# Shapley-Scarf Markets with Objective Indifferences

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**Abstract**

In many object allocation problems, some of the objects may be indistinguishable from each other, such as with dorm rooms or school seats. We call this setting "objective indifferences," describing situations where all agents are indifferent between identical copies. Thus matching mechanisms in such settings must account for indifferences. Top trading cycles (TTC) with fixed tie-breaking has been suggested and used in practice to deal with indifferences in object allocation problems. Under general indifferences, TTC with fixed tie-breaking is not Pareto efficient nor group strategy-proof. Furthermore, it may not select the core, even when it exists. However, under objective indifferences, agents are *always and only* indifferent between copies of the same object. In this setting, TTC with fixed tie-breaking maintains Pareto efficiency, group strategy-proofness, and core selection. Further, we show that objective indifferences is the most general setting where TTC with fixed tie-breaking maintains these important properties.

# 1 Introduction

Important markets including as living donor organ transplants, dorm assignments, and school choice are modeled as Shapley-Scarf markets: each agent is endowed with an indivisible object (which we call "houses") and has preferences over the set of objects. Monetary transfers are disallowed, and participants have property rights to their own endowments. The goal is to re-allocate these objects among the agents to achieve efficiency and stability. The usual stability notion is the core; an allocation is in the core if no subset of agents would prefer to trade their endowments among themselves. In the original setting of Shapley and Scarf (1974), agents have strict preferences over the houses, and Gale's *top trading cycles* (TTC) algorithm finds an

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allocation in the core. Roth and Postlewaite (1977) further show that the core is non-empty, unique, and Pareto efficient. Roth (1982) shows that TTC is strategy-proof; Bird (1984), Moulin (1995), Pápai (2000), and Sandholtz and Tai (2024) show it is group strategy-proof. These properties make TTC an attractive algorithm for practical applications.

However, the assumption that preferences are strict is quite strong. In particular, if the objects are not unique, agents should naturally be indifferent. We present a model of Shapley-Scarf markets where there are indistinguishable copies of house "types." The model restricts agents to be indifferent between copies of the same house type, but never indifferent between copies of different house types. We call these preferences "objective indifferences." This captures important situations where the Shapley-Scarf model is applied. For example, in dorm or public housing assignments, many units are effectively the same (e.g., two units with the same floor plan in the same building). Likewise in school assignments, different slots at the same school are indistinguishable. We see objective indifferences as a minimal model of indifferences, capturing the most basic and plausible form of indifferences.

In the fully general setting where agents' preferences may contain indifferences, *TTC with fixed tie-breaking* is often used in practice; ties in preference orders are broken by some external rule. For example, Abdulkadiroglu and Sönmez (2003) propose something similar in the setting of school choice with priorities. However, TTC with fixed tie-breaking is not Pareto efficient nor group strategy-proof. Indeed, Ehlers (2002) shows that these two properties are not compatible in Shapley-Scarf markets when agents have weak preferences. With weak preferences, the core of the market may be empty or non-unique. But even when core allocations exist, TTC with fixed tie-breaking may not select one.

Objective indifferences adds structure to the general case of indifferences by constraining any indifferences to be universal among agents. While the core still may not exist, it is essentially single-valued when it does exist. We show that in Shapley-Scarf markets with objective indifferences, TTC with fixed tie-breaking recovers Pareto efficiency and group strategy-proofness. It also selects the essentially unique core when it exists, and selects an element in the weak core otherwise. We also show that the objective indifferences setting is the most general setting such that TTC with fixed tie-breaking maintains any of these properties.

Others have have studied TTC under indifferences. In particular, Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) propose two generalizations of TTC for the general indifferences setting. Ehlers (2014) characterizes TTC in the general indifferences setting.

Our paper makes several important new contributions to the literature on Shapley-Scarf markets. First, it defines and explores a new domain of preferences that accurately capture many real-world scenarios where this model is applied. Second, it outlines the most general setting where TTC has no obvious drawbacks, in the sense that it retains all of the properties that make it so appealing under strict preferences. Third, it illustrates the underlying reason why weak preferences cause TTC to lose these properties: it is not indifferences per se, but *subjective* indifferences that may differ across agents.

Section 2 presents the formal notation. Section 3 explains TTC with fixed tie-breaking. Section 4 provides the main results. Section 5 concludes. Proofs of our results can be found in Appendix A.

# 2 Model

We present the model primitives. First we recount the classical Shapley and Scarf (1974) domain. Afterwards we introduce our "objective indifferences" domain.

We now present the general model of a Shapley-Scarf market. Let $N = \{1, \dots, n\}$ be a finite set of agents, with generic member $i$. Let $H = \{h_1, \dots, h_n\}$ be a set of houses, with generic member $h$. Every agent is endowed with one object, given by a bijection $w : N \to H$. The set of all endowments is $W(N, H)$ or $W$ for short. An allocation is an assignment of an object to each agent, given by a bijection $x : N \to H$. The set of all allocations is likewise $X(N, H)$ or $X$. We denote $x(i) = x_i$ and $w(i) = w_i$ for short.

Each agent has preferences $R_i$ over $H$. A preference profile is $R = (R_1, R_2, ..., R_n)$. Let $\mathcal{R}_i$ be the set of $i$'s possible preferences. A set $\mathcal{R} := \mathcal{R}_i^N$ of possible preference profiles is a **domain**. Note we restrict attention in this paper to domains that can be expressed as $\mathcal{R}_i^N$ for some $\mathcal{R}_i$. That is, every agent has the same set of possible preference orderings. If every $\mathcal{R}_i$ is the set of strict preference orderings, it is the classical **strict preferences domain**. If every $\mathcal{R}_i$ is the set of weak preference orderings, it is the classical **general indifferences domain**.

Our main domain is objective indifferences. Let $\mathcal{H} = \{H_1, H_2, \dots, H_K\}$ be a partition of $H$. An element $H_k$ of a partition is a **block**. Given $H$ and $\mathcal{H}$, denote $\eta : H \to \mathcal{H}$ as the mapping from a house to the partition element containing it; that is, $\eta(h) = H_k$ if $h \in H_k$. For a strict linear order $\geq$ over $\mathcal{H}$, we derive weak preferences $R_{\geq}$ over $H$. Formally, for $h, h' \in H$,

$$hR_{\geq}h' \iff \eta(h) \geq \eta(h')$$

The partition $\mathcal{H}$ defines the house types. Let $\mathcal{R}_i(\mathcal{H}) := \cup_{\geq} R_{\geq}$ be the set of all $R_{\geq}$ given an $\mathcal{H}$. Given $\mathcal{H}$, $\mathcal{R}(\mathcal{H}) := \mathcal{R}_i(\mathcal{H})^N$ is an **objective indifferences domain**. We sometimes suppress $(\mathcal{H})$ from the notation when context makes it clear. Note that all agents are indifferent between houses in the same block of $\mathcal{H}$ and have strict preferences between houses in different blocks. Because of this, we refer simply to "indifference classes" for the domain with the understanding that everyone shares the same indifference classes.

