(1) find $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} 5 & 0<t<3 \\ 0 & t>3 \end{cases}$

By definition,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^3 e^{-st}(5)\,dt + \int_3^\infty e^{-st}(0)\,dt$$

$$= 5\int_0^3 e^{-st}\,dt$$

$$= 5 \left.\frac{e^{-st}}{-s}\right|_0^3$$

$$= \frac{5(1-e^{-3s})}{s}$$

(2) Evaluate (a) $\mathcal{L}\{(5e^{2t}-3)^2\}$

~~$\mathcal{L}\{f(t)\} =$~~

$$\mathcal{L}\{(5e^{2t})^2 - 2.5e^{2t}.3 + 3^2\}$$

$$= \mathcal{L}(25e^{4t} - 30e^{2t} + 9)$$

$$= \mathcal{L}(25e^{4t}) - \mathcal{L}(30e^{2t}) + \mathcal{L}(9)$$

$$= 25.\frac{1}{s-4} - 30\frac{1}{s-2} + \frac{9}{s}$$

$$= \frac{25}{s-4} - \frac{30}{s-2} + \frac{9}{s}, \quad s>4$$

(b) $\mathcal{L}\{4\cos^2 2t\}$ Pg 31, Ques. 52(b)

3 (a) $\mathcal{L}\{t^3 e^{-3t}\}$

$\mathcal{L}(t^3) = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

Then,
$\mathcal{L}\{t^3 e^{-3t}\} = \frac{6}{(s+3)^4}$

(b) $\mathcal{L}\{e^{-t}\cos 2t\}$

$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$

If $\mathcal{L}\{F(t)\} = f(s)$, then, $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$

$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s-(-1)}{\{s-(-1)\}^2+4}$

$$= \frac{s+1}{(s+1)^2+4}$$

$$= \frac{s+1}{s^2+2s+1+4}$$

$$= \frac{s+1}{s^2+2s+5}$$

(c) $\mathcal{L}\{2e^{3t}\sin 4t\}$

$\mathcal{L}\{\sin 4t\} = \frac{4}{s^2+4^2} = \frac{4}{s^2+16}$

if $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$

$\mathcal{L}\{2e^{3t}\sin 4t\} = 2\mathcal{L}\{e^{3t}\sin 4t\}$

$$= 2 \frac{4\cancel{(s+3)}}{(s+3)^2+16}$$

$$= \frac{8}{s^2+6s+9+16}$$

$$= \frac{8}{s^2+6s+25}$$

(d) $\mathcal{L}\{(t+2)^2 e^t\}$ pg 31. Quer. 5~~8~~(d)

(e) $\mathcal{L}\{e^{2t}(3\sin 4t - 4\cos 4t)\}$

$\mathcal{L}\{3\sin 4t - 4\cos 4t\} = \mathcal{L}\{3\sin 4t\} - \mathcal{L}\{4\cos 4t\}$

$$= 3 \cdot \frac{4}{s^2+4^2} - 4\frac{s}{s^2+4^2}$$

$$= \frac{12}{s^2+16} - \frac{4s}{s^2+16}$$

$$= \frac{12-4s}{s^2+16}$$

If $\mathcal{L}\{F(t)\} = f(s)$, then, $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$

$$\mathcal{L}\{e^{2t}(3\sin 4t - 4\cos 4t)\} = \frac{12-4(s-2)}{(s-2)^2+16}$$

$$= \frac{12-4s+8}{s^2-4s+4+16}$$

$$= \frac{20-4s}{s^2-4s+20}$$

(f) $\mathcal{L}\{e^{-4t}\cosh 2t\}$

$$\mathcal{L}\{\cosh 2t\} = \frac{s}{s^2-2^2}$$

$$= \frac{s}{s^2-4}$$

If ~~F(t)~~ $\mathcal{L}\{F(t)\} = f(s)$, then, $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$

$$\mathcal{L}\{e^{-4t}\cosh 2t\} = \frac{s+4}{(s+4)^2-4}$$

$$= \frac{s+4}{\quad} = \frac{s+4}{\quad}$$

(g) $\mathcal{L}\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$

$$\mathcal{L}\{3\sinh 2t - 5\cosh 2t\} = \mathcal{L}\{3\sinh 2t\} - \mathcal{L}\{5\cosh 2t\}$$

$$= 3\frac{2}{s^2-2^2} - 5\frac{s}{s^2-2^2}$$

$$= \frac{6}{s^2-4} - \frac{5s}{s^2-4}$$

$$= \frac{6-5s}{s^2-4}$$

~~if $\mathcal{L}\{f(t)\} = f(s-a)$ then e^{at}~~

If $\mathcal{L}\{f(t)\} = f(s)$, then $\mathcal{L}\{e^{at}f(t)\} = f(s-a)$

$$\mathcal{L}\{e^{-t}(3\sinh 2t - 5\cosh 2t)\} = \frac{6-5(s+1)}{(s+1)^2-4}$$

$$= \frac{6-5s-5}{s^2+2s+1-4}$$

$1 - 5s$

(5) pg 32 Ques. 65

⑥ prove that, (a) $\mathcal{L}\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$

