Instantons-and-4-manifolds

Lyu JunZhe

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1 moduli space in line bundle

Theorem 1. *We proof that the YM moduli space of $U(1)$ bundle is $H^1(M,\mathbb{R})/H^1(M,\mathbb{Z})$, here M is the based $4-$manifold.(top of page 44)*

lemma 1. *The map $f \to \frac{i}{2\pi}\mathrm{dlog}f$ give the isomorphism $[M,S^1] \cong H^1_{DR}(M,\mathbb{Z})$,here the $H^1_{DR}(M,\mathbb{Z})$ means the integer subset of real coefficient De Rham cohomolgy.*

Proof. We know the isomorphism $[M,S^1] \cong H^1(M,\mathbb{Z})$ can be given as following:

pick a generator $\alpha \in H^1(M,\mathbb{Z})$, then the pullback $f^*\alpha$ give an element in $H^1(M,\mathbb{Z})$, so we establish a correspondence $f \to f^*\alpha$. Now pick $\alpha = \frac{i}{2\pi}\mathrm{dlog}z$. We have:

$$f^*\frac{i}{2\pi}\mathrm{dlog}z = \mathrm{d}f^*\frac{i}{2\pi}\mathrm{log}z$$
$$= \frac{i}{2\pi}\mathrm{dlog}f$$

$\square$

Now we prove the theorem.

Proof. First obverse that the gauge group $\mathcal{G}$ consist of map $M \to S^1$. So $H^0(\mathcal{G}) \cong [M, S^1] \cong H^1(M, \mathbb{Z})$. Consider the connected component $\mathcal{G}_0$ of $\mathcal{G}$ contains the identity, i.e. $exp(lie(\mathcal{G}))$. All $f \in \mathcal{G}_0$ in the form $e^{i\xi}$, here ξ a function on M. By the Gauge transformation:

$$\begin{aligned} f^*D &= D - (Df)f^{-1} \\ &= D - (\mathrm{d}f)f^{-1} \\ &= D - \mathrm{d}\xi \end{aligned}$$

So the connection space $\mathcal{A}$ mod the action of $\mathcal{G}_0$, that is $\mathcal{A}/\mathcal{G}_0 \cong \Omega^1(M)/d\Omega^0(M)$.Since we consider the YM moduli space, so the space is $H^1(M, \mathbb{R})$.

Now pick any $f \in \mathcal{G}$, by the gauge transformation we have:

$$f^*D = D - \mathrm{d}(\mathrm{log}f)$$

Although $\mathrm{log}f$ may not well-defined, $\mathrm{dlog}f$ a well-defined closed form. And suppose $[f] \in [M, S^1]$ represents the $[k\alpha]$ in $H^1(M, \mathbb{Z})$ via the isomorphism $[M, S^1] \cong H^1(M, \mathbb{Z})$, here α is the fundamental class. We have $[\frac{1}{2\pi}\mathrm{dlog}f] = [k\alpha] \in H^1(M, \mathbb{Z})$. Hence the YM moduli space is the torus $H^1(M, \mathbb{R})/2\pi * H^1(M, \mathbb{Z})$.

Remark: the proof above works also for manifold in every dimension(if we only consider the moduli space, do not request the connections satisfy YM equation,just purely connection quotient the gauge transformation). And the result will be $H^1(M, \mathbb{R})/2\pi * H^1(M, \mathbb{Z}) \oplus \mathrm{Imd}^*$ instead.

