Section 8: Repeated games

Econ C110 / PoliSci C135

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In this section, we will deal with repeated games. First, we'll define what a repeated game is. Then, we will calculate the payoffs to repeated strategy profiles. Finally, we use the Nash Folk Theorem to find NE of repeated games.

1 Introduction

For most of this class so far, we have been studying games that are played once, called a stage game or one-shot game. In **repeated games**, players interact by playing the same stage game multiple times, possibly infinitely.

Definition 1. A **repeated game** is an extensive form game consisting of repetitions of the **stage game** (or "**one-shot game**"). Payoffs each period are discounted with with a discount factor $\delta \in [0, 1)$.

Note. δ tells me how much I value next period's payoffs compared to this period. E.g., $\delta = 0.9$ means tomorrow's payoff is worth 90% of today's. (What does $\delta = 0$ mean?)

Example 1. Repeated prisoner's dilemma. Consider the following game, repeated infinite times, with a discount factor $\delta \in [0, 1)$.

Player 2

Player 1		L	R
	T	4,4	1,6
	B	6,1	2,2

2 Payoff streams

In repeated games, we need to calculate the value of a stream of payoffs. (In finance, this is often called **present value**). The most important formula is the **geometric series**.

Recall. Let

$$\begin{aligned} S &= 1 + \delta + \delta^2 + \cdots \\ &= \sum_{n=0}^{\infty} \delta^n \\ &= \frac{1}{1-\delta} \end{aligned}$$

Most other sums (including finite sums) can be calculated by manipulating this formula.

Exercise 1. Calculate the (present) value to the following outcomes for player 1 in the repeated prisoner's dilemma (in terms of δ):

1. (B, R) for all periods
2. (T, L) for the first period, then (B, R) every period after that.
3. $(B, L), (T, R), (B, L), (T, R), \ldots$
4. (B, R) for 100 periods.

3 Strategies and NE of repeated games

Definition 2. Player 1's **minmax strategy** is given by

$$\min_{s_1 \in S_1} \left\{ \max_{s_2 \in S_2} U_2 (s_1, s_2) \right\}$$

In words, this is the lowest payoff that I can force onto the other player. (Player 1 is minimizing player 2's maximum utility!) This is without regard to my own payoffs!

Definition 3. A strategy profile is **individually rational** if *every* player gets *strictly* more than her minmax payoff. (Note that player 1's minmax payoff is the payoff she gets from best responding to player 2's minmax strategy).

Theorem (Nash Folk Theorem). Let $\hat{s}$ be an individually rational strategy profile of a stage game. There is a Nash equilibrium of the infinitely repeated game in which the players play $\hat{s}$ in every period, as long as δ is high enough.

"Proof". The intuition is that I can threaten to play my minmax strategy forever if the other player doesn't cooperate on $\hat{s}$. This is called a **grim trigger strategy**. If the other player "betrays" me even once, I'll minimize her payoff from the next period onwards. As long as the other player is patient enough, this will be a deterrent against deviating. So when finding a Nash equilibrium that leads to cooperation on $\hat{s}$ every round, we try strategies like this:

- Player 1 plays $\hat{s}_1$ in the first round. In round $n > 1$, if $\hat{s}$ was the outcome in all $n - 1$ previous rounds, then player 1 plays $\hat{s}_1$ in round n. If not, then Player 1 plays the minmax strategy in n and all rounds after.

- Player 2 plays $\hat{s}_2$ in the first round. In round $n > 1$, if $\hat{s}$ was the outcome in all $n - 1$ previous rounds, then player 2 plays $\hat{s}_2$ in round n. If not, then Player 2 plays the minmax strategy in n and all rounds after.

This is a NE for high enough δ. To check what δ are valid, we calculate the payoffs of the grim trigger strategy and of the best possible deviation.

Note. A strategy in a repeated game has to specify a response *in every period* and *to every possibility.* The grim trigger strategy specifies every period (first round and $n > 1$) and every possibility (after the first round, continue $\hat{s}_1$ if player 2 played $\hat{s}_2$ last round; otherwise, play the minmax). There are other common strategies, like **tit-for-tat**, that we don't have time to cover here.

Exercise 2. Is the grim trigger strategy subgame perfect (in general)?

Exercise 3. Consider the repeated prisoner's dilemma (example 1, reprinted here) with some $\delta \in (0, 1)$.

		Player 2	
		L	R
Player 1	T	4,4	1,6
	B	6,1	2,2

1. What is player 1's minmax strategy in the stage game?
2. Is (B, R) in every period the outcome of a Nash equilibrium of the repeated game?
3. Is the following strategy profile a Nash equilibrium for some δ? Player 1 plays T every period, Player 2 plays L every period. If not, find a profitable deviation (and calculate the payoff).
4. Find a NE strategy profile in which the outcome is (T, L) every period for some δ. What values of δ can support this?

Exercise 4. Consider the following game:

Player 1 \ Player 2	W	X	Y	Z
A	6,6	-1, 10	-2,5	-5,-3
B	10,-1	3,3	-1,4	-1,-3
C	5,-2	4,-1	0,0	-3,-3
D	-3,-5	-3,-4	-3,-3	-2,-2

1. What are the minmax strategies of the stage game?
2. What are the individually rational strategy profiles of the stage game?
3. Construct an equilibrium that results in (B, X) every period by using grim trigger strategies. What values of δ can support this?