Introduction to Symplectic Topology

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1 Maslov Index for Lagrangian subspace

We give a definition of Maslov index of Lagrangian subspace for $(\mathbb{R}^{2n}, \omega_0)$,here ω_0 the standard symplectic form (And hence all symplectic vector space). My definition is same to the one in the text book, but I will give more details and a new proof.

Theorem 1. *The space of all Lagrangian subspace ($\mathcal{L}(n)$) admits a topology and differential structure such that it is diffeomorphism to $U(n)/O(n)$.*

lemma 1. *For any matrix(a linear map: $\mathbb{R}^n \to \mathbb{R}^{2n}$) in the form $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$, ImZ a Lagrangian subspace is equivalent to $X^TY = Y^TX$ and $rk(Z) = n$.*

Proof. Omit. The proof given in the book is clear enough.Note that since every dimension n subspace can be represent as ImZ,hence all Lagranigian subspace is represented. □

Proof. For all matrices satisfy $X^TY = Y^TX$, we give a another description of this matrix, consider: $\bar{Z} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$, $rk(Z = n \Rightarrow rk(X, -Y) = n)$,the condition $X^TY = Y^TX$ implys that : $\bar{Z}^T\bar{Z} = \begin{pmatrix} X^TX + Y^TY & 0 \\ 0 & X^TX + Y^TY \end{pmatrix}$, Since $rkX = rkX^TX$, for any $0 \neq u \in \mathbb{R}^n \Rightarrow u^T(X^TX + Y^TY)u > 0$, so $X^TX + Y^TY$ is positive definite. We choose $U \in GL_n(\mathbb{R}^n)$ such that $U^T(X^TX + Y^TY)U = Id$. Note that (Xu, Yu) and (XUu, YUu) represent the same space in $\mathbb{R}^{2n}$. So we can choose a representation of $\bar{Z}$(for all $\bar{Z}$ represent the same Lagrangian subspace) such that $X^TX + Y^TY = Id$, in another words $X + iY \in U(n)$.

Note that the stablizer is $O(n)$, hence we can give $\mathcal{L}(n)$ a differential structure as the orbit space $U(n)/O(n)$. □

Corollary 1. *Every two pairs of Lagrangian subspace Λ and Λ', there exist a symplectic map ψ such that $\Lambda = \psi\Lambda'$.*

Proof. It suffices to check the case $\Lambda' = \mathbb{R}^{2n}$. So we consider the matrix : $\begin{pmatrix} X \\ Y \end{pmatrix}$ represent the Λ such that $X + iY \in U(n) \subset Sp(2n)$. □

Remark: Since $U(n)$ is connected then so does $Sp(2n)$, so we can find a path from id to ψ, using this path we can prove that $\mathcal{L}(n)$ is path connected.

The Maslov index for Lagrangian subspace

Consider the map

$$S^1 \xrightarrow{I(t)} U(n)/O(n) \xrightarrow{det^2} U(1)$$

Which induce a map in the fundamental group, we call the degree of $det^2 \circ I(t)$ the Maslov index of path $I(t)$. It's easy to check the it is well-defined.

Theorem 2. *The Maslov index represent the isomorphism $\pi(\mathcal{L}(n)) \cong \mathbb{Z} : \mu(\Lambda(t)) \to \mathbb{Z}$.Here $\Lambda(t) \in \pi_1(\mathcal{L}(n))$.$\mu$ denotes the Maslov index.*

Proof. Consider the commutative diagram for principal bundle:

$$\begin{tikzcd}
O(n) \arrow[r, "det"'] \arrow[d] & O(1) \arrow[r, "\cong"'] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[d] & \\
U(n) \arrow[r, "det"'] \arrow[d] & U(1) \arrow[r, "\cong"'] \arrow[d] & S^1 \arrow[d, "z\to z^2"] & \\
\mathcal{L}(n) \arrow[r, "det^2"'] & U(1) \arrow[r, "\cong"'] & \mathbb{RP}^1 \arrow[r, "\cong"'] & S^1
\end{tikzcd}$$

