Signal Processing and Matched Filtering

How to find signals in noisy data when you have a template

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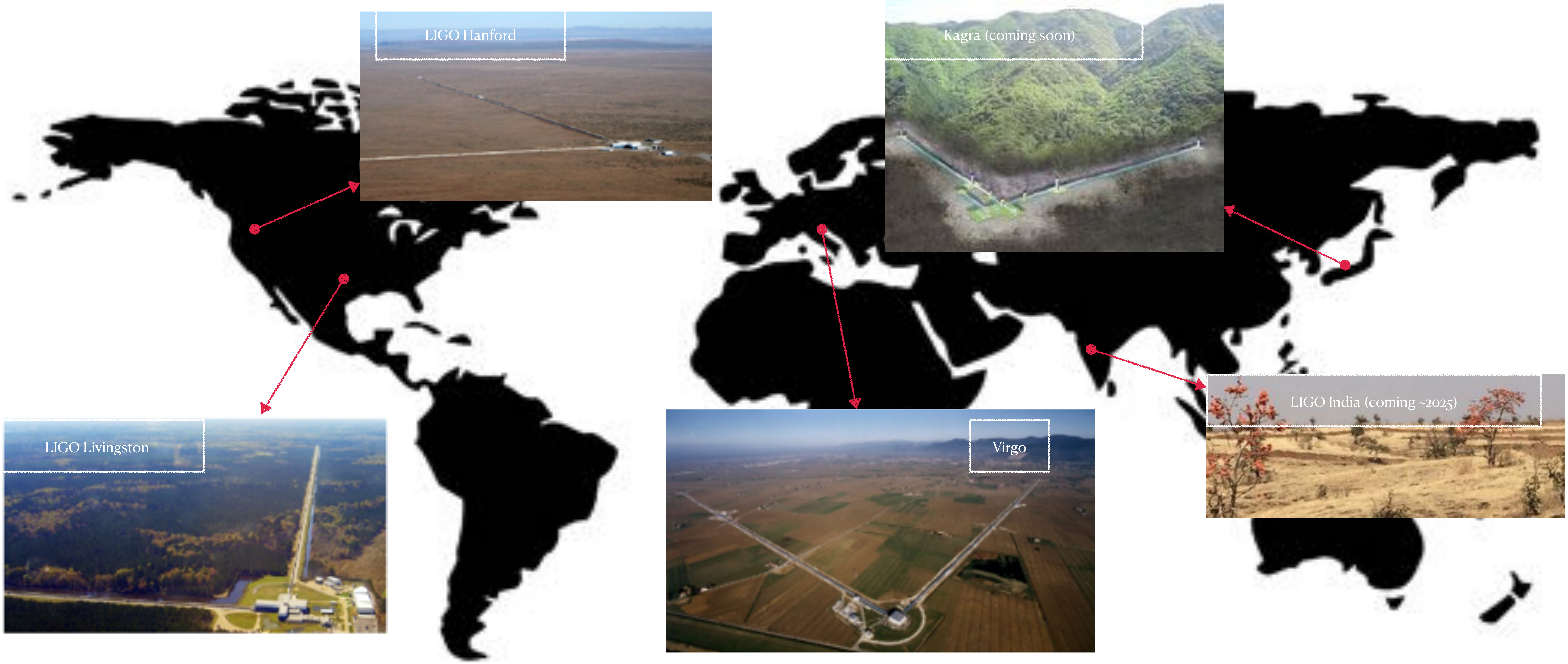
Today's lecture will cover:

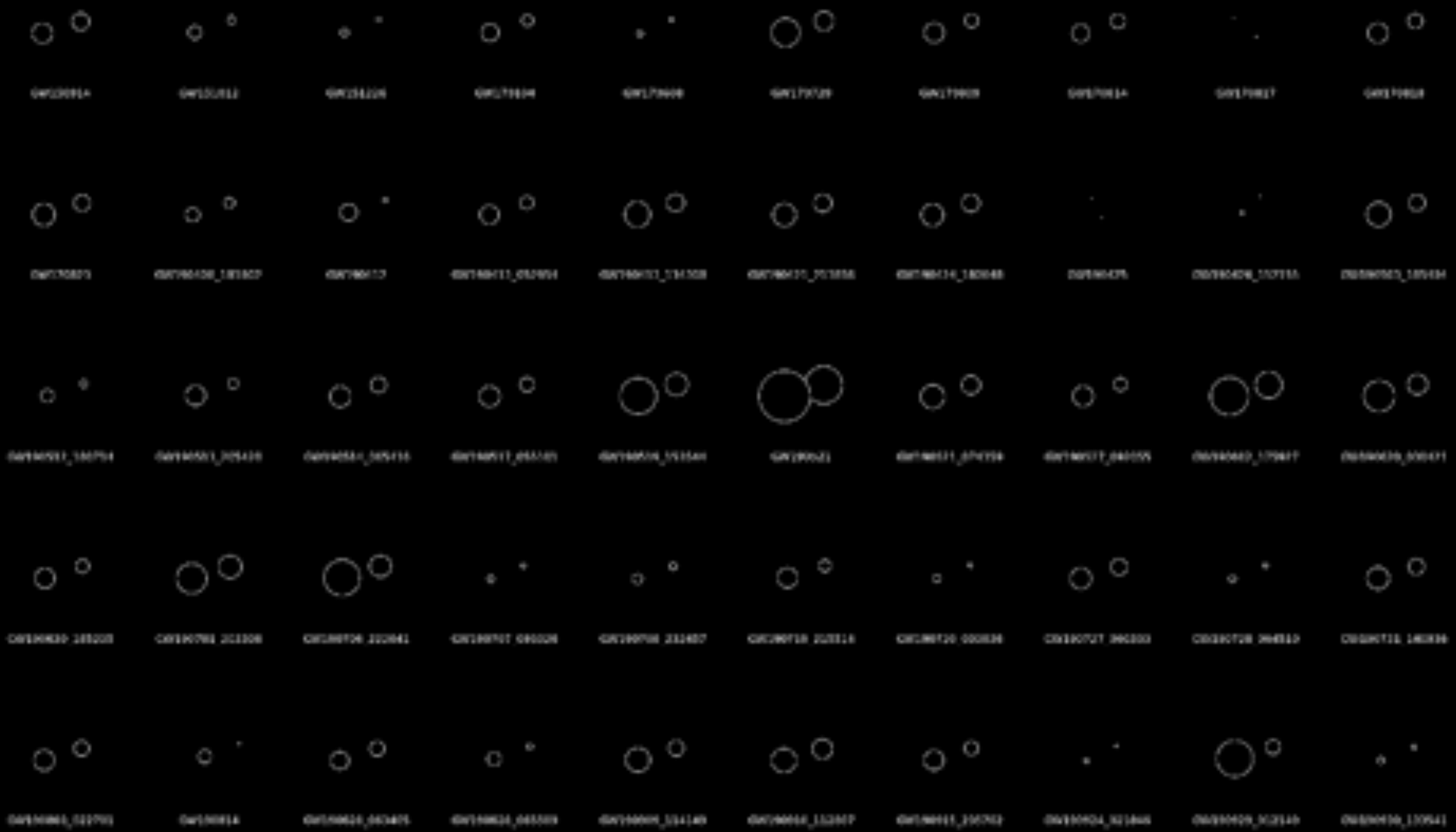
1. **Introduction to the problem**: detecting gravitational waves with LIGO and Virgo
2. **Detector noise:** stationary & Gaussian noise, amplitude spectral density, power spectral density
3. **Signal templates:** correlate datastream with templates
4. **Matched filter signal-to-noise ratio**: calculate the signal-to-noise ratio, optimal signal-to-noise ratio and template, signal-to-noise ratio timeseries
5. **Statistical significance of a trigger**: false alarm rate, false alarm probability

LIGO timeseries data (strain versus time)

