Section 3: Mixed Nash Equilibrium

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1 Objectives

- Calculate expected utility
- Understand how to specify and interpret mixed NE
- Understand how the verbal and formula definitions of mixed NE are the same
- Find mixed NE in formal games
- Find mixed NE in formal games with "degenerate" components

2 Expected (von Neumann-Morgenstern) Utility

Suppose we have a risky lottery. How do we value this? In standard game theory, we assume the players have **expected utility**, sometimes called **von Neumann-Morgenstern (vNM) utility**.

Definition 1. Suppose a lottery has potential outcomes x_i, with probabilities $p(x_i)$, and utility from the outcomes $u(x_i)$. **Expected utility** is given by:

$$
\begin{aligned}
U(lottery) &= EU(lottery) \\
&= \sum_i p(x_i)u(x_i)
\end{aligned}
$$

Example 1. Let $L1$ be a lottery with $\frac{1}{2}$ chance of \$25 and $\frac{1}{2}$ chance of \$36. Suppose utility for money is linear, so $u(m) = m$. Then the expected utility of the lottery is

$$
\begin{aligned}
EU(L1) &= \frac{1}{2} \times 25 + \frac{1}{2} \times 36 \\
&= 30.5
\end{aligned}
$$

3 Mixed Nash Equilibria

We now let players choose "mixed" strategies, in which they can "randomize" which strategy to take. This is often the best thing to do (for example, in rock paper scissors – think how badly you'd do if you always played the same strategy!).

Definition 2. A game consists of of players N, actions for each player (S_i), and utilities (u_i). A **mixed strategy** σ_i is a probability distribution over S_i. Notation: $\Delta(S_i)$ is called a "simplex over S_i". Formally, let $\Delta(S_i)$ be the set of possible probability distributions over S_i; a mixed strategy is an element $\sigma_i \in \Delta(S_i)$. Note that a mixed strategy could also be a pure strategy (a probability distribution with 1 on a single outcome is included).

Definition 3. A **mixed strategy NE** is a (mixed) strategy profile in which every player is playing a (mixed) best response to other players' strategies. I.e., the same as a Nash equilibrium, except that players may have mixed strategies. Formally, let

$$U_i(\sigma) = \sum_{a_i \in A_i} \sigma_i\left(a_i\right) U_i\left(a_i, \sigma_{-i}\right)$$

where $\sigma_i(a_i)$ is the probability of playing a_i. I.e.this is the expected utility for player i of the strategy profile σ. Then σ^* is a mixed NE if

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*)$$

where σ_i is any other mixed strategy for player i.

3.1 Finding mixed NE

We'll find a mixed NE step-by-step using an example.

Example 2. BoS (Bach or Stravinsky)

	L	R
T	2,1	0,0
B	0,0	1,2

Let's consider finding a BR from player 1's perspective.

1. Assume player 2 plays L with probability q and R with $1 - q$. Determine player 1's expected utility from each of her strategies, given player 2's strategy.

2. Determine player 1's best response in terms of q.

3. Write this out in terms of playing T with probability p and B with $1 - p$.

   Note that $p = 1$ means player 1 plays T with certainty (no randomization). Likewise, $p = 0$ means she plays B with certainty. If player 2 is playing $q = 1/3$, then player 1 is indifferent – this is the only case when she might mix. (Why?)

4. Repeat the same steps for player 2.

5. Where do they best respond to each other? Draw the BR graphs.
   In terms of (p, q), the Nash Equilibria are where the graphs intersect: $(1, 1), (0, 0), (2/3, 1/3)$.
   Note that the first two are the pure strategy NE: (T, L) and (B, R).

3.2 Exercises

Find the mixed strategy NE of the following games:[1]

Exercise 2.1 Matching pennies.

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Exercise 2.2

	L	R
T	6,0	0,6
B	3,2	6,0

Exercise 2.3

	L	R
T	0,1	0,2
B	2,2	0,1

[1] Exercises are from Osborne (2004).