

Revealed Preferences of One-Sided Matching

Andrew Tai*
UC Berkeley

July 2023

[Click here for most updated version.](#)

Abstract

Consider the object allocation (one-sided matching) model of Shapley and Scarf (1974). When final allocations are observed but agents' preferences are unknown, when might the allocation be in the core? This is a one-sided analogue of the model in Echenique, Lee, Shum, and Yenmez (2013). I build a model in which the strict core is testable – an allocation is “rationalizable” if there is a preference profile putting it in the core. In this manner, I develop a theory of the revealed preferences of one-sided matching. I study rationalizability in both non-transferrable and transferrable utility settings. In the non-transferrable utility setting, an allocation is rationalizable if and only if: whenever agents with the same preferences are in the same potential trading cycle, they receive the same allocation. In the transferrable utility setting, an allocation is rationalizable if and only if: there exists a price vector supporting the allocation as a competitive equilibrium; or equivalently, it satisfies a cyclic monotonicity condition. The proofs leverage simple graph theory and combinatorial optimization and tie together classic theories of consumer demand revealed preferences and competitive equilibrium.

1 Introduction

Matching literature is typically concerned with the construction of optimal or stable matchings. Exemplified by the canonical work of Gale and Shapley (1962) and Shapley and Scarf (1974), the agents' preferences are given, and the objective is to find stable matchings. In

*I am grateful to David Ahn, Federico Echenique, Haluk Ergin, and Chris Shannon for their guidance. I also thank Yuichiro Kamada and Shachar Kariv for helpful comments. All mistakes are my own.

real world economic settings however, market outcomes are typically observed and agents' preferences are not. Samuelson (1938) lays out a theory of revealed preferences for consumer theory. In this view, utility to be purely theoretical (and not observed) and choices are real (and observed). Given the consumer's choices, when is a utility maximizing agent falsified? The axioms of revealed preference give answers.

Work by Echenique, Lee, Shum, and Yenmez (2013) deals with the revealed preferences of two-sided matching. The matching (the market outcome) is observed, and the agents' preferences are not – can the matching be stable? My paper builds on such work and deals with the revealed preferences of one-sided matching. Similarly in my setup, the final allocation¹ is observed, but not the agents' preferences. My objective is to develop a model of house allocation where stability is testable.²

The outline of my model is as follows: consider the indivisible object assignment problem with endowments. We have a set of agents, each endowed with an indivisible object (“house”). In the original setting of Shapley and Scarf, the observer sees all the agents' preferences. The goal is to match agents to objects such that no coalition of agents would prefer to re-arrange among themselves (the “core”).

Now suppose the observer sees the allocation, but not the agents' preferences. Under what conditions can the allocation be *rationalized* as in the core? That is – is there a preference profile that makes the allocation in the core? Analogously to Echenique et al., I consider an aggregate matchings model, with *types* of agents and types of houses. There is some number of each type of agent, and agents of the same type have common preferences. Each agent is endowed with a house. In this paper, I derive necessary and sufficient conditions for an allocation to be possibly in the core in both non-transferrable and transferrable utility settings. While it is restrictive to impose types, this or a similar modeling choice is necessary to give the problem testable content.

While this is of course highly abstract, it provides a framework for a theory of object allocation as a market in the spirit of revealed preferences. In a typical observable economic market, only the set of agents and the outcome is observed, while the preferences and the market “mechanism” are unobserved. We would nevertheless like to infer stability properties of the market and perhaps back out the preferences. Alternatively, there may be no mechanism at all. For example, Roth and Xing (1997) study decentralized matching

¹To distinguish from two-sided matchings, I will refer to the matchings as “allocations.”

²An allocation is “stable” when it is in the core. Informally, means no subset (“coalition”) of agents would rather break off and trade their objects between themselves.

for clinical psychologists. In this interpretation, I develop a theory to test stability when there is no particular matching process.

This paper seeks to build on recent work in the revealed preferences of matching theory. As mentioned above, Echenique et al. find conditions on stable aggregate matchings when preferences are not observed in both transferable and non-transferable utility settings. Earlier work by Echenique (2008) finds conditions for stability when multiple matchings are observed for the same set of agents. I elaborate more on this in the next subsection.

There are two other ways to interpret this paper. Observers may deal with settings where the mechanism is unknown and therefore cannot be directly evaluated. In practice, many mechanisms are hidden, or no particular centralized mechanism is used at all. But we nevertheless want to determine whether these unknown mechanisms might be stable. Grigoryan and Möller (2023) develop a theory of “auditability”, where mechanism implementers may deviate for various reasons; auditability measures how much information the participants need to detect a deviation. This paper offers a way to evaluate mechanisms when essentially nothing is known about the matching process, but the analyst still wants to determine whether the allocation is may be stable.

Viewed another way, this paper provides a partial identification result for a one-sided matching model. Given an allocation presumed to be stable, I find a set of possible preference profiles. In a model with transferable utility, Choo and Siow (2006) studied aggregate matchings empirically in the marriage market. In the non-transferable utility case, analysts can use intermediate matching data to recover the agents’ preferences; Hitsch, Hortaçsu, and Ariely (2010) use rejections in online dating. Recent work by Galichon, Kominers, and Weber (2019) develops an intermediate case, where utility is imperfectly transferable. See Chiappori and Salanié (2016) for a survey of the econometrics of matching.

As will be apparent once the model is formally introduced, this paper is tied to the the object allocation problem with indifferences. Quint and Wako (2004) find the strict core of the Shapley-Scarf economy with indifferences. Alcalde-Unzu and Molis (2011) find Pareto efficient weak core allocations when the strict core is empty.

1.1 Revealed preferences

What do I mean by revealed preferences? A theoretical model is proposed, but it is not fully observable. Instead, the analyst observes data that may falsify the model, typically via some implication of the model. In classic consumer demand revealed preferences laid out by Samuelson, preferences or utility functions are purely theoretical – they cannot be

observed through any finite data. More philosophically, they may not even exist. However, observed choice data can be used to falsify a consumer being a utility-maximizing agent.

In the context of matching, the process may be unobserved or even unobservable, as in the case where there is no mechanism for trading. Likewise, participants' preferences may also be unobserved (or simply not exist). Then the idea of a stable matching is also not observable or may be purely theoretical. In the same way as revealed preferences of consumer demand asks whether an agent might be a utility maximizer, the revealed preferences of matching asks whether such a matching market might arrive at a stable allocation. I See Chambers and Echenique (2016) for a fuller discussion.

2 Model

The model is a object allocation analogue to Echenique, Lee, Shum, and Yenmez. The general setup is that of the Shapley-Scarf model, where agents are grouped into types. There are types of agents, and all agents within a type share the same preferences. Denote the set of agent types as $A = \{1, 2, \dots, \alpha\}$. Denote the number of each type $\mathbf{K} = (K_i)_{i \in \{1, \dots, \alpha\}}$ and the set of individual agents $\mathcal{A} = \{1_1, \dots, 1_{K_1}; \dots; \alpha_1, \dots, \alpha_{K_\alpha}\}$; I refer to these as “agents” or “individuals”. I refer to a generic agent type as i and a generic individual as ik . In examples and proofs when I refer to specific individuals, I will denote them as $1a, 1b, \dots, 2a, 2b, \dots$ and so on. That is, $1a$ and $1b$ are two individuals of the same type.

Analogously to the agents, let $H = \{h_1, \dots, h_\eta\}$ be the set of house types. I refer to a generic house type as h . I will not refer to individual houses – i.e., there is no house analogue of \mathcal{A} .

Each agent type i has a strict preference \succsim_i over H ; all agents of this type have the same preference. I will illustrate in the next section that this is the key restriction that makes the strict core testable. Let \succsim be the preference profile; with minimal consequence of confusion, this can be the profile of types or all individuals.

Each agent is endowed with a house, given by the endowment function $\mu^E : \mathcal{A} \rightarrow H$. An allocation is $\mu : \mathcal{A} \rightarrow H$ such that $|\mu^{-1}(h)| = |(\mu^E)^{-1}(h)|$ for all $h \in H$. That is, the number of agents allocated to h (demand) is the same as the number of agents endowed with it (supply). Individuals of the same type may have different endowments and allocations. Table 1 summarizes the notation. Given a subset of agents $A' \subseteq \mathcal{A}$, a sub-allocation μ' is an allocation among A' , $\mu' : A' \rightarrow H$ such that $|\mu'^{-1}(h)| = |(\mu^E)^{-1}(h) \cap A'|$.

Definition 1. An allocation μ is in the **(strict) core** for a preference profile \succsim , denoted

Table 1: Notation		
Object	Notation	Generic member
Agent types	A	i
Individuals/agents	\mathcal{A}	$ik; 1a, 2a, \dots (A-1)a, Aa$
Counters of agent types	$\mathbf{K} = (K_i)$	
Houses	H	h, h'
Endowment	$\mu^E(\cdot)$	
Allocation	$\mu(\cdot)$	

$\mu \in \text{core}(\succsim)$, if there is no blocking coalition $A' \subseteq \mathcal{A}$ and sub-allocation μ' such that:

1. For each $h \in H$, $\left| (\mu')^{-1}(h) \right| = \left| (\mu^E)^{-1}(h) \cap A' \right|$. That is, the number of each house required in the coalition is equal to the number endowed in the coalition.
2. $\mu'(ik) \succsim_i \mu(ik)$ for all $ik \in A'$, and $\mu'(ik) \succ_i \mu(ik)$ for at least one $ik \in A'$

By convention, when a blocking coalition A' is one individual, I say μ is not individually rational. A blocking coalition of one individual means he prefers his endowment to the allocation under μ .

2.1 Top Trading Cycles

I briefly describe the Shapley and Scarf setting and their algorithm (from David Gale), Top Trading Cycles (TTC). Familiar readers may skip this section. There are no agent types, just individuals. Every individual is endowed with an indivisible house and has strict preferences over the whole set of houses. All this is observed by the analyst. Every individual seeks exactly one house (they have no use for more than one.) The goal is to find the strict core allocation, meaning that no subset of individuals would prefer to rearrange their endowments among themselves. This includes single individuals, so the allocation must give a participant a house weakly preferred to his endowment. Equivalently, all Pareto improving trades have already been executed. As a side note, a competitive equilibrium allocations are in the strict core (though the converse is not true generally).

The TTC algorithm finds a strict core allocation, which furthermore always exists and is unique when preferences are strict. Informally, the algorithm is the following:

1. Draw a graph as follows: each agent is a vertex. Each agent points to the owner of his most preferred house.

2. There must be at least one cycle in this graph. In this cycle, implement the trades; i.e. each agent receives the house he points at. Remove these agents and houses.
3. If there are remaining agents, repeat from step 1.

Quint and Wako (2004), among others³, generalize the procedure to the setting where agents may have indifferences among the houses. The procedure is similar, looking for trading “segments”, then executing cycles within them if possible. The strict core may not exist and may not be unique when it does.

