Group Incentive Compatibility in a Market with Indivisible Goods: A Comment

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Abstract

We note that the proofs of Bird (1984), the first to show group strategy-proofness of top trading cycles (TTC), require correction. We provide a counterexample to a critical claim and present corrected proofs in the spirit of the originals.

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1 Introduction

Strategy-proof mechanisms are desirable because they are immune to an individual agent's misrepresentations. Agents' decisions are thus straightforward, because the optimal action for any individual is to report his or her true preferences. Group strategy-proof mechanisms ensure this is also true for coalitions of agents, protecting less sophisticated and less well-connected agents.

Therefore, in problems such as school assignment, housing assignment, and organ exchange, group strategy-proofness is valuable. These settings are applications of the canonical house swapping model, where each agent is endowed with a single indivisible good and has strict preferences over the set of goods. In this model, the top trading cycles (TTC) mechanism of Shapley and Scarf (1974) produces the strict core allocation. Roth and Postlewaite (1977) show that this is also the unique competitive equilibrium allocation.

Roth (1982) shows that TTC is strategy-proof. Bird (1984) presents a proof that TTC is weakly group strategy-proof. In this note, we show that Bird's proof requires correction. To our knowledge, we are the first to do so. While others have since provided alternative proofs that TTC is group strategy-proof,[1] we present new proofs in the spirit of the originals in Bird (1984). We also prove a non-obvious claim about strong group strategy-proofness, which Bird (1984) presents as a corollary.

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[1] See Moulin (1995) Lemma 3.3 and Pápai (2000).

2 Model and notation

We retain the notation in Bird (1984) and recount it briefly here. Let $N = \{1, ..., n\}$ be the set of agents and let $w = (w_1, ..., w_n)$ be the endowment, where agent i is endowed with w_i, which we call a house. Each agent i has strict preferences P_i over the houses. Let $P = (P_1, ..., P_n)$ be the preference profile of all agents. An allocation is a vector $x = (x_1, ..., x_n)$ where each x_i corresponds to some w_j.

Let $T(N, P)$ denote the allocation resulting from TTC applied to (N, P). For convenience, we use $TTC(N, P)$ to denote the procedure of TTC applied to (N, P). Let $S_k(P) \subseteq N$ be the set of agents in the kth trading cycle of $TTC(N, P)$, and let $S_0 = \emptyset$.[2] Define $R_k(P) = \cup_{i=1}^{k} S_i(P)$.

Suppose a subset Q of agents report their preferences as $P'_Q \neq P_Q$. Let $P' = (P'_Q, P_{-Q})$. Denote $x = T(N, P)$ and $x' = T(N, P')$.

We seek to show that TTC is weakly group strategy-proof: for any Q and P'_Q, there is some agent $i \in Q$ such that $x_i P_i x'_i$ or $x_i = x'_i$. That is, at least one agent is weakly worse off under the misrepresentation. Additionally, we show TTC is strongly group strategy-proof: for any Q and P'_Q, there is some agent $i \in Q$ such that $x_i P_i x'_i$. That is, at least one agent is strictly worse off under the misrepresentation.

3 Proofs

We first correct a critical claim, Lemma 1, which Bird (1984) uses to prove weak group strategy-proofness. We then revise his proof of weak group strategy-proofness. Finally, we present a new proof of strong group strategy-proofness using the corrected Lemma 1.

3.1 Lemma 1

Bird (1984) makes the following claim, which is critical to the main result.

Claim 1 (Bird, 1984. Lemma 1). If there is an $i \in S_k(P)$ such that $x'_i P_i x_i$, then there exists a $j \in R_{k-1}(P)$ and $h \in N - R_{k-1}(P)$ such that $w_h P'_j x_j$.

He gives the following intuition:

> [I]f any trader wants to get a more preferred good, he needs to get a trader in an earlier cycle to change his preference to a good that went in a later trading cycle. From this result, the group incentive compatibility follows easily.

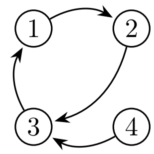
The lemma as stated requires correction. We first give a counterexample.

Example 1 (Counterexample to Claim 1)**.** *Let* $N = \{1, 2, 3, 4\}$ *with the following preferences.*

[2] The order of cycles generated by TTC is not generally unique. It is possible that two or more cycles are formed at the same step. However, the results carry through under any ordering of these cycles.

