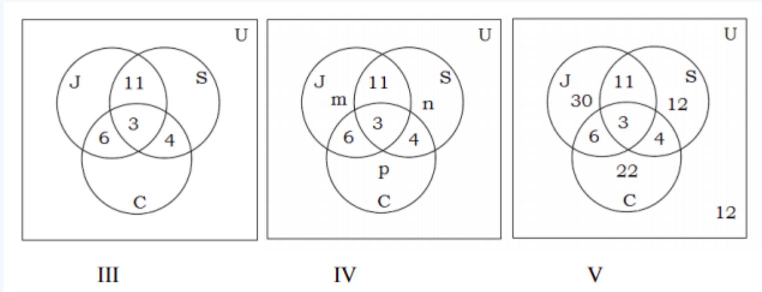
We place a 3 in the innermost region of figure I because it represents the number of people who participate in all three activities. Next we use figure II to compute x, y and z.

Since 14 people jog and swim, $x + 3 = 14$, or $x = 11$.

The fact that 9 people jog and cycle results in $y + 3 = 9$, or $y = 6$.

Since 7 people swim and cycle, z + 3 = 7, or z = 4.

This information is depicted in figure III.



Now we proceed to find the unknowns m, n and p, as shown in figure IV

Since 50 people jog, m + 11 + 6 + 3 = 50, or m = 30.

30 people swim, therefore, n + 11 + 4 + 3 = 30, or n = 12.

35 people cycle, therefore, p + 6 + 4 + 3 = 35, or p = 22.

By adding all the entries in all three sets, we get a sum of 88.
Since 100 people were surveyed, the number inside the universal set but outside of all three sets is 100 - 88, or 12.

In figure V, all the information is sorted out, and the questions can readily be answered.

a. The number of people who jog but do not swim or cycle is 30.
b. The number who take part in only one of these activities is 30 + 12 + 22 = 64.
c. The number of people who do not take part in any of these activities is 12.

6.1.1: Sets and Counting (Exercises)

SECTION 7.1 PROBLEM SET: SETS AND COUNTING

Find the indicated sets.

1. List all subsets of the following set. { Al, Bob }	2. List all subsets of the following set. { Al, Bob, Chris }
3. List the elements of the following set. { Al, Bob, Chris, Dave } ∩ { Bob, Chris, Dave, Ed }	4. List the elements of the following set. { Al, Bob, Chris, Dave } ∪ {Bob, Chris, Dave, Ed }

Problems 5 – 8: Let Universal set U = { a, b, c, d, e, f, g, h, i, j }, sets V = { a, e, i, f, h }, W = { a, c, e, g, i }. List the members of the following sets.

5. V ∪ W	6. V ∩ W
7. $\overline{V \cup W}$	8. $\overline{V} \cap \overline{W}$

Problems 9 – 12: Let Universal set U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 } and sets A = { 1, 2, 3, 4, 5 }, B = { 1, 3, 4, 6 }, and C = { 2, 4, 6 }.

List the members of the following sets.

9. A ∪ B	10. A ∩ C
11. $\overline{A \cup B} \cap C$	12. $\bar{A} \cup \overline{B \cap C}$

Use Venn Diagrams to find the number of elements in the following sets.

13. In Mrs. Yamamoto's class of 35 students, 12 students are taking history, 18 are taking English, and 4 are taking both. Draw a Venn diagram and use it determine how many students are taking neither history nor English.	14. In a survey of 1200 college students, 700 used Spotify to listen to music and 400 used iTunes to listen to music; of these, 100 used both. a. Draw a Venn Diagram and find the number of people in each region of the diagram. b. How many used either Spotify or iTunes?
15. A survey of athletes revealed that for their minor aches and pains, 30 used aspirin, 50 used ibuprofen, and 15 used both. How many athletes were surveyed?	16. In 2016, 80 college students were surveyed about what video services they subscribed to. Suppose the survey showed that 50 use Amazon Prime, 30 use Netflix, 20 use Hulu. Of those, 13 use Amazon Prime and Netflix, 9 use Amazon Prime and Hulu, 7 use Netflix and Hulu. 3 students use all three services. a. Draw a Venn Diagram and use it to determine the number of people in each region of the diagram. b. How many use at least one of these? c. How many use none of these?

17. A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.
    a. Find the number of students taking math but not the other two subjects.
    b. The number of students taking English or math but not history.
    c. The number of students taking none of these subjects.

18. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.
    a. Find the number of investors that participated in the survey.
    b. How many invested in stocks or mutual funds but not in bonds?
    c. How many invested in exactly one type of investment?

19. A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. (This question relates back to question #17.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Students who take at least one of these classes	b. Students who take exactly one of these classes
c. Students who take at least two of these classes	d. Students who take exactly two of these classes
e. Students who take at most two of these classes	f. Students who take English or Math but not both
g. Students who take Math or History but not English	h. Students who take all of these classes

20. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. (This question relates back to question #18.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Investors who invested in mutual funds only	b. Investors who invested in stocks and bonds but not mutual funds
c. Investors who invested in exactly one of these investments	d. Investors who invested in exactly two of these investments
e. Investors who invested in at least two of these investments	f. Investors who invested in at most two of these investments
g. Investors who did not invest in bonds	h. Investors who invested in all three investments

2.2: Tree Diagrams and the Multiplication Axiom

Learning Objectives

In this section you will learn to

1. Use trees to count possible outcomes in a multi-step process
2. Use the multiplication axiom to count possible outcomes in a multi-step process.

In this chapter, we are trying to develop counting techniques that will be used in the next chapter to study probability. One of the most fundamental of such techniques is called the Multiplication Axiom. Before we introduce the multiplication axiom, we first look at some examples.

Example 2.2.1

If a woman has two blouses and three skirts, how many different outfits consisting of a blouse and a skirt can she wear?

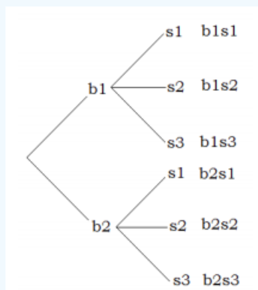
Solution

Suppose we call the blouses b_1 and b_2, and skirts s_1, s_2, and s_3.

We can have the following six outfits.

$$b_1 s_1, b_1 s_2, b_1 s_3, b_2 s_1, b_2 s_2, b_2 s_3$$

Alternatively, we can draw a tree diagram:



The tree diagram gives us all six possibilities. The method involves two steps. First the woman chooses a blouse. She has two choices: blouse one or blouse two. If she chooses blouse one, she has three skirts to match it with; skirt one, skirt two, or skirt three. Similarly if she chooses blouse two, she can match it with each of the three skirts, again. The tree diagram helps us visualize these possibilities.

The reader should note that the process involves two steps. For the first step of choosing a blouse, there are two choices, and for each choice of a blouse, there are three choices of choosing a skirt. So altogether there are $2 \cdot 3 = 6$ possibilities.

If, in the previous example, we add the shoes to the outfit, we have the following problem.

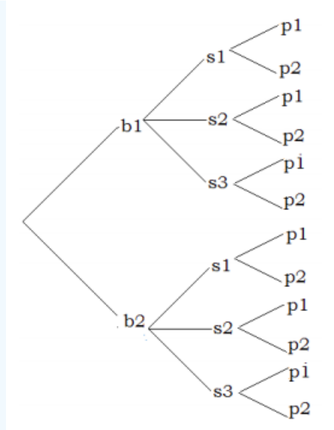
Example 2.2.2

If a woman has two blouses, three skirts, and two pumps, how many different outfits consisting of a blouse, a skirt, and a pair of pumps can she wear?

Solution

Suppose we call the blouses b_1 and b_2, the skirts s_1, s_2, and s_3, and the pumps p_1, and p_2.

The following tree diagram results.



We count the number of branches in the tree, and see that there are 12 different possibilities.

This time the method involves three steps. First, the woman chooses a blouse. She has two choices: blouse one or blouse two. Now suppose she chooses blouse one. This takes us to step two of the process which consists of choosing a skirt. She has three choices for a skirt, and let us suppose she chooses skirt two. Now that she has chosen a blouse and a skirt, we have moved to the third step of choosing a pair of pumps. Since she has two pairs of pumps, she has two choices for the last step. Let us suppose she chooses pumps two. She has chosen the outfit consisting of blouse one, skirt two, and pumps two, or $b_1 s_2 p_2$. By looking at the different branches on the tree, one can easily see the other possibilities.

The important thing to observe here, again, is that this is a three step process. There are two choices for the first step of choosing a blouse. For each choice of a blouse, there are three choices of choosing a skirt, and for each combination of a blouse and a skirt, there are two choices of selecting a pair of pumps.

All in all, we have $2 \cdot 3 \cdot 2 = 12$ different possibilities.

Tree diagrams help us visualize the different possibilities, but they are not practical when the possibilities are numerous. Besides, we are mostly interested in finding the number of elements in the set and not the actual list of all possibilities; once the problem is envisioned, we can solve it without a tree diagram. The two examples we just solved may have given us a clue to do just that.

Let us now try to solve Example $2.2.2$ without a tree diagram. The problem involves three steps: choosing a blouse, choosing a skirt, and choosing a pair of pumps. The number of ways of choosing each are listed below. By multiplying these three numbers we get 12, which is what we got when we did the problem using a tree diagram.

The number of ways of choosing a blouse	The number of ways of choosing a skirt	The number of ways of choosing pumps
2	3	2

The procedure we just employed is called the multiplication axiom.

The Multiplication Axiom

If a task can be done in m ways, and a second task can be done in n ways, then the operation involving the first task followed by the second can be performed in $m \cdot n$ ways.

The general multiplication axiom is not limited to just two tasks and can be used for any number of tasks.

Example 2.2.3

A truck license plate consists of a letter followed by four digits. How many such license plates are possible?

Solution

Since there are 26 letters and 10 digits, we have the following choices for each.

Letter	Digit	Digit	Digit	Digit
26	10	10	10	10

Therefore, the number of possible license plates is $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260,000$.

Example 2.2.4

In how many different ways can a 3-question true-false test be answered?

Solution

Since there are two choices for each question, we have

Question 1	Question 2	Question 3
2	2	2

Applying the multiplication axiom, we get $2 \cdot 2 \cdot 2 = 8$ different ways.

We list all eight possibilities: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF

The reader should note that the first letter in each possibility is the answer corresponding to the first question, the second letter corresponds to the answer to the second question, and so on. For example, TFF, says that the answer to the first question is given as true, and the answers to the second and third questions false.

Example 2.2.5

In how many different ways can four people be seated in a row?

Solution

Suppose we put four chairs in a row, and proceed to put four people in these seats.

There are four choices for the first chair we choose. Once a person sits down in that chair, there are only three choices for the second chair, and so on. We list as shown below.

4	3	2	1

So there are altogether $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different ways.

Example 2.2.6

How many three-letter word sequences can be formed using the letters { A, B, C } if no letter is to be repeated?

Solution

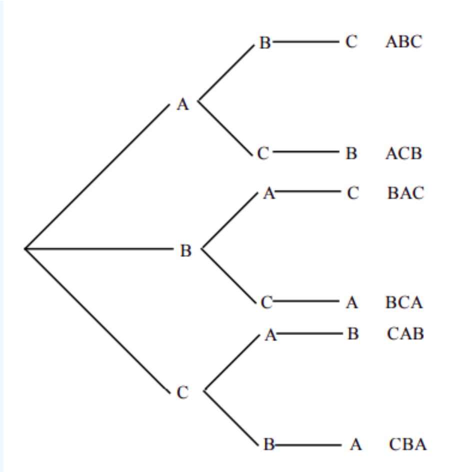
The problem is very similar to the previous example.

Imagine a child having three building blocks labeled A, B, and C. Suppose he puts these blocks on top of each other to make word sequences. For the first letter he has three choices, namely A, B, or C. Let us suppose he chooses the first letter to be a B, then for the second block which must go on top of the first, he has only two choices: A or C. And for the last letter he has only one choice. We list the choices below.

3	2	1

Therefore, 6 different word sequences can be formed.

Finally, we'd like to illustrate this with a tree diagram showing all six possibilities.


A
B
C
B
C
C
B
A
C
C
A
A
B
B
A
ABC
ACB
BAC
BCA
CAB
CBA

6.2.1: Tree Diagrams and the Multiplication Axiom (Exercises)

Do the following problems using a tree diagram or the multiplication axiom.

1. A man has 3 shirts, and 2 pairs of pants. Use a tree diagram to determine the number of possible outfits.	2. In a city election, there are 2 candidates for mayor, and 3 for supervisor. Use a tree diagram to find the number of ways to fill the two offices.
3. There are 4 roads from Town A to Town B, 2 roads from Town B to Town C. Use a tree diagram to find the number of ways one can travel from Town A to Town C.	4. Brown Home Construction offers a selection of 3 floor plans, 2 roof types, and 2 exterior wall types. Use a tree diagram to determine the number of possible homes available.
5. For lunch, a small restaurant offers 2 types of soups, three kinds of sandwiches, and two types of soft drinks. Use a tree diagram to determine the number of possible meals consisting of a soup, sandwich, and a soft drink.	6. A California license plate consists of a number from 1 to 5, then three letters followed by three digits. How many such plates are possible?

Do the following problems using the Multiplication Axiom

7. A license plate consists of three letters followed by three digits. How many license plates are possible if no letter may be repeated?	8. How many different 4-letter radio station call letters can be made if the first letter must be K or W and no letters can be repeated?
9. How many seven-digit telephone numbers are possible if the first two digits cannot be ones or zeros?	10. How many 3-letter word sequences can be formed using the letters {a, b, c, d} if no letter is to be repeated?

Use a tree diagram for questions 11 and 12:

11. A family has two children, use a tree diagram to determine all four possibilities of outcomes by gender.	12. A coin is tossed three times and the sequence of heads and tails is recorded. Use a tree diagram to list all the possible outcomes.

Do the following problems using the Multiplication Axiom

13. In how many ways can a 4-question true-false test be answered?	14. In how many ways can three people be arranged to stand in a straight line?
15. A combination lock is opened by first turning to the left, then to the right, and then to the left again. If there are 30 digits on the dial, how many possible combinations are there?	16. How many different answers are possible for a multiple-choice test with 10 questions and five possible answers for each question?
17. In the past, a college required students to use a 4 digit PIN (Personal Identification Number) as their password for its registration system. How many different PINs are possible if each must have 4 digits with no restrictions on selection or arrangement of the digits used?	18. The college decided that a more secure password system is needed. New passwords must have 3 numerical digits followed by 6 letters. There are no restrictions on the selection of the numerical digits. However, the letters I and O are not permitted. How many different passwords are possible?

2.3: Permutations

Learning Objectives

In this section you will learn to

1. Count the number of possible permutations (ordered arrangement) of n items taken r at a time
2. Count the number of possible permutations when there are conditions imposed on the arrangements
3. Perform calculations using factorials

In Example 7.2.6 of section 7.2, we were asked to find the word sequences formed by using the letters { A, B, C } if no letter is to be repeated. The tree diagram gave us the following six arrangements.

ABC, ACB, BAC, BCA, CAB, and CBA.

Arrangements like these, where order is important and no element is repeated, are called permutations.

Definition: Permutations

A permutation of a set of elements is an ordered arrangement where each element is used once.

Example 2.3.1

How many three-letter word sequences can be formed using the letters { A, B, C, D }?

Solution

There are four choices for the first letter of our word, three choices for the second letter, and two choices for the third.

4	3	2

Applying the multiplication axiom, we get $4 \cdot 3 \cdot 2 = 24$ different arrangements.

Example 2.3.2

How many permutations of the letters of the word ARTICLE have consonants in the first and last positions?

Solution

In the word ARTICLE, there are 4 consonants.

Since the first letter must be a consonant, we have four choices for the first position, and once we use up a consonant, there are only three consonants left for the last spot. We show as follows:

4						3

Since there are no more restrictions, we can go ahead and make the choices for the rest of the positions.

So far we have used up 2 letters, therefore, five remain. So for the next position there are five choices, for the position after that there are four choices, and so on. We get

4	5	4	3	2	1	3

So the total permutations are $4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 = 1440$.

Example 2.3.3

Given five letters { A, B, C, D, E }. Find the following:

a. The number of four-letter word sequences.
b. The number of three-letter word sequences.
c. The number of two-letter word sequences.

Solution

The problem is easily solved by the multiplication axiom, and answers are as follows:

a. The number of four-letter word sequences is $5 \cdot 4 \cdot 3 \cdot 2 = 120$.
b. The number of three-letter word sequences is $5 \cdot 4 \cdot 3 = 60$.
c. The number of two-letter word sequences is $5 \cdot 4 = 20$.

We often encounter situations where we have a set of n objects and we are selecting r objects to form permutations. We refer to this as **permutations of n objects taken r at a time**, and we write it as **nPr**.

Therefore, the above example can also be answered as listed below.

a. The number of four-letter word sequences is 5P4 = 120.
b. The number of three-letter word sequences is 5P3 = 60.
c. The number of two-letter word sequences is 5P2 = 20.

Before we give a formula for nPr, we'd like to introduce a symbol that we will use a great deal in this as well as in the next chapter.

Definition: Factorial

$$\mathrm{n}! = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots 3 \cdot 2 \cdot 1$$

where n is a natural number.

$$0! = 1$$

Now we define nPr.

Definition: nPr

The Number of Permutations of n Objects Taken r at a Time

$$\mathrm{nPr} = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots(\mathrm{n}-\mathrm{r}+1)$$

or

$$\mathrm{nPr} = \frac{\mathrm{n}!}{(\mathrm{n}-\mathrm{r})!}$$

where n and r are natural numbers.

The reader should become familiar with both formulas and should feel comfortable in applying either.

Example 2.3.4

Compute the following using both formulas.

a. 6P3
b. 7P2

Solution

We will identify n and r in each case and solve using the formulas provided.

a. 6P3 = $6 \cdot 5 \cdot 4 = 120$, alternately

$$6P3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120$$

b. 7P2 = $7 \cdot 6 = 42$, or

$$7P2 = \frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 42$$

Next we consider some more permutation problems to get further insight into these concepts.

Example 2.3.5

In how many different ways can 4 people be seated in a straight line if two of them insist on sitting next to each other?

Solution

Let us suppose we have four people A, B, C, and D. Further suppose that A and B want to sit together. For the sake of argument, we tie A and B together and treat them as one person.

The four people are $\boxed{AB}$ CD. Since $\boxed{AB}$ is treated as one person, we have the following possible arrangements.

$$\boxed{AB}CD, \boxed{AB}DC, C\boxed{AB}D, D\boxed{AB}C, CD\boxed{AB}, DC\boxed{AB}$$

Note that there are six more such permutations because A and B could also be tied in the order BA. And they are

$$\boxed{BA}CD, \boxed{BA}DC, C\boxed{BA}D, D\boxed{BA}C, CD\boxed{BA}, DC\boxed{BA}$$

So altogether there are 12 different permutations.

Let us now do the problem using the multiplication axiom.

After we tie two of the people together and treat them as one person, we can say we have only three people. The multiplication axiom tells us that three people can be seated in 3! ways. Since two people can be tied together 2! ways, there are 3! 2! = 12 different arrangements

Example 2.3.6

You have 4 math books and 5 history books to put on a shelf that has 5 slots. In how many ways can the books be shelved if the first three slots are filled with math books and the next two slots are filled with history books?

Solution

We first do the problem using the multiplication axiom.

Since the math books go in the first three slots, there are 4 choices for the first slot,
3 choices for the second and 2 choices for the third.

The fourth slot requires a history book, and has five choices. Once that choice is made, there are 4 history books left, and therefore, 4 choices for the last slot. The choices are shown below.

4	3	2	5	4

Therefore, the number of permutations are $4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 = 480$.

Alternately, we can see that $4 \cdot 3 \cdot 2$ is really same as 4P3, and $5 \cdot 4$ is 5P2.

So the answer can be written as (4P3) (5P2) = 480.

Clearly, this makes sense. For every permutation of three math books placed in the first three slots, there are 5P2 permutations of history books that can be placed in the last two slots. Hence the multiplication axiom applies, and we have the answer (4P3) (5P2).

We summarize the concepts of this section:

> Note
>
> **1. Permutations**
>
> A permutation of a set of elements is an ordered arrangement where each element is used once.
>
> **2. Factorial**
>
> $$\mathrm{n}! = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots 3\cdot 2\cdot 1$$
>
> Where n is a natural number.
>
> $$0! = 1$$
>
> **3. Permutations of n Objects Taken r at a Time**
>
> $$\mathrm{nPr} = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots(\mathrm{n}-\mathrm{r}+1)$$
>
> or
>
> $$\mathrm{nPr} = \frac{\mathrm{n}!}{(\mathrm{n}-\mathrm{r})!}$$
>
> where n and r are natural numbers.

6.3.1: Permutations (Exercises)

Do the following problems using permutations.

1. How many three-letter words can be made using the letters { a, b, c, d, e } if no repetitions are allowed?	2. A grocery store has five checkout counters, and seven clerks. How many different ways can the 7 clerks be assigned to the 5 counters?
3. A group of fifteen people who are members of an investment club wish to choose a president, and a secretary. How many different ways can this be done?	4. Compute the following. a. 9P2 b. 6P4 c. 8P3 d. 7P4
5. In how many ways can the letters of the word CUPERTINO be arranged if each letter is used only once in each arrangement?	6. How many permutations of the letters of the word PROBLEM end in a vowel?
7. How many permutations of the letters of the word SECURITY end in a consonant?	8. How many permutations of the letters PRODUCT have consonants in the second and third positions?
9. How many three-digit numbers are there?	10. How many three-digit odd numbers are there?
11. In how many different ways can five people be seated in a row if two of them insist on sitting next to each other?	12. In how many different ways can five people be seated in a row if two of them insist on not sitting next to each other?
13. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if the English books are set on the left, history books in the middle, and math books on the right?	14. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if they are grouped by subject?
15. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with math books and the next three with history books?	16. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with the books of one subject and the next three slots are to be filled with the books of the other subject?
17. A bakery has 9 different fancy cakes. In how many ways can 5 of the 9 fancy cakes be lined up in a row in the bakery display case?	18. A landscaper has 6 different flowering plants. She needs to plant 4 of them in a row in a garden. How many different ways can 4 of the 6 plants be arranged in a row?
19. At an auction of used construction vehicles, there are 7 different vehicles for sale. In how many orders could these 7 vehicles be listed in the auction program?	20. A landscaper has 6 different flowering plants and 4 different non-flowering bushes. She needs to plant a row of 6 plants in a garden. There must be a bush at each end, and four flowering plants in a row in between the bushes. How many different arrangements in a row are possible?
21. In how many ways can all 7 letters of the word QUIETLY be arranged if the letters Q and U must be next to each other in the order QU?	22. a. In how many ways can the letters ABCDEXY be arranged if the X and Y must be next to each other in either order XY or YX? b. In how many ways can the letters ABCDEXY be arranged if the X and Y can not be next to each other?

6.4: Circular Permutations and Permutations with Similar Elements

Learning Objectives

In this section you will learn to

1. Count the number of possible permutations of items arranged in a circle
2. Count the number of possible permutations when there are repeated items

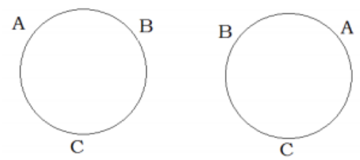
In this section we will address the following two problems.

1. In how many different ways can five people be seated in a circle?
2. In how many different ways can the letters of the word MISSISSIPPI be arranged?

The first problem comes under the category of Circular Permutations, and the second under Permutations with Similar Elements.

Circular Permutations

Suppose we have three people named A, B, and C. We have already determined that they can be seated in a straight line in 3! or 6 ways. Our next problem is to see how many ways these people can be seated in a circle. We draw a diagram.



It happens that there are only two ways we can seat three people in a circle, relative to each other's positions. This kind of permutation is called a circular permutation. In such cases, no matter where the first person sits, the permutation is not affected. Each person can shift as many places as they like, and the permutation will not be changed. We are interested in the position of each person in relation to the others. Imagine the people on a merry-go-round; the rotation of the permutation does not generate a new permutation. So in circular permutations, the first person is considered a place holder, and where he sits does not matter.

Definition: Circular Permutations

The number of permutations of n elements in a circle is $(n-1)!$

Example 6.4.1

In how many different ways can five people be seated at a circular table?

Solution

We have already determined that the first person is just a place holder. Therefore, there is only one choice for the first spot. We have

1	4	3	2	1

So the answer is 24.

Example 6.4.2

In how many ways can four couples be seated at a round table if the men and women want to sit alternately?

Solution

We again emphasize that the first person can sit anywhere without affecting the permutation.

So there is only one choice for the first spot. Suppose a man sat down first. The chair next to it must belong to a woman, and there are 4 choices. The next chair belongs to a man, so there are three choices and so on. We list the choices below.

1	4	3	3	2	2	1	1

So the answer is 144.

PERMUTATIONS WITH SIMILAR ELEMENTS

Let us determine the number of distinguishable permutations of the letters ELEMENT.

Suppose we make all the letters different by labeling the letters as follows.

$$E_1LE_2ME_3NT$$

Since all the letters are now different, there are 7! different permutations.

Let us now look at one such permutation, say

$$LE_1ME_2NE_3T$$

Suppose we form new permutations from this arrangement by only moving the E's. Clearly, there are 3! or 6 such arrangements. We list them below.

$$\begin{array}{l} \mathrm{LE_1ME_2NE_3} \\ \mathrm{LE_1ME_3NE_2} \\ \mathrm{LE_2ME_1NE_3T} \\ \mathrm{LE_2ME_3NE_1T} \\ \mathrm{LE_3ME_2NE_1T} \\ \mathrm{LE_3ME_{\mathit{I}}NE_2T} \end{array}$$

Because the E's are not different, there is only one arrangement LEMENET and not six. This is true for every permutation.

Let us suppose there are n different permutations of the letters ELEMENT.

Then there are $n \cdot 3!$ permutations of the letters $E_1LE_2ME_3NT$.

But we know there are 7! permutations of the letters $E_1LE_2ME_3NT$.

Therefore, $n \cdot 3! = 7!$

Or $n = \frac{7!}{3!}$.

This gives us the method we are looking for.

Definition: Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, is

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

Example 6.4.3

Find the number of different permutations of the letters of the word MISSISSIPPI.

Solution

The word MISSISSIPPI has 11 letters. If the letters were all different there would have been 11! different permutations. But MISSISSIPPI has 4 S's, 4 I's, and 2 P's that are alike.

So the answer is $\frac{11!}{4!4!2!} = 34,650$.

Example 6.4.4

If a coin is tossed six times, how many different outcomes consisting of 4 heads and 2 tails are there?

Solution

Again, we have permutations with similar elements.

We are looking for permutations for the letters HHHHTT.

The answer is $\frac{6!}{4!2!} = 15$.

Example 6.4.5

In how many different ways can 4 nickels, 3 dimes, and 2 quarters be arranged in a row?

Solution

Assuming that all nickels are similar, all dimes are similar, and all quarters are similar, we have permutations with similar elements. Therefore, the answer is

$$\frac{9!}{4!3!2!} = 1260$$

Example 6.4.6

A stock broker wants to assign 20 new clients equally to 4 of its salespeople. In how many different ways can this be done?

Solution

This means that each sales person gets 5 clients. The problem can be thought of as an ordered partitions problem. In that case, using the formula we get

$$\frac{20!}{5!5!5!5!} = 11,732,745,024$$

Example 6.4.7

A shopping mall has a straight row of 5 flagpoles at its main entrance plaza. It has 3 identical green flags and 2 identical yellow flags. How many distinct arrangements of flags on the flagpoles are possible?

Solution

The problem can be thought of as distinct permutations of the letters GGGYY; that is arrangements of 5 letters, where 3 letters are similar, and the remaining 2 letters are similar:

$$\frac{5!}{3!2!} = 10$$

Just to provide a little more insight into the solution, we list all 10 distinct permutations:

GGGYY, GGYGY, GGYYG, GYGGY, GYGYG, GYYGG, YGGGY, YGGYG, YGYGY, YYGGG

We summarize.

Summary

1. Circular Permutations

The number of permutations of n elements in a circle is

$$(n-1)!$$

2. Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, such that $\mathrm{n}=\mathrm{r}_1+\mathrm{r}_2+\ldots+\mathrm{r}_\mathrm{k}$ is

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

This is also referred to as **ordered partitions**.

6.4.1: Circular Permutations and Permutations with Similar Elements (Exercises)

Do the following problems using the techniques learned in this section.

1. In how many different ways can five children hold hands to play "Ring Around the Rosy"?	2. In how many ways can three people be made to sit at a round table?
3. In how many different ways can six children ride a "Merry Go Around" with six horses?	4. In how many ways can three couples be seated at a round table, so that men and women sit alternately?
5. In how many ways can six trinkets be arranged on a chain?	6. In how many ways can five keys be put on a key ring?
7. Find the number of different permutations of the letters of the word MASSACHUSETTS.	8. Find the number of different permutations of the letters of the word MATHEMATICS.
9. Seven flags are to be flown on seven poles: 3 flags are red, 2 are white, and 2 are blue,. How many different arrangements are possible?	10. How many different ways can 3 pennies, 2 nickels and 5 dimes be arranged in a row?
11. How many four-digit numbers can be made using two 2's and two 3's?	12. How many five-digit numbers can be made using two 6's and three 7's?
13. If a coin is tossed 5 times, how many different outcomes of 3 heads and 2 tails are possible?	14. If a coin is tossed 10 times, how many different outcomes of 7 heads and 3 tails are possible?
15. If a team plays ten games, how many different outcomes of 6 wins and 4 losses are possible?	16. If a team plays ten games, how many different ways can the team have a winning season?

6.5: Combinations

Learning Objectives

In this section you will learn to

1. Count the number of combinations of r out of n items (selections without regard to arrangement)
2. Use factorials to perform calculations involving combinations

Suppose we have a set of three letters { A, B, C }, and we are asked to make two-letter word sequences. We have the following six permutations.

AB BA BC CB AC CA

Now suppose we have a group of three people { A, B, C } as Al, Bob, and Chris, respectively, and we are asked to form committees of two people each. This time we have only three committees, namely,

AB BC AC

When forming committees, the order is not important, because the committee that has Al and Bob is no different than the committee that has Bob and Al. As a result, we have only three committees and not six.

Forming word sequences is an example of permutations, while forming committees is an example of **combinations** - the topic of this section.

Permutations are those arrangements where order is important, while combinations are those arrangements where order is not significant. From now on, this is how we will tell permutations and combinations apart.

In the above example, there were six permutations, but only three combinations.

Just as the symbol nPr represents the number of permutations of n objects taken r at a time, nCr represents the number of combinations of n objects taken r at a time.

So in the above example, 3P2 = 6, and 3C2 = 3.

Our next goal is to determine the relationship between the number of combinations and the number of permutations in a given situation.

In the above example, if we knew that there were three combinations, we could have found the number of permutations by multiplying this number by 2!. That is because each combination consists of two letters, and that makes 2! permutations.

Example 6.5.1

Given the set of letters { A, B, C, D }. Write the number of combinations of three letters, and then from these combinations determine the number of permutations.

Solution

We have the following four combinations.

ABC BCD CDA BDA

Since every combination has three letters, there are 3! permutations for every combination. We list them below.

$$\begin{array}{llll} ABC & BCD & CDA & BDA \\ ACB & BDC & CAD & BAD \\ BAC & CDB & DAC & DAB \\ BCA & CBD & DCA & DBA \\ CAB & DCB & ACD & ADB \\ CBA & DBC & ADC & ABD \end{array}$$

The number of permutations are 3! times the number of combinations; that is

4P3 = 3! · 4C3

or

$$4C3 = \frac{4P3}{3!}$$

In general,

$$nCr = \frac{nPr}{r!}$$

Since

$$nPr = \frac{n!}{(n-r)!}$$

We have,

$$nCr = \frac{n!}{(n-r)!r!}$$

Summarizing,

Note

1. Combinations

A combination of a set of elements is an arrangement where each element is used once, and order is not important.

2. The Number of Combinations of n Objects Taken r at a Time

$$nCr = \frac{n!}{(n-r)!r!}$$

where n and r are natural numbers.

Example \(\PageIndex{3}\)Example 6.5.2

Compute:

a. 5C3
b. 7C3

Solution

We use the above formula.

$$5C3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = 10$$

$$7C3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = 35$$

Example 6.5.3

In how many different ways can a student select to answer five questions from a test that has seven questions, if the order of the selection is not important?

Solution

Since the order is not important, it is a combination problem, and the answer is

7C5 = 21

Example 6.5.4

How many line segments can be drawn by connecting any two of the six points that lie on the circumference of a circle?

Solution

Since the line that goes from point A to point B is same as the one that goes from B to A, this is a combination problem.

It is a combination of 6 objects taken 2 at a time. Therefore, the answer is

$$6C2 = \frac{6!}{4!2!} = 15$$

Example 6.5.5

There are ten people at a party. If they all shake hands, how many hand-shakes are possible?

Solution

Note that between any two people there is only one hand shake. Therefore, we have

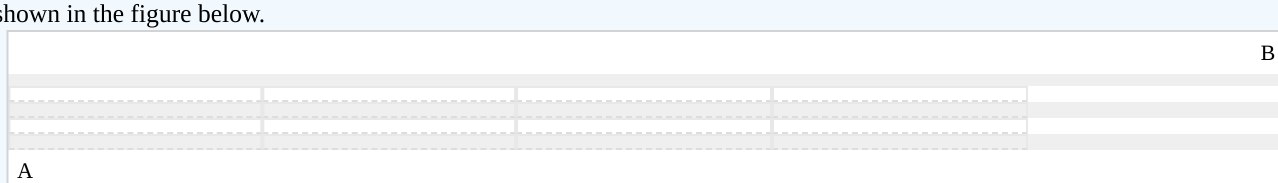
10C2 = 45 hand-shakes.

Example 6.5.6

The shopping area of a town is in the shape of square that is 5 blocks by 5 blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?

Solution

Let us suppose the taxi driver drives from the point A, the lower left hand corner, to the point B, the upper right hand corner as shown in the figure below.



To reach his destination, he has to travel ten blocks; five horizontal, and five vertical. So if out of the ten blocks he chooses any five horizontal, the other five will have to be the vertical blocks, and vice versa.

Therefore, all he has to do is to choose 5 out of ten to be the horizontal blocks

The answer is 10C5, or 252.

Alternately, the problem can be solved by permutations with similar elements.

The taxi driver's route consists of five horizontal and five vertical blocks. If we call a horizontal block H, and a vertical block a V, then one possible route may be as follows.

HHHHHVVVVV

Clearly there are $\frac{10!}{5!5!} = 252$ permutations.

Further note that by definition 10C5 = $\frac{10!}{5!5!}$.

Example 6.5.7

If a coin is tossed six times, in how many ways can it fall four heads and two tails?

Solution

First we solve this problem using section 6.5 technique-permutations with similar elements.

We need 4 heads and 2 tails, that is

HHHHTT

There are $\frac{6!}{4!2!} = 15$ permutations.

Now we solve this problem using combinations.

Suppose we have six spots to put the coins on. If we choose any four spots for heads, the other two will automatically be tails. So the problem is simply

6C4 = 15.

Incidentally, we could have easily chosen the two tails, instead. In that case, we would have gotten

6C2 = 15.

Further observe that by definition

$$6\mathrm{C}4 = \frac{6!}{2!4!}$$

and

$$6\mathrm{C}2 = \frac{6!}{4!2!}$$

Which implies 6C4 = 6C2.

6.5.1: Combinations (Exercises)

Do the following problems using combinations.

1. How many different 3-people committees can be chosen from ten people?	2. How many different 5-player teams can be chosen from eight players?
3. In how many ways can a person chose to vote for three out of five candidates on a ballot for a school board election?	4. Compute the following: a. 9C2 b. 6C4 c. 8C3 d. 7C4
5. How many 5-card hands can be chosen from a deck of cards?	6. How many 13-card bridge hands can be chosen from a deck of cards?
7. There are twelve people at a party. If they all shake hands, how many different hand-shakes are there?	8. In how many ways can a student choose to do four questions out of five on a test?
9. Five points lie on a circle. How many chords can be drawn through them?	10. How many diagonals does a hexagon have?
11. There are five team in a league. How many games are played if every team plays each other twice?	12. A team plays 15 games a season. In how many ways can it have 8 wins and 7 losses?
13. In how many different ways can a 4-child family have 2 boys and 2 girls?	14. A coin is tossed five times. In how many ways can it fall three heads and two tails?
15. The shopping area of a town is a square that is six blocks by six blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?	16. If the shopping area in the previous problem has a rectangular form of 5 blocks by 3 blocks, then how many different routes can a taxi driver take to drive from one end of the shopping area to the opposite kitty corner end?
17. A team of 7 workers is assigned to a project. In how many ways can 3 of the 7 workers be selected to make a presentation to the management about their progress on the project?	18. A real estate company has 12 houses listed for sale by their clients. In how many ways can 5 of the 12 houses be selected to be featured in an advertising brochures?
19. A frozen yogurt store has 9 toppings to choose from. In how many ways can 3 of the 9 toppings be selected ?	20. A kindergarten teacher has 14 books about a holiday. In how many ways can she select 4 of the books to read to her class in the week before the holiday?

6.6: Combinations- Involving Several Sets

Learning Objectives

In this section you will learn to

1. count the number of items selected from more than one set
2. count the number of items selected when there are restrictions on the selections

So far we have solved the basic combination problem of r objects chosen from n different objects. Now we will consider certain variations of this problem.

Example 6.6.1

How many five-people committees consisting of 2 men and 3 women can be chosen from a total of 4 men and 4 women?

Solution

We list 4 men and 4 women as follows:

$$M_1 M_2 M_3 M_4 W_1 W_2 W_3 W_4$$

Since we want 5-people committees consisting of 2 men and 3 women, we'll first form all possible two-man committees and all possible three-woman committees. Clearly there are 4C2 = 6 two-man committees, and 4C3 = 4 three-woman committees, we list them as follows:

2-Man Committees	3-Woman Committees
M_1M_2 M_1M_3 M_1M_4 M_2M_3 M_2M_4 M_3M_4	$W_1W_2W_3$ $W_1W_2W_4$ $W_1W_3W_4$ $W_2W_3W_4$

For every 2-man committee there are four 3-woman committees that can be chosen to make a 5-person committee. If we choose M_1M_2 as our 2-man committee, then we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees. As a result, we get

$$\boxed{M_1M_2}W_1W_2W_3, \boxed{M_1M_2}W_1W_2W_4, \boxed{M_1M_2}W_1W_3W_4, \boxed{M_1M_2}W_2W_3W_4$$

Similarly, if we choose M_1M_3 as our 2-man committee, then, again, we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees.

$$\boxed{M_1M_3}W_1W_2W_3, \boxed{M_1M_3}W_1W_2W_4, \boxed{M_1M_3}W_1W_3W_4, \boxed{M_1M_3}W_2W_3W_4$$

And so on.

Since there are six 2-man committees, and for every 2-man committee there are four 3-woman committees, there are altogether $6 \cdot 4 = 24$ five-people committees.

In essence, we are applying the multiplication axiom to the different combinations.

Example 6.6.2

A high school club consists of 4 freshmen, 5 sophomores, 5 juniors, and 6 seniors. How many ways can a committee of 4 people be chosen that includes

a. One student from each class?

b. All juniors?
c. Two freshmen and 2 seniors?
d. No freshmen?
e. At least three seniors?

Solution

a. Applying the multiplication axiom to the combinations involved, we get

$$(4C1)(5C1)(5C1)(6C1) = 600$$

b. We are choosing all 4 members from the 5 juniors, and none from the others.

$$5C4 = 5$$

c. $4C2 \cdot 6C2 = 90$

d. Since we don't want any freshmen on the committee, we need to choose all members from the remaining 16. That is

$$16C4 = 1820$$

e. Of the 4 people on the committee, we want at least three seniors. This can be done in two ways. We could have three seniors, and one non-senior, or all four seniors.

$$(6C3)(14C1) + 6C4 = 295$$

Example 6.6.3

How many five-letter word sequences consisting of 2 vowels and 3 consonants can be formed from the letters of the word INTRODUCE?

Solution

First we select a group of five letters consisting of 2 vowels and 3 consonants.
Since there are 4 vowels and 5 consonants, we have

$$(4C2)(5C3)$$

Since our next task is to make word sequences out of these letters, we multiply these by 5!.

$$(4C2)(5C3)(5!) = 7200.$$

Example 6.6.4

A standard deck of playing cards has 52 cards consisting of 4 suits each with 13 cards. In how many different ways can a 5-card hand consisting of four cards of one suit and one of another be drawn?

Solution

We will do the problem using the following steps.
Step 1. Select a suit.
Step 2. Select four cards from this suit.
Step 3. Select another suit.
Step 4. Select a card from that suit.

Applying the multiplication axiom, we have

Ways of selecting the first suit	Ways of selecting 4 cards from this suit	Ways of selecting the next suit	Ways of selecting a card from that suit
4C1	13C4	3C1	13C1

$(4C1)(13C4)(3C1)(13C1) = 111{,}540.$

A STANDARD DECK OF 52 PLAYING CARDS

As in the previous example, many examples and homework problems in this book refer to a standard deck of 52 playing cards. Before we end this section, we take a minute to describe a standard deck of playing cards, as some readers may not be familiar with this.

A standard deck of 52 playing cards has 4 suits with 13 cards in each suit.

♦ diamonds ♥ hearts ♠ spades ♣ clubs

Each suit is associated with a color, either black (spades, clubs) or red (diamonds, hearts)

Each suit contains 13 denominations (or values) for cards:

nine numbers 2, 3, 4, …., 10 and Jack(J), Queen (Q), King (K), Ace (A).

The Jack, Queen and King are called “face cards” because they have pictures on them. Therefore a standard deck has 12 face cards: (3 values JQK) x (4 suits ♦♥♠ ♣)

We can visualize the 52 cards by the following display

Suit	Color	Values (Denominations)
♦ Diamonds	Red	2 3 4 5 6 7 8 9 10 J Q K A
♥ Hearts	Red	2 3 4 5 6 7 8 9 10 J Q K A
♠ Spades	Black	2 3 4 5 6 7 8 9 10 J Q K A
♣ Clubs	Black	2 3 4 5 6 7 8 9 10 J Q K A

6.6.1: Combinations- Involving Several Sets (Exercises)

Following problems involve combinations from several different sets.

1. How many 5-people committees consisting of three boys and two girls can be chosen from a group of four boys and four girls?	2. A club has 4 men, 5 women, 8 boys and 10 girls as members. In how many ways can a group of 2 men, 3 women, 4 boys and 4 girls be chosen?
3. How many 4-people committees chosen from 4 men and 6 women will have at least 3 men?	4. A batch contains 10 transistors of which three are defective. If three are chosen, in how many ways can they be selected with two defective?
5. In how many ways can five counters labeled A, B, C, D and E at a store be staffed by two men and three women chosen from a group of four men and six women?	6. How many 4-letter word sequences consisting of two vowels and two consonants can be made from the letters of the word PHOENIX if no letter is repeated?

Three marbles are chosen from an urn that contains 5 red, 4 white, and 3 blue marbles. How many samples of the following type are possible?

7. All three white.	8. Two blue and one white
9. One of each color.	10. All three of the same color.
11. At least two red.	12. None red.

The following problems involve combinations from several different sets.

Five coins are chosen from a bag that contains 4 dimes, 5 nickels, and 6 pennies. How many samples of five coins of the following types are possible?

13. At least four nickels.	14. No pennies.
15. Five of a kind.	16. Four of a kind.
17. Two of one kind and two of another kind.	18. Three of one kind and two of another kind.

Find the number of different ways draw a 5-card hand from a deck to have the following combinations.

19. Three face cards.	20. A heart flush (all hearts).
21. Two hearts and three diamonds	22. Two cards of one suit, and three of another suit.
23. Two kings and three queesns.	24. 2 cards of one value and 3 of another value

The party affiliation of the 100 United States Senators in the 114th Congress, January 2015, was:

44 Democrats, 54 Republicans, and 2 Independents.

25. In how many ways could a 10 person committee be selected if it is to contain 4 Democrats, 5 Republicans, and 1 Independent?	26. In how many different ways could a 10 person committee be selected with 6 or 7 Republicans and the Democrats (with no Independents)?

The 100 United States Senators in the 114th Congress, January 2015, included 80 men and 20 women. Suppose a committee senators is working on legislation about wage discrimination by gender.

27. In how many ways could a 12 person committee be selected to contain equal numbers of men and women.	28. In how many ways could a 6 person committee be selected to contain fewer women than men?

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises. He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

How many different playlists of 6 songs can be selected that satisfy each of the following: (We care which songs are selected to be on the playlist, but not what order they are selected or listed in.)

29. Playlist has 2 songs of each type	30. Playlist has no country songs
31. Playlist has 3 rocks, 2 raps, and 1 country song	32. Playlist has 3 or 4 rock songs and all the rest are rap songs

6.7: Binomial Theorem

We end this chapter with one more application of combinations. Combinations are used in determining the coefficients of a binomial expansion such as $(x+y)^n$. Expanding a binomial expression by multiplying it out is a very tedious task, and is not practiced. Instead, a formula known as the Binomial Theorem is utilized to determine such an expansion. Before we introduce the Binomial Theorem, however, consider the following expansions.

$$\begin{aligned}
(x+y)^2 &= x^2+2xy+y^2 \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\
(x+y)^4 &= x^4+4x^2y+6x^2y^2+4xy^3+y^4 \\
(x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5 \\
(x+y)^6 &= x6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6
\end{aligned}$$

We make the following observations.

1. There are $n+1$ terms in the expansion $(x+y)^n$
2. The sum of the powers of x and y is n.
3. The powers of x begin with n and decrease by one with each successive term.
   The powers of y begin with 0 and increase by one with each successive term.

Suppose we want to expand $(x+y)^3$. We first write the expansion without the coefficients. We temporarily substitute a blank in place of the coefficients.

$$(x+y)^3 = \square x^3+\square x^2y+\square xy^2+\square y^3 \tag{6.7.1}$$

Our next job is to replace each of the blanks in equation (6.7.1) with the corresponding coefficients that belong to this expansion. Clearly,

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

If we multiply the right side and do not collect terms, we get the following.

$$xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy$$

Each product in the above expansion is the result of multiplying three variables by picking one from each of the factors $(x+y)(x+y)(x+y)$. For example, the product xxy is gotten by choosing x from the first factor, x from the second factor, and y from the third factor. There are three such products that simplify to x^2y, namely xxy, xyx, and yxx. These products take place when we choose an x from two of the factors and choose a y from the other factor. Clearly this can be done in 3C2, or 3 ways. Therefore, the coefficient of the term x^2y is 3. The coefficients of the other terms are obtained in a similar manner.

We now replace the blanks with the coefficients in equation (6.7.1), and we get

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

Example 6.7.1

Find the coefficient of the term x^2y^5 in the expansion $(x+y)^7$.

Solution

The expansion $(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$

In multiplying the right side, each product is gotten by picking an x or y from each of the seven factors $(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$.

The term x^2y^5 is obtained by choosing an x from two of the factors and a y from the other five factors. This can be done in 7C2, or 21 ways.

Therefore, the coefficient of the term x^2y^5 is 21.

Example 6.7.2

Expand $(x+y)^7$

Solution

We first write the expansion without the coefficients.

$$(x+y)^7 = \square x^7 + \square x^6 y + \square x^5 y^2 + \square x^4 y^3 + \square x^3 y^4 + \square x^2 y^5 + \square x y^6 + \square y^7$$

Now we determine the coefficient of each term as we did in Example 6.7.1.

The coefficient of the term x^7 is 7C7 or 7CO which equals 1.

The coefficient of the term $x^6 y$ is 7C6 or 7C1 which equals 7.

The coefficient of the term $x^5 y^2$ is 7C5 or 7C2 which equals 21.

The coefficient of the term $x^4 y^3$ is 7C4 or 7C3 which equals 35,

and so on.

Substituting, we get: $(x+y)^7 = x^7 + 7x^6 y + 21x^5 y^2 + 35x^4 y^3 + 35x^3 y^4 + 21x^2 y^5 + 7xy^6 + y^7$

We generalize the result.

Binomial Theorem

$$(x+y)^n =_n C_0 x^n +_n C_1 x^{n-1} y +_n C_2 x^{n-2} y^2 + \cdots +_n C_{n-1} x y_n^{n-1+} C_n y^n$$

Example 6.7.3

Expand $(3a-2b)^4$

Solution

If we let $x = 3a$ and $y = -2b$, and apply the Binomial Theorem, we get

$$\begin{aligned}(3a-2b)^4 &= 4\,\mathrm{Co}(3a)^4 + 4Cl(3a)^3(-2b) + 4C2(3a)^2(-2b)^2 + 4C3(3a)(-2b)^3 + 4C4(-2b)^4 \\ &= 1\left(81a^4\right) + 4\left(27a^3\right)(-2b) + 6\left(9a^2\right)\left(4b^2\right) + 4(3a)\left(-8b^3\right) + 1\left(16b^3\right) \\ &= 81a^4 - 216a^3 b + 216a^2 b^2 - 96ab^3 + 16b^4\end{aligned}$$

Example 6.7.4

Find the fifth term of the expansion $(3a-2b)^7$.

Solution

The Binomial theorem tells us that in the r-th term of an expansion, the exponent of the y term is always one less than r, and, the coefficient of the term is ${}_{\mathrm{n}}\mathrm{C}_{\mathrm{r}-1}$.

$n = 7$ and $r - 1 = 5 - 1 = 4$, so the coefficient is $7C4 = 35$

Thus, the fifth term is $(7\mathrm{C}4)(3\mathrm{a})^3(-2\mathrm{b})^4 = 35\left(27\mathrm{a}^3\right)\left(16\mathrm{b}^4\right) = 15120\mathrm{a}^3\mathrm{b}^4$

4.7.1: Binomial Theorem (Exercises)

Use the Binomial Theorem to do the following problems.

1. Expand $(a+b)^5$.	2. Expand $(a-b)^6$.
3. Expand $(x-2y)^5$.	4. Expand $(2x-3y)^4$.
5. Find the third term of $(2x-3y)^6$.	6. Find the sixth term of $(5x+y)^8$.
7. Find the coefficient of the x^3y^4 term in the expansion of $(2x+y)^7$.	8. Find the coefficient of the a^4b^6 term in the expansion of $(3a-b)^{10}$.
9. A coin is tossed 5 times, in how many ways is it possible to get three heads and two tails?	10. A coin is tossed 10 times, in how many ways is it possible to get seven heads and three tails?
11. How many subsets are there of a set that has 6 elements?	12. How many subsets are there of a set that has n elements?

2.8: Chapter Review

1. Suppose of the 4,000 freshmen at a college everyone is enrolled in a mathematics or an English class during a given quarter. If 2,000 are enrolled in a mathematics class, and 3,000 in an English class, how many are enrolled in both a mathematics class and an English class?
2. In a survey of 250 people, it was found that 125 had read Time magazine, 175 had read Newsweek, 100 had read U. S. News, 75 had read Time and Newsweek, 60 had read Newsweek and U. S. News, 55 had read Time and U. S. News, and 25 had read all three.
   a. How many had read Time but not the other two?
   b. How many had read Time or Newsweek but not the U. S. News And World Report?
   c. How many had read none of these three magazines?
3. At a manufacturing plant, a product goes through assembly, testing, and packing. If a plant has three assembly stations, two testing stations, and two packing stations, in how many different ways can a product achieve its completion?
4. Six people are to line up for a photograph. How many different lineups are possible if three of them insist on standing next to each other?
5. How many four-letter word sequences can be made from the letters of the word CUPERTINO?
6. In how many different ways can a 20-question multiple choice test be designed so that its answers contain 2 A's, 4 B's, 9 C's, 3 D's, and 2 E's?
7. The U. S. Supreme Court has nine judges. In how many different ways can the judges cast a six-tothree decision in favor of a ruling?
8. In how many different ways can a coach choose a linebacker, a guard, and a tackle from five players on the bench, if all five can play any of the three positions?
9. How many three digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if no repetitions are allowed?
10. Compute:
    a. $9 \mathrm{C} 4$
    b. $8 \mathrm{P} 3$
    c. $\frac{10!}{4!(10-4)!}$
11. In how many ways can 3 English, 3 Math, and 4 Spanish books be set on a shelf if the books are grouped by subject?
12. In how many ways can a 10-question multiple choice test with four possible answers for each question be answered?
13. On a soccer team three fullbacks can play any of the three fullback positions, left, center, and right. The three halfbacks can play any of the three halfback positions, the four forwards can play any of the four positions, and the goalkeeper plays only his position. How many different arrangements of the 11 players are possible?
14. From a group of 6 people, 3 are assigned to cleaning, 2 to hauling and one to garbage collecting. How many different ways can this be done?
15. How many three-letter word sequences can be made from the letters of the word OXYGEN?
16. In how many ways can 3 books be selected from 4 English and 2 History books if at least one English book must be chosen?
17. Five points lie on the rim of a circle. Choosing the points as vertices, how many different triangles can be drawn?
18. A club consists of six men and nine women. In how many ways can a president, a vice president and a treasurer be chosen if the two of the officers must be women?
19. Of its 12 sales people, a company wants to assign 4 to its Western territory, 5 to its Northern territory, and 3 to its Southern territory. How many ways can this be done?
20. How many permutations of the letters of the word OUTSIDE have consonants in the first and last place?
21. How many distinguishable permutations are there in the word COMMUNICATION?
22. How many five-card poker hands consisting of the following distribution are there?
    a. A flush(all five cards of a single suit)
    b. Three of a kind(e.g. three aces and two other cards)
    c. Two pairs(e.g. two aces, two kings and one other card)
    d. A straight(all five cards in a sequence)
23. Company stocks on an exchange are given symbols consisting of three letters. How many different three-letter symbols are possible?
24. How many four-digit odd numbers are there?