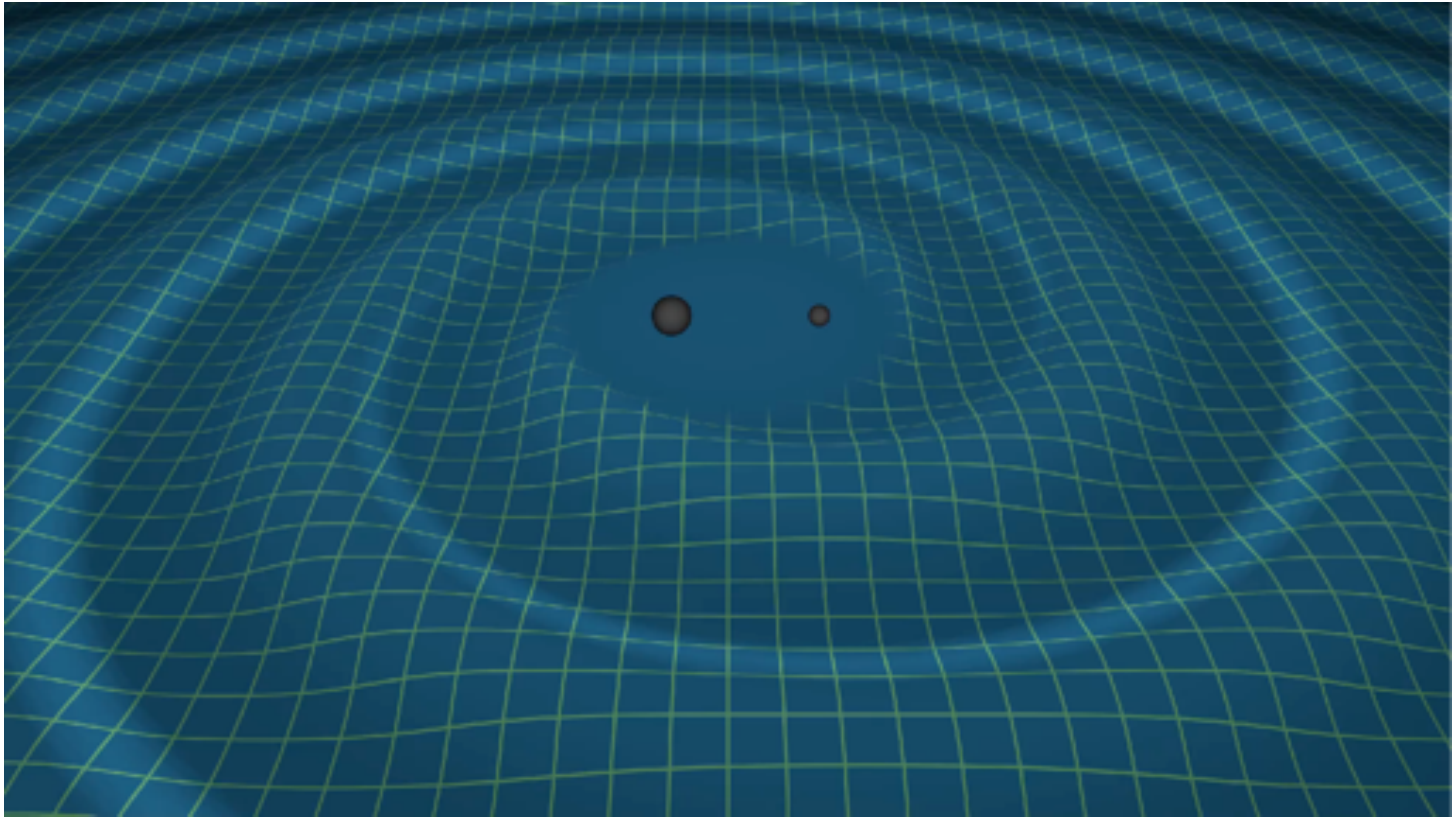
Worldwide network of interferometers



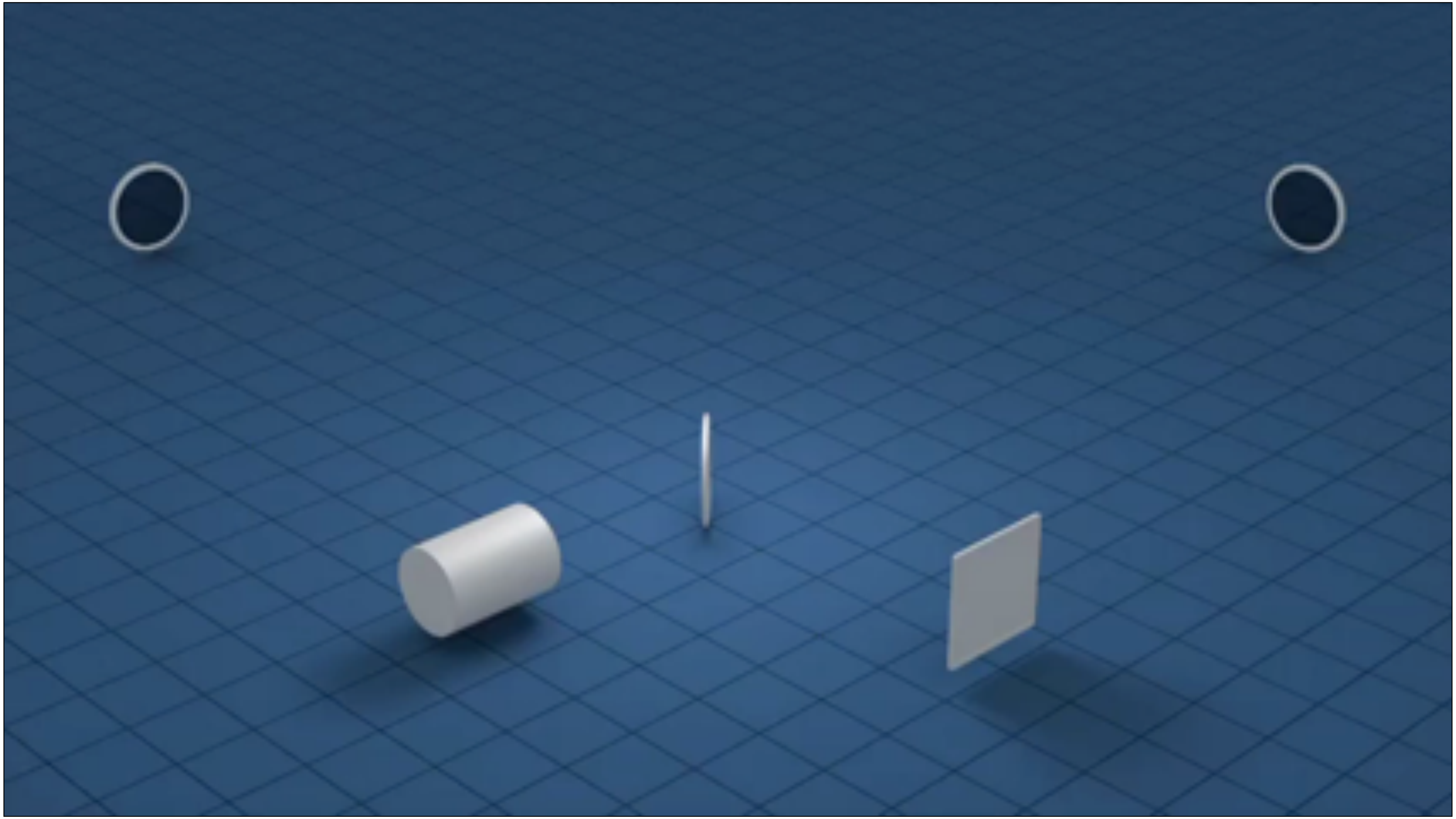
Listening for colliding black holes and neutron stars



