Section 11: Choice Under Uncertainty

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GSI: Andrew Tai

1 Objectives

- Understand the notation defining lotteries and compound lotteries
- Apply preferences to lotteries
- Prove simple results about preferences over lotteries

2 Lotteries

First, some notation for lotteries:

- **Consequences** $\mathcal{C} \subseteq \mathbb{R}^L$. Think of $\mathbb{R}^L$ as a bundle of goods. A particular consequence $x \in \mathcal{C}$ looks like $(x_1, x_2, ..., x_L)$ where each x_i is the amount of good i.
- Let there be N possible consequences, so $|\mathcal{C}| = N$. A **simple lottery** is a probability distrubution over the N consequences, $L = (p_1, ..., p_N)$. Each p_i is the probability of consequence x_i occurring. Of course, we need $\sum p_i = 1$. The set of all possible simple lotteries is $\mathcal{L}$; i.e., the set of all possible probability distributions over the N consequences.
- Consider K different lotteries, $\{L_1, ...L_K\}$. A **compound lottery** is a probability distribution over the K lotteries, $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$. Each α_i is the probability of lottery L_i occuring. Again, $\sum \alpha_i = 1$.

Think of the setup like this: you are in a room with a bunch of different prize wheels (the lotteries). Each prize wheel is associated with some possible prizes (the consequences). When you first enter a room, the attendant randomly assigns you to a prize wheel you get to spin. This whole situation is the compound lottery.

It's not hard to see that (if you can calculate probabilities correctly), you can reduce compound lotteries to simple lotteries.

Exercise 1. Let there be two goods, so $\mathcal{C} \subseteq \mathbb{R}^2$. Call them apples and oranges. Let there be three possible consequences:

$$\mathcal{C} = \{(1,0),(0,1),(-1,2)\}$$

Note that we are not yet defining any preference over these consequences. I.e., we don't have any preferences over bundles of apples and oranges. Consider a compound lottery over two simple lotteries,

$$\left\{(1,0,0),\left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right);\frac{2}{3},\frac{1}{3}\right\}$$

Write the reduced lottery.

3 Preferences over lotteries

We can also have preference relations over lotteries. For example, for the lotteries in exercise 1, we might have $(1,0,0) \succ \left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right)$. Again, note that we haven't defined any preference over the consequences yet!

Can we move from a preference over lotteries to a preference over outcomes? Yes, if these axioms are met:

Axiom 1. The preference relation $\succsim$ satisfies **continuity** if the upper and lower contour sets are closed.

Axiom 2. Let $L, L', L'' \in \mathcal{L}$ be lotteries. The preference relation $\succsim$ satisfies **independence** if

$$L \succsim L' \iff \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

Theorem. Van Neumann-Morgenstern expected utility. A preference relation $\succsim$ over lotteries $\mathcal{L}$ is rational, continuous, and independent if and only if it can be represented by a utility function

$$U(\pi) = \sum_{x \in \mathcal{C}} v(x)L(x)$$

where $v(x)$ is a utility function over outcomes and $L(x)$ is the probability of outcome x in lottery π.

Exercise 2. Let δ_x be the degenerate lottery giving outcome x for sure. Suppose $\mathcal{C}$ is finite, and $\succsim$ is an independent preference over $\mathcal{L}$. Prove the following: if there exist $L, L' \in \mathcal{L}$ such that $L \succ L'$, then there exist $x, y \in \mathcal{C}$ such that $\delta_x \succ \delta_y$. For simplicity, you may assume x is two-dimensional. **Hint**: note that $x \neq y$, so you can write any lottery as a convex combination of δ_x and δ_y.