Section 6: Repeated games

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GSI: Andrew Tai

1 Objectives

- Apply the geometric series to calculate streams of payoffs
- Specify strategies in repeated games, including grim trigger strategies
- Find NE of repeated games and required conditions on δ

2 Introduction

For most of this class so far, we have been studying games that are played once, called a stage game or one-shot game. In **repeated games**, players interact by playing the same stage game multiple times, possibly infinitely.

Definition 1. A **repeated game** is an extensive form game consisting of repetitions of the **stage game** (or "**one-shot game**"). Payoffs each period are discounted with with a discount factor $\delta \in [0, 1)$.

Note. δ tells me how much I value next period's payoffs compared to this period. E.g., $\delta = 0.9$ means tomorrow's payoff is worth 90% of today's. (What does $\delta = 0$ mean?)

Example 1. Repeated prisoner's dilemma. Consider the following game, repeated infinite times, with a discount factor $\delta \in (0, 1)$.

Player 1 \ Player 2	L	R
T	4,4	1,6
B	6,1	2,2

3 Payoff streams

In repeated games, we need to calculate the value of a stream of payoffs. (In finance, this is often called **present value**). The most important formula is the **geometric series**.

Recall. Let

$$\begin{aligned} S &= 1 + \delta + \delta^2 + \cdots \\ &= \sum_{n=0}^{\infty} \delta^n \\ &= \frac{1}{1-\delta} \end{aligned}$$

Most other sums (including finite sums) can be calculated by manipulating this formula.

Exercise 1. Calculate the (present) value to the following outcomes for player 1 in the repeated prisoner's dilemma (in terms of δ):

1. (B, R) for all periods
2. (T, L) for the first period, then (B, R) every period after that.
3. $(B, L), (T, R), (B, L), (T, R), \ldots$
4. (B, R) for 100 periods.

4 Strategies and NE of repeated games

Definition 2. Player 1's **minmax strategy** is given by

$$\min_{s_1 \in S_1} \left\{ \max_{s_2 \in S_2} U_2(s_1, s_2) \right\}$$

In words, this is the lowest payoff that I can force onto the other player. (Player 1 is minimizing player 2's maximum utility.) This is without regard to my own payoffs! Player 2's utility is his **minmax payoff**.

Theorem (Nash Folk Theorem). Let $\hat{s}$ be an strategy profile of a stage game in which both players get strictly more than his minmax payoff. There is a Nash equilibrium of the infinitely repeated game in which the players play $\hat{s}$ in every period, as long as δ is high enough.

Note. "Play $\hat{s}$" in every period is not the NE – it's the outcome/history (the strategies are more complex).

4.1 Constructing NE of repeated games

The intuition is that I can punish the player 2 by using the minmax strategy forever if he doesn't keep playing $\hat{s}$. This is called a **grim trigger strategy**. If the deviates even once, I'll minimize his payoff from the next period onwards. As long as the he's patient enough, this will be a deterrent. So when finding a Nash equilibrium that leads to $\hat{s}$ every round, we use strategies like this:

- Player 1 plays $\hat{s}_1$ in the first round. In round $n > 1$, if $\hat{s}$ was the outcome in all $n-1$ previous rounds, then player 1 plays $\hat{s}_1$ in round n. If not, then Player 1 plays the minmax strategy in n and all rounds after.
- Player 2 plays $\hat{s}_2$ in the first round. In round $n > 1$, if $\hat{s}$ was the outcome in all $n-1$ previous rounds, then player 2 plays $\hat{s}_2$ in round n. If not, then Player 2 plays the minmax strategy in n and all rounds after.

This is a NE for some high enough δ. In other words, if players are patient enough (they have high enough δ), this strategy profile is a NE. To check what δ are valid, we calculate the payoffs of the grim trigger strategy and of the best possible deviation.

Note. A strategy in a repeated game has to specify a response *in every period* and *to every possibility*. The grim trigger strategy specifies every period (first round and $n > 1$) and every possibility (after the first round, continue $\hat{s}_1$ if player 2 played $\hat{s}_2$ last round; otherwise, play the minmax). There are other common strategies, like **tit-for-tat**.

4.2 Step-by-step

1. Find the minmax payoffs. For each player, find the strategy that minimizes the maximum outcome of the other player. This is the most that this player can punish the other player for deviating! We can construct a NE for any strategy profile $\hat{s} = (\hat{s}_1, \hat{s}_2)$ with higher payoffs than the minmax payoffs.

2. Write the grim trigger strategies:

   (a) In the first period, player 1 plays the action $\hat{s}_1$.

   (b) In the 2nd period onward, player 1 plays $\hat{s}_1$ if player 2 played $\hat{s}_2$ in the previous period. Otherwise play the minmax in this and all future periods.

3. Check how high δ has to be for this to be a NE. For **each** player:

   (a) Calculate the payoff for the $\hat{s}$

   (b) Find the best possible deviation. Calculate the payoff for this. The player gets an immediate gain, and then has to accept his minmax payoff afterward (i.e., player 1 deviates and gets an immediate gain in the first period. Then player 2 starts playing his minmax in period 2 onwards, and player 1gets this payoff forever)

   (c) Check which δ makes (a) higher than (b). This means there is no profitable deviation.

If δ fulfills the condition in $4(c)$, then the strategy profile in 2 is a Nash equilibrium. **You should understand why we're doing each step.**

Exercise 2. Consider the repeated prisoner's dilemma (example 1, reprinted here) with some $\delta \in (0, 1)$.

		Player 2 C	Player 2 D
Player 1	C	4,4	1,6
Player 1	D	6,1	2,2

1. What is player 1's minmax strategy in the stage game?
2. Is (C, C) in every period the outcome of a Nash equilibrium of the repeated game?
3. Is the following strategy profile a Nash equilibrium for some δ? Player 1 plays C every period, Player 2 plays C every period. If not, find a profitable deviation (and calculate the payoff).
4. Find a NE strategy profile in which the outcome is (C, C) every period for some δ. What values of δ can support this?
5. What is the interpretation of (4)?

5 Other strategies

In general, to check whether a strategy profile is NE (or to find the δ that makes it NE):

1. Formally specify the strategies (if they aren't already). Recall: a strategy responds to every node and every possbility.
2. For each player, calculate the PV of the payoff stream for the proposed strategy.
3. For each player, find the best deviation from the proposed strategy (holding the other player fixed). Calculate the PV of the payoff stream for deviating for this player.
4. δ such that (2) $\geq$ (3) for both players makes the strategy profile in (1) NE. (Why?)

Exercise 3. (Prisoner's dilemma pt. 2). Consider the game in exercise 2. For what δ is $(\text{tit-for-tat}, \text{tit-for-tat})$ a NE?

Exercise 4. Consider the following game:

		Player 2			
		W	X	Y	Z
Player 1	A	6,6	-1, 10	-2,5	-5,-3
	B	10,-1	3,3	-1,4	-1,-3
	C	5,-2	4,-1	0,0	-3,-3
	D	-3,-5	-3,-4	-3,-3	-2,-2

1. What are the minmax strategies of the stage game?

2. Construct an equilibrium that results in (B, X) using grim trigger strategies. What values of δ can support this?