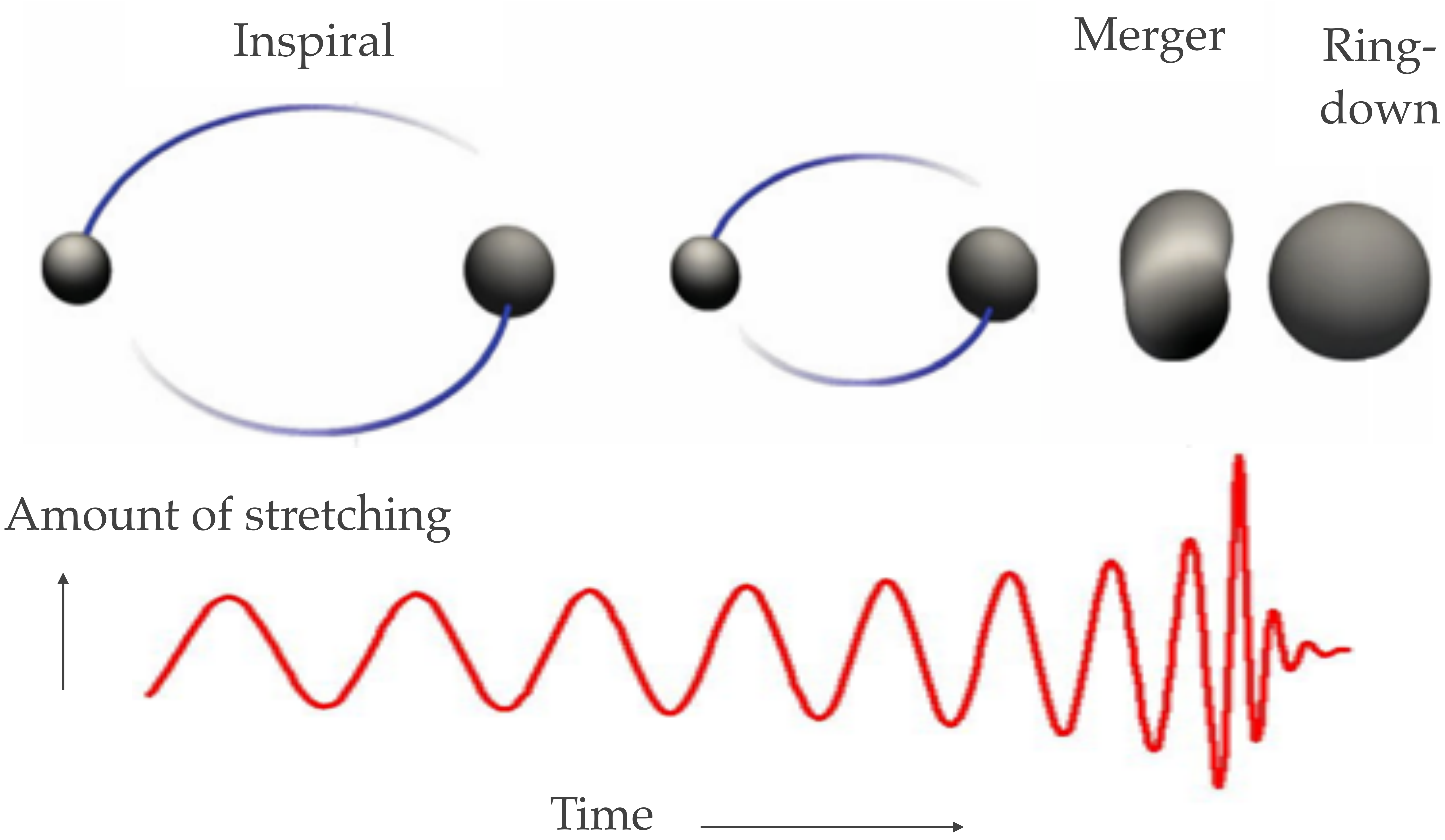
Compact binary coalescences emit ripples in spacetime known as gravitational waves



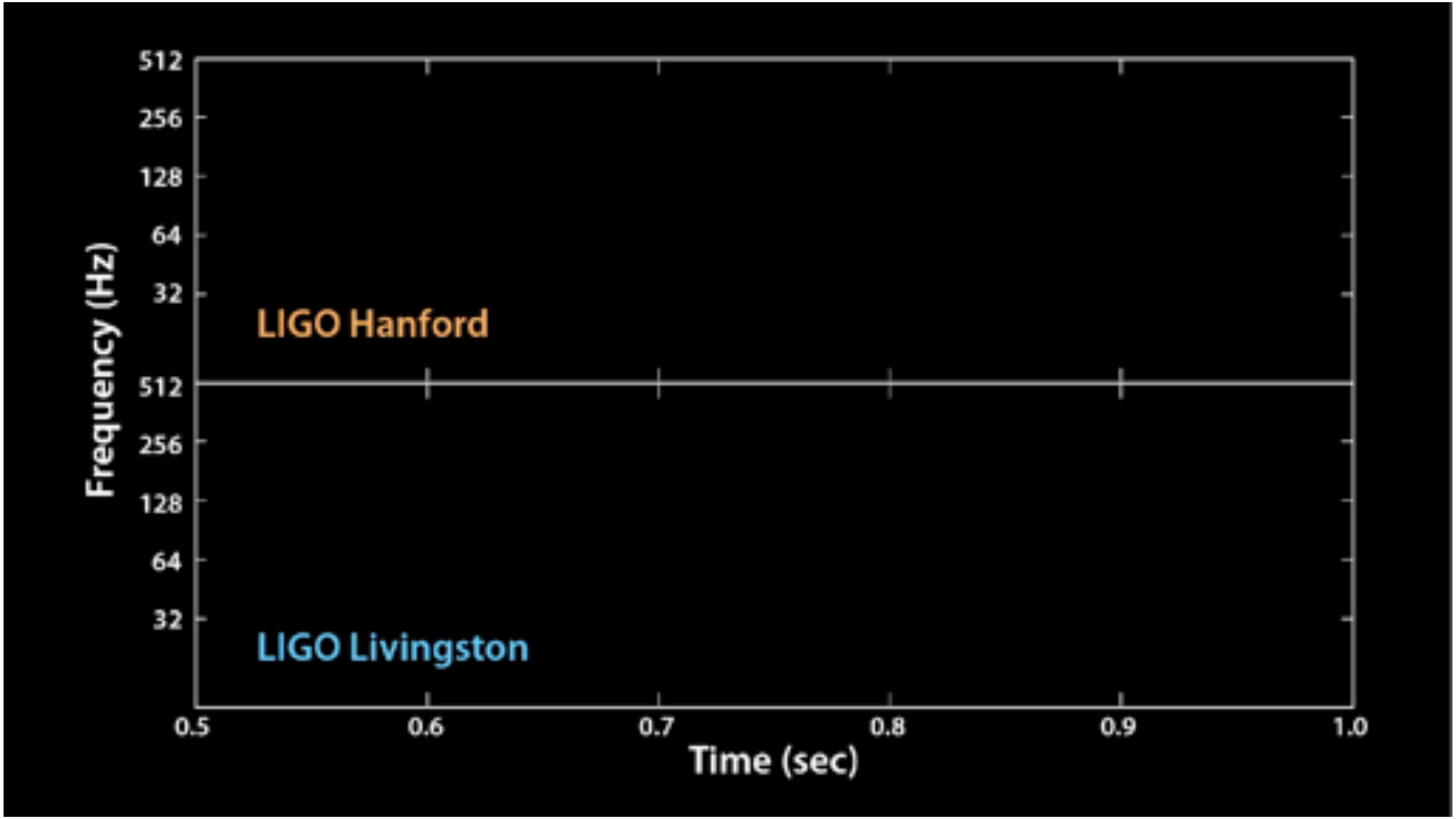
LIGO: Laser interferometer gravitational-wave observatory



Gravitational wave signal from merging black holes



The chirp of two black holes colliding



Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.*[*]
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with a peak gravitational-wave strain of 1.0×10^{-21}. It matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. The signal was observed with a matched-filter signal-to-noise ratio of 24 and a false alarm rate estimated to be less than 1 event per 203 000 years, equivalent to a significance greater than 5.1σ. The source lies at a luminosity distance of 410^{+160}_{-180} Mpc corresponding to a redshift $z = 0.09^{+0.03}_{-0.04}$. In the source frame, the initial black hole masses are $36^{+5}_{-4} M_\odot$ and $29^{+4}_{-4} M_\odot$, and the final black hole mass is $62^{+4}_{-4} M_\odot$, with $3.0^{+0.5}_{-0.5} M_\odot c^2$ radiated in gravitational waves. All uncertainties define 90% credible intervals. These observations demonstrate the existence of binary stellar-mass black hole systems. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger.