2.2 Rationalizability

Definition 2. Suppose we observe allocation μ , but not the agents’ preferences. A tuple $(A, \mathcal{A}, H, \mu^E, \mu)$ is a **problem**. Given a problem, the allocation μ is **rationalizable** if there exists a preference profile \succsim such that $\mu \in \text{core}(\succsim)$.

This paper derives necessary and sufficient conditions for an allocation to be rationalizable. This setting can be interpreted as the “reverse direction” of the classic house allocation problem with universally shared indifferences. That is, if we re-interpret this model in the typical positive direction, we have a market of house exchange where any indifferences are shared by all agents, and we would be looking for a TTC-like mechanism to find stable allocations. A number of papers deal with this; Quint and Wako (2004) is particularly important for this paper’s main result.

I have made an important modeling decision in restricting to common preferences within agent type. As noted above, this is necessary to give the problem testable content; with fully general preferences, any allocation is rationalizable.⁴ Suppose all agents are allowed to have unique preferences. Given an allocation μ , a preference profile such that each ik ’s favorite is $\mu(ik)$ rationalizes μ . Alternatively, suppose we allow for \succsim_i to have indifferences over H . Then making all agents indifferent over all houses rationalizes μ .

The reader might ask whether rationalizability is too weak a concept. The following simple example shows that this model indeed has testable content – there exist allocations that are not rationalizable.

Example 1. Let there be two agents of the same type endowed with different houses who trade. This is represented by the table below.

³See Aziz and Keijzer (2012); Alcalde-Unzu and Molis (2011); Jaramillo and Manjunath (2012) for examples.

⁴There are alternatives, such as repeated re-matchings as in Echenique (2008).

ik	$\mu(ik)$	$\mu^E(ik)$
$1a$	h_1	h_2
$1b$	h_2	h_1

This cannot be rationalized. Since $\mu(1a) = h_2$, we need $h_2 \succ_1 h_1$ for individual rationality. But then μ is not individually rational for $1b$.

Of course, a rationalizable allocation is not guaranteed to be in the core – it only *can* be. Should we check if allocations can be in the core under *all* preference profiles? The following proposition shows that this is too strict of a solution concept. For non-trivial problems, there always exists a preference profile \succsim such that $\mu \notin \text{core}(\succsim)$

Proposition 1. *Fix a problem $(A, \mathcal{A}, H, \mu^E, \mu)$ and let $|A|, |H| \geq 2$. For any allocation μ , there exists a preference profile \succsim such that $\mu \notin \text{core}(\succsim)$.*

Proof. Suppose there is an agent $1a$ who is not assigned to his own endowment. Let type 1's favorite be $\mu^E(1a)$. Then this allocation is not individually rational. Instead suppose all agents are assigned to their own endowments. Without loss of generality let $\mu^E(1a) = h_1 \neq h_2 = \mu^E(2a)$. Let 1's favorite be h_1 , and 2's favorite be h_2 . Then this allocation will not be in the core, as $1a$ and $2a$ form a blocking coalition. \square

3 Graphs

I first introduce some standard definitions for directed graphs that will be useful.

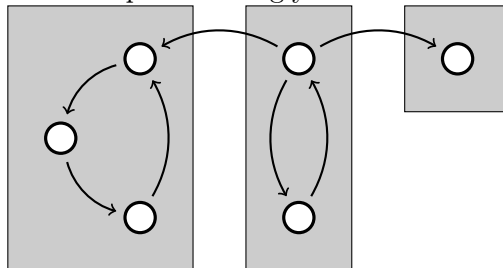
Definition 3. A **directed graph (digraph)** is $D = (V, E)$, where V is the set of vertices, and E is the set of arcs. An **arc** is an sequence of two vertices (v_i, v_j) ; here I allow for arcs of the form (v_i, v_i) , called a self-loop.⁵ A **(v_1, v_k) -path** is sequence of vertices (v_1, v_2, \dots, v_k) where each v_i is distinct, and $(v_{i-1}, v_i) \in E$ for each $i \in \{2, \dots, k\}$. A **cycle** is a path where $v_k = v_1$ is the only repeated vertex. I will also include self-loops as cycles. Equivalently, a path is a sequence of arcs $((v_1, v_2), \dots, (v_{k-1}, v_k))$, and a cycle is a path (in arcs) where $v_1 = v_k$. The **indegree** of a vertex $d^+(v_i) = |v_j : (v_j, v_i) \in E|$ is the number of arcs pointing at v_i . Likewise, the **outdegree** of a vertex $d^-(v_i) = |v_j : (v_i, v_j) \in E|$ is the number of arcs pointing from v_i .

The next definition is used in the main result and its discussion.

⁵This is more formally called a **directed pseudograph**.

Definition 4. A **strongly connected component (SCC)** of a digraph $D = (V, E)$ is a maximal set of vertices $S \subseteq V$ such that for all distinct vertices $v_i, v_j \in S$, there is a (v_i, v_j) -path and a (v_j, v_i) -path. By convention, there is always a path from v_i to itself, even if $(v_i, v_i) \notin E$; an isolated vertex is an SCC.

Figure 1: Example of strongly connected components



Every digraph can be uniquely partitioned (in vertices) into SCCs. An algorithm by Tarjan (1972) finds a partition in linear time, $O(|V| + |E|)$. Figure 1 illustrates a partition into SCCs.

I introduce a graph construction here that is important for the main result. Construct $\mathcal{G}^{big} = (\mathcal{A}, E)$ as follows: each individual is a vertex. Draw arcs from ik to *all* vertices $i'k'$ that are endowed with $\mu(ik)$. That is, let $(ik, i'k') \in E$ if $\mu(ik) = \mu^E(i'k')$. Let \mathcal{G}^{small} be a digraph representation of μ : draw one arc from ik to a vertex $i'k'$ endowed with $\mu(ik)$, such that every vertex has $d^+(ik) = d^-(ik) = 1$. An algorithm to construct this is in the appendix. From here, I will refer to a vertex alternatively as an agent or his endowed house, where context makes it clear.

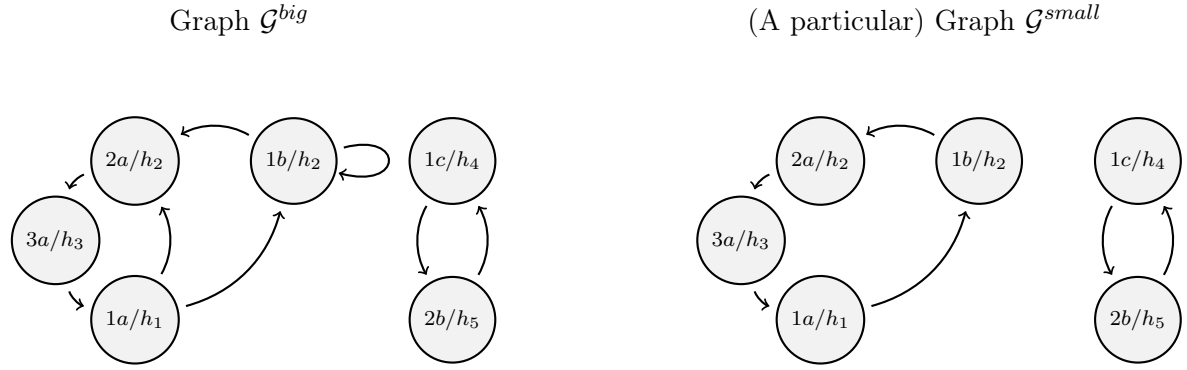
Example 2. Consider the (rationalizable) problem described below.

ik	$\mu^E(ik)$	$\mu(ik)$
1a	h_1	h_2
1b	h_2	h_2
1c	h_4	h_5
2a	h_2	h_3
2b	h_5	h_4
3a	h_3	h_1

That is, $\mu^E(1b) = \mu^E(2a)$, and other endowments are unique. The \mathcal{G}^{big} and an example

\mathcal{G}^{small} are given below in Figure 2.

Figure 2: Figure for Example 2



4 Rationalizability

I now give necessary and sufficient conditions for a problem to be rationalizable.

Theorem 1. *Fix a problem $(A, \mathcal{A}, H, \mu^E, \mu)$, and consider \mathcal{G}^{big} constructed from μ . The problem is rationalizable if and only if: for agents of the same type ik, ik' in the same SCC S , $\mu(ik) = \mu(ik')$. That is, if $ik, ik' \in S$ are the same type and in the same SCC, they receive the same house type.*

Proof. Appendix. □

Remark. I show in Corollary 6 in the appendix that two agents are in the same SCC in \mathcal{G}^{big} if and only if they are path connected. Then an equivalent statement is the following. The problem is rationalizable if and only if, for agents of the same type ik, ik' when there is a (ik, ik') -path, $\mu(ik) = \mu(ik')$.

The full proof is contained in the appendix. I give a sketch of the proof below.

Proof sketch of Theorem 1. To prove “if”: First, find the decomposition of \mathcal{G}^{big} into SCCs. Then assign an arbitrary order to the SCCs, and assign preferences in this order. That is, in the first SCC S_1 , let all agents’ allocated houses be their first preference. In S_2 , let all agents’ assigned houses be their first preference if possible, and the second preference if not.

The first key result is that all agents in the same SCC receive the same house type, so this is a well defined procedure at each step. The second key result is that all copies of the same house type are contained in the same SCC, so the procedure never attempts to “re-assign” a preference in a later step. The argument that this creates no blocking coalitions is similar to the argument behind Gale’s proof for TTC. To prove “only if”, I show that when the condition is violated, there is a blocking coalition for all preference profiles. \square

Example (Example 2 continued). The \mathcal{G}^{big} has two SCCs: the left component and the right component. To apply the theorem, select either arbitrary order. Let the left component be S_1 , and the right be S_2 .

1. In S_1 , assign all agents’ $\succsim_i(1) = \mu(i)$, so

i	$\succsim_i(1)$
1	h_2
2	h_3
3	h_1

2. In S_2 , assign all agents’ $\succsim_i(1) = \mu(i)$ if possible (this is not possible for anyone here). Otherwise, let $\succsim_i(2) = \mu(i)$.

i	$\succsim_i(2)$
1	h_5
2	h_4

3. Assign remaining preferences arbitrarily (omitted).

To check for a blocking coalition, note that S_1 all receive their favorite house. Only agents in S_2 are unsated. Then in any candidate blocking coalition (A', μ') , we require $\mu'(1c) = h_2$ or $\mu'(2b) = h_3$. This requires least one agent in $A' \cap S_1$ to receive either h_4 or h_5 , which are strictly dispreferred.

The condition required in Theorem 1 is easy to check; Tarjan’s algorithm finds the partition into SCCs in linear time. Within each SCC, checking for a non-repeated agent type-house type pair is linear in the number of agents.

I have said previously that this paper is tied to object allocation with indifferences. This is tightly connected to Quint and Wako (2004) in particular. Their paper establishes a mechanism to find core allocations through what they term “top trading segmentation.”

A segment is a smallest group of vertices whose neighbors are within the segment. SCCs coincide with segments in my setting, and my proof reverse engineers a partition of the agents into these segments when possible.