P_1	P_2	P_3	P_4
w_2	w_3	w_1	w_3
w_1	w_2	w_4	w_4
		w_3	

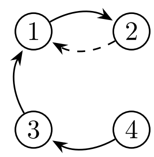
Figure 1: First step of $TTC(N, P)$



The TTC allocation is $x = (w_2, w_3, w_1, w_4)$. Now consider an alternative preference profile P':

P_1	P'_2	P_3	P_4
w_2	w_1	w_1	w_3
w_1	w_3	w_4	w_4
	w_2	w_3	

Figure 2: First step of $TTC(N, P')$



The new TTC allocation is $x' = (w_2, w_1, w_4, w_3)$.

In the notation of Claim 1, we have $i = 4$ and $k = 2$. That is, $4 \in S_2(P)$ and $x'_4 P_4 x_4$. Yet, there do not exist $j \in R_{k-1}(P) = S_1(P)$ and $h \in N - R_{k-1}(P) = S_2(P)$ such that $w_h P'_j x_j$. The only candidate for j is $2 \in R_1(P) = S_1(P)$, since she is the only agent whose preferences change under P'. But she does not rank any houses from $N - R_{k-1}(P) = S_2(P) = \{x_4\}$ above $x_2 = w_3$.

The error in Bird's proof of Lemma 1 stems from the following erroneous claim.

Claim 2. Let $x_m P'_m w_n$ for all $m \in R_{k-1}(P)$ and $n \in N - R_{k-1}(P)$. That is, all members of the first $k-1$ cycles rank their original assignments above all houses from cycles k and later. Then $R_{k-1}(P') = R_{k-1}(P)$. That is, the set of agents assigned in the first $k-1$ cycles of $TTC(N, P')$ is the same as the set of agents assigned in the first $k-1$ cycles of $TTC(N, P)$.

The above counterexample also serves as a counterexample to Claim 2, since $R_1(P) \neq R_1(P')$.

It is not necessary for an agent in an earlier cycle $\kappa < k$ to change her preference to a house in cycle k or later. She may change her preference to a house in cycle κ or later. This is the necessary addition; we present a corrected version.

Lemma 1 (Claim 1, Corrected)**.** *If there is an $i \in S_k(P)$ such that $x'_i P_i x_i$, then there exist $j \in S_\kappa(P)$ where $\kappa < k$ and $h \in N - R_{\kappa-1}(P)$ such that $w_h P'_j x_j$.*

That is, if an agent wants to get a more preferred good, he needs an agent in an earlier cycle to misrepresent her preference to favor a good that went in her own cycle or a later cycle. Using the notation of Lemma 1, h may be in the same cycle κ as j. In the case of Example 1, $\kappa = 1, j = 2, h = 1$, and $i = 4$.

Proof. Suppose there exists $i \in S_k(P)$ such that $x'_i P_i x_i$. Toward a contradiction, suppose that for each $\kappa < k$ and all $j \in S_\kappa(P)$, we have that $x_j P'_j w_h$ or $x_j = w_h$ for all $h \in N - R_{\kappa-1}(P)$. That is, all agents in cycles before k still rank their original allocation over any other house in their own cycle or later. We show by strong induction on the cycles t of $TTC(N, P)$ that $x'_j = x_j$ for all $j \in S_\kappa(P)$ where $\kappa < k$.

Step $t = 1$. For each $j \in S_1(P)$, x_j was top-ranked under P_j. By assumption, x_j is still top-ranked under P'_j. Then under TTC, the same cycle forms and $x'_j = x_j$ for $j \in S_1(P)$.

Step $t < k$. Suppose that $x'_j = x_j$ for all $j \in S_\tau(P)$ where $\tau < t$. Then $N - R_{t-1}(P') = N - R_{t-1}(P)$; i.e., the remaining houses at step t are the same under P and P'. By assumption, for any agent $j \in S_t(P)$, $x_j P'_j w_h$ for all $h \in N - R_{t-1}(P)$ where $w_h \neq x_j$. That is, x_j is still top-ranked under P'_j among remaining houses. Then under TTC, the same cycle forms and $x'_j = x_j$ for all $j \in S_t(P)$.

We have shown $x'_j = x_j$ for all $j \in S_\kappa(P)$ where $\kappa < k$. As a consequence, $R_{k-1}(P') = R_{k-1}(P)$. However, if $x'_i P_i x_i$, then $x'_i = w_j$ for some $j \in R_{k-1}(P) = R_{k-1}(P')$. But then $i \in R_{k-1}(P') = R_{k-1}(P)$, contradicting the assumption that $i \in S_k(P)$.