## 2.1 Rules

This subsection recounts formalities on rules (mechanisms) and top trading cycles. Familiar readers may safely skip this subsection.

A market is a tuple $(N, H, w, R)$. A **rule** is a function $f : \mathcal{R} \to X$; given a preference profile, it produces an allocation. When it is unimportant or clear from context, we suppress inputs from the notation. Denote $f_i(R)$ to be $i$'s allocation and let $f_Q = \{f_i : i \in Q\}$. Fix a rule $f$ and setting. We work with the following axioms.

A rule is Pareto efficient if it always produces Pareto efficient allocations.

**Pareto efficiency (PE)**. For all $R \in \mathcal{R}$, there is no other allocation $x \in X$ such that $x_i R_i f_i$ for all $i \in N$ and $x_i P_i f_i$ for at least one $i$.

Group strategy-proofness requires that no coalition of agents can collectively improve their outcomes by submitting false preferences. Note that in the following definition, we require both the true preferences and misreported preferences to come from the same set $\mathcal{R}$.

**Group strategy-proofness (GSP)**. For all $R \in \mathcal{R}$, there do not exist $Q \subseteq N$ and $R'_Q$ such that $(R'_Q, R_{-Q}) \in \mathcal{R}$ and $f_q(R'_Q, R_{-Q}) R_q f_q(R)$ for all $q \in Q$ with $f_q(R'_Q, R_{-Q}) P_q f_q(R)$ for at least one $q$.

Individual rationality models the constraint of voluntary participation. It requires that agents receive a house they weakly prefer to their endowment.

**Individual rationality (IR)**. For all $w$ and $R \in \mathcal{R}$, $f_i R_i w_i$.

We also define the core, which is a property of allocations. An allocation is in the core if there is no subset of agents who could benefit from trading their endowments among themselves.

**Definition 1.** An allocation $x$ is blocked if there exists a coalition $N' \subseteq N$ and allocation $x'$ such that $w_{N'} = x'_{N'}$ and for all $i \in N'$, $x'_i R_i x_i$, with $x'_i P_i x_i$ for at least one $i$. An allocation $x$ is in the **core** if it is not blocked.

The weak core requires that all members of a potential coalition are strictly better off.

**Definition 2.** An allocation $x$ is weakly blocked if there exists a coalition $N' \subseteq N$ and allocation $x'$ such that $w_{N'} = x'_{N'}$ and for all $i \in N'$, $x'_i P_i x_i$. An allocation $x$ is in the **weak core** if it is not weakly blocked.

The core property models the restriction imposed by property rights. Notice that individual rationality excludes blocking coalitions of size 1. The last axiom is Core-selecting.

**Core-selecting (CS).** For all $R \in \mathcal{R}$ and $w \in W$, if the core is nonempty, then $f(R)$ is in the core.

We will present characterization results of maximal domains on which all TTC$_\succ$ satisfy the axioms. By a "maximal" domain, we mean the following.

**Definition 3.** A domain $\mathcal{R}_i^N$ is **maximal** for an axiom $A$ and a class of rules $F$ if

1. each $f \in F$ is $A$ on $\mathcal{R}_i^N$, and
2. for any $\tilde{\mathcal{R}}_i^N \supset \mathcal{R}_i^N$, there is some $f \in F$ that is *not* $A$ on $\tilde{\mathcal{R}}_i^N$.

Note that this definition of maximality depends on both the axiom and the class of rules, which differs from elsewhere in the literature. Typically, a maximal domain for some property is the largest possible domain on which *some* rule exists which satisfies the desired property. We focus on a specific class of rules: top trading cycles with fixed tie-breaking. Again note that we only consider domains that can written as $\mathcal{R}_i^N$, which is common.

# 3 Top trading cycles with fixed tie-breaking

In this paper, we analyze top trading cycles with fixed tie-breaking in the settings defined in the previous section. For an extensive history, we refer the reader to Morill and Roth (2024). We briefly define TTC and TTC with fixed tie-breaking.

**Algorithm 1.** ***Top Trading Cycles***. *Consider a market $(N, H, w, R)$ under strict preferences. Draw a graph with $N$ as nodes.*

1. *Draw an arrow from each agent $i$ to the owner (endowee) of his favorite remaining object.*
2. *There must exist at least one cycle; select one of them. For each agent in this cycle, give him the object owned by the agent he is pointing at. Remove these agents from the graph.*
3. *If there are remaining agents, repeat from step 1.*

We denote this as $\text{TTC}(R)$.

TTC is well-defined only with strict preferences, as Step 1 requires a unique favorite object. In practice, a **fixed tie-breaking rule** $\succ$ is often used to resolve indifferences. Given $N$, let $\succ = (\succ_1, \ldots, \succ_n)$, where each $\succ_i$ is a strict linear order over $N$. This linear order will be used to break indifferences between objects (based on their owners). Then let $R_{i,\succ_i}$ be given by the following. For any $j \neq j'$, let $w_j P_{i,\succ_i} w_{j'}$ if either

1. $w_j P_i w_{j'}$, or
2. $w_j I_i w_{j'}$ and $j \succ_i j'$

Then $R_{i,\succ_i}$ is a strict linear order over the individual houses. Example 1 illustrates a tie-break rule. Let $R_\succ = (R_{1,\succ_1}, \ldots, R_{n,\succ_n})$. Given a fixed tie-breaking rule $\succ$, **TTC with fixed tie-breaking ($\text{TTC}_\succ$)** is $\text{TTC}_\succ(R) \equiv \text{TTC}(R_\succ)$. That is, the tie-breaking rule is used to generate strict preferences, and TTC is applied to the resulting strict preference profile. Formally, each tie-breaking profile $\succ$ generates a different $\text{TTC}_\succ$ rule.

**Example 1.** Let $N = \{1, 2, 3, 4\}$.

$$
\begin{array}{c}
R_1 \\ \hline
w_3, w_4 \\
w_1, w_2
\end{array}
\quad + \quad
\begin{array}{c}
\succ_1 \\ \hline
1 \\ 2 \\ 3 \\ 4
\end{array}
\quad \to \quad
\begin{array}{c}
R_{1,\succ_1} \\ \hline
w_3 \\ w_4 \\ w_1 \\ w_2
\end{array}
$$

# 4 Results

In the general indifferences domain, $\text{TTC}_\succ$ is not Pareto efficient, core-selecting, nor group strategy-proof. However, we show that in the objective indifferences domain, $\text{TTC}_\succ$ satisfies all three properties. Furthermore, we show that objective indifferences characterizes the set of maximal domains on which $\text{TTC}_\succ$ is PE and CS, and characterizes the set of "symmetric-maximal" domains on which $\text{TTC}_\succ$ is GSP.