The most direct interpretation of Theorem 1 is this: whenever agents with the same preferences are in the same “potential trading cycle”, they receive the same house type. In a house exchange market, any allocation can be broken up into trading cycles, where members trade their endowed objects among themselves only. The graph \mathcal{G}^{small} gives one particular arrangement of potential trading cycles. In the classic Shapley and Scarf setting, the TTC algorithm identifies the “correct” trading cycles to use. In the present setting, there are may be many potential trading cycles representing an allocation. The graph \mathcal{G}^{big} can be interpreted as representing all of them. The SCCs are the largest potential trading cycles, and we can focus attention to these. In Example 2, $\{1a, 1b, 2a, 3a\}$ forms a trading cycle, and $\{1c, 2b\}$ forms another. The former could instead be broken up into $\{1a, 2a, 3a\}$ and $\{1b\}$; however, it is only necessary to consider the largest ones. Within these largest potential trading cycles, two agents with the same preferences must receive the same house type. Continuing the above example, $1a$ and $1b$ must receive the same house type.

There are two related interpretations of the condition in the theorem. The first is physical: starting with any house in an SCC, it is possible to make a series of exchanges to obtain any other house in the same SCC. Then the necessity that two agents of the same type in an SCC receive the same house type is immediate. The less well-off agent could execute these exchanges to receive the better house, thus blocking the allocation. The second interpretation is in the context of a competitive equilibrium market. Famously, Roth and Postlewaite (1977) show that the strict core is a competitive equilibrium in the typical house exchange setting with no indifferences. Wako (1983) establishes that a strict core allocation is also a competitive equilibrium in a the setting with indifferences. So if μ is rationalizable and thus $\mu \in \text{core}(\succsim)$ for some \succsim , it is also a competitive equilibrium.

Lemma 1. *If a problem $(A, \mathcal{A}, H, \mu^E, \mu)$ is rationalizable by a preference profile \succsim , then μ is a competitive equilibrium.*

Similarly to Gale’s TTC proof, the supporting prices are descending in order of SCCs. Thus if two agents are in the same SCC, their endowments are worth the same in competitive equilibrium. Again, the necessity of the condition becomes immediate. Two agents with the same budget and preferences should purchase the same house type.

I present some corollaries. First, an important implication of Theorem 1 is the following

corollary:

Corollary 1. *Fix a problem $(A, \mathcal{A}, H, \mu^E, \mu)$. The problem is rationalizable only if: whenever agents ik, ik' are the same type and $\mu^E(ik) = \mu^E(ik')$, $\mu(ik) = \mu(ik')$.*

Proof. Appendix. □

That is, equal agents (of same type and same endowment) must receive the same house type. Briefly, the theorem requires equal treatment of equals. When types determine both preferences and endowments, this corollary gives us the condition for rationalizability.

Corollary 2. *Suppose $\mu^E(ik) = \mu^E(ik')$ for all $k, k' \in \{1, \dots, K_i\}$ and for all $i \in A$. That is, all agents of the same type have the same endowment. Then the problem $(A, \mathcal{A}, H, \mu^E, \mu)$ is rationalizable if and only if $\mu(ik) = \mu(ik')$ for all $k, k' \in \{1, \dots, K_i\}$ and for all $i \in A$. That is, if and only if all agents of the same type receive the same house type.*

Proof. “Only if” is a consequence of Corollary 1. To prove “if”, note that everyone of the same type receives the same house type, so we can let everyone’s favorite house be their allocated house. □

This resembles the Debreu and Scarf (1963) theorems for general equilibrium. Their model is an endowment economy with a finite number of goods, agent types, k copies of each type, and certain restrictions on preferences. Only allocations assigning the same bundle to all agents of the same type are in the core. While neither the Debreu-Scarf model nor my model contains the other, it would be interesting future work to investigate a whether deeper connection exists.

Another related question is: what is the minimum number of agent types necessary to rationalize an allocation? That is, suppose we are free to choose agent types. What is the minimum preference type heterogeneity required to put μ in the core? This question is sensible, since allowing every individual to be his own type always rationalizes an allocation.

Corollary 3. *Consider \mathcal{G}^{big} constructed from μ , and decompose this into SCCs, $\{S_1, \dots, S_M\}$. Let α_m be the number of distinct house types in SCC_m . The minimum number of types necessary to construct \succsim such that $\mu \in \text{core}(\succsim)$ is $\alpha = \min\{\alpha_1, \dots, \alpha_m\}$.*

Proof. In light of Theorem 1, individuals in the same SCC_m who receive different house types must not be the same agent type. There is no other restriction on agent types. □

The result also solves the analogous problem for two-sided matching in the strict core. That is, it solves a strict core analogue of Echenique, Lee, Shum, and Yenmez (2013) with non-transferable utility. There are types of men and women, and each type has a strict preference over potential partner types. Here, an agent's endowment is him- or her- self. An easy way to see this in the “only if” direction is to let agents' endowments be themselves and apply Corollary 1. In this model, agents of the same type always have the same endowment. Then “equal treatment” means it is all agents of the same type must be assigned the same type of partner. To see this is also sufficient, we can let all agents' first preference be their assigned partner. These teases out a larger idea – two-sided matching can be seen as one-sided matching where trading cycles must be size 2.

5 Partially transferable utility model

I now present an analogous model with partially transferable utility. That is, the setting is now an exchange economy with indivisible houses and money. There is unsurprisingly a deep connection to competitive equilibrium in this setting. This will become even clearer in the main result and its proof.

First, I introduce some new notation. It will be helpful to re-express some existing objects differently. As before, let A be the set of agent types, \mathcal{A} be the set of all agents. Let $H = \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\} \subset \mathbb{R}^\eta$ be the standard basis vectors, representing the house types. For example, $h_1 := (1, 0, \dots, 0)$ represents house type 1. Let $e_{ik} \in H$ be a standard basis vector representing ik 's endowed house. Every agent is endowed with some amount of money, $\omega_{ik} \in \mathbb{R}_+$. Similarly, let $x_{ik} \in H$ be the allocated house vector and $m_{ik} \in \mathbb{R}_+$ be the allocated money. Note that agents are restricted to have weakly positive amounts of money. An allocation is $(x, m) = (x_{ik}, m_{ik})_{ik \in \mathcal{A}}$ such that $\sum_{ik \in \mathcal{A}} e_{ik} = \sum_{ik \in \mathcal{A}} x_{ik}$ and $\sum_{ik \in \mathcal{A}} m_{ik} = \sum_{ik \in \mathcal{A}} \omega_{ik}$.⁶

Each agent type has quasilinear utility over his allocated house and money $V : H \times \mathbb{R}_+ \rightarrow \mathbb{R}$ given by $V_i(h, m) = v_i(h) + m$. As before, all members ik of a type i have a common utility function. Note that the $v_i(\cdot)$ can be interpreted as a utility index over H ; that is, it is an η -dimensional vector of real numbers representing an cardinal ranking of houses.

As is typical when dealing with exchange economies with money, I will deal with the weak core.⁷

⁶Implicitly, there is no free disposal of houses or money, but we presume everyone's own endowed house is acceptable to him and that money is desirable.

⁷The weak core and strict core are equivalent in this setting, except where an individual in a blocking

Table 2: Notation for transferable utility

Object	Notation	Generic member
Agent types	A	i
Individuals/agents	\mathcal{A}	$ik; 1a, 2a, \dots (A-1)a, Aa$
Utility	$V_i : H \times \mathbb{R}_+ \rightarrow \mathbb{R}$	$V_i(h, m) = v_i(h) + m$
Counters of agent types	$\mathbf{K} = (K_i)$	
House types	H	h, h'
Endowment	(e, ω)	
Allocation	(x, m)	

Definition 5. An allocation (x, m) is in the weak core if there is no blocking coalition $A' \subseteq \mathcal{A}$ and sub-allocation $(x', m')|_{A'}$ such that:

1. $\sum_{ik \in A'} x'_{ik} = \sum_{ik \in A'} e_{ik}$ and $\sum_{ik \in A'} m'_{ik} \leq \sum_{ik \in A'} \omega_{ik}$
2. $V_i(x'_{ik}, m'_{ik}) > V_i(x_{ik}, m_{ik})$ for all $ik \in A'$

Definition 6. A **transferable utility (TU) problem** is $(A, \mathcal{A}, H, x, m, e, \omega)$. A TU problem is **TU-rationalizable** if there exist some utility indices v_i for all $i \in A$ such that (x, m) is in the weak core. A TU problem is **strictly TU-rationalizable** if it is TU-rationalizable with some strict utility indices; that is, $v_i(h) = v_i(h')$ if and only if $h = h'$.

The main result will deal with TU-rationalizability, so will not impose that the $v_i(\cdot)$ are strict over H . However, I will discuss afterwards how strict TU-rationalizability is an intuitive corollary of the main result.

Remark 1. Even when $v_i(\cdot)$ are not strict, it is not the case that all agents can be trivially indifferent between all allocations, since m_{ik} are observed (not choice variables) and may differ.

Table 2 presents the current notation. Before the main result for transferable utility, I add a few new graph definitions.

Definition 7. A **weighted directed graph** is a directed graph $D = (V, E, \ell(\cdot))$, where $\ell : E \rightarrow \mathbb{R}$ is the **length** (or **weight**) function over arcs. The length of a path or cycle (v_1, v_2, \dots, v_k) is $\sum_{i=1}^{k-1} \ell(v_i, v_{i+1})$.

coalition spends all his money. It can be argued as in Kaneko (1982) and Quinzii (1984) that money is a bundle of goods outside the model, and it is not “normal” to consume only one indivisible good.

I now define a new weighted graph $\mathcal{G}_{TU}^{big} = (\mathcal{A}, E, \ell(\cdot))$ analogous to \mathcal{G}^{big} . As before, each individual is a vertex, and add arcs from ik to all vertices $i'k'$ endowed with x_{ik} . That is, let $(ik, i'k') \in E$ if $x_{ik} = e_{i'k'}$. This is the same construction as in the previous section. Now additionally, define the lengths of arcs $\ell(ik, i'k') = \omega_{ik} - m_{ik}$. (Note that this does not depend on $i'k'$).

5.1 Rationalizability

I derive necessary and sufficient conditions for a TU problem to be TU-rationalizable.