(b) $\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$

(a) $\mathcal{L}\{t\cos at\}$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

if $\mathcal{L}\{f(t)\} = f(s)$ then, $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

$$\mathcal{L}\{t\cos at\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= -\frac{(s^2+a^2)\frac{d}{ds}(s) - s\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2}$$

$$= -\frac{s^2+a^2-s.2s}{(s^2+a^2)^2}$$

$$= \frac{-s^2-a^2+2s^2}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$\therefore \mathcal{L}\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

(proved)

(b) $\mathcal{L}\{t\sin at\}$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

If $\mathcal{L}\{F(t)\} = f(s)$ then, $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

$$\mathcal{L}\{t\sin at\} = (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= -\frac{(s^2+a^2)\frac{d}{ds}(a) - a\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2}$$

$$= -\frac{-a.2s}{(s^2+a^2)^2}$$

$$= \frac{2as}{(s^2+a^2)^2}$$

$$\therefore \mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2} \quad \text{(proved)}$$

7) $\mathcal{L}\{t^2\sin t\} = \frac{6s^2-2}{(s^2+1)^3}$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = \frac{1}{s^2+1}$$

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

$$\mathcal{L}\{t^2\sin t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$

$$= \frac{d}{ds}\left\{(s^2+1)\frac{d}{ds}(1) - 1\frac{d}{ds}(s^2+1)\right.$$

$$= \frac{d}{ds}\left\{\frac{-2s}{(s^2+1)^2}\right\}$$

$$= \frac{(s^2+1)^2\frac{d}{ds}(-2s) - (-2s)\frac{d}{ds}(s^2+1)^2}{\{(s^2+1)^2\}^2}$$

$$= \frac{-2(s^2+1)^2 + 2s\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}$$

$$= \frac{(s^2+1)\{-2(s^2+1)+8s^2\}}{(s^2+1)^4}$$

$$= \frac{-2s^2-2+8s^2}{(s^2+1)^3}$$

$$= 6s^2-2$$

$$= -\frac{s^2-9-s \cdot 2s}{(s^2-9)^2}$$

$$= -\frac{s^2-9-2s^2}{(s^2-9)^2}$$

$$= -\frac{-s^2-9}{(s^2-9)^2}$$

$$= \frac{s^2+9}{(s^2-9)^2}$$

b) $\mathcal{L}\{t \sinh 2t\}$

$$\mathcal{L}\{\sinh 2t\} = \frac{2}{s^2-2^2} = \frac{2}{s^2-4}$$

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

$$\mathcal{L}\{t \sinh 2t\} = (-1)^1 \frac{d}{ds}\left(\frac{2}{s^2-4}\right)$$

$$= -\frac{(s^2-4)\frac{d}{ds}(2) - 2\frac{d}{ds}(s^2-4)}{(s^2-4)^2}$$

$$= -\frac{\cancel{0} - 2 \cdot 2s}{(s^2-4)^2}$$

$$= \frac{4s}{(s^2-4)^2}$$

(9) ~~(a)~~ find (a) $\mathcal{L}\{t^2 \cos t\}$

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1^2} = \frac{s}{s^2+1}$$

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

$$\mathcal{L}\{t^2 \cos t\} = (-1)^2 \frac{d^2}{ds^2} \left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right\}$$

$$= \frac{d}{ds}\left\{\frac{(s^2+1)\frac{d}{ds}(s) - s\cdot\frac{d}{ds}(s^2+1)}{(s^2+1)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{s^2+1-s\cdot 2s}{(s^2+1)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}$$

$$= \frac{(s^2+1)^2\frac{d}{ds}(1-s^2) - (1-s^2)\frac{d}{ds}(s^2+1)^2}{\{(s^2+1)^2\}^2}$$

$$= \frac{(s^2+1)^2(-2s) - (1-s^2)\,2(s^2+1)\cdot 2s}{(s^2+1)^4}$$

$$= \frac{(s^2+1)\{-2s(s^2+1) - 4s(1-s^2)\}}{(s^2+1)^4}$$

$$= \frac{-2s^3-2s-4s+4s^3}{\quad}$$

(b) $\mathcal{L}\{(t^2 - 3t + 2)\sin 3t\}$ pg 33, Ques. 78(b)