Detector noise

In the absence of a signal, the detector output is a noise time series *n(t)*

- Detector noise is often approximated as stationary
  - Autocorrelation function of the noise, $K_n(t, t') := \mathrm{E}[n(t)n(t')]$, depends only on the difference *t - t'*
- This motivates a frequency-domain analysis (***Fourier transform***). For stationary noise, the noise in each frequency bin is independent of the other frequencies.
- The noise is also often approximated as Gaussian, meaning the noise at each frequency follows a Gaussian distribution described by the *power spectral density*.
- Aside: both of these are only approximations that are used in signal searches. To calculate statistical significance, the noise background actually has to be estimated empirically

Stationary and Gaussian Noise

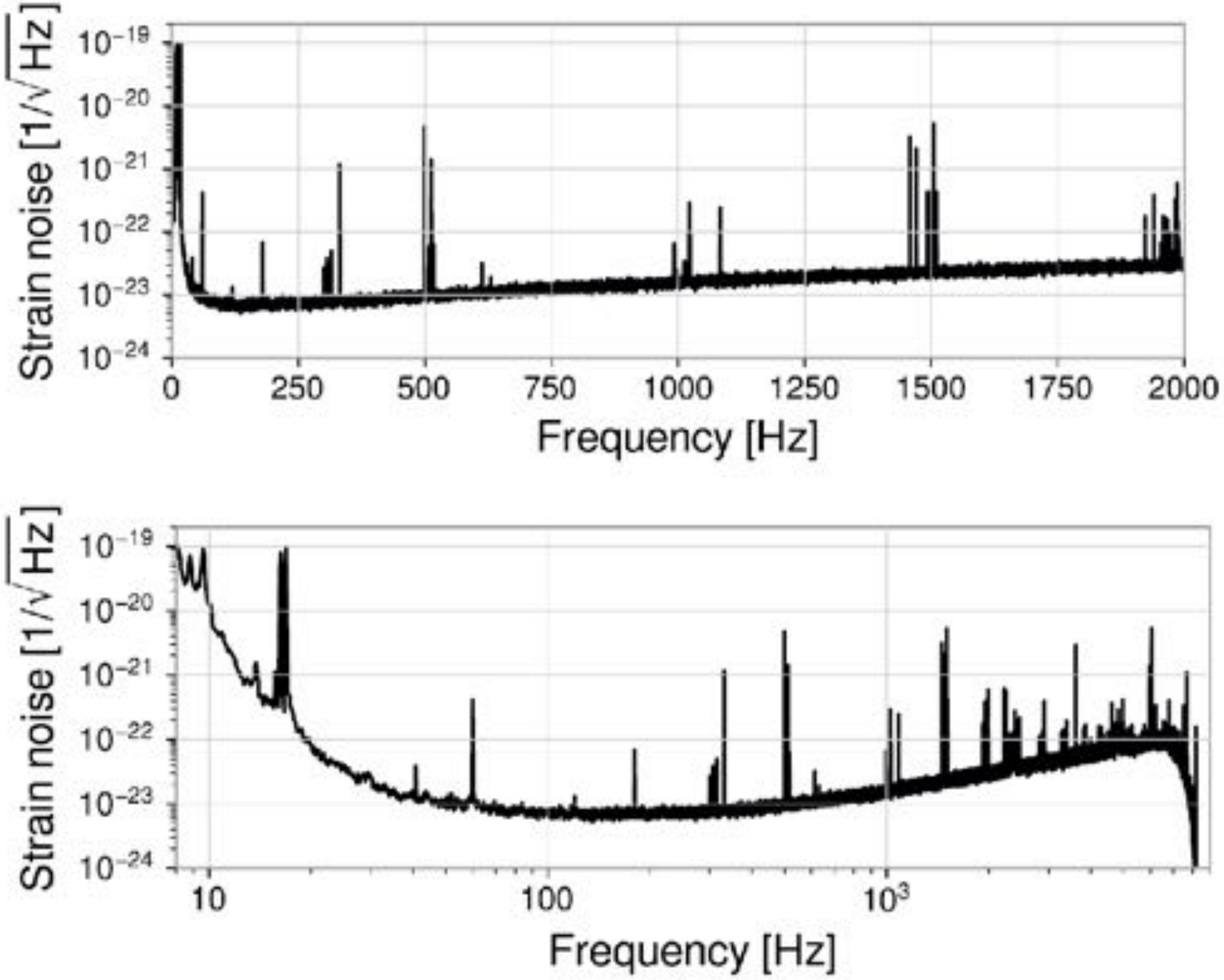
- We work with discrete samples from the time series data (for signal searches, sampling frequency ~16 kHz)
- For noise time series *n(t)*, can take discrete samples $n_i = n(t_i)$ that make up vector **n**
- Describe the noise as a stochastic process with a joint probability distribution ***p(n)***
- Gaussian noise: ***p(n)*** follows a multi-variate Gaussian, specified by the mean and the covariance
- Stationary noise implies that the covariance matrix C_{ij} depends only on the time lag $\tau = |t_i - t_j|$, so can be characterized by the correlation function $C(\tau)$.

Stationary and Gaussian Noise

In the frequency domain

- Applying a Fourier transform to a stationary time series, the covariance matrix between two frequencies f_i, f_j is diagonal: $C_{ij} = \delta_{ij} S_n(f_i)$. In other words, the noise is uncorrelated between frequency bins.
- This defines the power spectral density $S_n(f)$, the Fourier transform of the correlation function $C(\tau)$ (units: 1/Hz)
- The amplitude spectral density is the square root of the power spectral density (like standard deviation is to variance)
- White noise has a flat power spectral density, $C_{ij} = \delta_{ij} \sigma^2$