Theorem 2. *Fix a TU problem $(A, \mathcal{A}, H, x, m, e, \omega)$. Assume*

$$V_i(e_{ik}, \omega_{ik}) \geq V_i(h, 0) \quad \forall i \in A, \forall ik \in \mathcal{A}, h \in H \quad (A1)$$

Then the following are equivalent:

1. *The problem is TU-rationalizable.*
2. *There exists a vector $p \in \mathbb{R}_+^{|H|}$ such that*

$$(x_{ik} - e_{ik}) \cdot p = \omega_{ik} - m_{ik} \quad \forall ik \in \mathcal{A} \quad (P)$$

3. *The graph \mathcal{G}_{TU}^{big} has no cycles with length > 0 .*

Proof. Appendix. □

Assumption (A1) ensures that the weak and strict core coincide, and allows use of a theorem by Quinzii (1984). It says that it is never strictly preferable to spend the entire endowment on a single object. This can be justified as in Quinzii and Kaneko (1982) – money is a composite of all other goods, and it is not well behaved for an individual to consume one indivisible good but nothing else. It is also a “knife-edge” regularity condition. With even an infinitesimal positive ε amount of money, we are allowed to have $V_i(e_{ik}, \omega_{ik}) < V_i(h, \varepsilon)$. Alternatively, we can restrict the observed allocation to have $m_{ik} > 0$ for all $ik \in \mathcal{A}$.

The vector p in (P) (suggestively denoted) is interpretable as a price vector for houses. Indeed, the left side is the difference in price between the allocated and endowed houses, and the right side is the net payment from ik . This suggests an easy interpretation of the theorem: a problem is TU-rationalizable if and only if everyone who “buys” a house type

pays the same price for it. The reader may be surprised that the agent types appear to impose nothing; the intuition behind this is clear from the proof. I present a proof sketch here.

Proof sketch of Theorem 2. First, I show $(1) \implies (2)$, suppose a p satisfying (P) does not exist. Then there is no price vector supporting the allocation as a competitive equilibrium. Quinzii shows that in this setting, the set of competitive equilibria is equal to the set of weak core allocations, so (x, m) cannot be in the weak core for any utility indices v_i .

I show $(2) \implies (1)$. Given p , I want to find v_i such that (x, m) is a competitive equilibrium, which will then give us weak core. We are looking for utility indices v_i such that all agents ik are maximizing subject to their budget constraints, given by $e'_{ik} \cdot p + \omega_{ik}$. Then this becomes a classic consumer demand revealed preference problem. To see this, reinterpret an agent type i as a single consumer, and each individual ik as a demand data point:

$$\left\{ \underbrace{(x_{ik}, m_{ik})}_{\text{consumed good and money}}, \underbrace{(e'_{ik} \cdot p + \omega_{ik})}_{\text{budget}}, \underbrace{p}_{\text{price}} \right\}$$

In this structure, such demand data are always rationalizable (in the consumer demand revealed preference sense). The easiest way to show this is to let $v_i(x_{ik}) = x'_{ik} \cdot p$ for all i, ik , though I show in the full proof this knife-edge construction is not the only one. Then (x, m) is a competitive equilibrium supported by p , and thus (x, m) is in the weak core. We now have $(1) \iff (2)$.

I now show $((1) \iff) (2) \implies (3)$. To see this, note that a cycle C 's length $\sum_{ik \in C} \omega_{ik} - m_{ik}$ is its members' total net payment of money. If this is greater than 0, then this cycle net spends money. Its members can form a blocking coalition – they can allocate houses the same way as in (x, m) , but keep their full endowed money for themselves.

Finally, to show $(3) \implies (2)$, I use the shortest path length on \mathcal{G}_{TU}^{big} between two houses to construct the price difference between those houses. (We can choose an arbitrary base price high enough so that $p \geq 0$.) In the full proof, I show that this construction is consistent – the minimum path length between houses of the same type is always 0. This completes the proof. \square

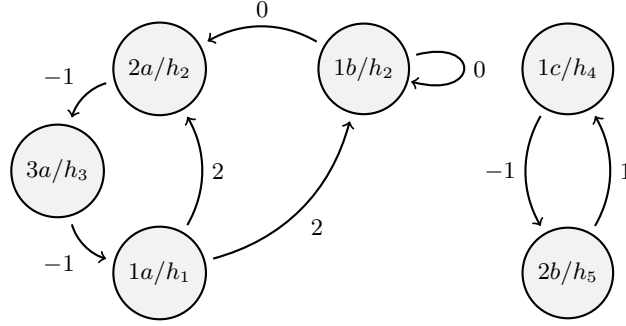
I give an example to illustrate TU-rationalizability.

Example 3. Consider the problem described in Example 2, adding the following payments:

\mathcal{A}	e_{ik}	x_{ik}	$\omega_{ik} - m_{ik}$
1a	h_1	h_2	2
1b	h_2	h_2	0
1c	h_4	h_5	1
2a	h_2	h_3	-1
2b	h_5	h_4	-1
3a	h_3	h_1	-1

For simplicity, let $\omega_{ik} = 3$ for all ik . It can be seen that all cycles have length 0, so this is rationalizable. Figure 3 shows the allocation, with $\omega_{ik} - m_{ik}$ as arc lengths.

Figure 3: Figure for Example 3, \mathcal{G}_{TU}^{big}



To construct utilities, set p the following way. In the left SCC, let $p_{h_1} = 3$ arbitrarily, and set the prices of other houses in this SCC by the minimum path length from h_2 plus 3, giving $p_{h_2} = 5, p_{h_3} = 4$. Notice that the path length between the two copies of h_1 is 0. In the right SCC, let $p_{h_4} = 1$ arbitrarily, and set $p_{h_5} = 2$ since the path length from h_4 to h_5 is 1. Altogether,

$$\begin{aligned}
 p_{h_1} &= 3 \\
 p_{h_2} &= 5 \\
 p_{h_3} &= 4 \\
 p_{h_4} &= 1 \\
 p_{h_5} &= 2
 \end{aligned}$$

The easiest way to construct Tu-rationalizing preferences is to let $v_i = p$ for all i . Though as mentioned above (and demonstrated in the full proof), this is not the only construction.

The theorem establishes a connection between TU-rationalizability, competitive equilibrium, and consumer demand rationalizability. The question of TU-rationalizability is equivalent to consumer demand rationalizability, à la Samuelson and Afriat. That is, an allocation is rationalizable if and only if each agent type, interpreted as demand data, is consumer demand rationalizable. That is, we are looking for utility indexes such that every agent type is optimizing in their demand. From here, it is a short hop to competitive equilibrium.

This yields the theorem's two equivalent and intuitive conditions for TU-rationalizability. The first condition is the existence of a price vector supporting the allocation as a competitive equilibrium. That is, an allocation is TU-rationalizable if and only if it can be supported as a competitive equilibrium. The second condition is reminiscent of cyclic monotonicity results common in revealed preference literature. It is also readily interpretable directly. A cycle having positive length means it net pays money. Then its members could implement the same house allocation while retaining more money, establishing a blocking coalition.

I now give some corollaries of Theorem 2. First, I give conditions for strict TU-rationalizability.

Corollary 4. *Fix a TU problem $(A, \mathcal{A}, H, x, m, e, \omega)$. Assume (A1). The problem is strictly TU-rationalizable if and only if both of the following are true:*

1. *The problem is TU-rationalizable.*
2. *If $ik, ik' \in S$ are the same type and in the same SCC in \mathcal{G}_{TU}^{big} , $x_{ik} = x_{ik'}$ OR the shortest path length from x_{ik} to $x_{ik'} \neq 0$.*

Proof. Appendix. □

This is the TU analogue to Theorem 1. The additional condition says that two individuals of the same type, in the same SCC, should either be allocated the same house or pay different amounts. This is because having a zero path length between x_{ik} and $x_{ik'}$ means their prices must be the same. Then if two different individuals type i purchase each one in competitive equilibrium, they must have the same utility. Conversely, having a nonzero path length allows us to construct different prices, and thus different utilities.

The following example illustrates the corollary.

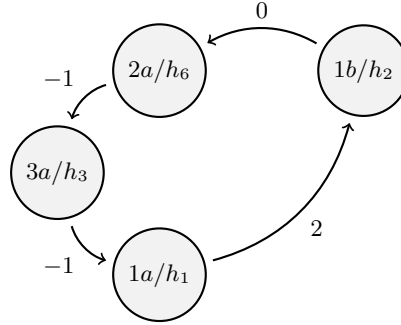
Example (Example 3 continued.). This example is strictly TU-rationalizable. The only thing to check is x_{1a} and x_{1b} . Since $x_{1a} = x_{1b}$, the problem is strictly TU-rationalizable – indeed, the utility given in the original example suffices.

Suppose instead $x_{1b} = e_{2a} = h_6$, a new house type, with no other changes. Focusing on the left SCC:

ik	e_{ik}	x_{ik}	$\omega_{ik} - m_{ik}$
1a	h_1	h_2	2
1b	h_2	h_6	0
2a	h_6	h_3	-1
3a	h_3	h_1	-1

This problem is TU-rationalizable, but not strictly TU-rationalizable. The minimum path

Figure 4: Figure for Example 3 continued.



length from $x_{1a} = h_2$ to $x_{1b} = h_6$ is 0, forcing $p_{h_2} = p_{h_6}$. If $v_1(h_2) > v_1(h_6)$, then 1b is not maximizing subject to his budget, so the allocation is not a competitive equilibrium and not in the weak core.

Corollary 5. Fix a TU problem $(A, \mathcal{A}, H, x, m, e, \omega)$. Assume (A1). A TU-rationalizable problem's solutions $v_i(\cdot)$ are characterized by solutions to the following linear system.

$$\begin{aligned}
& v_i(x_{ik}) \leq v_i(x_{ik'}) + p \cdot (x_{ik} - x_{ik'}) \quad \forall i, \forall ik, ik' \\
& \text{for any } h \text{ such that } h \neq x_{ik} \quad \forall x_{ik}, \text{ for any } ik \text{ such that } h \cdot p \leq e_{ik} \cdot p + \omega_{ik} : \\
& \quad v_i(h) - h \cdot p \leq v_i(x_{ik}) - x_{ik} \cdot p \\
& \text{s.t. } (x_{ik} - e_{ik}) \cdot p = \omega_{ik} - m_{ik} \quad \forall ik \in \mathcal{A} \\
& \text{and } v_i(e_{ik}) + \omega_{ik} \geq v_i(h) \quad \forall ik \in \mathcal{A}, h \in H
\end{aligned}$$

Proof. Appendix. □

The first line is the Afriat inequalities for quasilinear utility (with marginal utility of money equal to one). Given some valid price vector p , these give the restrictions of utilities for houses that are actually consumed by type i . The second line gives restrictions on utilities for any houses that are never consumed by type i . If a house h is never consumed but is affordable under some budget $e_{ik} \cdot p + \omega_{ik} := I_{ik}$, its consumption bundle $(h, I_{ik} - h \cdot p)$ must be dispreferred to the actual consumed bundle $(x_{ik}, I_{ik} - x_{ik} \cdot p)$. The third line characterizes valid price vectors. The fourth line is assumption (A1).

This linear system fully identifies possible values of (v_i) from the observed data. As is the case in consumer demand revealed preferences, these are joint restrictions rather than valid ranges for each $v_i(h)$. For example, there are infinite possible price vectors (e.g. $p + C$), leading to infinite possible v_i 's. Even more so, I show in the full proof of Theorem 2, relative prices are determined within an SCC but not across SCCs.⁸ Nevertheless, this corollary fully characterizes the joint restrictions for valid v_i 's.