(10) $\mathcal{L}\{t^3 \cos t\}$

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1^2} = \frac{s}{s^2+1}$$

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

$$\mathcal{L}\{t^3 \cos t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{s}{s^2+1}\right)$$

$$= -\frac{d^2}{ds^2}\left[\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]$$

$$= -\frac{d^2}{ds^2}\left\{\frac{(s^2+1)\frac{d}{ds}(s) - s\frac{d}{ds}(s^2+1)}{(s^2+1)^2}\right\}$$

$$= -\frac{d^2}{ds^2}\left\{\frac{s^2+1 - s.2s}{(s^2+1)^2}\right\}$$

$$= -\frac{d}{ds}\left[\frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}\right]$$

$$= -\frac{d}{ds}\left\{\frac{(s^2+1)^2(-2s) - (1-s^2)\,2(s^2+1).2s}{(s^2+1)^4}\right\}$$

$$= -\frac{d}{ds}\left[\frac{(s^2+1)\{(-2s)(s^2+1) - 4s(1-s^2)\}}{(s^2+1)^4}\right]$$

$$= -\frac{d}{ds}\left\{\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}\right\}$$

$$= -\frac{d}{ds}\left\{\frac{2s^3-6s}{(s^2+1)^3}\right\}$$

$$= -\left\{\frac{(s^2+1)^3\frac{d}{ds}(2s^3-6s) - (2s^3-6s)\frac{d}{ds}(s^2+1)^3}{\{(s^2+1)^3\}^2}\right\}$$

$$= -\frac{(s^2+1)^3(6s^2-6)-(2s^3-6s)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}$$

$$= -\frac{(s^2+1)^2\{(s^2+1)(6s^2-6)-6s(2s^3-6s)\}}{(s^2+1)^6}$$

$$= -\frac{6s^4-6s^2+6s^2-6-12s^4+36s^2}{(s^2+1)^4}$$

$$= -\frac{-6s^4+36s^2-6}{(s^2+1)^4}$$

$$= \frac{6s^4-36s^2+6}{(s^2+1)^4}$$

(11) Show that $\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \ln\left(\frac{s+b}{s+a}\right)$

$$\mathcal{L}\{e^{-at}-e^{-bt}\} = \mathcal{L}\{e^{-at}\} - \mathcal{L}\{e^{-bt}\}$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$

$$\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{u+a} - \frac{1}{u+b}\right)du$$

$$= \ln(u+a) - \ln(u+b)\Big|_s^\infty$$

$$= \ln\left(\frac{u+a}{u+b}\right)\Big|_s^\infty$$

$$= -\ln\left(\frac{s+a}{s+b}\right)$$

$$= \ln\left(\frac{s+b}{s+a}\right)$$

$$\therefore \mathcal{L}\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \ln\left(\frac{s+b}{s+a}\right)$$

Showed

pg 33, Ques, 82

(12) Show that, $\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2} \ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$

$$\mathcal{L}\{\cos at - \cos bt\} = \mathcal{L}\{\cos at\} - \mathcal{L}\{\cos bt\}$$

$$= \frac{a}{s^2+a^2} - \frac{b}{s^2+b^2}$$

if $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$

$$\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{a}{s^2+a^2} - \frac{b}{s^2+b^2}\right) du$$

$=$

(13) Find $\mathcal{L}\left\{\frac{\sinh t}{t}\right\}$

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1^2} = \frac{1}{s^2-1}$$

If $\mathcal{L}\{f(t)\} = f(s)$ then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(u)\,du$

$$\mathcal{L}\left\{\frac{\sinh t}{t}\right\} = \int_s^\infty \frac{1}{u^2-1}\,du$$

$$= \frac{1}{2\cdot 1} \ln\left(\frac{\cancel{u}-1}{\cancel{u}+1}\right)\Big|_s^\infty$$

$$= \frac{1}{2}\left\{-\ln\left(\frac{s-1}{s+1}\right)\right\}$$

$$= \frac{1}{2}\ln\frac{s+1}{s-1}$$

(14) (a) $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{\cancel{6t}\,6s-4}{(s^2-2)^2+16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6(s-2)}{(s-2)^2+16} + \frac{8}{(s-2)^2+16}\right\}$$

$$= 6\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$$

$$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t$$

$$= 2e^{2t}(3\cos 4t + \sin 4t)$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s+12}{s^2+8s+16}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{4s+12}{(s+4)^2}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{4(s+4)-4}{(s+4)^2}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{4}{s+4}\right\} - \mathcal{L}^{-1}\left\{\dfrac{4}{(s+4)^2}\right\}$

$= 4\mathcal{L}^{-1}\left\{\dfrac{1}{s+4}\right\} - 4\mathcal{L}^{-1}\left\{\dfrac{1}{(s+4)^2}\right\}$

$= 4e^{-4t} - 4te^{-4t}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{3s+7}{(s-1)^2-4}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{3(s-1)+10}{(s-1)^2-4}\right\}$

$\cancel{= 3\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}}$

$= 3\mathcal{L}^{-1}\left\{\dfrac{s-1}{(s-1)^2-4}\right\} + \cancel{10}\, 5\mathcal{L}^{-1}\left\{\dfrac{2}{(s-1)^2-4}\right\}$