## 4.1 Pareto efficiency and core-selecting

When we relax the assumption of strict preferences and allow for general indifferences, $\text{TTC}_\succ$ loses two of its most appealing properties: Pareto efficiency and core-selecting. However, in the intermediate case of objective indifferences, $\text{TTC}_\succ$ retains these two properties, regardless of the tie-breaking rule $\succ$ chosen. Moreover, on *any* larger domain, $\text{TTC}_\succ$ loses Pareto efficiency and core-selecting. Thus, we show that it is not indifferences per se, but rather *subjective* evaluations of indifferences, which cause $\text{TTC}_\succ$ to lose these properties.

We first demonstrate that $\text{TTC}_\succ$ is not Pareto efficient under general indifferences. Example 2 gives the simplest case.

**Example 2.** Let $N = \{1, 2\}$ and preferences be given by the following:

| $R_1$ | $R_2$ |
|---|---|
| $w_1, w_2$ | $w_1$ |
| | $w_2$ |

Let $\succ_i = (1, 2)$ for both agents. In the first step of $\text{TTC}_\succ(R)$, both agents point to agent 1. Agent 1 forms a self-cycle and is therefore assigned to $w_1$. In the second round, agent 2 forms a self-cycle and is assigned to $w_2$. Therefore, the $\text{TTC}_\succ$ allocation is $x = (w_1, w_2)$, which is Pareto dominated by $x' = (w_2, w_1)$.

The example illustrates the difference made by indifferences – some tie-breaking rules may not take advantage of Pareto gains made possible by the indifferences. However, under objective indifferences, if any agent is indifferent between two houses, then all agents are indifferent between those two houses. Therefore, objective indifferences rules out situations like in Example 2.

Under general indifferences, the set of core allocations may not be a singleton; there may be no core allocations or there may be multiple. As Example 2 demonstrates, even when the core of the market is non-empty, $\text{TTC}_\succ$ may still fail to select a core allocation.[1] Likewise, under objective indifferences, the set of core allocations may be empty or multi-valued, as Example 3 illustrates. However, under objective indifferences, if the core is nonempty then $\text{TTC}_\succ$ selects a core allocation for any tie-breaking rule $\succ$. This stands in contrast to the result from Ehlers (2014) for general indifferences, where $\text{TTC}_\succ$ is only guaranteed to select an allocation in the weak core.

[1] It is straightforward to see that $x' = (w_2, w_1)$ is in the core of the market.

**Example 3.** Let $R$ be given by the following.

| $R_1$ | $R_2$ | $R_3$ |
|---|---|---|
| $w_2, w_3$ | $w_1$ | $w_1$ |
| $w_1$ | $w_2, w_3$ | $w_2, w_3$ |

It is straight forward to verify that the core of the market is empty.

In fact, the objective indifferences setting characterizes the entire set of maximal domains on which $\text{TTC}_{\succ}$ is Pareto efficient and core-selecting for any tie-breaking rule $\succ$. That is, if all $\text{TTC}_{\succ}$ are PE/CS on some domain $\mathcal{R}_i^N$, then it must be a weak subset of some objective indifferences domain. Conversely, for any superset of an objective indifferences domain, there is some tie-break rule $\succ$ such that $\text{TTC}_{\succ}$ loses PE/CS.

**Theorem 1.** *The following are equivalent:*

1. $\mathcal{R}_i^N$ *is an objective indifferences domain.*
2. $\mathcal{R}_i^N$ *is a maximal domain on which all* $TTC_{\succ}$ *are Pareto efficient.*
3. $\mathcal{R}_i^N$ *is a maximal domain on which all* $TTC_{\succ}$ *are core-selecting.*

*Proof.* Appendix A.1. □

The full proof is in the appendix, but the intuition is simple. The objective differences domain precludes possibilities such as Example 2, and any larger domain inevitably introduces the possibility of such a pair.

Moreover, under objective indifferences, the core is *essentially single-valued* when it exists, in the sense that all agents are indifferent between their assignments under any core allocations (see Sönmez (1999)). In other words, the set of core allocations are just permutations of identical copies.

**Corollary 1.** *For any two allocations* $x \neq y$ *in the core of an objective indifferences market,* $x_i I_i y_i$ *for all* $i \in N$.

*Proof.* Appendix A.1. □

As Example 3 shows, the core of an objective indifferences market may be empty. However, when the core is empty, all $\text{TTC}_{\succ}$ select a weak core allocation.

**Proposition 1.** *For any market* $(N, H, w, R)$*, the weak core is nonempty and* $TTC_{\succ}$ *selects an allocation in the weak core.*

*Proof.* Appendix A.1. □

## 4.2 Group strategy-proofness

TTC$_\succ$ also loses group strategy-proofness once we move from strict preferences to weak preferences. However, in the intermediate case of objective indifferences, TTC$_\succ$ recovers group strategy-proofness. Further, TTC$_\succ$ is not GSP in any larger "symmetric" domain. We say that a domain is symmetric if, when $h_1 P_i h_2$ is allowed, then so is $h_2 P_i h_1$. We will informally argue that this is not an onerous modeling restriction.

First we present a simple example demonstrating that under general indifferences, TTC$_\succ$ is not group strategy-proof. Example 4 shows how an agent can break his own indifference to benefit a coalition member without harming himself.

**Example 4.** Let $R$ and $R'$ be given by the following, and let $Q = \{1, 3\}$.

| $R_1$ | $R_2$ | $R_3$ | $R'_1$ |
|---|---|---|---|
| $w_2, w_3$ | $w_1$ | $w_1$ | $w_3$ |
| $w_1$ | $w_2$ | $w_2$ | $w_2$ |
| | $w_3$ | $w_3$ | $w_1$ |

Let $\succ_i = (1, 2, 3)$ for all $i$. Then $TTC_\succ(R) = (w_2, w_1, w_3)$. But if 1 misreports $R'_1$, then $TTC_\succ(R') = (w_3, w_2, w_1)$. Then 1 is indifferent, and 3 is strictly better off.

Objective indifferences excludes situations like Example 4 in two ways. First, it eliminates the possibility that one agent is indifferent between two houses while another has a strict preference. Second, it constrains the possible set of misreports available to a manipulating coalition, since agents can *only* report indifference among all houses in the same indifference class given by $\mathcal{H}$.[2] Our next result characterizes the set of symmetric-maximal domains on which all TTC$_\succ$ are GSP.