25. In how many ways can 7 people be made to stand in a straight line? In a circle?
26. A United Nations delegation consists of 6 Americans, 5 Russians, and 4 Chinese. Answer the following questions.
    a. How many committees of five people are there?
    b. How many committees of three people consisting of at least one American are there?
    c. How many committees of four people having no Russians are there?
    d. How many committees of three people have more Americans than Russians?
    e. How many committees of three people do not have all three Americans?
27. If a coin is flipped five times, in how many different ways can it show up three heads?
28. To reach his destination, a man is to walk three blocks north and four blocks west. How many different routes are possible?
29. All three players of the women's beach volleyball team, and all three players of the men's beach volleyball team are to line up for a picture with all members of the women's team lined together and all members of men's team lined up together. How many ways can this be done?
30. From a group of 6 Americans, 5 Japanese and 4 German delegates, two Americans, two Japanese and a German are chosen to line up for a photograph. In how many different ways can this be done?
31. Find the fourth term of the expansion $(2x-3y)^8$.
32. Find the coefficient of the a^5b^4 term in the expansion of $(a–2b)^9$.

CHAPTER OVERVIEW

7: Probability

Learning Objectives

In this chapter, you will learn to:

1. Write sample spaces.
2. Determine whether two events are mutually exclusive.
3. Use the Addition Rule.
4. Calculate probabilities using both tree diagrams and combinations.
5. Do problems involving conditional probability.
6. Determine whether two events are independent.

7.1: Sample Spaces and Probability

Learning Objectives

In this section, you will learn to:

1. Write sample spaces.
2. Calculate probabilities by examining simple events in sample spaces.

If two coins are tossed, what is the probability that both coins will fall heads? The problem seems simple enough, but it is not uncommon to hear the incorrect answer 1/3. A student may incorrectly reason that if two coins are tossed there are three possibilities, one head, two heads, or no heads. Therefore, the probability of two heads is one out of three. The answer is wrong because if we toss two coins there are four possibilities and not three. For clarity, assume that one coin is a penny and the other a nickel. Then we have the following four possibilities.

HH HT TH TT

The possibility HT, for example, indicates a head on the penny and a tail on the nickel, while TH represents a tail on the penny and a head on the nickel. It is for this reason, we emphasize the need for understanding sample spaces.

Sample Spaces

An act of flipping coins, rolling dice, drawing cards, or surveying people are referred to as a probability **experiment**. A **sample space** of an experiment is the set of all possible outcomes.

Example 7.1.1

If a die is rolled, write a sample space.

Solution

A die has six faces each having an equally likely chance of appearing. Therefore, the set of all possible outcomes S is

{ 1, 2, 3, 4, 5, 6 }.

Example 7.1.2

A family has three children. Write a sample space.

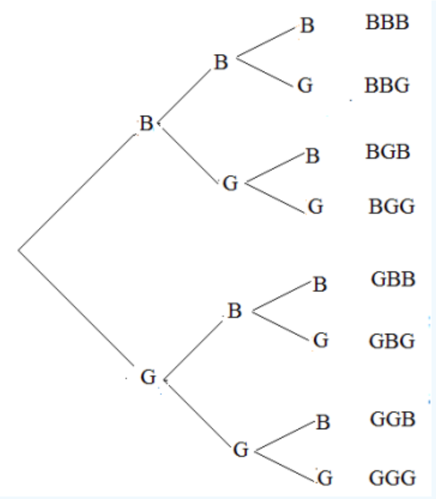
Solution

The sample space consists of eight possibilities.

{ BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG }

The possibility BGB, for example, indicates that the first born is a boy, the second born a girl, and the third a boy.

We illustrate these possibilities with a tree diagram.



Example 7.1.3

Two dice are rolled. Write the sample space.

Solution

We assume one of the dice is red, and the other green. We have the following 36 possibilities.

Green						
Red	**1**	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The entry (2, 5), for example, indicates that the red die shows a 2, and the green a 5.

Probability

Now that we understand the concept of a sample space, we will define probability.

Definition: Probability

For a sample space S, and an outcome A of S, the following two properties are satisfied.

1. If A is an outcome of a sample space, then the probability of A, denoted by $P(A)$, is between 0 and 1, inclusive.

$$0 \leq P(A) \leq 1$$

2. The sum of the probabilities of all the outcomes in S equals 1.

The probability $P(A)$ of an event A describes the chance or likelihood of that event occurring.

- If $P(A)=0$, event A is certain not to occur. If $P(A)=1$, event A is certain to occur.
- If $P(A)=0.5$, then event A is equally likely to occur or not occur.
- If we toss a fair coin that is equally likely to land on heads or tails, then P(Head) = 0.50.

- If the weather forecast says there is a 70% chance of rain today, then P(Rain) = 0.70, indicating is it more likely to rain than to not rain.

Example 7.1.4

If two dice, one red and one green, are rolled, find the probability that the red die shows a 3 and the green shows a six.

Solution

Since two dice are rolled, there are 36 possibilities. The probability of each outcome, listed in Example $7.1.3$, is equally likely.

Since (3, 6) is one such outcome, the probability of obtaining (3, 6) is 1/36.

The example we just considered consisted of only one outcome of the sample space. We are often interested in finding probabilities of several outcomes represented by an event.

An **event** is a subset of a sample space. If an event consists of only one outcome, it is called a **simple event.**

Example 7.1.5

If two dice are rolled, find the probability that the sum of the faces of the dice is 7.

Solution

Let E represent the event that the sum of the faces of two dice is 7.

The possible cases for the sum to be equal to 7 are: (1, 6), (2,5), (3, 4), (4, 3), (5, 2), and (6, 1), so event E is

E = {(1, 6), (2,5), (3, 4), (4, 3), (5, 2), (6, 1)}

The probability of the event E is

P(E) = 6/36 or 1/6.

Example 7.1.6

A jar contains 3 red, 4 white, and 3 blue marbles. If a marble is chosen at random, what is the probability that the marble is a red marble or a blue marble?

Solution

We assume the marbles are r_1, r_2, r_3, w_1, w_2, w_3, w_4, b_1, b_2, b_3. Let the event C represent that the marble is red or blue.

The sample space $\mathrm{S} = \{\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3, \mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3, \mathrm{w}_4, \mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3\}$.

And the event $\mathrm{C} = \{\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3, \mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3\}$

Therefore, the probability of C,

$$\mathrm{P(C)} = 6/10 \text{ or } 3/5$$

Example 7.1.7

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **without replacement**. That means that after a marble is drawn it is not replaced in the jar, and therefore is no longer available to select on the second draw.

Solution

Since two marbles are drawn without replacement, the sample space consists of the following six possibilities.

$$\mathrm{S} = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$$

Note that (1,1), (2,2) and (3,3) are not listed in the sample space. These outcomes are not possible when drawing without replacement, because once the first marble is drawn but not replaced into the jar, that marble is not available in the jar to be selected again on the second draw.

Let the event E represent that the sum of the numbers is five. Then

$$\mathrm{E}=\{(2,3),(3,2)\}$$

Therefore, the probability of E is

$$\mathrm{P}(\mathrm{E})=2/6 \text{ or } 1/3.$$

Example 7.1.8

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution

The sample space, as in Example $7.1.7$, consists of the following six possibilities.

$$\mathrm{S}=\{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$$

Let the event F represent that the sum of the numbers is at least four. Then

$$\mathrm{F}=\{(1,3),(3,1),(2,3),(3,2)\}$$

Therefore, the probability of F is

$$\mathrm{P}(\mathrm{F})=4/6 \text{ or } 2/3$$

Example 7.1.9

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **with replacement**. That means that after a marble is drawn it IS replaced in the jar, and therefore is available to select again on the second draw.

Solution

When two marbles are drawn with replacement, the sample space consists of the following nine possibilities.

$$\mathrm{S}=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

Note that (1,1), (2,2) and (3,3) are listed in the sample space. These outcomes are possible when drawing with replacement, because once the first marble is drawn and replaced, that marble is not available in the jar to be drawn again.

Let the event E represent that the sum of the numbers is four. Then

$$\mathrm{E}=(2,3),(3,2)$$

Therefore, the probability of F is $\mathrm{P}(\mathrm{E})=2/9$

Note that in Example $7.1.9$ when we selected marbles with replacement, the probability has changed from Example $7.1.7$ where we selected marbles without replacement.

Example 7.1.10

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution

The sample space when drawing with replacement consists of the following nine possibilities.

$$\mathrm{S}=(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$$

Let the event F represent that the sum of the numbers is at least four. Then

$$\mathrm{F}=(1,3),(3,1),(2,3),(3,2),(2,2),(3,3)$$

Therefore, the probability of F is

$$\mathrm{P}(\mathrm{F})=6/9 \text{ or } 2/3.$$

Note that in Example $7.1.10$ when we selected marbles with replacement, the probability is the same as in Example $7.1.8$ where we selected marbles without replacement.

Thus sampling with or without replacement MAY change the probabilities, but may not, depending on the situation in the particular problem under consideration. We'll re-examine the concepts of sampling with and without replacement in Section 8.3.

Example 7.1.11

One 6 sided die is rolled once. Find the probability that the result is greater than 4.

Solution

The sample space consists of the following six possibilities in set S: $\mathrm{S}=1,2,3,4,5,6$

Let E be the event that the number rolled is greater than four: $\mathrm{E}=5,6$

Therefore, the probability of E is: $\mathrm{P}(\mathrm{E})=2/6$ or $1/3$.

Page 3.1.1: Sample Spaces and Probability (Exercises)

SECTION 8.1 PROBLEM SET: SAMPLE SPACES AND PROBABILITY

In problems 1 - 6, write a sample space for the given experiment.

1) A die is rolled.	2) A penny and a nickel are tossed.
3) A die is rolled, and a coin is tossed.	4) Three coins are tossed.
5) Two dice are rolled.	6) A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn.

In problems 7 - 12, one card is randomly selected from a deck. Find the following probabilities.

7) P(an ace)	8) P(a red card)
9) P(a club)	10) P(a face card)
11) P(a jack or a spade)	12) P(a jack and a spade)

For problems 13 - 16: A jar contains 6 red, 7 white, and 7 blue marbles. If one marble is chosen at random, find the following probabilities.

13) P(red)	14) P(white)
15) P(red or blue)	16) P(red and blue)

For problems 17 - 22: Consider a family of three children. Find the following probabilities.

17) P(two boys and a girl)	18) P(at least one boy)
19) P(children of both sexes)	20) P(at most one girl)
21) P(first and third children are male)	22) P(all children are of the same gender)

For problems 23 - 27: Two dice are rolled. Find the following probabilities.

23) P(the sum of the dice is 5)	24) P(the sum of the dice is 8)
25) P(the sum is 3 or 6)	26) P(the sum is more than 10)
27) P(the result is a double) (Hint: a double means that both dice show the same value)	

For problems 28-31: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITHOUT REPLACEMENT. That means that after a marble is drawn it is NOT replaced in the jar before the second marble is selected. Find the following probabilities.

28) P(the sum of the numbers is 5)	29) P(the sum of the numbers is odd)
30) P(the sum of the numbers is 9)	31) P(one of the numbers is 3)

For problems 32-33: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITH REPLACEMENT. That means that after a marble is drawn it is replaced in the jar before the second marble is selected. Find the following probabilities.

32) P(the sum of the numbers is 5)	33) P(the sum of the numbers is 2)

3.2: Mutually Exclusive Events and the Addition Rule

Learning Objectives

In this section, you will learn to:

1. Define compound events using union, intersection, and complement.
2. Identify mutually exclusive events
3. Use the Addition Rule to calculate probability for unions of events.

In the last chapter, we learned to find the union, intersection, and complement of a set. We will now use these set operations to describe events.

- The **union** of two events E and F, E ∪ F, is the set of outcomes that are in E or in F or in both.
- The **intersection** of two events E and F, E ∩ F, is the set of outcomes that are in both E and F.
- The **complement** of an event E, denoted by E^c, is the set of outcomes in the sample space S that are not in E.

It is worth noting that $P(E^c) = 1 - P(E)$. This follows from the fact that if the sample space has n elements and E has k elements, then E^c has $n-k$ elements. Therefore,

$$P\left(E^{c}\right)=\frac{n-k}{n}=1-\frac{k}{n}=1-P(E)$$

Of particular interest to us are the events whose outcomes do not overlap. We call these events mutually exclusive.

Two events E and F are said to be **mutually exclusive** if they do not intersect: E ∩ F = ∅.

Next we'll determine whether a given pair of events are mutually exclusive.

Example 3.2.1

A card is drawn from a standard deck. Determine whether the pair of events given below is mutually exclusive.

E = {The card drawn is an Ace}

F = {The card drawn is a heart}

Solution

Clearly the ace of hearts belongs to both sets. That is

$$\mathrm{E} \cap \mathrm{F}=\{\text{ Ace of hearts }\} \neq \varnothing$$

Therefore, the events E and F are not mutually exclusive.

Example 3.2.2

Two dice are rolled. Determine whether the pair of events given below is mutually exclusive.

G = {The sum of the faces is six}

H = {One die shows a four}

Solution

For clarity, we list the elements of both sets.

G = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)} and H = {(2, 4), (4, 2)}

Clearly, G ∩ H = {(2, 4), (4, 2)} $\neq \varnothing$.

Therefore, the two sets are not mutually exclusive.

Example 3.2.3

A family has three children. Determine whether the following pair of events are mutually exclusive.

M = {The family has at least one boy}

N = {The family has all girls}

Solution

Although the answer may be clear, we list both the sets.

M = { BBB, BBG, BGB, BGG, GBB, GBG, GGB } and N = { GGG }

Clearly, M ∩ N = ∅

Therefore, events M and N are mutually exclusive.

We will now consider problems that involve the union of two events.

Given two events, E, F, then finding the probability of E ∪ F, is the same as finding the probability that E will happen, or F will happen, or both will happen.

Example 3.2.4

If a die is rolled, what is the probability of obtaining an even number or a number greater than four?

Solution

Let E be the event that the number shown on the die is an even number, and let F be the event that the number shown is greater than four.

The sample space S = { 1, 2, 3, 4, 5, 6 }. The event E = { 2, 4, 6 }, and event F = { 5, 6 }

We need to find P(E ∪ F).

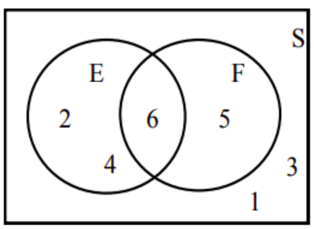
Since P(E) = 3/6, and P(F) = 2/6, a student may say P(E ∪ F) = 3/6 + 2/6. This will be incorrect because the element 6, which is in both E and F has been counted twice, once as an element of E and once as an element of F. In other words, the set E ∪ F has only four elements and not five: set E ∪ F = { 2,4,5,6 }

Therefore, P(E ∪ F) = 4/6 and not 5/6.

This can be illustrated by a Venn diagram. We'll use the Venn Diagram to re-examine Example $3.2.4$ and derive a probability rule that we can use to calculate probabilities for unions of events.

The sample space S, the events E and F, and E ∩ F are listed below.

$$\mathrm{S}=\{1,2,3,4,5,6\}, \mathrm{E}=\{2,4,6\}, \mathrm{F}=\{5,6\}, \text{ and } \mathrm{E} \cap \mathrm{F}=\{6\}.$$



The above figure shows S, E, F, and E ∩ F.

Finding the probability of E ∪ F, is the same as finding the probability that E will happen, or F will happen, or both will happen.

If we count the number of elements n(E) in E, and add to it the number of elements n(F) in F, the points in both E and F are counted twice, once as elements of E and once as elements of F. Now if we subtract from the sum, n(E) + n(F), the number n(E ∩ F), we remove the duplicity and get the correct answer. So as a rule,

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

By dividing the entire equation by n(S), we get

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

Since the probability of an event is the number of elements in that event divided by the number of all possible outcomes, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Applying the above for Example $3.2.4$, we get

$$P(E \cup F) = 3/6 + 2/6 - 1/6 = 4/6$$

This is because, when we add P(E) and P(F), we have added P(E ∩ F) twice. Therefore, we must subtract P(E ∩ F), once.

This gives us the general formula, called **the Addition Rule**, for finding the probability of the union of two events. Because event E ∪ F is the event that E will happen, OR F will happen, OR both will happen, we sometimes call this the **Addition Rule for OR Events.** It states

Addition Rule

$$\mathbf{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

If, and only if, two events E and F are mutually exclusive, then $E \cap F = \varnothing$ and $P(E \cap F) = 0$, and we get $P(E \cup F) = P(E) + P(F)$

Example $3.2.5$

If a card is drawn from a deck, use the addition rule to find the probability of obtaining an ace or a heart.

Solution

Let A be the event that the card is an ace, and H the event that it is a heart.

Since there are four aces, and thirteen hearts in the deck,

P(A) = 4/52 and P(H) = 13/52.

Furthermore, since the intersection of two events consists of only one card, the ace of hearts, we now have:

P(A ∩ H) = 1/52

We need to find P(A ∪ H):

$$\begin{aligned} P(A \cup H) &= P(A) + P(H) - P(A \cap H) \\ &= 4/52 + 13/52 - 1/52 = 16/52 \end{aligned}$$

Example $3.2.6$

Two dice are rolled, and the events F and T are as follows:

F = {The sum of the dice is four} and T = {At least one die shows a three}

Find P(F ∪ T).

Solution

We list F and T, and F ∩ T as follows:

F = {(1, 3), (2, 2), (3, 1)}

T = {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)}

F ∩ T = {(1, 3), (3, 1)}

Since P(F ∪ T) = P(F) + P(T) - P(F ∩ T)

We have P(F ∪ T) = 3/36 + 11/36 - 2/36 = 12/36.

Example 3.2.7

Mr. Washington is seeking a mathematics instructor's position at his favorite community college in Cupertino. His employment depends on two conditions: whether the board approves the position, and whether the hiring committee selects him. There is a 80% chance that the board will approve the position, and there is a 70% chance that the hiring committee will select him. If there is a 90% chance that at least one of the two conditions, the board approval or his selection, will be met, what is the probability that Mr. Washington will be hired?

Solution

Let A be the event that the board approves the position, and S be the event that Mr. Washington gets selected. We have,

P(A) = .80, P(S) = .70, and P(A ∪ S) = .90.

We need to find, P(A ∩ S).

The addition formula states that,

$$P(A \cup S)=P(A)+P(S)-P(A \cap S)$$

Substituting the known values, we get

$$.90=.80+.70-P(\mathrm{A} \cap \mathrm{S})$$

Therefore, P(A ∩ S) = .60.

Example 3.2.8

The probability that this weekend will be cold is .6, the probability that it will be rainy is .7, and probability that it will be both cold and rainy is .5. What is the probability that it will be neither cold nor rainy?

Solution

Let C be the event that the weekend will be cold, and R be event that it will be rainy. We are given that

$$\mathrm{P(C)}=.6, \quad \mathrm{P(R)}=.7, \quad \mathrm{P(C \cap R)}=.5$$

First we find P(C ∪ R) using the Addition Rule.

$$P(C \cup R)=P(C)+P(R)-P(C \cap R)=.6+.7-.5=.8$$

Then we find $P((C \cup R)^c)$ using the Complement Rule.

$$P\left((C \cup R)^c\right)=1-P(C \cup R)=1-.8=.2$$

We summarize this section by listing the important rules.

Summary

The Addition Rule

For Two Events E and F, $\mathrm{P(E \cup F)=P(E)+P(F)-P(E \cap F)}$

The Addition Rule for Mutually Exclusive Events

If Two Events E and F are Mutually Exclusive, then $\mathrm{P(E \cup F)=P(E)+P(F)}$

The Complement Rule

If E^c is the Complement of Event E, then $\mathrm{P}(\mathrm{E}^c)=1-\mathrm{P(E)}$

3.2.1: Mutually Exclusive Events and the Addition Rule (Exercises)

SECTION 8.2 PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE

Determine whether the following pair of events are mutually exclusive.

1) A = {A person earns more than $25,000} B = {A person earns less than $20,000}	2) A card is drawn from a deck. C = {It is a King} D = {It is a heart}.
3) A die is rolled. E = {An even number shows} F = {A number greater than 3 shows}	4) Two dice are rolled. G = {The sum of dice is 8} H = {One die shows a 6}
5) Three coins are tossed. I = {Two heads come up} J = {At least one tail comes up}	6) A family has three children. K = {First born is a boy} L = {The family has children of both sexes}

Use the Addition Rule to find the following probabilities.

7) A card is drawn from a deck. Events C and D are: C = {It is a king} D = {It is a heart} Find P(C or D).	8) A die is rolled. The events E and F are: E = {An even number shows} F = {A number greater than 3 shows} Find P(E or F).
9) Two dice are rolled. Events G and H are: G = {The sum of dice is 8} H ={Exactly one die shows a 6} Find P(G or H).	10) Three coins are tossed. Events I and J are: I = {Two heads come up} J = {At least one tail comes up} Find P(I or J).
11) At a college, 20% of the students take Finite Mathematics, 30% take Statistics and 10% take both. What percent of students take Finite Mathematics or Statistics?	12) This quarter, there is a 50% chance that Jason will pass Accounting, a 60% chance that he will pass English, and 80% chance that he will pass at least one of these two courses. What is the probability that he will pass both Accounting and English?

Questions 13 - 20 refer to the following: The table shows the distribution of Democratic and Republican U.S by gender in the 114^{th} Congress as of January 2015.

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities.

13) P(M and D)	14) P(F and R)
15) P(M or D)	16) P(F or R)
17) P(Mc or R)	18) P(M or F)
19) Are the events F, R mutually exclusive? Use probabilities to support your conclusions.	20) Are the events F, T mutually exclusive? Use probabilities to support your conclusion.

SECTION 8.2 PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE

Use the Addition Rule to find the following probabilities.

21) If P(E) = .5 , P(F) = .4 , E and F are mutually exclusive, find P(E and F).	22) If P(E) = .4 , P(F) = .2 , E and F are mutually exclusive, find P(E or F).
23) If P(E) = .3, P(E or F) = .6 , P(E and F) = .2, find P(F).	24) If P(E) = .4, P(F) = .5 , P(E or F) = .7, find P(E and F).

25) In a box of assorted cookies, 36% of cookies contain chocolate and 12% of cookies contain nuts. 8% of cookies have both chocolats and nuts. Sean is allergic to chocolate and nuts. Find the probability that a cookie has chocolate chips **or** nuts (he can't eat it).	26) At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper. Find the probability that a course requires a final exam **or** a research paper.

Questions 25 and 26 are adapted from Introductory Statistics from OpenStax under a creative Commons Attribution 3.0 Unported License, available for download free atcnx.org/content/col11562/latest u

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3.3: Probability Using Tree Diagrams and Combinations

Learning Objectives

In this section, you will learn to:

1. Use probability tree diagrams to calculate probabilities
2. Use combinations to calculate probabilities

In this section, we will apply previously learnt counting techniques in calculating probabilities, and use tree diagrams to help us gain a better understanding of what is involved.

USING TREE DIAGRAMS TO CALCULATE PROBABILITIES

We already used tree diagrams to list events in a sample space. Tree diagrams can be helpful in organizing information in probability problems; they help provide a structure for understanding probability. In this section we expand our previous use of tree diagrams to situations in which the events in the sample space are not all equally likely.

We assign the appropriate probabilities to the events shown on the branches of the tree.
By multiplying probabilities along a path through the tree, we can find probabilities for "and" events, which are intersections of events.

We begin with an example.

Example $3.3.1$

Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn with replacement, what is the probability that both marbles are red?

Solution

Let E be the event that the first marble drawn is red, and let F be the event that the second marble drawn is red.

We need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$.

By the statement, "two marbles are drawn with replacement," we mean that the first marble is replaced before the second marble is drawn.

There are 7 choices for the first draw. And since the first marble is replaced before the second is drawn, there are, again, seven choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 49 ordered pairs. Of the 49 ordered pairs, there are $3 \times 3=9$ ordered pairs that show red on the first draw and, also, red on the second draw. Therefore,

$$P(E \cap F)=\frac{9}{49}$$

Further note that in this particular case

$$P(E \cap F)=\frac{9}{49}=\frac{3}{7} \cdot \frac{3}{7}$$

giving us the result that in this example: $\mathbf{P}(\mathbf{E} \cap \mathbf{F})=\mathbf{P}(\mathbf{E}) \cdot \mathbf{P}(\mathbf{F})$

Example $3.3.2$

If in Example $3.3.1$, the two marbles are drawn without replacement, then what is the probability that both marbles are red?

Solution

By the statement, "two marbles are drawn without replacement," we mean that the first marble is not replaced before the second marble is drawn.

Again, we need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$.

There are, again, 7 choices for the first draw. And since the first marble is not replaced before the second is drawn, there are only six choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 42 ordered pairs. Of the 42 ordered pairs, there are $3 \times 2 = 6$ ordered pairs that show red on the first draw and red on the second draw. Therefore,

$$P(E \cap F) = \frac{6}{42}$$

Note that we can break this calculation down as

$$P(E \cap F) = \frac{6}{42} = \frac{3}{7} \cdot \frac{2}{6}$$

.

Here 3/7 represents $\mathrm{P(E)}$, and 2/6 represents the probability of drawing a red on the second draw, given that the first draw resulted in a red.

We write the latter as P(red on the second | red on first) or $\mathrm{P(F|E)}$. The "|" represents the word "given" or "if". This leads to the result that:

$$\mathbf{P(E \cap F) = P(E) \cdot P(F|E)}$$

The is an important result, called the **Multiplication Rule,** which will appear again in later sections.

We now demonstrate the above results with a tree diagram.

Example 3.3.3

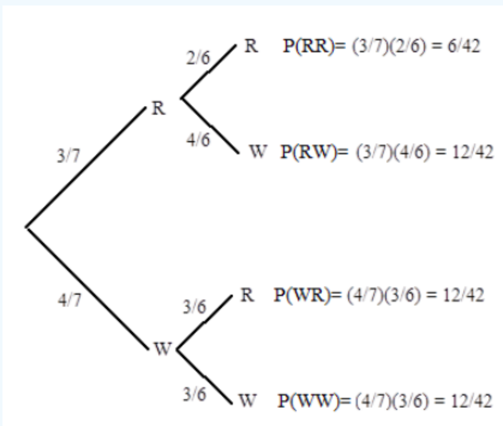
Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn without replacement, find the following probabilities using a tree diagram.

a. The probability that both marbles are red.
b. The probability that the first marble is red and the second white.
c. The probability that one marble is red and the other white.

Solution

Let R be the event that the marble drawn is red, and let W be the event that the marble drawn is white.

We draw the following tree diagram.



a. The probability that both marbles are red is $\mathrm{P(RR)} = 6/42$
b. The probability that the first marble is red and the second is white is $\mathrm{P(RW)} = 12/42$
c. For the probability that one marble is red and the other is white, we observe that this can be satisfied if the first is red and the second is white, **or** if the first is white and the second is red. The "or" tells us we'll be using the Addition Rule from Section 7.2.

Furthermore events RW and WR are mutually exclusive events, so we use the form of the Addition Rule that applies to mutually exclusive events.

Therefore

P(one marble is red and the other marble is white)

$$\begin{aligned}
&= \mathrm{P}(\mathrm{RW} \text{ or } \mathrm{WR}) \\
&= \mathrm{P}(\mathrm{RW}) + \mathrm{P}(\mathrm{WR}) \\
&= 12/42 + 12/42 = 24/42
\end{aligned}$$

USING COMBINATIONS TO FIND PROBABILITIES

Although the tree diagrams give us better insight into a problem, they are not practical for problems where more than two or three things are chosen. In such cases, we use the concept of combinations that we learned in the last chapter. This method is best suited for problems where the order in which the objects are chosen is not important, and the objects are chosen without replacement.

Example 3.3.4

Suppose a jar contains 3 red, 2 white, and 3 blue marbles. If three marbles are drawn without replacement, find the following probabilities.

a. P(Two red and one white)
b. P(One of each color)
c. P(None blue)
d. P(At least one blue)

Solution

Let us suppose the marbles are labeled as $R_1, R_2, R_3, W_1, W_2, B_1, B_2, B_3$.

a. P(Two red and one white)

Since we are choosing 3 marbles from a total of 8, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{C}2 \times 2\mathrm{C}1 = 6$ combinations consisting of 2 red and one white. Therefore,

$$P(\text{ Two red and one white }) = \frac{3\mathrm{C}2 \times 2\mathrm{C}1}{8\mathrm{C}3} = \frac{6}{56}.$$

b. P(One of each color)

Again, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{Cl} \times 2\mathrm{Cl} \times 3\mathrm{Cl} = 18$ combinations consisting of one red, one white, and one blue. Therefore,

$$P(\text{ One of each color }) = \frac{3\mathrm{C}1 \times 2\mathrm{C}1 \times 3\mathrm{C}1}{8\mathrm{C}3} = \frac{18}{56}$$

c. P(None blue)

There are 5 non-blue marbles, therefore

$$\mathrm{P}(\text{ None blue }) = \frac{5\mathrm{C}3}{8\mathrm{C}3} = \frac{10}{56} = \frac{5}{28}$$

d. P(At least one blue)

By "at least one blue marble," we mean the following: one blue marble and two non-blue marbles, *OR* two blue marbles and one non-blue marble, *OR* all three blue marbles. So we have to find the sum of the probabilities of all three cases.

$$\mathrm{P}(\text{At least one blue }) = \mathrm{P}(1 \text{ blue, } 2 \text{ non-blue }) + \mathrm{P}(2 \text{ blue, } 1 \text{ non-blue}) + \mathrm{P}(3 \text{ blue})$$

$$P(\text{ At least one blue }) = \frac{3\mathrm{C}1 \times 5\mathrm{C}2}{8\mathrm{C}3} + \frac{3\mathrm{C}2 \times 5\mathrm{C}1}{8\mathrm{C}3} + \frac{3\mathrm{C}3}{8\mathrm{C}3}$$

$$P(\text{ At least one blue }) = 30/56 + 15/56 + 1/56 = 46/56 = 23/28$$

Alternately, we can use the fact that $\mathrm{P(E)}=1-\mathrm{P(E^{c})}$. If the event E = At least one blue, then E^{c} = None blue.

But from part c of this example, we have $(\mathrm{E}^{c})=5/28$, so $\mathrm{P(E)}=1-5/28=23/28$.

Example 3.3.5

Five cards are drawn from a deck. Find the probability of obtaining two pairs, that is, two cards of one value, two of another value, and one other card.

Solution

Let us first do an easier problem-the probability of obtaining a pair of kings and queens.

Since there are four kings, and four queens in the deck, the probability of obtaining two kings, two queens and one other card is

$$\mathrm{P}(\text{A pair of kings and queens})=\frac{4\mathrm{C}2\times 4\mathrm{C}2\times 44\mathrm{C}1}{52\mathrm{C}5}$$

To find the probability of obtaining two pairs, we have to consider all possible pairs.

Since there are altogether 13 values, that is, aces, deuces, and so on, there are 13C2 different combinations of pairs.

$$P(\text{ Two pairs })=13\mathrm{C}2\cdot\frac{4\mathrm{C}2\times 4\mathrm{C}2\times 44\mathrm{C}1}{52\mathrm{C}5}=.04754$$

Example 3.3.6

A cell phone store receives a shipment of 15 cell phones that contains 8 iPhones and 7 Android phones. Suppose that 6 cell phones are randomly selected from this shipment. Find the probability that a randomly selected set of 6 cell phones consists of 2 iPhones and 4 Android phones.

Solution

There are 8C2 ways of selecting 2 out of the 8 iPhones.

and 7C4 ways of selecting 4 out of the 7 Android phones

But altogether there are 15C6 ways of selecting 6 out of 15 cell phones.

Therefore we have

$$P(\text{2 iPhones and 4 Android phones})=\frac{8\mathrm{C}2\times 7\mathrm{C}4}{15\mathrm{C}6}=\frac{(28)(35)}{5005}=\frac{980}{5005}=0.1958$$

Example 3.3.7

One afternoon, a bagel store still has 53 bagels remaining: 20 plain, 15 poppyseed, and 18 sesame seed bagels. Suppose that the store owner packages up a bag of 9 bagels to bring home for tomorrow's breakfast, and selects the bagels randomly. Find the probability that the bag contains 4 plain, 3 poppyseed, and 2 sesame seed.

Solution

There are 20C4 ways of selecting 4 out of the 20 plain bagels,

and 15C3 ways of selecting 3 out of the 15 poppyseed bagels,

and 18C2 ways of selecting 2 out of the 18 sesame seed bagels.

But altogether there are 53C9 ways of selecting 9 out of the 53 bagels.

$$\begin{aligned}\mathrm{P}(\text{4 plain, 3 poppyseed, and 2 sesame seed}) &=\frac{20\mathrm{C}4\times 15\mathrm{C}3\times 18\mathrm{C}2}{53\mathrm{C}9}\\ &=\frac{(4845)(455)(153)}{4431613550}\\ &=0.761\end{aligned}\tag{3.3.1}$$

We end the section by solving a famous problem called the **Birthday Problem**.

Example 3.3.8: Birthday Problem

If there are 25 people in a room, what is the probability that at least two people have the same birthday?

Solution

Let event E represent that at least two people have the same birthday.

We first find the probability that no two people have the same birthday.

We analyze as follows.

Suppose there are 365 days to every year. According to the multiplication axiom, there are 365^{25} possible birthdays for 25 people. Therefore, the sample space has 365^{25} elements. We are interested in the probability that no two people have the same birthday. There are 365 possible choices for the first person and since the second person must have a different birthday, there are 364 choices for the second, 363 for the third, and so on. Therefore,

$$\mathrm{P}(\text{No two have the same birthday }) = \frac{365 \cdot 364 \cdot 363 \cdots 341}{365^{25}} = \frac{365\mathrm{P}25}{365^{25}}$$

Since P(at least two people have the same birthday) = 1 - P(No two have the same birthday),

$$\mathrm{P}\text{ at least two people have the same birthday }) = 1 - \frac{365\mathrm{P}25}{365^{25}} = .5687$$

Page 3.3.1: Probability Using Tree Diagrams and Combinations (Exercises)

SECTION 8.3 PROBLEM SET: PROBABILITIES USING TREE DIAGRAMS AND COMBINATIONS

Two apples are chosen from a basket containing five red and three yellow apples.
Draw a tree diagram below, and find the following probabilities.

1) P(both red)	2) P(one red, one yellow)
3) P(both yellow)	4) P(First red and second yellow)

A basket contains six red and four blue marbles. Three marbles are drawn at random.
Find the following probabilities using the method shown in Example 8.3.2. Do not use combinations.

5) P(All three red)	6) P(two red, one blue)
7) P(one red, two blue)	8) P(first red, second blue, third red)

Three marbles are drawn from a jar containing five red, four white, and three blue marbles.
Find the following probabilities using combinations.

9) P(all three red)	10) P(two white and 1 blue)
11) P(none white)	12) P(at least one red)

A committee of four is selected from a total of 4 freshmen, 5 sophomores, and 6 juniors. Find the probabilities for the following events.

13) At least three freshmen.	14) No sophomores.
15) All four of the same class.	16) Not all four from the same class.
17) Exactly three of the same class.	18) More juniors than freshmen and sophomores combined.

Five cards are drawn from a deck. Find the probabilities for the following events.

19) Two hearts, two spades, and one club.	20) A flush of any suit (*all cards of a single suit*).
21) A full house of nines and tens (*3 nines and 2 tens*).	22) Any full house.
23) A pair of nines and a pair of tens (*and the fifth card is not a nine or ten*).	24) Any two pairs (*two cards of one value, two more cards of another value, and the fifth card does not have the same value as either pair*).

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises.
He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

Find the following probabilities:

25) P(playlist has 2 songs of each type)	26) P(playlist has no country songs)
27) P(playlist has 3 rock, 2 rap, and 1 country song)	28) P(playlist has 3 or 4 rock songs and the rest are rap songs)

A project is staffed 12 people: 5 engineers, 4 salespeople, and 3 customer service representatives.
A committee of 5 people is selected to make a presentation to senior management.

Find the probabilities of the following events.

29) The committee has 2 engineers, 2 salespeople, and 1 customer service representative.	30) The committee contains 3 engineer and 2 salespeople.
31) The committee has no engineers.	32) The committee has all salespeople.

Do the following birthday problems.

33) If there are 5 people in a room, what is the probability that no two have the same birthday?	34) If there are 5 people in a room, find the probability that at least 2 have the same birthday.

3.4: Conditional Probability

Learning Objectives

In this section, you will learn to:

1. recognize situations involving conditional probability
2. calculate conditional probabilities

Suppose a friend asks you the probability that it will snow today.

If you are in Boston, Massachusetts in the winter, the probability of snow today might be quite substantial. If you are in Cupertino, California in summer, the probability of snow today is very tiny, this probability is pretty much 0.

Let:

- A = the event that it will snow today
- B = the event that today you are in Boston in wintertime
- C = the event that today you are in Cupertino in summertime

Because the probability of snow is affected by the location and time of year, we can't just write $P(A)$ for the probability of snow. We need to indicate the other information we know -location and time of year. We need to use **conditional probability**.

The event we are interested in is event A for snow. The other event is called the condition, representing location and time of year in this case.

We represent conditional probability using a vertical line | that means "if", or "given that", or "if we know that". The event of interest appears on the left of the |. The condition appears on the right side of the |.

The probability it will snow given that (if) you are in Boston in the winter is represented by $\mathbf{P}\left(\mathbf{A} \mid \mathbf{B}\right)$. In this case, the condition is B.

The probability that it will snow given that (if) you are in Cupertino in the summer is represented by $\mathbf{P}(\mathbf{A} \mid \mathbf{C})$. In this case, the condition is C.

Now, let's examine a situation where we can calculate some probabilities.

Suppose you and a friend play a game that involves choosing a single card from a well-shuffled deck. Your friend deals you one card, face down, from the deck and offers you the following deal: If the card is a king, he will pay you \$5, otherwise, you pay him \$1. Should you play the game?

You reason in the following manner. Since there are four kings in the deck, the probability of obtaining a king is 4/52 or 1/13. So, probability of not obtaining a king is 12/13. This implies that the ratio of your winning to losing is 1 to 12, while the payoff ratio is only \$1 to \$5. Therefore, you determine that you should not play.

But consider the following scenario. While your friend was dealing the card, you happened to get a glance of it and noticed that the card was a face card. Should you, now, play the game?

Since there are 12 face cards in the deck, the total elements in the sample space are no longer 52, but just 12. This means the chance of obtaining a king is 4/12 or 1/3. So your chance of winning is 1/3 and of losing 2/3. This makes your winning to losing ratio 1 to 2 which fares much better with the payoff ratio of \$1 to \$5. This time, you determine that you should play.

In the second part of the above example, we were finding the probability of obtaining a king knowing that a face card had shown. This is an example of **conditional probability**. Whenever we are finding the probability of an event E under the condition that another event F has happened, we are finding conditional probability.

The symbol $P(E \mid F)$ denotes the problem of finding the probability of E given that F has occurred. We read $P(E \mid F)$ as "the probability of E, given F."

Example 3.4.1

A family has three children. Find the conditional probability of having two boys and a girl given that the first born is a boy.

Solution

Let event E be that the family has two boys and a girl, and F that the first born is a boy.

First, we the sample space for a family of three children as follows.

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$

Since we know the first born is a boy, our possibilities narrow down to four outcomes: BBB, BBG, BGB, and BGG.

Among the four, BBG and BGB represent two boys and a girl.

Therefore, $\mathrm{P}(\mathrm{E} | \mathrm{F})$ = 2/4 or 1/2.

Example 3.4.2

One six sided die is rolled once.

a. Find the probability that the result is even.
b. Find the probability that the result is even given that the result is greater than three.

Solution

The sample space is $\mathrm{S} = 1, 2, 3, 4, 5, 6$

Let event E be that the result is even and T be that the result is greater than 3.

a. $\mathrm{P}(\mathrm{E})$ = 3/6 because $\mathrm{E} = 2, 4, 6$

b. Because $T = 4, 5, 6$, we know that 1, 2, 3 cannot occur; only outcomes 4, 5, 6 are possible. Therefore of the values in E, only 4, 6 are possible.

Therefore, $\mathrm{P}(\mathrm{E} | \mathrm{T})$ = 2/3

Example 3.4.3

A fair coin is tossed twice.

a. Find the probability that the result is is two heads.
b. Find the probability that the result is two heads given that at least one head is obtained.

Solution

The sample space is $S = HH, HT, TH, TT$

Let event E be that the two heads are obtained and F be at least one head is obtained

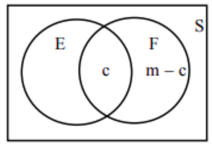
a. $\mathrm{P}(\mathrm{E})$ = 1/4 because $\mathrm{E} = HH$ and the sample space S has 4 outcomes.

b. $\mathrm{F} = HH, HT, TH$. Since at least one head was obtained, TT did not occur.
We are interested in the probability event $\mathrm{E} = HH$ out of the 3 outcomes in the reduced sample space F.

Therefore, $\mathrm{P}(\mathrm{E} | \mathrm{F})$ = 1/3

Let us now develop a formula for the conditional probability $\mathrm{P}(\mathrm{E} | \mathrm{F})$.

Suppose an experiment consists of n equally likely events. Further suppose that there are m elements in F, and c elements in $\mathrm{E} \cap \mathrm{F}$, as shown in the following Venn diagram.



If the event F has occurred, the set of all possible outcomes is no longer the entire sample space, but instead, the subset F. Therefore, we only look at the set F and at nothing outside of F. Since F has m elements, the denominator in the calculation of $\mathrm{P(E|F)}$ is m. We may think that the numerator for our conditional probability is the number of elements in E. But clearly we cannot consider the elements of E that are not in F. We can only count the elements of E that are in F, that is, the elements in $\mathrm{E \cap F}$. Therefore,

$$\mathrm{P(E|F)} = \frac{\mathrm{c}}{\mathrm{m}}$$

Dividing both the numerator and the denominator by n, we get

$$\mathrm{P(E|F)} = \frac{c/n}{m/n}$$

But $c/n = \mathrm{P(E \cap F)}$, and $m/n = \mathrm{P(F)}$.

Substituting, we derive the following formula for $\mathrm{P(E|F)}$.

Conditional Probability Rule

For two events E and F, the probability of "E Given F" is

$$\mathbf{P(E|F)} = \frac{\mathbf{P(E \cap F)}}{\mathbf{P(F)}}$$

Example $3.4.4$

A single die is rolled. Use the above formula to find the conditional probability of obtaining an even number given that a number greater than three has shown.

Solution

Let E be the event that an even number shows, and F be the event that a number greater than three shows. We want $\mathrm{P(E|F)}$.

$\mathrm{E} = 2, 4, 6$ and $\mathrm{F} = 4, 5, 6$. Which implies, $\mathrm{E \cap F} = 4, 6$

Therefore, $\mathrm{P(F)}$ = 3/6, and $\mathrm{P(E \cap F)}$ = 2/6

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

Example $3.4.5$

The following table shows the distribution by gender of students at a community college who take public transportation and the ones who drive to school.

	Male(M)	Female(F)	Total
Public Transportation(T)	8	13	21
Drive(D)	39	40	79
Total	47	53	100

The events M, F, T, and D are self explanatory. Find the following probabilities.

a. $\mathrm{P(D|M)}$
b. $\mathrm{P(F|D)}$
c. $\mathrm{P(M|T)}$

Solution 1

Conditional probabilities can often be found directly from a contingency table. If the condition corresponds to only one row or only one column in the table, then you can ignore the rest of the table and read the conditional probability directly from the row or column indicated by the condition.

a. The condition is event M; we can look at only the "Male" column of the table and ignore the rest of the table: $\mathrm{P(D|M)}=\frac{39}{47}$.
b. The condition is event D; we can look at only the "Drive" row of the table and ignore the rest of the table: $\mathrm{P(F|D)}=\frac{40}{79}$.
c. The condition is event T; we can look at only the "Public Transportation" row of the table and ignore the rest of the table: $\mathrm{P(M|T)}=\frac{8}{21}$.

Solution 2

We use the conditional probability formula $\mathrm{P(E|F)}=\frac{\mathrm{P(E\cap F)}}{\mathrm{P(F)}}$.

a.
$$P(D|M)=\frac{P(D\cap M)}{P(M)}=\frac{39/100}{47/100}=\frac{39}{47}.$$

b.
$$P(F|D)=\frac{P(F\cap D)}{P(D)}=\frac{40/100}{79/100}=\frac{40}{79}.$$

c.
$$\mathrm{P(M|T)}=\frac{P(M\cap T)}{P(T)}=\frac{8/100}{21/100}=\frac{8}{21}$$

✓ Example 3.4.6

Given $\mathrm{P(E)}$ = .5, $\mathrm{P(F)}$ = .7, and $\mathrm{P(E\cap F)}$ = .3. Find the following:

a. $\mathrm{P(E|F)}$
b. $\mathrm{P(F|E)}$

Solution

We use the conditional probability formula.

a. $P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{3}{7}=\frac{3}{7}$
b. $P(F|E)=\frac{P(E\cap F)}{P(E)}=.3/.5=3/5$

✓ Example 3.4.7

E and F are mutually exclusive events such that $\mathrm{P(E)}$ = .4, $\mathrm{P(F)}$ = .9. Find $\mathrm{P(E|F)}$.

Solution

E and F are mutually exclusive, so $\mathrm{P(E\cap F)}$ = 0.
Therefore $\mathrm{P(E|F)}=\frac{\mathrm{P(E\cap F)}}{\mathrm{P(F)}}=\frac{0}{9}=0$.

✓ Example 3.4.8

Given $\mathrm{P(F|E)}$ = .5, and $\mathrm{P(E\cap F)}$ = .3. Find $\mathrm{P(E)}$.

Solution

Using the conditional probability formula $\mathrm{P(E|F)}=\frac{\mathrm{P(E\cap F)}}{\mathrm{P(F)}}$, we get

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

Substituting and solving:

$$.5 = \frac{.3}{P(E)} \quad \text{or} \quad P(E) = 3/5$$

Example 3.4.9

In a family of three children, find the conditional probability of having two boys and a girl, given that the family has at least two boys.

Solution

Let event E be that the family has two boys and a girl, and let F be the probability that the family has at least two boys. We want $P(E|F)$.

We list the sample space along with the events E and F.

$$\begin{array}{l} S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\} \\ E = \{BBG, BGB, GBB\} \text{ and } F = \{BBB, BBG, BGB, GBB\} \\ E \cap F = \{BBG, BGB, GBB\} \end{array}$$

Therefore, $P(F)$ = 4/8, and $P(E \cap F)$ = 3/8, and

$$P(E|F) = \frac{P(E \cap F)}{P(E)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

Example 3.4.10

At a community college 65% of the students subscribe to Amazon Prime, 50% subscribe to Netflix, and 20% subscribe to both. If a student is chosen at random, find the following probabilities:

a. the student subscribes to Amazon Prime given that he subscribes to Netflix
b. the student subscribes to Netflix given that he subscribes to Amazon Prime

Solution

Let A be the event that the student subscribes to Amazon Prime, and N be the event that the student subscribes to Netflix.

First identify the probabilities and events given in the problem.

P(student subscribes to Amazon Prime) = $P(A)$ = 0.65

P(student subscribes to Netflix) = $P(N)$ = 0.50

P(student subscribes to both Amazon Prime and Netflix) = $P(A \cap N)$ = 0.20

Then use the conditional probability rule:

a. $P(A|N) = \frac{P(A \cap N)}{P(N)} = \frac{.20}{.50} = \frac{2}{5}$
b. $P(N|A) = \frac{P(A \cap N)}{P(A)} = \frac{.20}{.65} = \frac{4}{13}$

3.4.1: Conditional Probability (Exercises)

SECTION 8.4 PROBLEM SET: CONDITIONAL PROBABILITY

Questions 1 - 4: Do these problems using the conditional probability formula: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

1. A card is drawn from a deck. Find the conditional probability of P(a queen | a face card).
2. A card is drawn from a deck. Find the conditional probability of P(a queen | a club).
3. A die is rolled. Find the conditional probability that it shows a three if it is known that an odd number has shown.
4. If $P(A)$ = .3 , $P(B)$ = .4, $P(A$ and $B)$ = .12, find:
   a. $P(A|B)$
   b. $P(B|A)$

Questions 5 - 8 refer to the following: The table shows the distribution of Democratic and Republican U.S. Senators by gender in the 114th Congress as of January 2015.

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities:

5. $P(M|D)$
6. $P(D|M)$
7. $P(F|R)$
8. $P(R|F)$

Do the following conditional probability problems.

9. At a college, 20% of the students take Finite Math, 30% take History, and 5% take both Finite Math and History. If a student is chosen at random, find the following conditional probabilities.
   a. He is taking Finite Math given that he is taking History.
   b. He is taking History assuming that he is taking Finite Math.
10. At a college, 60% of the students pass Accounting, 70% pass English, and 30% pass both of these courses. If a student is selected at random, find the following conditional probabilities.
    a. He passes Accounting given that he passed English.
    b. He passes English assuming that he passed Accounting.
11. If $P(F) = .4$, $P(E|F) = .3$, find $P(E$ and $F)$.
12. $P(E) = .3$, $P(F) = .3$; E and F are mutually exclusive. Find $P(E|F)$.
13. If $P(E) = .6$, $P(E$ and $F)$ = .24, find $P(F|E)$.
14. If $P(E$ and $F)$ = .04, $P(E|F) = .1$, find $P(F)$.

At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper.

15. Find the probability that a course has a final exam given that it has a research paper.
16. Find the probability that a course has a research paper if it has a final exam.

SECTION 8.4 PROBLEM SET: CONDITIONAL PROBABILITY

Consider a family of three children. Find the following probabilities.

17. P(two boys | first born is a boy)	18. P(all girls | at least one girl is born)
19. P(children of both sexes | first born is a boy)	20. P(all boys | there are children of both sexes)

Questions 21 - 26 refer to the following:
The table shows highest attained educational status for a sample of US residents age 25 or over:

	(D) Did not Complete High School	(H) High School Graduate	(C) Some College	(A) Associate Degree	(B) Bachelor Degree	(G) Graduate Degree	TOTAL
25-44 (R)	95	228	143	81	188	61	796
45-64 (S)	83	256	136	80	150	67	772
65+ (T)	96	191	84	36	80	41	528
Total	274	675	363	197	418	169	2096

Use this table to determine the following probabilities:

21. $P(C\|T)$	22. $P(S\|A)$	23. $P(CandT)$
24. $P(R\|B)$	25. $P(B\|R)$	26. $P(G\|S)$

Page 3.5: Independent Events

Learning Objectives

In this section, you will:

1. Define independent events
2. Identify whether two events are independent or dependent

In the last section, we considered conditional probabilities. In some examples, the probability of an event changed when additional information was provided. This is not always the case. The additional information may or may not alter the probability of the event.

In Example $Page3.5.1$ we revisit the discussion at the beginning of the previous section and then contrast that with Example $Page3.5.2$.

Example $Page3.5.1$

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that the card is a face card.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | The card is a face card), we reason as follows:

There are 12 face cards in a deck of cards. There are 4 kings in a deck of cards.