Example Amplitude Spectral Density



Abbott et al 2020 Class. Quantum Grav. **37** 055002

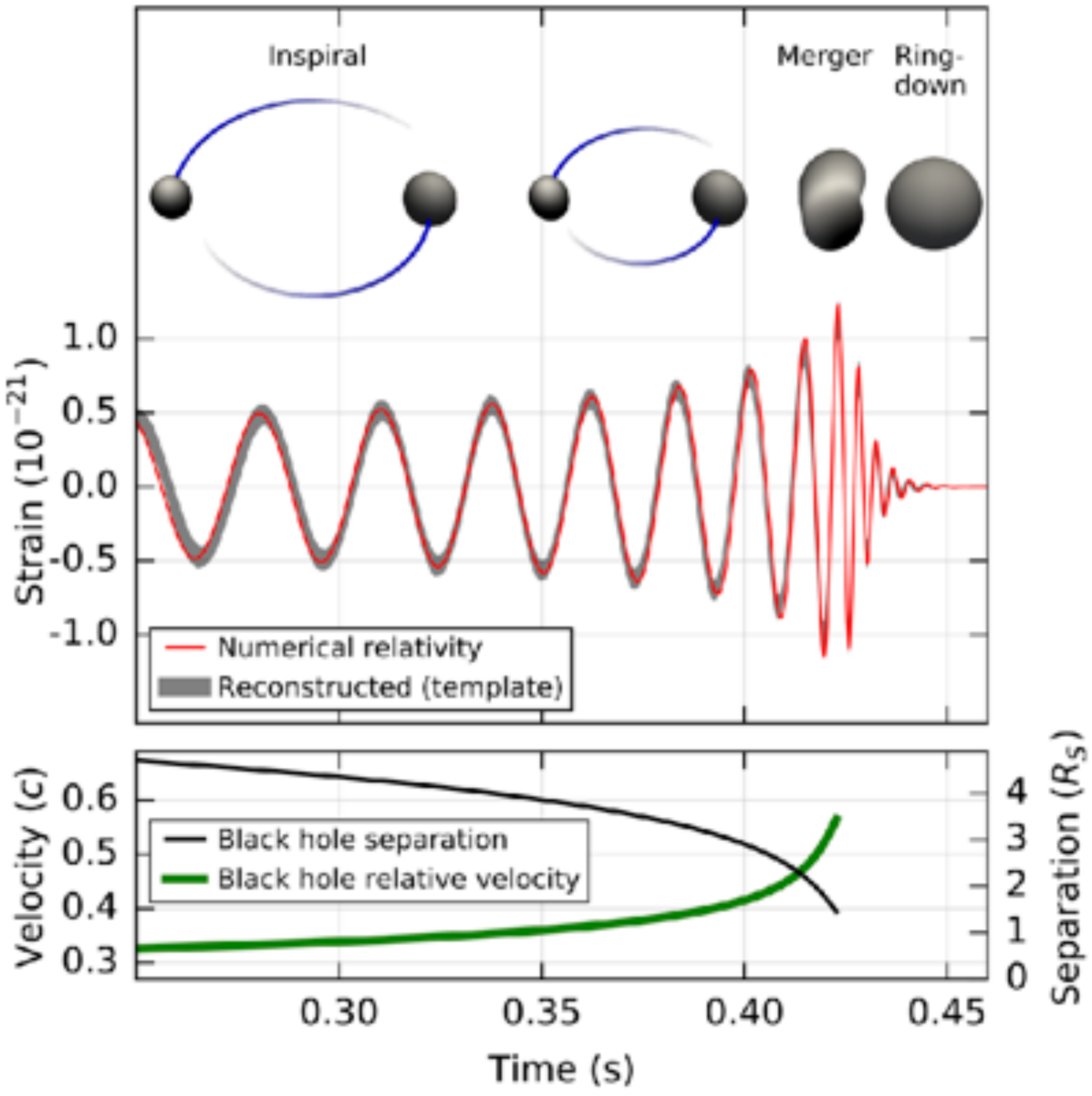
Fourier domain analysis

Fast Fourier transforms

- To work in the frequency domain, apply a Discrete Fourier Transform (typically Fast Fourier Transform, or FFT) to the time domain data
- Need to sample at a sufficiently high frequency: for a given sampling frequency f_s, can resolve frequencies up to $f_N = \frac{1}{2} f_s$ (the Nyquist frequency)
- If the data contains frequencies higher than twice the sampling frequency, we need to apply a low-pass filter (i.e. blur the higher frequencies)
- The spacing between two frequency bins is the inverse of the signal duration
- Given a discrete samples from a time series, the sampling frequency and the signal duration determine the frequency bins

Signal templates

Gravitational waves from merging black holes



Abbott+ 2016 PRL **116**, 061102

Template Bank

(Later, we will discuss the placing of templates in the bank)

