Revealed Preferences of One-Sided Matching

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Abstract

I study the testable implications of the core in an exchange economy with unit demand when agents' preferences are unobserved. I develop a model in which the core is testable; the identifying assumption is that agents' preferences are solely determined by observable characteristics. I present if and only if conditions for rationalizability of an observed allocation. These conditions are meaningful, intuitive, and tractable; they provide a nonparametric test for the core in the style of revealed preferences. The results formally link together the core, competitive equilibrium, and Afriat's theorem. I also develop a parametric method to estimate preference parameters from multiple observations of exchange economies. An allocation being in the core implies necessary moment inequalities, which I leverage to obtain partial identification.

1 Introduction

This paper studies the testable implications of the core in exchange economies with indivisible goods and unit demand. The setting coincides with the house-swapping matching model of Shapley and Scarf (1974). As in classical revealed preference theory, I take agents' identities, endowments, and allocations to be observable, but their preferences to be unobserved. I investigate the testable implications of the core in exchange economies where only this data are observed. This paper also develops a parametric method to estimate

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preference parameters from multiple observations of such data. In both models with monetary transfers and without, I find conditions that characterize when the observables are consistent with the core ("rationalizability"). Conversely, these conditions can falsify the market being in the core.

The exchange economies is a foundational model in economics for situations without explicit production. With unit demand and indivisible goods, these models correspond to exchanges or allocations of "large" objects. Furthermore, the process of allocation may be unknown or ambiguous – even in the setting with monetary transfers, competitive prices are not inherent to the model. Shapley and Scarf refer to these goods as "houses"; indeed, this is interpretable as a stylized model of (public) housing. The model is also applied to settings such as living donor organ exchange and school assignment.

The core is a game theoretic equilibrium concept. Informally, it captures group stability by requiring that no coalition of agents would prefer to break off and re-trade their endowments among themselves; it is also Pareto efficient. The core does not require competitive prices, which are not inherent to this model. However, I present equivalence results for the core and competitive equilibrium.

The conditions I present characterize restrictions on the observable data of core allocations. An analyst may wish to check for the core for a few reasons. Equilibrium itself may be the object of interest – an allocation which satisfies the conditions is plausibly stable and optimal. Other analysis may also require equilibrium, such as study of the preferences or welfare. The conditions for rationalizability also provide predictions for equilibrium market outcomes before the market is realized. Finally, even in settings with centralized mechanisms, we may wonder whether decentralized markets would select similar outcomes; the restrictions provide a way to test such outcomes.

To rationalize a market, it is sufficient to find a preference profile such that it is in the core. In classical consumer demand revealed preference theory, we infer that the chosen option is the best among affordable options. Afriat (1967) then proceeds from here to construct utility values. However, in an exchange economy, the available options are not exogenously determined by some budget. Stability in an exchange market is determined by all other agents' preferences. Further, the core is not equivalent to maximizing social utility, even when we allow monetary payments.

To gain traction in this setting, I deal with aggregate matchings, akin to Choo and Siow (2006)'s empirical work in marriage markets. Objects are grouped into types, equivalent within type and distinct across types. For instance, these may be apartments in the same development or houses in the same neighborhood, which can plausibly be regarded as essentially the same. I also assume that agents can be binned into "types" with the same preference, analogous to the assumption of Echenique, Lee, Shum, and Yenmez (2013). Stated another way, agents with the same observable characteristics (such as age, wealth, and socioeconomics) have the same preferences. This is a strong assumption as it rules out individual heterogeneity.[1] However, allowing for enough individual heterogeneity also allows any observed market to be rationalized.[2] Furthermore, the resulting test for rationalizability is nonparametric, in the spirit of revealed preferences.

To show the main results, I introduce a graph representation of allocations and develop graph theoretic results. This construction is extremely tractable, and it gives rise to intuitively appealing conditions for rationalizability. In the setting without transfers, rationalizability is equivalent to equal treatment within each type in each partition of the economy. In the setting with monetary transfers, there are two equivalent conditions: the existence of a price vector rationalizing the allocation as a competitive equilibrium, and a cyclic monotonicity condition similar to many in the revealed preference literature.

I also develop a parametric method to estimate utility parameters if the data consist of multiple aggregate matchings without transfers. The setting is similar to Fox (2010) and Echenique, Lee, and Shum (2013). Heterogeneity across aggregate matchings is allowed. Each aggregate matching can first be checked for stability by applying the conditions in the first part of the paper. Stability of the matching implies necessary moment inequalities, which I leverage to obtain partial identification. I illustrate the method by applying the method of Chernozhukov, Chetverikov, and Kato (2019) to the experimental data of Chen and Sönmez (2006).

This paper contributes to the study of the testable implications of equilibria. The Sonnenschein–Mantel–Debreu theorem (1972; 1974; 1974) gives a famous "anything goes" result on the excess demand function. In the same vein, Mas-Colell (1977) shows that there are essentially no restrictions on rationalizable prices in competitive equilibria. Brown and Matzkin (1996) apply revealed preference theory to find obtain restrictions on competitive equilibrium outcomes when a series of markets is observed. Bossert and Sprumont (2002) find conditions for core rationalizability in a two agent economy with divisible commodities. I study a distinct setting – exchange economies with indivisible goods and unit demand – and find tractable and intuitive restrictions on core allocations.

[1] In the model without transfers, rankings are purely ordinal, so small cardinal heterogeneity is allowed.

[2] Simply declare all agents' allocations to be their favorite things.

Additionally, I contribute to the growing literature on the revealed preferences of matching. Echenique, Lee, Shum, and Yenmez study the revealed preferences of matching in marriage markets with aggregate matching type data. Echenique (2008) finds testable implications for two-sided matching when individuals participate in a series of markets.

There are two other ways to interpret this paper. Observers may deal with settings where the mechanism is unknown and therefore cannot be directly evaluated. In practice, many mechanisms are hidden, or no particular centralized mechanism is used at all. But we nevertheless want to determine whether these unknown mechanisms might be stable. Grigoryan and Möller (2023) develop a theory of "auditability", where mechanism implementers may deviate for various reasons; auditability measures how much information the participants need to detect a deviation. This paper offers a way to evaluate mechanisms when essentially nothing is known about the matching process, but the analyst still wants to determine whether the allocation is may be stable. Alternatively, there may be no centralized mechanism at all. For example, Roth and Xing (1997) study decentralized matching for clinical psychologists. In this interpretation, I develop a theory to test stability when there is no particular matching process.

This paper provides a partial identification result for a one-sided matching model. Given allocations presumed to be stable, I find a set of possible utility parameters. In a model with transferable utility, Choo and Siow study aggregate matchings in the marriage market. In the non-transferable utility case, analysts can use intermediate matching data to recover the agents' preferences; Hitsch, Hortaçsu, and Ariely (2010) use rejections in online dating. Recent work by Galichon, Kominers, and Weber (2019) develops an intermediate case, where utility is imperfectly transferable. Echenique, Lee, and Shum develop moment conditions for aggregate two-sided matching data. I direct the reader to Chiappori and Salanié (2016) for a survey of the econometrics of matching.

2 Model

I will first present the model and notation for the case of no transfers. Then I will present the additions for the case of monetary transfers.

2.1 Without transfers

The basis of the model is the Shapley and Scarf house-swapping model with the addition that objects and agents are grouped into types. This will also turn out to be a pure exchange economy with unit demand. Agents of the same type share the same (unobserved to the analyst) preference. Let the set of agent types as $A = \{1, 2, ..., A\}$, where A denotes both the set and its cardinality at minimal risk of confusion; let $|A| < \infty$. Denote the set of individual agents as $\mathcal{A} = \{1a, 1b, ...; 2a, 2b, ...; Aa, Ab, ...\}$, and let $|\mathcal{A}| < \infty$. Implicitly, $\mathcal{A}$ also encodes the types of each individual; e.g., $1a$ and $1b$ are two individuals of the same type 1. I will refer to $i \in A$ as a "type", and $ik \in \mathcal{A}$ as an "agent".

Denote the set of object types H, also with cardinality H. I denote each object as a unit vector in $\mathbb{R}^H$; that is,

$$H = \{\underbrace{(1, 0, ..., 0)}_{:=h_1}, \underbrace{(0, 1, 0, ..., 0)}_{:=h_2}, \underbrace{(0, ..., 0, 1)}_{:=h_H}\} \subset \mathbb{R}^H$$

I will not refer to individual objects – i.e., there is no house analogue of $\mathcal{A}$.

Each agent is endowed with a object, denoted $e_{ik} \in H$. An endowment vector is $e = (e_{ik})_{ik \in \mathcal{A}}$. An allocation is $x = (x_{ik})_{ik \in \mathcal{A}}$ such that $\sum_{ik \in \mathcal{A}} x_{ik} = \sum_{ik \in \mathcal{A}} e_{ik}$. That is, the number of allocated objects of each type is equal to the number supplied.

A feasible sub-allocation for a coalition $A \subseteq \mathcal{A}'$ is $x' = (x'_{ik})_{ik \in A'}$ such that $\sum_{ik \in A'} x'_{ik} = \sum_{ik \in A'} e_{ik}$.

Each type i has a strict preference $\succsim_i$ over H; all ik of type i have the same preference. I will discuss this more after the model is finished. Denote $\succsim = (\succsim_i)_{i \in A}$ be the preference profile. With minimal risk (or consequence) of confusion, this could also be the profile of agents $\succsim = (\succsim_{ik})_{ik \in \mathcal{A}}$.

The equilibrium concept used in this paper is the core.

Definition 1. An allocation x is in the **strict core** for a preference profile $\succsim$ if there is no blocking coalition $A' \subseteq \mathcal{A}$ and feasible sub-allocation x' such that $x'_{ik} \succsim_i x_{ik}$ for all $ik \in A'$, and $x'_{ik} \succ_i x_{ik}$ for at least one $ik \in A'$.

By convention, when a blocking coalition A' is one individual, I say x is not individually rational.[3]

I can now state the main objective of the paper. If we observe individuals, types, endowments, and allocations, could the market be in the core? Formally, is there a preference profile such that x is in the strict core?

Definition 2. A tuple $(A, \mathcal{A}, H, e, x)$ is an **NT-problem** (non-transfers-problem). A problem is **NT-rationalizable** if there exists a preference profile $\succsim$ such that x is in the strict

[3] A blocking coalition of one individual ik means $e_{ik} \succ_i x_{ik}$.

core.

2.2 With transfers

I now introduce monetary transfers. The notation for types, agents, and objects remains the same. Endowments are now an object and amount of money, $(e, \omega) = (e_{ik}, \omega_{ik})_{ik \in \mathcal{A}}$, where $e_{ik} \in H$ and $\omega_{ik} \in \mathbb{R}_{++}$. Likewise, an allocation is an object and amount of money $(x, m) = (x_{ik}, m_{ik})_{ik \in \mathcal{A}}$, such that $m_{ik} \in \mathbb{R}_{++}$, $\sum_{ik \in \mathcal{A}} x_{ik} = \sum_{ik \in \mathcal{A}} e_{ik}$, and $\sum_{ik \in \mathcal{A}} m_{ik} \leq \sum_{ik \in \mathcal{A}} \omega_{ik}$. Note that endowed and allocated money are restricted to be strictly positive. Analogously, a feasible sub-allocation for a coalition A' is $(x', m') = (x'_{ik}, m'_{ik})_{ik \in A'}$ such that $\sum_{ik \in A'} x'_{ik} = \sum_{ik \in A'} e_{ik}$ and $\sum_{ik \in A'} m'_{ik} \leq \sum_{ik \in A'} \omega_{ik}$.