P(The card is a king | The card is a face card) = 4/12 = 1/3.

The reader should observe that in the above example,

P(The card is a king | The card is a face card) $\neq$ P(The card is a king)

In other words, the additional information, knowing that the card selected is a face card changed the probability of obtaining a king.

Example $Page3.5.2$

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that a red card has shown.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | A red card has shown), we reason as follows:

Since a red card has shown, there are only twenty six possibilities. Of the 26 red cards, there are two kings. Therefore,

P(The card is a king | A red card has shown) = 2/26 = 1/13.

The reader should observe that in the above example,

P(The card is a king | A red card has shown) = P(The card is a king)

In other words, the additional information, a red card has shown, did not affect the probability of obtaining a king.

Whenever the probability of an event E is not affected by the occurrence of another event F, and vice versa, we say that the two events E and F are **independent.** This leads to the following definition.

Definition: Independent

Two Events E and F are **independent** if and only if at least one of the following two conditions is true.

1. $\mathbf{P(E|F) = P(E)}$ or
2. $\mathbf{P(F|E) = P(F)}$

If the events are not independent, then they are dependent.

If one of these conditions is true, then both are true.

We can use the definition of independence to determine if two events are independent.

We can use that definition to develop another way to test whether two events are independent.

Recall the conditional probability formula:

$$\mathrm{P(E|F) = \frac{P(E \cap F)}{P(F)}}$$

Multiplying both sides by $\mathrm{P(F)}$, we get

$$\mathrm{P(E \cap F) = P(E|F)P(F)}$$

Now if the two events are independent, then by definition

$$\mathrm{P(E|F) = P(E)}$$

Substituting, $P(E \cap F) = P(E)P(F)$

We state it formally as follows.

Test For Independence

Two events E and F are independent if and only if

$$\mathbf{P(E \cap F) = P(E)P(F)}$$

In the Examples $Page 3.5.3$ and $Page 3.5.4$, we'll examine how to check for independence using both methods:

- Examine the probability of intersection of events to check whether $P(E \cap F) = P(E)P(F)$
- Examine conditional probabilities to check whether $P(E|F) = P(E)$ or $P(F|E) = P(F)$

We need to use only **one** of these methods. Both methods, if used properly, will always give results that are consistent with each other.

Use the method that seems easier based on the information given in the problem.

Example $Page 3.5.3$

The table below shows the distribution of color-blind people by gender.

	Male(M)	Female(F)	Total
Color-Blind(C)	6	1	7
Not Color-Blind(N)	46	47	93
Total	52	48	100

where M represents male, F represents female, C represents color-blind, and N represents not color-blind. Are the events color-blind and male independent?

Solution 1: According to the test for independence, C and M are independent if and only if $\mathrm{P(C \cap M) = P(C)P(M)}$.

From the table: $P(C)$ = 7/100, $P(M)$ = 52/100 and $P(C \cap M)$ = 6/100

So $P(C)P(M)$ = (7/100)(52/100) = .0364

which is **not** equal to $P(C \cap M)$ = 6/100 = .06

Therefore, the two events are not independent. We may say they are dependent.

Solution 2: C and M are independent if and only if $P(C|M) = P(C)$.

From the total column $P(C)$ = 7/100 = 0.07

From the male column $P(C|M)$ = 6/52= 0.1154

Therefore $P(C|M) \neq P(C)$, indicating that the two events are not independent.

Example $Page3.5.4$

In a city with two airports, 100 flights were surveyed. 20 of those flights departed late.

- 45 flights in the survey departed from airport A; 9 of those flights departed late.
- 55 flights in the survey departed from airport B; 11 flights departed late.

Are the events "depart from airport A" and "departed late" independent?

Solution 1

Let A be the event that a flight departs from airport A, and L the event that a flight departs late. We have

$P(A \cap L)$ = 9/100, $P(A)$ = 45/100 and $P(L)$ = 20/100

In order for two events to be independent, we must have $P(A \cap L) = P(A)P(L)$

Since $P(A \cap L)$ = 9/100 = 0.09

and $P(A)P(L)$ = (45/100)(20/100) = 900/10000 = 0.09

the two events "departing from airport A" and "departing late" are independent.

Solution 2

The definition of independent events states that two events are independent if $P(E|F) = P(E)$.

In this problem we are given that

$P(L|A)$ = 9/45= 0.2 and $P(L)$ = 20/100 = 0.2

$P(L|A) = P(L)$, so events "departing from airport A" and "departing late" are independent.

Example $Page3.5.5$

A coin is tossed three times, and the events E, F and G are defined as follows:

E: The coin shows a head on the first toss.

F: At least two heads appear.

G: Heads appear in two successive tosses.

Determine whether the following events are independent.

a. E and F
b. F and G
c. E and G

Solution

We list the sample space, the events, their intersections and the probabilities.

$$\begin{array}{ll} \mathrm{S}=\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT} & \\ \mathrm{E}=\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}\}, & \mathrm{P}(\mathrm{E})=4/8 \text{ or } 1/2 \\ \mathrm{F}=\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{THH}\}, & \mathrm{P}(\mathrm{F})=4/8 \text{ or } 1/2 \\ \mathrm{G}=\{\mathrm{HHT}, \mathrm{THH}\}, & \mathrm{P}(\mathrm{G})=2/8 \text{ or } 1/4 \\ \mathrm{E}\cap\mathrm{F}=\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}\}, & \mathrm{P}(\mathrm{E}\cap\mathrm{F})=3/8 \\ \mathrm{F}\cap\mathrm{G}=\{\mathrm{HHT}, \mathrm{THH}\}, & \mathrm{P}(\mathrm{F}\cap\mathrm{G})=2/8 \text{ or } 1/4 \\ \mathrm{E}\cap\mathrm{G}=\{\mathrm{HHT}\} & \mathrm{P}(\mathrm{E}\cap\mathrm{G})=1/8 \end{array}$$

a. E and F will be independent if and only if $P(E\cap F)=P(E)P(F)$

$$P(E\cap F)=3/8 \text{ and } P(E)P(F)=1/2\cdot 1/2=1/4.$$

Since 3/8 ≠ 1/4, we have $P(E\cap F)\neq P(E)P(F)$.

Events E and F are not independent.

b. F and G will be independent if and only if $P(F\cap G)=P(F)P(G)$.

$$P(F\cap G)=1/4 \text{ and } P(F)P(G)=1/2\cdot 1/4=1/8.$$

Since 3/8 ≠ 1/4, we have $P(F\cap G)\neq P(F)P(G)$.

Events F and G are not independent.

c. E and G will be independent if $P(E\cap G)=P(E)P(G)$

$$P(E\cap G)=1/8 \text{ and } P(E)P(G)=1/2\cdot 1/4=1/8$$

Events E and G are independent events because $P(E\cap G)=P(E)P(G)$

Example $Page3.5.6$

The probability that Jaime will visit his aunt in Baltimore this year is .30, and the probability that he will go river rafting on the Colorado river is .50. If the two events are independent, what is the probability that Jaime will do both?

Solution

Let A be the event that Jaime will visit his aunt this year, and R be the event that he will go river rafting.

We are given $P(A)$ = .30 and $P(R)$ = .50, and we want to find $P(A\cap R)$.

Since we are told that the events A and R are independent,

$$P(A\cap R)=P(A)P(R)=(.30)(.50)=.15$$

Example $Page3.5.7$

Given $P(B|A)=.4$. If A and B are independent, find $P(B)$.

Solution

If A and B are independent, then by definition $P(B|A)=P(B)$

Therefore, $P(B)=.4$

Example $Page3.5.8$

Given $P(A)=.7$, $P(B|A)=.5$. Find $P(A\cap B)$.

Solution 1

By definition $P(B|A)=\frac{P(A\cap B)}{P(A)}$

Substituting, we have

$$.5=\frac{\mathrm{P}(\mathrm{A}\cap\mathrm{B})}{.7}$$

Therefore, $P(A\cap B)=.35$

Solution 2

Again, start with $P(B|A)=\frac{P(A\cap B)}{P(A)}$

Multiplying both sides by $P(A)$ gives

$$P(A\cap B)=P(B|A)P(A)=(.5)(.7)=.35$$

Both solutions to Example $Page 3.5.8$ are actually the same, except that in Solution 2 we delayed substituting the values into the equation until after we solved the equation for $P(A\cap B)$. That gives the following result:

Multiplication Rule for events that are NOT independent

If events E and F are not independent

$$\mathbf{P(E\cap F)=P(E|F)P(F)}\quad\text{and}\quad\mathbf{P(E\cap F)=P(F|E)P(E)}$$

Example $Page 3.5.9$

Given $P(A)=.5$, $P(A\cup B)=.7$, if A and B are independent, find $P(B)$.

Solution

The addition rule states that

$$\mathrm{P(A\cup B)=P(A)+P(B)-P(A\cap B)}$$

Since A and B are independent, $P(A\cap B)=P(A)P(B)$

We substitute for $P(A\cap B)$ in the addition formula and get

$$\mathrm{P(A\cup B)=P(A)+P(B)-P(A)P(B)}$$

By letting $P(B)=x$, and substituting values, we get

$$\begin{aligned}.7&=.5+x-.5x\\.7&=.5+.5x\\.2&=.5x\\.4&=x\end{aligned}$$

Therefore, $P(B)=.4$

3.5.1: Independent Events (Exercises)

SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

The distribution of the number of fiction and non-fiction books checked out at a city's main library and at a smaller branch on a given day is as follows.

	MAIN (M)	BRANCH (B)	TOTAL
FICTION (F)	300	100	400
NON-FICTION (N)	150	50	200
TOTALS	450	150	600

Use this table to determine the following probabilities:

1. $P(F)$
2. $P(M \mid F)$
3. $P(N \mid B)$
4. Is the fact that a person checks out a fiction book independent of the main library? Use probabilities to justify your conclusion.

For a two-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

5. Find the following.
   a. $P(E)$
   b. $P(F)$
   c. $P(E \cap F)$
   d. Are E and F independent? Use probabilities to justify your conclusion.
6. Find the following.
   a. $P(F)$
   b. $P(G)$
   c. $P(F \cap G)$
   d. Are F and G independent? Use probabilities to justify your conclusion.

Do the following problems involving independence.

7. If $P(E)=.6$, $P(F)=.2$, and E and F are independent, find $P(E$ and $F)$.
8. If $P(E)=.6$, $P(F)=.2$, and E and F are independent, find $P(E$ or $F)$.
9. If $P(E)=.9$, $P(F \mid E)=.36$, and E and F are independent, find $P(F)$.
10. If $P(E)=.6$, $P(E$ or $F)$ = .8, and E and F are independent, find $P(F)$.
11. In a survey of 100 people, 40 were casual drinkers, and 60 did not drink. Of the ones who drank, 6 had minor headaches. Of the non-drinkers, 9 had minor headaches. Are the events "drinkers" and "had headaches" independent?
12. It is known that 80% of the people wear seat belts, and 5% of the people quit smoking last year. If 4% of the people who wear seat belts quit smoking, are the events, wearing a seat belt and quitting smoking, independent?
13. John's probability of passing statistics is 40%, and Linda's probability of passing the same course is 70%. If the two events are independent, find the following probabilities.
    a. $P($ both of them will pass statistics)
    b. $P($at least one of them will pass statistics)
14. Jane is flying home for the Christmas holidays. She has to change planes twice. There is an 80% chance that she will make the first connection, and a 90% chance that she will make the second connection. If the two events are independent, find the probabilities:
    a. $P($ Jane will make both connections)
    b. $P($Jane will make at least one connection)

For a three-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

15. Find the following.
    a. $P(E)$
    b. $P(F)$
    c. $P(E \cap F)$
    d. Are E and F independent?

16. Find the following.
    a. $P(F)$
    b. $P(G)$
    c. $P(F \cap G)$
    d. Are F and G independent?

SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

17. $P(K|D) = 0.7$, $P(D) = 0.25$ and $P(K) = 0.7$
    a. Are events K and D independent? Use probabilities to justify your conclusion.
    b. Find $P(K \cap D)$

18. $P(R|S) = 0.4$, $P(S) = 0.2$ and $P(R) = 0.3$
    a. Are events R and S independent? Use probabilities to justify your conclusion.
    b. Find $P(R \cap S)$

19. At a college:
    54% of students are female
    25% of students are majoring in engineering.
    15% of female students are majoring in engineering.
    Event E = student is majoring in engineering
    Event F = student is female
    a. Are events E and F independent? Use probabilities to justify your conclusion.
    b. Find $P(E \cap F)$

20. At a college:
    54% of all students are female
    60% of all students receive financial aid.
    60% of female students receive financial aid.
    Event A = student receives financial aid
    Event F = student is female
    a. Are events A and F independent? Use probabilities to justify your conclusion.
    b. Find $P(A \cap F)$

7.6: Binomial Probability

> **Learning Objectives**
>
> In this section, you will learn to:
>
> 1. Recognize when to use the binomial probability distribution
> 2. Derive the formula for the binomial probability distribution
> 3. Calculate probabilities for a binomial probability experiment

In this section, we consider problems that involve a sequence of trials, where each trial has only two outcomes, a *success* or a *failure*. These trials are independent, that is, the outcome of one does not affect the outcome of any other trial. The probability of success, p, and the probability of failure, $(1-p)$, remains the same throughout the experiment. These problems are called **binomial probability** problems. Since these problems were researched by Swiss mathematician Jacques Bernoulli around 1700, they are also called **Bernoulli trials.**

We give the following definition:

> **Binomial Experiment**
>
> A binomial experiment satisfies the following four conditions:
>
> 1. There are only two outcomes, a success or a failure, for each trial.
> 2. The same experiment is repeated several times.
> 3. The trials are independent; that is, the outcome of a particular trial does not affect the outcome of any other trial.
> 4. The probability of success remains the same for every trial.

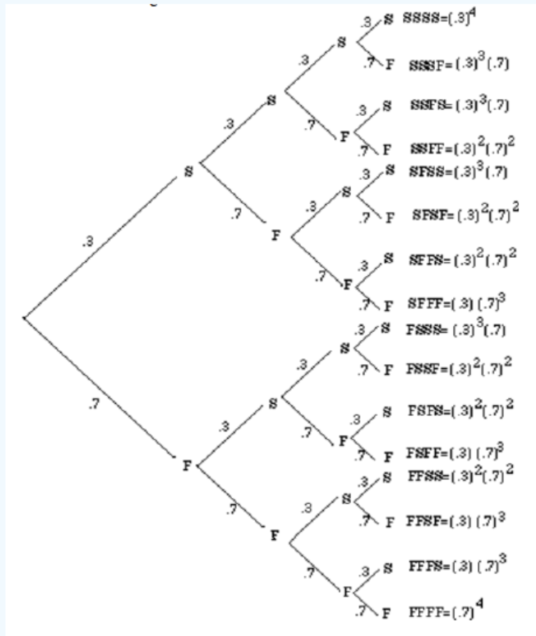
This probability model that will give us the tools to solve many real-life problems , such as:

1. If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?
2. If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 7 out of 10 free throws in a game?
3. If a medicine cures 80% of the people who take it, what is the probability that among the ten people who take the medicine, 6 will be cured?
4. If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?
5. If a telemarketing executive has determined that 15% of the people contacted will purchase the product, what is the probability that among the 12 people who are contacted, 2 will buy the product?

We now consider the following example to develop a formula for finding the probability of k successes in n Bernoulli trials.

> **Example 7.6.1**
>
> A baseball player has a batting average of .300. If he bats four times in a game, find the probability that he will have
>
> a. 4 hits
> b. 3 hits
> c. 2 hits
> d. 1 hit
> e. no hits.
>
> **Solution**
>
> Let S denote that the player gets a hit, and F denote that he does not get a hit.
>
> This is a binomial experiment because it meets all four conditions. First, there are only two outcomes, S or F. Clearly the experiment is repeated four times. Lastly, if we assume that the player's skillfulness to get a hit does not change each time he comes to bat, the trials are independent with a probability of .3 of getting a hit during each trial.

We draw a tree diagram to show all situations.



Let us first find the probability of getting, for example, two hits. We will have to consider the six possibilities, SSFF, SFSF, SFFS, FSSF, FSFS, FFSS, as shown in the above tree diagram. We list the probabilities of each below.

$$\begin{aligned}
\mathrm{P(SSFF)} &= (.3)(.3)(.7)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(SFSF)} &= (.3)(.7)(.3)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(SFFS)} &= (.3)(.7)(.7)(.3) = (.3)^2(.7)^2 \\
\mathrm{P(FSSF)} &= (.7)(.3)(.3)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(FSFS)} &= (.7)(.3)(.7)(.3) = (.3)^2(.7)^2 \\
\mathrm{P(FFSS)} &= (.7)(.7)(.3)(.3) = (.3)^2(.7)^2
\end{aligned}$$

Since the probability of each of these six outcomes is $(.3)^2(.7)^2$, the probability of obtaining two successes is $6(.3)^2(.7)^2$.

The probability of getting one hit can be obtained in the same way. Since each permutation has one S and three F's, there are four such outcomes: SFFF, FSFF, FFSF, and FFFS.

And since the probability of each of the four outcomes is $(.3)(.7)^3$, the probability of getting one hit is $4(.3)(.7)^3$.

The table below lists the probabilities for all cases, and shows a comparison with the binomial expansion of fourth degree. Again, p denotes the probability of success, and $q = (1-p)$ the probability of failure.

Outcome	Four Hits	Three Hits	Two Hits	One Hit	No Hits
Probability	$(.3)^4$	$4(.3)^3(.7)$	$\mathbf{6(.3)^2(.7)^2}$	$4(.3)(.7)^3$	$(.7)^4$

$$\begin{aligned}
(.3+.7)^4 &= (.3)^4 + 4(.3)^3(.7) + 6(.3)^2(.7)^2 + 4(.3)(.7)^3 + (.7)^4 \\
(p+q)^4 &= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4
\end{aligned}$$

This gives us the following theorem:

Binomial Probability Theorem

The probability of obtaining k successes in n independent Bernoulli trials is given by

$$\mathbf{P(n,k;p) = nCk\ p^k q^{n-k}}$$

where p denotes the probability of success and $q = (1-p)$ the probability of failure.

We use the binomial probability formula to solve the following examples.

Example 7.6.2

If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?

Solution

Let S denote the probability of obtaining a head, and F the probability of getting a tail.

Clearly, $n = 10$, $k = 3$, $p = 1/2$, and $q = 1/2$.

Therefore, $\mathrm{b}(10, 3; 1/2) = 10\mathrm{C}3\ (1/2)^3 (1/2)^7 = .1172$

Example 7.6.3

If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 6 out of 10 free throws in a game?

Solution

The probability of making a free throw is 3/4.
Therefore, $p = 3/4$, $q = 1/4$, $n = 10$, and $k = 6$.

Therefore, $b(10, 6; 3/4) = 10\mathrm{C}6\ (3/4)^6 (1/4)^4 = .1460$

Example 7.6.4

If a medicine cures 80% of the people who take it, what is the probability that of the eight people who take the medicine, 5 will be cured?

Solution

Here $p = .80$, $q = .20$, $n = 8$, and $k = 5$.

$$b(8, 5; .80) = 8\mathrm{C}5\ (.80)^5 (.20)^3 = .1468$$

Example 7.6.5

If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?

Solution

If S denotes the probability that the chip is defective, and F the probability that the chip is not defective, then $p = .04$, $q = .96$, $n = 60$, and $k = 3$.

$$b(60, 3; .04) = 60\mathrm{C}3\ (.04)^3 (.96)^{57} = .2138$$

Example 7.6.6

A telemarketing executive has determined that 15% of people contacted will purchase the product. 12 people are contacted about this product.

a. Find the probability that among 12 people contacted, 2 will buy the product.
b. Find the probability that among 12 people contacted, at most 2 will buy the product?

Solution

a. If S denoted the probability that a person will buy the product, and F the probability that the person will not buy the product, then $p = .15$, $q = .85$, $n = 12$, and $k = 2$.

$$b(12, 2; .15) = 12\text{C}2\ (.15)^2(.85)^{10} = .2924.$$

The probability that 2 people buy the product is 0.2924.

b. Again $p = .15$, $q = .85$, $n = 12$. But to find the probability that **at most 2** buy the product, we need to find the probabilities for $k = 0$, $k = 1$, $k = 2$ and add them together.

$$\begin{array}{l} \mathrm{b}(12, 0; .15) = 12\text{C}0\ (.15)^0(.85)^{12} = .1422 \\ \mathrm{b}(12, 1; .15) = 12\text{C}1\ (.15)^1(.85)^{11} = .3012 \end{array}$$

Adding all three probabilities gives: .1422 +0.3012 +.2924 = .7358

The probability that at most 2 people buy the product is 0.7358.

Page 3.6.1: Binomial Probability (Exercises)

SECTION 9.1 PROBLEM SET: BINOMIAL PROBABILITY

Do the following problems using the binomial probability formula.

1. A coin is tossed ten times. Find the probability of getting six heads and four tails.
2. A family has three children. Find the probability of having one boy and two girls.
3. What is the probability of getting three aces(ones) if a die is rolled five times?
4. A baseball player has a .250 batting average. What is the probability that he will have three hits in five times at bat?
5. A basketball player has an 80% chance of sinking a basket on a free throw. What is the probability that he will sink at least three baskets in five free throws?
6. With a new flu vaccination, 85% of the people in the high risk group can go through the entire winter without contracting the flu. In a group of six people who were vaccinated with this drug, what is the probabi
7. A transistor manufacturer has known that 5% of the transistors produced are defective. What is the probability that a batch of twenty five transistors will have two defective?
8. It has been determined that only 80% of the people wear seat belts. If a police officer stops a car with four people, what is the probability that at least one person will not be wearing a seat belt?
9. What is the probability that a family of five children will have at least three boys?
10. What is the probability that a toss of four coins will yield at most two heads?
11. A telemarketing executive has determined that for a particular product, 20% of the people contacted will purchase the product. If 10 people are contacted, what is the probability that at most 2 will buy the product?
12. To the problem: "Five cards are dealt from a deck of cards, find the probability that three of them are kings," the following incorrect answer was offered by a student.

    $$5C3\ (1/13)^3(12/13)^2$$

    What change would you make in the wording of the problem for the given answer to be correct?
13. 63% of all registered voters in a large city voted in the last election. 20 registered voters from this city are randomly selected. Find the probability that
    a. exactly half of them voted in the last election
    b. all of them voted
14. 30% of customers at BigMart pay cash for their purchases. Suppose that 15 customers are randomly selected. Find the probability that
    a. 5 or 6 of them pay cash
    b. at most 1 pays cash
15. 12% of all cars on Brighton Expressway exceed the speed limit. If 10 vehicles on this road are randomly selected and their speed is recorded by radar, find the probability that
    a. none of them are exceeding the speed limit
    b. 1 or 2 are exceeding the speed limit
16. Suppose that 73% of all people taking a professional certification exam pass the exam. If 12 people who take this exam are randomly selected, find the probability that
    a. exactly half of them pass the exam
    b. all of them pass the exam
    c. 8 or 9 of them pass the exam

3.7: Bayes' Formula

Learning Objectives

In this section, you will learn to:

1. Find probabilities using Bayes' formula
2. Use a probability tree to find and represent values needed when using Bayes' formula.

In this section, we will develop and use Bayes' Formula to solve an important type of probability problem. Bayes' formula is a method of calculating the conditional probability $P(F|E)$ from $P(E|F)$. The ideas involved here are not new, and most of these problems can be solved using a tree diagram. However, Bayes' formula does provide us with a tool with which we can solve these problems without a tree diagram.

We begin with an example.

Example 3.7.1

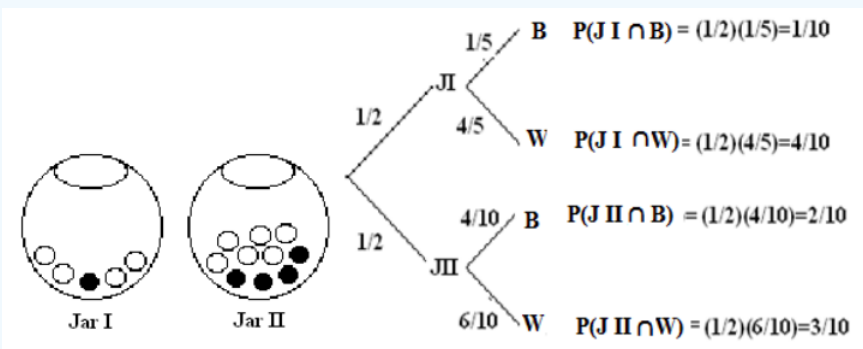
Suppose you are given two jars. Jar I contains one black and 4 white marbles, and Jar II contains 4 black and 6 white marbles. If a jar is selected at random and a marble is chosen,

a. What is the probability that the marble chosen is a black marble?
b. If the chosen marble is black, what is the probability that it came from Jar I?
c. If the chosen marble is black, what is the probability that it came from Jar II?

Solution

Let JI be the event that Jar I is chosen, JII be the event that Jar II is chosen, B be the event that a black marble is chosen and W the event that a white marble is chosen.

We illustrate using a tree diagram.



a. The probability that a black marble is chosen is $P(B)$ = 1/10 + 2/10 = 3/10.
b. To find $P(JI|B)$, we use the definition of conditional probability, and we get

$$P(JI|B) = \frac{P(JI \cap B)}{P(B)} = \frac{1/10}{3/10} = \frac{1}{3}$$

c. Similarly, $\mathrm{P(JII|B)} = \frac{\mathrm{P(JII \cap B)}}{\mathrm{P(B)}} = \frac{2/10}{3/10} = \frac{2}{3}$

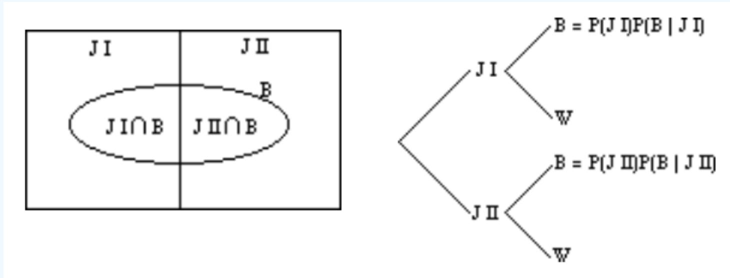
In parts b and c, the reader should note that the denominator is the sum of all probabilities of all branches of the tree that produce a black marble, while the numerator is the branch that is associated with the particular jar in question.

We will soon discover that this is a statement of Bayes' formula .

Let us first visualize the problem.

We are given a sample space S and two mutually exclusive events JI and JII. That is, the two events, JI and JII, divide the sample space into two parts such that $\mathrm{JI} \cup \mathrm{JII} = \mathrm{S}$. Furthermore, we are given an event B that has elements in both JI and

JII, as shown in the Venn diagram below.



From the Venn diagram, we can see that $\mathrm{B}=(\mathrm{B} \cap \mathrm{JI}) \cup(\mathrm{B} \cap \mathrm{JII})$ Therefore:

$$P(B)=P(B \cap JI)+P(B \cap JII) \tag{3.7.1}$$

But the product rule in chapter 7 gives us

$$P(B \cap JI)=P(JI) \cdot P(B \mid JI) \quad \text{and} \quad P(B \cap JII)=P(JII) \cdot P(B \mid JII)$$

Substituting in 3.7.1, we get

$$P(B)=P(JI) \cdot P(B \mid JI)+P(JII) \cdot P(B \mid JII)$$

The conditional probability formula gives us

$$P(JI \mid B)=\frac{P(JI \cap B)}{P(B)}$$

Therefore, $P(JI \mid B)=\frac{P(JI) \cdot P(B \mid JI)}{P(B)}$

or

$$P(JI \mid B)=\frac{P(JI) \cdot P(B \mid JI)}{P(JI) \cdot P(B \mid JI)+P(JII) \cdot P(B \mid JII)}$$

The last statement is Bayes' Formula for the case where the sample space is divided into two partitions.

The following is the generalization of Bayes’ formula for n partitions.

Bayes' Formula for n partitions

Let S be a sample space that is divided into n partitions, $A_1, A_2, \ldots A_n$. If E is any event in S, then

$$\mathbf{P}\left(\mathbf{A_i} \mid \mathbf{E}\right)=\frac{\mathbf{P}\left(\mathbf{A_i}\right) \mathbf{P}\left(\mathbf{E} \mid \mathbf{A_i}\right)}{\mathbf{P}\left(\mathbf{A_1}\right) \mathbf{P}\left(\mathbf{E} \mid \mathbf{A_1}\right)+\mathbf{P}\left(\mathbf{A_2}\right) \mathbf{P}\left(\mathbf{E} \mid \mathbf{A_2}\right)+\cdots+\mathbf{P}\left(\mathbf{A_n}\right) \mathbf{P}\left(\mathbf{E} \mid \mathbf{A_n}\right)}$$

We begin with the following example.

Example 3.7.2

A department store buys 50% of its appliances from Manufacturer A, 30% from Manufacturer B, and 20% from Manufacturer C. It is estimated that 6% of Manufacturer A's appliances, 5% of Manufacturer B's appliances, and 4% of Manufacturer C's appliances need repair before the warranty expires. An appliance is chosen at random. If the appliance chosen needed repair before the warranty expired, what is the probability that the appliance was manufactured by Manufacturer A? Manufacturer B? Manufacturer C?

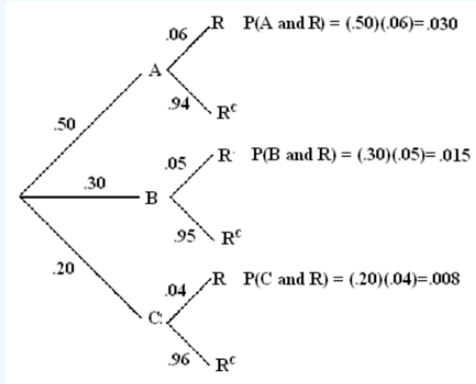
Solution

Let A, B and C be the events that the appliance is manufactured by Manufacturer A, Manufacturer B, and Manufacturer C, respectively. Further, suppose that the event R denotes that the appliance needs repair before the warranty expires.

We need to find P(A | R), P(B | R) and P(C | R).

We will do this problem both by using a tree diagram and by using Bayes' formula.

We draw a tree diagram.



The probability P(A | R), for example, is a fraction whose denominator is the sum of all probabilities of all branches of the tree that result in an appliance that needs repair before the warranty expires, and the numerator is the branch that is associated with Manufacturer A. P(B | R) and P(C | R) are found in the same way.

$$P(A|R) = \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566$$
$$P(B|R) = \frac{.015}{.053} = .283 \text{ and } P(C|R) = \frac{.008}{.053} = .151$$

Alternatively, using Bayes' formula,

$$\begin{aligned} \mathrm{P}(\mathrm{A}|\mathrm{R}) &= \frac{\mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{R}|\mathrm{A})}{\mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{R}|\mathrm{A})+\mathrm{P}(\mathrm{B})\mathrm{P}(\mathrm{R}|\mathrm{B})+\mathrm{P}(\mathrm{C})\mathrm{P}(\mathrm{R}|\mathrm{C})} \\ &= \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566 \end{aligned}$$

P(B | R) and P(C | R) can be determined in the same manner.

Example 3.7.3

There are five Jacy's department stores in San Jose. The distribution of number of employees by gender is given in the table below.

Store Number	Number of Employees	Percent of Women Employees
1	300	.40
2	150	.65
3	200	.60
4	250	.50
5	100	.70
	Total = 1000	

If an employee chosen at random is a woman, what is the probability that the employee works at store III?

Solution

Let k = 1, 2, . . . , 5 be the event that the employee worked at store k, and W be the event that the employee is a woman. Since there are a total of 1000 employees at the five stores,

$$P(1) = .30 \quad P(2) = .15 \quad P(3) = .20 \quad P(4) = .25 \quad P(5) = .10$$

Using Bayes' formula,

$$\begin{aligned} \mathrm{P}(3 \mid \mathrm{W}) &= \frac{\mathrm{P}(3)\mathrm{P}(\mathrm{W} \mid 3)}{\mathrm{P}(1)\mathrm{P}(\mathrm{W} \mid 1)+\mathrm{P}(2)\mathrm{P}(\mathrm{W} \mid 2)+\mathrm{P}(3)\mathrm{P}(\mathrm{W} \mid 3)+\mathrm{P}(4)\mathrm{P}(\mathrm{W} \mid 4)+\mathrm{P}(5)\mathrm{P}(\mathrm{W} \mid 5)} \\ &= \frac{(.20)(.60)}{(.30)(.40)+(.15)(.65)+(.20)(.60)+(.25)(.50)+(.10)(.70)} \\ &= .2254 \end{aligned}$$

7.7.1: Bayes' Formula (Exercises)

SECTION 9.2 PROBLEM SET: BAYES' FORMULA

1. Jar I contains five red and three white marbles, and Jar II contains four red and two white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram below, and find the following probabilities.
   a. P(marble is red)
   b. P(It came from Jar II | marble is white)
   c. P(Red | Jar I)

2. In Mr. Symons' class, if a student does homework most days, the chance of passing the course is 90%. On the other hand, if a student does not do homework most days, the chance of passing the course is only 20%.
   H = event that the student did homework
   C = event that the student passed the course
   Mr. Symons claims that 80% of his students do homework on a regular basis. If a student is chosen at random from Mr. Symons' class, find the following probabilities.
   a. P(C)
   b. P(H|C)
   c. P(C|H)

3. A city has 60% Democrats, and 40% Republicans. In the last mayoral election, 60% of the Democrats voted for their Democratic candidate while 95% of the Republicans voted for their candidate. Which party's mayor runs city hall?

4. In a certain population of 48% men and 52% women, 56% of the men and 8% of the women are color-blind.
   a. What percent of the people are color-blind?
   b. If a person is found to be color-blind, what is the probability that the person is a male?

5. A test for a certain disease gives a positive result 95% of the time if the person actually carries the disease. However, the test also gives a positive result 3% of the time when the individual is not carrying the disease. It is known that 10% of the population carries the disease. If a person tests positive, what is the probability that he or she has the disease?

6. A person has two coins: a fair coin and a two-headed coin. A coin is selected at random, and tossed. If the coin shows a head, what is the probability that the coin is fair?

7. A computer company buys its chips from three different manufacturers. Manufacturer I provides 60% of the chips and is known to produce 5% defective; Manufacturer II supplies 30% of the chips and makes 4% defective; while the rest are supplied by Manufacturer III with 3% defective chips. If a chip is chosen at random, find the following probabilities:
   a. P(the chip is defective)
   b. P(chip is from Manufacturer II | defective)
   c. P(defective |chip is from manufacturer III)

8. Lincoln Union High School District is made up of three high schools: Monterey, Fremont, and Kennedy, with an enrollment of 500, 300, and 200, respectively. On a given day, the percentage of students absent at Monterey High School is 6%, at Fremont 4%, and at Kennedy 5%. If a student is chosen at random, find the probabilities below: *Hint: Convert the enrollments into percentages.*
   a. P(the student is absent)
   b. P(student is from Kennedy | student is absent)
   c. P(student is absent | student is from Fremont)

9. At a retail store, 20% of customers use the store's online app to assist them when shopping in the store ; 80% of store shoppers don't use the app. Of those customers that use the online app while in the store, 50% are very satisfied with their purchases, 40% are moderately satisfied, and 10% are dissatisfied. Of those customers that do not use the online app while in the store, 30% are very satisfied with their purchases, 50% are moderately satisfied and 20% are dissatisfied. Indicate the events by the following: A = shopper uses the app in the store N = shopper does not use the app in the store V = very satisfied with purchase M = moderately satisfied D = dissatisfied a. Find P(A and D), the probability that a store customer uses the app and is dissatisfied b. Find P(A|D), the probability that a store customer uses the app if the customer is dissatisfied.	10. A medical clinic uses a pregnancy test to confirm pregnancy in patients who suspect they are pregnant. Historically data has shown that overall, 70% of the women at this clinic who are given the pregnancy test are pregnant, but 30% are not. The test's manufacturer indicates that if a woman is pregnant, the test will be positive 92% of the time. But if a woman is not pregnant, the test will be positive only 2% of the time and will be negative 98% of the time. a. Find the probability that a woman at this clinic is pregnant and tests positive. b. Find the probability that a woman at this clinic is actually pregnant given that she tests positive.

7.8: Expected Value

> **Learning Objectives**
>
> In this section, you will learn to:
>
> 1. Find the expected value of a discrete probability distribution
> 2. Interpret expected value as a long-run average

An expected gain or loss in a game of chance is called **Expected Value**. The concept of expected value is closely related to a *weighted average*. Consider the following situations.

1. Suppose you and your friend play a game that consists of rolling a die. Your friend offers you the following deal: If the die shows any number from 1 to 5, he will pay you the face value of the die in dollars, that is, if the die shows a 4, he will pay you $4. But if the die shows a 6, you will have to pay him $18.

Before you play the game you decide to find the expected value. You analyze as follows.

Since a die will show a number from 1 to 6, with an equal probability of 1/6, your chance of winning $1 is 1/6, winning $2 is 1/6, and so on up to the face value of 5. But if the die shows a 6, you will lose $18. You write the expected value.

$$\mathrm{E}=\$1(1/6)+\$2(1/6)+\$3(1/6)+\$4(1/6)+\$5(1/6)-\$18(1/6)=-\$.50$$

This means that every time you play this game, you can expect to lose 50 cents. In other words, if you play this game 100 times, theoretically you will lose $50. Obviously, it is not to your interest to play.

2. Suppose of the ten quizzes you took in a course, on eight quizzes you scored 80, and on two you scored 90. You wish to find the average of the ten quizzes.
The average is

$$\mathrm{A}=\frac{(80)(8)+(90)(2)}{10}=(80)\frac{8}{10}+(90)\frac{2}{10}=82$$

It should be observed that it would be incorrect to take the average of 80 and 90 because you scored 80 on eight quizzes, and 90 on only two of them. Therefore, you take a "weighted average" of 80 and 90. That is, the average of 8 parts of 80 and 2 parts of 90, which is 82.

In the first situation, to find the expected value, we multiplied each payoff by the probability of its occurrence, and then added up the amounts calculated for all possible cases. In the second example, if we consider our test score a payoff, we did the same. This leads us to the following definition.

> **Expected Value**
>
> If an experiment has the following probability distribution,
>
> | Payoff | x_1 | x_2 | x_3 | ... | x_n |
> |---|---|---|---|---|---|
> | Probability | $p(x_1)$ | $p(x_2)$ | $p(x_3)$ | ... | $p(x_n)$ |
>
> then the expected value of the experiment is
>
> $$\text{Expected Value }=x_1p\left(x_1\right)+x_2p\left(x_2\right)+x_3p\left(x_3\right)+\cdots+x_np\left(x_n\right)$$

> **Example 7.8.1**
>
> In a town, 10% of the families have three children, 60% of the families have two children, 20% of the families have one child, and 10% of the families have no children. What is the expected number of children to a family?
>
> **Solution**
>
> We list the information in the following table.

Number of children	3	2	1	0
Probability	0.10	0.60	0.20	0.10

Expected Value = $x_1p(x_1) + x_2p(x_2) + x_3p(x_3) + x_4p(x_4)$

$E = 3(.10) + 2(.60) + 1(.20) + 0(.10) = 1.7$

So on average, there are 1.7 children to a family.

Example 7.8.2

To sell an average house, a real estate broker spends \$1200 for advertisement expenses. If the house sells in three months, the broker makes \$8,000. Otherwise, the broker loses the listing. If there is a 40% chance that the house will sell in three months, what is the expected payoff for the real estate broker?

Solution

The broker makes \$8,000 with a probability of .40, but he loses \$1200 whether the house sells or not.

$$E = (\$8000)(.40) - (\$1200) = \$2,000.$$

Alternatively, the broker makes \$(8000 - 1200) with a probability of .40, but loses \$1200 with a probability of .60. Therefore,

$$E = (\$6800)(.40) - (\$1200)(.60) = \$2,000.$$

Example 7.8.3

In a town, the attendance at a football game depends on the weather. On a sunny day the attendance is 60,000, on a cold day the attendance is 40,000, and on a stormy day the attendance is 30,000. If for the next football season, the weatherman has predicted that 30% of the days will be sunny, 50% of the days will be cold, and 20% days will be stormy, what is the expected attendance for a single game?

Solution

Using the expected value formula, we get

$$E = (60,000)(.30) + (40,000)(.50) + (30,000)(.20) = 44,000.$$

Example 7.8.4

A lottery consists of choosing 6 numbers from a total of 51 numbers. The person who matches all six numbers wins \$2 million. If the lottery ticket costs \$1, what is the expected payoff?

Solution

Since there are $51C6 = 18,009,460$combinations of six numbers from a total of 51 numbers, the chance of choosing the winning number is 1 out of 18,009,460.

So the expected payoff is: $\mathrm{E} = (\$2 \text{ million }) \left(\frac{1}{18009460}\right) - \$1 = -\$0.89$

This means that every time a person spends \$1 to buy a ticket, he or she can expect to lose 89 cents.

7.8.1: Expected Value (Exercises)

SECTION 9.3 PROBLEM SET: EXPECTED VALUE

Do the following problems using the expected value concepts learned in this section,

1. You are about to make an investment which gives you a 30% chance of making $60,000 and 70% chance of losing $ 30,000. Should you invest? Explain.

2. In a town, 40% of the men and 30% of the women are overweight. If the town has 46% men and 54% women, what percent of the people are overweight?

3. A game involves rolling a Korean die (4 faces). If a one, two, or three shows, the player receives the face value of the die in dollars, but if a four shows, the player is obligated to pay $4. What is the expected value of the game?

4. A game involves rolling a single die. One receives the face value of the die in dollars. How much should one be willing to pay to roll the die to make the game fair?

5. In a European country, 20% of the families have three children, 40% have two children, 30% have one child, and 10% have no children. On average, how many children are there to a family?

6. A game involves drawing a single card from a standard deck. One receives 60 cents for an ace, 30 cents for a king, and 5 cents for a red card that is neither an ace nor a king. If the cost of each draw is 10 cents, should one play? Explain.

7. Hillview Church plans to raise money by raffling a television worth $500. A total of 3000 tickets are sold at $1 each. Find the expected value of the winnings for a person who buys a ticket in the raffle.

8. During her four years at college, Niki received A's in 30% of her courses, B's in 60% of her courses, and C's in the remaining 10%. If A = 4, B = 3, and C = 2, find her grade point average.

9. Attendance at a Stanford football game depends upon which team Stanford is playing against. If the game is against U. C. Berkeley,attendance will be 70,000; if it is against another California team, it will be 40,000; and if it is against an out of state team, it will be 30,000. If the probability of playing against U. C. Berkeley is 10%, against a California team 50% and against an out of state team 40%, how many fans are expected to attend a game?

10. A Texas oil drilling company has determined that it costs $25,000 to sink a test well. If oil is hit, the revenue for the company will be $500,000. If natural gas is found, the revenue will be $150,000. If the probability of hitting oil is 3% and of hitting gas is 6%, find the expected value of sinking a test well.

11. A $1 lottery ticket offers a grand prize of $10,000; 10 runner-up prizes each pay $1000; 100 third-place prizes each pay $100; and 1,000 fourth-place prizes each pay $10. Find the expected value of entering this contest if 1 million tickets are sold.

12. Assume that for the next heavyweight fight the odds of current champion winning are 15 to 2. A gambler bets $10 that the current champion will lose. If current champion loses, how much can the gambler hope to receive?

13. In a housing development, 35% of households have no school age children, 20% of households have 1 school age child, 25% of households have 2 school age children, 15% have 3, and 5% have 4 school age children.
    a. Find the average number of children per household
    b. If there are 300 homes in this housing development, what is the total number of children expected to attend school?

14. At a large community college, 30% of students take one course, 15% take two courses, 25% take three courses and 20% take four courses. The rest of the students take five courses.
    a. What percent of students take 5 courses?
    b. Find the average number of courses that students take.

Page 3.9: Probability Using Tree Diagrams

Learning Objectives

In this section, you will learn to:

1. Use probability trees to organize information in probability problems
2. Use probability trees to calculate probabilities

As we have already seen, tree diagrams play an important role in solving probability problems. A tree diagram helps us not only visualize, but also list all possible outcomes in a systematic fashion. Furthermore, when we list various outcomes of an experiment and their corresponding probabilities on a tree diagram, we gain a better understanding of when probabilities are multiplied and when they are added.

The meanings of the words *and* and *or* become clear when we learn to multiply probabilities horizontally across branches, and add probabilities vertically down the tree.

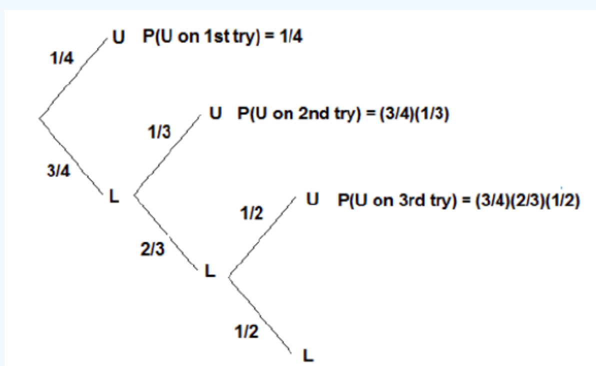
Although tree diagrams are not practical in situations where the possible outcomes become large, they are a significant tool in breaking the problem down in a schematic way. We consider some examples that may seem difficult at first, but with the help of a tree diagram, they can easily be solved.

Example $Page 3.9.1$

A person has four keys and only one key fits to the lock of a door. What is the probability that the locked door can be unlocked in at most three tries?

Solution

Let U be the event that the door has been unlocked and L be the event that the door has not been unlocked. We illustrate with a tree diagram.



The probability of unlocking the door in the first try = 1/4

The probability of unlocking the door in the second try = (3/4)(1/3) = 1/4

The probability of unlocking the door in the third try = (3/4)(2/3)(1/2) = 1/4

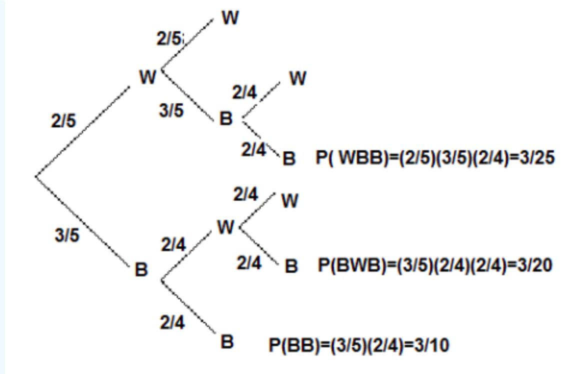
Therefore, the probability of unlocking the door in at most three tries = 1/4 + 1/4 + 1/4 = 3/4.

Example $Page 3.9.2$

A jar contains 3 black and 2 white marbles. We continue to draw marbles one at a time until two black marbles are drawn. If a white marble is drawn, the outcome is recorded and the marble is put back in the jar before drawing the next marble. What is the probability that we will get exactly two black marbles in at most three tries?

Solution

We illustrate using a tree diagram.



The probability that we will get two black marbles in the first two tries is listed adjacent to the lowest branch, and it = 3/10.

The probability of getting first black, second white, and third black = 3/20.

Similarly, the probability of getting first white, second black, and third black = 3/25.

Therefore, the probability of getting exactly two black marbles in at most three tries = 3/10 + 3/20 + 3/25 = 57/100.

Example $Page 3.9.3$

A circuit consists of three resistors: resistor R_1, resistor R_2, and resistor R_3, joined in a series. If one of the resistors fails, the circuit stops working. The probabilities that resistors R_1, R_2, or R_3 will fail are $.07$, $.10$, and $.08$, respectively. Find the probability that at least one of the resistors will fail?

Solution

The probability that at least one of the resistors fails = 1 - none of the resistors fails.

It is quite easy to find the probability of the event that none of the resistors fails.
We don't even need to draw a tree because we can visualize the only branch of the tree that assures this outcome.

The probabilities that R_1, R_2, R_3 will not fail are $.93$, $.90$, and $.92$ respectively. Therefore, the probability that none of the resistors fails = ($.93$)($.90$)($.92$) = $.77$.

Thus, the probability that at least one of them will fail = 1 - $.77$ = $.23$.

Page 3.9.1: Probability Using Tree Diagrams (Exercises)

SECTION 9.4 PROBLEM SET: PROBABILITY USING TREE DIAGRAM

Use a tree diagram to solve the following problems.

1. Suppose you have five keys and only one key fits to the lock of a door. What is the probability that you can open the door in at most three tries?	2. A coin is tossed until a head appears. What is the probability that a head will appear in at most three tries?
3. A basketball player has an 80% chance of making a basket on a free throw. If he makes the basket on the first throw, he has a 90% chance of making it on the second. However, if he misses on the first try, there is only a 70% chance he will make it on the second. If he gets two free throws, what is the probability that he will make at least one of them?	4. You are to play three games. In the first game, you draw a card, and you win if the card is a heart. In the second game, you toss two coins, and you win if one head and one tail are shown. In the third game, two dice are rolled and you win if the sum of the dice is 7 or 11. What is the probability that you win all three games? What is the probability that you win exactly two games?
5. John's car is in the garage, and he has to take a bus to get to school. He needs to make all three connections on time to get to his class. If the chance of making the first connection on time is 80%, the second 80%, and the third 70%, what is the chance that John will make it to his class on time?	6. For a real estate exam the probability of a person passing the test on the first try is .70. The probability that a person who fails on the first try will pass on each of the successive attempts is .80. What is the probability that a person passes the test in at most three attempts?
7. On a Christmas tree with lights, if one bulb goes out, the entire string goes out. If there are twelve bulbs on a string, and the probability of any one going out is .04, what is the probability that the string will not go out?	8. The Long Life Light Bulbs claims that the probability that a light bulb will go out when first used is 15%, but if it does not go out on the first use the probability that it will last the first year is 95%, and if it lasts the first year, there is a 90% probability that it will last two years. Find the probability that a new bulb will last 2 years.
9. A die is rolled until an ace (1) shows. What is the probability that an ace will show on the fourth try?	10. If there are four people in a room, what is the probability that no two have the same birthday?
11. Dan forgets to set his alarm 60% of the time. If he hears the alarm, he turns it off and goes back to sleep 20% of the time, and even if he does wake up on time, he is late getting ready 30% of the time. What is the probability that Dan will be late to school?	12. It has been estimated that 20% of the athletes take some type of drugs. A drug test is 90% accurate, that is, the probability of a falsenegative is 10%. Furthermore, for this test the probability of a false-positive is 20%. If an athlete tests positive, what is the probability that he is a drug user?

3.10: Chapter Review

SECTION 8.6 PROBLEM SET: CHAPTER REVIEW

1. Two dice are rolled. Find the probability that the sum of the dice is
   a. four
   b. five
2. A jar contains 3 red, 4 white, and 5 blue marbles. If a marble is chosen at random, find the following probabilities:
   a. P(red or blue)
   b. P(not blue)
3. A card is drawn from a standard deck. Find the following probabilities:
   a. P(a jack or a king)
   b. P(a jack or a spade)
4. A basket contains 3 red and 2 yellow apples. Two apples are chosen at random. Find the following probabilities:
   a. P(one red, one yellow)
   b. P(at least one red)
5. A basket contains 4 red, 3 white, and 3 blue marbles. Three marbles are chosen at random. Find the following probabilities:
   a. P(two red, one white)
   b. P(first red, second white, third blue)
   c. P(at least one red)
   d. P(none red)
6. Given a family of four children. Find the following probabilities:
   a. P(All boys)
   b. P(1 boy and 3 girls)
7. Consider a family of three children. Find the following:
   a. P(children of both sexes | first born is a boy)
   b. P(all girls | children of both sexes)
8. Mrs. Rossetti is flying from San Francisco to New York. On her way to the San Francisco Airport she encounters heavy traffic and determines that there is a 20% chance that she will be late to the airport and will miss her flight. Even if she makes her flight, there is a 10% chance that she will miss her connecting flight at Chicago. What is the probability that she will make it to New York as scheduled?
9. At a college, twenty percent of the students take history, thirty percent take math, and ten percent take both. What percent of the students take at least one of these two courses?
10. In a T-maze, a mouse may run to the right (R) or may run to the left (L). A mouse goes up the maze three times, and events E and F are described as follows:

    $$\text{E: Runs to the right on the first trial} \quad \text{F: Runs to the left two consecutive times}$$

    Determine whether the events E and F are independent.
11. A college has found that 20% of its students take advanced math courses, 40% take advanced English courses and 15% take both advanced math and advanced English courses. If a student is selected at random, what is the probability that
    a. he is taking English given that he is taking math?
    b. he is taking math or English?
12. If there are 35 students in a class, what is the probability that at least two have the same birthday?
13. A student feels that her probability of passing accounting is .62, of passing mathematics is .45, and her passing accounting or mathematics is .85. Find the probability that the student passes both accounting and math.
14. There are nine judges on the U. S. Supreme Court. Suppose that five are conservative and four are liberal. This year the court will act on six major cases. What is the probability that out of six cases the court will favor the conservatives in at least four?
15. Five cards are drawn from a deck. Find the probability of obtaining
    a. four cards of a single suit

b. two cards of one suit, two of another suit, and one from the remaining
c. a pair(e.g. two aces and three other cards)
d. a straight flush(five in a row of a single suit but not a royal flush)

16. The following table shows a distribution of drink preferences by gender.

	Coke(C)	Pepsi(P)	Seven Up(S)	TOTALS
Male(M)	60	50	22	132
Female(F)	50	40	18	108
TOTALS	110	90	40	240

The events M, F, C, P and S are defined as Male, Female, Coca Cola, Pepsi, and Seven Up, respectively. Find the following:

a. P(F | S)
b. P(P | F)
c. P(C | M)
d. P(M | P ∪ C)
e. Are the events F and S mutually exclusive?
f. Are the events F and S independent?