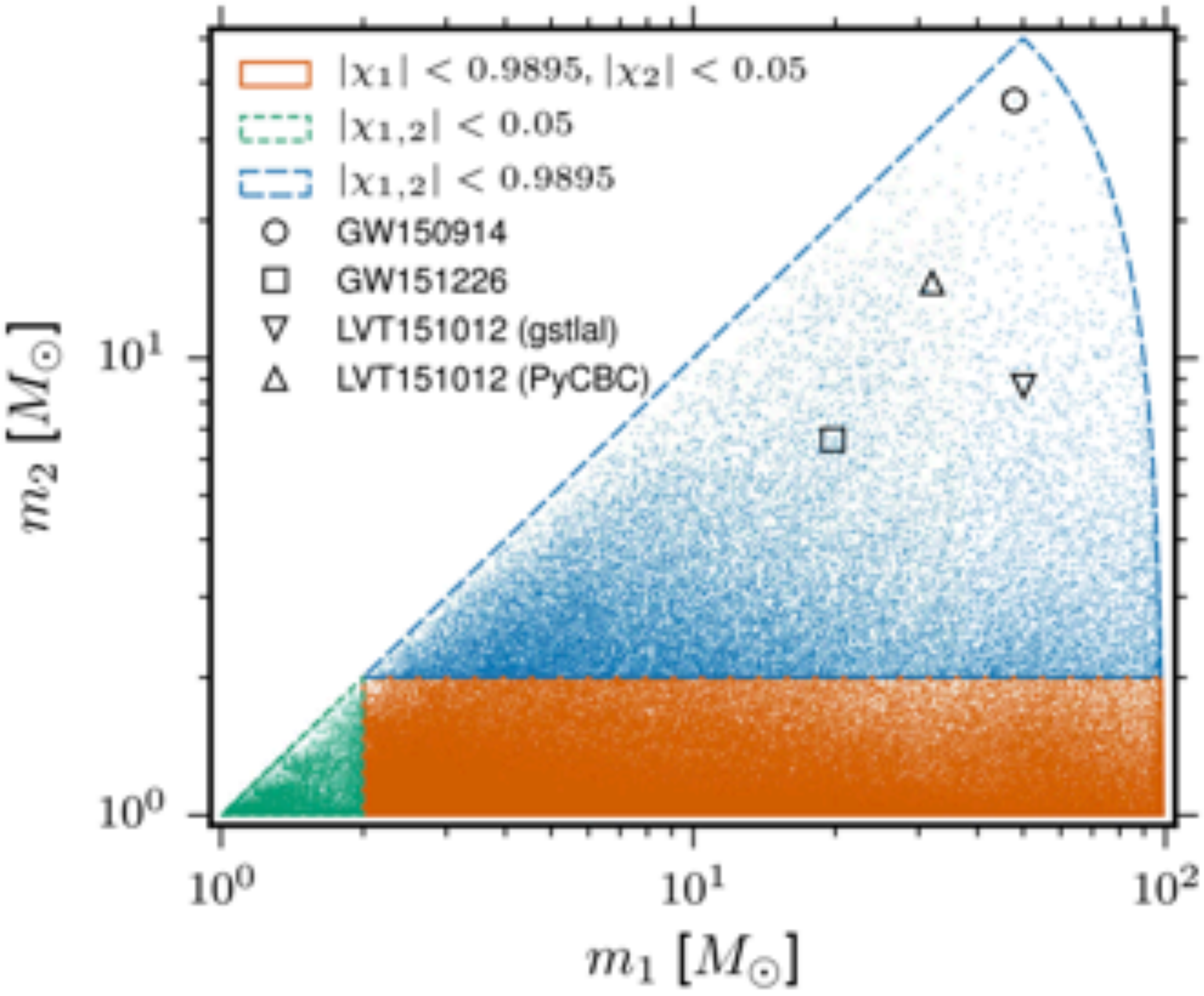


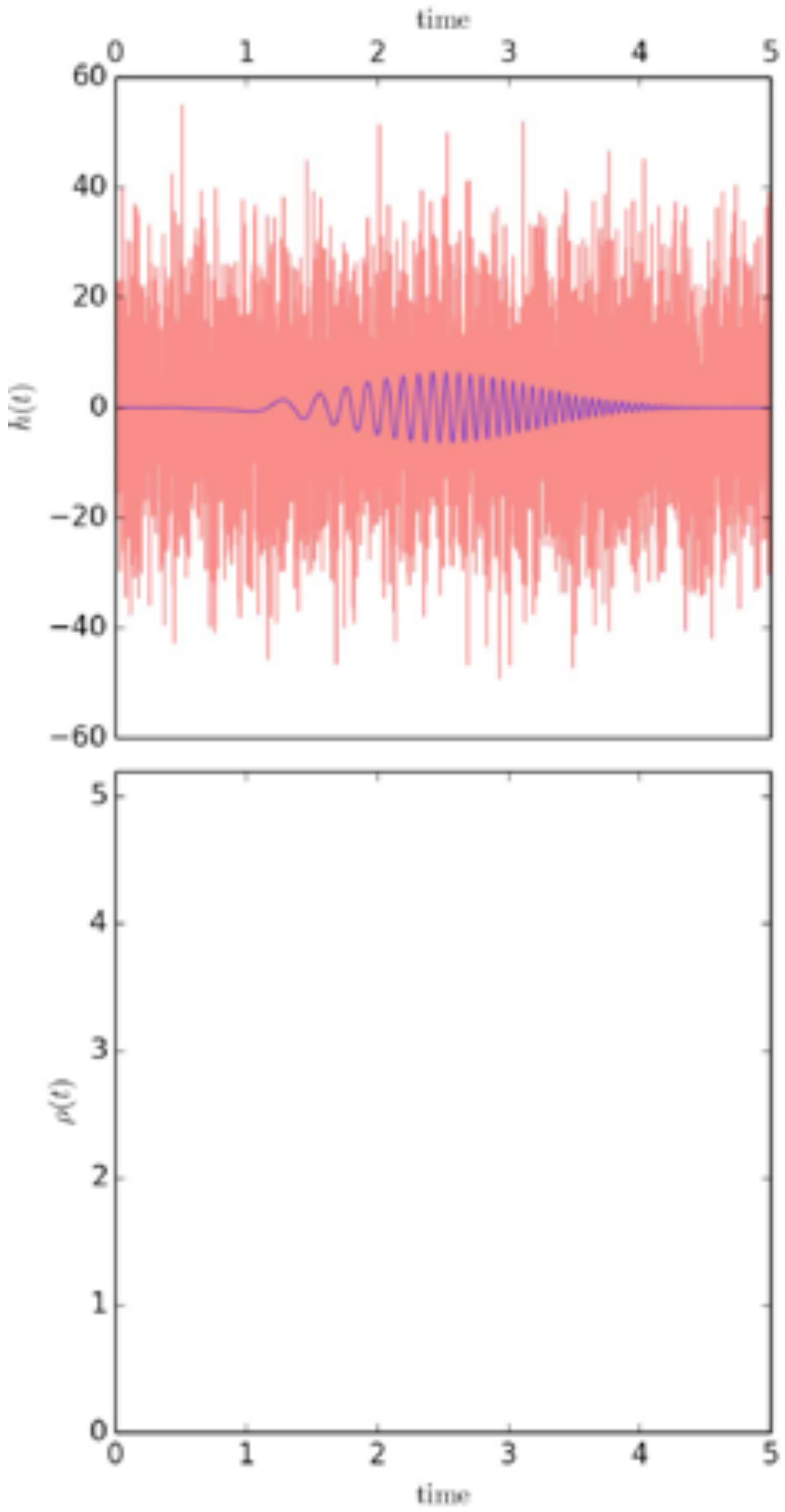
FIG. 2. The four-dimensional search parameter space covered by the template bank shown projected into the component-mass plane, using the convention $m_1 > m_2$. The colors indicate mass regions with different limits on the dimensionless spin parameters χ_1 and χ_2. Symbols indicate the best matching templates for GW150914, GW151226, and LVT151012. For GW150914 and GW151226, the templates were the same in the PyCBC and GstLAL searches, while for LVT151012 they differed. The parameters of the best matching templates are consistent, up to the discreteness of the template bank, with the detector frame mass ranges provided by detailed parameter estimation in Sec. IV.

LVC, Phys Rev X 6, 041015 (2016), arXiv:1606.04856

Matched Filter Signal-to-Noise Ratio (SNR)

Identifying Triggers

- Calculate new time series of the matched filter SNR(t)
- Local maxima in this times series are “triggers”
- Set an SNR threshold for recording triggers of interest



Movie credit: Reed C. Essick

Notation

Datastream $h(t)$

Noise $n(t)$

(Potential) signal $s(t)$

Templates $T(t \mid \lambda)$, λ are the source parameters like masses, spins

$h(t) = n(t) + s(t)$

Does this mean that $|s(t)|$ has to be larger than $|n(t)|$? Thankfully, no!

Calculating the Matched-Filter SNR

- Basic idea: we correlate h(t) with templates T to find the signal s(t)
- Here, the template T is the filter
- For any filter K(t), we can correlate our data stream $\hat{h} = \int_{-\infty}^{+\infty} dt h(t) K(t)$
- The value of $\hat{h}$ over different noise realizations follows a statistical distribution
- The **signal-to-noise ratio** is the ratio between *S = the expectation value of* $\hat{h}$ *when there is a signal* and *N = the rms value of* $\hat{h}$ *when there is no signal*
- "Expectation value", "rms" refers to averaging over noise realizations

The “S” in SNR = S/N:

$$S \equiv \langle \hat{h} \rangle = \int_{-\infty}^{+\infty} dt \langle h(t) \rangle K(t)$$

$$= \int_{-\infty}^{+\infty} dt s(t) K(t) + \int_{-\infty}^{+\infty} dt \langle n(t) \rangle K(t)$$

Because of Gaussian noise, $\langle n(t) \rangle$ is a constant (can take to be zero), so

$$S = \int_{-\infty}^{+\infty} dt s(t) K(t)$$

$$= \int_{-\infty}^{+\infty} df \tilde{s}(f) \tilde{K}^*(f) \quad \text{[property of Fourier transforms]}$$

The “N” in SNR = S/N:

$$N^2 \equiv [\langle \hat{h}^2(t)\rangle - \langle \hat{h}(t)\rangle^2]_{s=0}$$

$$= \left[\langle \hat{h}^2(t)\rangle - \left(\int_{-\infty}^{\infty} s(t)K(t) + \int_{-\infty}^{\infty} \langle n(t)\rangle K(t) \right)^2 \right]_{s=0}$$

Note both terms in (...) are zero, so

$N^2 = [\langle \hat{h}^2(t)\rangle]_{s=0} = \int_{-\infty}^{\infty} dt dt' K(t)K(t')\langle n(t)n(t')\rangle$, where the definition of the Fourier transform gives:

$\langle n(t)n(t')\rangle = \int_{-\infty}^{\infty} df df' e^{2\pi i f t - 2\pi i f' t'}\langle \tilde{n}^\star(f)\tilde{n}(f')\rangle$, and from the definition of the power spectral density

$\langle \tilde{n}^\star(f)\tilde{n}(f')\rangle \equiv \delta(f - f')\frac{1}{2}S_n(f)$, and some Fourier transform properties, we get:

$$N^2 = \int_{-\infty}^{\infty} df\, |\tilde{K}(f)|^2 \frac{1}{2} S_n(f)$$

Putting everything together, the matched filter SNR = S/N for any filter K is:

$$S/N = \frac{\int_{-\infty}^{+\infty} df \tilde{s}(f) \tilde{K}^*(f)}{\left[\int_{-\infty}^{\infty} df \,|\, \tilde{K}(f) \,|^2 \frac{1}{2} S_n(f) \right]^{1/2}}$$

What is the optimal filter?