The first two columns give the long exact sequences:

$$\begin{tikzcd}
\pi_1(S^1) \arrow[r] & \pi_1(S^1) \arrow[r] & \pi_0(O(0)) \arrow[r] & \pi_0(S^1) \arrow[r] & \pi_0(S^1) \\
\pi_1(U(n)) \arrow[r] \arrow[u, "det"'] & \pi_1(\mathcal{L}(n)) \arrow[r] \arrow[u, "det^2"'] & \pi_0(O(n)) \arrow[r] \arrow[u, "det"'] & \pi_0(U(n)) \arrow[r] \arrow[u, "det"'] & \pi_0(\mathcal{L}(n)) \arrow[u, "det^2"']
\end{tikzcd}$$

Since $\mathcal{L}(n)$ is path connected the fifth column is isomorphism, the 1st, 3rd and 4th quivers are all isomorphism. By five lemma we have the 2nd map is isomorphism. □

Theorem 3. *The Maslov index satisfy the product property, i.e. for $\psi(t) \in \pi_1(Sp(2n))$ and $\Lambda(t) \in \pi_1(\mathcal{L}(n))$,*

$$\mu(\psi(t)\Lambda(t)) = 2\mu(\psi(t)) + \mu(\Lambda(t))$$

(here $\mu(\psi)$ is the Maslov index of loop in $Sp(2n)$, which is explained clearly in the book.)

Maslov index satisfy some basic properties but we only prove this one.

Proof. It's easy to see that μ is a homotopy invariant, so write $\psi(s,t) = U(s,t)P(s,t)$, here $U(t) \in Sp(2n) \cap O(2n).\psi(1,t) = U(t); \psi(0,t) = \psi(t)$. So it suffices to calculate $\mu(U(t)\Lambda(t))$, which is $2\mu(U(t)) + \mu(\Lambda(t))$ by straight forward calculation. □

2 Affine Rigidity Theorem

Are symplectic morphisms flexible enough? It's an important question, we first show that the space of linear symplectic morphisms are a real subset with different topology to the space of linear map with determinant 1 by some homotopy theorem, then we introduce the affine nonsqueezing theorem to give a geometric explaination.

Theorem 4. *$Sp(2n)$ is not same to the space of linear map with determinant 1(we name it $A(2n)$).*

Proof. By the polar decomposition we can prove that $A(2n)$ is homotopy equivalent to $SO(2n)$ while $Sp(2n)$ is homotopy equivalent to $U(n)$.

Since $\pi_1(SO(2n)) = \mathbb{Z}/2\mathbb{Z}$ while $\pi_1(U(n)) = \mathbb{Z}$, it is the difference. $\square$

Now we state the affine nonsqueezing theorem.

Theorem 5. *Suppose $\mathbb{R}^{2n}$ the Eucildean space with symplectic coordinate $(x_1, y_1, \cdots\cdots, x_n, y_n)$. And we call a cylinder $Z(R)$ a symplectic cylinder if $Z(R) : x_1^2 + y_1^2 \leq R^2$. Then for any ball $B(r)$ with radius r and any symplectic map ψ, if $\psi B(r) \subset Z(R)$, then $r < R$.*

The proof in the book is clear, but there is a typo.

We should replace

$$sup_{|z|=1}(\langle e_1, \psi(z)\rangle^2 + \langle e_2, \psi(z)\rangle^2)$$

to

$$sup_{|z|=1}(\langle e_1, \psi(z)\rangle^2 + \langle f_1, \psi(z)\rangle^2)$$

Theorem 6. *Affine Rigidity Theorem : if ψ and ψ^{-1} both satisfies the nonsqueezing property, then ψ is symplectic or anti-symplectic.*

The proof is clear except a small gap:

Since $Ae_1 = \lambda e_1, Af_1 = \lambda f_1$, we write it down explicitly:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,2n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{2n,1} & a_{2n,2} & \dots & a_{2n,2n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Hence A has the form:

$$\begin{pmatrix} \lambda & 0 & ... & a_{1,2n} \\ 0 & \lambda & ... & a_{2,2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & a_{2n,2n} \end{pmatrix}$$

Thus A^T has the form:

$$\begin{pmatrix} \lambda & 0 & ... & 0 \\ 0 & \lambda & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,2n} & a_{2,2n} & ... & a_{2n,2n} \end{pmatrix}$$

Hence if $\langle v, e_1 \rangle = 0$ and $\langle v, f_1 \rangle = 0$,

$\langle A^T v, e_1 \rangle = 0$ and $\langle A^T v, f_1 \rangle = 0$.

So $A^T(B(1)) \subset Z(\lambda)$, which is contradicted to the non-sequeezing property , this is why we consider ψ^T instead of ψ as first.