6 Conclusion

In this paper, I build a model of aggregate matchings for one-sided matching with endowments in both non-transferrable and transferrable utility settings. Stability is falsifiable in this model, and I address the question of rationalizability: in a setting where the allocations are observed, but preferences are not, can the allocation be in the core? The main results give if and only if conditions for rationalizability. In the NTU setting, there is an easily checked condition based on the digraph \mathcal{G}^{big} constructed from μ – in any potential trading cycle, agents of the same type must receive the same house. The proof of the result leverages simple ideas from graph theory and techniques common in matching literature to construct a rationalizing preference profile. In the TU setting, I show that the question is connected to consumer demand revealed preferences, and rationalizability is equivalent to the existence of competitive equilibrium. Again, there is an easily checked condition – either a solution to a linear program or equivalently a cyclic monotonicity condition.

⁸For this reason I conjecture it is not possible to write a linear system without the existential statement of (P) .

References

- Afriat, S.N. 1967. "The Construction of Utility Functions from Expenditure Data." *International Economic Review*. 8(1): 67-77
- Jorge, Alcalde-Unzu and Elena Molis. 2011. "Exchange of Indivisible Goods and Indifferences: The Top Trading Absorbing Sets Mechanisms." *Games and Economic Behavior*. 73(1): 1-16.
- Aziz, Haris and Bart de Keijzer. 2012. "Housing markets with indifferences: A tale of two mechanisms." *Proceedings of the Twenty-Sixth AAAI Conference on Artificial Intelligence*.
- Bondy, J. Adrian and U.S.R. Murty. 2008. *Graph Theory*. Graduate Texts in Mathematics, Vol. 244. Springer, Berlin.
- Brown, Donald J. and Caterina Calsamiglia. 2006. "The nonparametric approach to applied welfare analysis." *Economic Theory*. 31: 183-188.
- Chambers, Christopher P. and Federico Echenique. 2016. *Revealed Preference Theory*. Econometric Society Monographs. Cambridge University Press, Cambridge.
- Chiappori, Pierre-André and Bernard Salanié. 2016. "The Econometrics of Matching Models." *Journal of Economic Literature*. 54(3): 832-861.
- Choo, Eugene and Aloysius Siow. 2006. "Who Marries Whom and Why." *Journal of Political Economy*. 114(1): 175-201.
- Debreu, Gerard and Herbert Scarf. 1963. "A Limit Theorem on the Core of an Economy." *International Economic Review*. 3(3): 235-246.
- Echenique, Federico. 2008. "What Matchings Can Be Stable? The Testable Implications of Matching Theory." *Mathematics of Operations Research*. 33(3): 757-768.
- Echenique, Federico, Sangmok Lee, Matthew Shum, and M. Bumin Yenmez. 2013. "The Revealed Preference Theory of Stable and Extremal Stable Matchings." *Econometrica*. 81(1): 153-171.
- Gale, David and Lloyd Shapley. 1962. "College Admissions and the Stability of Marriage." *The American Mathematical Monthly*. 69(1): 9-15.

- Galichon, Alfred, Scott Duke Kominers, and Simon Weber. 2019. "Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility." *Journal of Political Economy*. 127(6).
- Grigoryan, Aram and Markus Möller. 2023. "A Theory of Auditability for Allocation and Social Choice Mechanisms." Working paper.
- Hsieh, Yu-Wei. 2012. "Understanding Mate Preferences from Two-Sided Matching Markets: Identification, Estimation and Policy Analysis."
- Hitsch, Gunter J., Ali Hortasu, and Dan Ariely. 2010. "Matching and Sorting in Online Dating." *American Economic Review*, 100(1): 130-63.
- Jaramillo, Paula and Vikram Manjunath. 2012. "The difference indifference makes in strategy-proof allocation of objects." *Journal of Economic Theory*, 147 (5), 1913–1946.
- Kaneko, Mamoru. 1982. "The Central Assignment Game and the Assignment Markets." *Journal of Mathematical Economics*. 10(2-3): 1483-1504
- Quint, Thomas and Jun Wako. 2004. "On Houseswapping, the Strict Core, Segmentation, and Linear Programming." *Mathematics of Operations Research*, 29(4): 861-877.
- Quinzii, Martine. 1984. "Core and Competitive Equilibria with Indivisibilities." *International Journal of Game Theory*, 13(1): 41-60.
- Roth, Alvin E. and Andrew Postlewaite. 1977. "Weak Versus Strong Domination in a Market with Indivisible Goods." *Journal of Mathematical Economics* 4:131-137.
- Roth, Alvin E. and Xiaolin Xing. 1997. "Turnaround Time and Bottlenecks in Market Clearing: Decentralized Matching in the Market for Clinical Psychologists." *Journal of Political Economy*. 105(2).
- Samuelson, Paul A. 1938. "A note on the pure theory of consumers' behaviour." *Economica*, New Series. 5(17): 61-71.
- Shapley, Lloyd and Herbert Scarf. 1974. "On Cores and Indivisibility." *Journal of Mathematical Economics*. 1(1): 23-37.
- Tarjan, Robert. 1972. "Depth-first search and linear graph algorithms." *SIAM Journal on Computing*. 1(2): 146-160.

Veblen, Oswald. 1912. "An Application of Modular Equations in Analysis Situs." *Annals of Mathematics, Second Series*. 14(1): 86-94.

Wako, Jun. 1983. "A Note on the Strong Core of a Market with Indivisible Goods." *Journal of Mathematical Economics*. 13: 189-194.

A Proofs

First, I introduce the promised graph construction. Given an allocation μ , draw $\mathcal{G}^{small} = (\mathcal{A}, E')$ as follows:

Initialize. Draw all agents \mathcal{A} as vertices. The rest of the procedure takes $|H|$ steps.

Step m . Consider all agents receiving h_m , that is all ik such that $\mu(ik) = h_m$. Order them according to their index; refer to these as the “left” side. Similarly, order agents endowed with h_m according to their index; these are the “right” side. By construction, these two sets are the same cardinality. Draw one arc from the first agent on the left side to the first agent on the right side, and so on. If $m < \eta$, continue to step $m + 1$.

The graph produced after $|H|$ steps represents the allocation μ . Note that each agent has one out-arc and one in-arc. Recall the construction of $\mathcal{G}^{big} = (\mathcal{A}, E)$. Note also that $E \supseteq E'$; that is, \mathcal{G}^{big} is obtained by adding arcs to \mathcal{G}^{small} .

I now provide some intermediate results related to the constructed graphs \mathcal{G}^{small} and \mathcal{G}^{big} . These will be key for the proof of Theorem 1.

Proposition 2. *Consider $\mathcal{G}^{small} = (\mathcal{A}, E')$ constructed from μ . \mathcal{G}^{small} has a subgraph partition into cycles. That is, there are disjoint subgraphs C_1, \dots, C_N such that $\mathcal{G}^{small} = \cup_{n=1}^N C_n$, $C_m \cap C_n = \emptyset$ for all m, n , and each C_n is a cycle.*

Proof. Note each vertex i has $d^-(ik) = d^+(ik) = 1$. We can invoke a version of Veblen’s theorem:

(Veblen’s theorem) A directed graph $D = (V, E)$ admits a partition of arcs into cycles if and only if $d^-(v) = d^+(v)$ for all vertices $v \in V$. (Veblen, 1912; Bondy and Murty, 2008)

Since $d^-(ik) = d^+(ik)$, \mathcal{G}^{small} has a partition of arcs into cycles. There are no isolated vertices, so every vertex is in at least one cycle. Further, since $d^-(ik) = d^+(ik) = 1$ each vertex must be in at most one cycle. Thus the arc partition into cycles also partitions the vertices into cycles. \square

Proposition 3. *Consider \mathcal{G}^{big} constructed from μ . For every strongly connected component S of \mathcal{G}^{big} , there is a cycle covering all vertices in S .*

Proof. By Proposition 2, \mathcal{G}^{small} admits a partition of vertices into cycles. Recall $\mathcal{G}^{big} = (\mathcal{A}, E)$ and $\mathcal{G}^{small} = (\mathcal{A}, E')$, where $E \supseteq E'$, so these cycles also partition \mathcal{G}^{big} 's vertices. The SCC S in \mathcal{G}^{big} is composed of the vertices in a number of \mathcal{G}^{small} -cycles. It cannot include a strict subset of vertices in a \mathcal{G}^{small} -cycle since there is always a path between any two vertices in a cycle.

The remaining argument is by strong induction on the number K of \mathcal{G}^{small} -cycles contained in S . Assign an order to these cycles in the following way. Let the first cycle be any of these. Choose the k^{th} cycle such that it has the same house type as one of the first $k - 1$ cycles. It is always possible to do this – suppose at some point none of the remaining cycles has the same house type as the first k cycles. Then there are no paths in \mathcal{G}^{big} between the first k cycles and the remaining cycles (recall arcs are drawn from an agent to all agents whose endowment he receives), so they are not in the same SCC.

Claim. There is a cycle in \mathcal{G}^{big} covering all vertices in the first k \mathcal{G}^{small} -cycles in S . For convenience, I will call this the “big-cycle”, and the \mathcal{G}^{small} -cycles will be “small-cycles”.

Base claim. For $k = 1$, the claim is trivial.

k^{th} claim. Suppose the claim is true for the first $k - 1$ cycles. That is, there is a $k - 1^{th}$ big-cycle in \mathcal{G}^{big} covering all the vertices in the first $k - 1$ small-cycles. I show that there is a cycle covering all vertices in the $k - 1^{th}$ big-cycle and the k^{th} small-cycle. The following argument is illustrated in Figure 5. There are three cases, depending on whether either cycle is a self-loop.