Before presenting our result, we must define "symmetric" and "symmetric-maximal" domains.

**Definition 4.** A domain $\mathcal{R}$ is **symmetric** if for any $h_1, h_2 \in H$, if there exists $R_i \in \mathcal{R}_i$ such that $h_1 P_i h_2$, then there also exists $R'_i \in \mathcal{R}_i$ such that $h_2 P'_i h_1$.

**Definition 5.** A domain $\mathcal{R}_i^N$ is **symmetric-maximal** for an axiom $A$ and a class of rules $F$ if

1. $\mathcal{R}_i^N$ is symmetric,
2. each $f \in F$ is $A$ on $\mathcal{R}_i^N$, and
3. for any symmetric $\tilde{\mathcal{R}}_i^N \supset \mathcal{R}_i^N$, there is some $f \in F$ that is *not* $A$ on $\tilde{\mathcal{R}}_i^N$.

In practical applications, symmetry is a natural restriction to place on the domain; if it is possible that agents might report strictly preferring some house $h$ to another house $h'$, we should not preclude the possibility they strictly prefer $h'$ to $h$. Indeed, the point of mechanism design is that preferences are unknown and must be solicited. It is easy to see that objective indifferences domains are symmetric. Relative to

[2]The constraint on agents' reports is an important difference from Ehlers (2002).

maximality, symmetric-maximality restricts the possible expansions of objective indifferences domains that we must consider.

**Theorem 2.** *$\mathcal{R}_i^N$ is a symmetric-maximal domain on which all $TTC_\succ$ are group strategy-proof if and only if it is an objective indifferences domain.*

*Proof.* Appendix A.2. □

Our proof uses similar reasoning to the proof that TTC is group strategy-proof under strict preferences contained in Sandholtz and Tai (2024). Any coalition requires a "first mover" to misreport, but this agent must receive an inferior house to the one he originally received. In the following example, we note that objective indifferences domains are *not* maximal domains on which all $\text{TTC}_\succ$ are GSP.

**Example 5.** Consider $H = \{h_1, h_2\}$ and $\mathcal{H} = \{\{h_1, h_2\}\}$. Let $\mathcal{R}'_i = \mathcal{R}_i(\mathcal{H}) \cup (h_1 P h_2) = \{(h_1 I h_2), (h_1 P h_2)\}$. That is, expand the domain by including the ordering $(h_1 P h_2)$. It can be verified that $\text{TTC}_\succ$ is still group strategy-proof for any tie-breaking profile $\succ$. Note that this expanded domain is not symmetric, since $\mathcal{R}'_i$ does not contain the preference ordering $(h_2 P h_1)$.

If both agents have the same preferences, then there is clearly no possible group manipulation. Without loss of generality, assume $w_i = h_i$. Let $\succ_i$: $(1, 2)$ for both $i$. Consider two possible (true) preference profiles:

| $R_1$ | $R_2$ | | $R_1$ | $R_2$ |
|---|---|---|---|---|
| $h_1$ | $h_1, h_2$ | or | $h_1, h_2$ | $h_1$ |
| $h_2$ | | | | $h_2$ |

In the first case, there is no improving allocation since both agents receive a top-ranked house. In the second case, it would be advantageous for agent 1 to point at $h_2$ and leave $h_1$ for agent 2, but this is not possible, since this preference ranking is not available in $\mathcal{R}'$. It can also be verified that no other tie-breaking rule $\succ$ allows an improving coalition.

## 4.3 Discussion

Our main theorems show that objective indifferences are maximal domains where $\text{TTC}_\succ$ maintains Pareto efficiency, core selection, and group strategy-proofness. Therefore, in situations where one could reasonably assume that all agents have objective indifferences preferences, $\text{TTC}_\succ$ is a sensible choice of mechanism. Even when the context imposes constraints on the possible tie-breaking rules, it is guaranteed that $\text{TTC}_\succ$ will be PE, CS, and GSP regardless of which tie-breaking rule is chosen. Moreover, $\text{TTC}_\succ$ is efficient, as well as easy to explain and implement. However, we do not believe our results imply that $\text{TTC}_\succ$ should be avoided in settings beyond objective indifferences, or that market designers should only allow objective indifference preference rankings.

Consider school choice in San Francisco, which uses a lottery system to assign school seats at most public schools. While current details are not readily available, Abdulkadiroglu, Featherstone, Niederle, Pathak, and Roth designed a system using TTC.[3] Suppose families' preferences over school seats can be described by objective indifferences. That is, all families are indifferent between the roughly 120 seats for kindergarten at West Portal Elementary School (West Portal). Our results suggest that $\text{TTC}_{\succ}$ is an excellent candidate mechanism for this setting.

However, the real situation may be more complicated. West Portal's 120 seats are divided roughly into 30 Cantonese immersion and 90 general education seats and the Cantonese immersion program reserves roughly 2/3 of seats for already bilingual children.[4] This plausibly induces a setting where some families are indifferent between the two kinds of seats, while other families have strict preferences between them. For example, a family whose children are already bilingual in Cantonese may be indifferent between the two types of seats at West Portal, while another family may have a strict preference for cultural community through the Cantonese bilingual program.

The relative (non-)competitiveness of language immersion seats in the SFUSD appears to be a source of anxiety for parents.[5]. Indeed, while West Portal receives 7.7 requests per open seat, its Cantonese immersion program for already bilingual students receives 7.2 requests per open seat, and 13.3 requests per open seat for students not already bilingual. That is, parents who simply want a seat at West Portal may apply for and receive already-bilingual seats, preventing parents with true desires for these seats from receiving them. Perhaps noticing this, parents have been agitating for expansion of language immersion programs, particularly for Mandarin[6], although we can only speculate on private motives.

Similarly, we do not interpret our results as proscriptive. While the induced situation guarantees that $\text{TTC}_{\succ}$ loses each of PE, CS, and GSP, we of course would not suggest ending language immersion programs or preventing parents submitting preferences over them. Likewise, we would not state that $\text{TTC}_{\succ}$ should not be used; other mechanisms that address these issues (if any) may have other tradeoffs, such as increased computational or cognitive complexity.

Our results lay out exactly the situations where $\text{TTC}_{\succ}$ is Pareto efficient, core-selecting, and group strategy-proof. However, the results do not necessarily proscribe its use outside of these settings. Instead, one could view the results as rationalizing the use of $\text{TTC}_{\succ}$ in many settings where $\text{TTC}_{\succ}$ has actually been suggested and applied, such as in school choice, housing assignments[7], and organ exchange.