17. At a clothing outlet 20% of the clothes are irregular, 10% have at least a button missing and 4% are both irregular and have a button missing. If Martha found a dress that has a button missing, what is the probability that it is irregular?
18. A trade delegation consists of four Americans, three Japanese and two Germans. Three people are chosen at random. Find the following probabilities:

a. P(two Americans and one Japanese)
b. P(at least one American)
c. P(One of each nationality)
d. P(no German)

19. A coin is tossed three times, and the events E and F are as follows.

$$\text{E: It shows a head on the first toss} \qquad \text{F: Never turns up a tail}$$

Are the events E and F independent?
20. If $P(E) = .6$ and $P(F) = .4$ and E and F are mutually exclusive, find P(E and F).
21. If $P(E) = .5$ and $P(F) = .3$ and E and F are independent, find $P(E \cup F)$.
22. If $P(F) = .9$ and $P(E|F) = .36$ and E and F are independent, find $P(E)$.
23. If $P(E) = .4$ and P(E or F) =.9 and E and F are independent, find $P(F)$.
24. If $P(E) = .4$ and $P(F|E) = .5$, find P(E and F).
25. If $P(E) = .6$ and P(E and F) = .3, find $P(F|E)$.
26. If $P(E) = .3$ and $P(F) = .4$ and E and F are independent, find $P(E|F)$.

7.11: Chapter Review

SECTION 9.5 PROBLEM SET: CHAPTER REVIEW

1. A coin is tossed five times. Find the following
   a. P(2 heads and 3 tails)
   b. P(at least 4 tails)
2. A dandruff shampoo helps 80% of the people who use it. If 10 people apply this shampoo to their hair, what is the probability that 6 will be dandruff free?
3. A baseball player has a .250 batting average. What is the probability that he will have 2 hits in 4 times at bat?
4. Suppose that 60% of the voters in California intend to vote Democratic in the next election. If we choose five people at random, what is the probability that at least four will vote Democratic?
5. A basketball player has a .70 chance of sinking a basket on a free throw. What is the probability that he will sink at least 4 baskets in six shots?
6. During an archery competition, Stan has a 0.8 chance of hitting a target. If he shoots three times, what is the probability that he will hit the target all three times?
7. A company finds that one out of four new applicants overstate their work experience. If ten people apply for a job at this company, what is the probability that at most two will overstate their work experience?
8. A missile has a 70% chance of hitting a target. How many missiles should be fired to make sure that the target is destroyed with a probability of .99 or more?
9. Jar I contains 4 red and 5 white marbles, and Jar II contains 2 red and 4 white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram and find,
   a. P(Marble is red)
   b. P(It is white given that it came from Jar II)
   c. P(It came from Jar II knowing that the marble drawn is white)
10. Suppose a test is given to determine if a person is infected with HIV. If a person is infected with HIV, the test will detect it in 90% of the cases; and if the person is not infected with HIV, the test will show a positive result 3% of the time. If we assume that 2% of the population is actually infected with HIV, what is the probability that a person obtaining a positive result is actually infected with HIV?
11. A car dealer's inventory consists of 70% cars and 30% trucks. 20% of the cars and 10% of the trucks are used vehicles. If a vehicle chosen at random is used, find the probability that it is a car.
12. Two machines make all the products in a factory, with the first machine making 30% of the products and the second 70%. The first machine makes defective products 3% of the time and the second machine 5% of the time.
    a. Overall what percent of the products made are defective?
    b. If a defective product is found, what is the probability that it was made on the second machine?
    c. If it was made on the second machine, what is the probability that it is defective?
13. An instructor in a finite math course estimates that a student who does his homework has a 90% of chance of passing the course, while a student who does not do the homework has only a 20% chance of passing the course. It has been determined that 60% of the students in a large class do their homework.
    a. What percent of all the students will pass?
    b. If a student passes, what is the probability that he did the homework?
14. Cars are produced at three factories. Factory I produces 10% of the cars and it is known that 2% are defective. Factory II produces 20% of the cars and 3% are defective. Factory III produces 70% of the cars and 4% of those are defective. A car is chosen at random. Find the following probabilities:
    a. P(The car is defective)
    b. P(The car came from Factory III | the car is defective)
15. A stock has a 50% chance of a 10% gain, a 30% chance of no gain, and otherwise it will lose 8%. Find the expected return.
16. A game involves rolling a pair of dice. One receives the sum of the face value of both dice in dollars. How much should one be willing to pay to roll the dice to make the game fair?
17. A roulette wheel consists of numbers 1 through 36, 0, and 00. If the wheel shows an odd number you win a dollar, otherwise you lose a dollar. If you play the game ten times, what is your expectation?

18. A student takes a 100-question multiple-choice exam in which there are four choices to each question. If the student is just guessing the answers, what score can he expect?
19. Mr. Shaw invests 50% of his money in stocks, 30% in mutual funds, and the remaining 20% in bonds. If the annual yield from stocks is 10%, from mutual funds 12%, and from bonds 7%, what percent return can Mr. Shaw expect on his money?
20. An insurance company is planning to insure a group of surgeons against medical malpractice. Its research shows that two surgeons in every fifteen are involved in a medical malpractice suit each year where the average award to the victim is $450,000. How much minimum annual premium should the insurance company charge each doctor?
21. In an evening finite math class of 30 students, it was discovered that 5 students were of age 20, 8 students were about 25 years old, 10 students were close to 30, 4 students were 35, 2 students were 40 and one student 55. What is the average age of a student in this class?
22. Jar I contains 4 marbles of which one is red, and Jar II contains 6 marbles of which 3 are red. Katy selects a jar and then chooses a marble. If the marble is red, she gets paid 3 dollars, otherwise she loses a dollar. If she plays this game ten times, what is her expected payoff?
23. Jar I contains 1 red and 3 white, and Jar II contains 2 red and 3 white marbles. A marble is drawn from Jar I and put in Jar II. Now if one marble is drawn from Jar II, what is the probability that it is a red marble?
24. Let us suppose there are three traffic lights between your house and the school. The chance of finding the first light green is 60%, the second 50%, and the third 30%. What is the probability that on your way to school, you will find at least two lights green?
25. Sonya has just earned her law degree and is planning to take the bar exam. If her chance of passing the bar exam is 65% on each try, what is the probability that she will pass the exam in at least three tries?
26. Every time a particular baseball player is at bat, his probability of getting a hit is .3, his probability of walking is .1, and his probability of being struck out is .4. If he is at bat three times, what is the probability that he will get two hits and one walk?
27. Jar I contains 4 marbles of which none are red, and Jar II contains 6 marbles of which 4 are red. Juan first chooses a jar and then from it he chooses a marble. After the chosen marble is replaced, Mary repeats the same experiment. What is the probability that at least one of them chooses a red marble?
28. Andre and Pete are two tennis players with equal ability. Andre makes the following offer to Pete: We will not play more than four games, and anytime I win more games than you, I am declared a winner and we stop. Draw a tree diagram and determine Andre's probability of winning.

CHAPTER OVERVIEW

8: Probability Distributions and Statistics

7.1: Prelude to Discrete Random Variables

CHAPTER OBJECTIVES

By the end of this chapter, the student should be able to:

- Recognize and understand discrete probability distribution functions, in general.
- Calculate and interpret expected values.
- Recognize the binomial probability distribution and apply it appropriately.
- Recognize the Poisson probability distribution and apply it appropriately.
- Recognize the geometric probability distribution and apply it appropriately.
- Recognize the hypergeometric probability distribution and apply it appropriately.
- Classify discrete word problems by their distributions.

- A student takes a ten-question, true-false quiz. Because the student had such a busy schedule, he or she could not study and guesses randomly at each answer. What is the probability of the student passing the test with at least a 70%?
- Small companies might be interested in the number of long-distance phone calls their employees make during the peak time of the day. Suppose the average is 20 calls. What is the probability that the employees make more than 20 long-distance phone calls during the peak time?

These two examples illustrate two different types of probability problems involving discrete random variables. Recall that discrete data are data that you can count. A *random variable* describes the outcomes of a statistical experiment in words. The values of a random variable can vary with each repetition of an experiment.

Figure $7.1.1$You can use probability and discrete random variables to calculate the likelihood of lightning striking the ground five times during a half-hour thunderstorm. (Credit: Leszek Leszczynski)

Random Variable Notation

Upper case letters such as X or Y denote a random variable. Lower case letters like x or y denote the value of a random variable. If X **is a random variable, then** X **is written in words, and *x* is given as a number.**

For example, let $X =$ the number of heads you get when you toss three fair coins. The sample space for the toss of three fair coins is *TTT*; *THH*; *HTH*; *HHT*; *HTT*; *THT*; *TTH*; *HHH*. Then, $x = 0, 1, 2, 3$. X is in words and *x* is a number. Notice that for this example, the x values are countable outcomes. Because you can count the possible values that X can take on and the outcomes are random (the *x* values 0, 1, 2, 3), X is a discrete random variable.

Collaborative Exercise

Toss a coin ten times and record the number of heads. After all members of the class have completed the experiment (tossed a coin ten times and counted the number of heads), fill in Table. Let $X =$ the number of heads in ten tosses of the coin.

x	Frequency of x	Relative Frequency of x

x	Frequency of x	Relative Frequency of x

a. Which value(s) of x occurred most frequently?
b. If you tossed the coin 1,000 times, what values could x take on? Which value(s) of x do you think would occur most frequently?
c. What does the relative frequency column sum to?

Glossary

Random Variable (RV)

a characteristic of interest in a population being studied; common notation for variables are upper case Latin letters X, Y, Z,...; common notation for a specific value from the domain (set of all possible values of a variable) are lower case Latin letters x, y, and z. For example, if X is the number of children in a family, then x represents a specific integer 0, 1, 2, 3,.... Variables in statistics differ from variables in intermediate algebra in the two following ways.

- The domain of the random variable (RV) is not necessarily a numerical set; the domain may be expressed in words; for example, if $X =$ hair color then the domain is {black, blond, gray, green, orange}.
- We can tell what specific value x the random variable X takes only after performing the experiment.

7.2: Probability Distribution Function (PDF) for a Discrete Random Variable

A discrete probability distribution function has two characteristics:

a. Each probability is between zero and one, inclusive.
b. The sum of the probabilities is one.

Example 7.2.1

A child psychologist is interested in the number of times a newborn baby's crying wakes its mother after midnight. For a random sample of 50 mothers, the following information was obtained. Let $X =$ the number of times per week a newborn baby's crying wakes its mother after midnight. For this example, $x = 0, 1, 2, 3, 4, 5$

$P(x) =$ probability that X takes on a value x.

x	$P(x)$
0	$P(x=0) = \frac{2}{50}$
1	$P(x=1) = \frac{11}{50}$
2	$P(x=2) = \frac{23}{50}$
3	$P(x=3) = \frac{9}{50}$
4	$P(x=4) = \frac{4}{50}$
5	$P(x=5) = \frac{1}{50}$

X takes on the values 0, 1, 2, 3, 4, 5. This is a discrete PDF because:

a. Each $P(x)$ is between zero and one, inclusive.
b. The sum of the probabilities is one, that is,

$$\frac{2}{50} + \frac{11}{50} + \frac{23}{50} + \frac{9}{50} + \frac{4}{50} + \frac{1}{50} = 1 \tag{7.2.1}$$

Exercise 7.2.1

A hospital researcher is interested in the number of times the average post-op patient will ring the nurse during a 12-hour shift. For a random sample of 50 patients, the following information was obtained. Let $X =$ the number of times a patient rings the nurse during a 12-hour shift. For this exercise, $x = 0, 1, 2, 3, 4, 5$ $P(x) =$ the probability that X takes on value x. Why is this a discrete probability distribution function (two reasons)?

X	$P(x)$
0	$P(x=0) = \frac{4}{50}$
1	$P(x=1) = \frac{8}{50}$
2	$P(x=2) = \frac{16}{50}$
3	$P(x=3) = \frac{14}{50}$
4	$P(x=4) = \frac{6}{50}$
5	$P(x=5) = \frac{2}{50}$

Answer

Each $P(x)$ is between 0 and 1, inclusive, and the sum of the probabilities is 1, that is:

$$\frac{4}{50}+\frac{8}{50}+\frac{16}{50}+\frac{14}{50}+\frac{6}{50}+\frac{2}{50}=1 \tag{7.2.2}$$

✓ Example 7.2.2

Suppose Nancy has classes three days a week. She attends classes three days a week 80% of the time, two days 15% of the time, one day 4% of the time, and no days 1% of the time. Suppose one week is randomly selected.

a. Let X = the number of days Nancy ____________________.
b. X takes on what values?
c. Suppose one week is randomly chosen. Construct a probability distribution table (called a PDF table) like the one in Example. The table should have two columns labeled x and $P(x)$. What does the $P(x)$ column sum to?

Solutions

a. Let X = the number of days Nancy attends class per week.

b. 0, 1, 2, and 3

c

x	$P(x)$
0	0.01
1	0.04
2	0.15
3	0.80

? Exercise 7.2.2

Jeremiah has basketball practice two days a week. Ninety percent of the time, he attends both practices. Eight percent of the time, he attends one practice. Two percent of the time, he does not attend either practice. What is X and what values does it take on?

Answer

X is the number of days Jeremiah attends basketball practice per week. X takes on the values 0, 1, and 2.

Review

The characteristics of a probability distribution function (PDF) for a discrete random variable are as follows:

1. Each probability is between zero and one, inclusive (*inclusive* means to include zero and one).
2. The sum of the probabilities is one.

Use the following information to answer the next five exercises: A company wants to evaluate its attrition rate, in other words, how long new hires stay with the company. Over the years, they have established the following probability distribution.

Let $X =$ the number of years a new hire will stay with the company.

Let $P(x) =$ the probability that a new hire will stay with the company *x* years.

? Exercise 4.2.3

Complete Table using the data provided.

x	$P(x)$

x	$P(x)$
0	0.12
1	0.18
2	0.30
3	0.15
4	
5	0.10
6	0.05

Answer

x	$P(x)$
0	0.12
1	0.18
2	0.30
3	0.15
4	0.10
5	0.10
6	0.05

? Exercise 4.2.4

$P(x = 4) =$ _______

? Exercise 4.2.5

$P(x \geq 5) =$ _______

Answer

0.10 + 0.05 = 0.15

? Exercise 4.2.6

On average, how long would you expect a new hire to stay with the company?

? Exercise 4.2.7

What does the column "$P(x)$" sum to?

Answer

1

Use the following information to answer the next six exercises: A baker is deciding how many batches of muffins to make to sell in his bakery. He wants to make enough to sell every one and no fewer. Through observation, the baker has established a probability distribution.

x	$P(x)$

x	$P(x)$
1	0.15
2	0.35
3	0.40
4	0.10

Exercise 4.2.8

Define the random variable X.

Exercise 4.2.9

What is the probability the baker will sell more than one batch? $P(x>1)=$ _______

Answer

0.35 + 0.40 + 0.10 = 0.85

Exercise 4.2.10

What is the probability the baker will sell exactly one batch? $P(x=1)=$ _______

Exercise 4.2.11

On average, how many batches should the baker make?

Answer

1(0.15) + 2(0.35) + 3(0.40) + 4(0.10) = 0.15 + 0.70 + 1.20 + 0.40 = 2.45

Use the following information to answer the next four exercises: Ellen has music practice three days a week. She practices for all of the three days 85% of the time, two days 8% of the time, one day 4% of the time, and no days 3% of the time. One week is selected at random.

Exercise 4.2.12

Define the random variable X.

Exercise 4.2.13

Construct a probability distribution table for the data.

Answer

x	$P(x)$
0	0.03
1	0.04
2	0.08
3	0.85

? Exercise 4.2.14

We know that for a probability distribution function to be discrete, it must have two characteristics. One is that the sum of the probabilities is one. What is the other characteristic?

Use the following information to answer the next five exercises: Javier volunteers in community events each month. He does not do more than five events in a month. He attends exactly five events 35% of the time, four events 25% of the time, three events 20% of the time, two events 10% of the time, one event 5% of the time, and no events 5% of the time.

? Exercise 4.2.15

Define the random variable X.

Answer

Let $X =$ the number of events Javier volunteers for each month.

? Exercise 4.2.16

What values does $x$take on?

? Exercise 4.2.17

Construct a PDF table.

Answer

x	$P(x)$
0	0.05
1	0.05
2	0.10
3	0.20
4	0.25
5	0.35

? Exercise 4.2.18

Find the probability that Javier volunteers for less than three events each month. $P(x < 3) =$ _______

? Exercise 4.2.19

Find the probability that Javier volunteers for at least one event each month. $P(x > 0) =$ _______

Answer

1 – 0.05 = 0.95

Glossary

Probability Distribution Function (PDF)

a mathematical description of a discrete random variable (*RV*), given either in the form of an equation (formula) or in the form of a table listing all the possible outcomes of an experiment and the probability associated with each outcome.

7.3: Mean or Expected Value and Standard Deviation

The expected value is often referred to as the "long-term" average or mean. This means that over the long term of doing an experiment over and over, you would expect this average.

You toss a coin and record the result. What is the probability that the result is heads? If you flip a coin two times, does probability tell you that these flips will result in one heads and one tail? You might toss a fair coin ten times and record nine heads. As you learned in Chapter 3, probability does not describe the short-term results of an experiment. It gives information about what can be expected in the long term. To demonstrate this, Karl Pearson once tossed a fair coin 24,000 times! He recorded the results of each toss, obtaining heads 12,012 times. In his experiment, Pearson illustrated the Law of Large Numbers.

The Law of Large Numbers states that, as the number of trials in a probability experiment increases, the difference between the theoretical probability of an event and the relative frequency approaches zero (the theoretical probability and the relative frequency get closer and closer together). When evaluating the long-term results of statistical experiments, we often want to know the “average” outcome. This “long-term average” is known as the mean or expected value of the experiment and is denoted by the Greek letter μ. In other words, after conducting many trials of an experiment, you would expect this average value.

> To find the expected value or long term average, μ, simply multiply each value of the random variable by its probability and add the products.

Example 7.3.1

A men's soccer team plays soccer zero, one, or two days a week. The probability that they play zero days is 0.2, the probability that they play one day is 0.5, and the probability that they play two days is 0.3. Find the long-term average or expected value, μ, of the number of days per week the men's soccer team plays soccer.

Solution

To do the problem, first let the random variable $X =$ the number of days the men's soccer team plays soccer per week. X takes on the values 0, 1, 2. Construct a PDF table adding a column $x * P(x)$. In this column, you will multiply each x value by its probability.

Expected Value Table This table is called an expected value table. The table helps you calculate the expected value or long-term average.

x	$P(x)$	$x * P(x)$
0	0.2	(0)(0.2) = 0
1	0.5	(1)(0.5) = 0.5
2	0.3	(2)(0.3) = 0.6

Add the last column $x * P(x)$ to find the long term average or expected value:

$$(0)(0.2) + (1)(0.5) + (2)(0.3) = 0 + 0.5 + 0.6 = 1.1.$$

The expected value is 1.1. The men's soccer team would, on the average, expect to play soccer 1.1 days per week. The number 1.1 is the long-term average or expected value if the men's soccer team plays soccer week after week after week. We say $\mu = 1.1$.

Example 7.3.2

Find the expected value of the number of times a newborn baby's crying wakes its mother after midnight. The expected value is the expected number of times per week a newborn baby's crying wakes its mother after midnight. Calculate the standard deviation of the variable as well.

You expect a newborn to wake its mother after midnight 2.1 times per week, on the average.

x	$P(x)$	$x * P(x)$	$(x - \mu)^2 \cdot P(x)$
0	$P(x=0) = \frac{2}{50}$	$(0)\left(\frac{2}{50}\right) = 0$	$(0 - 2.1)^2 \cdot 0.04 = 0.1764$

x	$P(x)$	$x * P(x)$	$(x-\mu)^2 \cdot P(x)$
1	$P(x=1)=\frac{11}{50}$	$(1)\left(\frac{11}{50}\right)=\frac{11}{50}$	$(1-2.1)^2 \cdot 0.22 = 0.2662$
2	$P(x=2)=\frac{23}{50}$	$(2)\left(\frac{23}{50}\right)=\frac{46}{50}$	$(2-2.1)^2 \cdot 0.46 = 0.0046$
3	$P(x=3)=\frac{9}{50}$	$(3)\left(\frac{9}{50}\right)=\frac{27}{50}$	$(3-2.1)^2 \cdot 0.18 = 0.1458$
4	$P(x=4)=\frac{4}{50}$	$(4)\left(\frac{4}{50}\right)=\frac{16}{50}$	$(4-2.1)^2 \cdot 0.08 = 0.2888$
5	$P(x=5)=\frac{1}{50}$	$(5)\left(\frac{1}{50}\right)=\frac{5}{50}$	$(5-2.1)^2 \cdot 0.02 = 0.1682$

Add the values in the third column of the table to find the expected value of X:

$$\mu = \text{Expected Value} = \frac{105}{50} = 2.1$$

Use μ to complete the table. The fourth column of this table will provide the values you need to calculate the standard deviation. For each value x, multiply the square of its deviation by its probability. (Each deviation has the format $x-\mu$.

Add the values in the fourth column of the table:

$$0.1764 + 0.2662 + 0.0046 + 0.1458 + 0.2888 + 0.1682 = 1.05$$

The standard deviation of X is the square root of this sum: $\sigma = \sqrt{1.05} \approx 1.0247$

The mean, μ, of a discrete probability function is the expected value.

$$\mu = \sum (x \bullet P(x))$$

The standard deviation, Σ, of the PDF is the square root of the variance.

$$\sigma = \sqrt{\sum [(x-\mu)2 \bullet P(x)]}$$

When all outcomes in the probability distribution are equally likely, these formulas coincide with the mean and standard deviation of the set of possible outcomes.

? Exercise 7.3.2

A hospital researcher is interested in the number of times the average post-op patient will ring the nurse during a 12-hour shift. For a random sample of 50 patients, the following information was obtained. What is the expected value?

x	$P(x)$
0	$P(x=0)=\frac{4}{50}$
1	$P(x=1)=\frac{8}{50}$
2	$P(x=2)=\frac{16}{50}$
3	$P(x=3)=\frac{14}{50}$
4	$P(x=4)=\frac{6}{50}$
5	$P(x=5)=\frac{2}{50}$

Answer

The expected value is 2.24

$$(0)\frac{4}{50}+(1)\frac{8}{50}+(2)\frac{16}{50}+(3)\frac{14}{50}+(4)\frac{6}{50}+(5)\frac{2}{50}=0+\frac{8}{50}+\frac{32}{50}+\frac{42}{50}+\frac{24}{50}+\frac{10}{50}=\frac{116}{50}=2.32 \tag{7.3.1}$$

Example 7.3.2

Suppose you play a game of chance in which five numbers are chosen from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. A computer randomly selects five numbers from zero to nine with replacement. You pay \$2 to play and could profit \$100,000 if you match all five numbers in order (you get your \$2 back plus \$100,000). Over the long term, what is your **expected** profit of playing the game?

To do this problem, set up an expected value table for the amount of money you can profit.

Let $X =$ the amount of money you profit. The values of x are not 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Since you are interested in your profit (or loss), the values of x are 100,000 dollars and −2 dollars.

To win, you must get all five numbers correct, in order. The probability of choosing one correct number is $\frac{1}{10}$ because there are ten numbers. You may choose a number more than once. The probability of choosing all five numbers correctly and in order is

$$\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)=(1)(10^{-5})$$

$$=0.00001.$$

Therefore, the probability of winning is 0.00001 and the probability of losing is

$$1-0.00001=0.99999.1-0.00001=0.99999.$$

The expected value table is as follows:

Add the last column. –1.99998 + 1 = –0.99998

	x	$P(x)$	$xP(x)$
Loss	–2	0.99999	(–2)(0.99999) = –1.99998
Profit	100,000	0.00001	(100000)(0.00001) = 1

Since –0.99998 is about –1, you would, on average, expect to lose approximately \$1 for each game you play. However, each time you play, you either lose \$2 or profit \$100,000. The \$1 is the average or expected LOSS per game after playing this game over and over.

Exercise 7.3.3

You are playing a game of chance in which four cards are drawn from a standard deck of 52 cards. You guess the suit of each card before it is drawn. The cards are replaced in the deck on each draw. You pay \$1 to play. If you guess the right suit every time, you get your money back and \$256. What is your expected profit of playing the game over the long term?

Answer

Let $X =$ the amount of money you profit. The x-values are –\$1 and \$256.

The probability of guessing the right suit each time is $\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)=\frac{1}{256}=0.0039$

The probability of losing is $1-\frac{1}{256}=\frac{255}{256}=0.9961$

$(0.0039)256+(0.9961)(-1)=0.9984+(-0.9961)=0.0023$ or 0.23 cents.

Example 7.3.4

Suppose you play a game with a biased coin. You play each game by tossing the coin once. $P(\text{heads})=\frac{2}{3}$ and $P(\text{tails})=\frac{1}{3}$. If you toss a head, you pay \$6. If you toss a tail, you win \$10. If you play this game many times, will you come out ahead?

a. Define a random variable X.
b. Complete the following expected value table.
c. What is the expected value, μ? Do you come out ahead?

Solutions

a.

X = amount of profit

	x	____	____
WIN	10	$\frac{1}{3}$	____
LOSE	____	____	$\frac{-12}{3}$

b.

	x	$P(x)$	$xP(x)$
WIN	10	$\frac{1}{3}$	$\frac{10}{3}$
LOSE	–6	$\frac{2}{3}$	$\frac{-12}{3}$

c.

Add the last column of the table. The expected value $\mu = \frac{-2}{3}$. You lose, on average, about 67 cents each time you play the game so you do not come out ahead.

Exercise 7.3.4

Suppose you play a game with a spinner. You play each game by spinning the spinner once. $P(\text{red}) = \frac{2}{5}$, $P(\text{blue}) = \frac{2}{5}$, and $P(\text{green}) = \frac{1}{5}$. If you land on red, you pay \$10. If you land on blue, you don't pay or win anything. If you land on green, you win \$10. Complete the following expected value table.

	x	$P(x)$	
Red			$-\frac{20}{5}$
Blue		$\frac{2}{5}$	
Green	10		

Answer

	x	$P(x)$	$x * P(x)$
Red	–10	$\frac{2}{5}$	$-\frac{20}{5}$
Blue	0	$\frac{2}{5}$	$\frac{0}{5}$
Green	10	$\frac{1}{5}$	$\frac{1}{5}$

Like data, probability distributions have standard deviations. To calculate the standard deviation (σ) of a probability distribution, find each deviation from its expected value, square it, multiply it by its probability, add the products, and take the square root. To understand how to do the calculation, look at the table for the number of days per week a men's soccer team plays soccer. To find the standard deviation, add the entries in the column labeled $(x) – \mu^2 P(x)$ and take the square root.

x	$P(x)$	$x * P(x)$	$(x – \mu)^2 P(x)$
0	0.2	(0)(0.2) = 0	$(0 – 1.1)^2(0.2) = 0.242$
1	0.5	(1)(0.5) = 0.5	$(1 – 1.1)^2(0.5) = 0.005$
2	0.3	(2)(0.3) = 0.6	$(2 – 1.1)^2(0.3) = 0.243$

Add the last column in the table. $0.242 + 0.005 + 0.243 = 0.490$ The standard deviation is the square root of 0.49, or $\sigma = \sqrt{0.49} = 0.7$

Generally for probability distributions, we use a calculator or a computer to calculate μ and σ to reduce roundoff error. For some probability distributions, there are short-cut formulas for calculating μ and σ.

Example 7.3.5

Toss a fair, six-sided die twice. Let X = the number of faces that show an even number. Construct a table like Table and calculate the mean μ and standard deviation σ of X.

Solution

Tossing one fair six-sided die twice has the same sample space as tossing two fair six-sided dice. The sample space has 36 outcomes:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Use the sample space to complete the following table:

Calculating μ and σ.

x	$P(x)$	$xP(x)$	$(x – \mu)^2 \cdot P(x)$
0	$\frac{9}{36}$	0	$(0 – 1)^2 \cdot \frac{9}{36} = \frac{9}{36}$
1	$\frac{18}{36}$	$\frac{18}{36}$	$(1 – 1)^2 \cdot \frac{18}{36} = 0$
2	$\frac{9}{36}$	$\frac{18}{36}$	$(1 – 1)^2 \cdot \frac{9}{36} = \frac{9}{36}$

Add the values in the third column to find the expected value: $\mu = \frac{36}{36} = 1$. Use this value to complete the fourth column.

Add the values in the fourth column and take the square root of the sum:

$$\sigma = \sqrt{\frac{18}{36}} \approx 0.7071. \tag{7.3.2}$$

Example 7.3.6

On May 11, 2013 at 9:30 PM, the probability that moderate seismic activity (one moderate earthquake) would occur in the next 48 hours in Iran was about 21.42%. Suppose you make a bet that a moderate earthquake will occur in Iran during this period. If you win the bet, you win \$50. If you lose the bet, you pay \$20. Let X = the amount of profit from a bet.

$P(\text{win}) = P(\text{one moderate earthquake will occur}) = 21.42$

$P(\text{loss}) = P(\text{one moderate earthquake will not occur}) = 100$

If you bet many times, will you come out ahead? Explain your answer in a complete sentence using numbers. What is the standard deviation of X? Construct a table similar to Table and Table to help you answer these questions.

Answer

	x	$P(x)$	$xP(x)$	$(x–\mu^2)P(x)$
win	50	0.2142	10.71	$[50 – (–5.006)]^2(0.2142) = 648.0964$
loss	–20	0.7858	–15.716	$[–20 – (–5.006)]^2(0.7858) = 176.6636$

Mean = Expected Value = 10.71 + (–15.716) = –5.006.

If you make this bet many times under the same conditions, your long term outcome will be an average *loss* of $5.01 per bet.

Standard Deviation $= \sqrt{648.0964 + 176.6636} \approx 28.7186$

Exercise 7.3.6

On May 11, 2013 at 9:30 PM, the probability that moderate seismic activity (one moderate earthquake) would occur in the next 48 hours in Japan was about 1.08%. You bet that a moderate earthquake will occur in Japan during this period. If you win the bet, you win $100. If you lose the bet, you pay $10. Let X = the amount of profit from a bet. Find the mean and standard deviation of X.

Answer

	x	$P(x)$	$x \cdot P(x)$	$(x - \mu^2) \cdot P(x)$
win	100	0.0108	1.08	$[100 – (–8.812)]^2 \cdot 0.0108 = 127.8726$
loss	–10	0.9892	–9.892	$[–10 – (–8.812)]^2 \cdot 0.9892 = 1.3961$

Mean = Expected Value $= \mu = 1.08 + (-9.892) =- 8.812$

If you make this bet many times under the same conditions, your long term outcome will be an average loss of $8.81 per bet.

Standard Deviation $= \sqrt{127.7826 + 1.3961} \approx 11.3696$

Some of the more common discrete probability functions are binomial, geometric, hypergeometric, and Poisson. Most elementary courses do not cover the geometric, hypergeometric, and Poisson. Your instructor will let you know if he or she wishes to cover these distributions.

A probability distribution function is a pattern. You try to fit a probability problem into a **pattern** or distribution in order to perform the necessary calculations. These distributions are tools to make solving probability problems easier. Each distribution has its own special characteristics. Learning the characteristics enables you to distinguish among the different distributions.

Summary

The expected value, or mean, of a discrete random variable predicts the long-term results of a statistical experiment that has been repeated many times. The standard deviation of a probability distribution is used to measure the variability of possible outcomes.

Formula Review

1. Mean or Expected Value: $\mu = \sum_{x \in X} xP(x)$
2. Standard Deviation: $\sigma = \sqrt{\sum_{x \in X} (x - \mu)^2 P(x)}$

Glossary

Expected Value

expected arithmetic average when an experiment is repeated many times; also called the mean. Notations: μ. For a discrete random variable (RV) with probability distribution function $P(x)$,the definition can also be written in the form $\mu = \sum xP(x)$.

Mean

a number that measures the central tendency; a common name for mean is 'average.' The term 'mean' is a shortened form of 'arithmetic mean.' By definition, the mean for a sample (detonated by $\bar{x}$) is $\bar{x} = \frac{\text{Sum of all values in the sample}}{\text{Number of values in the sample}}$ and the mean for a population (denoted by μ) is $\mu = \frac{\text{Sum of all values in the population}}{\text{Number of values in the population}}$.

Mean of a Probability Distribution

the long-term average of many trials of a statistical experiment

Standard Deviation of a Probability Distribution

a number that measures how far the outcomes of a statistical experiment are from the mean of the distribution

The Law of Large Numbers

As the number of trials in a probability experiment increases, the difference between the theoretical probability of an event and the relative frequency probability approaches zero.

References

1. Class Catalogue at the Florida State University. Available online at apps.oti.fsu.edu/RegistrarCo...archFormLegacy (accessed May 15, 2013).
2. "World Earthquakes: Live Earthquake News and Highlights," World Earthquakes, 2012. www.world-earthquakes.com/ind...thq_prediction (accessed May 15, 2013).

7.4: Binomial Distribution

The binomial distribution is frequently used to model the number of successes in a sample of size n drawn *with replacement* from a population of size N.

Three characteristics of a binomial experiment

1. There are a fixed number of trials. Think of trials as repetitions of an experiment. The letter n denotes the number of trials.
2. There are only two possible outcomes, called "success" and "failure," for each trial. The letter p denotes the probability of a success on one trial, and q denotes the probability of a failure on one trial. $p+q=1$.
3. The n trials are independent and are repeated using identical conditions. Because the n trials are independent, the outcome of one trial does not help in predicting the outcome of another trial. Another way of saying this is that for each individual trial, the probability, p, of a success and probability, q, of a failure remain the same. For example, randomly guessing at a true-false statistics question has only two outcomes. If a success is guessing correctly, then a failure is guessing incorrectly. Suppose Joe always guesses correctly on any statistics true-false question with probability $p=0.6$. Then, $q=0.4$. This means that for every true-false statistics question Joe answers, his probability of success ($p=0.6$) and his probability of failure ($q=0.4$) remain the same.

The outcomes of a binomial experiment fit a **binomial probability distribution**. The random variable $X=$ the number of successes obtained in the n independent trials. The mean, μ, and variance, σ^2, for the binomial probability distribution are

$$\mu=np \tag{7.4.1}$$

and

$$\sigma^2=npq. \tag{7.4.2}$$

The standard deviation, σ, is then

$$\sigma=\sqrt{npq}. \tag{7.4.3}$$

Any experiment that has characteristics two and three and where $n=1$ is called a **Bernoulli Trial** (named after Jacob Bernoulli who, in the late 1600s, studied them extensively). A binomial experiment takes place when the number of successes is counted in one or more Bernoulli Trials.

Example 7.4.1

At ABC College, the withdrawal rate from an elementary physics course is 30% for any given term. This implies that, for any given term, 70% of the students stay in the class for the entire term. A "success" could be defined as an individual who withdrew. The random variable $X=$ the number of students who withdraw from the randomly selected elementary physics class.

Exercise 7.4.1

The state health board is concerned about the amount of fruit available in school lunches. Forty-eight percent of schools in the state offer fruit in their lunches every day. This implies that 52% do not. What would a "success" be in this case?

Answer

a school that offers fruit in their lunch every day

Example 7.4.2

Suppose you play a game that you can only either win or lose. The probability that you win any game is 55%, and the probability that you lose is 45%. Each game you play is independent. If you play the game 20 times, write the function that describes the probability that you win 15 of the 20 times. Here, if you define X as the number of wins, then X takes on the values 0, 1, 2, 3, ..., 20. The probability of a success is $p=0.55$. The probability of a failure is $q=0.45$. The number of trials is $n=20$. The probability question can be stated mathematically as $P(x=15)$.

Exercise 7.4.2

A trainer is teaching a dolphin to do tricks. The probability that the dolphin successfully performs the trick is 35%, and the probability that the dolphin does not successfully perform the trick is 65%. Out of 20 attempts, you want to find the probability that the dolphin succeeds 12 times. State the probability question mathematically.

Answer

$P(x=12)$

Example 7.4.3

A fair coin is flipped 15 times. Each flip is independent. What is the probability of getting more than ten heads? Let $X=$ the number of heads in 15 flips of the fair coin. Xtakes on the values 0, 1, 2, 3, ..., 15. Since the coin is fair, $p=0.5$ and $q=0.5$. The number of trials is $n=15$. State the probability question mathematically.

Solution

$P(x>10)$

Exercise 7.4.4

A fair, six-sided die is rolled ten times. Each roll is independent. You want to find the probability of rolling a one more than three times. State the probability question mathematically.

Answer

$P(x>3)$

Example 7.4.5

Approximately 70% of statistics students do their homework in time for it to be collected and graded. Each student does homework independently. In a statistics class of 50 students, what is the probability that at least 40 will do their homework on time? Students are selected randomly.

a. This is a binomial problem because there is only a success or a __________, there are a fixed number of trials, and the probability of a success is 0.70 for each trial.
b. If we are interested in the number of students who do their homework on time, then how do we define X?
c. What values does x take on?
d. What is a "failure," in words?
e. If $p+q=1$, then what is q?
f. The words "at least" translate as what kind of inequality for the probability question $P(x$ ____ 40).

Solution

a. failure
b. X = the number of statistics students who do their homework on time
c. 0, 1, 2, …, 50
d. Failure is defined as a student who does not complete his or her homework on time. The probability of a success is $p=0.70$. The number of trials is $n=50$.
e. $q=0.30$
f. greater than or equal to ($\geq$). The probability question is $P(x\geq 40)$.

Exercise 7.4.5

Sixty-five percent of people pass the state driver's exam on the first try. A group of 50 individuals who have taken the driver's exam is randomly selected. Give two reasons why this is a binomial problem.

Answer

This is a binomial problem because there is only a success or a failure, and there are a definite number of trials. The probability of a success stays the same for each trial.

Notation for the Binomial: $B =$ Binomial Probability Distribution Function

$$X \sim B(n, p) \tag{7.4.4}$$

Read this as "X is a random variable with a binomial distribution." The parameters are n and p; $n =$ number of trials, $p =$ probability of a success on each trial.

Example 7.4.6

It has been stated that about 41% of adult workers have a high school diploma but do not pursue any further education. If 20 adult workers are randomly selected, find the probability that at most 12 of them have a high school diploma but do not pursue any further education. How many adult workers do you expect to have a high school diploma but do not pursue any further education?

Let X = the number of workers who have a high school diploma but do not pursue any further education.

X takes on the values 0, 1, 2, ..., 20 where $n = 20, p = 0.41$, and $q = 1 - 0.41 = 0.59$. $X \sim B(20, 0.41)$

Find $P(x \leq 12)$. $P(x \leq 12) = 0.9738$. (calculator or computer)

Go into 2nd DISTR. The syntax for the instructions are as follows:

To calculate $(x =$ value$)$: binompdf$(\boldsymbol{n}, \boldsymbol{p},$ number $)$ if "number" is left out, the result is the binomial probability table.

To calculate $P(x \leq$ value$)$: binomcdf$(\boldsymbol{n}, \boldsymbol{p},$ number$)$ if "number" is left out, the result is the cumulative binomial probability table.

For this problem: After you are in 2nd DISTR, arrow down to binomcdf. Press ENTER. Enter 20,0.41,12). The result is $P(x \leq 12) = 0.9738$.

> If you want to find $P(x = 12)$, use the pdf (binompdf). If you want to find $P(x > 12)$, use $1 - \text{binomcdf}(20, 0.41, 12)$.

The probability that at most 12 workers have a high school diploma but do not pursue any further education is 0.9738.

The graph of $X \sim B(20, 0.41)$ is as follows:

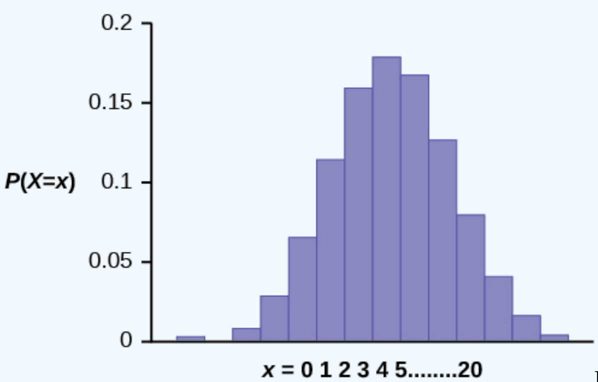


Figure 7.4.1: The graph of $X \sim B(20, 0.41)$.

The y-axis contains the probability of x, where $X =$ the number of workers who have only a high school diploma.

The number of adult workers that you expect to have a high school diploma but not pursue any further education is the mean, $\mu = np = (20)(0.41) = 8.2$.

The formula for the variance is $\sigma^2 = npq$. The standard deviation is $\sigma = \sqrt{npq}$.

$$\sigma = \sqrt{(20)(0.41)(0.59)} = 2.20. \tag{7.4.5}$$

Exercise 4.4.5

About 32% of students participate in a community volunteer program outside of school. If 30 students are selected at random, find the probability that at most 14 of them participate in a community volunteer program outside of school. Use the TI-83+ or TI-84 calculator to find the answer.

Answer

$P(x \leq 14) = 0.9695$

Example 7.4.7

In the 2013 *Jerry's Artarama* art supplies catalog, there are 560 pages. Eight of the pages feature signature artists. Suppose we randomly sample 100 pages. Let $X =$ the number of pages that feature signature artists.

a. What values does x take on?
b. What is the probability distribution? Find the following probabilities:
   i. the probability that two pages feature signature artists
   ii. the probability that at most six pages feature signature artists
   iii. the probability that more than three pages feature signature artists.
c. Using the formulas, calculate the (i) mean and (ii) standard deviation.

Answer

a. $x = 0, 1, 2, 3, 4, 5, 6, 7, 8$
b. $X \sim B(100, 8560)(100, 8560)$
   i. $P(x=2) = \text{binompdf}\left(100, \frac{8}{560}, 2\right) = 0.2466$
   ii. $P(x \leq 6) = \text{binomcdf}\left(100, \frac{8}{560}, 6\right) = 0.9994$
   iii. $P(x > 3) = 1 - P(x \leq 3) = 1 - \text{binomcdf}\left(100, \frac{8}{560}, 3\right) = 1 - 0.9443 = 0.0557$
c. i. Mean $= np = (100)\left(\frac{8}{560}\right) = \frac{800}{560} \approx 1.4286$
   ii. Standard Deviation $= \sqrt{npq} = \sqrt{(100)\left(\frac{8}{560}\right)\left(\frac{552}{560}\right)} \approx 1.1867$

Exercise 7.4.7

According to a Gallup poll, 60% of American adults prefer saving over spending. Let X = the number of American adults out of a random sample of 50 who prefer saving to spending.

a. What is the probability distribution for X?
b. Use your calculator to find the following probabilities:
   i. the probability that 25 adults in the sample prefer saving over spending
   ii. the probability that at most 20 adults prefer saving
   iii. the probability that more than 30 adults prefer saving
c. Using the formulas, calculate the (i) mean and (ii) standard deviation of X.

Answer

a. $X \sim B(50, 0.6)$
b. Using the TI-83, 83+, 84 calculator with instructions as provided in Example:
   i. $P(x = 25) = \text{binompdf}(50, 0.6, 25) = 0.0405$
   ii. $P(x \leq 20) = \text{binomcdf}(50, 0.6, 20) = 0.0034$

iii. $(x > 30) = 1 - \text{binomcdf}(50, 0.6, 30) = 1 – 0.5535 = 0.4465$

c. i. Mean $= np = 50(0.6) = 30$
   ii. Standard Deviation $= \sqrt{npq} = \sqrt{50(0.6)(0.4)} \approx 3.4641$

Example 7.4.8

The lifetime risk of developing pancreatic cancer is about one in 78 (1.28%). Suppose we randomly sample 200 people. Let X = the number of people who will develop pancreatic cancer.

a. What is the probability distribution for X?
b. Using the formulas, calculate the (i) mean and (ii) standard deviation of X.
c. Use your calculator to find the probability that at most eight people develop pancreatic cancer
d. Is it more likely that five or six people will develop pancreatic cancer? Justify your answer numerically.

Answer

a. $X \sim B(200, 0.0128)$
b. i. Mean $= np = 200(0.0128) = 2.56$
   ii. Standard Deviation $= \sqrt{npq} = \sqrt{(200)(0.0128)(0.9872)} \approx 1.5897$
c. Using the TI-83, 83+, 84 calculator with instructions as provided in Example:
   $P(x \leq 8) = \text{binomcdf}(200, 0.0128, 8) = 0.9988$
d. $P(x = 5) = \text{binompdf}(200, 0.0128, 5) = 0.0707$
   $P(x = 6) = \text{binompdf}(200, 0.0128, 6) = 0.0298$
   So $P(x = 5) > P(x = 6)$; it is more likely that five people will develop cancer than six.

Exercise 7.4.8

During the 2013 regular NBA season, DeAndre Jordan of the Los Angeles Clippers had the highest field goal completion rate in the league. DeAndre scored with 61.3% of his shots. Suppose you choose a random sample of 80 shots made by DeAndre during the 2013 season. Let $X =$ the number of shots that scored points.

a. What is the probability distribution for X?
b. Using the formulas, calculate the (i) mean and (ii) standard deviation of X.
c. Use your calculator to find the probability that DeAndre scored with 60 of these shots.
d. Find the probability that DeAndre scored with more than 50 of these shots.

Answer

a. $X \sim B(80, 0.613)$
b. i. Mean $= np = 80(0.613) = 49.04$
   ii. Standard Deviation $= \sqrt{npq} = \sqrt{80(0.613)(0.387)} \approx 4.3564$
c. Using the TI-83, 83+, 84 calculator with instructions as provided in Example:
   $P(x = 60) = \text{binompdf}(80, 0.613, 60) = 0.0036$
d. $P(x > 50) = 1 – P(x \leq 50) = 1 – \text{binomcdf}(80, 0.613, 50) = 1 – 0.6282 = 0.3718$

Example 7.4.9

The following example illustrates a problem that is **not** binomial. It violates the condition of independence. ABC College has a student advisory committee made up of ten staff members and six students. The committee wishes to choose a chairperson and a recorder. What is the probability that the chairperson and recorder are both students? The names of all committee members are put into a box, and two names are drawn **without replacement**. The first name drawn determines the chairperson and the second name the recorder. There are two trials. However, the trials are not independent because the outcome of the first trial affects the outcome of the second trial. The probability of a student on the first draw is $\frac{6}{16}$. The probability of a student on the second draw is $\frac{5}{15}$, when the first draw selects a student. The probability is $\frac{6}{15}$, when the first draw selects a staff member.

The probability of drawing a student's name changes for each of the trials and, therefore, violates the condition of independence.

Exercise 7.4.9

A lacrosse team is selecting a captain. The names of all the seniors are put into a hat, and the first three that are drawn will be the captains. The names are not replaced once they are drawn (one person cannot be two captains). You want to see if the captains all play the same position. State whether this is binomial or not and state why.

Answer

This is not binomial because the names are not replaced, which means the probability changes for each time a name is drawn. This violates the condition of independence.

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Review

A statistical experiment can be classified as a binomial experiment if the following conditions are met:

> There are a fixed number of trials, n.
>
> There are only two possible outcomes, called "success" and, "failure" for each trial. The letter p denotes the probability of a success on one trial and q denotes the probability of a failure on one trial.
>
> The n trials are independent and are repeated using identical conditions.

The outcomes of a binomial experiment fit a binomial probability distribution. The random variable $X =$ the number of successes obtained in the n independent trials. The mean of X can be calculated using the formula $\mu = np$, and the standard deviation is given by the formula $\sigma = \sqrt{npq}$.

Formula Review

- $X \sim B(n, p)$ means that the discrete random variable X has a binomial probability distribution with n trials and probability of success p.
- $X =$ the number of successes in n independent trials
- $n =$ the number of independent trials
- X takes on the values $x = 0, 1, 2, 3, \ldots, n$
- $p =$ the probability of a success for any trial
- $q =$ the probability of a failure for any trial
- $p + q = 1$
- $q = 1 - p$

The mean of X is $\mu = np$. The standard deviation of X is $\sigma = \sqrt{npq}$.

Use the following information to answer the next eight exercises: The Higher Education Research Institute at UCLA collected data from 203,967 incoming first-time, full-time freshmen from 270 four-year colleges and universities in the U.S. 71.3% of those students replied that, yes, they believe that same-sex couples should have the right to legal marital status. Suppose that you randomly pick eight first-time, full-time freshmen from the survey. You are interested in the number that believes that same sex-couples should have the right to legal marital status.

Exercise 4.4.9

In words, define the random variable X.

Answer

$X =$ the number that reply "yes"

Exercise 4.4.10

$X \sim$ _____(_____,_____)

Exercise 4.4.11

What values does the random variable X take on?

Answer

0, 1, 2, 3, 4, 5, 6, 7, 8

Exercise 4.4.12

Construct the probability distribution function (PDF).

x	$P(x)$

Exercise 4.4.13

On average (μ), how many would you expect to answer yes?

Answer

5.7

❓ Exercise 4.4.14

What is the standard deviation (σ)?

❓ Exercise 4.4.15

What is the probability that at most five of the freshmen reply “yes”?

Answer

0.4151

❓ Exercise 4.4.16

What is the probability that at least two of the freshmen reply “yes”?

Glossary

Binomial Experiment

a statistical experiment that satisfies the following three conditions:

1. There are a fixed number of trials, n.
2. There are only two possible outcomes, called "success" and, "failure," for each trial. The letter p denotes the probability of a success on one trial, and q denotes the probability of a failure on one trial.
3. The n trials are independent and are repeated using identical conditions.

Bernoulli Trials

an experiment with the following characteristics:

1. There are only two possible outcomes called “success” and “failure” for each trial.
2. The probability p of a success is the same for any trial (so the probability $q = 1 - p$ of a failure is the same for any trial).

Binomial Probability Distribution

a discrete random variable (RV) that arises from Bernoulli trials; there are a fixed number, n, of independent trials. “Independent” means that the result of any trial (for example, trial one) does not affect the results of the following trials, and all trials are conducted under the same conditions. Under these circumstances the binomial RV X is defined as the number of successes in n trials. The notation is: $X\ B(n,p)$. The mean is $\mu = np$ and the standard deviation is $\sigma = \sqrt{npq}$. The probability of exactly x successes in n trials is
$P(X = x) = \binom{n}{x} p^x q^{n-x}$.

7.5: Introduction

CHAPTER OBJECTIVES

By the end of this chapter, the student should be able to:

- Recognize and understand continuous probability density functions in general.
- Recognize the uniform probability distribution and apply it appropriately.
- Recognize the exponential probability distribution and apply it appropriately.

Continuous random variables have many applications. Baseball batting averages, IQ scores, the length of time a long distance telephone call lasts, the amount of money a person carries, the length of time a computer chip lasts, and SAT scores are just a few. The field of reliability depends on a variety of continuous random variables.

> The values of discrete and continuous random variables can be ambiguous. For example, if X is equal to the number of miles (to the nearest mile) you drive to work, then X is a discrete random variable. You count the miles. If X is the distance you drive to work, then you measure values of X and X is a continuous random variable. For a second example, if X is equal to the number of books in a backpack, then X is a discrete random variable. If X is the weight of a book, then X is a continuous random variable because weights are measured. How the random variable is defined is very important.