Let utility $V_i : H \times \mathbb{R}_+ \to \mathbb{R}$ be quasilinear, given by $V_i(h, m) = v_i(h) + m$. Notice that the subscript is on types. The $v_i(\cdot)$ can be interpreted as a utility index over H; that is, it is an H-dimensional vector of real numbers representing an cardinal utility for objects. We can regard this model as a partial equilibrium analysis, where all other goods are grouped into money. This is also a common assumption in market design and matching (e.g. Gul, Pesendorfer, and Zhang, 2018; Shapley and Shubik, 1971).

The equilibrium concept in the transfers model is the weak core.

Definition 3. An allocation (x, m) is in the **weak core** if there is no blocking coalition $A' \subseteq \mathcal{A}$ and sub-allocation $(x', m')|_{A'}$ such that $V_i(x'_{ik}, m'_{ik}) > V_i(x_{ik}, m_{ik})$ for all $ik \in A'$.

The weak core and strict core usually coincide, as any strictly better off members can give ε payments to any indifferent members. The exception is when all strictly better off members exhaust their money. The assumption that $\omega_{ik}, m_{ik} > 0$ ensures that money truly enters the model and that the weak core and strict core coincide for rationalizable allocations.[4]

The definition of rationalizability is completely analogous. The analyst observes individuals, types, endowments, and allocations (the latter two including money). I seek a preference profile such that (x, m) is in the core.

Definition 4. A tuple $(A, \mathcal{A}, H, (e, \omega), (x, m))$ is a **T-problem** (transfers-problem). A problem is **T-rationalizable** (transfers-rationalizable) if there exists a preference profile $\succsim$ such that (x, m) is in the weak core. It is **strictly T-rationalizable** if it is T-rationalizable with some strict utility indices; that is, $v_i(h) = v_i(h')$ if and only if $h = h'$ for all i.

[4]It can be argued as in Kaneko (1982) and Quinzii (1984) that money is a bundle of goods outside the model, and it is not "normal" to consume only one indivisible good.

Table 1: Notation

Object	Without transfers	With transfers	Generic member
Agent types	A		i
Individuals/agents	$\mathcal{A}$		ik
Houses	H		h
Endowment	e	(e, ω)	
Allocation	x	(x, m)	
Preferences	$\succsim$	$V_i(h, m) = v_i(h) + m$	

The main result for T-problems will deal with T-rationalizability, so will not impose that the $v_i(\cdot)$ are strict over H. However, I will discuss afterwards how strict T-rationalizability is an intuitive corollary of the main result.

2.3 Discussion

This paper derives necessary and sufficient conditions for a problem to be rationalizable. Stated another way, I characterize allocations which are compatible with the core. As mentioned earlier, this characterization can be used to check for equilibrium; this may be of interest in and of itself or be necessary for further analysis.

Mechanically, this problem is the "reverse direction" of the classic house swapping problem. That is, we have a house swapping market as in Shapley and Scarf (1974) where there are potentially multiple copies of each object. Given an allocation, we are seeking preferences generating it.

The key identifying assumption is common preferences within agent type. As noted above, this is necessary to give the problem testable content; with enough individual heterogeneity, any problem is rationalizable.[5] While not explicitly modeled, this means that preferences are solely functions of agents' observable characteristics. For the non-transfers case, the resulting characterizations are completely nonparametric. For transfers case, I have imposed quasilinear utility; but the utility for objects $v_i(h)$ is otherwise nonparametric. Since the non-transfers preferences are purely ordinal, some *cardinal* heterogeneity is allowed, as long as the same *ordinal* rankings are generated.

If types are constructed from binned variables, the analyst has some degree of choice. Coarser bins result in stronger implications on the allocation, and finer bins result in weaker implications. The "correct" tradeoff is outside of the model of this paper, but the analyst

[5]There are alternatives, such as repeated re-matchings as in Echenique (2008).

can decide on the most reasonable choice.

Finally, rationalizability is not an vacuous concept; it is not hard to construct problems that are not rationalizable. Indeed, the formal results characterize such problems.

2.4 Graph theory

I first introduce some standard definitions for directed graphs that will be useful. Familiar readers can skip this subsection.

Definition 5. A **directed graph (digraph)** is $D = (V, E)$, where V is the set of vertices, and E is the set of arcs. An **arc** is an sequence of two vertices (v_i, v_j); here I allow for arcs of the form (v_i, v_i), called a self-loop.[6] A (v_1, v_k)**-path** is sequence of vertices $(v_1, v_2, ..., v_k)$ where each v_i is distinct, and $(v_{i-1}, v_i) \in E$ for each $i \in \{2, ..., k\}$. A **cycle** is a sequence of vertices $(v_1, v_2, ..., v_k, v_1)$, where each v_i is distinct except for the first and last, and $(v_{i-1}, v_i) \in E$ for each $i \in \{2, ..., k\}$. I will also include self-loops (v_1, v_1) as cycles. Equivalently, a path is a sequence of arcs $((v_1, v_2), ..., (v_{k-1}, v_k))$, and analogously for cycles. The **indegree** of a vertex $d^-(v_i) = |v_j : (v_j, v_i) \in E|$ is the number of arcs pointing at v_i. Likewise, the **outdegree** of a vertex $d^+(v_i) = |v_j : (v_i, v_j) \in E|$ is the number of arcs pointing from v_i.

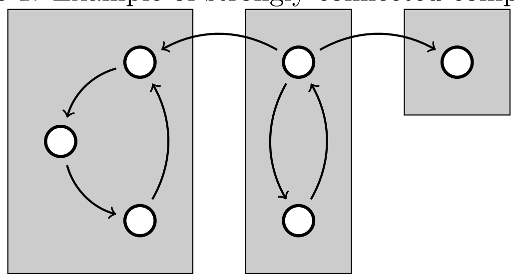
Definition 6. A **weighted directed graph** is a directed graph $D = (V, E, \ell(\cdot))$, where $\ell : E \to \mathbb{R}$ is the **length** (or **weight**) function over arcs. The length of a path or cycle $(v_1, v_2, ..., v_k)$ is $\sum_{i=1}^{k-1} \ell(v_i, v_{i+1})$.

The next definition is used in the main result and its discussion.

Definition 7. A **strongly connected component (SCC)** of a digraph $D = (V, E)$ is a maximal set of vertices $S \subseteq V$ such that for all distinct vertices $v_i, v_j \in S$, there is a (v_i, v_j)-path and a (v_j, v_i)-path. By convention, there is always a path from v_i to itself, even if $(v_i, v_i) \notin E$; an isolated vertex is an SCC.

[6] This is more formally called a **directed pseudograph**.

Figure 1: Example of strongly connected components



The vertices of any digraph can be uniquely partitioned into SCCs. An algorithm by Tarjan (1972) finds a partition in linear time, $O(|V| + |E|)$. Figure 1 illustrates a partition into SCCs.

3 Rationalizability

I now give necessary and sufficient conditions for a problem to be rationalizable.

3.1 Without transfers

First, I introduce a graph construction that is important for the main results. Construct the digraph $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x) = (\mathcal{A}, E)$ as follows: each individual is a vertex. Draw arcs from ik to *all* vertices $i'k'$ that are endowed with x_{ik}. That is, let $(ik, i'k') \in E$ if $x_{ik} = e_{i'k'}$.

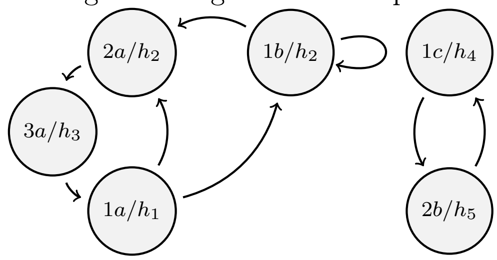
Example 1. Consider the problem described below.

ik	e_{ik}	x_{ik}
$1a$	h_1	h_2
$1b$	h_2	h_2
$1c$	h_4	h_5
$2a$	h_2	h_3
$2b$	h_5	h_4
$3a$	h_3	h_1

That is, $e_{1b} = e_{2a}$, and other endowments are unique. The graph $\mathcal{G}_{NT}$ is given below in Figure 2.

The SCCs of $\mathcal{G}_{NT}$ are the focus of the main result. In the context of this paper's setting, the SCCs are interpretable as partitioning the market into segments that trade

Figure 2: Figure for Example 1



among themselves. I will refer to these informally as *market segments.* Readers familiar with matching may know that any allocation can be decomposed into trading cycles.[7] In the setting of Shapley and Scarf without indifferences, this decomposition is unique. In the present setting, these cycles may not be unique; however, $\mathcal{G}_{NT}$ superimposes all such trading cycles onto one graph.

I now present the main result for the NT-problem.

Theorem 1. *Fix an NT-problem* $(A, \mathcal{A}, H, e, x)$*, and consider* $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$*. The problem is NT-rationalizable if and only if: for agents of the same type* ik, ik' *in the same SCC* S*,* $x_{ik} = x_{ik'}$*. That is, if* $ik, ik' \in S$ *are the same type and in the same SCC, they receive the same house type.*

Proof. Appendix. □

The full proof is contained in the appendix. I give a sketch of the proof below.

Proof sketch of Theorem 1. A key feature of $\mathcal{G}_{NT}$ is that all objects of the same type are contained in the same SCC. The proof of this claim is under 7.

To prove "if": First, find the decomposition of $\mathcal{G}^{big}$ into SCCs. Then assign an arbitrary order to the SCCs, and assign preferences in this order. That is, in the first SCC S_1, let all agents' allocated houses be their first preference. In S_2, let all agents' assigned houses be their first preference if possible, and the second preference if not. By assumption, all agents of the same type in the same SCC receive the same object, so this is a well defined procedure at each step. Since all objects of the same type are in the same SCC, the procedure never attempts to "re-assign" a preference in a later step. The argument that this creates no blocking coalitions is similar to the argument behind Gale's proof for TTC.

[7] Not necessarily Gale's TTC cycles – no claim on optimality is made here yet.

To prove "only if", I show that when the condition is violated, there is a blocking coalition for all preference profiles. One of the two agents of the same type must be worse off; this one can form a blocking coalition with a subset of other members of the SCC. □

Example (Example 1 continued.)**.** The $\mathcal{G}^{big}$ has two SCCs: the left component and the right component. To apply the theorem, select either order arbitrarily. Let the left component be S_1, and the right be S_2. Let $\succsim_i (k)$ denote type i's k^{th} favorite object.

1. In S_1, assign all agents' $\succsim_i (1) = \mu(i)$, so

i	$\succsim_i (1)$
1	h_2
2	h_3
3	h_1

2. In S_2, assign all agents' $\succsim_i (1) = \mu(i)$ if possible (this is not possible for anyone here). Otherwise, let $\succsim_i (2) = \mu(i)$.

i	$\succsim_i (2)$
1	h_5
2	h_4

3. Assign remaining preferences arbitrarily (omitted).

To check for a blocking coalition, note that S_1 all receive their favorite objects. Only agents in S_2 are unsated. Then in any candidate blocking coalition (A', μ'), we require $\mu'(1c) = h_2$ or $\mu'(2b) = h_3$. This requires least one agent in $A' \cap S_1$ to receive either h_4 or h_5, which are strictly dispreferred.