[3] See the blog post by Al Roth: https://marketdesigner.blogspot.com/2010/09/san-francisco-school-choice-goes-in.html. As he notes, the team were not privy to the implementation or resulting data.

[4] https://web.archive.org/web/20250422170224/https://www.sfchronicle.com/bayarea/article/sfusd-competitive-public-schools-20252957.php

[5] https://web.archive.org/web/20250316121046/https://sfparents.org/parent-guide-to-applying-to-sfusd/

[6] e.g. SF Parents for Mandarin: https://docs.google.com/forms/d/e/1FAIpQLSeDtB8qhjm3e6vt6fXsCrLo4YKmVREJFb-OaM36FwBLs2tjGg/viewform

[7] As a low stakes example, at time of writing, the Math department at UC Berkeley uses TTC to assign graduate students to offices, which can be reasonably grouped into identical layouts.

# 5 Conclusion

The Shapley-Scarf market is a classic model in economic theory with applications to important markets like housing assignment, school choice, and organ exchange. When agents may be indifferent between objects, TTC with fixed tie-breaking is a commonly proposed mechanism. Unfortunately, it does not retain Pareto efficiency, group strategy-proofness, nor core selection.

We introduce a new domain of preferences, "objective indifferences," which captures situations where there are identical, indistinguishable copies of objects. Objective indifferences reflects many of the real-life applications of Shapley-Scarf markets. For example, consider a house assignment market with many indistinguishable dorm rooms. We show that TTC with fixed tie-breaking preserves the aforementioned properties – Pareto efficiency, group strategy-proofness, and core selection – on objective indifferences domains. Moreover, Pareto efficiency and core selection fail on any more general domains. While group strategy-proofness is preserved on some more general domains, it fails on any more general domain that is "symmetric."

It is remarkable that the maximal domains on which TTC satisfies these three distinct properties (essentially) coincide. We therefore view objective indifferences domains as the most general possible setting where TTC can be applied without any tradeoffs. Moreover, we interpret our results as showing that it *subjective* indifferences, not indifferences themselves, which cause issues for TTC when we relax the assumption of strict preferences.

Our paper opens interesting new lines of inquiry. First, we believe that studying matching markets with constrained indifferences is an exciting avenue for future research. In many real-world matching markets, agents have indifferences, but often with a certain structure imposed by the specific market. Understanding how adding structure to the case of general indifferences may affect matching problems is not only theoretically interesting, but could improve policy choices. For instance, tradeoffs in the selection of the partition $\mathcal{H}$ given the set of objects $H$. In some cases, there may be some ambiguity: are two dorms with the same floor plan but on different floors of the same building equivalent? Inappropriately combining indifference classes can lead to efficiency losses in the spirit of Example 2. On the other hand, splitting indifference classes can allow group manipulations like in Example 4. We leave formal results as future work. We also leave an axiomatic characterization of $\text{TTC}_{\succ}$ on objective indifferences domains as future work.

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# Appendix A Proofs

We provide proofs for the results in the main text. Given a market and $\text{TTC}_{\succ}(R)$, denote $S_k(R)$ as the $k$th cycle executed in $\text{TTC}_{\succ}(R)$.[8] Note that individual rationality (IR) of $\text{TTC}_{\succ}$ follows immediately from IR of TTC and the fact that $TTC_{\succ}(R) \equiv TTC(R_{\succ})$.

We will appeal to the following fact: let $x = \text{TTC}_{\succ}(R)$; if $i \in S_\ell(R)$ and $hP_ix_i$, then $h$ must have been assigned at some step before step $\ell$. This follows from the definitions – $hP_ix_i$ implies $hP_{i,\succ}x_i$, and $\text{TTC}_{\succ}(R) \equiv \text{TTC}(R_{\succ})$. Under $\text{TTC}(R_{\succ})$, an object $h$ such that $hP_{i,\succ}x_i$ must have been assigned prior to step $\ell$, otherwise $i$ would have pointed to $h$'s owner.

## Appendix A.1 Pareto efficiency and core-selecting

**Theorem 1.** *The following are equivalent:*

*(1) $\mathcal{R}_i^N$ is an objective indifferences domain.*

*(2) $\mathcal{R}_i^N$ is a maximal domain on which all $TTC_{\succ}$ are Pareto efficient.*

*(3) $\mathcal{R}_i^N$ is a maximal domain on which all $TTC_{\succ}$ are core-selecting.*

The result is trivial for $|N| = 1$, so assume $|N| \geq 2$. First we show that statements (1) and (2) are equivalent.

*Proof of (1) $\iff$ (2).* First we show that for any objective indifferences domain, all $\text{TTC}_{\succ}$ are PE. Consider any $(N, H, w)$ and fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of $H$ and let $R \in \mathcal{R}(\mathcal{H})$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks. Let $x = TTC_{\succ}(R)$, and suppose that some feasible allocation $y$ Pareto dominates $x$. Let $W = \{i : y_iP_ix_i\}$ be the set of agents who strictly improve under $y$, which must be nonempty. Let $i \in W$ be the first agent in $W$ assigned during the process of $TTC_{\succ}(R)$. If $i \in S_k(R)$ and $y_iP_ix_i$, then i) $\eta(y_i) \neq \eta(x_i)$, and ii) $y_i$ was assigned prior to step $k$. Therefore, there must be an agent $j$ in $\cup_{\ell=1}^{k-1} S_\ell(R)$ for whom $x_j \in \eta(y_i)$ but $y_j \notin \eta(y_i)$. Since $y$ Pareto dominates $x$, this implies $y_jP_jx_j$. But then $j \in W$, a contradiction.

Next we show that for any domain $\tilde{\mathcal{R}}_i^N$ where $\tilde{\mathcal{R}}_i \nsubseteq \mathcal{R}_i(\mathcal{H})$ for any $\mathcal{H}$, $\text{TTC}_{\succ}$ is not PE on $\tilde{\mathcal{R}}_i^N$. Fix $(N, H)$. Without loss of generality, assume $w_i = h_i$. If $\tilde{R}_i \nsubseteq \mathcal{R}_i(\mathcal{H})$ for any $\mathcal{H}$, it must contain two orderings $R_*, R_{**}$, such that for some $h_1, h_2 \in H$, we have $h_1I_*h_2$ but $h_1P_{**}h_2$.

Taking only the existence of $R_*, R_{**} \in \tilde{\mathcal{R}}_i$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}_i^N$ and tie-breaking profile $\succ$ such that $TTC_{\succ}(R)$ is not Pareto efficient. Define $A = \{i : w_iR_*w_1\} \setminus \{2\}$; note that $1 \in A$ and $2 \in A^c$. Define the preference profile $R$ such that

$$R_i = \begin{cases} R_* & \text{if } i \in A \\ R_{**} & \text{if } i \in A^c \end{cases}$$

[8]Note that $S_k$ may not be unique, since multiple cycles may appear in step 2 of Algorithm 1.