Figure $7.5.1$: The heights of these radish plants are continuous random variables. (Credit: Rev Stan)

Properties of Continuous Probability Distributions

The graph of a continuous probability distribution is a curve. Probability is represented by area under the curve. The curve is called the probability density function (abbreviated as pdf). We use the symbol $f(x)$ to represent the curve. $f(x)$ is the function that corresponds to the graph; we use the density function $f(x)$ to draw the graph of the probability distribution. Area under the curve is given by a different function called the cumulative distribution function(abbreviated as cdf). The cumulative distribution function is used to evaluate probability as area.

- The outcomes are measured, not counted.
- The entire area under the curve and above the x-axis is equal to one.
- Probability is found for intervals of x values rather than for individual x values.
- $P(c < x < d)$ is the probability that the random variable X is in the interval between the values c and d. $P(c < x < d)$ is the area under the curve, above the *x*-axis, to the right of c and the left of d.
- $P(x = c) = 0$ The probability that x takes on any single individual value is zero. The area below the curve, above the *x*-axis, and between $x = c$ and $x = c$ has no width, and therefore no area (area = 0). Since the probability is equal to the area, the probability is also zero.
- $P(c < x < d)$ is the same as $P(c \leq x \leq d)$ because probability is equal to area.

We will find the area that represents probability by using geometry, formulas, technology, or probability tables. In general, calculus is needed to find the area under the curve for many probability density functions. When we use formulas to find the area in this textbook, the formulas were found by using the techniques of integral calculus. However, because most students taking this course have not studied calculus, we will not be using calculus in this textbook. There are many continuous probability distributions. When using a continuous probability distribution to model probability, the distribution used is selected to model and fit the particular situation in the best way.

In this chapter and the next, we will study the uniform distribution, the exponential distribution, and the normal distribution. The following graphs illustrate these distributions.

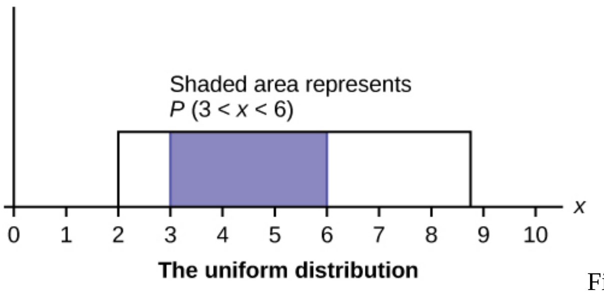

Figure $7.5.2$ The graph shows a Uniform Distribution with the area between $x=3$ and $x=6$ shaded to represent the probability that the value of the random variable X is in the interval between three and six.

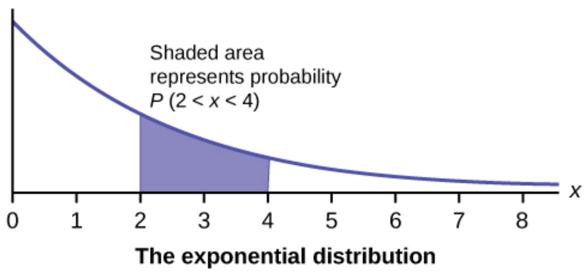

Figure $7.5.3$ The graph shows an Exponential Distribution with the area between $x=2$ and $x=4$ shaded to represent the probability that the value of the random variable X is in the interval between two and four.

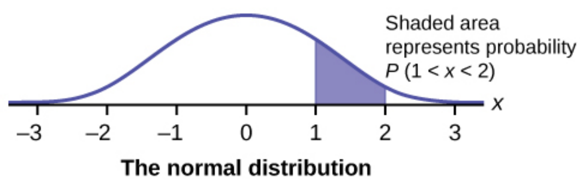

Figure $7.5.4$ The graph shows the Standard Normal Distribution with the area between $x=1$ and $x=2$ shaded to represent the probability that the value of the random variable X is in the interval between one and two.

Glossary

Uniform Distribution

a continuous random variable (RV) that has equally likely outcomes over the domain, $a<x<b$; it is often referred as the rectangular distribution because the graph of the pdf has the form of a rectangle. Notation: $X \sim U(a,b)$. The mean is $\mu=\frac{a+b}{2}$ and the standard deviation is $\sigma=\sqrt{\frac{(b-a)^2}{12}}$. The probability density function is $f(x)=\frac{1}{b-a}$ for $a<x<b$ or $a \leq x \leq b$. The cumulative distribution is $P(X \leq x)=\frac{x-a}{b-a}$.

Exponential Distribution

a continuous random variable (RV) that appears when we are interested in the intervals of time between some random events, for example, the length of time between emergency arrivals at a hospital; the notation is $X \sim \operatorname{Exp}(m)$. The mean is $\mu=\frac{1}{m}$ and the standard deviation is $\sigma=\frac{1}{m}$. The probability density function is $f(x)=me^{-mx}$, $x \geq 0$ and the cumulative distribution function is $P(X \leq x)=1-e^{mx}$.

7.6: Continuous Probability Functions

We begin by defining a continuous probability density function. We use the function notation $f(x)$. Intermediate algebra may have been your first formal introduction to functions. In the study of probability, the functions we study are special. We define the function $f(x)$ so that the area between it and the x-axis is equal to a probability. Since the maximum probability is one, the maximum area is also one. **For continuous probability distributions, PROBABILITY = AREA.**

Example 7.6.1

Consider the function $f(x) = \frac{1}{20}$ for $0 \leq x \leq 20$. $x =$ a real number. The graph of $f(x) = \frac{1}{20}$ is a horizontal line. However, since $0 \leq x \leq 20$, $f(x)$ is restricted to the portion between $x = 0$ and $x = 20$, inclusive.

Figure 7.6.1

$$f(x) = \frac{1}{20} \text{ for } 0 \leq x \leq 20. \tag{7.6.1}$$

The graph of $f(x) = \frac{1}{20}$ is a horizontal line segment when $0 \leq x \leq 20$.

The area between $f(x) = \frac{1}{20}$ where $0 \leq x \leq 20$ and the *x*-axis is the area of a rectangle with base = 20 and height = $\frac{1}{20}$.

$$AREA = 20\left(\frac{1}{20}\right) = 1 \tag{7.6.2}$$

Suppose we want to find the area between $f(x) = \frac{1}{20}$ and the *x*-axis where $0 < x < 2$.

Figure 7.6.2

$$AREA = (2-0)\left(\frac{1}{20}\right) = 0.1 \tag{7.6.3}$$

$$(2-0) = 2 = \text{base of a rectangle}$$

REMINDER: area of a rectangle = (base)(height).

The area corresponds to a probability. The probability that *x* is between zero and two is 0.1, which can be written mathematically as $P(0 < x < 2) = P(x < 2) = 0.1$.

Suppose we want to find the area between $f(x) = \frac{1}{20}$ and the *x*-axis where $4 < x < 15$.

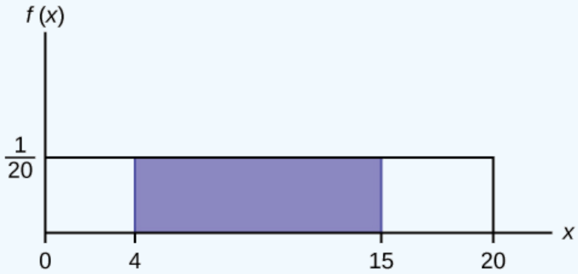

Figure 7.6.3

$\text{AREA} = (15–4)(\frac{1}{20}) = 0.55$

$\text{AREA} = (15–4)(\frac{1}{20}) = 0.55$

$(15–4) = 11 = \text{the base of a rectangle}(15–4) = 11 = \text{the base of a rectangle}$

The area corresponds to the probability $P(4 < x < 15) = 0.55$.

Suppose we want to find $P(x = 15)$. On an x-y graph, $x = 15$ is a vertical line. A vertical line has no width (or zero width). Therefore, $P(x = 15) = (\text{base})(\text{height}) = (0)\left(\frac{1}{20}\right) = 0$

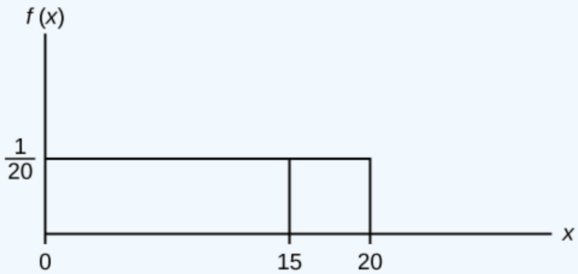

Figure 7.6.4

$P(X \leq x)$ (can be written as $P(X < x)$ for continuous distributions) is called the cumulative distribution function or CDF. Notice the "less than or equal to" symbol. We can use the CDF to calculate $P(X > x)$. The CDF gives "area to the left" and $P(X > x)$ gives "area to the right." We calculate $P(X > x)$ for continuous distributions as follows: $P(X > x) = 1 – P(X < x)$.

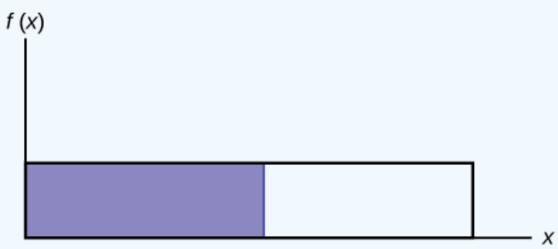

Figure 7.6.5

Label the graph with $f(x)$ and x. Scale the x and y axes with the maximum x and y values. $f(x) = \frac{1}{20}$, $0 \leq x \leq 20$.

To calculate the probability that x is between two values, look at the following graph. Shade the region between $x = 2.3$ and $x = 12.7$. Then calculate the shaded area of a rectangle.

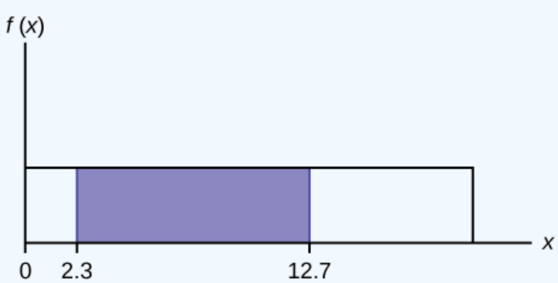

Figure 7.6.6

$$P(2.3 < x < 12.7) = (\text{base})(\text{height}) = (12.7 - 2.3)\left(\frac{1}{20}\right) = 0.52 \tag{7.6.4}$$

? Exercise 7.6.1

Consider the function $f(x) = \frac{1}{8}$ for $0 \leq x \leq 8$. Draw the graph of $f(x)$ and find $P(2.5 < x < 7.5)$.

Answer

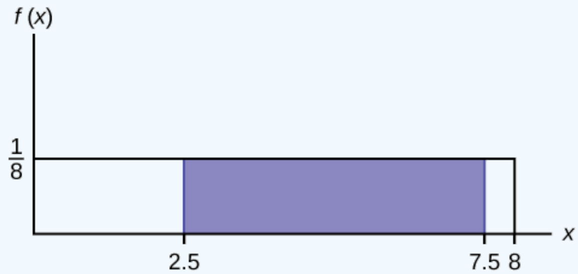

Figure 7.6.7

$P(2.5 < x < 7.5) = 0.625$

Summary

The probability density function (pdf) is used to describe probabilities for continuous random variables. The area under the density curve between two points corresponds to the probability that the variable falls between those two values. In other words, the area under the density curve between points a and b is equal to $P(a < x < b)$. The cumulative distribution function (cdf) gives the probability as an area. If X is a continuous random variable, the probability density function (pdf), $f(x)$, is used to draw the graph of the probability distribution. The total area under the graph of $f(x)$ is one. The area under the graph of $f(x)$ and between values a and b gives the probability $P(a < x < b)$.

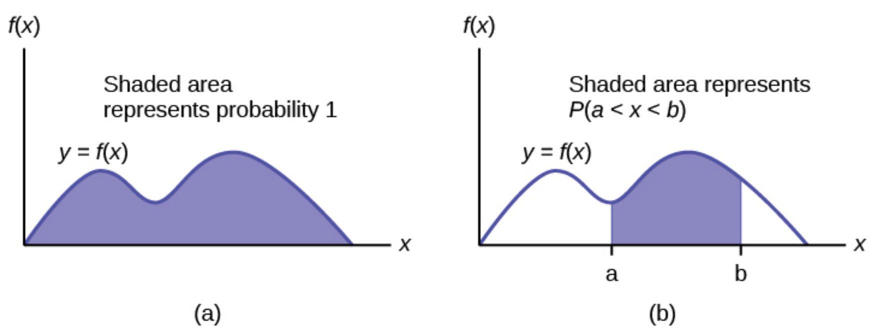

Figure 7.6.8

The cumulative distribution function (cdf) of X is defined by $P(X \leq x)$. It is a function of x that gives the probability that the random variable is less than or equal to x.

Formula Review

Probability density function (pdf) $f(x)$:

- $f(x) \geq 0$
- The total area under the curve $f(x)$ is one.

Cumulative distribution function (cdf): $P(X \leq x)$

7.7: Prelude to The Normal Distribution

Learning Objectives

By the end of this chapter, the student should be able to:

- Recognize the normal probability distribution and apply it appropriately.
- Recognize the standard normal probability distribution and apply it appropriately.
- Compare normal probabilities by converting to the standard normal distribution.

The normal, a continuous distribution, is the most important of all the distributions. It is widely used and even more widely abused. Its graph is bell-shaped. You see the bell curve in almost all disciplines. Some of these include psychology, business, economics, the sciences, nursing, and, of course, mathematics. Some of your instructors may use the normal distribution to help determine your grade. Most IQ scores are normally distributed. Often real-estate prices fit a normal distribution. The normal distribution is extremely important, but it cannot be applied to everything in the real world.

Figure $7.7.1$: If you ask enough people about their shoe size, you will find that your graphed data is shaped like a bell curve and can be described as normally distributed. (credit: Ömer Ünlü)

In this chapter, you will study the normal distribution, the standard normal distribution, and applications associated with them. The normal distribution has two parameters (two numerical descriptive measures), the mean (μ) and the standard deviation (σ). If X is a quantity to be measured that has a normal distribution with mean (μ) and standard deviation (σ), we designate this by writing

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{\left(-\frac{1}{2}\right) \cdot \left(\frac{x-\mu}{\sigma}\right)^2} \tag{7.7.1}$$

The probability density function is a rather complicated function. **Do not memorize it**. It is not necessary.

The cumulative distribution function is $P(X < x)$. It is calculated either by a calculator or a computer, or it is looked up in a table. Technology has made the tables virtually obsolete. For that reason, as well as the fact that there are various table formats, we are not including table instructions.

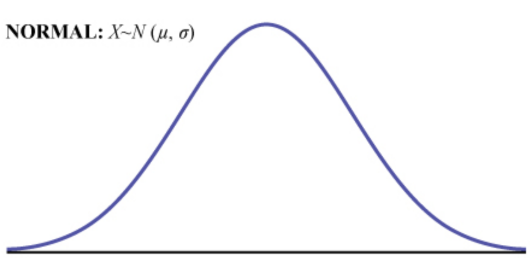

Figure $7.7.2$ The standard normal distribution

The curve is symmetrical about a vertical line drawn through the mean, μ. In theory, the mean is the same as the median, because the graph is symmetric about μ. As the notation indicates, the normal distribution depends only on the mean and the standard deviation. Since the area under the curve must equal one, a change in the standard deviation, σ, causes a change in the shape of the curve; the curve becomes fatter or skinnier depending on σ. A change in μ causes the graph to shift to the left or right. This means there are an infinite number of normal probability distributions. One of special interest is called the **standard normal distribution**.

COLLABORATIVE CLASSROOM ACTIVITY

Your instructor will record the heights of both men and women in your class, separately. Draw histograms of your data. Then draw a smooth curve through each histogram. Is each curve somewhat bell-shaped? Do you think that if you had recorded 200 data values for men and 200 for women that the curves would look bell-shaped? Calculate the mean for each data set. Write the means on the *x*-axis of the appropriate graph below the peak. Shade the approximate area that represents the probability that one randomly chosen male is taller than 72 inches. Shade the approximate area that represents the probability that one randomly chosen female is shorter than 60 inches. If the total area under each curve is one, does either probability appear to be more than 0.5?

Formula Review

- $X \sim N(\mu, \sigma)$
- $\mu =$ the mean $\sigma =$ the standard deviation

Glossary

Normal Distribution

a continuous random variable (RV) with pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x \cdot \mu)^2}{2\sigma^2}} \tag{7.7.2}$$

, where μ is the mean of the distribution and σ is the standard deviation; notation: $X \sim N(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, the RV is called the **standard normal distribution**.

7.8: The Standard Normal Distribution

Z-Scores

The standard normal distribution is a normal distribution of standardized values called *z-scores*. A z-score is measured in units of the standard deviation.

Definition: Z-Score

If X is a normally distributed random variable and $X \sim N(\mu, \sigma)$, then the *z*-score is:

$$z=\frac{x-\mu}{\sigma} \tag{7.8.1}$$

The *z*-score tells you how many standard deviations the value x is above (to the right of) or below (to the left of) the mean, μ. Values of x that are larger than the mean have positive z-scores, and values of x that are smaller than the mean have negative z-scores. If x equals the mean, then x has a z-score of zero. For example, if the mean of a normal distribution is five and the standard deviation is two, the value 11 is three standard deviations above (or to the right of) the mean. The calculation is as follows:

$$\begin{aligned} x &= \mu+(z)(\sigma) \\ &= 5+(3)(2)=11 \end{aligned}$$

The *z*-score is three.

Since the mean for the standard normal distribution is zero and the standard deviation is one, then the transformation in Equation 7.8.1 produces the distribution $Z \sim N(0,1)$. The value x comes from a normal distribution with mean μ and standard deviation σ.

> *A z-score is measured in units of the standard deviation.*

Example 7.8.1

Suppose $X \sim N(5,6)$. This says that x is a normally distributed random variable with mean $\mu=5$ and standard deviation $\sigma=6$. Suppose $x=17$. Then (via Equation 7.8.1):

$$z=\frac{x-\mu}{\sigma}=\frac{17-5}{6}=2$$

This means that $x=17$ is **two** standard deviations (2σ) above or to the right of the mean $\mu=5$. The standard deviation is $\sigma=6$.

Notice that: $5+(2)(6)=17$ (The pattern is $\mu+z\sigma=x$)

Now suppose $x=1$. Then:

$$z=\frac{x-\mu}{\sigma}=\frac{1-5}{6}=-0.67$$

(rounded to two decimal places)

This means that $x=1$ is 0.67 standard deviations (-0.67σ) below or to the left of the mean $\mu=5$. Notice that: $5+(-0.67)(6)$ is approximately equal to one (This has the pattern $\mu+(-0.67)\sigma=1$)

Summarizing, when z is positive, x is above or to the right of μ and when z is negative, x is to the left of or below μ. Or, when z is positive, x is greater than μ, and when z is negative x is less than μ.

Exercise 7.8.1

What is the z-score of x, when $x=1$ and $X \sim N(12,3)$?

Answer

$$z = \frac{1-12}{3} \approx -3.67$$

Example 7.8.2

Some doctors believe that a person can lose five pounds, on the average, in a month by reducing his or her fat intake and by exercising consistently. Suppose weight loss has a normal distribution. Let $X =$ the amount of weight lost(in pounds) by a person in a month. Use a standard deviation of two pounds. $X \sim N(5, 2)$. Fill in the blanks.

a. Suppose a person **lost** ten pounds in a month. The z-score when $x = 10$ pounds is $z = 2.5$ (verify). This z-score tells you that $x = 10$ is ________ standard deviations to the ________ (right or left) of the mean _____ (What is the mean?).
b. Suppose a person **gained** three pounds (a negative weight loss). Then $z =$ __________. This z-score tells you that $x = -3$ is ________ standard deviations to the __________ (right or left) of the mean.

Answers

a. This z-score tells you that $x = 10$ is 2.5 standard deviations to the right of the mean five.

b. Suppose the random variables X and Y have the following normal distributions: $X \sim N(5, 6)$ and $Y \sim N(2, 1)$. If $x = 17$, then $z = 2$. (This was previously shown.) If $y = 4$, what is z?

$$z = \frac{y-\mu}{\sigma} = \frac{4-2}{1} = 2$$

where $\mu = 2$ and $\sigma = 1$.

The z-score for $y = 4$ is $z = 2$. This means that four is $z = 2$ standard deviations to the right of the mean. Therefore, $x = 17$ and $y = 4$ are both two (of their own) standard deviations to the right of their respective means.

The z-score allows us to compare data that are scaled differently. To understand the concept, suppose $X \sim N(5, 6)$ represents weight gains for one group of people who are trying to gain weight in a six week period and $Y \sim N(2, 1)$ measures the same weight gain for a second group of people. A negative weight gain would be a weight loss. Since $x = 17$ and $y = 4$ are each two standard deviations to the right of their means, they represent the same, standardized weight gain **relative to their means**.

Exercise 7.8.2

Fill in the blanks.

Jerome averages 16 points a game with a standard deviation of four points. $X \sim N(16, 4)$. Suppose Jerome scores ten points in a game. The z–score when $x = 10$ is -1.5. This score tells you that $x = 10$ is _____ standard deviations to the ______(right or left) of the mean______(What is the mean?).

Answer

1.5, left, 16

The Empirical Rule

If X is a random variable and has a normal distribution with mean μ and standard deviation σ, then the *Empirical Rule* says the following:

- About 68% of the x values lie between –1σ and +1σ of the mean μ (within one standard deviation of the mean).
- About 95% of the x values lie between –2σ and +2σ of the mean μ (within two standard deviations of the mean).
- About 99.7% of the x values lie between –3σ and +3σ of the mean μ (within three standard deviations of the mean). Notice that almost all the x values lie within three standard deviations of the mean.
- The z-scores for +1σ and –1σ are +1 and –1, respectively.
- The z-scores for +2σ and –2σ are +2 and –2, respectively.
- The z-scores for +3σ and –3σ are +3 and –3 respectively.

The empirical rule is also known as the 68-95-99.7 rule.

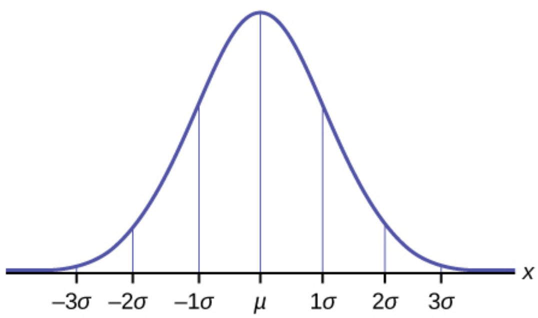

Figure $7.8.1$

Example $7.8.3$

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let $X =$ the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then $X \sim N(170, 6.28)$.

a. Suppose a 15 to 18-year-old male from Chile was 168 cm tall from 2009 to 2010. The z-score when $x = 168$ cm is $z =$ _______. This z-score tells you that $x = 168$ is ________ standard deviations to the ________ (right or left) of the mean _____ (What is the mean?).
b. Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a z-score of $z = 1.27$. What is the male's height? The z-score ($z = 1.27$) tells you that the male's height is ________ standard deviations to the __________ (right or left) of the mean.

Answers

a. –0.32, 0.32, left, 170
b. 177.98, 1.27, right

Exercise $7.8.3$

Use the information in Example $7.8.3$ to answer the following questions.

a. Suppose a 15 to 18-year-old male from Chile was 176 cm tall from 2009 to 2010. The z-score when $x = 176$ cm is $z =$ _______. This z-score tells you that $x = 176$ cm is ________ standard deviations to the ________ (right or left) of the mean _____ (What is the mean?).
b. Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a z-score of $z =- 2$. What is the male's height? The z-score ($z =- 2$) tells you that the male's height is ________ standard deviations to the __________ (right or left) of the mean.

Answer

Solve the equation $z = \dfrac{x-\mu}{\sigma}$ for z. $x = \mu + (z)(\sigma)$

$z = \dfrac{176-170}{6.28}$, This z-score tells you that $x = 176$ cm is 0.96 standard deviations to the right of the mean 170 cm.

Answer

Solve the equation $z = \dfrac{x-\mu}{\sigma}$ for z. $x = \mu + (z)(\sigma)$

$X = 157.44$ cm, The z-score($z =- 2$) tells you that the male's height is two standard deviations to the left of the mean.

Example $7.8.4$

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let $Y =$ the height of 15 to 18-year-old males from 1984 to 1985. Then $Y \sim N(172.36, 6.34)$.

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let $X =$ the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then $X \sim N(170, 6.28)$.

Find the z-scores for $x = 160.58$ cm and $y = 162.85$ cm. Interpret each z-score. What can you say about $x = 160.58$ cm and $y = 162.85$ cm?

Answer

- The z-score (Equation 7.8.1) for $x = 160.58$ is $z = -1.5$.
- The z-score for $y = 162.85$ is $z = -1.5$.

Both $x = 160.58$ and $y = 162.85$ deviate the same number of standard deviations from their respective means and in the same direction.

Exercise 7.8.4

In 2012, 1,664,479 students took the SAT exam. The distribution of scores in the verbal section of the SAT had a mean $\mu = 496$ and a standard deviation $\sigma = 114$. Let $X =$ a SAT exam verbal section score in 2012. Then $X \sim N(496, 114)$.

Find the z-scores for $x_1 = 325$ and $x_2 = 366.21$. Interpret each z-score. What can you say about $x_1 = 325$ and $x_2 = 366.21$?

Answer

The z-score (Equation 7.8.1) for $x_1 = 325$ is $z_1 = -1.15$.

The z-score (Equation 7.8.1) for $x_2 = 366.21$ is $z_2 = -1.14$.

Student 2 scored closer to the mean than Student 1 and, since they both had negative z-scores, Student 2 had the better score.

Example 7.8.5

Suppose *x* has a normal distribution with mean 50 and standard deviation 6.

- About 68% of the *x* values lie within one standard deviation of the mean. Therefore, about 68% of the *x* values lie between –1σ = (–1)(6) = –6 and 1σ = (1)(6) = 6 of the mean 50. The values 50 – 6 = 44 and 50 + 6 = 56 are within one standard deviation from the mean 50. The z-scores are –1 and +1 for 44 and 56, respectively.
- About 95% of the x values lie within two standard deviations of the mean. Therefore, about 95% of the x values lie between –2σ = (–2)(6) = –12 and 2σ = (2)(6) = 12. The values 50 – 12 = 38 and 50 + 12 = 62 are within two standard deviations from the mean 50. The z-scores are –2 and +2 for 38 and 62, respectively.
- About 99.7% of the x values lie within three standard deviations of the mean. Therefore, about 99.7% of the x values lie between –3σ = (–3)(6) = –18 and 3σ = (3)(6) = 18 from the mean 50. The values 50 – 18 = 32 and 50 + 18 = 68 are within three standard deviations of the mean 50. The z-scores are –3 and +3 for 32 and 68, respectively.

Exercise 7.8.5

Suppose X has a normal distribution with mean 25 and standard deviation five. Between what values of x do 68% of the values lie?

Answer

between 20 and 30.

Example 7.8.6

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let $Y =$ the height of 15 to 18-year-old males in 1984 to 1985. Then $Y \sim N(172.36, 6.34)$.

a. About 68% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.
b. About 95% of the y values lie between what two values? These values are ________________. The z-scores are ________________ respectively.
c. About 99.7% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.

Answer

a. About 68% of the values lie between 166.02 and 178.7. The z-scores are –1 and 1.
b. About 95% of the values lie between 159.68 and 185.04. The z-scores are –2 and 2.
c. About 99.7% of the values lie between 153.34 and 191.38. The z-scores are –3 and 3.

Exercise 7.8.6

The scores on a college entrance exam have an approximate normal distribution with mean, $\mu = 52$ points and a standard deviation, $\sigma = 11$ points.

a. About 68% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.
b. About 95% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.
c. About 99.7% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.

Answer a

About 68% of the values lie between the values 41 and 63. The z-scores are –1 and 1, respectively.

Answer b

About 95% of the values lie between the values 30 and 74. The z-scores are –2 and 2, respectively.

Answer c

About 99.7% of the values lie between the values 19 and 85. The z-scores are –3 and 3, respectively.

Summary

A z-score is a standardized value. Its distribution is the standard normal, $Z \sim N(0,1)$. The mean of the z-scores is zero and the standard deviation is one. If y is the z-score for a value x from the normal distribution $N(\mu,\sigma)$ then z tells you how many standard deviations x is above (greater than) or below (less than) μ.

Formula Review

$Z \sim N(0,1)$

$z = a$ standardized value (z-score)

mean = 0; standard deviation = 1

To find the K^{th} percentile of X when the z-scores is known:

$k = \mu + (z)\sigma$

z-score: $z = \dfrac{x-\mu}{\sigma}$

$Z =$ the random variable for z-scores

$Z \sim N(0,1)$

Glossary

Standard Normal Distribution

a continuous random variable (RV) $X \sim N(0,1)$; when X follows the standard normal distribution, it is often noted as \(Z \sim N(0, 1)\.

***z*-score**

the linear transformation of the form $z = \frac{x-\mu}{\sigma}$; if this transformation is applied to any normal distribution $X \sim N(\mu, \sigma$ the result is the standard normal distribution $Z \sim N(0,1)$. If this transformation is applied to any specific value x of the RV with mean μ and standard deviation σ, the result is called the z-score of x. The z-score allows us to compare data that are normally distributed but scaled differently.

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7.8E: The Standard Normal Distribution (Exercises)

? Exercise $7.8E.7$

A bottle of water contains 12.05 fluid ounces with a standard deviation of 0.01 ounces. Define the random variable X in words. $X =$ ____________.

Answer

ounces of water in a bottle

? Exercise $7.8E.8$

A normal distribution has a mean of 61 and a standard deviation of 15. What is the median?

? Exercise $7.8E.9$

$X \sim N(1, 2)$

$\sigma =$ _______

Answer

2

? Exercise $7.8E.10$

A company manufactures rubber balls. The mean diameter of a ball is 12 cm with a standard deviation of 0.2 cm. Define the random variable X in words. $X =$ ______________.

? Exercise $7.8E.11$

$X \sim N(-4, 1)$

What is the median?

Answer

–4

? Exercise $7.8E.12$

$X \sim N(3, 5)$

$\sigma =$ _______

? Exercise $7.8E.13$

$X \sim N(-2, 1)$

$\mu =$ _______

Answer

–2

Exercise $7.8E.14$

What does a z-score measure?

Exercise $7.8E.15$

What does standardizing a normal distribution do to the mean?

Answer

The mean becomes zero.

Exercise $7.8E.16$

Is $X \sim N(0,1)$ a standardized normal distribution? Why or why not?

Exercise $7.8E.17$

What is the z-score of $x = 12$, if it is two standard deviations to the right of the mean?

Answer

$z = 2$

Exercise $7.8E.18$

What is the z-score of $x = 9$, if it is 1.5 standard deviations to the left of the mean?

Exercise $7.8E.19$

What is the z-score of $x = -2$, if it is 2.78 standard deviations to the right of the mean?

Answer

$z = 2.78$

Exercise $7.8E.20$

What is the z-score of $x = 7$, if it is 0.133 standard deviations to the left of the mean?

Exercise $7.8E.21$

Suppose $X \sim N(2,6)$. What value of *x* has a *z*-score of three?

Answer

$x = 20$

Exercise $7.8E.22$

Suppose $X \sim N(8,1)$. What value of x has a z-score of –2.25?

Exercise $7.8E.23$

Suppose $X \sim N(9,5)$. What value of x has a z-score of –0.5?

Answer

$x = 6.5$

Exercise $7.8E.24$

Suppose $X \sim N(2,3)$. What value of x has a z-score of –0.67?

Exercise $7.8E.25$

Suppose $X \sim N(4,2)$. What value of x is 1.5 standard deviations to the left of the mean?

Answer

$x=1$

Exercise $7.8E.26$

Suppose $X \sim N(4,2)$. What value of x is two standard deviations to the right of the mean?

Exercise $7.8E.27$

Suppose $X \sim N(8,9)$. What value of x is 0.67 standard deviations to the left of the mean?

Answer

$x=1.97$

Exercise $7.8E.28$

Suppose $X \sim N(-1,12)$. What is the z-score of $x=2$?

Exercise $7.8E.29$

Suppose $X \sim N(12,6)$. What is the z-score of $x=2$?

Answer

$z=-1.67$

Exercise $7.8E.30$

Suppose $X \sim N(9,3)$. What is the z-score of $x=9$?

Exercise $7.8E.31$

Suppose a normal distribution has a mean of six and a standard deviation of 1.5. What is the z-score of $x=5.5$?

Answer

$z \approx -0.33$

Exercise $7.8E.32$

In a normal distribution, $x=5$ and $z=-1.25$. This tells you that $x=5$ is ____ standard deviations to the ____ (right or left) of the mean.

Exercise $7.8E.33$

In a normal distribution, $x=3$ and $z=0.67$. This tells you that $x=3$ is ____ standard deviations to the ____ (right or left) of the mean.

Answer

0.67, right

Exercise $7.8E.34$

In a normal distribution, $x = -2$ and $z = 6$. This tells you that $z = -2$ is ____ standard deviations to the ____ (right or left) of the mean.

Exercise $7.8E.35$

In a normal distribution, $x = -5$ and $z = -3.14$. This tells you that $x = -5$ is ____ standard deviations to the ____ (right or left) of the mean.

Answer

3.14, left

Exercise $7.8E.36$

In a normal distribution, $x = 6$ and $z = -1.7$. This tells you that $x = 6$ is ____ standard deviations to the ____ (right or left) of the mean.

Exercise $7.8E.37$

About what percent of x values from a normal distribution lie within one standard deviation (left and right) of the mean of that distribution?

Answer

about 68%

Exercise $7.8E.38$

About what percent of the x values from a normal distribution lie within two standard deviations (left and right) of the mean of that distribution?

Exercise $7.8E.39$

About what percent of x values lie between the second and third standard deviations (both sides)?

Answer

about 4%

Exercise $7.8E.40$

Suppose $X \sim N(15, 3)$. Between what x values does 68.27% of the data lie? The range of $x$values is centered at the mean of the distribution (i.e., 15).

Exercise $7.8E.41$

Suppose $X \sim N(-3, 1)$. Between what x values does 95.45% of the data lie? The range of $x$values is centered at the mean of the distribution (i.e., –3).

Answer

between –5 and –1

? Exercise 7.8E.42

Suppose $X \sim N(-3,1)$. Between what x values does 34.14% of the data lie?

? Exercise 7.8E.43

About what percent of x values lie between the mean and three standard deviations?

Answer

about 50%

? Exercise 7.8E.44

About what percent of x values lie between the mean and one standard deviation?

? Exercise 7.8E.45

About what percent of x values lie between the first and second standard deviations from the mean (both sides)?

Answer

about 27%

? Exercise 7.8E.46

About what percent of x values lie between the first and third standard deviations(both sides)?

Use the following information to answer the next two exercises: The life of Sunshine CD players is normally distributed with mean of 4.1 years and a standard deviation of 1.3 years. A CD player is guaranteed for three years. We are interested in the length of time a CD player lasts.

? Exercise 7.8E.47

Define the random variable X in words. $X =$ _______________.

Answer

The lifetime of a Sunshine CD player measured in years.

? Exercise 7.8E.48

$X \sim$ _____(_____,_____)

7.9: Using the Normal Distribution

The shaded area in the following graph indicates the area to the left of x. This area is represented by the probability $P(X < x)$. Normal tables, computers, and calculators provide or calculate the probability $P(X < x)$.

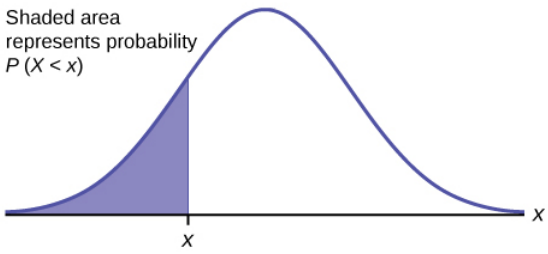

Figure $7.9.1$

The area to the right is then $P(X > x) = 1 – P(X < x)$. Remember, $P(X < x) =$ **Area to the left** of the vertical line through x. $P(X > x) = 1 – P(X < x) =$ **Area to the right** of the vertical line through x. $P(X < x)$ is the same as $P(X \leq x)$ and $P(X > x)$ is the same as $P(X \geq x)$ for continuous distributions.

Calculations of Probabilities

Probabilities are calculated using technology. There are instructions given as necessary for the TI-83+ and TI-84 calculators.To calculate the probability, use the probability tables provided in [link] without the use of technology. The tables include instructions for how to use them.

Example $7.9.1$

If the area to the left is 0.0228, then the area to the right is $1 - 0.0228 = 0.9772$.

Exercise $7.9.1$

If the area to the left of x is 0.012, then what is the area to the right?

Answer

$1 - 0.012 = 0.988$

Example $7.9.2$

The final exam scores in a statistics class were normally distributed with a mean of 63 and a standard deviation of five.

a. Find the probability that a randomly selected student scored more than 65 on the exam.
b. Find the probability that a randomly selected student scored less than 85.
c. Find the 90$^{\text{th}}$ percentile (that is, find the score k that has 90% of the scores below k and 10% of the scores above k).
d. Find the 70$^{\text{th}}$ percentile (that is, find the score k such that 70% of scores are below $k$and 30% of the scores are above k).

Answer

a. Let X = a score on the final exam. $X \sim N(63, 5)$, where $\mu = 63$ and $\sigma = 5$

Draw a graph.

Then, find $P(x > 65)$.

$$P(x > 65) = 0.3446$$

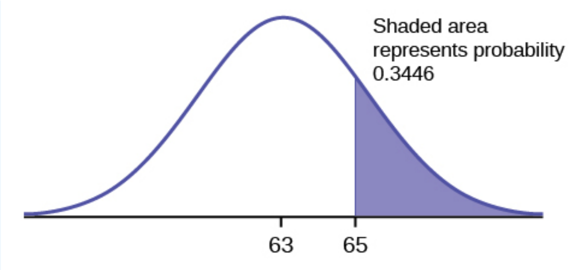

Figure $7.9.2$

The probability that any student selected at random scores more than 65 is 0.3446.

USING THE TI-83, 83+, 84, 84+ CALCULATOR

Go into `2nd DISTR` .

After pressing `2nd DISTR` , press `2:normalcdf` .

The syntax for the instructions are as follows:

normalcdf(lower value, upper value, mean, standard deviation) For this problem: normalcdf(65,1E99,63,5) = 0.3446. You get 1E99 (= 10^{99}) by pressing `1` , the `EE` key (a 2nd key) and then `99` . Or, you can enter `10^ 99` instead. The number 10^{99} is way out in the right tail of the normal curve. We are calculating the area between 65 and 10^{99}. In some instances, the lower number of the area might be –1E99 (= -10^{99}). The number -10^{99} is way out in the left tail of the normal curve.

Historical Note

The TI probability program calculates a z-score and then the probability from the z-score. Before technology, the z-score was looked up in a standard normal probability table (because the math involved is too cumbersome) to find the probability. In this example, a standard normal table with area to the left of the z-score was used. You calculate the z-score and look up the area to the left. The probability is the area to the right.

$$z = 65 - 63565 - 635 = 0.4$$

Area to the left is 0.6554.

$$P(x > 65) = P(z > 0.4) = 1 - 0.6554 = 0.3446$$

USING THE TI-83, 83+, 84, 84+ CALCULATOR

Find the percentile for a student scoring 65:

*Press `2nd Distr`
*Press `2:normalcdf` (
*Enter lower bound, upper bound, mean, standard deviation followed by)
*Press `ENTER` .
For this Example, the steps are
`2nd Distr`
`2:normalcdf` (65,1,2nd EE,99,63,5) `ENTER`

The probability that a selected student scored more than 65 is 0.3446.
To find the probability that a selected student scored *more than* 65, subtract the percentile from 1.

Answer

b. Draw a graph.

Then find $P(x < 85)$, and shade the graph.

Using a computer or calculator, find $P(x < 85) = 1$.

$\text{normalcdf}(0, 85, 63, 5) = 1 \text{(rounds to one)}$

The probability that one student scores less than 85 is approximately one (or 100%).

Answer

c. Find the 90th percentile. For each problem or part of a problem, draw a new graph. Draw the x-axis. Shade the area that corresponds to the 90th percentile.

Let $k =$ the 90th percentile. The variable k is located on the x-axis. $P(x < k)$ is the area to the left of k. The 90th percentile k separates the exam scores into those that are the same or lower than k and those that are the same or higher. Ninety percent of the test scores are the same or lower than k, and ten percent are the same or higher. The variable k is often called a critical value.

$k = 69.4$

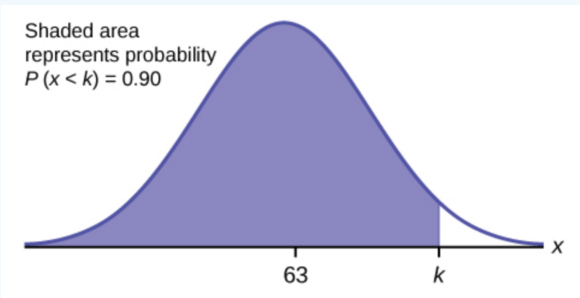

Figure 7.9.3

The 90th percentile is 69.4. This means that 90% of the test scores fall at or below 69.4 and 10% fall at or above. To get this answer on the calculator, follow this step:

`invNorm` in `2nd DISTR`. invNorm(area to the left, mean, standard deviation)

For this problem, $\text{invNorm}(0.90, 63, 5) = 69.4$

Answer

d. Find the 70th percentile.

Draw a new graph and label it appropriately. $k = 65.6$

The 70th percentile is 65.6. This means that 70% of the test scores fall at or below 65.6 and 30% fall at or above.

$$\text{invNorm}(0.70, 63, 5) = 65.6$$

Exercise 7.9.2

The golf scores for a school team were normally distributed with a mean of 68 and a standard deviation of three. Find the probability that a randomly selected golfer scored less than 65.

Answer

$\text{normalcdf}(10^{99}, 65, 68, 3) = 0.1587$

Example 7.9.3

A personal computer is used for office work at home, research, communication, personal finances, education, entertainment, social networking, and a myriad of other things. Suppose that the average number of hours a household personal computer is used for entertainment is two hours per day. Assume the times for entertainment are normally distributed and the standard deviation for the times is half an hour.

a. Find the probability that a household personal computer is used for entertainment between 1.8 and 2.75 hours per day.

b. Find the maximum number of hours per day that the bottom quartile of households uses a personal computer for entertainment.

Answer

a. Let $X =$ the amount of time (in hours) a household personal computer is used for entertainment. $X \sim N(2, 0.5)$ where $\mu = 2$ and $\sigma = 0.5$.

Find $P(1.8 < x < 2.75)$.

The probability for which you are looking is the area **between** $x = 1.8$ and $x = 2.75$. $P(1.8 < x < 2.75) = 0.5886$

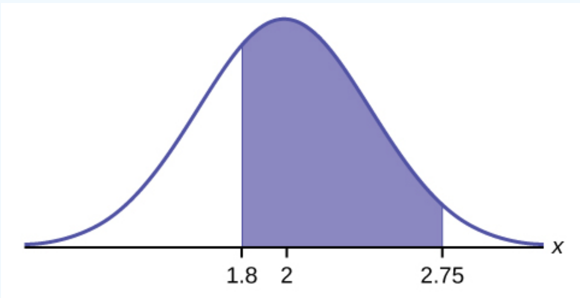

Figure $7.9.4$

$$\text{normalcdf}(1.8, 2.75, 2, 0.5) = 0.5886$$

The probability that a household personal computer is used between 1.8 and 2.75 hours per day for entertainment is 0.5886.

b.

To find the maximum number of hours per day that the bottom quartile of households uses a personal computer for entertainment, **find the 25th percentile,** k, where $P(x < k) = 0.25$.

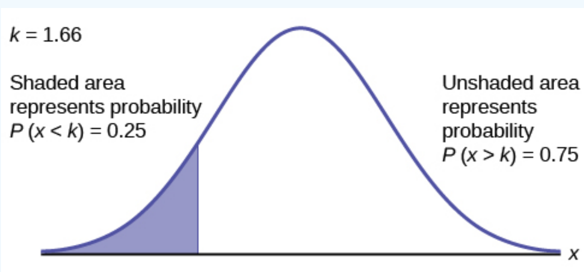

Figure $7.9.5$

$$\text{invNorm}(0.25, 2, 0.5) = 1.66$$

The maximum number of hours per day that the bottom quartile of households uses a personal computer for entertainment is 1.66 hours.

Exercise $7.9.3$

The golf scores for a school team were normally distributed with a mean of 68 and a standard deviation of three. Find the probability that a golfer scored between 66 and 70.

Answer

$$\text{normalcdf}(66, 70, 68, 3) = 0.4950$$

Example $7.9.4$

There are approximately one billion smartphone users in the world today. In the United States the ages 13 to 55+ of smartphone users approximately follow a normal distribution with approximate mean and standard deviation of 36.9 years and 13.9 years, respectively.

a. Determine the probability that a random smartphone user in the age range 13 to 55+ is between 23 and 64.7 years old.
b. Determine the probability that a randomly selected smartphone user in the age range 13 to 55+ is at most 50.8 years old.
c. Find the 80th percentile of this distribution, and interpret it in a complete sentence.

Answer

a. $\text{normalcdf}(23, 64.7, 36.9, 13.9) = 0.8186$
b. $\text{normalcdf}(-10^{99}, 50.8, 36.9, 13.9) = 0.8413$
c. $\text{invNorm}(0.80, 36.9, 13.9) = 48.6$

The 80th percentile is 48.6 years.

80% of the smartphone users in the age range 13 – 55+ are 48.6 years old or less.

Use the information in Example to answer the following questions.

Exercise 7.9.4

a. Find the 30th percentile, and interpret it in a complete sentence.
b. What is the probability that the age of a randomly selected smartphone user in the range 13 to 55+ is less than 27 years old and at least 0 years old?

70.

Answer

Let $X =$ a smart phone user whose age is 13 to 55+. $X \sim N(36.9, 13.9)$

To find the 30th percentile, find k such that $P(x < k) = 0.30$.
$\text{invNorm}(0.30, 36.9, 13.9) = 29.6$ years
Thirty percent of smartphone users 13 to 55+ are at most 29.6 years and 70% are at least 29.6 years. Find $P(x < 27)$
(Note that $\text{normalcdf}(-10^{99}, 27, 36.9, 13.9) = 0.2382$. The two answers differ only by 0.0040.)

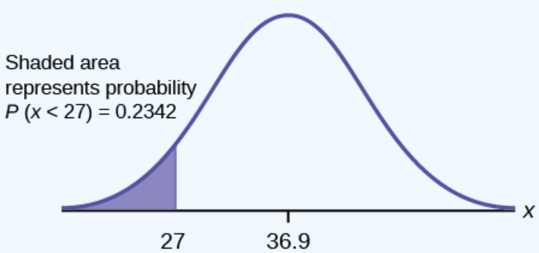

Figure 7.9.6

$$\text{normalcdf}(0, 27, 36.9, 13.9) = 0.2342$$

Example 7.9.5

In the United States the ages 13 to 55+ of smartphone users approximately follow a normal distribution with approximate mean and standard deviation of 36.9 years and 13.9 years respectively. Using this information, answer the following questions (round answers to one decimal place).

a. Calculate the interquartile range (IQR).
b. Forty percent of the ages that range from 13 to 55+ are at least what age?

Answer

a.

$$IQR = Q_3 - Q_1$$

Calculate $Q_3 = 75^{\text{th}}$ percentile and $Q_1 = 25^{\text{th}}$ percentile.

$$\text{invNorm}(0.75, 36.9, 13.9) = Q_3 = 46.2754$$

$$\text{invNorm}(0.25, 36.9, 13.9) = Q_1 = 27.5246$$

$$IQR = Q_3 - Q_1 = 18.7508$$

b.

Find k where $P(x > k) = 0.40$ ("At least" translates to "greater than or equal to.")

$0.40 =$ the area to the right.

Area to the left $= 1 - 0.40 = 0.60$.

The area to the left of $k = 0.60$.

$\text{invNorm}(0.60, 36.9, 13.9) = 40.4215$

$k = 40.42$.

Forty percent of the smartphone users from 13 to 55+ are at least 40.4 years.

Exercise 7.9.5

Two thousand students took an exam. The scores on the exam have an approximate normal distribution with a mean $\mu = 81$ points and standard deviation $\sigma = 15$ points.

a. Calculate the first- and third-quartile scores for this exam.
b. The middle 50% of the exam scores are between what two values?

Answer

a. $Q_1 = 25^{\text{th}}$ percentile $= \text{invNorm}(0.25, 81, 15) = 70.9$
   $Q_3 = 75^{\text{th}}$ percentile $= \text{invNorm}(0.75, 81, 15) = 91.1$
b. The middle 50% of the scores are between 70.9 and 91.1.

Example 7.9.6

A citrus farmer who grows mandarin oranges finds that the diameters of mandarin oranges harvested on his farm follow a normal distribution with a mean diameter of 5.85 cm and a standard deviation of 0.24 cm.

a. Find the probability that a randomly selected mandarin orange from this farm has a diameter larger than 6.0 cm. Sketch the graph.
b. The middle 20% of mandarin oranges from this farm have diameters between ______ and ______.
c. Find the 90^{th} percentile for the diameters of mandarin oranges, and interpret it in a complete sentence.

Answer

a. $\text{normalcdf}(6, 10^{99}, 5.85, 0.24) = 0.2660$

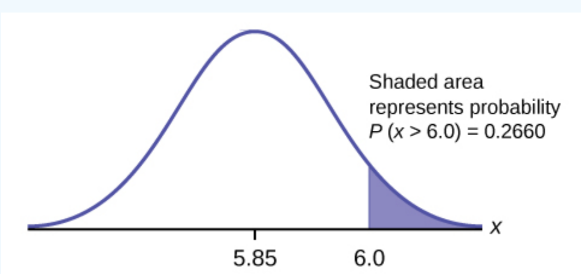

Figure 7.9.7.

Answer

b.

$1 - 0.20 = 0.80$

The tails of the graph of the normal distribution each have an area of 0.40.

Find $k1$, the 40$^{\text{th}}$ percentile, and $k2$, the 60$^{\text{th}}$ percentile $(0.40+0.20=0.60)$.

$k1=\text{invNorm}(0.40, 5.85, 0.24)=5.79\text{cm}$

$k2=\text{invNorm}(0.60, 5.85, 0.24)=5.91\text{cm}$

Answer

c. 6.16: Ninety percent of the diameter of the mandarin oranges is at most 6.15 cm.

Exercise 7.9.6

Using the information from Example, answer the following:

a. The middle 45% of mandarin oranges from this farm are between ______ and ______.
b. Find the 16$^{\text{th}}$ percentile and interpret it in a complete sentence.

Answer a

The middle area $=0.40$, so each tail has an area of 0.30.

$-0.40=0.60$

The tails of the graph of the normal distribution each have an area of 0.30.

Find $k1$, the 30$^{\text{th}}$ percentile and $k2$, the 70$^{\text{th}}$ percentile $(0.40+0.30=0.70)$.

$k1=\text{invNorm}(0.30, 5.85, 0.24)=5.72\text{cm}$

$k2=\text{invNorm}(0.70, 5.85, 0.24)=5.98\text{cm}$

Answer b

$\text{normalcdf}(5, 10^{99}, 5.85, 0.24)=0.9998$

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Review

The normal distribution, which is continuous, is the most important of all the probability distributions. Its graph is bell-shaped. This bell-shaped curve is used in almost all disciplines. Since it is a continuous distribution, the total area under the curve is one. The parameters of the normal are the mean μ and the standard deviation σ. A special normal distribution, called the standard normal distribution is the distribution of *z*-scores. Its mean is zero, and its standard deviation is one.

Formula Review

- Normal Distribution: $X \sim N(\mu, \sigma)$ where μ is the mean and σ is the standard deviation.
- Standard Normal Distribution: $Z \sim N(0, 1)$.
- Calculator function for probability: normalcdf (lower x value of the area, upper x value of the area, mean, standard deviation)
- Calculator function for the k^{th} percentile: $k=\text{invNorm}$ (area to the left of k, mean, standard deviation)

? Exercise 7.9.7

How would you represent the area to the left of one in a probability statement?