Case 1. Suppose neither is a self-loop. Let the big-cycle be $(1a, \dots, 2a, 1a)$, and the k^{th} small-cycle be $(3a, 4a, \dots, 3a)$. That is, $\mu(2a) = \mu^E(1a)$ and so on. I do not require that the denoted agents are all different types; e.g. $2a$ can be $1b$. By the ordering of the cycles, the k^{th} small-cycle and the $k - 1^{th}$ big-cycle have at least one of the same house type. Without loss of generality let $\mu^E(1a) = \mu^E(4a)$. This gives $\mu(2a) = \mu^E(1a) = \mu^E(4a)$, so we have the arc $(2a, 4a) \in E$. Similarly, $\mu(3a) = \mu^E(4a) = \mu^E(1a)$, so we have the arc $(3a, 1a) \in E$. This gives us a new big-cycle across all the vertices in the first k small-cycles.:

$$(\underbrace{1a, \dots, 2a}_{\text{big-cycle } k-1}, \underbrace{4a, \dots, 3a}_{k^{th} \text{ cycle}}, 1a).$$

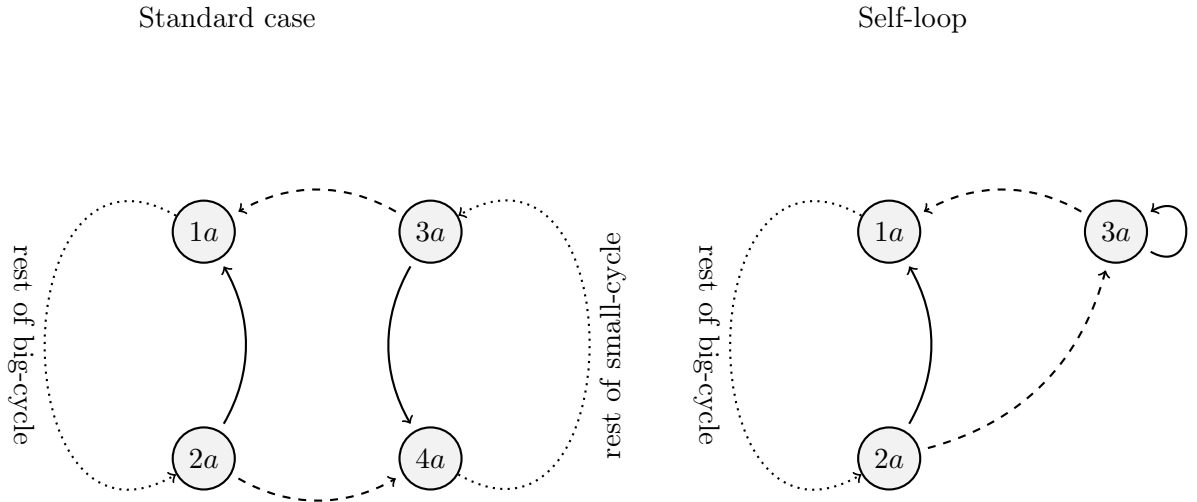
Case 2. Suppose the k^{th} small-cycle is a self-loop, but the $k - 1^{th}$ big-cycle is not. Then let the big-cycle be $(1a, \dots, 2a, 1a)$, and the k^{th} small-cycle be $(3a, 3a)$. Again,

let $\mu^E(1a) = \mu^E(3a)$ without loss of generality. Then $\mu(2a) = \mu^E(1a) = \mu^E(3a)$ implies $(2a, 3a) \in E$. Likewise, $\mu(3a) = \mu^E(3a) = \mu^E(1a)$ implies $(3a, 1a) \in E$. So we have a new big-cycle $(\underbrace{1a, \dots, 2a}_{\text{big-cycle } k-1}, 3a, 1a)$. The case if the big-cycle is a self-loop is the same (this may occur in the $k = 2$ claim).

Case 3. Suppose both are self-loops. Then let the big-cycle be $(1a, 1a)$ and the k^{th} small-cycle be $(3a, 3a)$. Again, we suppose $\mu^E(1a) = \mu^E(3a)$. Then $\mu(1a) = \mu^E(1a) = \mu^E(3a)$ implies $(1a, 3a) \in E$, and likewise $(3a, 1a) \in E$. So we have a new big-cycle $(1a, 3a, 1a)$.

This completes the proof. \square

Figure 5: Illustration of Proposition 3



The following lemma is derived from Proposition 3 and its proof.

Lemma 2. *Consider \mathcal{G}^{big} constructed from μ . Every strongly connected component S has no in- or out- arcs. That is, if $ik \in S$ and $(ik, i'k') \in E$ or $(i'k', ik) \in E$, then $i'k' \in S$.*

Proof. There is a cycle covering all vertices of S by Proposition 3. Suppose there is an out-arc from S pointing to a vertex in a different SCC S' . S' also has a cycle covering

all its vertices. The same argument as in the induction part of the proof of Proposition 3 establishes an arc from S' to S . Thus there are paths from between any vertices in S and S' , and they are in the same SCC, a contradiction. The case for no in-arcs is a relabeling of S and S' . \square

The following is a corollary of Lemma 2.

Corollary 6. *Consider \mathcal{G}^{big} constructed from μ . Let ik and $i'k'$ be distinct vertices. There exists a $(ik, i'k')$ -path if and only if ik and $i'k'$ are in the same SCC. Equivalently, there exists a $(ik, i'k')$ -path if and only if there exists a $(i'k', ik)$ -path.*

Proof. If ik and $i'k'$ are in the same SCC, there exists a $(ik, i'k')$ -path by definition. Suppose there exists a $(ik, i'k')$ -path. By Lemma 2, there are no paths between different SCCs, so ik and $i'k'$ must be in the same SCC. \square

Corollary 7. *Consider \mathcal{G}^{big} constructed from μ . All copies of the same house type are in the same SCC. That is, if $\mu^E(ik) = \mu^E(i'k')$ and $ik \in S$, then $i'k' \in S$.*

Proof. Let $\mu^E(ik) = \mu^E(i'k')$. There is at least one agent pointing to ik , so $\exists a \in \mathcal{A}$ such that $(a, ik) \in E$. Then $(a, i'k') \in E$ as well by construction. By Corollary 6, there are (ik, a) - and $(i'k', a)$ - paths. Then there are $(ik, i'k')$ - and $(i'k', ik)$ - paths (through a), so ik and $i'k'$ are in the same SCC. \square

The above results give us significant information about the SCCs of \mathcal{G}^{big} . The following is a summary of these results. From Proposition 3, each SCC contains a cycle covering all its vertices. From Lemma 2 and Corollary 6, \mathcal{G}^{big} can be vertex- and arc- partitioned into its SCCs. That is, \mathcal{G}^{big} consists of SCCs with no links between them. Finally, Corollary 7 tells us all copies of a given house type are in the same SCC.

If we take Theorem 1 as given for now, we can use the above result to prove Corollary 1.

Proof of Corollary 1. If $\mu^E(ik) = \mu^E(i'k')$, then ik and $i'k'$ are in the same SCC. Then apply Theorem 1 to get the desired result. \square

A.1 Proof of Theorem 1

Proof of Theorem 1. (“If”) Let the supposition be true: whenever agents of the same type are in the same SCC, they receive the same house type. I find a preference profile \succsim that

such that $\mu \in \text{core}(\succsim)$. First find the partition of vertices into SCCs. Then assign an arbitrary order to the SCCs, and denote them S_1, \dots, S_M . Construct the preferences by the following procedure. As helpful notation, let $\succsim_i(n)$ denote type i 's n^{th} favorite house.

Step 1. In S_1 , for all $i \in S_1$, let $\succsim_i(1) = \mu(i)$. This is well defined since if there are multiple agents of the same type in S_1 , they all receive the same house type.

Step 2. In S_2 , for all $i \in S_2$, let $\succsim_i(1) = \mu(i)$ if possible. This is possible if there were no type i 's in S_1 . Otherwise, let $\succsim_i(2) = \mu(i)$. By Corollary 7, a house never reappears in a later step, so this never assigns a house to two places in the same preference.

Step m . In S_m for $m = 2, \dots, M$, for all $i \in S_k$, let $\succsim_i(m') = \mu(i)$ for the lowest unassigned $m' = 2, \dots, m$. Again by the same argument above, this never assigns two houses to the same type; it also never assigns the same house type to multiple places in the same preference.

Step $M + 1$. Assign remaining preferences in any order, if necessary.

I now show this preference profile admits no blocking coalition. Suppose that there is a coalition of agents $A' \subseteq \mathcal{A}$ and sub-allocation μ' such that $|\mu'^{-1}(h)| = |(\mu^E)^{-1}(h) \cap A'|$ and for all $ik \in A' : \mu'(ik) \succsim_i \mu(ik)$. The argument is by strong induction on the number of SCCs M . In each SCC S_k , the claim is that $\mu'(ik) = \mu(ik)$ for all $ik \in A' \cap S_m$.

Base case. In S_1 , all agents receive their favorite house. Then $\mu'(ik) \sim_i \mu(ik)$ for all $i \in A' \cap S_1$. The only indifferences are between copies of the same house type, so this implies $\mu'(ik) = \mu(ik)$.

m^{th} case. Suppose the claim is true for all agents in $A' \cap (S_1 \cup \dots \cup S_{m-1})$. This implies that μ' allocates all agents in $A' \cap (S_1 \cup \dots \cup S_{m-1})$ houses in their own SCC. That is, $\mu'(ik) \in \mu^E(A' \cap S_1)$ for all $ik \in A' \cap S_1$, and so on. Toward a contradiction, suppose that $\exists ik \in S_m$ such that $\mu'(ik) := h \succ_i \mu(ik)$. Then it must be $h \in \mu^E(S_1 \cup \dots \cup S_{m-1})$, since all strictly preferred houses are in earlier SCCs. Further, since μ' reallocates within A' , it must be $h \in \mu^E(A' \cap (S_1 \cup \dots \cup S_{m-1}))$. But then it must be that an agent in $A' \cap (S_1 \cup \dots \cup S_{m-1})$ receives a house in $\mu^E(A' \cap (S_m \cup \dots \cup S_M))$. This contradicts the supposition, so it must be that $\mu'(ik) \sim \mu(ik)$ for $ik \in A' \cap S_m$, which implies $\mu'(ik) = \mu(ik)$.

Thus $\mu'(ik) = \mu(ik)$ for all $ik \in A'$, and A' is not a blocking coalition.

(“Only if”) Toward the contrapositive, suppose there is a SCC S with two agents of the same type who receive different houses. By Proposition 3, there is a cycle covering all vertices in S . I now construct a blocking coalition using this cycle. Note that two of these vertices represent agents of the same type who receive different houses. Let these two agents be $1a$ and $1b$; I consider cases based on their relative positions in the cycle.

1. Suppose the cycle is $1a \rightarrow \underbrace{2a \rightarrow \cdots \rightarrow 1b}_{:=c} \rightarrow 3a \rightarrow \cdots \rightarrow 1a$, and $\mu^E(2a) \neq \mu^E(3a)$.
 Suppose $\mu^E(2a) \succ_1 \mu^E(3a)$. Then $1b \rightarrow \underbrace{2a \rightarrow \cdots \rightarrow 1b}_c$ represents a blocking coalition. Note that this is a feasible sub-allocation; it contains its own endowment, and $1b$ is strictly better off. The case $\mu^E(2a) \prec_1 \mu^E(3a)$ is a rotation and relabeling of the cycle.
2. Suppose the cycle is $1a \rightarrow 1b \rightarrow \underbrace{2a \rightarrow \cdots \rightarrow 1a}_{:=c}$. If $\mu^E(2a) \succ_1 \mu^E(1b)$, then $1a \rightarrow \underbrace{2a \rightarrow \cdots \rightarrow 1a}_c$ is a blocking coalition. If instead $\mu^E(1b) \succ_1 \mu^E(2a)$, then μ is not individually rational for $1b$.
3. If the cycle is $1a \rightarrow 1b \rightarrow 1a$ and $\mu^E(1a) \neq \mu^E(1b)$, then μ is not individually rational.

This completes the proof. \square

Remark. For readers familiar with the result in Quint and Wako (2004), it suffices to show that executing their “*STRICTCORE*” algorithm on the above constructed preferences results in the allocation μ . This is readily apparent, and a formal proof is omitted.

