Section 4: Pareto Efficiency and Equilibria of Dynamic Games

Econ 104, Spring 2021

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1 Objectives

- Define Pareto efficiency and identify Pareto efficient outcomes
- Explain why Pareto efficiency is a positive (not a normative) concept
- Understand the difference between Pareto efficiency and equilibrium
- Solve for NE of dynamic games
- Solve for SPNE of dynamic games using backward induction

2 Pareto efficiency

Definition 1. An allocation (outcome) is **Pareto efficient** (or **Pareto optimal**) if no agent can be made strictly better off without making another agent worse off. In other words, making an agent better off would *require* making another agent worse off. An allocation is **Pareto inefficient** if at least one agent can be made better off without making any agents worse off.

Table 1

	L	R
T	0,0	6,3
B	5,4	5,5

Example 1. (B,R) and (T,R) are both Pareto efficient.

- At (B,R), any other allocation makes player 2 worse off – we can improve player 1's outcome by moving to (T,R), but this makes player 2's outcome worse.
- At (T,R), any other allocation makes player 1 worse off.
- At (T,L), we can improve *both* players' outcomes by choosing any other outcome.
- At (B,L), we can improve player 2's outcome by moving to (B,R) without harming player 1.

Exercise 1. What are the Nash Equilibria of the following game? Which outcomes if any are Pareto efficient?

Table 4

	L	R
T	-1,-1	-3,0
B	0,-3	-2,-2

Example 2.

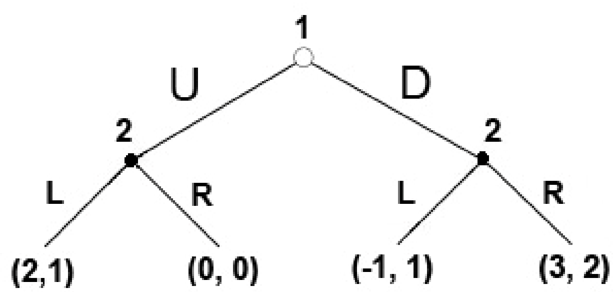
- Andrew and Zheng both like apples. Andrew's utility function is $u_{Andrew}(a) = 1000a$, where a is the number of apples he gets. Andrew really loves apples. Zheng's utility function is $u_{Zheng}(a) = a$. We have 100 apples to divide up between them. We give all 100 to Zheng. This is a Pareto efficient outcome (why?).
- We think up with an amazing new policy. If we implement it, it can double Jeff Bezos's wealth at no cost to anyone else. It is Pareto **in**efficient if we don't implement this.
  - But maybe it's bad for society if we do it. Again, Pareto efficiency is not a normative concept. It may not even be a good one sometimes.

3 Nash equilibrium of extensive form games

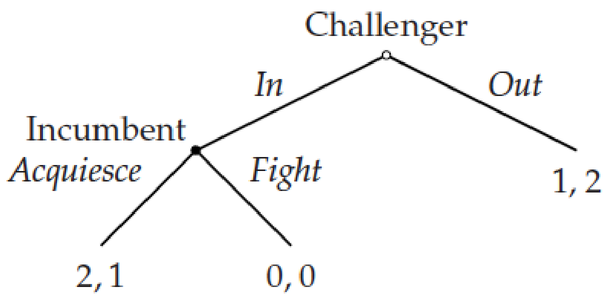
Definition 2. A **Nash equilibrium** of a dynamic game is a strategy profile in which all players are playing best responses to the other players' strategies. I.e., this is the same definition as in static games.

Proposition 1. The Nash equilibria of an extensive form game are the same as the NE of its static representation. (Why?) This includes the mixed strategy NE. To find the NE of an extensive form game, represent it in matrix form (with a row/column for each strategy).

Exercise 2. Consider the following extensive form game. What are the NE?



Exercise 3. (Osborne) Find the NE of the following game. Is there something unappealing about the NE?



4 Subgame perfect Nash equilibrium

Definition 3. A **subgame** is a collection of nodes and branches that satisfies three properties:

1. It starts at a single decision node.
2. It contains every alternative at this node and all nodes after this node.
3. It doesn't split up any information sets. (For now, we won't worry about information sets)

Note that the entire game is a subgame.

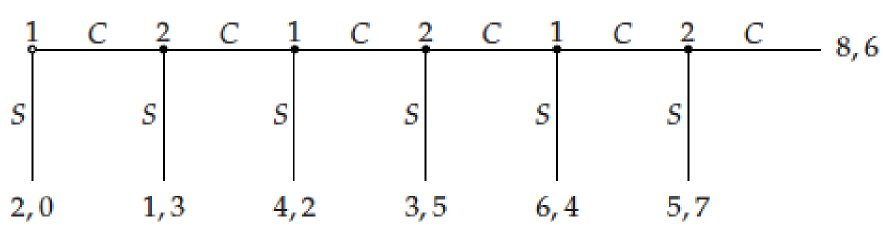
Definition 4. A **subgame perfect Nash equilibrium (SPNE or SPE)** is an equilibrium in an extensive form game. The strategy profile is a Nash equilibrium in every subgame, even ones that aren't reached. Subgame perfect Nash equilibria are a subset of Nash equilibria.

4.1 Finding SPNE

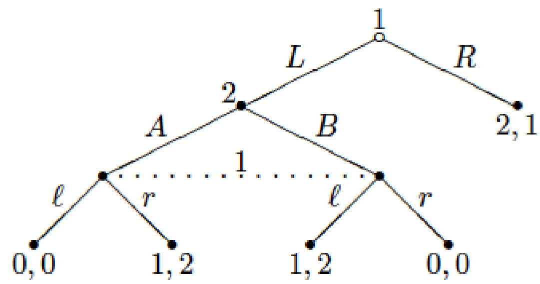
There are two ways of finding SPNE.

1. In a finite game of perfect information, solve via backward induction. Start at the last node; what would the player do at this node? Given this, what would happen at the second to last node(s)? Etc.
2. Check all the Nash equilibria.
   (a) Find all the NE of the whole game.
   (b) Find all subgames of the extensive form game.
   (c) Find the NE of each subgame.
   (d) Check which NE of the whole game are also NE in every subgame. I.e., check that each players' strategies are consistent with the subgame NE.

Exercise 4. (Selten's Horse) Find the SPNE of the following game.



Exercise 5. (Osborne & Rubinstein) Find the subgames of the following game. Find the NE and SPNE.



Exercise 6. (Hungry lions)[1] There are n hungry lions in a hierarchical pack who encounter a prey. Lion 1 can choose either to eat the prey or not eat it. If he doesn't eat it, the prey escapes and the game ends. If he eats, he becomes fat and slow, and Lion 2 can eat Lion 1. If Lion 2 doesn't eat Lion 1, the game ends. If he eats Lion 1, then Lion 3 can eat Lion 2, and so on. Each lion prefers being full over being hungry over being eaten. Find the SPNE for a generic n number of lions. *Hint: Start with* $n = 1$ *and* $n = 2$.

[1] Osborne, 2004. Ex 202.1

Exercise 7. (Ultimatum game)[2] Two people are dividing a dollar. Player 1 makes an offer $x \in [0,1]$. Then player 2 either accepts or rejects the offer. If she accepts, then the players get $(1-x, x)$. If she rejects, then both players get nothing.

- What are the values of x that can be offered in a Nash equilibrium?
- Suppose x can only be in cents, so $x \in [0.00, 0.01, 0.02, ...0.99, 1.00]$. What is (are?) the SPNE?

[2]Osborne, 2004. Ex 183