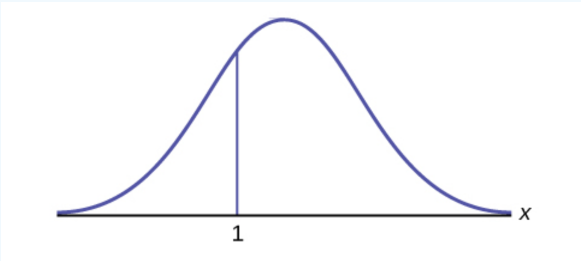


Figure 7.9.8

Answer

$P(x < 1)$

? Exercise 7.9.8

Is $P(x < 1)$ equal to $P(x \leq 1)$? Why?

Answer

Yes, because they are the same in a continuous distribution: $P(x = 1) = 0$

? Exercise 7.9.9

How would you represent the area to the left of three in a probability statement?

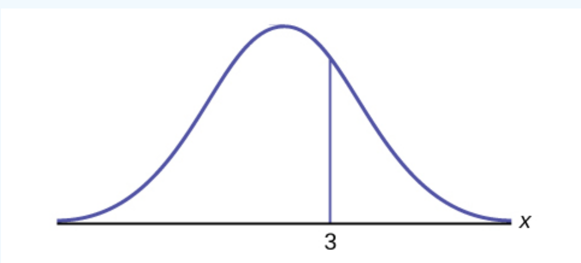


Figure 7.9.10

? Exercise 7.9.10

What is the area to the right of three?

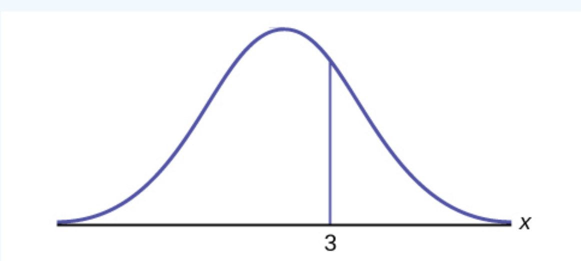


Figure 7.9.11

Answer

$1 - P(x < 3)$ or $P(x > 3)$

Exercise 7.9.11

If the area to the left of x in a normal distribution is 0.123, what is the area to the right of x?

Exercise 7.9.12

If the area to the right of x in a normal distribution is 0.543, what is the area to the left of x?

Answer

$1 - 0.543 = 0.457$

Use the following information to answer the next four exercises:

$X \sim N(54, 8)$

Exercise 7.9.13

Find the probability that $x > 56$.

Exercise 7.9.14

Find the probability that $x < 30$.

Answer

0.0013

Exercise 7.9.15

Find the 80$^{\text{th}}$ percentile.

Exercise 7.9.16

Find the 60$^{\text{th}}$ percentile.

Answer

56.03

Exercise 7.9.17

$X \sim N(6, 2)$

Find the probability that x is between three and nine.

Exercise 7.9.18

$X \sim N(-3, 4)$

Find the probability that x is between one and four.

Answer

0.1186

Exercise 7.9.19

$X \sim N(4, 5)$

Find the maximum of x in the bottom quartile.

Exercise 7.9.20

Use the following information to answer the next three exercise: The life of Sunshine CD players is normally distributed with a mean of 4.1 years and a standard deviation of 1.3 years. A CD player is guaranteed for three years. We are interested in the length of time a CD player lasts. Find the probability that a CD player will break down during the guarantee period.

a. Sketch the situation. Label and scale the axes. Shade the region corresponding to the probability.

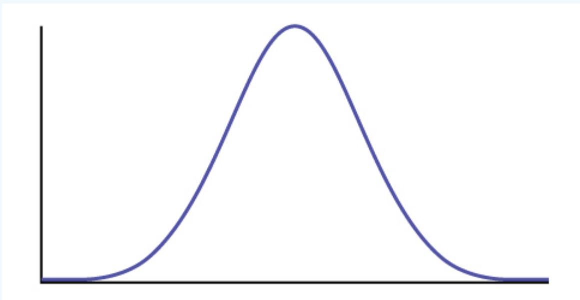


***Figure* 7.9.12.**

$P(0 < x <$ ____________$)=$ ___________ (Use zero for the minimum value of x.)

Answer

a. Check student's solution.
b. 3, 0.1979

Exercise 7.9.21

Find the probability that a CD player will last between 2.8 and six years.

a. Sketch the situation. Label and scale the axes. Shade the region corresponding to the probability.

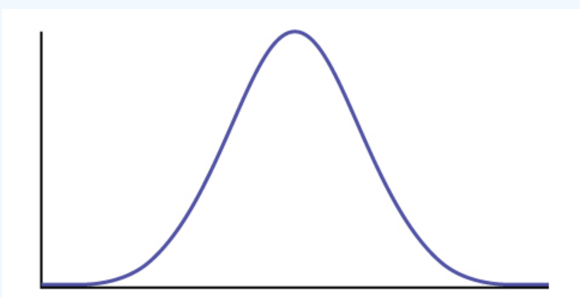


***Figure* 7.9.13.**

$P($__________ $< x <$ __________) = __________

Exercise 7.9.22

Find the 70th percentile of the distribution for the time a CD player lasts.

a. Sketch the situation. Label and scale the axes. Shade the region corresponding to the lower 70%.

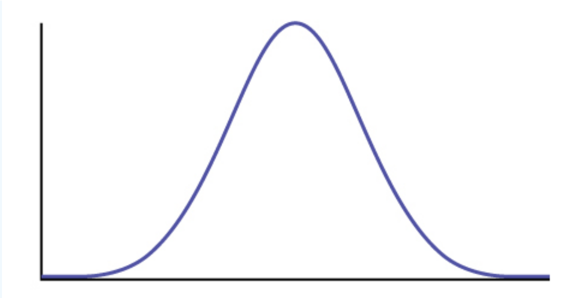


***Figure* $7.9.14.$**

$P(x < k) =$ __________ Therefore, $k =$ _________

Answer

a. Check student's solution.
b. 0.70, 4.78 years

7.E: Continuous Random Variables (Exercises)

These are homework exercises to accompany the Textmap created for "Introductory Statistics" by OpenStax.

Continuous Probability Functions

For each probability and percentile problem, draw the picture.

Q 5.2.1

Consider the following experiment. You are one of 100 people enlisted to take part in a study to determine the percent of nurses in America with an R.N. (registered nurse) degree. You ask nurses if they have an R.N. degree. The nurses answer “yes” or “no.” You then calculate the percentage of nurses with an R.N. degree. You give that percentage to your supervisor.

a. What part of the experiment will yield discrete data?
b. What part of the experiment will yield continuous data?

Q 5.2.2

When age is rounded to the nearest year, do the data stay continuous, or do they become discrete? Why?

S 5.2.2

Age is a measurement, regardless of the accuracy used.

Exercise 5.2.2

Which type of distribution does the graph illustrate?

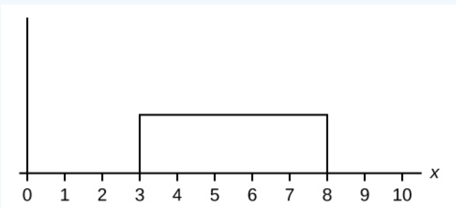


Figure 5.2.9.

Answer

Uniform Distribution

Exercise 5.2.3

Which type of distribution does the graph illustrate?

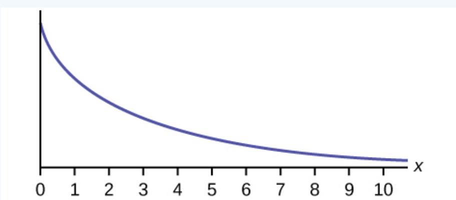


Figure 5.2.10.

Exercise 5.2.4

Which type of distribution does the graph illustrate?

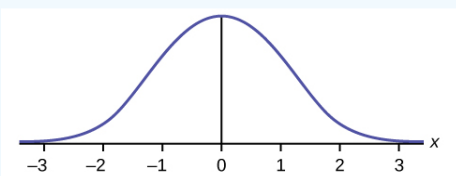


Figure 5.2.11.

Answer

Exercise 5.2.5

What does the shaded area represent? $P(\_\_\_< x < \_\_\_)$

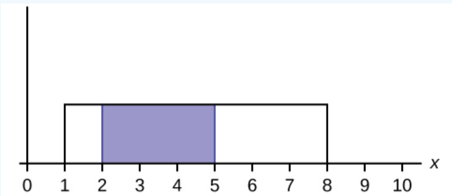


Figure 5.2.12.

Exercise 5.2.6

What does the shaded area represent? $P(\_\_\_< x < \_\_\_)$

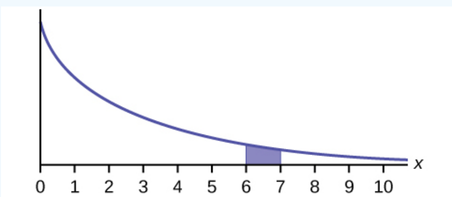


Figure 5.2.13.

Answer

$P(6 < x < 7)$

Exercise 5.2.7

For a continuous probability distribution, $0 \leq x \leq 15$. What is $P(x > 15)$?

Exercise 5.2.8

What is the area under $f(x)$ if the function is a continuous probability density function?

Answer

one

Exercise 5.2.9

For a continuous probability distribution, $0 \leq x \leq 10$. What is $P(x = 7)$?

Exercise 5.2.11

A **continuous** probability function is restricted to the portion between $x = 0$ and 7. What is $P(x = 10)$?

Answer

zero

Exercise 5.2.12

$f(x)$ for a continuous probability function is $\frac{1}{5}$, and the function is restricted to $0 \leq x \leq 5$. What is $P(x < 0)$?

Exercise 5.2.13

$f(x)$, a continuous probability function, is equal to $\frac{1}{12}$, and the function is restricted to $0 \leq x \leq 12$. What is $P(0 < x) < 12)$?

Answer

one

Exercise 5.2.14

Find the probability that x falls in the shaded area.

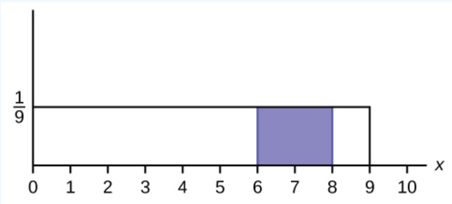


Figure 5.2.14.

Exercise 5.2.15

Find the probability that x falls in the shaded area.

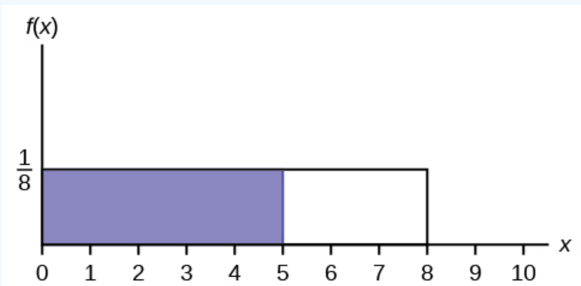


Figure 5.2.15.

Answer

0.625

Exercise 5.2.16

Find the probability that x falls in the shaded area.

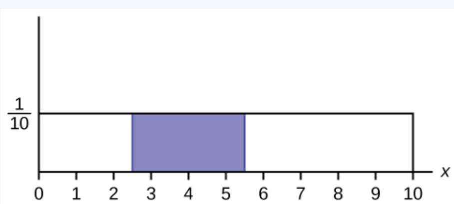


Figure 5.2.16.

Exercise 5.2.17

$f(x)$, a continuous probability function, is equal to $\frac{1}{3}$ and the function is restricted to $1 \leq x \leq 4$. Describe $P(x > \frac{3}{2})$.

Answer

The probability is equal to the area from $x = \frac{3}{2}$ to $x = 4$ above the x-axis and up to $f(x) = \frac{1}{3}$.

The Uniform Distribution

For each probability and percentile problem, draw the picture.

Q 5.3.1

Births are approximately uniformly distributed between the 52 weeks of the year. They can be said to follow a uniform distribution from one to 53 (spread of 52 weeks).

a. $X \sim$ _________
b. Graph the probability distribution.
c. $f(x) =$ _________

d. $\mu =$ ________
e. $\sigma =$ ________
f. Find the probability that a person is born at the exact moment week 19 starts. That is, find $P(x = 19) =$ ________
g. $P(2 < x < 31) =$ ________
h. Find the probability that a person is born after week 40.
i. $P(12 < x | x < 28) =$ ________
j. Find the 70th percentile.
k. Find the minimum for the upper quarter.

Q 5.3.2

A random number generator picks a number from one to nine in a uniform manner.

a. $X \sim$ ________
b. Graph the probability distribution.
c. $f(x) =$ ________
d. $\mu =$ ________
e. $\mu =$ ________
f. $P(3.5 < x < 7.25) =$ ________
g. $P(x > 5.67) =$ ________
h. $P(x > 5 | x > 3) =$ ________
i. Find the 90th percentile.

S 5.3.2

a. $X \sim U(1, 9)$
b. Check student's solution.
c. $f(x) = 18$ where $1 \leq x \leq 9$
d. five
e. 2.3
f. $\frac{15}{32}$
g. $\frac{333}{800}$
h. $\frac{2}{3}$
i. 8.2

Q 5.3.3

According to a study by Dr. John McDougall of his live-in weight loss program at St. Helena Hospital, the people who follow his program lose between six and 15 pounds a month until they approach trim body weight. Let's suppose that the weight loss is uniformly distributed. We are interested in the weight loss of a randomly selected individual following the program for one month.

a. Define the random variable. $X =$ ________
b. $X \sim$ ________
c. Graph the probability distribution.
d. $f(x) =$ ________
e. $\mu =$ ________
f. $\sigma =$ ________
g. Find the probability that the individual lost more than ten pounds in a month.
h. Suppose it is known that the individual lost more than ten pounds in a month. Find the probability that he lost less than 12 pounds in the month.
i. $P(7 < x < 13 | x > 9) =$ __________. State this in a probability question, similarly to parts g and h, draw the picture, and find the probability.

Q 5.3.4

A subway train on the Red Line arrives every eight minutes during rush hour. We are interested in the length of time a commuter must wait for a train to arrive. The time follows a uniform distribution.

a. Define the random variable. $X =$ _______

b. $X \sim$ _______
c. Graph the probability distribution.
d. $f(x) =$ _______
e. $\mu =$ _______
f. $\sigma =$ _______
g. Find the probability that the commuter waits less than one minute.
h. Find the probability that the commuter waits between three and four minutes.
i. Sixty percent of commuters wait more than how long for the train? State this in a probability question, similarly to parts g and h, draw the picture, and find the probability.

S 5.3.5

a. X represents the length of time a commuter must wait for a train to arrive on the Red Line.
b. $X \sim U(0, 8)$
c. $f(x) = \frac{1}{8}$ where $leqxleq8$
d. four
e. 2.31
f. $\frac{1}{8}$
g. $\frac{1}{8}$
h. 3.2

Q 5.3.6

The age of a first grader on September 1 at Garden Elementary School is uniformly distributed from 5.8 to 6.8 years. We randomly select one first grader from the class.

a. Define the random variable. $X =$ _________
b. $X \sim$ _________
c. Graph the probability distribution.
d. $f(x) =$ _________
e. $\mu =$ _________
f. $\sigma =$ _________
g. Find the probability that she is over 6.5 years old.
h. Find the probability that she is between four and six years old.
i. Find the 70^{th} percentile for the age of first graders on September 1 at Garden Elementary School.

Use the following information to answer the next three exercises. The Sky Train from the terminal to the rental–car and long–term parking center is supposed to arrive every eight minutes. The waiting times for the train are known to follow a uniform distribution.

Q 5.3.7

What is the average waiting time (in minutes)?

a. zero
b. two
c. three
d. four

S 5.3.7

d

Q 5.3.8

Find the 30^{th} percentile for the waiting times (in minutes).

a. two
b. 2.4
c. 2.75
d. three

Q 5.3.9

The probability of waiting more than seven minutes given a person has waited more than four minutes is?

a. 0.125
b. 0.25
c. 0.5
d. 0.75

S 5.3.9

b

Q 5.3.10

The time (in minutes) until the next bus departs a major bus depot follows a distribution with $f(x) = \frac{1}{20}$ where x goes from 25 to 45 minutes.

a. Define the random variable. $X =$ ________
b. $X \sim$ ________
c. Graph the probability distribution.
d. The distribution is ______________ (name of distribution). It is _____________ (discrete or continuous).
e. $\mu =$ ________
f. $\sigma =$ ________
g. Find the probability that the time is at most 30 minutes. Sketch and label a graph of the distribution. Shade the area of interest. Write the answer in a probability statement.
h. Find the probability that the time is between 30 and 40 minutes. Sketch and label a graph of the distribution. Shade the area of interest. Write the answer in a probability statement.
i. $P(25 < x < 55) =$ _________. State this in a probability statement, similarly to parts g and h, draw the picture, and find the probability.
j. Find the 90^{th} percentile. This means that 90% of the time, the time is less than _____ minutes.
k. Find the 75^{th} percentile. In a complete sentence, state what this means. (See part j.)
l. Find the probability that the time is more than 40 minutes given (or knowing that) it is at least 30 minutes.

Q 5.3.11

Suppose that the value of a stock varies each day from $16 to $25 with a uniform distribution.

a. Find the probability that the value of the stock is more than $19.
b. Find the probability that the value of the stock is between $19 and $22.
c. Find the upper quartile - 25% of all days the stock is above what value? Draw the graph.
d. Given that the stock is greater than $18, find the probability that the stock is more than $21.

S 5.3.11

a. The probability density function of X is $\frac{1}{25-16} = \frac{1}{9}$.
   $P(X > 19) = (25 - 19)\left(\frac{1}{9}\right) = \frac{6}{9} = \frac{2}{3}$. alt Figure 5.3.2.
b. The area must be 0.25, and $0.25 = (\text{width})\left(\frac{1}{9}\right)$, so $\text{width} = (0.25)(9) = 2.25$. Thus, the value is $25 - 2.25 = 22.75$
c. This is a conditional probability question. $P(x > 21 | x > 18)$. You can do this two ways:
   - Draw the graph where a is now 18 and b is still 25. The height is $\frac{1}{25-18} = \frac{1}{7}$
     So, $P(x > 21 | x > 18) = (25 - 21)\left(\frac{1}{7}\right) = \frac{4}{7}$.
   - Use the formula: $P(x > 21 | x > 18) = \frac{P(x>21 \text{AND} x>18)}{P(x>18)} = \frac{P(x>21)}{P(x>18)} = \frac{(25-21)}{25-18} = \frac{4}{7}$.

Q 5.3.12

A fireworks show is designed so that the time between fireworks is between one and five seconds, and follows a uniform distribution.

a. Find the average time between fireworks.
b. Find probability that the time between fireworks is greater than four seconds.

Q 5.3.13

The number of miles driven by a truck driver falls between 300 and 700, and follows a uniform distribution.

a. Find the probability that the truck driver goes more than 650 miles in a day.
b. Find the probability that the truck drivers goes between 400 and 650 miles in a day.
c. At least how many miles does the truck driver travel on the furthest 10% of days?

S 5.3.13

a. $P(X > 650) = \frac{700-650}{700-300} = \frac{500}{400} = \frac{1}{8} = 0.125$.
b. $P(400 < X < 650) = \frac{700-650}{700-300} = \frac{250}{400} = 0.625$
c. $0.10 = \frac{\text{width}}{700-300}$, so $\text{width} = 400(0.10) = 40$. Since $700 - 40 = 660$, the drivers travel at least 660 miles on the furthest 10% of days.

The Exponential Distribution

Use the following information to answer the next ten exercises. A customer service representative must spend different amounts of time with each customer to resolve various concerns. The amount of time spent with each customer can be modeled by the following distribution: $X \sim Exp(0.2)$

Exercise 5.4.8

What type of distribution is this?

Exercise 5.4.9

Are outcomes equally likely in this distribution? Why or why not?

Answer

No, outcomes are not equally likely. In this distribution, more people require a little bit of time, and fewer people require a lot of time, so it is more likely that someone will require less time.

Exercise 5.4.10

What is m? What does it represent?

Exercise 5.4.11

What is the mean?

Answer

five

Exercise 5.4.12

What is the standard deviation?

Exercise 5.4.13

State the probability density function.

Answer

$f(x) = 0.2e^{-0.2x}$

Exercise 5.4.14

Graph the distribution.

Exercise 5.4.15

Find $P(2 < x < 10)$.

Answer

0.5350

Exercise 5.4.16

Find $P(x > 6)$.

Exercise 5.4.17

Find the 70th percentile.

Answer

6.02

Use the following information to answer the next seven exercises. A distribution is given as $X \sim Exp(0.75)$.

Exercise 5.4.18

What is m?

Exercise 5.4.19

What is the probability density function?

Answer

$f(x) = 0.75e^{-0.75x}$

Exercise 5.4.20

What is the cumulative distribution function?

Exercise 5.4.21

Draw the distribution.

Answer

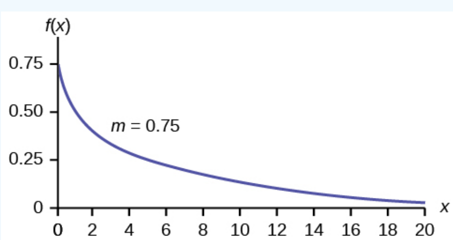


Figure 5.4.4

Exercise 5.4.22

Find $P(x < 4)$.

Exercise 5.4.23

Find the 30th percentile.

Answer

0.4756

Exercise 5.4.24

Find the median.

Exercise 5.4.25

Which is larger, the mean or the median?

Answer

The mean is larger. The mean is $\frac{1}{m} = \frac{1}{0.75} \approx 1.33$, which is greater than 0.9242

Use the following information to answer the next 16 exercises. Carbon-14 is a radioactive element with a half-life of about 5,730 years. Carbon-14 is said to decay exponentially. The decay rate is 0.000121. We start with one gram of carbon-14. We are interested in the time (years) it takes to decay carbon-14.

Exercise 5.4.26

What is being measured here?

Exercise 5.4.27

Are the data discrete or continuous?

Answer

continuous

Exercise 5.4.28

In words, define the random variable X.

Exercise 5.4.29

What is the decay rate (m)?

Answer

$m = 0.000121$

Exercise 5.4.30

The distribution for X is ______.

Exercise 5.4.31

Find the amount (percent of one gram) of carbon-14 lasting less than 5,730 years. This means, find $P(x < 5,730)$.

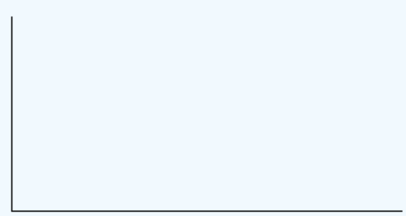


Figure 5.4.5

a. Sketch the graph, and shade the area of interest.
b. Find the probability. $P(x < 5,730) =$_ __________

Answer

a. Check student's solution
b. $P(x < 5,730) = 0.5001$

Exercise 5.4.32

Find the percentage of carbon-14 lasting longer than 10,000 years.

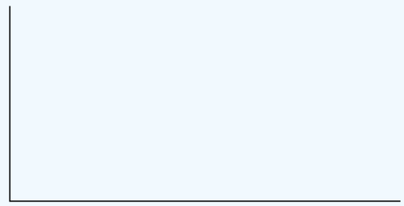


Figure 5.4.6

a. Sketch the graph, and shade the area of interest.
b. Find the probability. $P(x > 10{,}000) =$ ________

Exercise 5.2.33

Thirty percent (30%) of carbon-14 will decay within how many years?

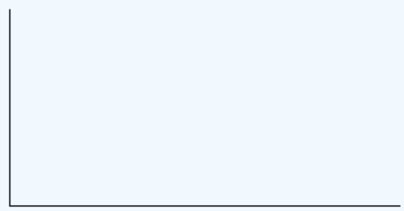


Figure 5.4.7

a. Sketch the graph, and shade the area of interest.
b. Find the value k such that $P(x < k) = 0.30$.

Answer

a. Check student's solution.
b. $k = 2947.73$

Q 5.4.1

Suppose that the length of long distance phone calls, measured in minutes, is known to have an exponential distribution with the average length of a call equal to eight minutes.

a. Define the random variable. $X =$ ________________.
b. Is X continuous or discrete?
c. $X \sim$ ________
d. $\mu =$ ________
e. $\sigma =$ ________
f. Draw a graph of the probability distribution. Label the axes.
g. Find the probability that a phone call lasts less than nine minutes.
h. Find the probability that a phone call lasts more than nine minutes.
i. Find the probability that a phone call lasts between seven and nine minutes.
j. If 25 phone calls are made one after another, on average, what would you expect the total to be? Why?

Q 5.4.2

Suppose that the useful life of a particular car battery, measured in months, decays with parameter 0.025. We are interested in the life of the battery.

a. Define the random variable. $X =$ _________________________________.
b. Is X continuous or discrete?
c. $X \sim$ ________
d. On average, how long would you expect one car battery to last?
e. On average, how long would you expect nine car batteries to last, if they are used one after another?
f. Find the probability that a car battery lasts more than 36 months.

g. Seventy percent of the batteries last at least how long?

S 5.4.2

a. $X =$ the useful life of a particular car battery, measured in months.
b. X is continuous.
c. $X \sim \text{Exp}(0.025)$
d. 40 months
e. 360 months
f. 0.4066
g. 14.27

Q 5.4.3

The percent of persons (ages five and older) in each state who speak a language at home other than English is approximately exponentially distributed with a mean of 9.848. Suppose we randomly pick a state.

a. Define the random variable. $X =$ _________________________________.
b. Is X continuous or discrete?
c. $X \sim$ ________
d. $\mu =$ ________
e. $\sigma =$ ________
f. Draw a graph of the probability distribution. Label the axes.
g. Find the probability that the percent is less than 12.
h. Find the probability that the percent is between eight and 14.
i. The percent of all individuals living in the United States who speak a language at home other than English is 13.8.
   i. Why is this number different from 9.848%?
   ii. What would make this number higher than 9.848%?

Q 5.4.4

The time (in years) **after** reaching age 60 that it takes an individual to retire is approximately exponentially distributed with a mean of about five years. Suppose we randomly pick one retired individual. We are interested in the time after age 60 to retirement.

a. Define the random variable. $X =$ _________________________________.
b. Is X continuous or discrete?
c. $X \sim$ ________
d. $\mu =$ ________
e. $\sigma =$ ________
f. Draw a graph of the probability distribution. Label the axes.
g. Find the probability that the person retired after age 70.
h. Do more people retire before age 65 or after age 65?
i. In a room of 1,000 people over age 80, how many do you expect will NOT have retired yet?

S 5.4.4

a. $X =$ the time (in years) after reaching age 60 that it takes an individual to retire
b. X is continuous.
c. $X \text{ Exp}\left(\frac{1}{5}\right)$
d. five
e. five
f. Check student's solution.
g. 0.1353
h. before
i. 18.3

Q 5.4.5

The cost of all maintenance for a car during its first year is approximately exponentially distributed with a mean of $150.

a. Define the random variable. $X =$ ________________________________.
b. $X \sim$ ________
c. $\mu =$ ________
d. $\sigma =$ ________
e. Draw a graph of the probability distribution. Label the axes.
f. Find the probability that a car required over $300 for maintenance during its first year.

Use the following information to answer the next three exercises. The average lifetime of a certain new cell phone is three years. The manufacturer will replace any cell phone failing within two years of the date of purchase. The lifetime of these cell phones is known to follow an exponential distribution.

Q 5.4.6

The decay rate is:

a. 0.3333
b. 0.5000
c. 2
d. 3

S 5.4.6

a

Q 5.4.7

What is the probability that a phone will fail within two years of the date of purchase?

a. 0.8647
b. 0.4866
c. 0.2212
d. 0.9997

Q 5.4.8

What is the median lifetime of these phones (in years)?

a. 0.1941
b. 1.3863
c. 2.0794
d. 5.5452

S 5.4.8

c

Q 5.4.9

Let $X \sim \text{Exp}(0.1)$.

a. decay rate = ________
b. $\mu =$ ________
c. Graph the probability distribution function.
d. On the graph, shade the area corresponding to $P(x < 6)$ and find the probability.
e. Sketch a new graph, shade the area corresponding to $P(3 < x < 6)$ and find the probability.
f. Sketch a new graph, shade the area corresponding to $P(x < 7)$ and find the probability.
g. Sketch a new graph, shade the area corresponding to the 40^{th} percentile and find the value.
h. Find the average value of x.

Q 5.4.10

Suppose that the longevity of a light bulb is exponential with a mean lifetime of eight years.

a. Find the probability that a light bulb lasts less than one year.
b. Find the probability that a light bulb lasts between six and ten years.
c. Seventy percent of all light bulbs last at least how long?
d. A company decides to offer a warranty to give refunds to light bulbs whose lifetime is among the lowest two percent of all bulbs. To the nearest month, what should be the cutoff lifetime for the warranty to take place?
e. If a light bulb has lasted seven years, what is the probability that it fails within the 8^{th} year.

S 5.4.10

Let $T =$ the life time of a light bulb.

The decay parameter is $m = \frac{1}{8}$, and $T \sim \text{Exp}(\frac{1}{8})$. The cumulative distribution function is $P(T < t) = 1 - e^{-\frac{t}{8}}$

a. Therefore, $P(T < t) = 1 - {}^{-\frac{1}{8}} \approx 0.1175$.
b. We want to find $P(6 < t < 10)$.
   To do this, $P(6 < t < 10) - P(t < 6) = \left(1 - e^{-\frac{1}{8}*10}\right) - \left(1 - e^{\frac{1}{8}*6}\right) \approx 0.7135 - 0.5276 = 0.1859$

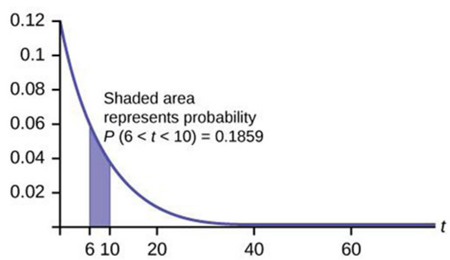
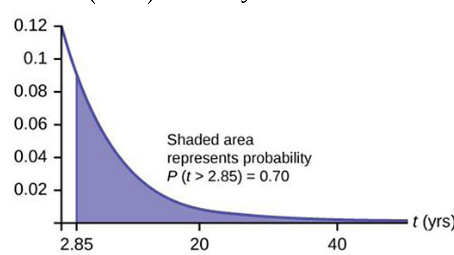


Figure 5.4.1.

c. We want to find $0.70 = P(T < t) = 1 - \left(1 - e^{-\frac{t}{8}}\right) = e^{-\frac{t}{8}}$.
   Solving for t, $e^{-\frac{t}{8}} = 0.70$, so $-\frac{t}{8} = \ln(0.70)$ and $t = -8 \cdot \ln(0.70) \approx 2.85$ years.



   Or use $t = \frac{\ln(\text{area\_to\_the\_right})}{(-m)} = \frac{\ln(0.70)}{-\frac{1}{8}} \approx 2.85$ years

Figure 5.4.2.

d. We want to find $0.02 = P(T < t) = 1 - e^{-\frac{t}{8}}$
   Solving for t, $e^{-\frac{t}{8}} = 0.98$, so $\frac{t}{8} = \ln(0.98)$, and $t = -8 \cdot \ln(0.98) \approx 0.1616$ years, or roughly two months.
   The warranty should cover light bulbs that last less than 2 months.
   Or use $\frac{\ln(\text{area\_to\_the\_right})}{(-m)} = \frac{\ln(1-0.2)}{-\frac{1}{8}} = 0.1616$.
e. We must find $P(T < 8|T > 7)$.
   Notice that by the rule of complement events, $P(T < 8|T > 7) = 1 - P(T > 8|T > 7)$.
   By the memoryless property $(P(X > r + t|X > r) = P(X > t))$.
   So $P(T > 8|T > 7) = P(T > 1) = 1 - \left(1 - e^{-\frac{1}{8}}\right) = e^{-\frac{1}{8}} \approx 0.8825$
   Therefore, $P(T < 8|T > 7) = 1 - 0.8825 = 0.1175$.

Q 5.4.11

At a 911 call center, calls come in at an average rate of one call every two minutes. Assume that the time that elapses from one call to the next has the exponential distribution.

a. On average, how much time occurs between five consecutive calls?

b. Find the probability that after a call is received, it takes more than three minutes for the next call to occur.
c. Ninety-percent of all calls occur within how many minutes of the previous call?
d. Suppose that two minutes have elapsed since the last call. Find the probability that the next call will occur within the next minute.
e. Find the probability that less than 20 calls occur within an hour.

Q 5.4.12

In major league baseball, a no-hitter is a game in which a pitcher, or pitchers, doesn't give up any hits throughout the game. No-hitters occur at a rate of about three per season. Assume that the duration of time between no-hitters is exponential.

a. What is the probability that an entire season elapses with a single no-hitter?
b. If an entire season elapses without any no-hitters, what is the probability that there are no no-hitters in the following season?
c. What is the probability that there are more than 3 no-hitters in a single season?

S 5.4.12

Let $X =$ the number of no-hitters throughout a season. Since the duration of time between no-hitters is exponential, the number of no-hitters per season is Poisson with mean $\lambda = 3$.

Therefore, $(X = 0) = \frac{3^0 e^3}{0!} = e^{-3} \approx 0.0498$

NOTE

You could let $T =$ duration of time between no-hitters. Since the time is exponential and there are 3 no-hitters per season, then the time between no-hitters is $\frac{1}{3}$ season. For the exponential, $\mu = \frac{1}{3}$.

Therefore, $m = \frac{1}{\mu} = 3$ and $T \sim \text{Exp}(3)$.

a. The desired probability is $P(T > 1) = 1 - P(T < 1) = 1 - (1 - e^{-3}) = e^{-3} \approx 0.0498$.
b. Let $T =$ duration of time between no-hitters. We find $P(T > 2 | T > 1)$, and by the **memoryless property** this is simply $P(T > 1)$, which we found to be 0.0498 in part a.
c. Let $X =$ the number of no-hitters is a season. Assume that X is Poisson with mean $\lambda = 3$. Then $P(X > 3) = 1 - P(X \leq 3) = 0.3528$.

Q 5.4.13

During the years 1998–2012, a total of 29 earthquakes of magnitude greater than 6.5 have occurred in Papua New Guinea. Assume that the time spent waiting between earthquakes is exponential.

1. What is the probability that the next earthquake occurs within the next three months?
2. Given that six months has passed without an earthquake in Papua New Guinea, what is the probability that the next three months will be free of earthquakes?
3. What is the probability of zero earthquakes occurring in 2014?
4. What is the probability that at least two earthquakes will occur in 2014?

Q 5.4.14

According to the American Red Cross, about one out of nine people in the U.S. have Type B blood. Suppose the blood types of people arriving at a blood drive are independent. In this case, the number of Type B blood types that arrive roughly follows the Poisson distribution.

a. If 100 people arrive, how many on average would be expected to have Type B blood?
b. What is the probability that over 10 people out of these 100 have type B blood?
c. What is the probability that more than 20 people arrive before a person with type B blood is found?

S 5.4.14

a. $\frac{100}{9} = 11.11$
b. $P(X > 10) = 1 - P(X \leq 10) = 1 - \text{Poissoncdf}(11.11, 10) \approx 0.5532$.
c. The number of people with Type B blood encountered roughly follows the Poisson distribution, so the number of people X who arrive between successive Type B arrivals is roughly exponential with mean $\mu = 9$ and $m = \frac{1}{9}$. The cumulative distribution

function of X is $P(X < x) = 1 - e^{-\frac{x}{9}}$. Thus, $P(X > 20) = 1 - P(X \leq 20) = 1 - (1 - e^{-\frac{20}{9}}) \approx 0.1084.$.

NOTE

We could also deduce that each person arriving has a 8/9 chance of not having Type B blood. So the probability that none of the first 20 people arrive have Type B blood is $\left(\frac{8}{9}\right)^{20} \approx 0.0948$. (The geometric distribution is more appropriate than the exponential because the number of people between Type B people is discrete instead of continuous.)

Q 5.4.15

A web site experiences traffic during normal working hours at a rate of 12 visits per hour. Assume that the duration between visits has the exponential distribution.

a. Find the probability that the duration between two successive visits to the web site is more than ten minutes.
b. The top 25% of durations between visits are at least how long?
c. Suppose that 20 minutes have passed since the last visit to the web site. What is the probability that the next visit will occur within the next 5 minutes?
d. Find the probability that less than 7 visits occur within a one-hour period.

Q 5.4.16

At an urgent care facility, patients arrive at an average rate of one patient every seven minutes. Assume that the duration between arrivals is exponentially distributed.

a. Find the probability that the time between two successive visits to the urgent care facility is less than 2 minutes.
b. Find the probability that the time between two successive visits to the urgent care facility is more than 15 minutes.
c. If 10 minutes have passed since the last arrival, what is the probability that the next person will arrive within the next five minutes?
d. Find the probability that more than eight patients arrive during a half-hour period.

S 5.4.17

Let $T =$ duration (in minutes) between successive visits. Since patients arrive at a rate of one patient every seven minutes, $\mu = 7$ and the decay constant is $m = \frac{1}{7}$. The cdf is $P(T < t) = 1 - e^{\frac{t}{7}}$

a. $P(T < 2) = 1 - 1 - e^{-\frac{2}{7}} \approx 0.2485$.
b. $P(T > 15) = 1 - P(T < 15) = 1 - \left(1 - e^{-\frac{15}{7}}\right) \approx e^{-\frac{15}{7}} \approx 0.1173$.
c. $P(T > 15|T > 10) = P(T > 5) = 1 - \left(1 - e^{\frac{5}{7}}\right) = e^{-\frac{5}{7}} \approx 0.4895$.
d. Let $X =$ # of patients arriving during a half-hour period. Then X has the Poisson distribution with a mean of $\frac{30}{7}$, $X \sim \text{Poisson}\left(\frac{30}{7}\right)$. Find $P(X > 8) = 1 – P(X \leq 8) \approx 0.0311$.

Continuous Distribution

7.E: Discrete Random Variables (Exercises)

These are homework exercises to accompany the Textmap created for "Introductory Statistics" by OpenStax.

4.1: Introduction

4.2: Probability Distribution Function (PDF) for a Discrete Random Variable

Q 4.2.1

Suppose that the PDF for the number of years it takes to earn a Bachelor of Science (B.S.) degree is given in Table.

x	$P(x)$
3	0.05
4	0.40
5	0.30
6	0.15
7	0.10

a. In words, define the random variable X.
b. What does it mean that the values zero, one, and two are not included for x in the PDF?

Exercise 4.3.5

Complete the expected value table.

x	$P(x)$	$x * P(x)$
0	0.2	
1	0.2	
2	0.4	
3	0.2	

Exercise 4.3.6

Find the expected value from the expected value table.

x	$P(x)$	$x * P(x)$
2	0.1	
4	0.3	4(0.3) = 1.2
6	0.4	6(0.4) = 2.4
8	0.2	8(0.2) = 1.6

Answer

$0.2 + 1.2 + 2.4 + 1.6 = 5.4$

Exercise 4.3.7

Find the standard deviation.

x	$P(x)$	$x * P(x)$	$(x-\mu)^2 P(x)$
2	0.1	2(0.1) = 0.2	$(2–5.4)^2(0.1) = 1.156$

x	$P(x)$	$x*P(x)$	$(x–\mu)^2P(x)$
4	0.3	4(0.3) = 1.2	$(4–5.4)^2(0.3) = 0.588$
6	0.4	6(0.4) = 2.4	$(6–5.4)^2(0.4) = 0.144$
8	0.2	8(0.2) = 1.6	$(8–5.4)^2(0.2) = 1.352$

Exercise 4.3.8

Identify the mistake in the probability distribution table.

x	$P(x)$	$x*P(x)$
1	0.15	0.15
2	0.25	0.50
3	0.30	0.90
4	0.20	0.80
5	0.15	0.75

Answer

The values of $P(x)$ do not sum to one.

Exercise 4.3.9

Identify the mistake in the probability distribution table.

x	$P(x)$	$x*P(x)$
1	0.15	0.15
2	0.25	0.40
3	0.25	0.65
4	0.20	0.85
5	0.15	1

Use the following information to answer the next five exercises: A physics professor wants to know what percent of physics majors will spend the next several years doing post-graduate research. He has the following probability distribution.

x	$P(x)$	$x*P(x)$
1	0.35	
2	0.20	
3	0.15	
4		
5	0.10	
6	0.05	

Exercise 4.3.10

Define the random variable X.

Answer

Let $X =$ the number of years a physics major will spend doing post-graduate research.

Exercise 4.3.11

Define $P(x)$, or the probability of x.

Exercise 4.3.12

Find the probability that a physics major will do post-graduate research for four years. $P(x=4)=$ _______

Answer

$1–0.35–0.20–0.15–0.10–0.05=0.15$

Exercise 4.3.13

Find the probability that a physics major will do post-graduate research for at most three years. $P(x \leq 3)=$ _______

Exercise 4.3.14

On average, how many years would you expect a physics major to spend doing post-graduate research?

Answer

$1(0.35)+2(0.20)+3(0.15)+4(0.15)+5(0.10)+6(0.05)=0.35+0.40+0.45+0.60+0.50+0.30=2.6$ years

Use the following information to answer the next seven exercises: A ballet instructor is interested in knowing what percent of each year's class will continue on to the next, so that she can plan what classes to offer. Over the years, she has established the following probability distribution.

- Let $X=$ the number of years a student will study ballet with the teacher.
- Let $P(x)=$ the probability that a student will study ballet x years.

Exercise 4.3.15

Complete Table using the data provided.

x	$P(x)$	$x * P(x)$
1	0.10	
2	0.05	
3	0.10	
4		
5	0.30	
6	0.20	
7	0.10	

Exercise 4.3.16

In words, define the random variable X.

Answer

X is the number of years a student studies ballet with the teacher.

Exercise 4.3.17

$P(x=4)=$ _______

Exercise 4.3.18

$P(x<4)=$ _______

Answer

$0.10 + 0.05 + 0.10 = 0.25$

Exercise 4.3.19

On average, how many years would you expect a child to study ballet with this teacher?

Exercise 4.3.20

What does the column "$P(x)$" sum to and why?

Answer

The sum of the probabilities sum to one because it is a probability distribution.

Exercise 4.3.21

What does the column "$x * P(x)$" sum to and why?

Exercise 4.3.22

You are playing a game by drawing a card from a standard deck and replacing it. If the card is a face card, you win $30. If it is not a face card, you pay $2. There are 12 face cards in a deck of 52 cards. What is the expected value of playing the game?

Answer

$$-2\left(\frac{40}{52}\right) + 30\left(\frac{12}{52}\right) = -1.54 + 6.92 = 5.38$$

Exercise 4.3.23

You are playing a game by drawing a card from a standard deck and replacing it. If the card is a face card, you win $30. If it is not a face card, you pay $2. There are 12 face cards in a deck of 52 cards. Should you play the game?

4.3: Mean or Expected Value and Standard Deviation

Q 4.3.1

A theater group holds a fund-raiser. It sells 100 raffle tickets for $5 apiece. Suppose you purchase four tickets. The prize is two passes to a Broadway show, worth a total of $150.

a. What are you interested in here?
b. In words, define the random variable X.
c. List the values that X may take on.
d. Construct a PDF.
e. If this fund-raiser is repeated often and you always purchase four tickets, what would be your expected average winnings per raffle?

Q 4.3.2

A game involves selecting a card from a regular 52-card deck and tossing a coin. The coin is a fair coin and is equally likely to land on heads or tails.

- If the card is a face card, and the coin lands on Heads, you win $6
- If the card is a face card, and the coin lands on Tails, you win $2
- If the card is not a face card, you lose $2, no matter what the coin shows.

a. Find the expected value for this game (expected net gain or loss).
b. Explain what your calculations indicate about your long-term average profits and losses on this game.
c. Should you play this game to win money?

S 4.3.2

The variable of interest is X, or the gain or loss, in dollars.

The face cards jack, queen, and king. There are $(3)(4) = 12$ face cards and $52 – 12 = 40$ cards that are not face cards.

We first need to construct the probability distribution for X. We use the card and coin events to determine the probability for each outcome, but we use the monetary value of X to determine the expected value.

Card Event	X **net gain/loss**	$P(X)$
Face Card and Heads	6	$\left(\frac{12}{52}\right)\left(\frac{1}{2}\right)=\left(\frac{6}{52}\right)$
Face Card and Tails	2	$\left(\frac{12}{52}\right)\left(\frac{1}{2}\right)=\left(\frac{6}{52}\right)$
(Not Face Card) and (H or T)	–2	$\left(\frac{40}{52}\right)(1)=\left(\frac{40}{52}\right)$

- $\text{Expected value} = (6)\left(\frac{6}{52}\right)+(2)\left(\frac{6}{52}\right)+(-2)\left(\frac{40}{52}\right)=-\frac{32}{52}$
- $\text{Expected value} = -\$0.62$, rounded to the nearest cent
- If you play this game repeatedly, over a long string of games, you would expect to lose 62 cents per game, on average.
- You should not play this game to win money because the expected value indicates an expected average loss.

Q 4.3.3

You buy a lottery ticket to a lottery that costs $10 per ticket. There are only 100 tickets available to be sold in this lottery. In this lottery there are one $500 prize, two $100 prizes, and four $25 prizes. Find your expected gain or loss.

Q 4.3.4

Complete the PDF and answer the questions.

x	$P(x)$	$xP(x)$
0	0.3	
1	0.2	
2		
3	0.4	

a. Find the probability that $x=2$.
b. Find the expected value.

S 4.3.4

a. 0.1
b. 1.6

Q 4.3.5

Suppose that you are offered the following “deal.” You roll a die. If you roll a six, you win $10. If you roll a four or five, you win $5. If you roll a one, two, or three, you pay $6.

a. What are you ultimately interested in here (the value of the roll or the money you win)?
b. In words, define the Random Variable X.
c. List the values that X may take on.
d. Construct a PDF.
e. Over the long run of playing this game, what are your expected average winnings per game?
f. Based on numerical values, should you take the deal? Explain your decision in complete sentences.

Q 4.3.6

A venture capitalist, willing to invest $1,000,000, has three investments to choose from. The first investment, a software company, has a 10% chance of returning $5,000,000 profit, a 30% chance of returning $1,000,000 profit, and a 60% chance of losing the million dollars. The second company, a hardware company, has a 20% chance of returning $3,000,000 profit, a 40% chance of returning $1,000,000 profit, and a 40% chance of losing the million dollars. The third company, a biotech firm, has a 10% chance of returning $6,000,000 profit, a 70% of no profit or loss, and a 20% chance of losing the million dollars.

a. Construct a PDF for each investment.

b. Find the expected value for each investment.
c. Which is the safest investment? Why do you think so?
d. Which is the riskiest investment? Why do you think so?
e. Which investment has the highest expected return, on average?

S 4.3.6

a.

Software Company	
x	$P(x)$
5,000,000	0.10
1,000,000	0.30
–1,000,000	0.60

Hardware Company	
x	$P(x)$
3,000,000	0.20
1,000,000	0.40
–1,000,00	0.40

Biotech Firm	
x	$P(x)$
6,00,000	0.10
0	0.70
–1,000,000	0.20

b. $200,000; $600,000; $400,000
c. third investment because it has the lowest probability of loss
d. first investment because it has the highest probability of loss
e. second investment

Q 4.3.7

Suppose that 20,000 married adults in the United States were randomly surveyed as to the number of children they have. The results are compiled and are used as theoretical probabilities. Let $X =$ the number of children married people have.

x	$P(x)$	$xP(x)$
0	0.10	
1	0.20	
2	0.30	
3		
4	0.10	
5	0.05	
6 (or more)	0.05	

a. Find the probability that a married adult has three children.
b. In words, what does the expected value in this example represent?
c. Find the expected value.

d. Is it more likely that a married adult will have two to three children or four to six children? How do you know?

Q 4.3.8

Suppose that the PDF for the number of years it takes to earn a Bachelor of Science (B.S.) degree is given as in the Table.

x	$P(x)$
3	0.05
4	0.40
5	0.30
6	0.15
7	0.10

On average, how many years do you expect it to take for an individual to earn a B.S.?

S 4.3.8

4.85 years

Q 4.3.9

People visiting video rental stores often rent more than one DVD at a time. The probability distribution for DVD rentals per customer at Video To Go is given in the following table. There is a five-video limit per customer at this store, so nobody ever rents more than five DVDs.

x	$P(x)$
0	0.03
1	0.50
2	0.24
3	
4	0.70
5	0.04

a. Describe the random variable X in words.
b. Find the probability that a customer rents three DVDs.
c. Find the probability that a customer rents at least four DVDs.
d. Find the probability that a customer rents at most two DVDs. Another shop, Entertainment Headquarters, rents DVDs and video games. The probability distribution for DVD rentals per customer at this shop is given as follows. They also have a five-DVD limit per customer.

x	$P(x)$
0	0.35
1	0.25
2	0.20
3	0.10
4	0.05
5	0.05

e. At which store is the expected number of DVDs rented per customer higher?
f. If Video to Go estimates that they will have 300 customers next week, how many DVDs do they expect to rent next week? Answer in sentence form.

g. If Video to Go expects 300 customers next week, and Entertainment HQ projects that they will have 420 customers, for which store is the expected number of DVD rentals for next week higher? Explain.
h. Which of the two video stores experiences more variation in the number of DVD rentals per customer? How do you know that?

Q 4.3.10

A “friend” offers you the following “deal.” For a $10 fee, you may pick an envelope from a box containing 100 seemingly identical envelopes. However, each envelope contains a coupon for a free gift.

- Ten of the coupons are for a free gift worth $6.
- Eighty of the coupons are for a free gift worth $8.
- Six of the coupons are for a free gift worth $12.
- Four of the coupons are for a free gift worth $40.

Based upon the financial gain or loss over the long run, should you play the game?

a. Yes, I expect to come out ahead in money.
b. No, I expect to come out behind in money.
c. It doesn’t matter. I expect to break even.

S 4.3.10

b

Q 4.3.11

Florida State University has 14 statistics classes scheduled for its Summer 2013 term. One class has space available for 30 students, eight classes have space for 60 students, one class has space for 70 students, and four classes have space for 100 students.

a. What is the average class size assuming each class is filled to capacity?
b. Space is available for 980 students. Suppose that each class is filled to capacity and select a statistics student at random. Let the random variable X equal the size of the student’s class. Define the PDF for X.
c. Find the mean of X.
d. Find the standard deviation of X.

Q 4.3.12

In a lottery, there are 250 prizes of $5, 50 prizes of $25, and ten prizes of $100. Assuming that 10,000 tickets are to be issued and sold, what is a fair price to charge to break even?

S 4.3.12

Let $X =$ the amount of money to be won on a ticket. The following table shows the PDF for X.

x	$P(x)$
0	0.969
5	$\frac{250}{10,000} = 0.025$
25	$\frac{50}{10,000} = 0.005$
100	$\frac{10}{10,000} = 0.001$

Calculate the expected value of X.

$$0(0.969)+5(0.025)+25(0.005)+100(0.001)=0.35 \tag{7.E.1}$$

A fair price for a ticket is $0.35. Any price over $0.35 will enable the lottery to raise money.

4.4: Binomial Distribution

Q 4.4.1

According to a recent article the average number of babies born with significant hearing loss (deafness) is approximately two per 1,000 babies in a healthy baby nursery. The number climbs to an average of 30 per 1,000 babies in an intensive care nursery.

Suppose that 1,000 babies from healthy baby nurseries were randomly surveyed. Find the probability that exactly two babies were born deaf.

Use the following information to answer the next four exercises. Recently, a nurse commented that when a patient calls the medical advice line claiming to have the flu, the chance that he or she truly has the flu (and not just a nasty cold) is only about 4%. Of the next 25 patients calling in claiming to have the flu, we are interested in how many actually have the flu.

Q 4.4.2

Define the random variable and list its possible values.

S 4.4.2

$X =$ the number of patients calling in claiming to have the flu, who actually have the flu.

$X = 0, 1, 2, \ldots 25$

Q 4.4.3

State the distribution of X.

Q 4.4.4

Find the probability that at least four of the 25 patients actually have the flu.

S 4.4.4

0.0165

Q 4.4.5

On average, for every 25 patients calling in, how many do you expect to have the flu?

Q 4.4.6

People visiting video rental stores often rent more than one DVD at a time. The probability distribution for DVD rentals per customer at Video To Go is given Table. There is five-video limit per customer at this store, so nobody ever rents more than five DVDs.

x	$P(x)$
0	0.03
1	0.50
2	0.24
3	
4	0.07
5	0.04

a. Describe the random variable X in words.
b. Find the probability that a customer rents three DVDs.
c. Find the probability that a customer rents at least four DVDs.
d. Find the probability that a customer rents at most two DVDs.