$= 3e^{t}\cosh 2t + 5e^{t}\sinh 2t$

$= e^{t}(3\cosh 2t + 5\sinh 2t)$

Scanned by CamScanner

(d) $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$

$= \frac{1}{\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{3}{2}\right)^{\frac{1}{2}}}\right\}$

$= \frac{1}{\sqrt{2}} e^{-3t/2} \frac{t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}$

$= \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-3t/2}$

(15) If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $\mathcal{L}^{-1}\{e^{-as} f(s)\} = G(t)$

where $G(t) = \begin{cases} F(t-a) & t>a \\ 0 & t<a \end{cases}$

Since, $f(s) = \int_0^\infty e^{-st} F(t)\, dt$, we have

$e^{-as} f(s) = \int_0^\infty e^{-as} e^{-st} F(t)\, dt$

$= \int_0^\infty e^{-s(t+a)} F(t)\, dt$

$= \int_0^\infty e^{-su} F(u-a)\, du$ [letting $t+a=u$]

$= \int_0^a e^{-st} (0)\, dt + \int_a^\infty e^{-st} F(t-a)\, dt$

$= \int_0^\infty e^{-st} G(t)\, dt$,

from which the required result follows.

(16) (a) $\mathcal{L}^{-1}\left\{\dfrac{e^{-5s}}{(s-2)^4}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= \frac{t^3 e^{2t}}{3!} = \frac{1}{6}t^3 e^{2t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = \begin{cases} \frac{1}{6}(t-5)^3 e^{2(t-5)} & t>5 \\ 0 & t<5 \end{cases}$$

$$= \frac{1}{6}(t-5)^3 e^{2(t-5)} u(t-5)$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{se^{-4\pi s/5}}{s^2+25}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} = \cos 5t$$

$$\mathcal{L}^{-1}\left\{\frac{se^{-4\pi s/5}}{s^2+25}\right\} = \begin{cases} \cos 5(t-4\pi/5) & t>4\pi/5 \\ 0 & t<4\pi/5 \end{cases}$$

$$= \begin{cases} \cos 5t & t>4\pi/5 \\ 0 & t<4\pi/5 \end{cases}$$

$$= \cos 5t\, u(t-4\pi/5)$$

(c) $\mathcal{L}^{-1}\left\{\dfrac{(s+1)\,e^{-\pi s}}{s^2+s+1}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} + \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}t}{2}$$

$$= \frac{e^{-\frac{1}{2}t}}{\sqrt{3}}\left(\sqrt{3}\cos\frac{\sqrt{3}t}{2} + \sin\frac{\sqrt{3}t}{2}\right)$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{(s+1)\,e^{-\pi s}}{s^2+s+1}\right\} = \begin{cases} \dfrac{e^{-\frac{1}{2}(t-\pi)}}{\sqrt{3}}\left(\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi) + \sin\frac{\sqrt{3}}{2}(t-\pi)\right) & t>\pi \\ 0 & t<\pi \end{cases}$$

$$= \frac{e^{-\frac{1}{2}(t-\pi)}}{\sqrt{3}}\left\{\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi) + \sin\frac{\sqrt{3}}{2}(t-\pi)\right\}u(t-\pi)$$

(d) $L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\}$

$$= e^{-4t} L^{-1}\left\{\frac{1}{s^{5/2}}\right\}$$

$$= e^{-4t}\frac{t^{3/2}}{\Gamma(5/2)} = \frac{4t^{3/2}e^{-4t}}{3\sqrt{\pi}}$$

Thus,

$$L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^{4} L^{-1}\left\{\frac{e^{-3s}}{(s+4)^{5/2}}\right\}$$

$$= \begin{cases} \frac{4e^{4}(t-3)^{3/2}e^{-4(t-3)}}{3\sqrt{\pi}} & t>3 \\ 0 & t<3 \end{cases}$$

$$= \begin{cases} \frac{4(t-3)^{3/2}e^{-4}(t-4)}{3\sqrt{\pi}} & t>3 \\ 0 & t<3 \end{cases}$$

$$= \frac{4(t-3)^{3/2}e^{-4(t-4)}}{3\sqrt{\pi}}u(t-3)$$

(17) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

we have, $\frac{d}{ds}\left\{\frac{1}{s^2+a^2}\right\} = \frac{-2s}{(s^2+a^2)^2}$

Thus, $\frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{d}{ds}\left\{\frac{1}{s^2+a^2}\right\}$