$\square$

2 What a reducible connection?

Theorem 2. *Let $\mathcal{G}_D$ the stabilizer of connection D, assuming D is not flat(it's automatically if we consider instantons number $k = 1$),then the following is equivalent:*

a) D is irreducible

b) $\mathcal{G}_D/(\mathbb{Z}/2\mathbb{Z}) \neq 1$

c) $\mathcal{G}_D/(\mathbb{Z}/2\mathbb{Z}) = \mathrm{U}(1)$

d) $D : \Omega^0(adP) \to \Omega^1(adP)$ *has a kernel.*

All argument is clear except the part $(d) \to (a)$*:*$KerD \neq \emptyset$ *imply* D *a reducible connection.*

Proof. Taking any associated vector bundle E with complex rank 2 w.r.t. our SU(2) bundle η. Then pick an element u in $kerD$ and fix a neighborhood U_i of based manifold M s.t.$u|_{U_i} = A$ a traceless skew-hermitian matrix, hence we can choose an eigenvector e with length 1 of A in E(under a fixed trivialization) satisfied $Ae = i\lambda e$,here $\lambda \in \mathbb{R}$. Then we have the following local calculation.

$$D(Ae) = (DA)e + ADe = ADe = iD(\lambda e) = i((d\lambda)e + \lambda De) = i(d\lambda)e + i\lambda De$$

By taking inner product with e we have

$$(ADe, e) = id\lambda(e, e) + i\lambda(De, e)$$

Combine with the relation $(e, e) = 1 \Rightarrow \mathrm{Re}(De, e) = 0$, we obtain

$$\mathrm{Im}(ADe, e) = d\lambda$$

Since A is skew-Hermitian, the following holds

$$\mathrm{Im}(ADe, e) = \mathrm{Im}(De, A^*e) = -\mathrm{Im}(De, Ae) = -\mathrm{Im}(De, i\lambda e) = \lambda\mathrm{Re}(De, e) = 0$$

hence we have $d\lambda = 0$, so λ a constant eigenvalue.

However it does not mean that we obtain a global-defined eigenvector, since for any $\theta \in \mathrm{U}(1)$, we have $A\theta e = i\lambda\theta e$ so the eigenvector is not uniquely determined.But thanks to the fact that eigenvector space of $i\lambda$ is 1-dimensional, If we pick another open set(small enough to admit a trivialization of adη) U_j and $U_j \cap U_i \neq \emptyset$, in $U_j \cap U_i \subseteq U_i$, Suppose $u|_{U_i} e_i = i\lambda e_i$ and $u|_{U_j} e_j = i\lambda e_j$(we use the pointwise matrix multiplication),we have $e_j = f_{ij} e_i$(in $U_i \cap U_j$ and $f_{ij} \in \mathrm{U}(1)$).

Note these f_{ij} satisfied cocycle condition and define a U(1) bundle L over M,and they together define a global eigenvector e, which give the embedding $L \to \eta$.Similarly, If we consider the eigenvalue $-i\lambda$ we will also obtain a line bundle named $\hat{L}$ with global eigenvector $\hat{e}$ and the embedding $\hat{e} : \hat{L} \to E$. Thus we have the isomorphism $L \oplus \hat{L} \to E$ via $(e, \hat{e})$.Since the first Chern class of SU(2) bundle is trivial.BY the spiltting $E \cong L \oplus \hat{L}$(topologically), we have $c_1(L) = -c_1(\hat{L})$.We know that complex line bundle is classified by the first Chern class, hence $\hat{L} \cong L^{-1}$.

By the equation $De = 0$.Consider the connection d_1, d_2 on L and L^{-1} make e covariantly constants on bundle $Hom(L; E)$ and $Hom(L^{-1}; E)$. take section $f_1 e + f_2 \hat{e}$ in

$E(f_1, f_2$ the section of $L, \hat{L})$. We have $D(f_1 e + f_2 \hat{e}) = d_1(f_1)e + d_2(f_2)\hat{e}$ thus the connection exactly spilt.

Remark: One can see that these eigenvectors $\in Hom(L, E)$ or $\in Hom(L^{-1}, E)$ from the transition function viewpoint, consider U_i and U_j two open neighborhood and $\phi_i : E|_{U_i} \to U_i \times \mathbb{C}^2$, $\phi_j : E|_{U_j} \to U_j \times \mathbb{C}^2$ two trivialization and $\hat{e}_i, \hat{e}_j$ two eigenvector(under the correspondent trivialization). We have the transition function(from $\hat{e}_i$ to $\hat{e}_j$):$\phi_i^{-1} f_{ij} \phi_j = g_{ij} f_{ij}$, here g_{ij} the transition function of vector bundle E.

