Section 10: Producer Theory

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1 Objectives

- Define various production function concepts and solve for them
- Be comfortable working with production set notation
- Prove simple properties of production sets

2 Production functions

Definition 1. A **production function** $f : \mathbb{R}^n \to \mathbb{R}^m$ represents the maximum output (goods of dimension m) for a set of inputs (goods of dimension n)

- the **marginal product of input** i is $\frac{\partial f(x)}{\partial x_i}$
- the **technical rate of substitution** between goods i and j is $\frac{dx_j}{dx_i} = -\frac{\partial y/\partial x_i}{\partial y/\partial x_j}$
- the production function has **decreasing returns to scale** if $f(kx) < kf(x)$...analogously for increasing and constant

This is probably familiar from intro/intermediate micro. Either way, solving problems is very similar to consumer theory.

Example 1. Cobb Douglas production. Let $f(x) = x_1^{1/3} x_2^{2/3}$. To add specifics, let the production good be bookshelves, x_1 be planks, and x_2 be nails.

3 Production sets

Production sets are generalizations of production functions. They can show the same information, but also allow for more possibilities.

Definition 2. Consider an n good economy (any good might be an input, output, or both). A production set is a subset $Y \subseteq \mathbb{R}^n$ showing feasible production possibilities.

Example 2. Consider example 1. We can represent this as a production set as the following: $\left\{(-x_1, -x_2, x_3) : x_1, x_2 \in \mathbb{R}_+, x_3 = x_1^{1/3} x_2^{2/3}\right\}$, where the third good is bookshelves.

Example 3. Let x_4 be tables, and suppose we can also produce tables with the same materials: $x_4 = f(x) = x_1^{1/3} x_2^{2/3}$. Now the production set is

$$\left\{(-x_1, -x_2, x_3, x_4) : x_1, x_2 \in \mathbb{R}_+, x_3 + x_4 = x_1^{1/3} x_2^{2/3}\right\}$$

Now suppose we can also convert tables to shelves, and vice versa. Then the production set is

$$\left\{(-x_1, -x_2, x_3, x_4) : x_1, x_2 \in \mathbb{R}_+, x_3 + x_4 = x_1^{1/3} x_2^{2/3}\right\} \bigcup \{(0, 0, -x_3, x_4) : x_3 = -x_4\}$$

Definition 3. In the follwing, note that $y \in Y$ is multidimensional. $\geq, =$ apply to every dimension. A non-empty production set Y is satisfies...

- **closed** if it contains its boundary (this is the same as the definition for sets in general)
- **free disposal** if $\forall y \in Y$, if $y' \leq y$ then $y' \in Y$ (what's the intuition?)
- **no free lunch** if $\forall y \in Y$, if $y \geq 0$, then $y = 0$ (what's the intuition?)
- admits **inaction** if $0 \in Y$
- **non-decreasing returns to scale** if $\forall y \in Y$ and $\forall \alpha \geq 1$, $\alpha y \in Y$ (note: this is not the same definition as increasing returns to scale for production functions)
- **non-increasing returns to scale** if $\forall y \in Y$ and $\forall \alpha \in [0, 1]$, $\alpha y \in Y$
- **constant returns to scale** if $\forall y \in Y$ and $\forall \alpha \geq 0$, $\alpha y \in Y$
- **additivity** if $\forall y, y' \in Y, y + y' \in Y$
- **convexity** if $\alpha y + (1 - \alpha) y' \in Y$ (this is the same as the definition for sets in general)

Exercise 1. Suppose the firm has a production function $f : \mathbb{R}_+^n \to \mathbb{R}_+^m$. I.e., there are n inputs and m outputs. Then the production set is

$$Y = \left\{(-x, y) \in \mathbb{R}_-^n \times \mathbb{R}_+^m : y \leq f(x)\right\}$$

Suppose $f(\mathbf{0}) = \mathbf{0}$ and $f(x) \geq f(x')$ if $x \geq x'$.

- Prove Y satisfies no free lunch, admits inaction, and free disposal.
- Suppose $m = 1$, so there is only one output. Prove Y satisfies constant returns to scale if and only if f is homogenous degree 1 (i.e. $f(\alpha x) = \alpha f(x)$ for all $\alpha \geq 0$).