Then since, $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\}$$

$$= -\frac{1}{2}t\left(\frac{\sin at}{a}\right) = \frac{-t\sin at}{2a}$$

(18) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\,\frac{s}{(s^2+1)^2}\right\}$$

$$= \int_0^t \frac{1}{2}u\sin u$$

$$= \left(\frac{1}{2}u\right)(-\cos u) - \left(\frac{1}{2}\right)(-\sin u)\Big|_0^t$$

$$= \frac{1}{2}(\sin t - t\cos t)$$

(19) pg 55 - Ques. 20

(a) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

$$\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}, \text{ then, since, } \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} =$$

$\cos at$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au \, \frac{\sin a(t-u)}{a} \, du$$

$$= \frac{1}{a}\int_0^t (\cos au)(\sin at \cos u - \cos at \sin au)\, du$$

$$= \frac{1}{a}\sin at \int_0^t \cos^2 au\, du - \frac{1}{a}\cos at \int_0^t \sin au \cos au \, du$$

$$= \frac{1}{a}\sin at \int_0^t \left(\frac{1+\cos 2au}{2}\right) du - \frac{1}{a}\cos at \int_0^t \frac{\sin 2au}{2}\, du$$

$$= \frac{1}{a}\sin at\left(\frac{t}{2} + \frac{\sin 2at}{4a}\right) - \frac{1}{a}\cos at\left(\frac{1-\cos 2at}{4a}\right)$$

$$= \frac{1}{a}\sin at\left(\frac{t}{2} + \frac{\sin at \cos at}{4a}\right) - \frac{1}{a}\cos at\left(\frac{\sin^2 at}{2a}\right)$$

$$= \frac{t \sin at}{2a}$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t, \quad \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\} = te^{-t},$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t (ue^{-u})(t-u)\,du$$

$$= \int_0^t (ut-u^2)\,e^{-u}\,du$$

$$= \left[(ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\right]_0^t$$

$$= te^{-t} + 2e^{-t} + t - 2$$

<u>Check:</u> $\mathcal{L}\{te^{-t}+2e^{-t}+t-2\} = \dfrac{1}{(s+1)^2} + \dfrac{2}{s+1} + \dfrac{1}{s^2} - \dfrac{2}{s}$

$$= \frac{s^2+2s^2(s+1)+(s+1)^2-2s(s+1)^2}{s^2(s+1)^2}$$

$$= \frac{1}{s^2(s+1)^2}$$

(21) $\mathcal{L}^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$

$$\frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

Multiplying by $(s-3)(s+1)$ we obtain

$3s+7 = A(s+1) + B(s-3) = (A+B)s + A - 3B$

Equating coefficients,

$A+B = 3$ and $A-3b = 7$ then $A=4$, $B=-1$

$$\frac{3s+7}{(s-3)(s+1)} = \frac{4}{s-3} - \frac{1}{s+1}$$

$$\mathcal{L}^{-1}\left\{\frac{3s+7}{(s-3)(s+1)}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 4e^{3t} - e^{-t}$$

(22) $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

we have, $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$

Multiply both sider by $s+1$ and let $s \to -1$, then,

$$A = \lim_{s\to -1} \frac{2s^2-4}{(s-2)(s-3)} = -\frac{1}{6}$$

Multiply both sider by $s-2$ and let ~~t~~ $s \to 2$, then,

$$B = \lim_{s\to 2} \frac{2s^2-4}{(s+1)(s-3)} = -\frac{4}{3}$$

Multiply both sider by $s-3$ and let $s \to 3$, then,

$$C = \lim_{s\to 3} \frac{2s^2-4}{(s+1)(s-2)} = \frac{7}{2}$$

Thus,

$\mathcal{L}^{-1}\{ 2s^2-4 \quad \} \quad \mathcal{L}^{-1}\{ -\frac{1}{6} \quad -\frac{4}{3} \quad \frac{7}{2} \}$

(23) $\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{s-2}$$

Multiply both sides by $s+1$ and let $s \to -1$, then

$$A = \lim_{s\to -1} \frac{5s^2-15s-11}{(s-2)^3} = -\frac{1}{3}$$

Multiply both sides by $(s-2)^3$ and let $s \to 2$, then

$$B = \lim_{s\to 2} \frac{5s^2-15s-11}{s+1} = -7$$

The method fails to determine C and D. However, since we know A and B, we have,

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{-\frac{1}{3}}{s+1} + \frac{-7}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{s-2}$$

To determine C and D we can substitute two values for s, say $s=0$ and $s=1$ from which we find respectively,

$$\frac{11}{8} = -\frac{1}{3} + \frac{7}{8} + \frac{C}{4} - \frac{D}{2}$$

$$\frac{21}{2} = -\frac{1}{6} + 7 + C - D$$

i.e. $3C - 6D = 10$ and $3C - 3D = 11$, from which

$C = 4$, $D = \frac{1}{3}$, thus,

$$\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{-\frac{1}{3}}{s+1} + \frac{-7}{(s-2)^3} + \frac{4}{(s-2)^2} + \frac{\frac{1}{3}}{s-2}\right\}$$

$$= -\frac{1}{3}e^{-t} - \frac{7}{2}t^2 e^{2t} + 4t e^{2t} + \frac{1}{3}e^{2t}$$

(24) $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

Multiply both sides by $s-1$ and let $s \to 1$, then

$$A = \lim_{s \to 1} \frac{3s+1}{s^2+1} = 2 \text{ and}$$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{Bs+C}{s^2+1}$$