□

In the rest of note we will use A instead of D to emphasize that we are identitying the space of connection to $\Omega(adP)$ by fix a connection A.

3 local model of moduli space

Theorem 3. *For a connection A in $\mathcal{A}$, pick the isotopy group Γ_A of d_A in $\mathcal{G}$, we proof that, there exist an open set $T_{A,\epsilon}$ around A in the coloumb slice ($T_A : a \in \Omega^1 g_E, d_A^* a = 0$) ($T_{A,\epsilon} : a \in \Omega^1 g_E, d_A^* a = 0, |a| < \epsilon$)such that $(T_{A,\epsilon} \times \mathcal{G})/\Gamma_A \cong U$,here U a open set in $\mathcal{A}$ around A.(Theorem 3.2 in page 57)*

To prove this theorem, we need a lemma(which can be found in Donaldson's book Geometry of four manifolds).

Lemma: pick two sequence A_n and B_n of connection s.t. $\lim A_n = A; \lim B_n = B$ and there exists gauge transformation g_n satisfy $g_n^* A_n = B_n$, then there exist $g \in \mathcal{G}$ s.t. $g^* A = B$.

Proof. We first consider a map $\phi : kerd_A^* \times \mathcal{G} \to \mathcal{A}$ given by $\phi(a, g) = g^*(A + a)$, let's explain why $(T_{A,\epsilon} \times \mathcal{G})/\Gamma_A$ makes sense,pick $a \in kerd^* A, g \in \mathcal{G}$:

$$d_A^* g^*(a) = d_A^* gag^{-1} = - * d_A * gag^{-1}$$

$$= - * d_A g * ag^{-1} = - * (d_A g) * ag^{-1} - g * d_A * ag^{-1}$$

By the condition $g \in \Gamma_A \Rightarrow d_A g = 0$ and $d_A^* a = 0$, we have $d_A^* g^*(a) = 0$, hence $kerd^*$ is invariant under the action Γ_A. So we can define Γ_A acts on $kerd^* \times \mathcal{G}$ in the following rules:

$$h(a, g) = (h^*(a), gh^{-1})$$

Which is a free and properly discontinuous actions, thus $(kerd_A^* \times \mathcal{G}/)\Gamma_A$ a bannach manifold.(For a proof, One can read David G.Ebin's paper: On the space of riemannian metrics, it's similar to the case we consider)

Secondly we proof that this map ϕ is a diffeomorphism.

Note that the derivative of ϕ at $(0, id)$ is (id, d_A) and the kernel is $(0, kerd_A)$, also the tangent space of the orbits passed $(0,1)$, which is $0 \times \Gamma_A$ is also $(0, kerd_A)$, since $\Omega^1(g_E) = Imd_A \oplus kerd^*$ and (id, d_A) acts on elements (a, ξ) in tangent space $kerd^* \times \Omega^0(g_E)$ of $kerd^* \times \mathcal{G}$ give $a + d_A\xi$, which give a surjection onto $\mathcal{A} \cong \Omega^1(g_E)$. By the inverse function theorem we have $(kerd^* \times \mathcal{G})/\Gamma_A$ has a local diffeomorphism onto $\mathcal{A}$ around $(0,1)$.

finally we should extend this diffeomorphism to a global one for some positive number ϵ and the space $T_{A,\epsilon}$.