The condition required in Theorem 1 is easy to check; Tarjan's algorithm finds the partition into SCCs in linear time, $O(|\mathcal{A}|)$. Within each SCC, checking for a non-repeated agent type-house type pair is linear in the number of agents.

The most direct interpretation of Theorem 1 is this: whenever agents with the same preferences are in the same market segment, they receive the same house type. Informally, agents in the same market segment have similar market power; if there are multiple agents of the same type, one should not be worse off. Within a market segment, any agent can make a series trades to receive any object in this segment; the formal proof of Theorem 1 implements this to find a blocking coalition.

More formally, a second interpretation is in the context of a competitive equilibrium market. Roth and Postlewaite (1977) show that the strict core is a competitive equilibrium in the typical house exchange setting with no indifferences. Wako (1984) establishes that a strict core allocation is also a competitive equilibrium in a the setting with indifferences. If x is in the core for some preference profile $\succsim$, it is also a competitive equilibrium allocation for some price vector. A supporting price vector is descending in the (arbitrarily selected) order of SCCs. Thus if two agents are in the same SCC, their endowments are worth the same in competitive equilibrium. The necessity of the condition becomes immediate; two agents with the same budget and preferences should purchase the same house type.

I now present some related results. First, an important implication of Theorem 1 is the following corollary:

Corollary 1. *Fix a problem $(A, \mathcal{A}, H, e, x)$. The problem is rationalizable only if: whenever agents ik, ik' are the same type and $e_{ik} = e_{ik'}$, $x_{ik} = x_{ik'}$.*

Proof. Appendix. □

That is, equal agents (of same type and same endowment) must receive the same house type. Briefly, the theorem requires equal treatment of equals. When types determine both preferences and endowments, this corollary gives us the condition for rationalizability.

Corollary 2. *Suppose $e_{ik} = e_{ik'}$ for all k, k' and for all $i \in A$. That is, all agents of the same type have the same endowment. Then the problem $(A, \mathcal{A}, H, e, x)$ is rationalizable if and only if $x_{ik} = x_{ik'}$ for all k, k' and for all $i \in A$. That is, if and only if all agents of the same type receive the same object.*

Proof. "Only if" is a consequence of Corollary 1. To prove "if", note that everyone of the same type receives the same object, so we can let everyone's favorite object be their allocated object. □

This resembles the Debreu and Scarf (1963) theorems for general equilibrium. Their model is an endowment economy with a finite number of goods, agent types, k copies of each type, and certain restrictions on preferences. Only allocations assigning the same bundle to all agents of the same type are in the core. While neither the Debreu and Scarf model nor my model contains the other, it would be interesting future work to investigate a whether deeper connection exists.

Another related question is: what is the minimum number of agent types necessary to rationalize an allocation? That is, suppose we are free to choose agent types. What is the minimum preference type heterogeneity required to put x in the core? This question is sensible, since allowing every individual to be his own type always rationalizes an allocation.

Let $\tilde{A}$ be the set of individual agents, without encoding information on types. With this data, we can construct a graph $\tilde{\mathcal{G}}_{NT}\left(\tilde{\mathcal{A}}, H, e, x\right)$ in the same way as $\mathcal{G}_{NT}\left(A, \mathcal{A}, H, e, x\right)$.

Corollary 3. *Consider $\tilde{\mathcal{G}}_{NT}\left(\mathcal{A}, H, e, x\right)$, and decompose this into SCCs, $\{S_1, ..., S_M\}$. Let α_m be the number of distinct object types in S_m. The minimum number of types necessary to construct $\succsim$ such that x is in the core is $\alpha = \min\{\alpha_1, ..., \alpha_m\}$.*

Proof. In light of Theorem 1, individuals in the same SCC_m who receive different objects must be different agent types. There is no other restriction on agent types.

Within S_m, there are α_m distinct objects; order them arbitrarily. Let everyone who receives object 1 be type 1, and so on. By Theorem 1, this will be rationalizable. It is also clear that having fewer than α types will make the problem not NT-rationalizable. $\square$

The result also solves the analogous problem for two-sided matching in the strict core. That is, it solves a strict stability analogue of Echenique, Lee, Shum, and Yenmez with non-transferable utility. The result follows from inserting one-sided matching with endowments into two-sided matching in the usual way. There are types of men and women, and each type has a strict preference over potential partner types. Each agent type has a unique endowment (him- or her- self). Apply Corollary 1; an observed market is rationalizable if and only if all agents of the same type must be assigned the same type of partner.

3.2 With transfers

I derive necessary and sufficient conditions for a T-problem to be T-rationalizable. First, I introduce a new weighted digraph $\mathcal{G}_T\left(A, \mathcal{A}, H, (e, \omega), (x, m)\right) = (\mathcal{A}, E, \ell(\cdot))$. Draw vertices and arcs as in $\mathcal{G}_{NT}$; let each agent be a node, and draw arcs from ik to *all* vertices $i'k'$ that are endowed with x_{ik}. In addition, define the lengths arcs by $\ell\left(ik, i'k'\right) = \omega_{ik} - m_{ik}$. Note that this depends only on the first vertex, not the second.

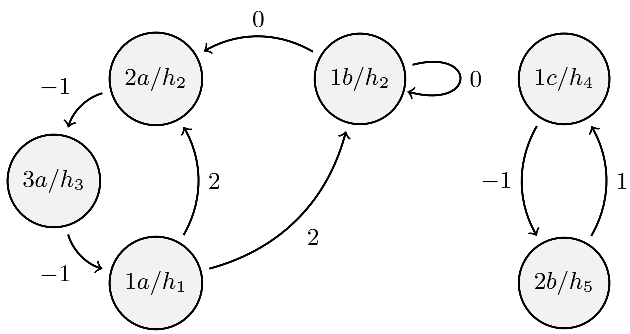
The following example adds to 1.

Example 2. Consider the problem described in Example 1, adding the following payments:

$\mathcal{A}$	e_{ik}	x_{ik}	$\omega_{ik} - m_{ik}$
$1a$	h_1	h_2	2
$1b$	h_2	h_2	0
$1c$	h_4	h_5	1
$2a$	h_2	h_3	-1
$2b$	h_5	h_4	-1
$3a$	h_3	h_1	-1

The following figure illustrates $\mathcal{G}_T$.

Figure 3: Figure for Example 2, $\mathcal{G}_T$



I now give the main result for T-rationalizability.

Theorem 2. *Fix a T problem* $(A, \mathcal{A}, H, (e, \omega), (x, m))$. *The following are equivalent:*

1. *The problem is T-rationalizable.*
2. *There exists a vector* $p \in \mathbb{R}_+^{|H|}$ *such that*

$$(x_{ik} - e_{ik}) \cdot p = \omega_{ik} - m_{ik} \quad \forall ik \in \mathcal{A} \qquad (P)$$

3. *The graph* $\mathcal{G}_T\,(A, \mathcal{A}, H, (e, \omega), (x, m))$ *has no cycles with length* > 0.

Proof. Appendix. □

The vector p in (P) (suggestively denoted) is interpretable is a price vector for houses. Indeed, the left side is the difference in price between the allocated and endowed houses, and the right side is the net payment from ik. This suggests an easy interpretation of the theorem: a problem is TU-rationalizable if and only if everyone who "buys" a house type pays the same price for it. I present a sketch of the proof here.

Proof sketch of Theorem 2. I show (2) $\implies$ (1). Given p, I seek v_i such that (x, m) is a competitive equilibrium, which will then give us weak core by the typical arguments. We are looking for utility indices v_i such that all agents ik are maximizing subject to their budget constraints, given by $e_{ik} \cdot p + \omega_{ik}$. Then this becomes a classic consumer demand revealed preference problem. To see this, reinterpret a type i as a single consumer, and each individual ik as a demand data point:

$$\left\{ \underbrace{(x_{ik}, m_{ik})}_{\text{consumed good and money}}, \underbrace{(e_{ik} \cdot p + \omega_{ik})}_{\text{budget}}, \underbrace{p}_{\text{price}} \right\}$$

In this structure, such demand data are always rationalizable (in the consumer demand revealed preference sense). The easiest way to show this is to let $v_i(x_{ik}) = x_{ik} \cdot p$ for all i, ik, though I show in the full proof this knife-edge construction is not the only one. Then (x, m) is a competitive equilibrium supported by p, and thus (x, m) is in the weak core.

I now show (1) $\implies$ (3). To see this, note that a cycle C's length $\sum_{ik \in C} \omega_{ik} - m_{ik}$ is its members' total net payment of money. If this is greater than 0, then this cycle net spends money. Its members can form a blocking coalition – they can allocate houses the same way as in (x, m), but keep their full endowed money for themselves.

Finally, to show (3) $\implies$ (2), I use the shortest path length on $\mathcal{G}_T$ between two houses to construct the price difference between those houses. The construction is similar to that in Quinzii (1984). (We can choose an arbitrary base price high enough so that $p \geq 0$.) In the full proof, I show that this construction is consistent – the minimum path length between houses of the same type is always 0. This completes the proof. □

I give an example to illustrate T-rationalizability.

Example (Example 2 continued.)**.** For simplicity, let $\omega_{ik} = 3$ for all ik. It can be seen that all cycles have length 0, so this is rationalizable. Figure 3 shows $\mathcal{G}_T$, with $\omega_{ik} - m_{ik}$ as arc lengths.

Example 3. To construct utilities, set p the following way. In the left SCC, let $p_{h_1} = 3$ arbitrarily, and set the prices of other houses in this SCC by the minimum path length from

h_2 plus 3, giving $p_{h_2} = 5, p_{h_3} = 4$. Notice that the path length between the two copies of h_1 is 0. In the right SCC, let $p_{h_4} = 1$ arbitrarily, and set $p_{h_5} = 2$ since the path length from h_4 to h_5 is 1. Altogether,

$$
\begin{aligned}
p_{h_1} &= 3 \\
p_{h_2} &= 5 \\
p_{h_3} &= 4 \\
p_{h_4} &= 1 \\
p_{h_5} &= 2
\end{aligned}
$$

The easiest way to construct T-rationalizing preferences is to let $v_i = p$ for all i. Though as mentioned above (and demonstrated in the full proof), this is not the only construction.

The theorem establishes a connection between T-rationalizability, competitive equilibrium, and consumer demand rationalizability. The question of T-rationalizability is equivalent to consumer demand rationalizability, à la Samuelson and Afriat. That is, an allocation is rationalizable if and only if each agent type, interpreted as demand data, is consumer demand rationalizable. Thus, we are looking for utility indexes such that every agent type is optimizing in their demand. Competitive equilibrium follows.

This yields the theorem's two equivalent and intuitive conditions for T-rationalizability. The first condition is the existence of a price vector supporting the allocation as a competitive equilibrium. That is, an allocation is T-rationalizable if and only if it can be supported as a competitive equilibrium. The second condition is reminiscent of cyclic monotonicity results common in revealed preference literature. It is readily interpretable directly; a cycle having positive length means it net transfers money outwards. Then its members can implement the same object allocation while retaining its money, establishing a blocking coalition.

I now give some corollaries of Theorem 2. First, I give conditions for strict T-rationalizability.