To determine B and C, let $s = 0$ and 2, then

$-1 = -2 + C$, $\frac{7}{5} = 2 + \frac{2B+C}{5}$, from which

$C = 1$ and $B = -2$

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-1} + \frac{-2s+1}{s^2+1}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 2e^{t} - 2\cos t + \sin t$$

(25) $L^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

we have $P(s) = 2s^2-4$, $Q(s) = (s+1)(s-2)(s-3)$

$= s^3-4s^2+s+6$

$Q'(s) = 3s^2-8s+1$

$a_1 = -1,\ a_2 = 2,\ a_3 = 3$

$$\frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{3t}$$

$$= \frac{-2}{12}e^{-t} + \frac{4}{-3}e^{2t} + \frac{14}{4}e^{3t}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

(26) (a) $L^{-1}\left\{\dfrac{3}{s+4}\right\}$

$= 3L^{-1}\left\{\dfrac{1}{s+4}\right\}$

$= 3e^{-4t}$

(b) $L^{-1}\left\{\dfrac{8s}{s^2+16}\right\}$

$= 8L^{-1}\left\{\dfrac{s}{s^2+4^2}\right\}$

$= 8\cos 4t$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{2s-5}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{2\left(s-\frac{5}{2}\right)}\right\}$

$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{5}{2}}\right\}$

$= \frac{1}{2}e^{5t/2}$

(d) $\mathcal{L}^{-1}\left\{\frac{6}{s^2+4}\right\}$

$= 6\mathcal{L}^{-1}\left\{\frac{\frac{1}{2}\cdot 2}{s^2+2^2}\right\} = \cancel{6\sin}\ 6\cdot\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = 3\sin 2t$

(e) $\mathcal{L}^{-1}\left\{\frac{3s-12}{s^2+8}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{3s}{s^2+8}\right\} - \mathcal{L}^{-1}\left\{\frac{12}{s^2+8}\right\}$

$= 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+(2\sqrt{2})^2}\right\} - 12\mathcal{L}^{-1}\left\{\frac{\frac{1}{2\sqrt{2}}\cdot 2\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$

$= 3\cos 2\sqrt{2}t - 12\sin 2\sqrt{2}t\cdot\frac{1}{2\sqrt{2}}$

$= 3\cos 2\sqrt{2}t - 3\sqrt{2}\sin 2\sqrt{2}t$

(f) $\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2-9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{2s}{s^2-9}\right\} - \mathcal{L}^{-1}\left\{\frac{5}{s^2-9}\right\}$

$= 2\cosh 3t - 5\cdot\frac{1}{3}\sinh 3t$

(g) $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

$= \frac{t^4}{4!} = \frac{t^4}{24}$

(h) $\mathcal{L}^{-1}\left\{\frac{1}{s^{7/2}}\right\}$ pg 68 - Ques 47 (h)

$=$

(i) $\mathcal{L}^{-1}\left\{\frac{12}{4-3s}\right\}$

$= -12\ \mathcal{L}^{-1}\left\{\frac{1}{3s-4}\right\}$

$= -12 \cdot \frac{1}{3}\ \mathcal{L}^{-1}\left\{\frac{1}{s-\frac{4}{3}}\right\} = -4e^{4t/3}$

(j) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^{4/3}}\right\}$ pg 68, Ques 47 (j)

$=$

27(a) $\mathcal{L}^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{3s}{4s^2+25}\right\} - \mathcal{L}^{-1}\left\{\frac{8}{4s^2+25}\right\}$

$= 3\cdot\frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+\left(\frac{5}{2}\right)^2}\right\} - 8\cdot\mathcal{L}^{-1}\left\{\frac{1}{4\left(s^2+\frac{25}{4}\right)}\right\}$

$= \frac{3}{4}\cos\frac{5}{2}t - 8\cdot\frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{\frac{2}{5}\cdot\frac{5}{2}}{s^2+\left(\frac{5}{2}\right)^2}\right\}$

$= \frac{3}{4}\cos\frac{5}{2}t - \cancel{2\mathcal{L}^{-1}}\, 2\cdot\frac{2}{5}\sin\frac{5}{2}t$

$= \frac{3}{4}\cos\frac{5t}{2} - \frac{4}{5}\sin\frac{5t}{2}$

(b) $\mathcal{L}^{-1}\left\{\frac{5s+10}{9s^2-16}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{5s}{9s^2-16}\right\} + \mathcal{L}^{-1}\left\{\frac{10}{9s^2-16}\right\}$