A.2 Proof of Theorem 2

I present a theorem by Quinzii (1984), which I will use in the proof of the main result. There are no “types” in her model, but I retain my present notation for consistency. I first give a formal definition of competitive equilibrium in an exchange economy setting.

Definition 8. Let $E = \{(\omega_{ik}, e_{ik}), (u_{ik})\}_{ik \in \mathcal{A}}$ be an exchange economy. A **competitive equilibrium** is a price vector $p \in \mathbb{R}^H$ and a feasible allocation $(x_{ik}, m_{ik})_{ik \in \mathcal{A}}$ such that for all $ik \in \mathcal{A}$:

- $m_{ik} + p \cdot x_{ik} \leq \omega_{ik} + p \cdot e_{ik}$
- $(u_{ik}(h, m) \geq u_{ik}(x_{ik}, m_{ik})) \implies (m + p \cdot h > \omega_{ik} + p \cdot e_{ik})$

That is, all agents' allocations are affordable for them, and any better allocation is unaffordable. A **competitive equilibrium allocation** is $(x_{ik}, m_{ik})_{ik \in \mathcal{A}}$ for which there exists a price vector supporting it as a competitive equilibrium.

Theorem 3. (*Quinzii, 1984, pg. 54*) Let $E = \{(\omega_{ik}, e_{ik}), (u_{ik})\}_{ik \in \mathcal{A}}$ be an exchange economy. Assume u_{ik} are utility functions such that:

1. u_{ik} are increasing with respect to money, and $\lim_{m \rightarrow \infty} u_{ik}(h, m) = \infty$ for all $ik \in \mathcal{A}$
2. $u_{ik}(e_{ik}, \omega_{ik}) \geq u_{ik}(h, 0)$ for all $ik \in \mathcal{A}, h \in H$. That is, the endowment (both house and money) is preferred to consuming any house and 0 money.

Then the set of weak core allocations and the set of competitive equilibrium allocations of E coincide.

In the present paper's setting, this theorem gives us equivalence of the weak core and competitive equilibrium allocations. Thus to show TU-rationalizability, it is equivalent to find $(v_i)_{i \in \mathcal{A}}$ (with the restriction that these are common within agent types) and a price vector $p \in \mathbb{R}_+^{|H|}$ supporting (x, m) as a competitive equilibrium.

I briefly leave the exchange economy setting and consider the consumer demand setting. I give a definition for consumer demand quasilinear rationalizability, then I present a well-known theorem for classic consumer demand revealed preferences due to Brown and Calsamiglia (2007).

Definition 9. Let $(x_r, m_r, p_r), r = 1, \dots, N$ be observed demand and price data, where $x_r \in \mathbb{R}_+^H; p_r \in \mathbb{R}_{++}^H$. The data is **quasilinear rationalizable** if for some $I > 0, \forall r (x_r, m_r)$ solves

$$\begin{aligned} & \max_{x \in \mathbb{R}_{++}^n} v(x) + m \\ & \text{s.t. } p_r x + m = I \end{aligned}$$

for some concave v .

Theorem 4. (*Brown and Calsamiglia, 2007*) Let $(x_r, m_r, p_r), r = 1, \dots, N$ be observed demand and price data, where $x_r \in \mathbb{R}_+^H; p_r \in \mathbb{R}_{++}^H$. The following are equivalent:

1. The data (x_r, m_r, p_r) are quasilinear rationalizable by a continuous, concave, strictly monotone utility function v .

2. The data (x_r, m_r, p_r) satisfy Afriat's inequalities with constant marginal utilities of income. That is, there exist $v_r, v_l > 0 \forall r$ such that

$$v_r \leq v_l + p_l \cdot (x_r - x_l) \quad \forall r, l = 1, \dots, N \quad (A)$$

3. The data (x_r, m_r, p_r) are “cyclically monotone”, that is, if for any given subset of the data $\{(x_s, p_s)\}_{s=1}^m$:

$$p_1 \cdot (x_2 - x_1) + p_2 \cdot (x_3 - x_2) + \dots + p_m \cdot (x_1 - x_m) \geq 0 \quad (C)$$

The last condition is known as “cyclic monotonicity.” While it is probably not obvious how I will apply Theorem 4, I will show that there is a deep connection between the my present setting and consumer demand revealed preferences.

I now give the full proof for Theorem 2.

Proof of Theorem 2. I first show that (1) \iff (2), then (2) \iff (3).

First, (1) \implies (2). Suppose the TU problem $(A, \mathcal{A}, H, x, m, e, \omega)$ is TU-rationalizable. That is, there is some profile of utility indices $(v_i)_{i \in A}$ such that (x, m) is in the weak core. By Theorem 3, there is some price vector p supporting (x, m) as a competitive equilibrium. So p satisfies $m_{ik} + p \cdot x_{ik} \leq \omega_{ik} + p \cdot e_{ik}$. With quasilinear utility, money always enters utility, so this holds with equality: $m_{ik} + p \cdot x_{ik} = \omega_{ik} + p \cdot e_{ik}$. Then p must satisfy equation (P). Theorem 3 allows negative prices, but adding any positive constant $p + C$ will also satisfy (P), so we can let $p \geq 0$.

I now show (2) \implies (1). Suppose there exists a vector p satisfying equation (P). I seek to show that this p supports (x, m) as a competitive equilibrium for some utility indices (v_i) . That is, I want to construct v_i such that all agents ik are maximizing utility subject to their budget constraints $e'_{ik} \cdot p + \omega_{ik}$.⁹ This becomes a classic consumer demand revealed preference problem. To see this, reinterpret an agent type i as a single consumer, and each individual agent ik as a demand data point from this consumer:

$$\left(\underbrace{(x_{ik}, m_{ik})}_{\text{consumed good and money}}, \underbrace{(e'_{ik} \cdot p + \omega_{ik}) := I_{ik}}_{\text{budget}}, \underbrace{p}_{\text{price}} \right)_{k \in \{1, \dots, K_i\}}$$

⁹Agent ik sells his endowment e'_{ik} at price p and is additionally endowed with ω_{ik} money.

That is, i is a consumer, and each ik is a single observation of demand at a particular budget. There are $|A|$ consumers and K_i demand points for each consumer i . We seek to rationalize the demand data in a consumer revealed demand sense by constructing (v_i) such that each consumer i is maximizing utility $V_i(h, m) = v_i(h) + m$ in each consumption bundle-budget pair.

The easiest way to do this is to let $v_i(x_{ik}) = x'_{ik} \cdot p$, making all agents indifferent to any possible consumption bundle while still satisfying assumption (A2). However, I show these data are rationalizable in a deeper sense than this knife-edge construction.

I will apply Theorem 4. Notice that cyclic monotonicity (C) is trivially fulfilled when $p_s \equiv p$ is constant. Thus the consumption data with some sufficient constant budget

$$(x_{ik}, m_{ik}, I, p)_{k \in \{1, \dots, K_i\}}$$

are always quasilinear rationalizable. Our consumption data has varying budgets instead

$$(x_{ik}, m_{ik}, I_{ik}, p)_{k \in \{1, \dots, K_i\}}$$

However, the quasilinear utility

$$V(x) = \sum_{n=1}^X x_n p_n + m_{ik}$$

is concave, continuous, and strictly increasing, and rationalizes either set of data. Thus we can also apply Theorem 4 to see that utility indices fulfilling Afriat's inequalities (A) will also suffice for (v_i) .

I now show $((1) \iff) (2) \implies (3)$. Toward a contradiction, suppose \mathcal{G}_{TU}^{big} has a cycle C with positive length; i.e. $\sum_{ik \in C} \omega_{ik} - m_{ik} > 0$. The members of C can form a blocking coalition for (x, m) by allocating to each $ik \in C$

$$\left(x_{ik}, m + \frac{\sum_{ik \in C} \omega_{ik} - m_{ik}}{|C|} \right)$$

That is, each agent receives the same house and receives more money from the excess endowment. This is of course feasible for C and strictly preferred by all $ik \in C$.

Finally, I prove $(3) \implies (2)$. Suppose \mathcal{G}_{TU}^{big} has no cycles with length > 0 . I construct a price p satisfying (P) via path lengths on \mathcal{G}_{TU}^{big} . Note that Proposition 3, Lemma 2, and

Corollary 7 still apply to \mathcal{G}_{TU}^{big} . Every SCC has a cycle covering all its vertices; there are no paths between two SCCs; and all houses of the same type are in the same SCC. Denote p_h as the price of house type $h \in H$. Construct p as follows:

1. For each SCC, choose any house type h in this SCC and set p_h to be any number.
2. For all houses h' in this SCC, set $p_{h'} - p_h$ to be length of the shortest path from h to h' . That is, the shortest path between an agent endowed with h to an agent endowed with h' determines the price difference.
3. Repeat steps 1 and 2 for all SCCs.
4. Add a constant to p to ensure $p \geq 0$.

I will show that all paths between two vertices are the same length, then that the path length between a house type h and itself is always 0, so that the construction is consistent, i.e. $p_h - p_{h'} = 0$ when $h = h'$. The rest of the proof will immediately follow.

Note the whole economy is budget balanced; we have $\sum_{ik \in \mathcal{A}} \omega_{ik} = \sum_{ik \in \mathcal{A}} m_{ik}$. For any cycles that form a vertex-partition of \mathcal{G}_{TU}^{big} : these cycles must have length 0. A negative length cycle that is in a partition of the overall economy implies a positive length cycle elsewhere by budget balancedness, a contradiction.

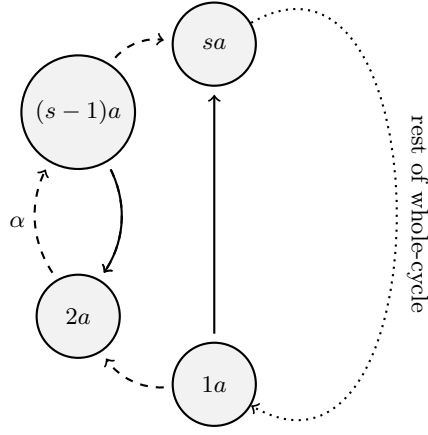
In particular, by Proposition 3, each SCC has a cycle containing all its vertices; call this the “whole-cycle” as shorthand. These partition the whole economy, so each whole-cycle must have length 0. For the following claims, assume the SCC has at least three vertices. I will show the cases for one or two vertices separately. Enumerate the whole-cycle as $(1a, 2a, \dots, sa, \dots, (S-1)a, Sa, 1a)$. (Allowing any of these agents to be of the same type – this is unimportant.) Now consider $1a$ and sa distinct and in the same SCC (recall there are no paths between SCCs), and consider the path $(1a, \dots, sa)$ via the whole-cycle. Denote this path $(1a, \underbrace{2a, \dots, (s-1)a}_{:=\alpha}, sa)$, and call it the “whole-cycle path” as shorthand.