To do this we need the lemma above,suppose the opposite direction,if ϕ restrict to $(T_{A,\epsilon} \times \mathcal{G})/\Gamma_A$ not injective for every $\epsilon > 0$, then there exist two series in $\Omega^1(g_E)$ a_n and b_n s.t. $\lim a_n = \lim b_n = 0$ and correspondent gauge transformation $g_n; \hat{g}_n$ s.t. $g_n^*(A+a_n) = \hat{g}_n^*(A + b_n)$ but $[a_n, g_n] \neq [b_n, \hat{g}_n]$ in $(T_{A,\epsilon} \times \mathcal{G})/\Gamma_A$. Then we have:

$$\hat{g}_n^{-1} g_n^*(A + a_n) = A + b_n$$

By lemma there exist g and $\hat{g}$ s.t. $\hat{g}^{-1}g^*A = A$ hence $\hat{g}^{-1}g \in \Gamma_A$.

However, $\lim[b_n, id] = [0, id]$ and $\lim[a_n, \hat{g}_n^{-1}g_n] = [0, \hat{g}^{-1}g] = [0, id]$. The local diffeomorphism property forces $[a_n, \hat{g}_n^{-1}g_n] = [b_n, id]$, which also means that $[a_n, g_n] = [b_n, \hat{g}_n]$ and leads to a contradiction.

For the surjection part, at point $(0, g) \in kerd^* \times \mathcal{G}$, by calculation:

$$\frac{d}{dt}(ge^{t\xi^*}A) = \frac{d}{dt}(A - (d_A ge^{t\xi})ge^{-t\xi}) = gd_A\xi g^{-1}$$

Hence $kerD\phi|_{(0,g)} = (0, gkerd_A)$. However the tangent space of orbit $(0, gh^{-1})$ is $(0, gker_A)$, thus the local diffeomorphism property hold for any point $(0, g)$, we denotes the open set around $(0, g)$ such that the diffeomorphism property holds as U_g. By the compactness of group G and based manifold M, the gauge group is compact,we can find finite subset of U_g such that $P_2(U_g)$ give the open cover $\mathcal{G}$,here P_2 means projection to the second coordinate. Using the injection above we can find a $\epsilon > 0$ such that $(T_{A,\epsilon} \times \mathcal{G})/\Gamma_A$ give the local model of $\mathcal{A}$ around d_A.(Here I omit all the sobolev subscript) □

Theorem 4. *Dimension of ASD moduli space*

We follow the proof of Atiyah-Hitchin-Singer and only consider the case SU(2) bundle, to calculate the dimension of ASD moduli space, we only need to calculate the dimension of tangent space.(Note here we admits ASD moduli space is a manifold for generic metric, and we only consider the case $H^2(A) = 0$ for given metric)

Proof. To visualize the moduli space, first note that ASD connection is invariant under gauge transformation, i.e. $F_A^+ = 0 \Leftrightarrow F_{A^g}^+ = 0$. To see this, we consider a one parameter family of gauge transformation $g = e^{t\xi} \in \Omega^0(AdP)|\xi \in \Omega^0(adP)$, then we calculate the differentiate:

$$F_{A^g}^+ = \frac{d}{dt}P^+(d_A - (d_A g)g^{-1})(d_A - (d_A g)g^{-1}) = -P^+ d_A d_A \xi$$

Since A is ASD connection, which means that $F_A^+ = P^+ d_A d_A = 0$, so $\frac{d}{dt}P^+(d_A - (d_A g)g^{-1})(d_A - (d_A g)g^{-1}) = 0$,by ODE we have $F_{A^g}^+ = 0$.

Now for a sufficient small neighborhood of irreducible ASD connecion A,we have the moduli space is exactly $a \in \Omega^1(adP)|d_A^* a = 0 \cap a \in \omega^1(adP)|F_{A+a}^+ = 0$. consider a one parameter family of ASD connection(not necessarily irreducible) $F_{\Phi(t)}^+$,here $\Phi(t)$ represent a curve with initial value 0 and initial derivative $\Phi'(0) = a$. By

$$0 = \frac{d}{dt}F_{A+a}^+|_{t=0} = \frac{d}{dt}P^+(F_A + t d_A a + t^2 a^a) = P^+ d_A a$$

