1 Preliminaries

This is a supplement to the section 8 notes. Please let me know ASAP if you find or suspect a mistake (email atai1@berkeley.edu).

2 Finding NE in infinitely repeated games

To find Nash equilibria in infinitely repeated games, we invoke the **Nash folk theorem**. This tells us for any individually rational (IR) outcome $\hat{s}$ of the stage game, there is a NE that *results in the players taking these actions.* This does not mean that the strategies are just to play the IR actions. They involve some threat – this is where the grim trigger strategy comes in. Here's how we find the NE in infinitely repeated games:

1. Find the minmax payoffs. For each player, find the strategy that minimizes the maximum outcome of the other player. This is the most that this player can punish the other player for deviating!
2. Find the IR outcomes. These are the outcomes in which the payoffs are strictly higher than the minmax payoff for both players.
3. Write the grim trigger strategies. It usually goes like this:
   (a) In the first period, player 1 plays the IR outcome $\hat{s}_1$.
   (b) In the 2nd period onward, player 1 plays $\hat{s}_1$ if the history was $\hat{s}$ in all prior periods. If the history was something else, then play the minmax.
4. Check how high δ has to be for this to be a NE. For **each** player:
   (a) Calculate the payoff for the equilibrium (the IR outcome)
   (b) Calculate the payoff for deviating. The player gets an immediate gain, and then has to accept his minmax payoff afterward (i.e., player 1 deviates and gets an immediate gain in the first period. Then player 2 starts playing his minmax in period 2 onwards, and player 1gets this payoff forever)
   (c) Check which δ makes (a) higher than (b).

3 Finding SPNE in infinitely repeated games

Finding SPNE is similar, but we need to make sure threats are credible. The grim trigger strategy is generally not credible (why?), and so it's not SPNE. The Nash folk theorem for SPNE is a slight alteration. Let $\hat{s}$ be an outcome with strictly higher payoffs than a NE of the stage game (which always exists, but it may not be pure). There is a SPNE that results in $\hat{s}$. The process for finding these SPNE is similar:

1. Find the NE of the stage game. **If there are multiple NE, which ones do we care about?** The lowest payoff NEs give us the biggest range for δ (why?). There may be no single NE that's lowest for both players, so you'd have to check this for multiple to see which gives the widest range for δ!
2. Write the modified "grim trigger" strategies:
   (a) In the first period, player 1 plays the better-than-NE strategy $\hat{s}_1$. Pick the worst NE (or pick one and do the rest of this process, then check the other one)
   (b) In the 2nd period onward, player 1 plays $\hat{s}_1$ if the history was $\hat{s}$ in all prior periods. If the history was something else, then play the NE from step (a).
3. Check how high δ has to be for this to be a SPNE. This process is the same as before.

4 Example

Exercise 3. Consider the following game, repeated infinite times, with a discount rate $\delta \in [0, 1)$.

Player 2

		W	X	Y	Z
Player 1	A	6,6	-1, 10	-2,5	-5,-3
	B	10,-1	3,3	-1,4	-1,-3
	C	5,-2	4,-1	0,0	-3,-3
	D	-3,-5	-3,-4	-3,-3	-2,-2

4.1 Finding a NE

First, find the minmax strategies.

1. Consider player 1's strategies. If he plays A, then 2's max is 10. If he plays B, then 2's maximum is 4. If he plays C, then 2's maximum is 0. If he plays D, then 2's maximum is -2. So player 1's minmax strategy is D. Player 2's minmax payoff is -2.
2. Consider player 2's strategies. If he plays W, then 1's max is 10. If he plays X, then 1's max is 4. If he plays Y, then 1's max is 0. If he plays Z, then 1's max is -1. So player 2's minmax strategy is Z. Player 1's minmax payoff is -1.

The Nash folk theorem says that any outcome higher than $(-1, -2)$ can be sustained in a NE, as long as δ is high enough. So (A, W), (B, X), (C, Y), (B, W), and (C, X) can be sustained as NE. [1]

[1] I can't guarantee I didn't make a mistake here

Let's find the NE that results in (B, X) every round. The strategies are:

- Player 1 plays B in the first round.
- In the second round and onwards, player 1 plays B as long as the history was (B, X) in every past round. Otherwise, play D.
- Player 2's strategy is analogous (except change D for Z).

For what δ is this a NE?

1. Consider player 1.
   (a) In the equilibrium, he gets $3 + 3\delta + 3\delta^2 + \cdots = \frac{3}{1-\delta}$.
   (b) What if he deviates? He can get 4 immediately from playing C. Afterwards, player 2 will play the minmax strategy of Z. Then player 1 can get at most -1. So player 1 will get

$$4 - 1\delta - 1\delta^2 - \cdots = 4 - \frac{\delta}{1-\delta}$$

   (c) How high does δ have to be to make sure player 1 doesn't deviate?

$$\begin{aligned} \frac{3}{1-\delta} &\geq 4 - \frac{\delta}{1-\delta} \\ 3 &\geq 4 - 4\delta - \delta \\ 5\delta &\geq 1 \\ \delta &\geq 1/5 \end{aligned}$$

2. Consider player 2.
   (a) In the equilibrium, he also gets $\frac{3}{1-\delta}$.
   (b) If he deviates, he can get 4 by playing Y. Then player 1 will minmax him, so he can get at most -2. So this is worth

$$4 - 2\delta - 2\delta^2 - \cdots = 4 - \frac{2\delta}{1-\delta}$$

   (c) How high does δ have to be to make sure player 2 doesn't deviate? We could stop here, because I know that it will be lower than player 1's δ. (why?). But let's finish it out anyway:

$$\begin{aligned} \frac{3}{1-\delta} &\geq 4 - \frac{2\delta}{1-\delta} \\ 3 &\geq 4 - 4\delta - 2\delta \\ 6\delta &\geq 1 \\ \delta &\geq 1/6 \end{aligned}$$

So $\delta \geq 1/5$ makes the above a NE.

4.2 Finding a SPNE

Now let's find a SPNE.

First, find the NE of the stage game. We actually already did this implicitly when we were looking for the minmax. The pure NE is (C, Y), which has payoffs $(0, 0)$. The Nash folk theorem for SPNE says that any outcome better than $(0, 0)$ can be sustained in a SPNE. So (A, W) and (B, X) are possible.

Let's find the SPNE that results in (B, X). The strategies are:

- Player 1 plays B in the first round.
- In the second round and onwards, player 1 plays B as long as the history was (B, X) in every past round. Otherwise, play C.
- Player 2 is analogous (switch out C for Y)

Note that the "threat" is now changed to play the NE instead of the minmax. (Why is this subgame perfect now?)

For what δ is this a SPNE?

1. Consider player 1.
   (a) As before, in equilibrium he gets $\frac{3}{1-\delta}$.
   (b) If he deviates, he can get 4 immediately, but now he is punished by player 2 with the NE, so he gets 0 afterwards. So this is worth 4.
   (c) He won't deviate as long as

$$\begin{aligned} \frac{3}{1-\delta} &\geq 4 \\ 3 &\geq 4 - 4\delta \\ \delta &\geq 1/4 \end{aligned}$$

2. Consider player 2. This time, the payoffs are symmetric, so the payoffs from equilibrium and deviating are the same as player 1's.

So $\delta \geq 1/4$ sustains the SPNE. **Note that this is higher than the NE we found.** (Why?)

Exercise. Find a SPNE in which (A, W) is the outcome in every period. What δ makes this possible?