Claim 1. If the arc $(1a, sa)$ exists, it is the same length as the whole-cycle path. That is, $\ell(1a, sa) = \ell(1a, 2a, \dots, (s-1)a, sa)$.

Figure 6 illustrates the following argument. If the arc $(1a, sa)$ exists, then $e_{2a} = e_{sa}$, so there is an arc $((s-1)a, 2a)$. Then $(\underbrace{2a, \dots, (s-1)a}_{:=\alpha}, 2a)$ forms a cycle, and $(1a, sa, \underbrace{\dots}_{\text{rest of whole-cycle}}, 1a)$ also forms a cycle. Since the two cycles partition the SCC, they are part of a partition of the overall economy; thus both cycles must have length 0. If $\ell(1a, sa) > \ell(1a, 2a, \dots, (s-1)a, sa)$,

then the latter cycle has positive length, a contradiction. This is because the whole-cycle has length 0 as established, and we have found a cycle with shorter length. If instead $\ell(1a, sa) < \ell(1a, 2a, \dots, (s-1)a, sa)$, then the latter cycle has negative length, also a contradiction. Note the same argument carries through if $2a = (s-1)a$ – the first cycle is a self-loop, and $1a = (s-1)a$ is symmetric.

Figure 6: Illustration of Claim 1



Claim 2. If the arc $(sa, 1a)$ exists, it has length negative of the whole-cycle path from $1a$ to sa . That is, $\ell(sa, 1a) = -\ell(1a, 2a, \dots, (s-1)a, sa)$.

From Claim 1, $\ell(sa, 1a) = \ell(sa, (s+1)a, \dots, Sa, 1a)$. Notice that $(sa, (s+1)a, \dots, Sa, 1a)$ and $(1a, 2a, \dots, (s-1)a, sa)$ form the whole cycle, so their lengths sum to 0. That is, $\ell(sa, 1a) + \ell(1a, 2a, \dots, (s-1)a, sa) = 0$, and the claim follows.

Remark 2. The indexing of $1a$ and sa in Claims 1 and 2 is not important. Since the whole-cycle is a cycle, $1a$ can be any vertex. (It is convenient to have $1 \leq s \leq S$.)

Claim 3. Any $(1a, sa)$ -path is the same length as the whole-cycle path $(1a, \underbrace{2a, \dots, (s-1)a}_{:=\alpha}, sa)$.

The $(1a, sa)$ -path is some permutation of a subset of vertices of the SCC. Denote this

$(\underbrace{\sigma_1 a}_{=1a}, \sigma_2 a, \dots, \sigma_{j-1} a, \underbrace{\sigma_j a}_{=sa})$, where $j \leq S$. I will show

$$\ell(\sigma_1 a, \dots, \sigma_{j-1} a, \sigma_j a) = \underbrace{\ell(1a, 2a) + \dots + \ell((\sigma_j - 1)a, \sigma_j a)}_{\text{whole-cycle path}} \equiv \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a)$$

Note that $\sigma_{j-1} \neq \sigma_j - 1$ in general.

I will show the claim by strong induction on the length of j . The base case of $j = 1$ is Claim 1. Now suppose the claim is true for j ; that is, $\ell(1a, \dots, \sigma_{j-1} a, \sigma_j a) = \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a)$. Now consider $j + 1$. We have $\ell(1a, \sigma_{j+1} a) = \ell(1a, \sigma_j a) + \ell(\sigma_j a, \sigma_{j+1} a)$. If $\sigma_{j+1} > \sigma_j$, then by Claim 1 write

$$\ell(\sigma_j a, \sigma_{j+1} a) = \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)$$

So

$$\begin{aligned} \ell(1a, \dots, \sigma_j a, \sigma_{j+1} a) &= \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) + \ell(\sigma_j a, \sigma_{j+1} a) \\ &= \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) + \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) \\ &= \sum_{i=1}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) \end{aligned}$$

If $\sigma_{j+1} < \sigma_j$, then by Claim 2 write

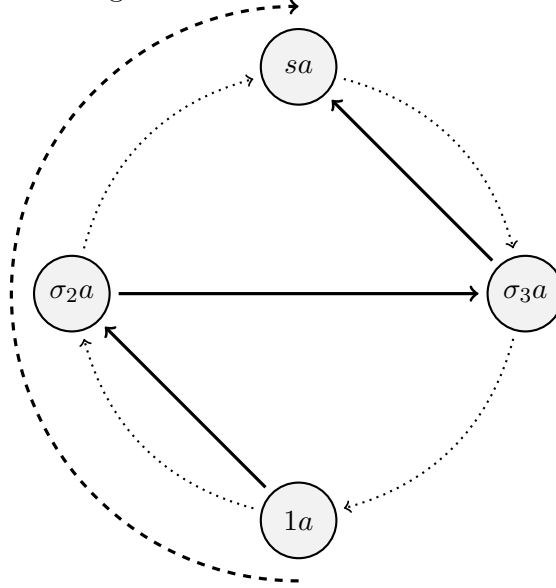
$$\ell(\sigma_j a, \sigma_{j+1} a) = - \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)$$

So

$$\begin{aligned}
\ell(1a, \dots, \sigma_j a, \sigma_{j+1} a) &= \sum_{i=1}^{\sigma_j-1} \ell(ia, (i+1)a) + \ell(\sigma_j a, \sigma_{j+1} a) \\
&= \sum_{i=1}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) + \sum_{i=1}^{\sigma_j-1} \ell(ia, (i+1)a) - \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) \\
&= \sum_{i=1}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)
\end{aligned}$$

as desired.

Figure 7: Illustration of Claim 3



Claim 4. The length of any path between a house type h and itself is 0.

Figure 8 illustrates the following argument. Note that two vertices (agents) may be endowed with the same house type, so these can be distinct nodes. Recall that all copies of the same house type are contained in the same SCC. The path length from a vertex to itself is 0 since the whole-cycle has length 0, and any other path is the same length. Now suppose h is contained in two distinct vertices, $1a$ and $2a$. Consider a node sa such that $x_{sa} = h$. (This may be $1a$ or $2a$.) Then the arcs $(sa, 1a)$ and $(sa, 2a)$ exist. These have

the same length, $\omega_{sa} - m_{sa}$, by construction of \mathcal{G}_{TU}^{big} . Denote $\ell(sa, 1a) = \ell(sa, 2a) = \ell_1$. I show the length of the path from $1a$ to $2a$ is 0. Denote this path $(1a, \dots, 2a)$, and let $\ell(1a, \dots, 2a) = \ell_2$. Both $(sa, 1a, \dots, 2a)$ and $(sa, 2a)$ are paths from sa to $2a$, so must have the same length. Then $\ell_1 = \ell_1 + \ell_2$, giving us $\ell_2 = 0$ as desired.

Figure 8: Illustration of Claim 4



I have shown the above claims for SCCs of size at least three. Now consider an SCC of only one vertex. The only arc must be $(1a, 1a)$, which constitutes the whole-cycle and must have length 0, and the path length from this house type to itself is 0.

Now consider an SCC of two vertices, $1a$ and $2a$. If they are endowed with distinct house types, the arcs $(1a, 2a)$ and $(2a, 1a)$ are the only arcs, and the claims are true trivially. If they are endowed with the same house type, the self loops are also present. The two self-loops partition the SCC, so have length 0. We have $\ell(1a, 1a) = \ell(1a, 2a)$ by construction, so $\ell(1a, 2a) = 0$, and similarly $\ell(2a, 1a) = 0$. Then all arcs have length 0 in this SCC, so the claims are again true.

The rest of the proof follows easily. The path length between any house type h and itself is 0 (so the minimum path length is 0), ensuring it is possible to construct prices this way. Next, for any $ik \in \mathcal{A}$, the path length from $e_{ik} := h$ to $x_{ik} := h'$ is $m_{ik} - \omega_{ik}$, so that $p_{h'} - p_h = m_{ik} - \omega_{ik}$. This gives

$$(x_{ik} - e_{ik}) \cdot p = p_{h'} - p_h = m_{ik} - \omega_{ik}$$

as desired.

This completes the proof of the theorem. \square

Proof of Corollary 4. As argued in the proof of Theorem 2, any price must satisfy $(x_{ik} -$

$e_{ik} \cdot p = \omega_{ik} - m_{ik}$ for all $ik \in \mathcal{A}$. By the construction of \mathcal{G}_{TU}^{big} , $x_{ik} - e_{ik}$ is an arc from e_{ik} to x_{ik} with length $\omega_{ik} - m_{ik}$, which is also the price difference between these houses. Inductively (I will omit the full formality), a path from x_{ik} to $x_{ik'}$ has path length 0 if and only if the price difference between them is 0. (Note that by Claim 2, there also must be a path from $x_{ik'}$ to x_{ik} , and it has length 0 as well.)

(“If”) Let both conditions be true. As in the main theorem, it is sufficient to set $v_i(x_{ik}) = p \cdot x_{ik}$. Since prices can be set arbitrarily across SCCs, we can ensure no two houses in different SCCs have the same price.

(“Only if”) Toward a contradiction, suppose the problem is not TU-rationalizable. Then it is of course not strictly TU-rationalizable. Now suppose the second condition is false. That is, there are ik, ik' in the same SCC such that $x_{ik} \neq x_{ik'}$, but the shortest path length between them is 0. Then $p_{x_{ik}} = p_{x_{ik'}}$. Suppose $v_i(x_{ik}) > v_i(x_{ik'})$ without loss of generality. Then ik' can afford $(x_{ik}, m_{ik'})$, which is preferable to $(x_{ik'}, m_{ik'})$. Thus (x, m) is not a competitive equilibrium, so is not strictly TU-rationalizable. \square

Proof of Corollary 5. This comes from the proof of Theorem 2. The first inequality is (A) from the result by Brown and Calsamiglia (2007). This is exactly Afriat’s inequalities when the marginal utility of money is 1. These give joint restrictions on any the utility for houses actually consumed by agent type i given some p . Necessity and sufficiency are from Afriat’s theorem.

The second inequality gives restrictions on the utility for houses not consumed by type i . A house h that is affordable under some ik ’s budget must have $V(h, e_{ik} \cdot p + \omega_{ik} - p \cdot h) \leq V(x_{ik}, e_{ik} \cdot p + \omega_{ik} - p \cdot x_{ik})$, else (x, m) is not a competitive equilibrium. This gives the inequality in the corollary:

$$\begin{aligned} v_i(h) + (e_{ik} \cdot p + \omega_{ik} - h \cdot p) &\leq v_i(x_{ik}) + (e_{ik} \cdot p + \omega_{ik} - x_{ik} \cdot p) \\ v_i(h) - h \cdot p &\leq v_i(x_{ik}) - x_{ik} \cdot p \end{aligned}$$

That is, if h is affordable to ik , then its utility (including leftover money) must be less than that of x_{ik} . Note that a house that is too expensive for all ik is allowed to have any utility. Again, necessity and sufficiency are immediate.

The third inequality defines valid vectors p , which comes from Theorem 2 and its proof. The fourth inequality is (A1). \square