S 4.4.6

a. $X =$ the number of DVDs a Video to Go customer rents
b. 0.12
c. 0.11
d. 0.77

Q 4.4.7

A school newspaper reporter decides to randomly survey 12 students to see if they will attend Tet (Vietnamese New Year) festivities this year. Based on past years, she knows that 18% of students attend Tet festivities. We are interested in the number of students who will attend the festivities.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many of the 12 students do we expect to attend the festivities?
e. Find the probability that at most four students will attend.
f. Find the probability that more than two students will attend.

Use the following information to answer the next three exercises: The probability that the San Jose Sharks will win any given game is 0.3694 based on a 13-year win history of 382 wins out of 1,034 games played (as of a certain date). An upcoming monthly schedule contains 12 games.

Q 4.4.8

The expected number of wins for that upcoming month is:

a. 1.67
b. 12
c. $\frac{382}{1043}$
d. 4.43

S 4.4.8

d. 4.43

Let $X =$ the number of games won in that upcoming month.

Q 4.4.9

What is the probability that the San Jose Sharks win six games in that upcoming month?

a. 0.1476
b. 0.2336
c. 0.7664
d. 0.8903

Q 4.4.10

What is the probability that the San Jose Sharks win at least five games in that upcoming month?

a. 0.3694
b. 0.5266
c. 0.4734
d. 0.2305

S 4.4.10

c

Q 4.4.11

A student takes a ten-question true-false quiz, but did not study and randomly guesses each answer. Find the probability that the student passes the quiz with a grade of at least 70% of the questions correct.

Q 4.4.12

A student takes a 32-question multiple-choice exam, but did not study and randomly guesses each answer. Each question has three possible choices for the answer. Find the probability that the student guesses **more than** 75% of the questions correctly.

S 4.4.13

- $X =$ number of questions answered correctly
- $X \sim B(32, \frac{1}{3})$
- We are interested in MORE THAN 75% of 32 questions correct. 75% of 32 is 24. We want to find $P(x > 24)$. The event "more than 24" is the complement of "less than or equal to 24."
- Using your calculator's distribution menu: $1-\text{binomcdf}(32, \frac{1}{3}, 24)$
- $P(x > 24) = 0$
- The probability of getting more than 75% of the 32 questions correct when randomly guessing is very small and practically zero.

Q 4.4.14

Six different colored dice are rolled. Of interest is the number of dice that show a one.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. On average, how many dice would you expect to show a one?
e. Find the probability that all six dice show a one.
f. Is it more likely that three or that four dice will show a one? Use numbers to justify your answer numerically.

Q 4.4.15

More than 96 percent of the very largest colleges and universities (more than 15,000 total enrollments) have some online offerings. Suppose you randomly pick 13 such institutions. We are interested in the number that offer distance learning courses.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. On average, how many schools would you expect to offer such courses?
e. Find the probability that at most ten offer such courses.
f. Is it more likely that 12 or that 13 will offer such courses? Use numbers to justify your answer numerically and answer in a complete sentence.

S 4.4.15

a. $X =$ the number of college and universities that offer online offerings.
b. 0, 1, 2, …, 13
c. $X \sim B(13, 0.96)$
d. 12.48
e. 0.0135
f. $P(x = 12) = 0.3186 P(x = 13) = 0.5882$ More likely to get 13.

Q 4.4.16

Suppose that about 85% of graduating students attend their graduation. A group of 22 graduating students is randomly chosen.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many are expected to attend their graduation?
e. Find the probability that 17 or 18 attend.
f. Based on numerical values, would you be surprised if all 22 attended graduation? Justify your answer numerically.

Q 4.4.17

At The Fencing Center, 60% of the fencers use the foil as their main weapon. We randomly survey 25 fencers at The Fencing Center. We are interested in the number of fencers who do **not** use the foil as their main weapon.

a. In words, define the random variable X.

b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many are expected to **not** to use the foil as their main weapon?
e. Find the probability that six do **not** use the foil as their main weapon.
f. Based on numerical values, would you be surprised if all 25 did **not** use foil as their main weapon? Justify your answer numerically.

S 4.4.17

a. $X =$ the number of fencers who do **not** use the foil as their main weapon
b. 0, 1, 2, 3,... 25
c. $X \sim B(25, 0.40)$
d. 10
e. 0.0442
f. The probability that all 25 not use the foil is almost zero. Therefore, it would be very surprising.

Q 4.4.18

Approximately 8% of students at a local high school participate in after-school sports all four years of high school. A group of 60 seniors is randomly chosen. Of interest is the number who participated in after-school sports all four years of high school.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many seniors are expected to have participated in after-school sports all four years of high school?
e. Based on numerical values, would you be surprised if none of the seniors participated in after-school sports all four years of high school? Justify your answer numerically.
f. Based upon numerical values, is it more likely that four or that five of the seniors participated in after-school sports all four years of high school? Justify your answer numerically.

Q 4.4.19

The chance of an IRS audit for a tax return with over $25,000 in income is about 2% per year. We are interested in the expected number of audits a person with that income has in a 20-year period. Assume each year is independent.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many audits are expected in a 20-year period?
e. Find the probability that a person is not audited at all.
f. Find the probability that a person is audited more than twice.

S 4.4.19

a. $X =$ the number of audits in a 20-year period
b. 0, 1, 2, …, 20
c. $X \sim B(20, 0.02)$
d. 0.4
e. 0.6676
f. 0.0071

Q 4.4.20

It has been estimated that only about 30% of California residents have adequate earthquake supplies. Suppose you randomly survey 11 California residents. We are interested in the number who have adequate earthquake supplies.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. What is the probability that at least eight have adequate earthquake supplies?

e. Is it more likely that none or that all of the residents surveyed will have adequate earthquake supplies? Why?
f. How many residents do you expect will have adequate earthquake supplies?

Q 4.4.21

There are two similar games played for Chinese New Year and Vietnamese New Year. In the Chinese version, fair dice with numbers 1, 2, 3, 4, 5, and 6 are used, along with a board with those numbers. In the Vietnamese version, fair dice with pictures of a gourd, fish, rooster, crab, crayfish, and deer are used. The board has those six objects on it, also. We will play with bets being $1. The player places a bet on a number or object. The "house" rolls three dice. If none of the dice show the number or object that was bet, the house keeps the $1 bet. If one of the dice shows the number or object bet (and the other two do not show it), the player gets back his or her $1 bet, plus $1 profit. If two of the dice show the number or object bet (and the third die does not show it), the player gets back his or her $1 bet, plus $2 profit. If all three dice show the number or object bet, the player gets back his or her $1 bet, plus $3 profit. Let $X =$ number of matches and $Y =$ profit per game.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. List the values that Y may take on. Then, construct one PDF table that includes both $X$and Y and their probabilities.
e. Calculate the average expected matches over the long run of playing this game for the player.
f. Calculate the average expected earnings over the long run of playing this game for the player.
g. Determine who has the advantage, the player or the house.

S 4.4.21

a. $X =$ the number of matches
b. 0, 1, 2, 3
c. $X \sim B(3, 16)(3, 16)$
d. In dollars: −1, 1, 2, 3
e. $\frac{1}{2}$
f. Multiply each Y value by the corresponding X probability from the PDF table. The answer is −0.0787. You lose about eight cents, on average, per game.
g. The house has the advantage.

Q 4.4.22

According to The World Bank, only 9% of the population of Uganda had access to electricity as of 2009. Suppose we randomly sample 150 people in Uganda. Let $X =$ the number of people who have access to electricity.

a. What is the probability distribution for X?
b. Using the formulas, calculate the mean and standard deviation of X.
c. Use your calculator to find the probability that 15 people in the sample have access to electricity.
d. Find the probability that at most ten people in the sample have access to electricity.
e. Find the probability that more than 25 people in the sample have access to electricity.

Q 4.4.23

The literacy rate for a nation measures the proportion of people age 15 and over that can read and write. The literacy rate in Afghanistan is 28.1%. Suppose you choose 15 people in Afghanistan at random. Let $X =$ the number of people who are literate.

1. Sketch a graph of the probability distribution of X.
2. Using the formulas, calculate the (i) mean and (ii) standard deviation of X.
3. Find the probability that more than five people in the sample are literate. Is it is more likely that three people or four people are literate.

S 4.4.23

1. $X \sim B(15, 0.281)$

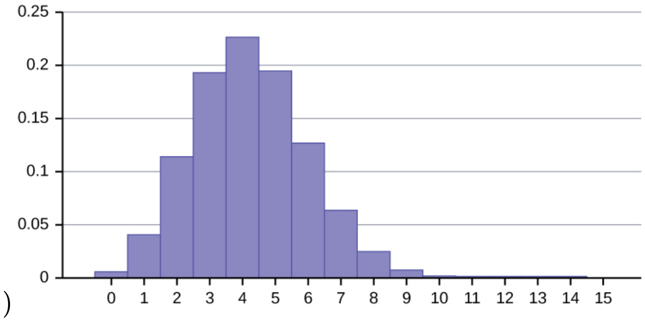


Figure 4.4.1.

2. 1. Mean $= \mu = np = 15(0.281) = 4.215$
   2. Standard Deviation $= \sigma = \sqrt{npq} = \sqrt{15(0.281)(0.719)} = 1.7409$
3. $P(x > 5) = 1 - P(x \leq 5) = 1 - \text{binomcdf}(15, 0.281, 5) = 1 - 0.7754 = 0.2246$
   $P(x = 3) = \text{binompdf}(15, 0.281, 3) = 0.1927$
   $P(x = 4) = \text{binompdf}(15, 0.281, 4) = 0.2259$
   It is more likely that four people are literate that three people are.

4.5: Geometric Distribution

Q 4.5.1

A consumer looking to buy a used red Miata car will call dealerships until she finds a dealership that carries the car. She estimates the probability that any independent dealership will have the car will be 28%. We are interested in the number of dealerships she must call.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. On average, how many dealerships would we expect her to have to call until she finds one that has the car?
e. Find the probability that she must call at most four dealerships.
f. Find the probability that she must call three or four dealerships.

Q 4.5.2

Suppose that the probability that an adult in America will watch the Super Bowl is 40%. Each person is considered independent. We are interested in the number of adults in America we must survey until we find one who will watch the Super Bowl.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many adults in America do you expect to survey until you find one who will watch the Super Bowl?
e. Find the probability that you must ask seven people.
f. Find the probability that you must ask three or four people.

S 4.5.2

a. $X =$ the number of adults in America who are surveyed until one says he or she will watch the Super Bowl.
b. $X \sim G(0.40)$
c. 2.5
d. 0.0187
e. 0.2304

Q 4.5.3

It has been estimated that only about 30% of California residents have adequate earthquake supplies. Suppose we are interested in the number of California residents we must survey until we find a resident who does **not** have adequate earthquake supplies.

a. In words, define the random variable X.
b. List the values that X may take on.

c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. What is the probability that we must survey just one or two residents until we find a California resident who does not have adequate earthquake supplies?
e. What is the probability that we must survey at least three California residents until we find a California resident who does not have adequate earthquake supplies?
f. How many California residents do you expect to need to survey until you find a California resident who **does not** have adequate earthquake supplies?
g. How many California residents do you expect to need to survey until you find a California resident who **does** have adequate earthquake supplies?

Q 4.5.4

In one of its Spring catalogs, L.L. Bean® advertised footwear on 29 of its 192 catalog pages. Suppose we randomly survey 20 pages. We are interested in the number of pages that advertise footwear. Each page may be picked more than once.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many pages do you expect to advertise footwear on them?
e. Is it probable that all twenty will advertise footwear on them? Why or why not?
f. What is the probability that fewer than ten will advertise footwear on them?
g. Reminder: A page may be picked more than once. We are interested in the number of pages that we must randomly survey until we find one that has footwear advertised on it. Define the random variable X and give its distribution.
h. What is the probability that you only need to survey at most three pages in order to find one that advertises footwear on it?
i. How many pages do you expect to need to survey in order to find one that advertises footwear?

S 4.5.4

a. $X =$ the number of pages that advertise footwear
b. X takes on the values 0, 1, 2, ..., 20
c. $X \sim B(20, \frac{29}{192})$
d. 3.02
e. No
f. 0.9997
g. $X =$ the number of pages we must survey until we find one that advertises footwear. $X \sim G(\frac{29}{192})$
h. 0.3881
i. 6.6207 pages

Q 4.5.5

Suppose that you are performing the probability experiment of rolling one fair six-sided die. Let F be the event of rolling a four or a five. You are interested in how many times you need to roll the die in order to obtain the first four or five as the outcome.

- $p =$ probability of success (event F occurs)
- $q =$ probability of failure (event F does not occur)

a. Write the description of the random variable X.
b. What are the values that X can take on?
c. Find the values of p and q.
d. Find the probability that the first occurrence of event F (rolling a four or five) is on the second trial.

Q 4.5.5

Ellen has music practice three days a week. She practices for all of the three days 85% of the time, two days 8% of the time, one day 4% of the time, and no days 3% of the time. One week is selected at random. What values does X take on?

S 4.5.5

0, 1, 2, and 3

Q 4.5.6

The World Bank records the prevalence of HIV in countries around the world. According to their data, “Prevalence of HIV refers to the percentage of people ages 15 to 49 who are infected with HIV.”[1] In South Africa, the prevalence of HIV is 17.3%. Let $X =$ the number of people you test until you find a person infected with HIV.

a. Sketch a graph of the distribution of the discrete random variable X.
b. What is the probability that you must test 30 people to find one with HIV?
c. What is the probability that you must ask ten people?
d. Find the (i) mean and (ii) standard deviation of the distribution of X.

Q 4.5.7

According to a recent Pew Research poll, 75% of millenials (people born between 1981 and 1995) have a profile on a social networking site. Let $X =$ the number of millenials you ask until you find a person without a profile on a social networking site.

a. Describe the distribution of X.
b. Find the (i) mean and (ii) standard deviation of X.
c. What is the probability that you must ask ten people to find one person without a social networking site?
d. What is the probability that you must ask 20 people to find one person without a social networking site?
e. What is the probability that you must ask *at most* five people?

S 4.5.7

a. $X \sim \mathrm{G}(0.25)$
b. i. Mean $= \mu = \frac{1}{p} = \frac{1}{0.25} = 4$
   ii. Standard Deviation $= \sigma = \sqrt{\frac{1-p}{p^2}} = \sqrt{\frac{1-0.25}{0.25^2}} \approx 3.4641$
c. $P(x = 10) = \text{geometpdf}(0.25, 10) = 0.0188$
d. $P(x = 20) = \text{geometpdf}(0.25, 20) = 0.0011$
e. $P(x \leq 5) = \text{geometcdf}(0.25, 5) = 0.7627$

Footnotes

1. ”Prevalence of HIV, total (% of populations ages 15-49),” The World Bank, 2013. Available online at http://data.worldbank.org/indicator/...last&sort=desc (accessed May 15, 2013).

4.6: Hypergeometric Distribution

Q 4.6.1

A group of Martial Arts students is planning on participating in an upcoming demonstration. Six are students of Tae Kwon Do; seven are students of Shotokan Karate. Suppose that eight students are randomly picked to be in the first demonstration. We are interested in the number of Shotokan Karate students in that first demonstration.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many Shotokan Karate students do we expect to be in that first demonstration?

Q 4.6.2

In one of its Spring catalogs, L.L. Bean® advertised footwear on 29 of its 192 catalog pages. Suppose we randomly survey 20 pages. We are interested in the number of pages that advertise footwear. Each page may be picked at most once.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many pages do you expect to advertise footwear on them?
e. Calculate the standard deviation.

S 4.6.2

a. $X =$ the number of pages that advertise footwear
b. 0, 1, 2, 3, ..., 20
c. $X \sim \mathrm{H}(29, 163, 20); r = 29, b = 163, n = 20$
d. 3.03
e. 1.5197

Q 4.6.3

Suppose that a technology task force is being formed to study technology awareness among instructors. Assume that ten people will be randomly chosen to be on the committee from a group of 28 volunteers, 20 who are technically proficient and eight who are not. We are interested in the number on the committee who are **not** technically proficient.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many instructors do you expect on the committee who are **not** technically proficient?
e. Find the probability that at least five on the committee are not technically proficient.
f. Find the probability that at most three on the committee are not technically proficient.

Q 4.6.4

Suppose that nine Massachusetts athletes are scheduled to appear at a charity benefit. The nine are randomly chosen from eight volunteers from the Boston Celtics and four volunteers from the New England Patriots. We are interested in the number of Patriots picked.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. Are you choosing the nine athletes with or without replacement?

S 4.6.4

a. $X =$ the number of Patriots picked
b. 0, 1, 2, 3, 4
c. $X \sim H(4, 8, 9)$
d. Without replacement

Q 4.6.5

A bridge hand is defined as 13 cards selected at random and without replacement from a deck of 52 cards. In a standard deck of cards, there are 13 cards from each suit: hearts, spades, clubs, and diamonds. What is the probability of being dealt a hand that does not contain a heart?

a. What is the group of interest?
b. How many are in the group of interest?
c. How many are in the other group?
d. Let $X =$ _________. What values does X take on?
e. The probability question is $P($_______$)$.
f. Find the probability in question.
g. Find the (i) mean and (ii) standard deviation of X.

4.7: Poisson Distribution

Q 4.7.1

The switchboard in a Minneapolis law office gets an average of 5.5 incoming phone calls during the noon hour on Mondays. Experience shows that the existing staff can handle up to six calls in an hour. Let $X =$ the number of calls received at noon.

a. Find the mean and standard deviation of X.
b. What is the probability that the office receives at most six calls at noon on Monday?

c. Find the probability that the law office receives six calls at noon. What does this mean to the law office staff who get, on average, 5.5 incoming phone calls at noon?
d. What is the probability that the office receives more than eight calls at noon?

S 4.7.1

a. $X \sim P(5.5); \mu = 5.5; \sigma = \sqrt{5.5} \approx 2.3452$
b. $P(x \leq 6) = \text{poissoncdf}(5.5, 6) \approx 0.6860$
c. There is a 15.7% probability that the law staff will receive more calls than they can handle.
d. $P(x > 8) = 1 - P(x \leq 8) = 1 - \text{poissoncdf}(5.5, 8) \approx 1 - 0.8944 = 0.1056$

Q 4.7.2

The maternity ward at Dr. Jose Fabella Memorial Hospital in Manila in the Philippines is one of the busiest in the world with an average of 60 births per day. Let $X =$ the number of births in an hour.

a. Find the mean and standard deviation of X.
b. Sketch a graph of the probability distribution of X.
c. What is the probability that the maternity ward will deliver three babies in one hour?
d. What is the probability that the maternity ward will deliver at most three babies in one hour?
e. What is the probability that the maternity ward will deliver more than five babies in one hour?

Q 4.7.3

A manufacturer of Christmas tree light bulbs knows that 3% of its bulbs are defective. Find the probability that a string of 100 lights contains at most four defective bulbs using both the binomial and Poisson distributions.

S 4.7.3

Let $X =$ the number of defective bulbs in a string.

Using the Poisson distribution:

- $\mu = np = 100(0.03) = 3$
- $X \sim P(3)$
- $P(x \leq 4) = \text{poissoncdf}(3, 4) \approx 0.8153$

Using the binomial distribution:

- $X \sim \text{B}(100, 0.03)$
- $P(x \leq 4) = \text{binomcdf}(100, 0.03, 4) \approx 0.8179$

The Poisson approximation is very good—the difference between the probabilities is only 0.0026.

Q 4.7.4

The average number of children a Japanese woman has in her lifetime is 1.37. Suppose that one Japanese woman is randomly chosen.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. Find the probability that she has no children.
e. Find the probability that she has fewer children than the Japanese average.
f. Find the probability that she has more children than the Japanese average.

Q 4.7.5

The average number of children a Spanish woman has in her lifetime is 1.47. Suppose that one Spanish woman is randomly chosen.

a. In words, define the Random Variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)

d. Find the probability that she has no children.
e. Find the probability that she has fewer children than the Spanish average.
f. Find the probability that she has more children than the Spanish average .

S 4.7.5

a. $X =$ the number of children for a Spanish woman
b. 0, 1, 2, 3,...
c. $X \sim P(1.47)$
d. 0.2299
e. 0.5679
f. 0.4321

Q 4.7.6

Fertile, female cats produce an average of three litters per year. Suppose that one fertile, female cat is randomly chosen. In one year, find the probability she produces:

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _______
d. Find the probability that she has no litters in one year.
e. Find the probability that she has at least two litters in one year.
f. Find the probability that she has exactly three litters in one year.

Q 4.7.7

he chance of having an extra fortune in a fortune cookie is about 3%. Given a bag of 144 fortune cookies, we are interested in the number of cookies with an extra fortune. Two distributions may be used to solve this problem, but only use one distribution to solve the problem.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many cookies do we expect to have an extra fortune?
e. Find the probability that none of the cookies have an extra fortune.
f. Find the probability that more than three have an extra fortune.
g. As n increases, what happens involving the probabilities using the two distributions? Explain in complete sentences.

S 4.7.7

a. $X =$ the number of fortune cookies that have an extra fortune
b. 0, 1, 2, 3,... 144
c. $X \sim B(144, 0.03)$ or $P(4.32)$
d. 4.32
e. 0.0124 or 0.0133
f. 0.6300 or 0.6264
g. As n gets larger, the probabilities get closer together.

Q 4.7.8

According to the South Carolina Department of Mental Health web site, for every 200 U.S. women, the average number who suffer from anorexia is one. Out of a randomly chosen group of 600 U.S. women determine the following.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many are expected to suffer from anorexia?
e. Find the probability that no one suffers from anorexia.
f. Find the probability that more than four suffer from anorexia.

Q 4.7.9

The chance of an IRS audit for a tax return with over $25,000 in income is about 2% per year. Suppose that 100 people with tax returns over $25,000 are randomly picked. We are interested in the number of people audited in one year. Use a Poisson distribution to answer the following questions.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many are expected to be audited?
e. Find the probability that no one was audited.
f. Find the probability that at least three were audited.

S 4.7.9

a. $X =$ the number of people audited in one year
b. 0, 1, 2, ..., 100
c. $X \sim P(2)$
d. 2
e. 0.1353
f. 0.3233

Q 4.7.10

Approximately 8% of students at a local high school participate in after-school sports all four years of high school. A group of 60 seniors is randomly chosen. Of interest is the number that participated in after-school sports all four years of high school.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. How many seniors are expected to have participated in after-school sports all four years of high school?
e. Based on numerical values, would you be surprised if none of the seniors participated in after-school sports all four years of high school? Justify your answer numerically.
f. Based on numerical values, is it more likely that four or that five of the seniors participated in after-school sports all four years of high school? Justify your answer numerically.

Q 4.7.11

On average, Pierre, an amateur chef, drops three pieces of egg shell into every two cake batters he makes. Suppose that you buy one of his cakes.

a. In words, define the random variable X.
b. List the values that X may take on.
c. Give the distribution of X. $X \sim$ _____(_____,_____)
d. On average, how many pieces of egg shell do you expect to be in the cake?
e. What is the probability that there will not be any pieces of egg shell in the cake?
f. Let's say that you buy one of Pierre's cakes each week for six weeks. What is the probability that there will not be any egg shell in any of the cakes?
g. Based upon the average given for Pierre, is it possible for there to be seven pieces of shell in the cake? Why?

S 4.7.11

a. $X =$ the number of shell pieces in one cake
b. 0, 1, 2, 3,...
c. $X \sim P(1.5)$
d. 1.5
e. 0.2231
f. 0.0001
g. Yes

Use the following information to answer the next two exercises: The average number of times per week that Mrs. Plum's cats wake her up at night because they want to play is ten. We are interested in the number of times her cats wake her up each week.

Q 4.7.12

In words, the random variable $X =$ _________________

a. the number of times Mrs. Plum's cats wake her up each week.
b. the number of times Mrs. Plum's cats wake her up each hour.
c. the number of times Mrs. Plum's cats wake her up each night.
d. the number of times Mrs. Plum's cats wake her up.

Q 4.7.13

Find the probability that her cats will wake her up no more than five times next week.

a. 0.5000
b. 0.9329
c. 0.0378
d. 0.0671

S 4.7.13

d

4.8: Discrete Distribution (Playing Card Experiment)

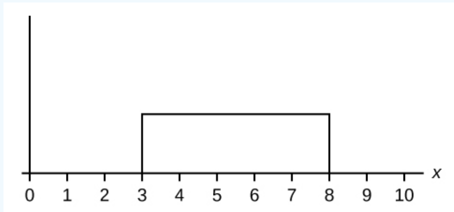
4.9: Discrete Distribution (Lucky Dice Experiment)

7.E: Exercises

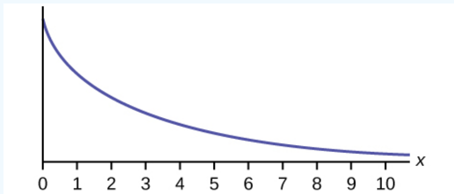
5.1: Introduction

5.2: Continuous Probability Functions

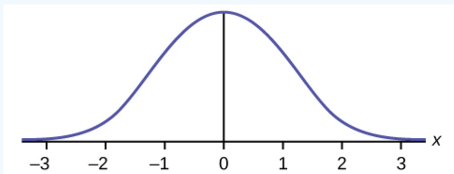
Which type of distribution does the graph illustrate?



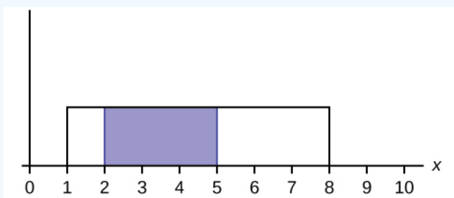
Which type of distribution does the graph illustrate?



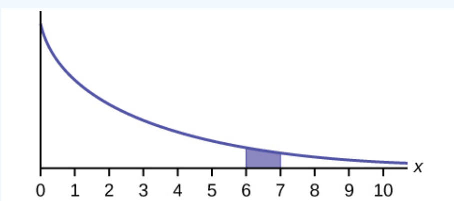
Which type of distribution does the graph illustrate?



What does the shaded area represent? $P($___$< x <$ ___$)$



What does the shaded area represent? $P($___$< x <$ ___$)$



For a continuous probability distribution, $0 \leq x \leq 15$. What is $P(x > 15)$?

What is the area under $f(x)$ if the function is a continuous probability density function?

For a continuous probability distribution, $0 \le x \le 10$. What is $P(x = 7)$?

A **continuous** probability function is restricted to the portion between $x = 0$ and 7. What is $P(x = 10)$?

$f(x)$ for a continuous probability function is

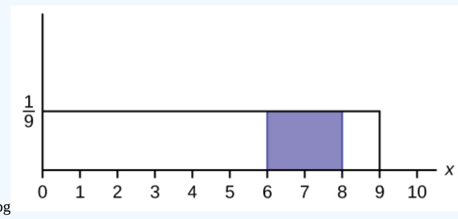
, and the function is restricted to $0 \le x \le 5$. What is $P(x < 0)$?

$f(x)$, a continuous probability function, is equal to

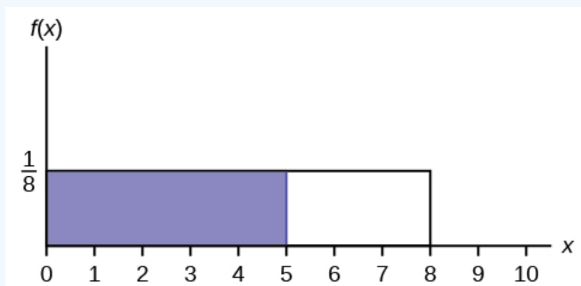
, and the function is restricted to $0 \le x \le 12$. What is $P(0 < x < 12)$?

one

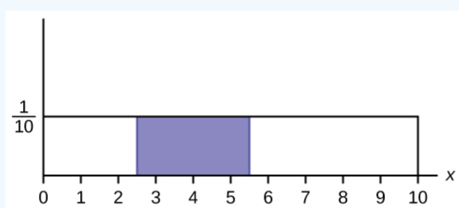
Find the probability that x falls in the shaded area.

CNX_Stats_C05_M02_item001.jpg 

Find the probability that x falls in the shaded area.



Find the probability that x falls in the shaded area.

CNX_Stats_C05_M02_item003.jpg 

$f(x)$, a continuous probability function, is equal to

and the function is restricted to $1 \le x \le 4$. Describe

The probability is equal to the area from $x =$ to $x = 4$ above the x-axis and up to $f(x) =$

Homework

For each probability and percentile problem, draw the picture.

Consider the following experiment. You are one of 100 people enlisted to take part in a study to determine the percent of nurses in America with an R.N. (registered nurse) degree. You ask nurses if they have an R.N. degree. The nurses answer “yes” or "no." You then calculate the percentage of nurses with an R.N. degree. You give that percentage to your supervisor.

1. What part of the experiment will yield discrete data?
2. What part of the experiment will yield continuous data?

When age is rounded to the nearest year, do the data stay continuous, or do they become discrete? Why?

Age is a measurement, regardless of the accuracy used.

5.3: The Uniform Distribution

5.4: The Exponential Distribution

5.5: Continuous Distribution

7.E: The Normal Distribution (Exercises)

These are homework exercises to accompany the Textmap created for "Introductory Statistics" by OpenStax.

6.1: Introduction

6.2: The Standard Normal Distribution

Use the following information to answer the next two exercises: The patient recovery time from a particular surgical procedure is normally distributed with a mean of 5.3 days and a standard deviation of 2.1 days.

Q 6.2.1

What is the median recovery time?

a. 2.7
b. 5.3
c. 7.4
d. 2.1

Q 6.2.2

What is the *z*-score for a patient who takes ten days to recover?

a. 1.5
b. 0.2
c. 2.2
d. 7.3

S 6.2.2

c

Q 6.2.3

The length of time to find a parking space at 9 A.M. follows a normal distribution with a mean of five minutes and a standard deviation of two minutes. If the mean is significantly greater than the standard deviation, which of the following statements is true?

I. The data cannot follow the uniform distribution.
II. The data cannot follow the exponential distribution..
III. The data cannot follow the normal distribution.

a. I only
b. II only
c. III only
d. I, II, and III

Q 6.2.4

The heights of the 430 National Basketball Association players were listed on team rosters at the start of the 2005–2006 season. The heights of basketball players have an approximate normal distribution with mean, μ = 79 inches and a standard deviation, σ = 3.89 inches. For each of the following heights, calculate the *z*-score and interpret it using complete sentences.

a. 77 inches
b. 85 inches
c. If an NBA player reported his height had a *z*-score of 3.5, would you believe him? Explain your answer.

S 6.2.4

a. Use the z-score formula. $z = -0.5141$. The height of 77 inches is 0.5141 standard deviations below the mean. An NBA player whose height is 77 inches is shorter than average.
b. Use the z-score formula. $z = 1.5424$. The height 85 inches is 1.5424 standard deviations above the mean. An NBA player whose height is 85 inches is taller than average.

c. Height $= 79 + 3.5(3.89) = 90.67$ inches, which is over 7.7 feet tall. There are very few NBA players this tall so the answer is no, not likely.

Q 6.2.5

The systolic blood pressure (given in millimeters) of males has an approximately normal distribution with mean $\mu = 125$ and standard deviation $\sigma = 14$. Systolic blood pressure for males follows a normal distribution.

a. Calculate the *z*-scores for the male systolic blood pressures 100 and 150 millimeters.
b. If a male friend of yours said he thought his systolic blood pressure was 2.5 standard deviations below the mean, but that he believed his blood pressure was between 100 and 150 millimeters, what would you say to him?

Q 6.2.6

Kyle's doctor told him that the *z*-score for his systolic blood pressure is 1.75. Which of the following is the best interpretation of this standardized score? The systolic blood pressure (given in millimeters) of males has an approximately normal distribution with mean $\mu = 125$ and standard deviation $\sigma = 14$. If $X =$ a systolic blood pressure score then $X \sim N(125, 14)$.

a. Which answer(s) **is/are** correct?
    i. Kyle's systolic blood pressure is 175.
    ii. Kyle's systolic blood pressure is 1.75 times the average blood pressure of men his age.
    iii. Kyle's systolic blood pressure is 1.75 above the average systolic blood pressure of men his age.
    iv. Kyles's systolic blood pressure is 1.75 standard deviations above the average systolic blood pressure for men.
b. Calculate Kyle's blood pressure.

S 6.2.6

a. iv
b. Kyle's blood pressure is equal to $125 + (1.75)(14) = 149.5$.

Q 6.2.7

Height and weight are two measurements used to track a child's development. The World Health Organization measures child development by comparing the weights of children who are the same height and the same gender. In 2009, weights for all 80 cm girls in the reference population had a mean $\mu = 10.2$ kg and standard deviation $\sigma = 0.8$ kg. Weights are normally distributed. $X \sim N(10.2, 0.8)$. Calculate the *z*-scores that correspond to the following weights and interpret them.

a. 11 kg
b. 7.9 kg
c. 12.2 kg

Q 6.2.8

In 2005, 1,475,623 students heading to college took the SAT. The distribution of scores in the math section of the SAT follows a normal distribution with mean $\mu = 520$ and standard deviation $\sigma = 115$.

a. Calculate the z-score for an SAT score of 720. Interpret it using a complete sentence.
b. What math SAT score is 1.5 standard deviations above the mean? What can you say about this SAT score?
c. For 2012, the SAT math test had a mean of 514 and standard deviation 117. The ACT math test is an alternate to the SAT and is approximately normally distributed with mean 21 and standard deviation 5.3. If one person took the SAT math test and scored 700 and a second person took the ACT math test and scored 30, who did better with respect to the test they took?

S 6.2.8

Let $X =$ an SAT math score and $Y =$ an ACT math score.

a. $X = 720 \frac{720-520}{15} = 1.74$ The exam score of 720 is 1.74 standard deviations above the mean of 520.
b. $z = 1.5$
   The math SAT score is $520 + 1.5(115) \approx 692.5$ The exam score of 692.5 is 1.5 standard deviations above the mean of 520.
c. $\frac{X-\mu}{\sigma} = \frac{700-514}{117} \approx 1.59$, the *z*-score for the SAT. $\frac{Y-\mu}{\sigma} = \frac{30-21}{5.3} \approx 1.70$, the *z*-scores for the ACT. With respect to the test they took, the person who took the ACT did better (has the higher *z*-score).

6.3: Using the Normal Distribution

Use the following information to answer the next two exercises: The patient recovery time from a particular surgical procedure is normally distributed with a mean of 5.3 days and a standard deviation of 2.1 days.

Q 6.3.1

What is the probability of spending more than two days in recovery?

a. 0.0580
b. 0.8447
c. 0.0553
d. 0.9420

Q 6.3.2

The 90th percentile for recovery times is?

a. 8.89
b. 7.07
c. 7.99
d. 4.32

S 6.3.2

c

Use the following information to answer the next three exercises: The length of time it takes to find a parking space at 9 A.M. follows a normal distribution with a mean of five minutes and a standard deviation of two minutes.

Q 6.3.3

Based upon the given information and numerically justified, would you be surprised if it took less than one minute to find a parking space?

a. Yes
b. No
c. Unable to determine

Q 6.3.4

Find the probability that it takes at least eight minutes to find a parking space.

a. 0.0001
b. 0.9270
c. 0.1862
d. 0.0668

S 6.3.4

d

Q 6.3.5

Seventy percent of the time, it takes more than how many minutes to find a parking space?

a. 1.24
b. 2.41
c. 3.95
d. 6.05

Q 6.3.6

According to a study done by De Anza students, the height for Asian adult males is normally distributed with an average of 66 inches and a standard deviation of 2.5 inches. Suppose one Asian adult male is randomly chosen. Let $X =$ height of the individual.

a. $X \sim$ _____(_____,_____)
b. Find the probability that the person is between 65 and 69 inches. Include a sketch of the graph, and write a probability statement.
c. Would you expect to meet many Asian adult males over 72 inches? Explain why or why not, and justify your answer numerically.
d. The middle 40% of heights fall between what two values? Sketch the graph, and write the probability statement.

S 6.3.6

a. $X \sim N(66, 2.5)$
b. 0.5404
c. No, the probability that an Asian male is over 72 inches tall is 0.0082

Q 6.3.7

IQ is normally distributed with a mean of 100 and a standard deviation of 15. Suppose one individual is randomly chosen. Let $X =$ IQ of an individual.

a. $X \sim$ _____(_____,_____)
b. Find the probability that the person has an IQ greater than 120. Include a sketch of the graph, and write a probability statement.
c. MENSA is an organization whose members have the top 2% of all IQs. Find the minimum IQ needed to qualify for the MENSA organization. Sketch the graph, and write the probability statement.
d. The middle 50% of IQs fall between what two values? Sketch the graph and write the probability statement.

Q 6.3.8

The percent of fat calories that a person in America consumes each day is normally distributed with a mean of about 36 and a standard deviation of 10. Suppose that one individual is randomly chosen. Let $X =$ percent of fat calories.

a. $X \sim$ _____(_____,_____)
b. Find the probability that the percent of fat calories a person consumes is more than 40. Graph the situation. Shade in the area to be determined.
c. Find the maximum number for the lower quarter of percent of fat calories. Sketch the graph and write the probability statement.

S 6.3.8

a. $X \sim N(36, 10)$
b. The probability that a person consumes more than 40% of their calories as fat is 0.3446.
c. Approximately 25% of people consume less than 29.26% of their calories as fat.

Q 6.3.9

Suppose that the distance of fly balls hit to the outfield (in baseball) is normally distributed with a mean of 250 feet and a standard deviation of 50 feet.

a. If $X =$ distance in feet for a fly ball, then $X \sim$ _____(_____,_____)
b. If one fly ball is randomly chosen from this distribution, what is the probability that this ball traveled fewer than 220 feet? Sketch the graph. Scale the horizontal axis *X*. Shade the region corresponding to the probability. Find the probability.
c. Find the 80$^{\text{th}}$ percentile of the distribution of fly balls. Sketch the graph, and write the probability statement.

Q 6.3.10

In China, four-year-olds average three hours a day unsupervised. Most of the unsupervised children live in rural areas, considered safe. Suppose that the standard deviation is 1.5 hours and the amount of time spent alone is normally distributed. We randomly select one Chinese four-year-old living in a rural area. We are interested in the amount of time the child spends alone per day.

a. In words, define the random variable X.
b. $X \sim$ _____(_____,_____)
c. Find the probability that the child spends less than one hour per day unsupervised. Sketch the graph, and write the probability statement.
d. What percent of the children spend over ten hours per day unsupervised?
e. Seventy percent of the children spend at least how long per day unsupervised?

S 6.3.10

a. $X =$ number of hours that a Chinese four-year-old in a rural area is unsupervised during the day.
b. $X\ N(3, 1.5)$
c. The probability that the child spends less than one hour a day unsupervised is 0.0918.
d. The probability that a child spends over ten hours a day unsupervised is less than 0.0001.
e. 2.21 hours

Q 6.3.11

In the 1992 presidential election, Alaska's 40 election districts averaged 1,956.8 votes per district for President Clinton. The standard deviation was 572.3. (There are only 40 election districts in Alaska.) The distribution of the votes per district for President Clinton was bell-shaped. Let $X =$ number of votes for President Clinton for an election district.

a. State the approximate distribution of X.
b. Is 1,956.8 a population mean or a sample mean? How do you know?
c. Find the probability that a randomly selected district had fewer than 1,600 votes for President Clinton. Sketch the graph and write the probability statement.
d. Find the probability that a randomly selected district had between 1,800 and 2,000 votes for President Clinton.
e. Find the third quartile for votes for President Clinton.

Q 6.3.12

Suppose that the duration of a particular type of criminal trial is known to be normally distributed with a mean of 21 days and a standard deviation of seven days.

a. In words, define the random variable X.
b. $X \sim$ _____(_____,_____)
c. If one of the trials is randomly chosen, find the probability that it lasted at least 24 days. Sketch the graph and write the probability statement.
d. Sixty percent of all trials of this type are completed within how many days?

S 6.3.12

a. $X =$ the distribution of the number of days a particular type of criminal trial will take
b. $X \sim N(21, 7)$
c. The probability that a randomly selected trial will last more than 24 days is 0.3336.
d. 22.77

Q 6.3.13

Terri Vogel, an amateur motorcycle racer, averages 129.71 seconds per 2.5 mile lap (in a seven-lap race) with a standard deviation of 2.28 seconds. The distribution of her race times is normally distributed. We are interested in one of her randomly selected laps.

a. In words, define the random variable X.
b. $X \sim$ _____(_____,_____)
c. Find the percent of her laps that are completed in less than 130 seconds.
d. The fastest 3% of her laps are under _____.
e. The middle 80% of her laps are from _______ seconds to _______ seconds.

Q 6.3.14

Thuy Dau, Ngoc Bui, Sam Su, and Lan Voung conducted a survey as to how long customers at Lucky claimed to wait in the checkout line until their turn. Let $X =$ time in line. Table displays the ordered real data (in minutes):

0.50	4.25	5	6	7.25
1.75	4.25	5.25	6	7.25
2	4.25	5.25	6.25	7.25
2.25	4.25	5.5	6.25	7.75
2.25	4.5	5.5	6.5	8

2.5	4.75	5.5	6.5	8.25
2.75	4.75	5.75	6.5	9.5
3.25	4.75	5.75	6.75	9.5
3.75	5	6	6.75	9.75
3.75	5	6	6.75	10.75

a. Calculate the sample mean and the sample standard deviation.
b. Construct a histogram.
c. Draw a smooth curve through the midpoints of the tops of the bars.
d. In words, describe the shape of your histogram and smooth curve.
e. Let the sample mean approximate μ and the sample standard deviation approximate σ. The distribution of X can then be approximated by $X \sim$ _____(_____,_____)
f. Use the distribution in part e to calculate the probability that a person will wait fewer than 6.1 minutes.
g. Determine the cumulative relative frequency for waiting less than 6.1 minutes.
h. Why aren't the answers to part f and part g exactly the same?
i. Why are the answers to part f and part g as close as they are?
j. If only ten customers has been surveyed rather than 50, do you think the answers to part f and part g would have been closer together or farther apart? Explain your conclusion.

S 6.3.14

a. $\text{mean} = 5.51$, $s = 2.15$
b. Check student's solution.
c. Check student's solution.
d. Check student's solution.
e. $X \sim N(5.51, 2.15)$
f. 0.6029
g. The cumulative frequency for less than 6.1 minutes is 0.64.
h. The answers to part f and part g are not exactly the same, because the normal distribution is only an approximation to the real one.
i. The answers to part f and part g are close, because a normal distribution is an excellent approximation when the sample size is greater than 30.
j. The approximation would have been less accurate, because the smaller sample size means that the data does not fit normal curve as well.

Q 6.3.15

Suppose that Ricardo and Anita attend different colleges. Ricardo's GPA is the same as the average GPA at his school. Anita's GPA is 0.70 standard deviations above her school average. In complete sentences, explain why each of the following statements may be false.

a. Ricardo's actual GPA is lower than Anita's actual GPA.
b. Ricardo is not passing because his *z*-score is zero.
c. Anita is in the 70$^{\text{th}}$ percentile of students at her college.

Q 6.3.16

Table shows a sample of the maximum capacity (maximum number of spectators) of sports stadiums. The table does not include horse-racing or motor-racing stadiums.

40,000	40,000	45,050	45,500	46,249	48,134
49,133	50,071	50,096	50,466	50,832	51,100
51,500	51,900	52,000	52,132	52,200	52,530
52,692	53,864	54,000	55,000	55,000	55,000

55,000	55,000	55,000	55,082	57,000	58,008
59,680	60,000	60,000	60,492	60,580	62,380
62,872	64,035	65,000	65,050	65,647	66,000
66,161	67,428	68,349	68,976	69,372	70,107
70,585	71,594	72,000	72,922	73,379	74,500
75,025	76,212	78,000	80,000	80,000	82,300

a. Calculate the sample mean and the sample standard deviation for the maximum capacity of sports stadiums (the data).
b. Construct a histogram.
c. Draw a smooth curve through the midpoints of the tops of the bars of the histogram.
d. In words, describe the shape of your histogram and smooth curve.
e. Let the sample mean approximate μ and the sample standard deviation approximate σ. The distribution of X can then be approximated by $X \sim$ _____(_____,_____).
f. Use the distribution in part e to calculate the probability that the maximum capacity of sports stadiums is less than 67,000 spectators.
g. Determine the cumulative relative frequency that the maximum capacity of sports stadiums is less than 67,000 spectators. Hint: Order the data and count the sports stadiums that have a maximum capacity less than 67,000. Divide by the total number of sports stadiums in the sample.
h. Why aren't the answers to part f and part g exactly the same?

S 6.3.16

a. $\text{mean} = 60,136$, $s = 10,468$
b. Answers will vary.
c. Answers will vary.
d. Answers will vary.
e. $X \sim N(60136, 10468)$
f. 0.7440
g. The cumulative relative frequency is $\frac{43}{60} = 0.717$.
h. The answers for part f and part g are not the same, because the normal distribution is only an approximation.

Q 6.3.17

An expert witness for a paternity lawsuit testifies that the length of a pregnancy is normally distributed with a mean of 280 days and a standard deviation of 13 days. An alleged father was out of the country from 240 to 306 days before the birth of the child, so the pregnancy would have been less than 240 days or more than 306 days long if he was the father. The birth was uncomplicated, and the child needed no medical intervention. What is the probability that he was NOT the father? What is the probability that he could be the father? Calculate the *z*-scores first, and then use those to calculate the probability.

Q 6.3.18

A NUMMI assembly line, which has been operating since 1984, has built an average of 6,000 cars and trucks a week. Generally, 10% of the cars were defective coming off the assembly line. Suppose we draw a random sample of n = 100 cars. Let X represent the number of defective cars in the sample. What can we say about X in regard to the 68-95-99.7 empirical rule (one standard deviation, two standard deviations and three standard deviations from the mean are being referred to)? Assume a normal distribution for the defective cars in the sample.

S 6.3.18

- $n = 100; p = 0.1; q = 0.9$
- $\mu = np = (100)(0.10) = 10$
- $\sigma = \sqrt{npq} = \sqrt{(100)(0.1)(0.9)} = 3$

a. $z = \pm : x_1 = \mu + z\sigma = 10 + 1(3) = 13$ and $x_2 = \mu = z\sigma = 10 - 1(3) = 7.68$ of the defective cars will fall between seven and 13.

b. $z = \pm : x_1 = \mu + z\sigma = 10 + 2(3) = 16$ and $x_2 = \mu = z\sigma = 10 - 2(3) = 4.95$ of the defective cars will fall between four and 16
c. $z = \pm : x_1 = \mu + z\sigma = 10 + 3(3) = 19$ and $x_2 = \mu = z\sigma = 10 - 3(3) = 1.997$ of the defective cars will fall between one and 19.

Q 6.3.19

We flip a coin 100 times ($n = 100$) and note that it only comes up heads 20% ($p = 0.20$) of the time. The mean and standard deviation for the number of times the coin lands on heads is $\mu = 20$ and $\sigma = 4$ (verify the mean and standard deviation). Solve the following:

a. There is about a 68% chance that the number of heads will be somewhere between ___ and ___.
b. There is about a ____chance that the number of heads will be somewhere between 12 and 28.
c. There is about a ____ chance that the number of heads will be somewhere between eight and 32.

Q 6.3.20

A $1 scratch off lotto ticket will be a winner one out of five times. Out of a shipment of $n = 190$ lotto tickets, find the probability for the lotto tickets that there are

a. somewhere between 34 and 54 prizes.
b. somewhere between 54 and 64 prizes.
c. more than 64 prizes.

S 6.3.21

- $n = 190; p = 1515 = 0.2; q = 0.8$
- $\mu = np = (190)(0.2) = 38$
- $\sigma = \sqrt{npq} = \sqrt{(190)(0.2)(0.8)} = 5.5136$

a. For this problem: $P(34 < x < 54) = \text{normalcdf}(34, 54, 48, 5.5136) = 0.7641$
b. For this problem: $P(54 < x < 64) = \text{normalcdf}(54, 64, 48, 5.5136) = 0.0018$
c. For this problem: $P(x > 64) = \text{normalcdf}(64, 10^{99}, 48, 5.5136) = 0.0000012$(approximately 0)

Q 6.3.22

Facebook provides a variety of statistics on its Web site that detail the growth and popularity of the site.

On average, 28 percent of 18 to 34 year olds check their Facebook profiles before getting out of bed in the morning. Suppose this percentage follows a normal distribution with a standard deviation of five percent.

a. Find the probability that the percent of 18 to 34-year-olds who check Facebook before getting out of bed in the morning is at least 30.
b. Find the 95$^{\text{th}}$ percentile, and express it in a sentence.

6.4: Normal Distribution (Lap Times)

6.5: Normal Distribution (Pinkie Length)

CHAPTER OVERVIEW

9: Problem Solving

Thumbnail: pixabay.com/photos/tax-form-irs-tax-taxes-finance-4080693/

9.1: Introduction

In previous math courses, you've no doubt run into the infamous "word problems." Unfortunately, these problems rarely resemble the type of problems we actually encounter in everyday life. In math books, you usually are told exactly which formula or procedure to use, and are given exactly the information you need to answer the question. In real life, problem solving requires identifying an appropriate formula or procedure, and determining what information you will need (and won't need) to answer the question.

In this chapter, we will review several basic but powerful algebraic ideas: percents, rates, and proportions. We will then focus on the problem solving process, and explore how to use these ideas to solve problems where we don't have perfect information.

8.2: Percents

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties[1]. Who is correct? How can we make sense of these numbers?

Percent literally means "per 100," or "parts per hundred." When we write 40%, this is equivalent to the fraction $\frac{40}{100}$ or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since $\frac{80}{200}=\frac{10}{25}=\frac{40}{100}$.

Example 1

243 people out of 400 state that they like dogs. What percent is this?

Solution

$\frac{243}{400}=0.6075=\frac{60.75}{100}$. This is 60.75%.

Notice that the percent can be found from the equivalent decimal by moving the decimal point two places to the right.

Example 2

Write each as a percent: a) $\frac{1}{4}$ b) 0.02 c) 2.35

Solution

a) $\frac{1}{4}=0.25=25\%$ b) $0.02=2\%$ c) $2.35=235\%$

Percents

If we have a *part* that is some *percent* of a *whole*, then

$\text{percent }=\frac{\text{part}}{\text{whole}}$, or equivalently, $\text{part }=\text{ percent}\cdot\text{whole}$

To do the calculations, we write the percent as a decimal.

Example 3

The sales tax in a town is 9.4%. How much tax will you pay on a \$140 purchase?

Solution

Here, \$140 is the whole, and we want to find 9.4% *of* \$140. We start by writing the percent as a decimal by moving the decimal point two places to the left (which is equivalent to dividing by 100). We can then compute:

$\text{tax}=0.094(140)=\$13.16$in tax.

Example 4

In the news, you hear "tuition is expected to increase by 7% next year." If tuition this year was \$1200 per quarter, what will it be next year?

Solution

The tuition next year will be the current tuition plus an additional 7%, so it will be 107% of this year's tuition:

$\$1200(1.07)=\$1284$

Alternatively, we could have first calculated 7% of \$1200: $\$1200(0.07)=\$84$

Notice this is *not* the expected tuition for next year (we could only wish). Instead, this is the expected *increase*, so to calculate the expected tuition, we'll need to add this change to the previous year's tuition:

$\$1200+\$84=\$1284$

Try it Now 1

A TV originally priced at $799 is on sale for 30% off. There is then a 9.2% sales tax. Find the price after including the discount and sales tax.

Answer

The sale price is $\$799(0.70)=\$559.30$ After tax, the price is $\$559.30(1.092)=\$610.76$

Example 5

The value of a car dropped from $7400 to $6800 over the last year. What percent decrease is this?

Solution

To compute the percent change, we first need to find the dollar value change: $\$6800-\$7400=-\$600$ Often we will take the absolute value of this amount, which is called the **absolute change**: $|-600|=600$.

Since we are computing the decrease relative to the starting value, we compute this percent out of $\$7400$:

$\frac{600}{7400}=0.081=8.1\%$ decrease. This is called a **relative change**.

Absolute and Relative Change

Given two quantities,

Absolute change = $|\text{ending quantity } - \text{ starting quantity}|$

Relative change: $\frac{\text{absolute change}}{\text{starting quantity}}$

Absolute change has the same units as the original quantity.

Relative change gives a percent change.

The starting quantity is called the **base** of the percent change.