Let the tie-breaking rule profile $\succ$ be such that $i \succ_i j$ for all $i$ and for all $j \neq i$. That is, each agent's tie-breaking rule top-prioritizes himself. Denote $x = \text{TTC}_\succ(R)$. We first prove that $x = w$.

**Claim 1.** *$x = w$.*

By construction of $\succ$, if $x_i I_i w_i$ then $x_i = w_i$. This is because $i$ will always point to himself before any other $j$ such that $w_j I_i w_i$.

We first show that $x_i = w_i$ for all $i \in A$. Toward a contradiction, suppose there exists $i \in A$ such that $x_i \neq w_i$, so $x_i P_* w_i$. Without loss of generality, let $x_i$ be (one of) the highest ranked house(s) under $R_*$ such that this is true. Denote $w_j = x_i$. Since $w_j P_* w_i P_* w_1$, $j \in A$ as well. Now since $x_j \neq w_j$, and by individual rationality, $x_j P_* w_j$. But this contradicts that $x_i$ was the highest ranked house under $R_*$ such that $x_i \neq w_i$. So $x_i = w_i$ for all $i \in A$.

Now consider $i \in A^c$. Note $x_i \in w(A^c)$ and $R_i = R_{**}$ for all $i \in A^c$. That is, $\text{TTC}_\succ$ must rearrange $w(A^c)$ among $A^c$, and all agents in $A^c$ have the same preferences. Thus individual rationality requires $x_i I_i w_i$ for all $i \in A^c$. Then $x_i = w_i$ for all $i \in A^c$.

We have $\text{TTC}_\succ(R) = w$. However, note that $w_1 P_2 w_2$ and $w_1 I_1 w_2$, so $TTC_\succ(R)$ is Pareto dominated by $(w_2, w_1, w_3, ..., w_n)$. $\square$

We now turn to the second part of the proof.

*Proof of (1) $\iff$ (3).* First we show that for any objective indifferences domain, all $\text{TTC}_\succ$ are CS. Consider any $(N, H, w)$ and fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of $H$ and let $R \in \mathcal{R}(\mathcal{H})$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks.

Suppose that the core of $(N, H, w, R)$ is non-empty and contains some allocation $y$. Denote $x = TTC_\succ(R)$. We will show that $x_i I_i y_i$ $(\star)$ for all $i$ by induction on the steps of $TTC_\succ(R)$.

> $\boxed{\text{Step 1}}$ All $i \in S_1(R)$ received one of their top-ranked objects, so $x_i R_i y_i$. Suppose $(\star)$ is not true for $S_1(R)$. Then there is some $i \in S_1(R)$ such that $x_i P_i y_i$. But then $S_1(R)$ and $x$ block against $y$, a contradiction.
>
> $\boxed{\text{Step k}}$ Suppose that $(\star)$ is true for all steps before $k$. Suppose for some $i \in S_k(R)$ we have $y_i P_i x_i$. Then $y_i$ was assigned before step $k$. Further, $\eta(y_i) \neq \eta(x_i)$. (Otherwise, it could not be that $y_i P_i x_i$.) So if $y_i$ is assigned to $i$ under $y$, there must be an agent $j$ in $\cup_{\ell=1}^{k-1} S_\ell(R)$ for whom $x_j \in \eta(y_i)$ but $y_j \notin \eta(y_i)$. But then it cannot be that $y_j I_j x_j$, a contradiction.[9] Thus we have that $x_i R_i y_i$ for all $i \in S_k(R)$. Suppose $(\star)$ is not true for $S_k(R)$. Then there is some $i \in S_k(R)$ such that $x_i P_i y_i$. But then $S_k(R)$ and $x$ block against $y$, a contradiction.

Thus $x_i I_i y_i$ for all $i$. (Since $y$ was an arbitrary allocation in the core, this also proves Corollary 1.)

[9] This is where objective indifferences is used – this claim fails in general indifferences.

Next we show that for any domain $\tilde{\mathcal{R}}_i^N$ where $\tilde{\mathcal{R}}_i \not\subseteq \mathcal{R}_i(\mathcal{H})$ for any $\mathcal{H}$, $\text{TTC}_\succ$ is not CS on $\tilde{\mathcal{R}}_i^N$. Fix $(N, H)$. Without loss of generality, assume $w_i = h_i$. If $\tilde{R}_i \not\subseteq \mathcal{R}_i(\mathcal{H})$ for any $\mathcal{H}$, it must contain two orderings $R_*, R_{**}$, such that for some $h_1, h_2 \in H$, we have $h_1 I_* h_2$ but $h_1 P_{**} h_2$.

Define $A \subseteq N$ and $R \in \tilde{\mathcal{R}}_i^N$ exactly as we did in Part 1. By Claim 1, $TTC_\succ(R) = w$. However, $TTC_\succ(R)$ is blocked by $x' = (w_2, w_1, w_3, ..., w_n)$. It remains to show that $x'$ is in the core.

Suppose there is a coalition $Q$ and allocation $x''$ that blocks $x' = (w_2, w_1, w_3, ..., w_N)$. Let $W = \{i \in Q : x''_i P_i x'_i\}$, which must be nonempty. Suppose $W_A := W \cap A$ is nonempty and take any $i \in W_A$ such that $x''_i R_* x''_j$ for all $j \in W_A$.

**Version 1:** Note that since $i \in W_A$ and by individual rationality, $x''_i P_* w_i R_* w_1$. It must be $x''_i = w_{q_1}$ for some $q_1 \in Q$. In order for $q_1 \in Q$, we must have $x''_{q_1} I_{q_1} w_{q_1}$. Note that $q_1 \in A$ as well, since $w_{q_1} P_* x''_i P_* w_1$. Thus we need $q_1 \in Q$ such that $x''_{q_1} I_* w_{q_1}$. Then it must be $x''_{q_1} = w_{q_2}$ such that $w_{q_2} I_* w_{q_1}$. We repeat the argument to show that all for all $\{q_i \in Q : w_{q_i} I_* x''_i\}$, we must have $x_{q_i} I_* w_{q_i}$. That is, for all $q_i \in Q$ endowed with $w_{q_i} I_* x''_i$, $q_i$ must be assigned a house inside this indifference class, and $q_i \in A$ by construction. But then for all such agents, $x''$ must re-arrange their endowments among themselves, and it cannot be that $x''_i = w_{q_1}$, a contradiction. Thus $W_A = \emptyset$.