□

3.2 Weak group strategy-proofness

We now update the proof of Bird's main theorem using the corrected lemma. The argument proceeds in the same manner as the original.

Theorem (Bird, 1984)**.** *TTC is weakly group strategy-proof.*

Proof. Suppose there is a subset $Q \subseteq N$ reporting $P'_Q \neq P_Q$. Let agent $i \in S_k(P)$ be the first agent in Q to enter a trading cycle in $TTC(N, P)$. If there are multiple such agents, i.e. $|S_k(P) \cap Q| \geq 2$, let i be any such agent. We will show that i cannot strictly improve.

Toward a contradiction, suppose that $x'_i P_i x_i$. Note this requires $k \geq 2$, since agents in $S_1(P)$ could not have improved. By Lemma 1, there exists $j \in S_\kappa(P)$ and $h \in N - R_{\kappa-1}(P)$, where $\kappa < k$, such that $w_h P'_j x_j$. Then $P'_j \neq P_j$ and $j \in Q$. But then i could not have been the first agent (or one of the first) in Q to enter a trading cycle in $TTC(N, P)$, a contradiction. □

3.3 Strong group strategy-proofness

Bird (1984) also presents strong group strategy-proofness as a corollary.

Corollary (Bird, 1984)**.** *TTC is strongly group strategy-proof.*

He gives the following justification:

> [The corollary] follows directly. Trader j must misrepresent his preferences if trader i is to do better. Since the preferences are strict, trader j forms a cycle and receives a good that he does not prefer to the one he would receive under the original top trading cycle.

We feel this requires more elucidation. It is not immediate from strict preferences that j forms a cycle while pointing at the worse house. Thus we provide a proof of strong group strategy-proofness as a corollary of Lemma 1. While other proofs are available,[3] we provide a new one following the ideas laid out here.

We first state the following lemma.

Lemma 2. *Let $x = T(N, P)$, and denote $Q \subseteq P$ and $P = (P_Q, P_{-Q})$. Let P''_Q be such that for all $q \in Q$, P''_q top-ranks x_q and preserves the rest of the rankings in P_q. Denote $P'' = (P''_Q, P_{-Q})$. Then $T(N, P'') = x$.*

That is, if a subset of agents deviate and top-rank the houses they receive in TTC, the resulting allocation is the same. Similar claims are proven in Miyagawa (2002) and Pápai (2000), but we provide a short proof for convenience.

Proof. Note that x is in the strong core for (N, P). Then x is also in the strong core for (N, P''). To see this, suppose there is a blocking coalition against x under (N, P''). Denote this coalition $M \subseteq N$ and the blocking allocation y. For any $m \in Q \cap M$, it must be that $y_q = x_q$, since x_q is the favorite house under P''_q. At least one $i \in Q^c \cap M$ is strictly better off under y, and the rest are weakly better off. But $P''_{Q^c} = P_{Q^c}$, so this is true under P as well. Then (M, y) also blocks under P. We have shown x is also in the strong core for (N, P''). Since TTC finds the unique strong core allocation, $x = TTC(N, P'')$. $\square$

We now prove strong group strategy-proofness by applying Lemmas 1 and 2.

Proof of Corollary (Bird, 1984). Let $Q \subseteq N$ and P'_Q be a misreport. Denote $P' = (P'_Q, P_{-Q})$, $x = T(N, P)$, and $x' = T(N, P')$. Suppose there exists $i \in Q$ such that $x'_i P_i x_i$. We seek to show that some $j \in Q$ is strictly worse off under x'.

Let P''_Q be such that for each $q \in Q$, P''_q top-ranks x'_q and preserves the rest of the rankings in P_q. By Lemma 2, $T(N, P'') = x'$, where $P'' = (P''_Q, P_{-Q})$.

Applying Lemma 1 to (N, P'') and x', there exists $j \in Q$ such that $j \in S_\kappa(P)$ and $w_h P''_j x_j$ for some $h \in N - R_{\kappa-1}(P)$. However, the only change in P''_j from P_j is to top-rank x'_j, so it must be that $w_h = x'_j$. Since x'_j is from cycle κ or later, it was available when j was assigned to x_j. Then $x_j P_j x'_j$ as desired. $\square$

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

[3] Such as Moulin (1995).

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