$= 5\cdot\frac{1}{9}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-\left(\frac{4}{3}\right)^2}\right\} + 10\cdot\frac{1}{9}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-\left(\frac{4}{3}\right)^2}\right\}$

$= \frac{5}{9}\cos\frac{4t}{3} + \frac{10}{9}\cdot\frac{3}{4}\sin\frac{44}{3}$

$= \frac{5}{9}\cos\frac{4t}{3} + \frac{5}{6}\sin\frac{4t}{3}$

(28) $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$

Taking the laplace transforms of both sides of the differential equation and using the given condition, we have,

$$\mathcal{L}\{Y''\} + \mathcal{L}\{Y\} = \mathcal{L}\{t\}$$

$$s^2 y - sY(0) - Y'(0) + y = \frac{1}{s^2}$$

$$s^2 y - s + 2 + y = \frac{1}{s^2}$$

Then $y = \mathcal{L}\{Y\} = \dfrac{1}{s^2(s^2+1)} + \dfrac{s-2}{s^2+1}$

$$= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

and, $Y = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2} + \dfrac{s}{s^2+1} - \dfrac{3}{s^2+1}\right\}$

$$= t + \cos t - 3\sin t$$

$Y = t + \cos t - 3\sin t$, $Y' = 1 - \sin t - 3\cos t$,

$Y'' = -\cos t + 3\sin t$, Then $Y'' + Y = t$

$Y(0) = 1$, $Y'(0) = -2$ and the function obtained is the required solution.

(29) $Y'' - 3Y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$

$\mathcal{L}\{Y''\} - 3\mathcal{L}\{Y'\} + 2\mathcal{L}\{Y\} = 4\mathcal{L}\{e^{2t}\}$

$s^2y - sY(0) - Y'(0) - 3\{sy - Y(0)\} + 2y = \frac{4}{s-2}$

$\{s^2y + 3s - 5\} - 3\{sy + 3\} + 2y = \frac{4}{s-2}$

$(s^2 - 3s + 2)y + 3s - 14 = \frac{4}{s-2}$

$y = \frac{4}{(s^2-3s+2)(s-2)} + \frac{14-3s}{s^2-3s+2}$

$= \frac{-3s^2+20s-24}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

Thus, $Y = \mathcal{L}^{-1}\left\{\frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}\right\}$

$= -7e^t + 4e^{2t} + 4te^{2t}$, which can be verified as the solution.

(30) $Y'' + 2Y' + 5Y = e^{-t}\sin t$, $Y(0) = 0$, $Y'(0) = 1$

we have, $\mathcal{L}\{Y''\} + 2\mathcal{L}\{Y'\} + 5\mathcal{L}\{Y\} = \mathcal{L}\{e^{-t}\sin t\}$

$\{s^2y - sY(0) - Y'(0)\} + 2\{sy - Y(0)\} + 5y =$

$\frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$

$\{s^2y - s(0) - 1\} + 2\{sy - 0\} + 5y = \frac{1}{s^2+2s+2}$

$$(s^2+2s+5)y-1 = \frac{1}{s^2+2s+2}$$

$$y = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+2)(s^2+2s+5)}$$

$$= \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$Y = \mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = \frac{1}{3}e^{-t}(\sin t+\sin 2t)$$

(31) $Y''' - 3Y'' + 3Y' - Y = t^2e^t$, $Y(0)=1$, $Y'(0)=0$, $Y''(0)=-2$

we have,

$$\mathcal{L}\{Y'''\} - 3\mathcal{L}\{Y''\} + 3\mathcal{L}\{Y'\} - \mathcal{L}\{Y\} = \mathcal{L}\{t^2e^t\}$$

$$\{s^3y - s^2Y(0) - sY'(0) - Y''(0)\} - 3\{s^2y - sY(0) - Y'(0)\} +$$
$$3\{sy - Y(0)\} - y = \frac{2}{(s-1)^3}$$

Thus,

$$(s^3-3s^2+3s-1)y - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$

$$y = \frac{s^2-3s+1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{s^2-2s+1-s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{(s-1)^2-(s-1)-1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$Y = e^t - te^t - \frac{t^2e^t}{2} + \frac{t^5e^t}{60}$$

Scanned by CamScanner

$y''' - 3y'' + 3y' - y = t^2 e^t$, ~~$y(0) = 1, y'(0) = 0, y''(0) = -2$~~

In case of finding the general solution of the equation, the initial conditions are arbitrary. If we assume $y(0) = A$, $y'(0) = B$, $y''(0) = C$,

$$\mathcal{L}\{y'''\} - 3\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{t^2 e^t\}$$

$$\{s^3 y - s^2 y(0) - s y'(0) - y''(0)\} - 3\{s^2 y - s y(0) - y'(0)\} + 3\{sy - y(0)\} - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow (s^3 y - As^2 - Bs - C) - 3\{s^2 y - As - B\} + 3\{sy - A\} - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow y = \frac{As^2 + (B-3A)s + 3A - 3B + C}{(s-1)^3} + \frac{2}{(s-1)^3}$$