Hence to calculate the dimension of tangent vector space, the trick is viewing $d_A^* a = 0$ and $P^+ d_A a = 0$ the equation in tangent bundle, to calculate the dimension of some subspace satisfy the these 2 equation. We want to treat this two equation as elliptic operator $d_A^* + P^+ d_A$ for sutiable bundles, if $coker(d_A^* + P^+ d_A)$ is trivial, then we can calculate the dimension by Atiyah-Singer index theorem, so now we turn to consider the following elliptic complex

$$0 \longrightarrow \Omega^0(adP) \xrightarrow{d_A} \Omega^1(adP) \xrightarrow{P^+ d_A} \Omega^2_+(adP) \longrightarrow 0$$

We consider $\Omega^2_+(adP)$ the section space of vector bundle of the space of ASD form, named ASD for short.

In fact we have the spilting $\Omega^1(adP) \cong \Omega^0(adP) \oplus \Omega^2(adP)$ by the map $d_A^* + P^+ d_A$. Since $\dim(adP \otimes T^*M) = 12$ and $\dim(ASD) = 3 \times (\frac{6}{2}) = 9$ and $\dim(adP) = 3$,to check $d_A^* + P^+ d_A$ is elliptic we only need to check that it is injective. And write $d_A = d + A$ locally, we only need to check $d^* + P^+ d$ is elliptic since we care about the principal symbol, which is obvious if you write it down in local coordinate.

Together with the materials above, for the case A is irreducible, for the complex $H^0_A = kerd_A = 0$(since $d_A\xi = 0 \Leftrightarrow e^{t\xi}A = A$,or you can use theorem 2), so $cokerd^*_A = H^0_A = 0$.By the Atiyah -Singer index theorem(**I don't really understand**), we have the index is $P_1(adP)[X] - \frac{1}{2}\mathrm{G}(\chi - \tau)$(one can read AHS for detail).Here $P_1(adP)$ the 1st Pontrjagin class,$[X]$ the fundamental class of based manifold X, G the lie group associate to $P.\chi$ the Euler characteristic of X and τ the signature of X.

For the case $G = SU(2)$ and instantons number k=1(i.e.second Chern class valued on [X] equal to -1). We have $P_1(adP) = 8; dim(G) = 3$, suppose $rkH^1(X) = b_1; rkH^2(X) = b_2$ and the manifold is connected, using the Poincare duality the index equal to $8 - 3(1 - b_1 + b_2^+)$,b_2^+ the dimension of subspace in $H^2(M)$ that the intersection form is positive definite. For the case X is simply connected and the intersection form is negative definite, more expicitly, $b_1 = 0; b_2^+ = 0$, the dimension of moduli space is 5. □

Theorem 5. *Calculate the dimension by excision and gluing*

Now we perform another way to calculate the dimension for arbitrary $4-$ manifold(of course compact and oriented), the method is simple, if we know the dimension of simple model(e.g. sphere), then we can patch the simple model into an arbitrary manifold. Let me state the proof below instead of those vague words.First we introduce a lemma and Ulhenbeck will prove it in section 6.

lemma 2. *If we have 2 manifold $X_1 = U_1 \cup V_1$ and $X_2 = U_2 \cup V_2$,in addition*

$U_1 \cap V_1 = W_1 \cong W_2 = U_2 \cap V_2$

We have two elliptic operatorsD_1 and D_2(w.r.t. bundle $E_1 \to F_1$ over X_1 and $E_2 \to F_2$ over X_2),

moreover there exist bundle isomorphism $\phi : E_1|_{W_1} \to E_2|_{W_2}$ and $\psi : F_1|_{W_1} \to F_2|_{W_2}$ over the diffeomorphism $U_1 \cap V_1 = W_1 \cong W_2 = U_2 \cap V_2$ satisfied $D_2 = \psi D_1 \phi^{-1}$ on W_2.

then we define $X_3 = U_1 \cup V_2; X_4 = U_2 \cup V_2$ using the diffeomorphism, then we obtain $E_3; F_3$ w.r.t. X_3 and $E_4; F_4$ w.r.t. X_4,

what's more we have $D_3 : E_3 \to F_3$ and $D_4 : E_4\ toF_4$.