Corollary 4. *Fix a TU problem $(A, \mathcal{A}, H, x, m, e, \omega)$. Assume (A1). The problem is strictly T-rationalizable if and only if* ***both*** *of the following are true:*

1. *The problem is T-rationalizable.*
2. *If $ik, ik' \in S$ are the same type and in the same SCC in $\mathcal{G}_T$, then $x_{ik} = x_{ik'}$ OR the shortest path length from x_{ik} to $x_{ik'} \neq 0$.*

Proof. Appendix. □

This is the T-rationalizability analogue to Theorem 1. The additional condition says that two individuals of the same type, in the same SCC, should either be allocated the same object or pay different amounts. Having a zero path length between x_{ik} and $x_{ik'}$ means their prices must be the same. Then if two different individuals type i purchase each one in competitive equilibrium, they must have the same utility. Conversely, having a nonzero path length allows us to construct different prices, and thus different utilities.

The following examples illustrate the corollary.

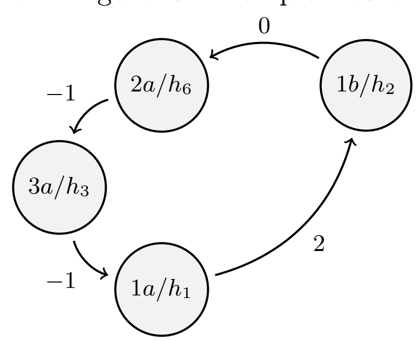
Example (Example 2 continued.)**.** This example is strictly T-rationalizable. The only thing to check is x_{1a} and x_{1b}. Since $x_{1a} = x_{1b}$, the problem is strictly TU-rationalizable – indeed, the utility given in the original example suffices.

Example 4. Suppose instead $x_{1b} = e_{2a} = h_6$, a new house type, with no other changes. Focusing on the left SCC:

ik	e_{ik}	x_{ik}	$\omega_{ik} - m_{ik}$
$1a$	h_1	h_2	2
$1b$	h_2	h_6	0
$2a$	h_6	h_3	-1
$3a$	h_3	h_1	-1

This problem is T-rationalizable, but not strictly T-rationalizable. The minimum path

Figure 4: Figure for Example 2 continued.



length from $x_{1a} = h_2$ to $x_{1b} = h_6$ is 0, forcing $p_{h_2} = p_{h_6}$. If $v_1(h_2) > v_1(h_6)$, then $1b$ is not maximizing subject to his budget, so the allocation is not a competitive equilibrium and not in the weak core.

The next corollary characterizes possible utility indexes $v_i(\cdot)$ that a T-problem.

Corollary 5. *Fix a T-problem $(A, \mathcal{A}, H, (e, \omega), (x, m))$. A T-rationalizable problem's solutions $v_i(\cdot)$ are characterized by solutions to the following linear system.*

1. $v_i(x_{ik}) \leq v_i(x_{ik'}) + p \cdot (x_{ik} - x_{ik'}) \quad \forall i, \forall ik, ik'$
2. *for any h such that $h \neq x_{ik}$ $\forall x_{ik}$, for any ik such that $h \cdot p \leq e_{ik} \cdot p + \omega_{ik}$:*
$$v_i(h) - h \cdot p \leq v_i(x_{ik}) - x_{ik} \cdot p$$

for p s.t. $(x_{ik} - e_{ik}) \cdot p = \omega_{ik} - m_{ik} \quad \forall ik \in \mathcal{A}$

Proof. Appendix. □

The first line is the Afriat inequalities for quasilinear utility (with marginal utility of money equal to one). Given some valid price vector p, these give the restrictions of utilities for houses that are actually consumed by type i. The second line gives restrictions on utilities for any houses that are never consumed by type i. If a house h is never consumed but is affordable under some budget $e_{ik} \cdot p + \omega_{ik} := I_{ik}$, its consumption bundle $(h, I_k - h \cdot p)$ must be dispreferred to the actually consumed bundle $(x_{ik}, I_k - x_{ik} \cdot p)$. The third line characterizes valid price vectors.

This linear system identifies possible values of (v_i) from the observed data. As is the case in consumer demand revealed preferences, these are joint restrictions rather than valid ranges for each $v_i(h)$. For example, there are infinite possible price vectors, each leading to a range of possible utility indexes v_i's. I also show in the full proof of Theorem 2, relative prices are determined within an SCC but not across SCCs.[8] Nevertheless, this corollary fully characterizes the joint restrictions for valid v_is.

4 Estimating utility parameters from aggregate matching data

I turn to the task of estimating preferences from aggregate matchings without transfers. In the original setting, it is hard to determine rationalizing preference profiles. The proof Theorem 1 shows that many preference profiles rationalize a problem, and they are "dissimilar" due to the arbitrary order of SCCs. However, with a series of observations involving

[8] For this reason I conjecture it is not possible to write a linear system without the existential statement of (P).

the same agent types and object types; and if we assume a parametric form of utility; it is possible to estimate utility parameters.

In this section, I derive an econometric method to estimate a confidence region for utility parameters from multiple stable matchings. The setup is similar to Fox (2010), though the resulting method is distinct. In the absence of perfectly transferable utility, we cannot assume utility maximizing choices. Thus, my objective is to estimate utility parameters from revealed preferences-type data. I will derive necessary moment inequalities for stability, then apply the method of Chernozhukov, Chetverikov, and Kato (2019).

4.1 Setup

The basic setup is the same as the exchange economy without transfers in Section 2.1.

Definition 8. The **aggregate matching matrix** X is the matrix with $A \times H$ rows, representing agent type-endowed object pairs; and H columns, representing allocated objects. Entry $X_{ie,h}$ is the number of type i endowed with e allocated h.

Now we observe $t = 1, ..., T$ rationalizable economies with the same types, each represented by X_t. Given a series of aggregate matchings, we can first apply the condition in Theorem 1 to check for rationalizability. Additionally, let preferences be given by utility $u_i(h; \beta, \varepsilon_{iht})$, a function of observable characteristics of the object, unknown parameter β, and heterogeneity ε_{iht} with known distribution. This heterogeneity term is not necessarily *individual* heterogeneity, but it allows *types* to have heterogenous utility for objects across aggregate matchings t.

4.2 Moments and identification

An aggregate matching being in the core implies moment inequalities we can use to estimate the parameter β. First, the allocations must respect individual rationality. For $e, h \in H$ and $e \neq h$, an individual of type i must prefer his allocation to his endowment

$$\mathbb{1}\left(X_{ie,h} > 0\right) \Longrightarrow \left[\mathbb{1}\left(h \succ_i e\right) = 1\right]$$

Giving

$$\mathbb{P}_t\left(X_{ie,h} > 0\right) \leq \mathbb{P}\left[\mathbb{1}\left(h \succ_i e; \beta\right) = 1\right]$$

and moment inequality

$$\mathbb{E}_t\left[\mathbb{1}\left(X_{ie,h}>0\right)\right]-\mathbb{P}\left[\mathbb{1}\left(h\succ_i e;\beta\right)=1\right]\leq 0 \tag{1}$$

Likewise, the core implies no blocking coalitions of size 2. For $e\neq h', e'\neq h, e\neq e'$, and $i\neq i'$, we have

$$\mathbb{1}\left(X_{ie,h}>0, X_{i'e',h'}>0\right)\Longrightarrow\left[\mathbb{1}\left(e\succ_{i'} h'\right)\mathbb{1}\left(e'\succ_i h\right)=0\right]$$

That is, it cannot be that there is an individual of type i and one of type i' that prefer each other's endowments. Then

$$\mathbb{P}_t\left[X_{ie,h}>0, X_{i'e',h'}>0\right]\leq\mathbb{P}\left[\mathbb{1}\left(e\succ_{i'} h'\right)\mathbb{1}\left(e'\succ_i h\right)=0;\beta\right]$$

This gives the analogous moment inequality

$$\mathbb{E}_t\left[\mathbb{1}\left(X_{ie,h}>0, X_{i'e',h'}>0\right)\right]-\mathbb{P}\left[\mathbb{1}\left(e\succ_{i'} h'\right)\mathbb{1}\left(e'\succ_i h\right)=0;\beta\right]\leq 0 \tag{2}$$

I use inequalities in 1 and 2 to estimate β. The identified set is given by

$$\left\{\beta:\begin{cases}\mathbb{E}_t\left[\mathbb{1}\left(X_{ie,h}>0\right)\right]-\mathbb{P}\left[\mathbb{1}\left(h\succ_i e;\beta\right)=1\right]\leq 0 & \forall i, e\neq h'\\ \mathbb{E}_t\left[\mathbb{1}\left(X_{ie,h}>0, X_{i'e',h'}>0\right)\right]-\mathbb{P}\left[\mathbb{1}\left(e\succ_{i'} h'\right)\mathbb{1}\left(e'\succ_i h\right)=0;\beta\right]\leq 0 & \forall i\neq i', e\neq h', e'\neq h, e\neq e'\end{cases}\right\}$$

These are necessary conditions for the core; they form an outer bound for the true β. It is also possible to add analogous inequalities for coalitions of size ≥ 3. However, the number of inequalities grows combinatorially, so the trade-off in tractability is unlikely to be favorable. Note also that this is a partially identified model.

These conditions do not come from utility maximization in a choice set, as in Choo and Siow (2006) or Fox (2010). In this setting, agents' choice sets are functions of the matching process. Indeed, in the setting without transfers, the allocation is not the social welfare maximizing allocation in general.

There is a substantial econometric literature on estimating confidence sets from moment inequalities; see Chernozhukov, Hong, and Tamer (2007); Chernozhukov, Chetverikov, and Kato (2019); Canay, Gaston, and Velez (2023). A number of methods are possible to estimate the given model. I will apply Chernozhukov, Chetverikov, and Kato (2019), who develop an analytic test when the number of moment inequalities p is large (as it is here). Their setting allows p to potentially grow with T, and in the present setting p is fixed with the number of types. However, we are likely to have $p \gg T$ in any plausible data.

4.3 Empirical application

I use data from Chen and Sönmez (2006) to illustrate the method. In their experiment, they have subjects participate with each other in Deferred Acceptance, Top Trading Cycles, or the Boston mechanism in a school assignment setting. In six treatments of 36 participants each, they induce a true ranking over 7 (A-G) schools via a "designed" utility.

Participants are assigned an ID number 1-36, which determines the nonrandom component of utility. Each ID number is assigned to a home district (e.g. 1-3 are assigned to A). Participants receive 10 utility for being assigned to their home district; this also functions as the endowment.[9] School A is a top school with arts specialty; school B is a top school with a science specialty. Odd numbered participants receive 40 utility for school B (top school and good fit) and 20 utility for school A (top school and bad fit). Even numbered participants receive 40 utility for school A and 20 utility for school B. ID numbers thus determine types; utility of type i for school S can be written as

$$\begin{aligned} u_i(S) = \beta_1 \times \mathbb{1}\{S = \text{home}\} \\ + \beta_2 \times \mathbb{1}\{S = \text{top school, good fit}\} \\ + \beta_3 \times \mathbb{1}\{S = \text{top school, bad fit}\} \\ + \varepsilon_{i,s} \end{aligned}$$

where $\varepsilon_{i,s} \sim_{iid} \text{Norm}(\mu = 20, \sigma = 11.5)$[10] and $\beta = (10, 40, 20)$. Given true rankings induced by this procedure, participants submit rankings for one of the three mechanisms.