**Version 2:** If $i \in A$, individual rationality implies $x''_i P_* w_i R_* w_1$. Therefore, since $x''_Q = w_Q$ and $x'_j = w_j$ for all $j$ such that $w_j P_* w_1$, there exists some $j \in Q$ such that i) $w_j I_* x''_i$, implying $j \in A$ and $R_j = R_*$; and ii) $\neg(x''_j I_* w_j)$. Individual rationality requires $x''_j P_* w_j$, but then $j \in W_A$, a contradiction.

**V1:** Then $Q \subseteq A^c$. Recall that $x_2 = w_1$ and $w_1 P_{**} w_i$ for all $i \in A^c$. Thus it is impossible to include agent 2 in any blocking coalition, so $Q \subseteq A^c \backslash \{2\}$. Also recall that $x_i = w_i$ and $R_i = R_{**}$ for all $i \in A^c \backslash \{2\}$. Thus it is impossible for members of $A^c \backslash \{2\}$ to form a blocking coalition against $x'$, the desired contradiction.

**V2:** So $W \subseteq B$.

Moreover, note that $w_1 P_* x''_i$ for all $i \in W$. If not, then

Take any $i \in W$ such that $x''_i R_{**} x''_j$ for all other $j \in W$.

There must be some agent $j \in Q$ such that $x'_j I_{**} x''_i$ but $\neg(x''_j I_{**} x'_j)$. Since $x''_i I_* x'_i$ for all $i \in Q \cap A$, $j \in Q \cap B$. Moreover, since no agent in $Q$ is worse off under $x''$, it must that $x''_j P_{**} x'_j$. But then $j \in W_B$ and $x''_j P_{**} x'_j I_{**} x''_i$, contradicting that $x''_i R_{**} x''_j$ for all other $j \in W_B$. So $W_B = \emptyset$. Then $W = W_A \cup W_B = \emptyset$, contradicting that $Q$ and $x''$ block $x'$. □

We show a proof that $\text{TTC}_\succ$ selects the weak core.

**Proposition 1**. *For any market $(N, H, w, R)$, the weak core is nonempty and $TTC_\succ$ selects an allocation in the weak core.*

*Proof.* Denote $x = \text{TTC}_\succ(R)$. Since $x = \text{TTC}(R_\succ)$, $x$ is in the strict core of the tie-broken preference profile $R_\succ$. That is, there is no blocking coalition according to $R_\succ$. Suppose there exists $Q \subseteq N$ and $x'$ such that

$x_i' P_i x_i$, which implies $x_i' P_{i,\succ} x_i$. Then there exists some $j$ such that $x_j P_{j,\succ} x_j'$, which implies $x_j R_j x_j'$. Thus $Q$ cannot be a weak blocking coalition. $\square$

## Appendix A.2 Group strategy-proofness

We first review an important property of TTC$_\succ$ and state a useful lemma. Let $L(h, R_i) = \{h' \in H : hR_i h'\}$ be the lower contour set of a preference ranking $R_i$ at house $h$.

**Monotonicity (MON)**. A rule $f$ is **monotone** if, for any $R$ and $R'$ such that $L(f_i(R), R_i) \subseteq L(f_i(R), R_i')$ for all $i$, then $f(R) = f(R')$.

That is, a rule $f$ is monotone if, whenever any set of agents move up their allocations in their rankings, the allocation remains the same. It is straightforward to show that TTC is monotone for strict preferences; e.g. Takamiya (2001). Then, since $TTC_\succ(R) \equiv TTC(R_\succ)$ for any $R$ and $\succ$, it follows directly that TTC$_\succ$ is monotone.

The following result is adapted from Sandholtz and Tai (2024), who show it for TTC with strict preferences.

**Lemma 1** (Sandholtz and Tai, 2024)**.** *For any $R, R'$, let $x = TTC_\succ(R)$ and $x' = TTC_\succ(R')$. Suppose there is some $i$ such that $x_i' P_{i,\succ} x_i$. Then there exists some agent $j$ and house $h$ such that $hP'_{j,\succ} x_j$ and $x_j P_{j,\succ} h$.*

**Theorem 2.** *$\mathcal{R}_i^N$ is a symmetric-maximal domain on which all TTC$_\succ$ are group strategy-proof if and only if it is an objective indifferences domain.*

*Proof.* First we show that for any objective indifferences domain, all TTC$_\succ$ are GSP. Consider any $(N, H, w)$ and fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of $H$ and let $R \in \mathcal{R}(\mathcal{H})$. If $|N| = 1$ or $\mathcal{H} = \{H\}$, the result is trivial, so suppose that $|N| \geq 2$ and that the partition has at least two blocks. Without loss of generality, assume $w_i = h_i$ for all $i$.

Suppose $Q \subseteq N$ reports $R_Q'$ where $(R_Q', R_{-Q}) \in \mathcal{R}(\mathcal{H})$. Denote $R' = (R_Q', R_{-Q})$ and $x' = \text{TTC}_\succ(R')$. We will show that if $x_i' P_i x_i$ for some $i \in Q$, then $x_j P_j x_j'$ for some $j \in Q$.

Let $R''$ be the preference profile in $\mathcal{R}(\mathcal{H})$ such that each $R_i''$ top-ranks $\eta(x_i')$ and otherwise preserves the ordering of $R_i$. Let $x'' = TTC_\succ(R'')$. By monotonicity of TTC$_\succ$ , $x'' = x'$. Therefore $x_i'' P_i x_i$, and consequently, $x_i'' P_{i,\succ_i} x_i$. Applying Lemma 1, there must be some $j \in Q$ and $h \in H$ such that $x_j P_{j,\succ_j} h$ but $hP''_{j,\succ_j} x_j$. Note that $h \notin \eta(x_j)$; if it were, then for any $R, R'' \in \mathcal{R}(\mathcal{H})$, $x_j P_{j,\succ_j} h$ if and only if $x_j P''_{j,\succ_j} h$. Therefore, $x_j P_j h$ and $hP_j'' x_j$.[10] The only change from $R_j$ to $R_j''$ is to top-rank $\eta(x_j')$, so it must be that $h \in \eta(x_j')$. But then $x_j P_j x_j'$, as desired.