By optimal filter, we mean the one that maximizes the signal-to-noise ratio for a given signal

We define an inner product between any two timeseries A(t) and B(t):

$$\left(A \mid B\right) \equiv \mathrm{Re}\int_{-\infty}^{\infty} df\, \frac{\tilde{A}^{\star}(f)\tilde{B}(f)}{\frac{1}{2}S_n(f)} = 4\mathrm{Re}\int_{0}^{\infty} df\, \frac{\tilde{A}^{\star}(f)\tilde{B}(f)}{S_n(f)}$$

So defining $\tilde{u}(f) \equiv \dfrac{1}{2}S_n(f)\tilde{K}(f)$ for our filter K, we can write S/N as:

$$S/N = \frac{(u\,|\,s)}{(u\,|\,u)^{1/2}}$$

From this, we can see that S/N is maximized for u “parallel” to the signal s, so the **optimal filter *K* is proportional to the *signal itself* weighted by the PSD*:***

$\tilde{K}(f) = \mathrm{const} \times \dfrac{\tilde{s}(f)}{S_n(f)}$, and the optimal SNR for any signal s will be:

$$(S/N)_{\mathrm{optimal}} = \left[4\mathrm{Re}\int_{0}^{\infty} df\, \frac{|\,\tilde{s}(f)\,|^2}{S_n(f)}\right]^{1/2}$$

Matched-filter SNR timeseries

- When we do a search, we don't know the exact signal *s* or when it occurs, all we have is our datastream h(t) and a template bank consisting of templates T(t | ƛ) for different ƛ
- We "slide" the templates over the data and calculate the SNR as a function of the *time shift* between each template and the data
- A spike in the SNR timeseries is a "trigger" — a potential signal of interest
- Luckily, we can get all possible time shifts between the template and the signal at once with a single FFT

Matched-filter SNR timeseries: Fourier transform

We want to calculate the SNR = $\dfrac{(T|h)}{(T|T)^{1/2}}(\tau)$ as a function of the time shift τ between the template T and the data h

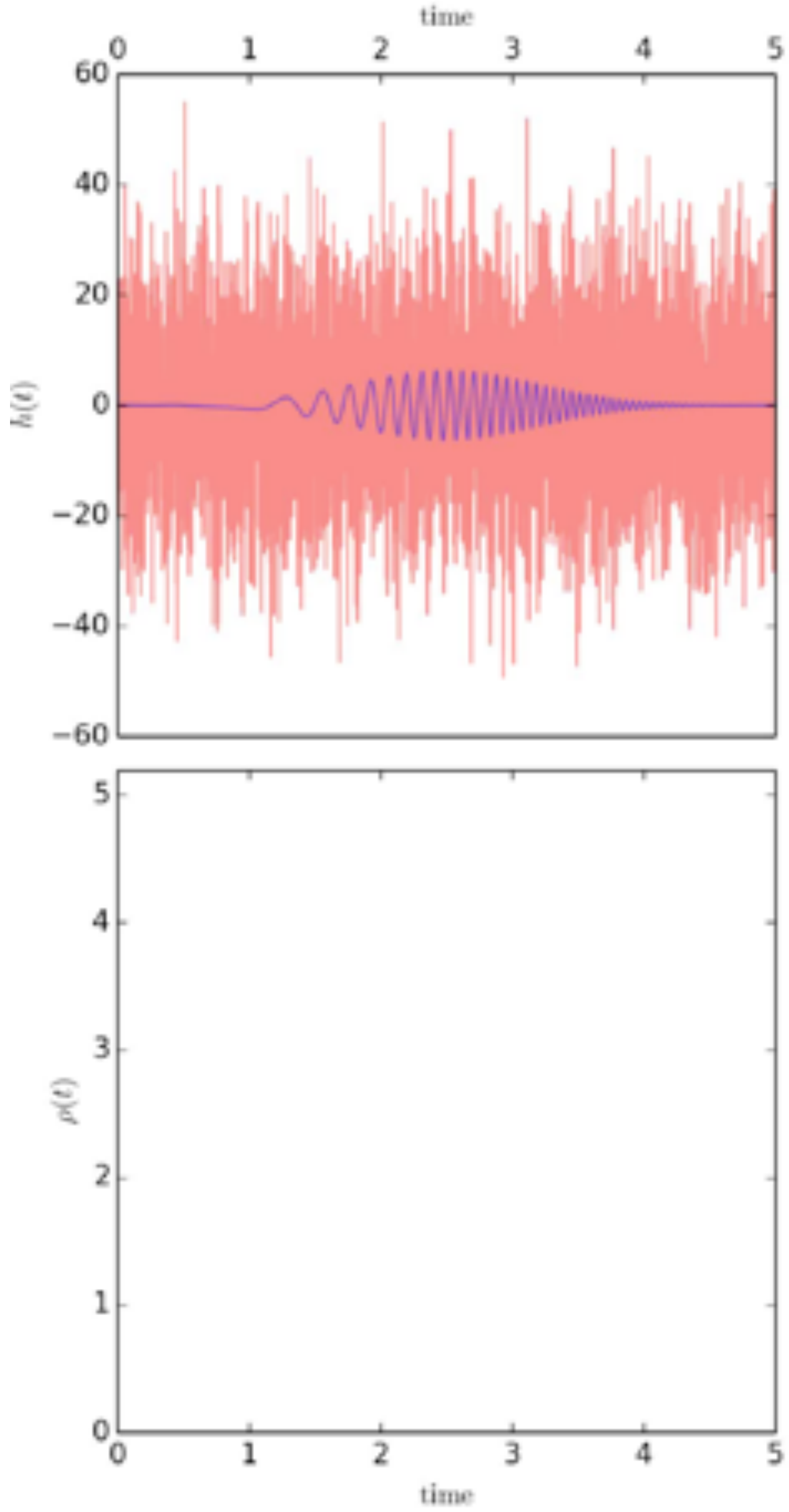
The template T is a function of the time shift, $T(t\,|\,\tau, m_1, m_2, \ldots)$

If $\tilde{T}(f)$ is the Fourier transform of $T(t\,|\,\tau' = 0,\ldots)$, the Fourier transform of $T(t\,|\,\tau' = \tau)$ is $\tilde{T}(f)e^{i2\pi f\tau}$, so:

$$(T|h)(\tau) = 4\mathrm{Re}\int_0^\infty df\,\frac{\tilde{T}^\star(f)\tilde{h}(f)}{S_n(f)}e^{i2\pi f\tau}$$

This is the inverse Fourier transform of $\dfrac{\tilde{T}^\star(f)\tilde{h}(f)}{S_n(f)}$

Thus, we can compute the inverse FFT of the above expression to get a [unnormalized] time series SNR(τ) and easily find the τ corresponding to a local maximum



Movie credit: Reed C. Essick

How do we construct the template bank?

We want to limit the SNR loss due to the discreteness of the bank for any signal $s(\lambda)$ in the parameter space

We have some templates T_λ in the bank. The maximum SNR over the bank is:

$$\left[\frac{(s\,|\,T_\lambda)}{(T_\lambda\,|\,T_\lambda)^{1/2}}\right]_{\text{max over bank}} \times \frac{(s\,|\,s)^{1/2}}{(s\,|\,s)^{1/2}}$$

$$= \left[\frac{(s\,|\,T_\lambda)}{(T_\lambda\,|\,T_\lambda)^{1/2}(s\,|\,s)^{1/2}}\right]_{\text{max over bank}} \times (s\,|\,s)^{1/2}$$

The second factor in the above is the **optimal SNR**, and the first factor is the factor by which the maximum SNR over the bank is degraded with respect to the optimal SNR. We call this the **fitting factor**.