Chen and Sönmez use this setup to test rates of truth-telling in strategy-proof mechanisms. I will not directly address the same question as their experiment – their procedure induces a setting that can be used to illustrate this paper's method. I apply my moment inequalities to produce confidence regions for β from the resulting allocations. The probabilities in 1 and 2 can be calculated analytically given the known distribution of $\varepsilon_{i,s}$. For

[9]In each mechanism used, the participant is guaranteed no worse than his home district (by reported rankings).

[10]Chen and Sönmez actually use Unif$(0, 40)$, but I will assume a normal distribution with the same mean and variance for tractability.

example,

$$\begin{aligned}
\mathbb{P}\left[\mathbb{1}\left(h \succ_i e;\beta\right)=1\right] &= \mathbb{P}[\beta_1\left\{h=\text{home}\right\}+\beta_2\left\{h=\text{top school, good fit}\right\}+\beta_3\left\{h=\text{top school, bad fit}\right\}+\varepsilon_{i,h} \\
&\qquad > \beta_1\left\{e=\text{home}\right\}+\beta_2\left\{e=\text{top school, good fit}\right\}+\beta_3\left\{e=\text{top school, bad fit}\right\}+\varepsilon_{i,e}] \\
&= \mathbb{P}\left[\beta_1\Delta\{\text{home}\}+\beta_2\Delta\left\{\text{top school, good fit}\right\}+\beta_3\Delta\left\{\text{top school, bad fit}\right\}>\varepsilon_{i,e}-\varepsilon_{i,h}\right] \\
&= \Phi_{\mu,\sigma}\left(\beta_1\Delta\{\text{home}\}+\beta_2\Delta\left\{\text{top school, good fit}\right\}+\beta_3\Delta\left\{\text{top school, bad fit}\right\}\right)
\end{aligned}$$

where $\Phi_{\mu,\sigma}(\cdot)$ is the CDF of a Norm$(0, 16.33)$ distribution.

Since their data only includes 6 treatments of this kind (and thus 6 observations), I supplement the data by simulating additional allocations. I randomly sample 36 individuals – one of each type across the 6 treatments – then interact them in TTC to produce another observation. I repeat this procedure 24 times to produce 30 total aggregate matchings.

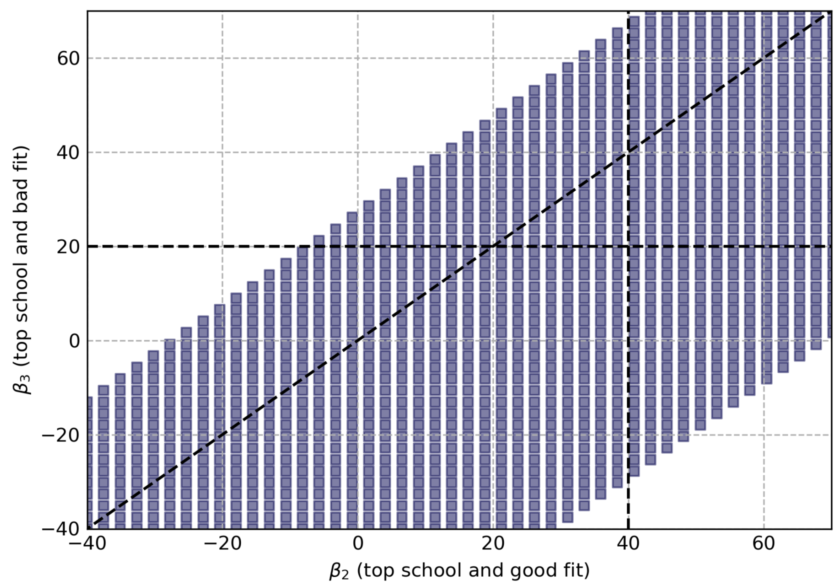
There are a few features of the data to note. The first is that the aggregate matchings are produced using *submitted* rankings, not the true induced rankings. The second is that the random component of utility is the same magnitude as the nonrandom component.

We first check the conditions of Theorem 1 to confirm each of the 30 aggregate matchings are rationalizable, and thus possibly in the core. (Since the 6 real treatments have one individual of each type, this passes. The 24 simulated treatments are generated via TTC, which is guaranteed to produce a core allocation according to submitted rankings. Note that the allocations are not necessarily stable according to the *true* rankings.) Then I estimate a confidence set for β using the moments in 1 and 2. Figure 5 shows a 80% confidence set by cross-sections of β_1.

While the confidence set is large, this is to be expected given the nature of the data. I also cannot pin down the scale of the parameters β with the given data, so I fix the range to that shown in Figure 5.

Some interesting patterns from the experiment emerge in this estimation. The true β_0 is contained well inside the confidence set. The confidence set also reflects some common "strategic behavior" that Chen and Sönmez observe in their experiment. While most of the confidence set is in $\beta_2 > \beta_3$, a sizable portion is not; this is consistent with top-two switching, a very common non-truth-telling strategy. The region $\beta_2, \beta_3 < \beta_1 = 10$ is also consistent with home district bias, another common deviation from the truth. The findings illustrate the method I have derived here.

Figure 5: Estimated 80% confidence set for β_2, β_3
PLACEHOLDER:



5 Conclusion

I present testable implications of the core in exchange economies with and without monetary transfers. The key identifying assumption is on agent types – that preferences are solely a function of observable characteristics of the agents. The analyst observes these types, the endowment, and allocation, but not the preference. Given this, I derive tractable and intuitive conditions for the core to be rationalizable.

These conditions characterize markets which are compatible with the core. That is, they can falsify a market being in the core; they also serve as ex ante predictions for market outcomes. The results can also be applied to audit mechanisms when the matching procedure is unknown.

I also develop a parametric method to estimate parameters of utility generating core allocations. Given a series of core allocations ("aggregate matchings"), the core implies a series of moment conditions, which I use to obtain partial identification.

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3.1

A Results for $\mathcal{G}_{NT}$

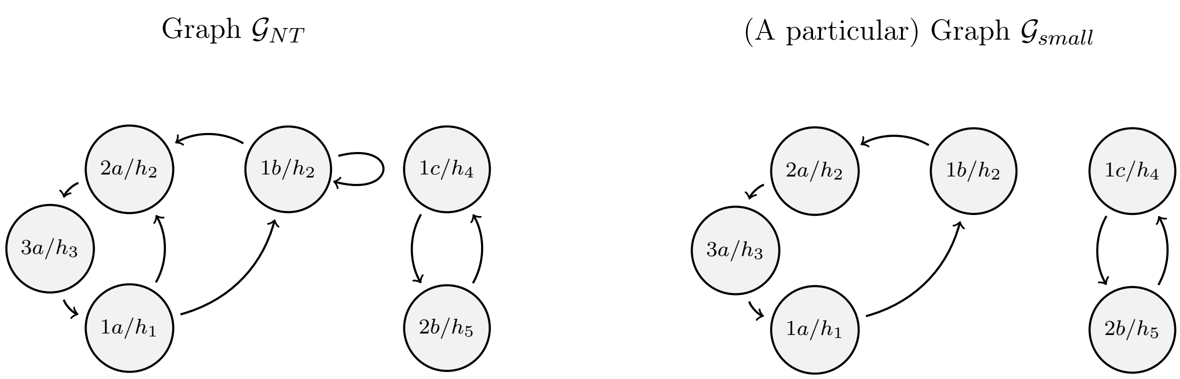
First, I introduce another graph construction. Given a NT-problem[11], draw $\mathcal{G}_{small}\left(A, \mathcal{A}, H, e, x\right) = \left(\mathcal{A}, E'\right)$ as follows:

Initialize. Draw all agents $\mathcal{A}$ as vertices.

Step m. Consider all agents receiving h_m, that is all ik such that $x_{ik} = h_m$. Order them according to their index; refer to these as the “left” side. Similarly, order agents endowed with h_m according to their index; these are the “right” side. By construction, these two sets are the same cardinality. Draw one arc from the first agent on the left side to the first agent on the right side, and so on. If $m < \eta$, continue to step $m+1$.

The graph produced after $|H|$ steps represents the allocation μ. Note that each agent has one out-arc and one in-arc. Recall the construction of $\mathcal{G}_{NT} = (\mathcal{A}, E)$. Note also that $E \supseteq E'$; that is, $\mathcal{G}_{NT}$ can be obtained by adding arcs to $\mathcal{G}_{small}$. Figure 6 shows both constructinos for Example 1.

Figure 6: Figure for Example 1



I now provide some intermediate results related to the constructed graphs $\mathcal{G}_{small}$ and $\mathcal{G}_{NT}$. These will be key for the proof of Theorem 1.

[11] Or, if given a T-problem, discard ω and m.

Proposition 1. *Consider* $\mathcal{G}_{small}\,(A, \mathcal{A}, H, e, x) = (\mathcal{A}, E')$*.* $\mathcal{G}_{small}$ *has a subgraph partition into cycles. That is, there are disjoint subgraphs* $C_1, ..., C_N$ *such that* $\mathcal{G}_{small} = \cup_{n=1}^{N} C_n$*,* $C_m \cap C_n = \emptyset$ *for all* m, n*, and each* C_n *is a cycle.*

Proof. Note each vertex i has $d^-(ik) = d^+(ik) = 1$. We can invoke a version of Veblen's theorem:

> (Veblen's theorem) A directed graph $D = (V, E)$ admits a partition of arcs into cycles if and only if $d^-(v) = d^+(v)$ for all vertices $v \in V$. (Veblen, 1912; Bondy and Murty, 2008)

Since $d^-(ik) = d^+(ik)$, $\mathcal{G}_{small}$ has a partition of arcs into cycles. There are no isolated vertices, so every vertex is in at least one cycle. Further, since $d^-(ik) = d^+(ik) = 1$ each vertex must be in at most one cycle. Thus the arc partition into cycles also partitions the vertices into cycles. □

Proposition 2. *Consider* $\mathcal{G}_{NT}\,(A, \mathcal{A}, H, e, x)$*. For every strongly connected component* S *of* $\mathcal{G}_{NT}$*, there is a cycle including all vertices in* S*.*

Proof. By Proposition 1, $\mathcal{G}_{small}$ admits a partition of vertices into cycles. Recall $\mathcal{G}_{NT} = (\mathcal{A}, E)$ and $\mathcal{G}_{small} = (\mathcal{A}, E')$, where $E \supseteq E'$. Then these cycles also partition $\mathcal{G}_{NT}$'s vertices. The SCC S in $\mathcal{G}_{NT}$ is composed of the vertices in a number of $\mathcal{G}_{small}$-cycles. It cannot include a strict subset of vertices in a $\mathcal{G}_{small}$-cycle since there is always a path between any two vertices in a cycle.

The remaining argument is by strong induction on the number K of $\mathcal{G}_{small}$-cycles contained in S. Assign an order to these cycles in the following way. Let the first cycle be any of these. Choose the k^{th} cycle such that it has the same house type as one of the first $k-1$ cycles. It is always possible to do this – suppose at some point none of the remaining cycles has the same house type as the first k cycles. Then there are no paths in $\mathcal{G}_{NT}$ between the first k cycles and the remaining cycles (recall arcs are drawn from an agent to all agents whose endowment he receives), so they are not in the same SCC.

Claim. There is a cycle in $\mathcal{G}_{NT}$ covering all vertices in the first k $\mathcal{G}_{small}$-cycles in S. As shorthand, I will call this the "big-cycle", and the $\mathcal{G}_{small}$-cycles will be "small-cycles".