Next we show that for any symmetric domain $\tilde{\mathcal{R}}_i^N$ where $\tilde{\mathcal{R}}_i \not\subseteq \mathcal{R}_i(\mathcal{H})$ for any $\mathcal{H}$, TTC$_\succ$ is not GSP on $\tilde{\mathcal{R}}_i^N$. Fix $(N, H)$. Without loss of generality, let $w_i = h_i$ for all $i$. If $\tilde{R}_i \not\subseteq \mathcal{R}_i(\mathcal{H})$, then it must contain two orderings $R_*, R_{**}$ such that for some $h_1, h_2 \in H$ we have $h_1 I_* h_2$ but $h_1 P_{**} h_2$. The symmetric

[10] This is where the restriction to objective indifferences is used. Under general indifferences, this is not necessarily true.

requirement also necessitates that $\tilde{\mathcal{R}}_i$ contains some $R_{***}$ such that $h_2 P_{***} h_1$. Taking only the existence of $R_*, R_{**}, R_{***} \in \tilde{\mathcal{R}}_i$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}_i^N$ and tie-breaking profile $\succ$ such that $TTC_\succ(R)$ is not group strategy-proof.

Define $A = \{i : w_i R_* w_1\} \setminus \{2\}$, $B = \{i : w_1 P_* w_i \text{ and } w_i R_{**} w_1\} \cup \{2\}$, and $C = N \setminus (A \cup B)$. Note that $1 \in A$ and $2 \in B$.

$$R_i = \begin{cases} R_* & \text{if } i \in A \\ R_{**} & \text{if } i \in B \\ R_{***} & \text{if } i \in C \end{cases}$$

Note that for $i \neq 2, i \in A \cup B$ if and only if $w_i P_i w_1$. Let $\succ$ be any tie-breaking profile such that $i \succ_i j$ for all $i \neq j$.

**Claim 2.** $TTC_\succ(R) = w$.

The proof is similar to the proof of Claim 1. Denote $x = TTC_\succ(R)$. As in the proof of Theorem 1, by construction of $\succ$, if $x_i I_i w_i$ then $x_i = w_i$. Individual rationality additionally requires if $x_i \neq w_i$, then $x_i P_i w_i$.

That all $x_i = w_i$ for $i \in A$ is shown in exactly the same way as in the preceding proof for Claim 1.

Next we show that $x_i = w_i$ for all $i \in B$, which is similar. Toward a contradiction, suppose there exists $i \in B$ such that $x_i \neq w_i$, so $x_i P_i w_i$. Without loss of generality, let $x_i$ be the most favored house under $R_{**}$ such that this is true. Denote $w_j = x_i$. Since $w_j P_{**} w_i$ and either $i = 2$ or $w_i R_{**} w_1$, $j \in B$ as well. Since $x_j \neq w_j$, we must have $x_j P_{**} w_j$. But this contradicts that $x_i$ was the most favored house. So $x_i = w_i$ for all $i \in B$.

Finally, consider $C = (A \cup B)^c$. As before in Claim 1, TTC$_\succ$ rearranges $w(C)$ among $C$, who all have the same preferences. Then $x_i = w_i$ for all $i \in C$.

Now we consider a misreport.

**Claim 3.** *Let $R'_1 = R_{***}$ and $R' = (R'_i, R_{-i})$. $TTC_\succ(R') = (w_2, w_1, w_3, ..., w_n)$.*

Denote TTC$_\succ(R') = x'$. Consider the steps of $TTC_\succ(R)$. Without loss of generality, suppose agent 1's (self-)cycle was executed when there were no other possible (self-)cycles to execute. That is, agent 1 was assigned in the latest possible step $k$ of TTC$_\succ(R)$. Then $w_1 P_i w_i$ for all $i$ who remained at step $k$. Therefore, by construction of $R$, the set of remaining agents at step $k$ was $N_k = \{1, 2\} \cup C$. Thus at step $k$, agents 1 and 2 pointed at agent 1, while all other agents pointed at agent 2.[11] After, at step $k+1$, agent 2 formed a self-cycle and was assigned to his endowment.

Now consider the steps of $TTC_\succ(R')$. Assume without loss of generality cycles are executed in the same order, if duplicated. Since only agent 1's preferences change from $R$ to $R'$, steps 1 through $k-1$ proceed as

[11] Recall that $w_2 P_{***} w_1$.

before. Then at step $k$, the same set $N_k$ of agents remains. $R'_1 = R_{***}$, so agent 1 now points at agent 2 and forms a trading cycle. Agents 1 and 2 swap houses. Thus $x'_1 = w_2$ and $x'_2 = w_1$.

Recall $x = w$, $R_1 = R_*$, and $R_2 = R_{**}$. Then $w_1 I_1 w_2$ and $w_1 P_2 w_2$, so$x'$ is improving for $Q = 1, 2$. $\square$

# Appendix B Relation to school choice with priorities

We briefly note that TTC in the objective indifferences setting is not identical to TTC in the school choice with priorities setting. Intuitively, in objective indifferences the fixed tie-breaking rule determines for $i$ whom to point at; conversely, a school priority determines who points at $i$. Consider an example with 3 schools and 4 students.

**Example 6.** Let the set of schools (objects) be $H = \{A, B, C\}$, with $C$ having two slots. Let the students be $N = \{a, b, c_1, c_2\}$, where $a$ is "endowed" with $A$, and so on.

Let the school priorities be given by

| $A$ | $B$ | $C$ |
|---|---|---|
| $a$ | $b$ | $c_1$ |
| $b$ | $a$ | $c_2$ |
| $c_1$ | $c_2$ | $a$ |
| $c_1$ | $c_1$ | $b$ |

Alternatively, let a fixed tie-breaking rule $\succ$ be given by

| $\succ_a$ | $\succ_b$ | $\succ_{c_1}$ | $\succ_{c_2}$ |
|---|---|---|---|
| $c_1$ | $c_2$ | $c_1$ | $c_1$ |
| $c_2$ | $c_1$ | $c_2$ | $c_2$ |

Finally, compare two alternatives for student preferences

| $R_a$ | $R_b$ | $R_{c_1}$ | $R_{c_2}$ |
|---|---|---|---|
| $C$ | $C$ | $A$ | $A$ |
| $A$ | $A$ | $B$ | $B$ |
| $B$ | $B$ | $C$ | $C$ |

and

| $R'_a$ | $R'_b$ | $R'_{c_1}$ | $R'_{c_2}$ |
|---|---|---|---|
| $C$ | $C$ | $B$ | $B$ |
| $A$ | $A$ | $A$ | $A$ |
| $B$ | $B$ | $C$ | $C$ |

TTC with school priorities results in $Ac_1, Bc_2, Cab$ and $Ac_2, Bc_1, Cab$ under $R$ and $R'$ respectively. Crucially, $c_1$ gets the preferred school in either case, since it depends on school $C$'s priority. TTC$_\succ$ results in $Ac_1, Bc_2, Cab$ and $Ac_1, Bc_2, Cab$ under $R$ and $R'$ respectively. Either $c_1$ or $c_2$ will get the more preferred school, since it depends on $a$'s or $b$'s tie-breaking rule.