Template bank discreteness and event rate loss

In LIGO-Virgo gravitational-wave searches, we typically pick the fitting factor to be no less than 0.97

This SNR loss translates to an event rate loss of ~10%:

Rate of detections R $\propto$ sensitive volume V $\propto$ Distance^3 $\propto$ $(\mathrm{SNR})^{-3}$

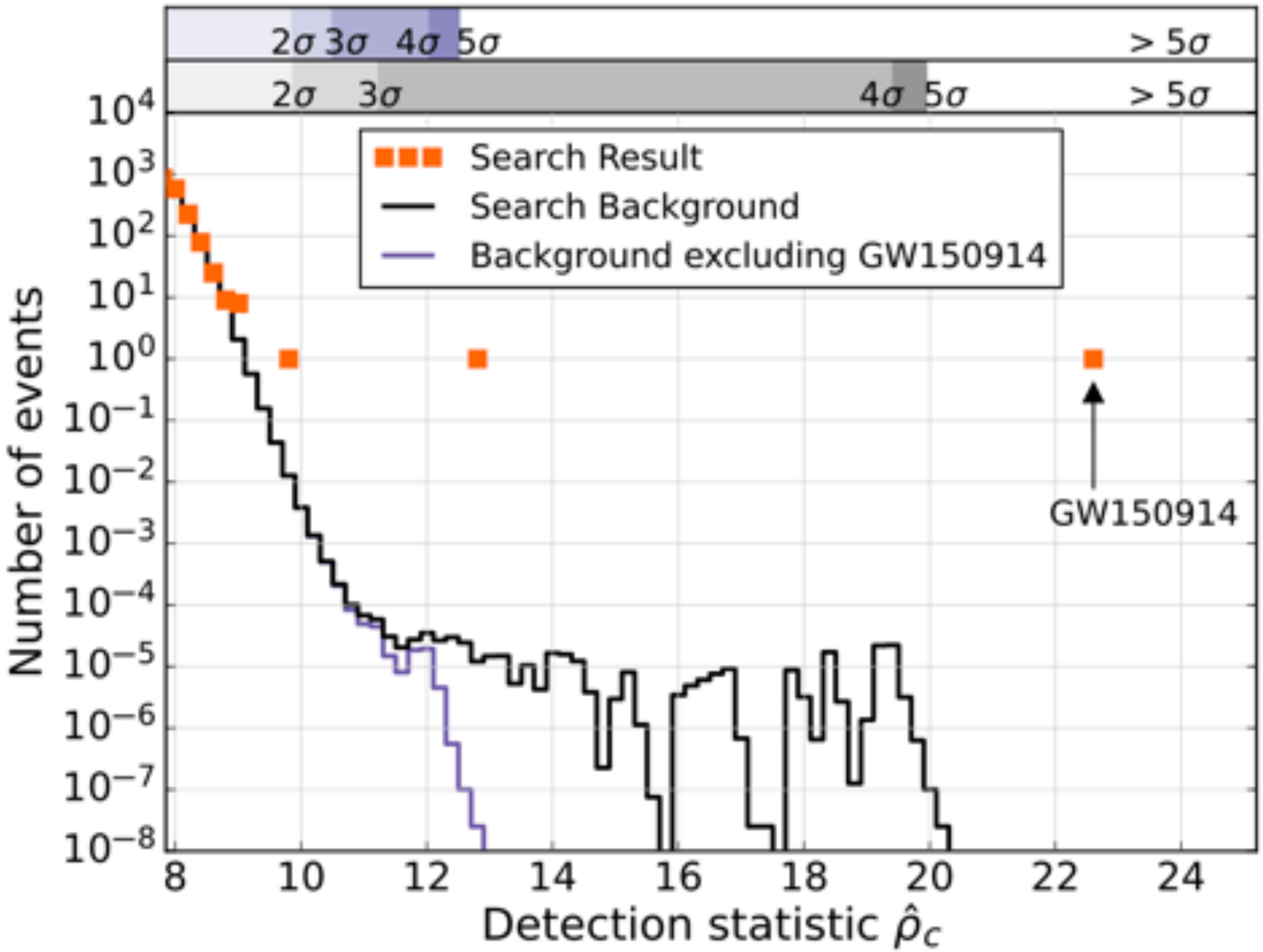
So if $(\mathrm{SNR})^{\mathrm{bank}} = 0.97\,(\mathrm{SNR})^{\mathrm{ideal}}$,

$$\frac{R^{\mathrm{bank}}}{R^{\mathrm{ideal}}} = \left(\frac{\mathrm{SNR}^{\mathrm{bank}}}{\mathrm{SNR}^{\mathrm{ideal}}}\right)^{-3} = 0.97^3 \approx 0.9$$

Statistical significance of a trigger

Search significance

- Assign a ranking statistic to triggers, usually resembles the matched-filter signal-to-noise ratio
- How often can noise produce trigger of similar or higher ranking statistic?
- Noise background has to be determined empirically — challenging because we only have a finite observing time and we cannot isolate the noise and collect data without gravitational-waves!
- Determine background using time slides — shift data between detectors by 0.1 seconds, greater than the 10 ms intersite propagation time



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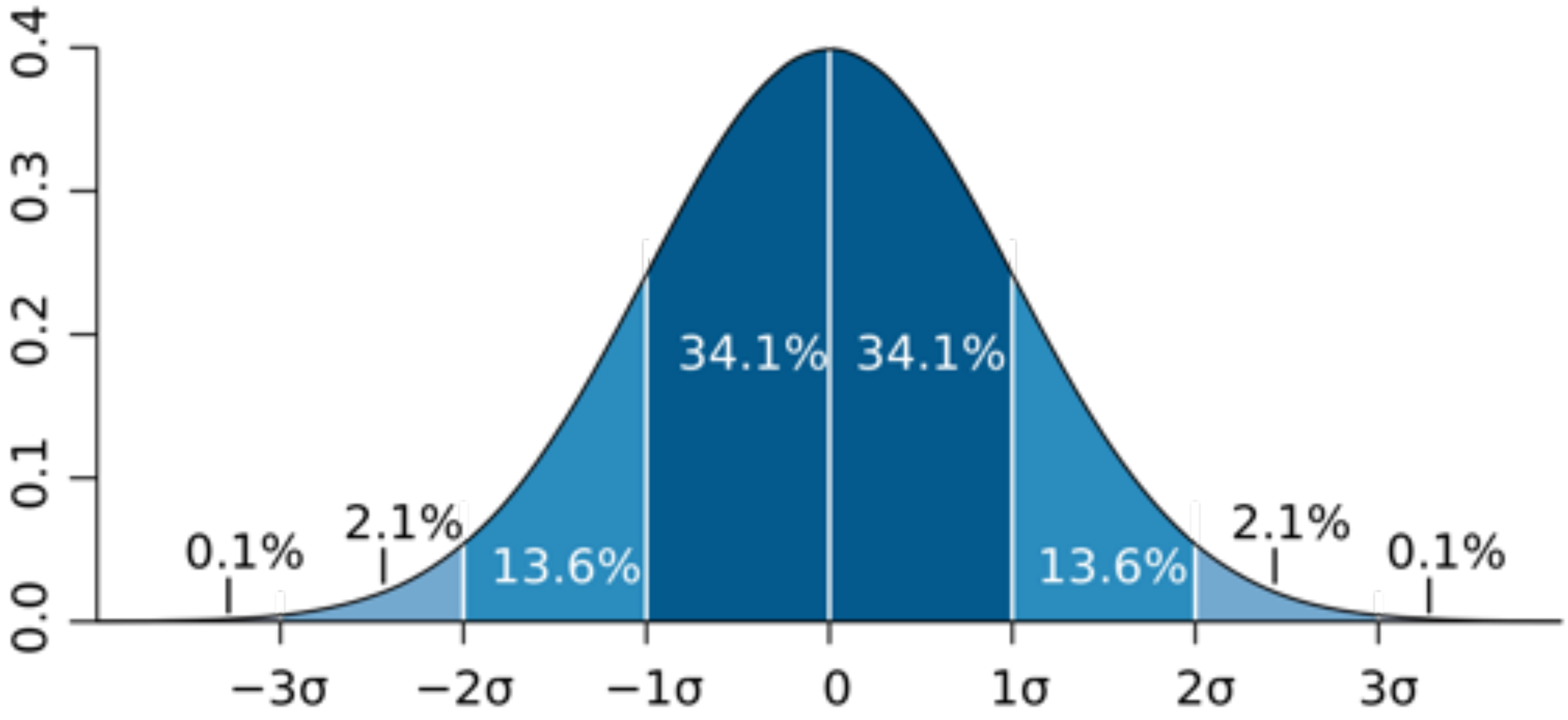
False alarm rate (FAR) and false alarm probability (FAP)

- Null hypothesis: dataset has no GW signal
- If the null hypothesis is true, the rate at which our detectors produce noise triggers of equal to or higher ranking statistics is the FAR (in units of 1/time)
- False alarm probability: takes into account how long we were observing, multiply FAR by observing time.

From FAP to a “sigma” statement



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