Base claim. For $k = 1$, the claim is trivial.

k^{th} claim. Suppose the claim is true for the first $k-1$ cycles. That is, there is a $k-1^{th}$ big-cycle in $\mathcal{G}_{NT}$ covering all the vertices in the first $k-1$ small-cycles. I show that there is a cycle covering all vertices in the $k-1^{th}$ big-cycle and the k^{th} small-cycle. The following argument is illustrated in Figure 7. There are three cases, depending on whether either cycle is a self-loop.

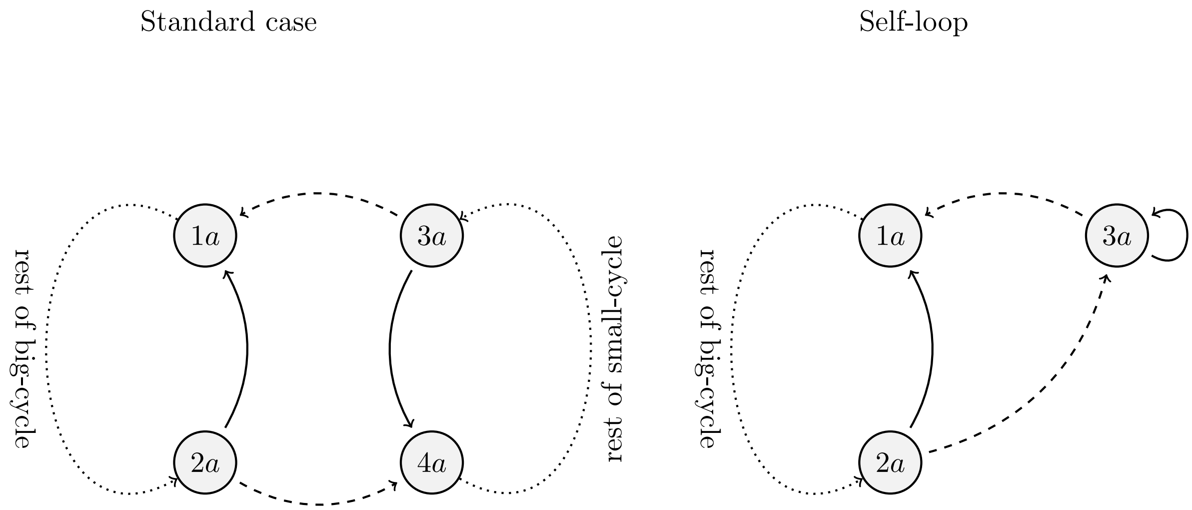
Case 1. Suppose neither is a self-loop. Let the big-cycle be $(1a, ..., 2a, 1a)$, and the k^{th} small-cycle be $(3a, 4a, ..., 3a)$. That is, $x_{2a} = e_{1a}$ and so on. I do not require that the denoted agents are all different types; e.g. $2a$ can be $1b$. By the ordering of the cycles, the k^{th} small-cycle and the $k-1^{th}$ big-cycle have at least one of the same house type. Without loss of generality let $e_{1a} = e_{4a}$. This gives $x_{2a} = e_{1a} = e_{4a}$, so we have the arc $(2a, 4a) \in E$. Similarly, $x_{3a} = e_{4a} = e_{1a}$, so we have the arc $(3a, 1a) \in E$. This gives us a new big-cycle across all the vertices in the first k small-cycles: $(\underbrace{1a, ..., 2a}_{\text{big-cycle } k-1}, \underbrace{4a, ..., 3a}_{k^{th} \text{ cycle}}, 1a)$.

Case 2. Suppose the k^{th} small-cycle is a self-loop, but the $k-1^{th}$ big-cycle is not. Then let the big-cycle be $(1a, ..., 2a, 1a)$, and the k^{th} small-cycle be $(3a, 3a)$. Again, let $e_{1a} = e_{3a}$ without loss of generality. Then $x_{2a} = e_{1a} = e_{3a}$ implies $(2a, 3a) \in E$. Likewise, $x_{3a} = e_{3a} = e_{1a}$ implies $(3a, 1a) \in E$. So we have a new big-cycle $(\underbrace{1a, ..., 2a}_{\text{big-cycle } k-1}, 3a, 1a)$. The case if the big-cycle is a self-loop is the same (this may occur in the $k = 2$ claim).

Case 3. Suppose both are self-loops. Then let the big-cycle be $(1a, 1a)$ and the k^{th} small-cycle be $(3a, 3a)$. Again, we suppose $e_{1a} = e_{3a}$. Then $x_{1a} = e_{1a} = e_{3a}$ implies $(1a, 3a) \in E$, and likewise $(3a, 1a) \in E$. So we have a new big-cycle $(1a, 3a, 1a)$.

This completes the proof. $\square$

Figure 7: Illustration of Proposition 2



The following lemma is derived from Proposition 2 and its proof.

Lemma 1. *Consider $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$. Every strongly connected component S has no in- or out- arcs. That is, if $ik \in S$ and $(ik, i'k') \in E$ or $(i'k', ik) \in E$, then $i'k' \in S$.*

Proof. There is a cycle covering all vertices of S by Proposition 2. Suppose there is an out-arc from S pointing to a vertex in a different SCC S'. S' also has a cycle covering all its vertices. The same argument as in the induction part of the proof of Proposition 2 establishes an arc from S' to S. Thus there are paths from between any vertices in S and S', and they are in the same SCC, a contradiction. The case for no in-arcs is a relabeling of S and S'. □

The following is a corollary of Lemma 1.

Corollary 6. *Consider $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$. Let ik and $i'k'$ be distinct vertices. There exists a $(ik, i'k')$-path if and only if ik and $i'k'$ are in the same SCC. Equivalently, there exists a $(ik, i'k')$-path if and only if there exists a $(i'k', ik)$-path.*

Proof. If ik and $i'k'$ are in the same SCC, there exists a $(ik, i'k')$-path by definition. Suppose there exists a $(ik, i'k')$-path. By Lemma 1, there are no paths between different SCCs, so ik and $i'k'$ must be in the same SCC. □

Corollary 7. *Consider $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$. All copies of the same house type are in the same SCC. That is, if $e_{ik} = e_{i'k'}$ and $ik \in S$, then $i'k' \in S$.*

Proof. Let $e_{ik} = e_{i'k'}$. There is at least one agent pointing to ik, so $\exists a \in \mathcal{A}$ such that $(a, ik) \in E$. Then $(a, i'k') \in E$ as well by construction. By Corollary 6, there are (ik, a)- and $(i'k', a)$- paths. Then there are $(ik, i'k')$- and $(i'k', ik)$- paths (through a), so ik and $i'k'$ are in the same SCC. □

The above results give us significant information about the SCCs of $\mathcal{G}_{NT}$. The following is a summary of these results. From Proposition 2, each each SCC contains a cycle covering all its vertices. From Lemma 1 and Corollary 6, $\mathcal{G}_{NT}$ can be vertex- *and* arc- partitioned into its SCCs. That is, $\mathcal{G}_{NT}$ consists of SCCs with no links between them. Finally, Corollary 7 tells us all copies of a given house type are in the same SCC.

If we take Theorem 1 as given for now, we can use the above result to prove Corollary 1.

Proof of Corollary 1. If if $e_{ik} = e_{ik'}$, then ik and ik' are in the same SCC. Then apply Theorem 1 to get the desired result. □

B Proof of Theorem 1

Proof of Theorem 1. ("If") Let the supposition be true: whenever agents of the same type are in the same SCC, they receive the same house type. I find a preference profile $\succsim$ that such that x is in the core. First find the partition of vertices into SCCs. Then assign an arbitrary order to the SCCs, and denote them $S_1, ...S_M$. Construct the preferences by the following procedure. Let $\succsim_i (n)$ denote type i's n^{th} favorite house.

Step 1. In S_1, for all $i \in S_1$, let $\succsim_i (1) = x_i$. This is well defined since if there are multiple agents of the same type in S_1, they all receive the same house type.

Step 2. In S_2, for all $i \in S_2$, let $\succsim_i (1) = x_i$ if possible. This is possible if there were no type i's in S_1. Otherwise, let $\succsim_i (2) = x_i$. By Corollary 7, a house never reappears in a later step, so this never assigns a house to two places in the same preference.

Step m. In S_m for $m = 2, ..., M$, for all $i \in S_k$, let $\succsim_i (m') = x_i$ for the lowest unassigned $m' = 2, ..., m$. Again by the same argument above, this never assigns two houses to the same type; it also never assigns the same house type to multiple places in the same preference.

Step $M+1$. Assign remaining preferences in any order, if necessary.

I now show this preference profile admits no blocking coalition. Suppose that there is a coalition of agents $A' \subseteq \mathcal{A}$ and feasible sub-allocation μ' such that for all $ik \in A' : x'_{ik} \succsim_i x_{ik}$. The argument is by induction on the number of SCCs M. In each SCC S_m, the claim to demonstrate is that $x'_{ik} = x_{ik}$ for all $ik \in A' \cap S_m$.

Base case. In S_1, all agents receive their favorite house. Then $x'_{ik} \sim x_{ik}$ for all $i \in A' \cap S_1$. The only indifferences are between copies of the same house type, so this gives $x'_{ik} = x_{ik}$.

m^{th} case. Suppose the claim is true for all agents in $A' \cap (S_1 \cup \cdots \cup S_{m-1})$. This implies that x' allocates all agents in $A' \cap (S_1 \cup \cdots \cup S_{m-1})$ houses in their own SCC. That is, $x'_{ik} \in \cup_{ik \in A' \cap S_m} e_{ik}$.

Toward a contradiction, suppose that $\exists ik \in S_m$ such that $x'_{ik} := h \succ_i x_{ik}$. Then it must be $h \in \cup_{ik \in S_1 \cup \cdots \cup S_{m-1}} e_{ik}$, since all strictly preferred houses are in earlier SCCs. Further, since x' reallocates within A', it must be $h \in \cup_{ik \in A' \cap (S_1 \cup \cdots \cup S_{m-1})} e_{ik}$. Then it must be that an agent in $A' \cap (S_1 \cup \cdots \cup S_{m-1})$ receives a house in $\cup_{ik \in A' \cap (S_1 \cup \cdots \cup S_{m-1})} e_{ik}$. This contradicts the supposition, so it must be that $x'_{ik} \sim x_{ik}$ for $ik \in A' \cap S_m$, giving $x'_{ik} = x_{ik}$.

Thus $x'_{ik} = x_{ik}$ for all $ik \in A'$, and A' is not a blocking coalition.

("Only if") Toward the contrapositive, suppose there is a SCC S with two agents of the same type who receive different houses. By Proposition 2, there is a cycle covering all vertices in S. I now construct a blocking coalition using this cycle. Note that two of these vertices represent agents of the same type who receive different houses. Let these two agents be $1a$ and $1b$; I consider cases based on their relative positions in the cycle.

1. Suppose the cycle is $1a \to \underbrace{2a \to \cdots \to 1b}_{:=c} \to 3a \to \cdots \to 1a$, and $e_{2a} \neq e_{3a}$. Suppose $e_{2a} \succ_1 e_{3a}$. Then $1b \to \underbrace{2a \to \cdots \to 1b}_{c}$ represents a blocking coalition. Note that this is a feasible sub-allocation; it contains its own endowment, and $1b$ is strictly better off. The case $e_{2a} \prec_1 e_{3a}$ is a rotation and relabeling of the cycle.