Since A, B, C are arbitrary, so, also is the polynomial in the numerator of the first term on the right. Thus, we can write,

$$y = \frac{c_1}{(s-1)^3} + \frac{c_2}{(s-1)^2} + \frac{c_3}{s-1} + \frac{2}{(s-1)^6}$$

and invert to find the required general solution,

$$y = \frac{c_1 t^2 e^t}{2} + c_2 t e^t + c_3 e^t + \frac{t^5 e^t}{60}$$

$$= c_4 t^2 + c_5 t e^t + c_6 e^t + \frac{t^5 e^t}{60}$$

Scanned by CamScanner

(33) $Y'' + 9Y = \cos 2t$ if $Y(0) = 1$, $Y(\pi/2) = -1$

since, $Y'(0)$ is not known, let $Y'(0) = c$, then

$\mathcal{L}\{Y''\} + 9\mathcal{L}\{Y\} = \mathcal{L}\{\cos 2t\}$

$s^2y - sY(0) - Y'(0) + 9y = \frac{s}{s^2+4}$

$(s^2+9)y - s - c = \frac{s}{s^2+4}$

$$y = \frac{s+c}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

$$= \frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}$$

$$= \frac{4}{5}\left(\frac{s}{s^2+9}\right) + \frac{c}{s^2+9} + \frac{s}{5(s^2+4)}$$

$$Y = \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t$$

since, $Y(\pi/2) = -1$, so, $-1 = -\frac{c}{3} - \frac{1}{5}$ or, $c = \frac{12}{5}$

Then, $Y = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t$

(34) $\cancel{y'' + a^2y}$ $Y'' + a^2Y = F(t)$, $Y(0) = 1$, $Y'(0) = -2$

we have,

$\mathcal{L}'\{Y''\} + a^2\mathcal{L}\{Y\} = \mathcal{L}\{F(t)\} = f(s)$

$\Rightarrow s^2y - sY(0) - Y'(0) + a^2y = f(s)$

$\Rightarrow s^2y - s + 2 + a^2y = f(s)$

$$\Rightarrow y = \frac{s-2}{s^2+a^2} + \frac{f(s)}{s^2+a^2}$$

using the convulation theorem,

$$Y = \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{f(s)}{s^2+a^2}\right\}$$

$$= \cos at - \frac{2\sin at}{a} + F(t) * \frac{\sin at}{a}$$

$$= \cos at - \frac{2\sin at}{a} + \frac{1}{a}\int_0^t F(u) - \sin a(t-u)\, du$$

(35) $tY'' + Y' + 4tY = 0, \quad Y(0) = 3, \quad Y'(0) = 0$

we have,

$$\mathcal{L}\{tY''\} + \mathcal{L}\{Y'\} + \mathcal{L}\{4tY\} = 0$$

$$\Rightarrow -\frac{d}{ds}\{s^2y - sY(0) - Y'(0)\} + \{sy - Y(0)\} - 4\frac{dy}{ds} = 0$$

$$\Rightarrow (s^2+4)\frac{dy}{ds} + sy = 0$$

Then, $\frac{dy}{y} + \frac{s\,ds}{s^2+4} = 0$

and integrating. $\ln y + \frac{1}{2}\ln(s^2+4) = c_1$

$$\Rightarrow y = \frac{c}{\sqrt{s^2+4}}$$

Inverting we find, $Y = cJ_0(2t)$

To determine c,

$$Y(0) = cJ_0(0) = c = 3$$

Thus, $Y = 3J_0(2t)$

(6) $tY'' + 2Y' + tY = 0, \quad Y(0+) = 1, \quad Y(\pi) = 0$

Let, $Y'(0+) = c$,

Then taking laplace transform of each term

$-\frac{d}{ds}\{s^2y - sY(0+) - Y'(0+)\} + 2\{sy - Y(0+)\} - \frac{d}{ds}(y) = 0$

$\Rightarrow -s^2y' - 2sy + 1 + 2sy - 2 - y' = 0$

$\Rightarrow -(s^2+1)y' - 1 = 0$

$\Rightarrow y' = \frac{-1}{s^2+1}$

Integrating,

$y = -\tan^{-1}s + A$

Since, $y \to 0$ as $s \to \infty$, we must have $A = \pi/2$.

Thus, $y = \frac{\pi}{2} - \tan^{-1}s = \tan^{-1}\frac{1}{s}$

$Y = \mathcal{L}^{-1}\left\{\tan^{-1}\frac{1}{s}\right\} = \frac{\sin t}{t}$

This satisfies $Y(\pi) = 0$ and is the required solution.