The base of a percent is very important. For example, while Nixon was president, it was argued that marijuana was a "gateway" drug, claiming that 80% of marijuana smokers went on to use harder drugs like cocaine. The problem is, this isn't true. The true claim is that 80% of harder drug users first smoked marijuana. The difference is one of base: 80% of marijuana smokers using hard drugs, vs. 80% of hard drug users having smoked marijuana. These numbers are not equivalent. As it turns out, only one in 2,400 marijuana users actually go on to use harder drugs[2].

Example 6

There are about 75 QFC supermarkets in the U.S. Albertsons has about 215 stores. Compare the size of the two companies.

Solution

When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is $215-75=140$. From this, we could say "Albertsons has 140 more stores than QFC." However, if you wrote this in an article or paper, that number does not mean much. The relative difference may be more meaningful. There are two different relative changes we could calculate, depending on which store we use as the base:

Using QFC as the base, $\frac{140}{75}=1.867$.

This tells us Albertsons is 186.7% larger than QFC.

Using Albertsons as the base, $\frac{140}{215}=0.651$.

This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*. We could also calculate the size of Albertsons relative to QFC: , which tells us Albertsons is 2.867 times the size of QFC. Likewise, we could calculate the size of QFC relative to Albertsons: , which tells us that QFC is 34.9% of the size of Albertsons.

Example 7

Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

Solution

To answer this question, suppose the value started at $100. After one week, the value dropped by 60%:

$\$100 - \$100(0.60) = \$100 - \$60 = \$40.$

In the next week, notice that base of the percent has changed to the new value, $40. Computing the 75% increase:

$\$40 + \$40(0.75) = \$40 + \$30 = \$70.$

In the end, the stock is still $30 lower, or $\frac{\$30}{\$100} = 30\%$ lower, valued than it started.

Try it Now 2

The U.S. federal debt at the end of 2001 was $5.77 trillion, and grew to $6.20 trillion by the end of 2002. At the end of 2005 it was $7.91 trillion, and grew to $8.45 trillion by the end of 2006[3]. Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger increase in federal debt?

Answer

2001-2002: Absolute change: $0.43 trillion. Relative change: 7.45%

2005-2006: Absolute change: $0.54 trillion. Relative change: 6.83%

2005-2006 saw a larger absolute increase, but a smaller relative increase.

Example 8

A Seattle Times article on high school graduation rates reported “The number of schools graduating 60 percent or fewer students in four years – sometimes referred to as “dropout factories” – decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half.”

a) Is the “decrease by 17” number a useful comparison?

b) Considering the last sentence, can we conclude that the number of “dropout factories” was originally 34?

Solution

a) This number is hard to evaluate, since we have no basis for judging whether this is a larger or small change. If the number of “dropout factories” dropped from 20 to 3, that’d be a very significant change, but if the number dropped from 217 to 200, that’d be less of an improvement.

b) The last sentence provides relative change which helps put the first sentence in perspective. We can estimate that the number of “dropout factories” was probably previously around 34. However, it’s possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

Example 9

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. Who is correct?

Solution

Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percents are talking about different things, so one does not necessarily contradict the other. Edward's claim was a percent with coalition forces as the base of the percent, while Cheney's claim was a percent with both coalition and Iraqi security forces as the base of the percent. It turns out both statistics are in fact fairly accurate.

Try it Now 3

In the 2012 presidential elections, one candidate argued that "the president's plan will cut $716 billion from Medicare, leading to fewer services for seniors," while the other candidate rebuts that "our plan does not cut current spending and actually expands benefits for seniors, while implementing cost saving measures." Are these claims in conflict, in agreement, or not comparable because they're talking about different things?

Answer

Without more information, it is hard to judge these arguments. This is compounded by the complexity of Medicare. As it turns out, the $716 billion is not a cut in current spending, but a cut in future increases in spending, largely reducing future growth in health care payments. In this case, at least the numerical claims in both statements could be considered at least partially true. Here is one source of more information if you're interested: http://factcheck.org/2012/08/a-campaign-full-of-mediscare/

We'll wrap up our review of percents with a couple cautions. First, when talking about a change of quantities that are already measured in percents, we have to be careful in how we describe the change.

Example 10

A politician's support increases from 40% of voters to 50% of voters. Describe the change.

Solution

We could describe this using an absolute change: $|50\%-40\%|=10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of 10 **percentage points**.

In contrast, we could compute the percent change: $\frac{10\%}{40\%}=0.25=25\%$ increase. This is the relative change, and we'd say the politician's support has increased by 25%.

Lastly, a caution against averaging percents.

Example 11

A basketball player scores on 40% of 2-point field goal attempts, and on 30% of 3-point of field goal attempts. Find the player's overall field goal percentage.

Solution

It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more 2-point attempts than 3-point attempts. We don't actually have enough information to answer the question. Suppose the player attempted 200 2-point field goals and 100 3-point field goals. Then they made 200(0.40) = 80 2-point shots and $100(0.30)=30$ 3-point shots. Overall, they made 110 shots out of 300, for a $\frac{110}{300}=0.367=36.7\%$ overall field goal percentage.

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[1] www.factcheck.org/cheney_edwards_mangle_facts.html

[2] http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics

[3] www.whitehouse.gov/sites/defa...s/hist07z1.xls

- **1.2: Percents** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

9.3: Proportions and Rates

If you wanted to power the city of Seattle using wind power, how many windmills would you need to install? Questions like these can be answered using rates and proportions.

Rates

A rate is the ratio (fraction) of two quantities.

A **unit rate** is a rate with a denominator of one.

Example 12

Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate.

Solution

Expressed as a rate, $\frac{300 \text{ miles}}{15 \text{ gallons}}$. We can divide to find a unit rate: $\frac{20 \text{ miles}}{1 \text{gallon}}$, which we could also write as $20 \frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon.

Proportion Equation

A proportion equation is an equation showing the equivalence of two rates or ratios.

Example 13

Solve the proportion $\frac{5}{3} = \frac{x}{6}$ for the unknown value x.

Solution

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\frac{5}{3}$. We can solve this by multiplying both sides of the equation by 6, giving $x = \frac{5}{3} \cdot 6 = 10$.

Example 14

A map scale indicates that $\frac{1}{2}$ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $2\frac{1}{4}$ inches apart on the map?

Solution

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, x, to represent the unknown quantity – the mile distance between the cities.

$$\begin{array}{ll} \frac{\frac{1}{2} \text{ map inch}}{3 \text{ miles}} = \frac{2\frac{1}{4} \text{ map inches}}{x \text{ miles}} & \text{Multiply both sides by } x \text{ and rewriting the mixed number} \\ \frac{\frac{1}{2}}{3} \cdot x = \frac{9}{4} & \text{Multiply both sides by 3} \\ \frac{1}{2}x = \frac{27}{4} & \text{Multiply both sides by 2 (or divide by } \frac{1}{2}\text{)} \\ x = \frac{27}{2} = 13\frac{1}{2} \text{ miles} & \end{array}$$

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units.

Example 15

Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons?

Solution

We could certainly answer this question using a proportion: $\$\frac{300 \text{ miles}}{15 \text{ gallons}} = \frac{x \text{ miles}}{40 \text{ gallons}}\$$.

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit "cancels" and we're left with a number of miles:

$$40\text{gallons} \cdot \frac{20 \text{ miles}}{\text{gallon}} = \frac{40 \text{ gallons}}{1} \cdot \frac{20 \text{ miles}}{\text{gallon}} = 800 \text{ miles}$$

Notice if instead we were asked "how many gallons are needed to drive 50 miles?" we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we're left with gallons:

$$50\text{miles} \cdot \frac{1\text{gallon}}{20 \text{ miles}} = \frac{50 \text{ miles}}{1} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ gallons}}{20} = 2.5 \text{ gallons}$$

Dimensional analysis can also be used to do unit conversions. Here are some unit conversions for reference.

Unit Conversions

Length

$1 \text{ foot (ft)} = 12 \text{ inches (in)}$ $\quad$ $1 \text{ yard (yd)} = 3 \text{ feet (ft)}$

$1 \text{ mile} = 5,280 \text{ feet}$

$1000 \text{ millimeters } mm = 1 \text{ meter (m)}$ $\quad$ $100 \text{ centimeters (cm)} = 1 \text{ meter}$

$1000 \text{ meters (m)} = 1 \text{ kilometer (km)}$ $\quad$ $2.54 \text{ centimeters (cm)} = 1 \text{ inch}$

Weight and Mass

$1 \text{ pound (lb)} = 16 \text{ ounces (oz)}$ $\quad$ $1 \text{ ton} = 2000 \text{ pounds}$

$1000 \text{ milligrams (mg)} = 1 \text{ gram (g)}$ $\quad$ $1000 \text{ grams} = 1 \text{ kilogram (kg)}$

$1 \text{ kilogram} = 2.2 \text{ pounds (on earth)}$

Capacity

$1 \text{ cup} = 8 \text{ fluid ounces (fl oz)}^*$ $\quad$ $1 \text{ pint} = 2 \text{ cups}$

$1 \text{ quart} = 2 \text{ pints} = 4 \text{ cups}$ $\quad$ $1 \text{ gallon} = 4 \text{ quarts} = 16 \text{ cups}$

$1000 \text{ milliliters (ml)} = 1 \text{ liter (L)}$

*Fluid ounces are a capacity measurement for liquids. 1 fluid ounce ≈ 1 ounce (weight) for water only.

Example 16

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Solution

To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don't, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{180} \text{ hour} \qquad \text{Now we can multiply by the 15 miles/hr}$$

$$\frac{1}{180} \text{ hour} \cdot \frac{15 \text{ miles}}{1\text{hour}} = \frac{1}{12} \text{ mile} \qquad \text{Now we can convert to feet}$$

$$\frac{1}{12} \text{ mile} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

We could have also done this entire calculation in one long set of products:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

Try it Now 4

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

Answer

$$18 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{19.8 \text{ pounds}}{1000 \text{ feet}} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} \approx 0.475 \text{ ounces}$$

Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

Example 17

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

Solution

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:

$$\frac{100 \text{ tiles}}{100\text{ft}^2} = \frac{n \text{ tiles}}{400\text{ft}^2}$$

Other quantities just don't scale proportionally at all.

Example 18

Suppose a small company spends $1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend $10,000?

Solution

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

Example 19

Compare the 2010 U.S. military budget of $683.7 billion to other quantities.

Here we have a very large number, about $683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over $488,000.

There are about 300 million people in the U.S. The military budget is about $2,200 per person.

If you were to put $683.7 billion in $100 bills, and count out 1 per second, it would take 216 years to finish counting it.

Example 20

Compare the electricity consumption per capita in China to the rate in Japan.

To address this question, we will first need data. From the CIA[1] website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank[2], we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

Solution

Computing the consumption per capita for each country:

China: $\frac{4{,}693{,}000{,}000{,}000 \mathrm{KWH}}{1{,}344{,}130{,}000 \text{ people}} \approx 3491.5$ KWH per person

Japan: $\frac{859{,}700{,}000{,}000 \mathrm{KWH}}{127{,}817{,}277 \text{ people}} \approx 6726$ KWH per person

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

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[1] www.cia.gov/library/publicat.../2042rank.html

[2] http://data.worldbank.org/indicator/SP.POP.TOT

Page 1.4: Geometry

Geometric shapes, as well as area and volumes, can often be important in problem solving.

Example 21

You are curious how tall a tree is, but don't have any way to climb it. Describe a method for determining the height.

There are several approaches we could take. We'll use one based on triangles, which requires that it's a sunny day. Suppose the tree is casting a shadow, say 15 ft long. I can then have a friend help me measure my own shadow. Suppose I am 6 ft tall, and cast a 1.5 ft shadow. Since the triangle formed by the tree and its shadow has the same angles as the triangle formed by me and my shadow, these triangles are called **similar triangles** and their sides will scale proportionally. In other words, the ratio of height to width will be the same in both triangles. Using this, we can find the height of the tree, which we'll denote by h:

Solution

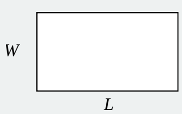
$$\frac{6\text{ft tall}}{1.5\text{ft shadow}} = \frac{h\text{ft tall}}{15\text{ft shadow}}$$

Multiplying both sides by 15, we get $h = 60$. The tree is about 60 ft tall.

It may be helpful to recall some formulas for areas and volumes of a few basic shapes.

Areas

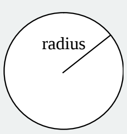
Rectangle



Area: $L \cdot W$

Perimeter: $2L + 2W$

Circle, radius r

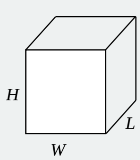


Area: πr^2

Circumference = $2\pi r$

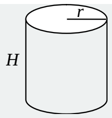
Volumes

Rectangular Box



Volume: $L \cdot W \cdot H$

Cylinder



Volume: $\pi r^2 H$

Example 22

If a 12 inch diameter pizza requires 10 ounces of dough, how much dough is needed for a 16 inch pizza?

Solution

To answer this question, we need to consider how the weight of the dough will scale. The weight will be based on the volume of the dough. However, since both pizzas will be about the same thickness, the weight will scale with the area of the top of the pizza. We can find the area of each pizza using the formula for area of a circle, $A = \pi r^2$:

A 12" pizza has radius 6 inches, so the area will be $\pi 6^2$ = about 113 square inches.

A 16" pizza has radius 8 inches, so the area will be $\pi 8^2$ = about 201 square inches.

Notice that if both pizzas were 1 inch thick, the volumes would be 113 in^3 and 201 in^3 respectively, which are at the same ratio as the areas. As mentioned earlier, since the thickness is the same for both pizzas, we can safely ignore it.

We can now set up a proportion to find the weight of the dough for a 16" pizza:

$\frac{10 \text{ ounces}}{113 \text{in}^2} = \frac{x \text{ ounces}}{201 \text{in}^2}$ Multiply both sides by 201

$x = 201 \cdot \frac{10}{113} =$ about 17.8 ounces of dough for a 16" pizza.

It is interesting to note that while the diameter is $\frac{16}{12} = 1.33$ times larger, the dough required, which scales with area, is $1.33^2 = 1.78$ times larger.

Example 23

A company makes regular and jumbo marshmallows. The regular marshmallow has 25 calories. How many calories will the jumbo marshmallow have?

Photo courtesy Christopher Danielson

We would expect the calories to scale with volume. Since the marshmallows have cylindrical shapes, we can use that formula to find the volume. From the grid in the image, we can estimate the radius and height of each marshmallow.

Solution

The regular marshmallow appears to have a diameter of about 3.5 units, giving a radius of 1.75 units, and a height of about 3.5 units. The volume is about $\pi(1.75)^2(3.5) = 33.7 \text{ units}^3$.

The jumbo marshmallow appears to have a diameter of about 5.5 units, giving a radius of 2.75 units, and a height of about 5 units. The volume is about $\pi(2.75)^2(5) = 118.8 \text{ units}^3$.

We could now set up a proportion, or use rates. The regular marshmallow has 25 calories for 33.7 cubic units of volume. The jumbo marshmallow will have:

$118.8 \text{ units}^3 \cdot \frac{25 \text{ calories}}{33.7 \text{ units}^3} = 88.1 \text{ calories}$

It is interesting to note that while the diameter and height are about 1.5 times larger for the jumbo marshmallow, the volume and calories are about $1.5^3 = 3.375$ times larger.

Try it Now 5

A website says that you'll need 48 fifty-pound bags of sand to fill a sandbox that measure 8ft by 8ft by 1ft. How many bags would you need for a sandbox 6ft by 4ft by 1ft?

Answer

The original sandbox has volume $64\mathrm{ft}^3$. The smaller sandbox has volume $24\mathrm{ft}^3$.

$\frac{48\mathrm{bags}}{64\mathrm{ft}^3} = \frac{x\mathrm{bags}}{24\mathrm{ft}^3}$ results in $x = 18$ bags.

1.5: Problem Solving and Estimating

Finally, we will bring together the mathematical tools we've reviewed, and use them to approach more complex problems. In many problems, it is tempting to take the given information, plug it into whatever formulas you have handy, and hope that the result is what you were supposed to find. Chances are, this approach has served you well in other math classes.

This approach does not work well with real life problems. Instead, problem solving is best approached by first starting at the end: identifying exactly what you are looking for. From there, you then work backwards, asking "what information and procedures will I need to find this?" Very few interesting questions can be answered in one mathematical step; often times you will need to chain together a solution pathway, a series of steps that will allow you to answer the question.

Problem Solving Process

1. Identify the question you're trying to answer.
2. Work backwards, identifying the information you will need and the relationships you will use to answer that question.
3. Continue working backwards, creating a solution pathway.
4. If you are missing necessary information, look it up or estimate it. If you have unnecessary information, ignore it.
5. Solve the problem, following your solution pathway.

In most problems we work, we will be approximating a solution, because we will not have perfect information. We will begin with a few examples where we will be able to approximate the solution using basic knowledge from our lives.

Example 24

How many times does your heart beat in a year?

Solution

This question is asking for the rate of heart beats per year. Since a year is a long time to measure heart beats for, if we knew the rate of heart beats per minute, we could scale that quantity up to a year. So the information we need to answer this question is heart beats per minute. This is something you can easily measure by counting your pulse while watching a clock for a minute.

Suppose you count 80 beats in a minute. To convert this beats per year:

$\frac{80 \text{ beats}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ year}}=42,048,000$ beats per year

Example 25

How thick is a single sheet of paper? How much does it weigh?

Solution

While you might have a sheet of paper handy, trying to measure it would be tricky. Instead we might imagine a stack of paper, and then scale the thickness and weight to a single sheet. If you've ever bought paper for a printer or copier, you probably bought a ream, which contains 500 sheets. We could estimate that a ream of paper is about 2 inches thick and weighs about 5 pounds. Scaling these down,

$\frac{2 \text{ inches}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}}=0.004$ inches per sheet

$\frac{5 \text{ pounds}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}}=0.01$ pounds per sheet, or 0.16 ounces per sheet.

Example 26

A recipe for zucchini muffins states that it yields 12 muffins, with 250 calories per muffin. You instead decide to make mini-muffins, and the recipe yields 20 muffins. If you eat 4, how many calories will you consume?

Solution

There are several possible solution pathways to answer this question. We will explore one.

To answer the question of how many calories 4 mini-muffins will contain, we would want to know the number of calories in each mini-muffin. To find the calories in each mini-muffin, we could first find the total calories for the entire recipe, then divide it by the number of mini-muffins produced. To find the total calories for the recipe, we could multiply the calories per standard muffin by the number per muffin. Notice that this produces a multi-step solution pathway. It is often easier to solve a problem in small steps, rather than trying to find a way to jump directly from the given information to the solution.

We can now execute our plan:

$12 \text{muffins\$} \cdot \frac{250 \text{ calories}}{\text{muffin}} = 3000\$$ calories for the whole recipe

$\frac{3000 \text{ calories}}{20 \text{ mini} - \text{muffins}}$ gives 150 calories per mini-muffin

$4 \text{ mini muffins } \cdot \frac{150 \text{ calories}}{\text{mini - muffin}}$ totals 600 calories consumed.

Example 27

You need to replace the boards on your deck. About how much will the materials cost?

Solution

There are two approaches we could take to this problem: 1) estimate the number of boards we will need and find the cost per board, or 2) estimate the area of the deck and find the approximate cost per square foot for deck boards. We will take the latter approach.

For this solution pathway, we will be able to answer the question if we know the cost per square foot for decking boards and the square footage of the deck. To find the cost per square foot for decking boards, we could compute the area of a single board, and divide it into the cost for that board. We can compute the square footage of the deck using geometric formulas. So first we need information: the dimensions of the deck, and the cost and dimensions of a single deck board.

Suppose that measuring the deck, it is rectangular, measuring 16 ft by 24 ft, for a total area of $384 \mathrm{ft}^2$.

From a visit to the local home store, you find that an 8 foot by 4 inch cedar deck board costs about $7.50. The area of this board, doing the necessary conversion from inches to feet, is:

$8 \text{ feet } \cdot 4 \text{ inches } \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = 2.667 \mathrm{ft}^2$. The cost per square foot is then

$\frac{\$7.50}{2.667 \mathrm{ft}^2} = \$2.8125 \text{ per } \mathrm{ft}^2$.

This will allow us to estimate the material cost for the whole $384 \mathrm{ft}^2$ deck

$\$384 \mathrm{ft}^2 \cdot \frac{\$2.8125}{\mathrm{ft}^2} = \$1080$ total cost.

Of course, this cost estimate assumes that there is no waste, which is rarely the case. It is common to add at least 10% to the cost estimate to account for waste.

Example 28

Is it worth buying a Hyundai Sonata hybrid instead the regular Hyundai Sonata?

Solution

To make this decision, we must first decide what our basis for comparison will be. For the purposes of this example, we'll focus on fuel and purchase costs, but environmental impacts and maintenance costs are other factors a buyer might consider.

It might be interesting to compare the cost of gas to run both cars for a year. To determine this, we will need to know the miles per gallon both cars get, as well as the number of miles we expect to drive in a year. From that information, we can find the number of gallons required from a year. Using the price of gas per gallon, we can find the running cost.

From Hyundai's website, the 2013 Sonata will get 24 miles per gallon (mpg) in the city, and 35 mpg on the highway. The hybrid will get 35 mpg in the city, and 40 mpg on the highway.

An average driver drives about 12,000 miles a year. Suppose that you expect to drive about 75% of that in the city, so 9,000 city miles a year, and 3,000 highway miles a year.

We can then find the number of gallons each car would require for the year.

Sonata:

$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{24 \text{ city miles}} + 3000 \text{ hightway miles.} \frac{1 \text{ gallon}}{35 \text{ highway miles}} = 460.7 \text{ gallons}$

Hybrid:

$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{35 \text{ city miles}} + 3000 \text{ hightway miles.} \frac{1 \text{ gallon}}{40 \text{ highway miles}} = 332.1 \text{ gallons}$

If gas in your area averages about $3.50 per gallon, we can use that to find the running cost:

Sonata: $460.7 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1612.45$

Hybrid: $332.1 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1162.35$

The hybrid will save $450.10 a year. The gas costs for the hybrid are about $\frac{\$450.10}{\$1612.45} = 0.279 = 27.9\%$ lower than the costs for the standard Sonata.

While both the absolute and relative comparisons are useful here, they still make it hard to answer the original question, since "is it worth it" implies there is some tradeoff for the gas savings. Indeed, the hybrid Sonata costs about $25,850, compared to the base model for the regular Sonata, at $20,895.

To better answer the "is it worth it" question, we might explore how long it will take the gas savings to make up for the additional initial cost. The hybrid costs $4965 more. With gas savings of $451.10 a year, it will take about 11 years for the gas savings to make up for the higher initial costs.

We can conclude that if you expect to own the car 11 years, the hybrid is indeed worth it. If you plan to own the car for less than 11 years, it may still be worth it, since the resale value of the hybrid may be higher, or for other non-monetary reasons. This is a case where math can help guide your decision, but it can't make it for you.

Try it Now 6

If traveling from Seattle, WA to Spokane WA for a three-day conference, does it make more sense to drive or fly?

Answer

There is not enough information provided to answer the question, so we will have to make some assumptions, and look up some values.

Assumptions:

a) We own a car. Suppose it gets 24 miles to the gallon. We will only consider gas cost.

b) We will not need to rent a car in Spokane, but will need to get a taxi from the airport to the conference hotel downtown and back.

c) We can get someone to drop us off at the airport, so we don't need to consider airport parking.

d) We will not consider whether we will lose money by having to take time off work to drive.

Values looked up (your values may be different)

a) Flight cost: $\$184$

b) Taxi cost: $\$25$ each way (estimate, according to hotel website)

c) Driving distance: 280 miles each way

d) Gas cost: $\$3.79$ a gallon

Cost for flying: $\$184 \text{ flight cost } + \$50 \text{ in taxi fares } = \$234$.

Cost for driving: 560 miles round trip will require 23.3 gallons of gas, costing $\$88.31$

Based on these assumptions, driving is cheaper. However, our assumption that we only include gas cost may not be a good one. Tax law allows you deduct $\$0.55$ (in 2012) for each mile driven, a value that accounts for gas as well as a portion of the car cost, insurance, maintenance, etc. Based on this number, the cost of driving would be $\$319$.

8.6: Exercises

1. Out of 230 racers who started the marathon, 212 completed the race, 14 gave up, and 4 were disqualified. What percentage did not complete the marathon?

2. Patrick left an $8 tip on a $50 restaurant bill. What percent tip is that?

3. Ireland has a 23% VAT (value-added tax, similar to a sales tax). How much will the VAT be on a purchase of a €250 item?

4. Employees in 2012 paid 4.2% of their gross wages towards social security (FICA tax), while employers paid another 6.2%. How much will someone earning $45,000 a year pay towards social security out of their gross wages?

5. A project on Kickstarter.com was aiming to raise $15,000 for a precision coffee press. They ended up with 714 supporters, raising 557% of their goal. How much did they raise?

6. Another project on Kickstarter for an iPad stylus raised 1,253% of their goal, raising a total of $313,490 from 7,511 supporters. What was their original goal?

7. The population of a town increased from 3,250 in 2008 to 4,300 in 2010. Find the absolute and relative (percent) increase.

8. The number of CDs sold in 2010 was 114 million, down from 147 million the previous year[1]. Find the absolute and relative (percent) decrease.

9. A company wants to decrease their energy use by 15%.

 a. If their electric bill is currently $2,200 a month, what will their bill be if they're successful?

 b. If their next bill is $1,700 a month, were they successful? Why or why not?

10. A store is hoping an advertising campaign will increase their number of customers by 30%. They currently have about 80 customers a day.

 a. How many customers will they have if their campaign is successful?

 b. If they increase to 120 customers a day, were they successful? Why or why not?

11. An article reports "attendance dropped 6% this year, to 300." What was the attendance before the drop?

12. An article reports "sales have grown by 30% this year, to $200 million." What were sales before the growth?

13. The Walden University had 47,456 students in 2010, while Kaplan University had 77,966 students. Complete the following statements:

 a. Kaplan's enrollment was ___% larger than Walden's.

 b. Walden's enrollment was ___% smaller than Kaplan's.

 c. Walden's enrollment was ___% of Kaplan's.

14. In the 2012 Olympics, Usain Bolt ran the 100m dash in 9.63 seconds. Jim Hines won the 1968 Olympic gold with a time of 9.95 seconds.

 a. Bolt's time was ___% faster than Hines'.

 b. Hine' time was ___% slower than Bolt's.

 c. Hine' time was ___% of Bolt's.

15. A store has clearance items that have been marked down by 60%. They are having a sale, advertising an additional 30% off clearance items. What percent of the original price do you end up paying?

16. Which is better: having a stock that goes up 30% on Monday than drops 30% on Tuesday, or a stock that drops 30% on Monday and goes up 30% on Tuesday? In each case, what is the net percent gain or loss?

17. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?

 a. "16.3% of Americans are without health insurance"[2]

 b. "only 55.9% of adults receive employer provided health insurance"[3]

18. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?

a. "We mark up the wholesale price by 33% to come up with the retail price"

b. "The store has a 25% profit margin"

19. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?

a. "Every year since 1950, the number of American children gunned down has doubled."

b. "The number of child gunshot deaths has doubled from 1950 to 1994."

20. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?[4]

a. "75 percent of the federal health care law's taxes would be paid by those earning less than $120,000 a year"

b. "76 percent of those who would pay the penalty [health care law's taxes] for not having insurance in 2016 would earn under $120,000"

21. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?

a. "The school levy is only a 0.1% increase of the property tax rate."

b. "This new levy is a 12% tax hike, raising our total rate to $9.33 per $1000 of value."

22. Are the values compared in this statement comparable or not comparable? "Guns have murdered more Americans here at home in recent years than have died on the battlefields of Iraq and Afghanistan. In support of the two wars, more than 6,500 American soldiers have lost their lives. During the same period, however, guns have been used to murder about 100,000 people on American soil"[5]

23. A high school currently has a 30% dropout rate. They've been tasked to decrease that rate by 20%. Find the equivalent percentage point drop.

24. A politician's support grew from 42% by 3 percentage points to 45%. What percent (relative) change is this?

25. Marcy has a 70% average in her class going into the final exam. She says "I need to get a 100% on this final so I can raise my score to 85%." Is she correct?

26. Suppose you have one quart of water/juice mix that is 50% juice, and you add 2 quarts of juice. What percent juice is the final mix?

27. Find a unit rate: You bought 10 pounds of potatoes for $4.

28. Find a unit rate: Joel ran 1500 meters in 4 minutes, 45 seconds.

29. Solve: $\frac{2}{5}=\frac{6}{x}$.

30. Solve: $\frac{n}{5}=\frac{16}{20}$.

31. A crepe recipe calls for 2 eggs, 1 cup of flour, and 1 cup of milk. How much flour would you need if you use 5 eggs?

32. An 8ft length of 4 inch wide crown molding costs $14. How much will it cost to buy 40ft of crown molding?

33. Four 3-megawatt wind turbines can supply enough electricity to power 3000 homes. How many turbines would be required to power 55,000 homes?

34. A highway had a landslide, where 3,000 cubic yards of material fell on the road, requiring 200 dump truck loads to clear. On another highway, a slide left 40,000 cubic yards on the road. How many dump truck loads would be needed to clear this slide?

35. Convert 8 feet to inches.

36. Convert 6 kilograms to grams.

37. A wire costs $2 per meter. How much will 3 kilometers of wire cost?

38. Sugar contains 15 calories per teaspoon. How many calories are in 1 cup of sugar?

39. A car is driving at 100 kilometers per hour. How far does it travel in 2 seconds?

40. A chain weighs 10 pounds per foot. How many ounces will 4 inches weigh?

41. The table below gives data on three movies. Gross earnings is the amount of money the movie brings in. Compare the net earnings (money made after expenses) for the three movies.[6]

Movie Earnings

Movie	Release Date	Budget	Gross Earnings
Saw	10/29/2004	$1,200,000	$103,096,345
Titanic	10/29/2004	$200,000,000	$1,842,879,955
Jurassic Park	6/11/1993	$63,000,000	$923,863,984

42. For the movies in the previous problem, which provided the best return on investment?

43. The population of the U.S. is about 309,975,000, covering a land area of 3,717,000 square miles. The population of India is about 1,184,639,000, covering a land area of 1,269,000 square miles. Compare the population densities of the two countries.

44. The GDP (Gross Domestic Product) of China was $5,739 billion in 2010, and the GDP of Sweden was $435 billion. The population of China is about 1,347 million, while the population of Sweden is about 9.5 million. Compare the GDP per capita of the two countries.

45. In June 2012, Twitter was reporting 400 million tweets per day. Each tweet can consist of up to 140 characters (letter, numbers, etc.). Create a comparison to help understand the amount of tweets in a year by imagining each character was a drop of water and comparing to filling something up.

46. The photo sharing site Flickr had 2.7 billion photos in June 2012. Create a comparison to understand this number by assuming each picture is about 2 megabytes in size, and comparing to the data stored on other media like DVDs, iPods, or flash drives.

47. Your chocolate milk mix says to use 4 scoops of mix for 2 cups of milk. After pouring in the milk, you start adding the mix, but get distracted and accidentally put in 5 scoops of mix. How can you adjust the mix if:

a. There is still room in the cup?

b. The cup is already full?

48. A recipe for sabayon calls for 2 egg yolks, 3 tablespoons of sugar, and $\frac{1}{4}$ cup of white wine. After cracking the eggs, you start measuring the sugar, but accidentally put in 4 tablespoons of sugar. How can you compensate?

49. The Deepwater Horizon oil spill resulted in 4.9 million barrels of oil spilling into the Gulf of Mexico. Each barrel of oil can be processed into about 19 gallons of gasoline. How many cars could this have fueled for a year? Assume an average car gets 20 miles to the gallon, and drives about 12,000 miles in a year.

50. The store is selling lemons at 2 for $1. Each yields about 2 tablespoons of juice. How much will it cost to buy enough lemons to make a 9-inch lemon pie requiring $\frac{1}{2}$ cup of lemon juice?

51. A piece of paper can be made into a cylinder in two ways: by joining the short sides together, or by joining the long sides together[7]. Which cylinder would hold more? How much more?

52. Which of these glasses contains more liquid? How much more?

In the next 4 questions, estimate the values by making reasonable approximations for unknown values, or by doing some research to find reasonable values.

53. Estimate how many gallons of water you drink in a year.

54. Estimate how many times you blink in a day.

55. How much does the water in a 6-person hot tub weigh?

56. How many gallons of paint would be needed to paint a two-story house 40 ft long and 30 ft wide?

57. During the landing of the Mars Science Laboratory *Curiosity*, it was reported that the signal from the rover would take 14 minutes to reach earth. Radio signals travel at the speed of light, about 186,000 miles per second. How far was Mars from Earth when *Curiosity* landed?

58. It is estimated that a driver takes, on average, 1.5 seconds from seeing an obstacle to reacting by applying the brake or swerving. How far will a car traveling at 60 miles per hour travel (in feet) before the driver reacts to an obstacle?

59. The flash of lightning travels at the speed of light, which is about 186,000 miles per second. The sound of lightning (thunder) travels at the speed of sound, which is about 750 miles per hour.

a. If you see a flash of lightning, then hear the thunder 4 seconds later, how far away is the lightning?

b. Now let's generalize that result. Suppose it takes n seconds to hear the thunder after a flash of lightning. How far away is the lightning, in terms of n?

60. Sound travels about 750 miles per hour. If you stand in a parking lot near a building and sound a horn, you will hear an echo.

a. Suppose it takes about $\frac{1}{2}$ a second to hear the echo. How far away is the building[8]?

b. Now let's generalize that result. Suppose it takes n seconds to hear the echo. How far away is the building, in terms of n?

61. It takes an air pump 5 minutes to fill a twin sized air mattress (39 by 8.75 by 75 inches). How long will it take to fill a queen sized mattress (60 by 8.75 by 80 inches)?

62. It takes your garden hose 20 seconds to fill your 2-gallon watering can. How long will it take to fill

a. An inflatable pool measuring 3 feet wide, 8 feet long, and 1 foot deep.[9]

b. A circular inflatable pool 13 feet in diameter and 3 feet deep.[10]

63. You want to put a 2" thick layer of topsoil for a new 20'x30' garden. The dirt store sells by the cubic yards. How many cubic yards will you need to order?

64. A box of Jell-O costs $0.50, and makes 2 cups. How much would it cost to fill a swimming pool 4 feet deep, 8 feet wide, and 12 feet long with Jell-O? (1 cubic foot is about 7.5 gallons)

65. You read online that a 15 ft by 20 ft brick patio would cost about $2,275 to have professionally installed. Estimate the cost of having a 18 by 22 ft brick patio installed.

66. I was at the store, and saw two sizes of avocados being sold. The regular size sold for $0.88 each, while the jumbo ones sold for $1.68 each. Which is the better deal?

67. The grocery store has bulk pecans on sale, which is great since you're planning on making 10 pecan pies for a wedding. Your recipe calls for 1¾ cups pecans per pie. However, in the bulk section there's only a scale available, not a measuring cup. You run over to the baking aisle and find a bag of pecans, and look at the nutrition label to gather some info. How many pounds of pecans should you buy?

Nutrition Facts

Serving Size: 1 cup, halves (99 g)
Servings per Container: about 2

Amount Per Serving	
Calories 684	Calories from Fat 596
	% Daily Value*
Total Fat 71g	110%
Saturated Fat 6g	31%
Trans Fat	
Cholesterol 0mg	0%

68. Soda is often sold in 20 ounce bottles. The nutrition label for one of these bottles is shown to the right. A packet of sugar (the kind they have at restaurants for your coffee or tea) typically contain 4 grams of sugar in the U.S. Drinking a 20 oz soda is equivalent to eating how many packets of sugar?[11]

Nutrition Facts

Serving Size: 8 fl oz (240 mL)
Servings Per Container: about 2.5

Amount Per Serving	
Calories 110	
	% Daily Value*
Total Fat 0g	0%
Sodium 70mg	3%
Total Carbohydrate 31g	10%
Sugars 30g	
Protein 0g	

For the next set of questions, *first* identify the information you need to answer the question, and *then* turn to the end of the section to find that information. The details may be imprecise; answer the question the best you can with the provided information. Be sure to justify your decision.

69. You're planning on making 6 meatloafs for a party. You go to the store to buy breadcrumbs, and see they are sold by the canister. How many canisters do you need to buy?

70. Your friend wants to cover their car in bottle caps, like in this picture.[12] How many bottle caps are you going to need?

71. You need to buy some chicken for dinner tonight. You found an ad showing that the store across town has it on sale for $2.99 a pound, which is cheaper than your usual neighborhood store, which sells it for $3.79 a pound. Is it worth the extra drive?

72. I have an old gas furnace, and am considering replacing it with a new, high efficiency model. Is upgrading worth it?

73. Janine is considering buying a water filter and a reusable water bottle rather than buying bottled water. Will doing so save her money?

74. Marcus is considering going car-free to save money and be more environmentally friendly. Is this financially a good decision?

For the next set of problems, research or make educated estimates for any unknown quantities needed to answer the question.

75. You want to travel from Tacoma, WA to Chico, CA for a wedding. Compare the costs and time involved with driving, flying, and taking a train. Assume that if you fly or take the train you'll need to rent a car while you're there. Which option is best?

76. You want to paint the walls of a 6ft by 9ft storage room that has one door and one window. You want to put on two coats of paint. How many gallons and/or quarts of paint should you buy to paint the room as cheaply as possible?

77. A restaurant in New York tiled their floor with pennies[13]. Just for the materials, is this more expensive than using a more traditional material like ceramic tiles? If each penny has to be laid by hand, estimate how long it would take to lay the pennies for a 12ft by 10ft room. Considering material and labor costs, are pennies a cost-effective replacement for ceramic tiles?

78. You are considering taking up part of your back yard and turning it into a vegetable garden, to grow broccoli, tomatoes, and zucchini. Will doing so save you money, or cost you more than buying vegetables from the store?

79. Barry is trying to decide whether to keep his 1993 Honda Civic with 140,000 miles, or trade it in for a used 2008 Honda Civic. Consider gas, maintenance, and insurance costs in helping him make a decision.

80. Some people claim it costs more to eat vegetarian, while some claim it costs less. Examine your own grocery habits, and compare your current costs to the costs of switching your diet (from omnivore to vegetarian or vice versa as appropriate). Which

diet is more cost effective based on your eating habits?

Info for the breadcrumbs question

How much breadcrumbs does the recipe call for?

It calls for 1½ cups of breadcrumbs.

How many meatloafs does the recipe make?

It makes 1 meatloaf.

How many servings does that recipe make?

It says it serves 8.

How big is the canister?

It is cylindrical, 3.5 inches across and 7 inches tall.

What is the net weight of the contents of 1 canister?

15 ounces.

How much does a cup of breadcrumbs weigh?

I’m not sure, but maybe something from the nutritional label will help.

Nutrition Facts	
Serving Size: 1/3 cup (30g)	
Servings per Container: about 14	
Amount Per Serving	
Calories 110	Calories from Fat 15
	% Daily Value*
Total Fat 1.5g	2%

How much does a canister cost?

$2.39

Info for bottle cap car

What kind of car is that?

A 1993 Honda Accord.

How big is that car / what are the dimensions? Here is some details from MSN autos:

Weight: 2800lb Length: 185.2 in Width: 67.1 in Height: 55.2 in

How much of the car was covered with caps?

Everything but the windows and the underside.

How big is a bottle cap?

Caps are 1 inch in diameter.

Info for chicken problem

How much chicken will you be buying?

Four pounds

How far are the two stores?

My neighborhood store is 2.2 miles away, and takes about 7 minutes. The store across town is 8.9 miles away, and takes about 25 minutes.

What kind of mileage does your car get?

It averages about 24 miles per gallon in the city.

How many gallons does your car hold?

About 14 gallons

How much is gas?

About $3.69/gallon right now.

Info for furnace problem

How efficient is the current furnace?

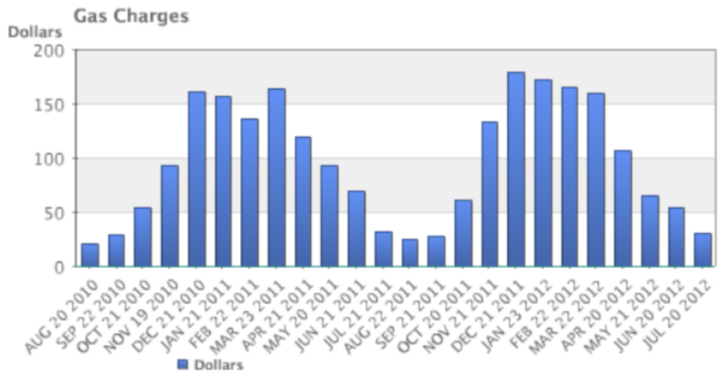
It is a 60% efficient furnace.

How efficient is the new furnace?

It is 94% efficient.

What is your gas bill?

Here is the history for 2 years:

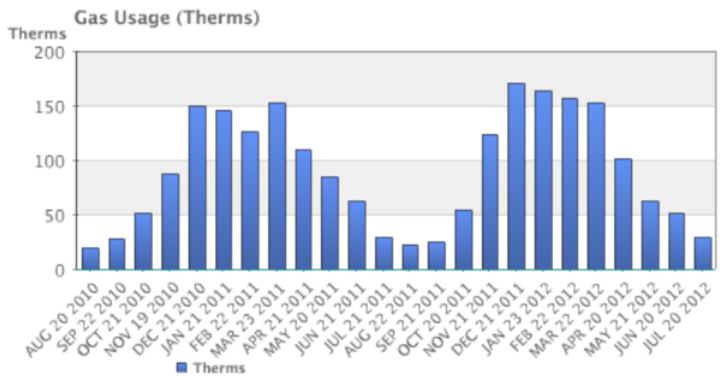


How much do you pay for gas?

There is $10.34 base charge, plus $0.39097 per Therm for a delivery charge, and $0.65195 per Therm for cost of gas.

How much gas do you use?

Here is the history for 2 years:



How much does the new furnace cost?

It will cost $7,450.

How long do you plan to live in the house?

Probably at least 15 years.

Info for water filter problem

How much water does Janine drink in a day?

She normally drinks 3 bottles a day, each 16.9 ounces.

How much does a bottle of water cost?

She buys 24-packs of 16.9 ounce bottles for $3.99.

How much does a reusable water bottle cost?

About $10.

How long does a reusable water bottle last?

Basically forever (or until you lose it).

How much does a water filter cost? How much water will they filter?

- A faucet-mounted filter costs about $28. Refill filters cost about $33 for a 3-pack. The box says each filter will filter up to 100 gallons (378 liters)
- A water filter pitcher costs about $22. Refill filters cost about $20 for a 4-pack. The box says each filter lasts for 40 gallons or 2 months
- An under-sink filter costs $130. Refill filters cost about $60 each. The filter lasts for 500 gallons.

Info for car-free problem

Where does Marcus currently drive? He:

- Drives to work 5 days a week, located 4 miles from his house.
- Drives to the store twice a week, located 7 miles from his house.
- Drives to other locations on average 5 days a week, with locations ranging from 1 mile to 20 miles.
- Drives to his parent's house 80 miles away once a month.

How will he get to these locations without a car?

- For work, he can walk when it's sunny and he gets up early enough. Otherwise he can take a bus, which takes about 20 minutes
- For the store, he can take a bus, which takes about 35 minutes.
- Some of the other locations he can bus to. Sometimes he'll be able to get a friend to pick him up. A few locations he is able to walk to. A couple locations are hard to get to by bus, but there is a ZipCar (short term car rental) location within a few blocks.
- He'll need to get a ZipCar to visit his parents.

How much does gas cost?

About $3.69/gallon.

How much does he pay for insurance and maintenance?

- He pays $95/month for insurance.
- He pays $30 every 3 months for an oil change, and has averaged about $300/year for other maintenance costs.

How much is he paying for the car?

- He's paying $220/month on his car loan right now, and has 3 years left on the loan.
- If he sold the car, he'd be able to make enough to pay off the loan.
- If he keeps the car, he's planning on trading the car in for a newer model in a couple years.

What mileage does his car get?

About 26 miles per gallon on average.

How much does a bus ride cost?

$2.50 per trip, or $90 for an unlimited monthly pass.

How much does a ZipCar rental cost?

- The "occasional driving plan": $25 application fee and $60 annual fee, with no monthly commitment. Monday-Thursday the cost is $8/hour, or $72 per day. Friday-Sunday the cost is $8/hour or $78/day. Gas, insurance, and 180 miles are included in the cost. Additional miles are $0.45/mile.
- The "extra value plan": Same as above, but with a $50 monthly commitment, getting you a 10% discount on the usage costs.

[1] http://www.cnn.com/2010/SHOWBIZ/Musi...les/index.html

[2] http://www.cnn.com/2012/06/27/politi...are/index.html

[3] http://www.politico.com/news/stories/0712/78134.html

[4] http://factcheck.org/2012/07/twistin...th-care-taxes/

[5] www.northjersey.com/news/opin...tml?c=y&page=2

[6] http://www.the-numbers.com/movies/records/budgets.php

[7] vimeo.com/42501010

[8] vimeo.com/40377128

[9] http://www.youtube.com/watch?v=DIkwefReHZc

[10] http://www.youtube.com/watch?v=p9SABH7Yg9M

[11] http://www.youtube.com/watch?v=62JMfv0tf3Q

[12] Photo credit: http://www.flickr.com/photos/swayze/, CC-BY

[13] http://www.notcot.com/archives/2009/...-of-pennie.php

9.7: Extension - Taxes

Governments collect taxes to pay for the services they provide. In the United States, federal income taxes help fund the military, the environmental protection agency, and thousands of other programs. Property taxes help fund schools. Gasoline taxes help pay for road improvements. While very few people enjoy paying taxes, they are necessary to pay for the services we all depend upon.

Taxes can be computed in a variety of ways, but are typically computed as a percentage of a sale, of one's income, or of one's assets.

✓ Example 1

The sales tax rate in a city is 9.3%. How much sales tax will you pay on a $140 purchase?

Solution

The sales tax will be 9.3% of $140. To compute this, we multiply $140 by the percent written as a decimal: $\$140(0.093) = \$13.02$

When taxes are not given as a fixed percentage rate, sometimes it is necessary to calculate the **effective rate.**

Effective rate

The effective tax rate is the equivalent percent rate of the tax paid out of the dollar amount the tax is based on.

✓ Example 2

Joan paid $3,200 in property taxes on her house valued at $215,000 last year. What is the effective tax rate?

Solution

We can compute the equivalent percentage: $3200/215000 = 0.01488$ or about 1.49% effective rate.

Taxes are often referred to as progressive, regressive, or flat.

Tax categories

A **flat tax**, or proportional tax, charges a constant percentage rate.

A **progressive tax** increases the percent rate as the base amount increases.

A **regressive tax** decreases the percent rate as the base amount increases.

✓ Example 3

The United States federal income tax on earned wages is an example of a progressive tax. People with a higher wage income pay a higher percent tax on their income.

Solution

For a single person in 2011, adjusted gross income (income after deductions) under $8,500 was taxed at 10%. Income over $8,500 but under $34,500 was taxed at 15%.

A person earning $10,000 would pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

$$\begin{array}{ll} 8500(0.10) = 850 & 10\% \text{ of } 8500 \\ 1500(0.15) = 225 & 15\% \text{ of the remaining } \$1500 \text{ of income} \\ \text{Total tax: } = \$1075 & \end{array}$$

The effective tax rate paid is $1075/10000 = 10.75\%$

A person earning \$30,000 would also pay 10% on the portion of their income under \$8,500, and 15% on the income over \$8,500, so they'd pay:

$$\begin{aligned}
&8500(0.10)=850 && 10\% \text{ of } 8500\\
&21500(0.15)=3225 && 15\% \text{ of the remaining } \$21500 \text{ of income}\\
&\text{Total tax: } =\$4075
\end{aligned}$$

The effective tax rate paid is $4075/30000=13.58\%$

Notice that the effective rate has increased with income, showing this is a progressive tax.

✔ Example 4

A gasoline tax is a flat tax when considered in terms of consumption, a tax of, say, \$0.30 per gallon is proportional to the amount of gasoline purchased. Someone buying 10 gallons of gas at \$4 a gallon would pay \$3 in tax, which is $\$3/\$40=7.5\%$ Someone buying 30 gallons of gas at \$4 a gallon would pay \$9 in tax, which is $\$9/\$120=7.5\%$ the same effective rate.

Solution

However, in terms of income, a gasoline tax is often considered a regressive tax. It is likely that someone earning \$30,000 a year and someone earning \$60,000 a year will drive about the same amount. If both pay \$60 in gasoline taxes over a year, the person earning \$30,000 has paid 0.2% of their income, while the person earning \$60,000 has paid 0.1% of their income in gas taxes.

? Try it Now 1

A sales tax is a fixed percentage tax on a person's purchases. Is this a flat, progressive, or regressive tax?

Answer

While sales tax is a flat percentage rate, it is often considered a regressive tax for the same reasons as the gasoline tax.

9.8: Income Taxation

Many people have proposed various revisions to the income tax collection in the United States. Some, for example, have claimed that a flat tax would be fairer. Others call for revisions to how different types of income are taxed, since currently investment income is taxed at a different rate than wage income.

The following two projects will allow you to explore some of these ideas and draw your own conclusions.

Project 1: Flat tax, Modified Flat Tax, and Progressive Tax.

Imagine the country is made up of 100 households. The federal government needs to collect $800,000 in income taxes to be able to function. The population consists of 6 groups:

Group A: 20 households that earn $12,000 each

Group B: 20 households that earn $29,000 each

Group C: 20 households that earn $50,000 each

Group D: 20 households that earn $79,000 each

Group E: 15 households that earn $129,000 each

Group F: 5 households that earn $295,000 each

This scenario is roughly proportional to the actual United States population and tax needs. We are going to determine new income tax rates.

The first proposal we'll consider is a flat tax – one where every income group is taxed at the same percentage tax rate.

1) Determine the total income for the population (all 100 people together)

2) Determine what flat tax rate would be necessary to collect enough money.

The second proposal we'll consider is a modified flat-tax plan, where everyone only pays taxes on any income over $20,000. So, everyone in group A will pay no taxes. Everyone in group B will pay taxes only on $9,000.

3) Determine the total *taxable* income for the whole population

4) Determine what flat tax rate would be necessary to collect enough money in this modified system

5) Complete this table for both the plans

		Flat Tax Plan		Modified Flat Tax Plan	
Group	**Income per household**	**Income tax per household**	**Income after taxes**	**Income tax per household**	**Income after taxes**
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				

The third proposal we'll consider is a progressive tax, where lower income groups are taxed at a lower percent rate, and higher income groups are taxed at a higher percent rate. For simplicity, we're going to assume that a household is taxed at the same rate on *all* their income.

6) Set progressive tax rates for each income group to bring in enough money. There is no one right answer here – just make sure you bring in enough money!

Group	Income per household	Tax rate (%)	Income tax per household	Total tax collected for all households	Income after taxes per household
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				
				This better total to $800,000	

7) Discretionary income is the income people have left over after paying for necessities like rent, food, transportation, etc. The cost of basic expenses does increase with income, since housing and car costs are higher, however usually not proportionally. For each income group, estimate their essential expenses, and calculate their discretionary income. Then compute the effective tax rate for each plan relative to discretionary income rather than income.

Group	Income per household	Discretionary Income (estimated)	Effective rate, flat	Effective rate, modified	Effective rate, progressive
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				

8) Which plan seems the most fair to you? Which plan seems the least fair to you? Why?

Project 2: Calculating Taxes.

Visit www.irs.gov, and download the most recent version of forms 1040, and schedules A, B, C, and D.

Scenario 1: Calculate the taxes for someone who earned $60,000 in standard wage income (W-2 income), has no dependents, and takes the standard deduction.

Scenario 2: Calculate the taxes for someone who earned $20,000 in standard wage income, $40,000 in qualified dividends, has no dependents, and takes the standard deduction. (Qualified dividends are earnings on certain investments such as stocks.)

Scenario 3: Calculate the taxes for someone who earned $60,000 in small business income, has no dependents, and takes the standard deduction.

Based on these three scenarios, what are your impressions of how the income tax system treats these different forms of income (wage, dividends, and business income)?

Scenario 4: To get a more realistic sense for calculating taxes, you'll need to consider itemized deductions. Calculate the income taxes for someone with the income and expenses listed below.

Married with 2 children, filing jointly

Wage income: $50,000 combined

Paid sales tax in Washington State

Property taxes paid: $3200

Home mortgage interest paid: $4800

Charitable gifts: $1200

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