2. Suppose the cycle is $1a \to 1b \to \underbrace{2a \to \cdots \to 1a}_{:=c}$. If $e_{2a} \succ_1 e_{1b}$, then $1a \to \underbrace{2a \to \cdots \to 1a}_{c}$ is a blocking coalition. If instead $e_{1b} \succ_1 e_{2a}$, then x is not individually rational for $1b$.

3. If the cycle is $1a \to 1b \to 1a$ and $e_{1a} \neq e_{1b}$, then μ is not individually rational.

This completes the proof. □

Remark. For readers familiar with the result in Quint and Wako (2004), it suffices to show that executing their "$\mathcal{STRICTCORE}$" algorithm on the above constructed preferences results in the allocation μ. This is readily apparent, and a formal proof is omitted.

C Proof of Theorem 2

I present a theorem by Quinzii (1984), which I will use in the proof of the main result. There are no "types" in her model, but I retain my present notation for consistency. I first give a formal definition of competitive equilibrium in an exchange economy setting.

Definition 9. Let $E = \{(\omega_{ik}, e_{ik}), (u_{ik})\}_{ik \in \mathcal{A}}$ be an exchange economy. A **competitive equilibrium** is a price vector $p \in \mathbb{R}^H$ and a feasible allocation $(x_{ik}, m_{ik})_{ik \in \mathcal{A}}$ such that for all $ik \in \mathcal{A}$:

- $m_{ik} + p \cdot x_{ik} \leq \omega_{ik} + p \cdot e_{ik}$
- $(u_{ik}(h, m) \geq u_{ik}(x_{ik}, m_{ik})) \implies (m + p \cdot h > \omega_{ik} + p \cdot e_{ik})$

That is, all agents' allocations are affordable for them, and any better allocation is unaffordable. A **competitive equilibrium allocation** is $(x_{ik}, m_{ik})_{ik \in \mathcal{A}}$ for which there exists a price vector supporting it as a competitive equilibrium.

Theorem 3. *(Quinzii, 1984, pg. 54) Let $E = \{(\omega_{ik}, e_{ik}), (u_{ik})\}_{ik \in \mathcal{A}}$ be an exchange economy. Assume u_{ik} are utility functions such that:*

1. *u_{ik} are increasing with respect to money, and $\lim_{m \to \infty} u_{ik}(h, m) = \infty$ for all $ik \in \mathcal{A}$*
2. *$u_{ik}(e_{ik}, \omega_{ik}) \geq u_{ik}(h, 0)$ for all $ik \in \mathcal{A}, h \in H$. That is, the endowment (both house and money) is preferred to consuming any house and 0 money.*

Then the set of weak core allocations and the set of competitive equilibrium allocations of E coincide.

In the present paper's setting, this theorem gives us equivalence of the weak core and competitive equilibrium allocations. Thus to show TU-rationalizability, it is equivalent to find $(v_i)_{i\in A}$ (with the restriction that these are common within agent types) and a price vector $p \in \mathbb{R}_+^{|H|}$ supporting (x, m) as a competitive equilibrium.

I briefly leave the exchange economy setting and consider the consumer demand setting. I give a definition for consumer demand quasilinear rationalizability, then I present a well-known theorem for classic consumer demand revealed preferences due to Brown and Calsamiglia (2007).

Definition 10. Let $(x_r, m_r, p_r), r = 1, ..., N$ be observed demand and price data, where $x_r \in \mathbb{R}_+^H; p_r \in \mathbb{R}_{++}^H$. The data is **quasilinear rationalizable** if for some $I > 0$, $\forall r$ (x_r, m_r) solves

$$\max_{x\in\mathbb{R}_{++}^n} v(x) + m$$
$$\text{s.t. } p_r x + m = I$$

for some concave v.

Theorem 4. *(Brown and Calsamiglia, 2007) Let $(x_r, m_r, p_r), r = 1, ..., N$ be observed demand and price data, where $x_r \in \mathbb{R}_+^H; p_r \in \mathbb{R}_{++}^H$. The following are equivalent:*

1. *The data (x_r, m_r, p_r) are quasilinear rationalizable by a continuous, concave, strictly monotone utility function v.*

2. *The data (x_r, m_r, p_r) satisfy Afriat's inequalities with constant marginal utilities of income. That is, there exist $v_r, v_l > 0$ $\forall r$ such that*

$$v_r \leq v_l + p_l \cdot (x_r - x_l) \ \forall r, l = 1, ..., N \qquad (A)$$

3. *The data (x_r, m_r, p_r) are "cyclically monotone", that is, if for any given subset of the data $\{(x_s, p_s)\}_{s=1}^m$:*

$$p_1 \cdot (x_2 - x_1) + p_2 \cdot (x_3 - x_2) + \cdots + p_m \cdot (x_1 - x_m) \geq 0 \qquad (C)$$

The last condition is known as "cyclic monotonicity." While it is probably not obvious how I will apply Theorem 4, I will show that there is a deep connection between the my present setting and consumer demand revealed preferences.

I now give the full proof for Theorem 2.

Proof of Theorem 2. I first show that (1) $\iff$ (2), then (2) $\iff$ (3).

First, (1) $\implies$ (2). Suppose the TU problem $(A, \mathcal{A}, H, x, m, e, \omega)$ is TU-rationalizable. That is, there is some profile of utility indices $(v_i)_{i \in A}$ such that (x, m) is in the weak core. By Theorem 3, there is some price vector p supporting (x, m) as a competitive equilibrium. So p satisfies $m_{ik} + p \cdot x_{ik} \leq \omega_{ik} + p \cdot e_{ik}$. With quasilinear utility, money always enters utility, so this holds with equality: $m_{ik} + p \cdot x_{ik} = \omega_{ik} + p \cdot e_{ik}$. Then p must satisfy equation (P). Theorem 3 allows negative prices, but adding any positive constant $p + C$ will also satisfy (P), so we can let $p \geq 0$.

I now show (2) $\implies$ (1). Suppose there exists a vector p satisfying equation (P). I seek to show that this p supports (x, m) as a competitive equilibrium for some utility indices (v_i). That is, I want to construct v_i such that all agents ik are maximizing utility subject to their budget constraints $e'_{ik} \cdot p + \omega_{ik}$.[12] This becomes a classic consumer demand revealed preference problem. To see this, reinterpret an agent type i as a single consumer, and each individual agent ik as a demand data point from this consumer:

$$\left(\underbrace{(x_{ik}, m_{ik})}_{\text{consumed good and money}}, \underbrace{(e'_{ik} \cdot p + \omega_{ik}) := I_{ik}}_{\text{budget}}, \underbrace{p}_{\text{price}} \right)_{k \in \{1, \ldots, K_i\}}$$

That is, i is a consumer, and each ik is a single observation of demand at a particular budget. There are $|A|$ consumers and K_i demand points for each consumer i. We seek to rationalize the demand data in a consumer revealed demand sense by constructing (v_i) such that each consumer i is maximizing utility $V_i(h, m) = v_i(h) + m$ in each consumption bundle-budget pair.

The easiest way to do this is to let $v_i(x_{ik}) = x'_{ik} \cdot p$, making all agents indifferent to any possible consumption bundle while still satisfying assumption $(A2)$. However, I show these data are rationalizable in a deeper sense than this knife-edge construction.

I will apply Theorem 4. Notice that cyclic monotonicity (C) is trivially fulfilled when $p_s \equiv p$ is constant. Thus the consumption data with some sufficient constant budget

$$(x_{ik}, m_{ik}, I, p)_{k \in \{1, \ldots, K_i\}}$$

[12] Agent ik sells his endowment e'_{ik} at price p and is additionally endowed with ω_{ik} money.

are always quasilinear rationalizable. Our consumption data has varying budgets instead

$$(x_{ik}, m_{ik}, I_{ik}, p)_{k\in\{1,\ldots,K_i\}}$$

However, the quasilinear utility

$$V(x) = \sum_{n=1}^{X} x_n p_n + m_{ik}$$

is concave, continuous, and strictly increasing, and rationalizes either set of data. Thus we can also apply Theorem 4 to see that utility indices fulfilling Afriat's inequalities (A) will also suffice for (v_i).

I now show ((1) $\iff$) (2) $\implies$ (3). Toward a contradiction, suppose $\mathcal{G}_{TU}^{big}$ has a cycle C with positive length; i.e. $\sum_{ik\in C} \omega_{ik} - m_{ik} > 0$. The members of C can form a blocking coalition for (x, m) by allocating to each $ik \in C$

$$\left(x_{ik}, m + \frac{\sum_{ik\in C} \omega_{ik} - m_{ik}}{|C|}\right)$$

That is, each agent receives the same house and receives more money from the excess endowment. This is of course feasible for C and strictly preferred by all $ik \in C$.

Finally, I prove (3) $\implies$ (2). Suppose $\mathcal{G}_{TU}^{big}$ has no cycles with length > 0. I construct a price p satisfying (P) via path lengths on $\mathcal{G}_{TU}^{big}$. Note that Proposition 2, Lemma 1, and Corollary 7 still apply to $\mathcal{G}_{TU}^{big}$. Every SCC has a cycle covering all its vertices; there are no paths between two SCCs; and all houses of the same type are in the same SCC. Denote p_h as the price of house type $h \in H$. Construct p as follows:

1. For each SCC, choose any house type h in this SCC and set p_h to be any number.
2. For all houses h' in this SCC, set $p_{h'} - p_h$ to be length of the shortest path from h to h'. That is, the shortest path between an agent endowed with h to an agent endowed with h' determines the price difference.
3. Repeat steps 1 and 2 for all SCCs.
4. Add a constant to p to ensure $p \geq 0$.

I will show that all paths between two vertices are the same length, then that the path length between a house type h and itself is always 0, so that the construction is consistent, i.e. $p_h - p_{h'} = 0$ when $h = h'$. The rest of the proof will immediately follow.

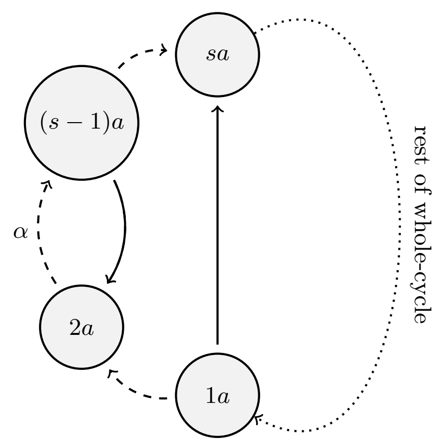
Note the whole economy is budget balanced; we have $\sum_{ik \in \mathcal{A}} \omega_{ik} = \sum_{ik \in A} m_{ik}$. For any cycles that form a vertex-partition of $\mathcal{G}_{TU}^{big}$: these cycles must have length 0. A negative length cycle that is in a partition of the overall economy implies a positive length cycle elsewhere by budget balancedness, a contradiction.

In particular, by Proposition 2, each SCC has a cycle containing all its vertices; call this the "whole-cycle" as shorthand. These partition the whole economy, so each whole-cycle must have length 0. For the following claims, assume the SCC has at least three vertices. I will show the cases for one or two vertices separately. Enumerate the whole-cycle as $(1a, 2a, ..., sa, ...(S-1)a, Sa, 1a)$. (Allowing any of these agents to be of the same type – this is unimportant.) Now consider $1a$ and sa distinct and in the same SCC (recall there are no paths between SCCs), and consider the path $(1a, ..., sa)$ via the whole-cycle. Denote this path $(1a, \underbrace{2a, ..., (s-1)\,a}_{:=\alpha}, sa)$, and call it the "whole-cycle path" as shorthand.

Claim 1. If the arc $(1a, sa)$ exists, it is the same length as the whole-cycle path. That is, $\ell(1a, sa) = \ell(1a, 2a, ..., (s-1)\,a, sa)$.

Figure 8 illustrates the following argument. If the arc $(1a, sa)$ exists, then $e_{2a} = e_{sa}$, so there is an arc $((s-1)a, 2a)$. Then $(\underbrace{2a, ..., (s-1)a}_{:=\alpha}, 2a)$ forms a cycle, and $(1a, sa, \underbrace{\quad ... \quad}_{\text{rest of whole-cycle}}, 1a)$ also forms a cycle. Since the two cycles partition the SCC, they are part of a partition of the overall economy; thus both cycles must have length 0. If $\ell(1a, sa) > \ell(1a, 2a, ..., (s-1)\,a, sa)$, then the latter cycle has positive length, a contradiction. This is because the whole-cycle has length 0 as established, and we have found a cycle with shorter length. If instead $\ell(1a, sa) < \ell(1a, 2a, ..., (s-1)\,a, sa)$, then the latter cycle has negative length, also a contradiction. Note the same argument carries through if $2a = (s-1)a$ – the first cycle is a self-loop, and $1a = (s-1)a$ is symmetric.

Figure 8: Illustration of Claim 1



Claim 2. If the arc $(sa, 1a)$ exists, it has length negative of the whole-cycle path from $1a$ to sa. That is, $\ell(sa, 1a) = -\ell(1a, 2a, ..., (s-1)\,a, sa)$.

From Claim 1, $\ell(sa, 1a) = \ell(sa, (s+1)a, ..., Sa, 1a)$. Notice that $(sa, (s+1)a, ..., Sa, 1a)$ and $(1a, 2a, ..., (s-1)a, sa)$ form the whole cycle, so their lengths sum to 0. That is, $\ell(sa, 1a) + \ell(1a, 2a, ..., (s-1)a, sa) = 0$, and the claim follows.

Remark 1. The indexing of $1a$ and sa in Claims 1 and 2 is not important. Since the whole-cycle is a cycle, $1a$ can be any vertex. (It is convenient to have $1 \leq s \leq S$.)

Claim 3. Any $(1a, sa)$-path is the same length as the whole-cycle path $(1a, \underbrace{2a, ..., (s-1)\,a}_{:=\alpha}, sa)$.

The $(1a, sa)$-path is some permutation of a subset of vertices of the SCC. Denote this $(\underbrace{\sigma_1 a}_{=1a}, \sigma_2 a, ..., \sigma_{j-1} a, \underbrace{\sigma_j a}_{=sa})$, where $j \leq S$. I will show

$$\ell(\sigma_1 a, ..., \sigma_{j-1} a, \sigma_j a) = \underbrace{\ell(1a, 2a) + \cdots + \ell((\sigma_j - 1)a, \sigma_j a)}_{\text{whole-cycle path}} \equiv \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a)$$

Note that $\sigma_{j-1} \neq \sigma_j - 1$ in general.

I will show the claim by strong induction on the length of j. The base case of $j = 1$ is Claim 1. Now suppose the claim is true for j; that is, $\ell(1a, ..., \sigma_{j-1}a, \sigma_j a) = \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a)$. Now consider $j+1$. We have $\ell(1a, \sigma_{j+1}a) = \ell(1a, \sigma_j a) + \ell(\sigma_j a, \sigma_{j+1}a)$. If $\sigma_{j+1} > \sigma_j$, then by Claim 1 write

$$\ell(\sigma_j a, \sigma_{j+1}a) = \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)$$

So

$$\begin{aligned}
\ell(1a, ..., \sigma_j a, \sigma_{j+1}a) &= \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) + \ell(\sigma_j a, \sigma_{j+1}a) \\
&= \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) + \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) \\
&= \sum_{i=1}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)
\end{aligned}$$

If $\sigma_{j+1} < \sigma_j$, then by Claim 2 write

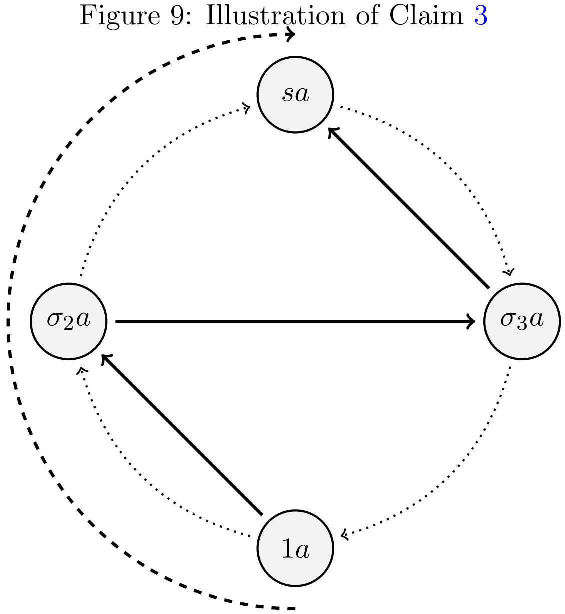
$$\ell(\sigma_j a, \sigma_{j+1}a) = -\sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)$$

So

$$\begin{aligned}
\ell(1a, ..., \sigma_j a, \sigma_{j+1}a) &= \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) + \ell(\sigma_j a, \sigma_{j+1}a) \\
&= \sum_{i=1}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) + \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) - \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) \\
&= \sum_{i=1}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)
\end{aligned}$$

as desired.

Figure 9: Illustration of Claim 3



Claim 4. The length of any path between a house type h and itself is 0.

Figure 10 illustrates the following argument. Note that two vertices (agents) may be endowed with the same house type, so these can be distinct nodes. Recall that all copies of the same house type are contained in the same SCC. The path length from a vertex to itself is 0 since the whole-cycle has length 0, and any other path is the same length. Now suppose h is contained in two distinct vertices, $1a$ and $2a$. Consider a node sa such that $x_{sa} = h$. (This may be $1a$ or $2a$.) Then the arcs $(sa, 1a)$ and $(sa, 2a)$ exist. These have the same length, $\omega_{sa} - m_{sa}$, by construction of $\mathcal{G}_{TU}^{big}$. Denote $\ell(sa, 1a) = \ell(sa, 2a) = \ell_1$. I show the length of the path from $1a$ to $2a$ is 0. Denote this path $(1a, ..., 2a)$, and let $\ell(1a, ..., 2a) = \ell_2$. Both $(sa, 1a, ..., 2a)$ and $(sa, 2a)$ are paths from sa to $2a$, so must have the same length. Then $\ell_1 = \ell_1 + \ell_2$, giving us $\ell_2 = 0$ as desired.

Figure 10: Illustration of Claim 4

I have shown the above claims for SCCs of size at least three. Now consider an SCC of only one vertex. The only arc must be $(1a, 1a)$, which constitutes the whole-cycle and must have length 0, and the path length from this house type to itself is 0.

Now consider an SCC of two vertices, $1a$ and $2a$. If they are endowed with distinct house types, the arcs $(1a, 2a)$ and $(2a, 1a)$ are the only arcs, and the claims are true trivially. If they are endowed with the same house type, the self loops are also present. The two self-loops partition the SCC, so have length 0. We have $\ell(1a, 1a) = \ell(1a, 2a)$ by construction, so $\ell(1a, 2a) = 0$, and similarly $\ell(2a, 1a) = 0$. Then all arcs have length 0 in this SCC, so the claims are again true.

The rest of the proof follows easily. The path length between any house type h and itself is 0 (so the minimum path length is 0), ensuring it is possible to construct prices this way. Next, for any $ik \in \mathcal{A}$, the path length from $e_{ik} := h$ to $x_{ik} := h'$ is $m_{ik} - \omega_{ik}$, so that $p_{h'} - p_h = m_{ik} - \omega_{ik}$. This gives

$$(x_{ik} - e_{ik}) \cdot p = p_{h'} - p_h = m_{ik} - \omega_{ik}$$

as desired.

This completes the proof of the theorem. $\square$

Proof of Corollary 4. As argued in the proof of Theorem 2, any price must satisfy $(x_{ik} - e_{ik}) \cdot p = \omega_{ik} - m_{ik}$ for all $ik \in \mathcal{A}$. By the construction of $\mathcal{G}_{TU}^{big}$, $x_{ik} - e_{ik}$ is an arc from e_{ik} to x_{ik} with length $\omega_{ik} - m_{ik}$, which is also the price difference between these houses. Inductively (I will omit the full formality), a path from x_{ik} to $x_{ik'}$ has path length 0 if and only if the price difference between them is 0. (Note that by Claim 2, there also must be a path from $x_{ik'}$ to x_{ik}, and it has length 0 as well.)

("If") Let both conditions be true. As in the main theorem, it is sufficient to set $v_i(x_{ik}) = p \cdot x_{ik}$. Since prices can be set arbitrarily across SCCs, we can ensure no two houses in different SCCs have the same price.

("Only if") Toward a contradiction, suppose the problem is not T-rationalizable. Then it is of course not strictly T-rationalizable. Now suppose the second condition is false. That is, there are ik, ik' in the same SCC such that $x_{ik} \neq x_{ik'}$, but the shortest path length between them is 0. Then $p_{x_{ik}} = p_{x_{ik'}}$. Suppose $v_i(x_{ik}) > v_i(x_{ik'})$ without loss of generality. Then ik' can afford $(x_{ik}, m_{ik'})$, which is preferable to $(x_{ik'}, m_{ik'})$. Thus (x, m) is not a competitive equilibrium, so is not strictly T-rationalizable.

In particular, x_{ik} can purchase $x_{ik'}$ instead. Since $m_{ik} > 0$ by assumption, ik can form a blocking coalition by compensating other members of the blocking coalition. □

Proof of Corollary 5. This comes from the proof of Theorem 2. The first inequality is (A) from the result by Brown and Calsamiglia (2007). This is exactly Afriat's inequalities when the marginal utility of money is 1. These give joint restrictions on any the utility for houses actually consumed by agent type i given some p. Necessity and sufficiency are from Afriat's theorem.

The second inequality gives restrictions on the utility for houses not consumed by type i. A house h that is affordable under some ik's budget must have $V(h, e_{ik} \cdot p + \omega_{ik} - p \cdot h) \leq V(x_{ik}, e_{ik} \cdot p + \omega_{ik} - p \cdot x_{ik})$, else (x, m) is not a competitive equilibrium. This gives the inequality in the corollary:

$$v_i(h) + (e_{ik} \cdot p + \omega_{ik} - h \cdot p) \leq v_i(x_{ik}) + (e_{ik} \cdot p + \omega_{ik} - x_{ik} \cdot p)$$
$$v_i(h) - h \cdot p \leq v_i(x_{ik}) - x_{ik} \cdot p$$

That is, if h is affordable to ik, then its utility (including leftover money) must be less than that of x_{ik}. Note that a house that is too expensive for all ik is allowed to have any utility. Again, necessity and sufficiency are immediate.

The third inequality defines valid vectors p, which comes from Theorem 2 and its proof. The fourth inequality is $(A1)$. □