The theorem is, $IndD_1 + IndD_2 = IndD_3 = IndD_4$.

Proof. We now using the lemma without proof, First we know that the dimension of moduli space of sphere S^4 is 5(by the construction that I will give in the future too, or you can follow the AHS for detail).

Let $X_1 = X, c_2(P_1) = -1, X_2 = S^4, c_2(P_2) = 0,$

$X_3 = X, c_2(P_3) = 0, X_4 = S^4, c_2(P_4) = -1.$

The reason we can assume $c_2(P_3) = 0$ is that we can find a bundle over X such that it is trivial over $X\ B^4$, here B^4 a small open ball. The construction of these kind of bundle will be given later.

More explicitly, we excise a small ball B_1 of X(with bundle P_1) and paste a ball B_2 come from S^4(with bundle P_2) to obtain bundle P_3, meanwhile we paste the small B_1 into S^4 to obtain bundle P_4.

Now $IndD_1 = IndD_3 + IndD_4 - IndD_2$. For the trivial bundle, our complex will be

$$0 \longrightarrow \Omega^0() \otimes \mathfrak{g} \xrightarrow{d} \Omega^1() \otimes \mathfrak{g} \xrightarrow{P^+d} \Omega^2() \otimes \mathfrak{g} \longrightarrow 0$$

The () can be both X and S^4. So the index of $d + P^+d$ will be

$$-\frac{3}{2}(\chi() + \sigma())$$

Here the χ means Euler characteristic and σ means the signature, for D_2 case it will be -3.

So we have $IndD_1 = 5$. □

Now we construct the bundle such that trivial in a extremely big range but its 2nd Chern class(when evaluate at fundamental class) $= -1$.The method is, bundle is determined by the transition function, so we consider two open set $p \in U \cong B^4$ and $V = X - p$, and two trivial bundle over them, if we give a suitable transition function over the intersection $U - p$ then it is possible to construct a non trivial bundle which trivial in a large range.

lemma 3. *Their exist such bundle with second Chern class(evaluate in fundamental class) is* -1

Proof. Consider $g : \mathbb{R}^4 - 0 \to SU(2)$:$y = (y_1, y_2, y_3, y_4) \to |y|^{-1} \begin{pmatrix} y_4 + iy_3 & iy_1 - y_2 \\ iy_1 + y_2 & y_4 - y_3 \end{pmatrix}$ For $\psi : U \to \mathbb{R}^4$, we define $g(\psi) : U - 0 \to SU(2)$ as transition function, Consider $\chi : \mathbb{R} \to \mathbb{R}$ such that χ is zero near 0 and $\chi = 1$ for all $x \geq 1$.

we define the connection compatible to the trnasition function as ${\psi^{-1}}^* a_p = \chi(|y|)g^{-1}dg$ and ${\psi^{-1}}^* a_\infty = \chi(|y| - 1)(dg)g^{-1}$. Note this two thing is same under local gauge transformation(by transition function), they patch together to give a global connection.

Now to calculate the second Chern class it suffices to calculate $\frac{1}{8\pi^2} tr(F_a \wedge F_a)$, by the local equation(it's enough since our connection does not vanishing only for a local region)$F_a = da + a \wedge a$(we write ${\psi^{-1}}^* a_p$ as a for short now),

$$\frac{1}{8\pi^2} tr(F_a \wedge F_a)$$

$$= \frac{1}{8\pi^2} tr(da \wedge da + da \wedge a \wedge a + a \wedge a \wedge da + a \wedge a \wedge a \wedge a)$$

Note that $tr(\omega_1\omega_2) = (-1)^{deg(\omega_1)deg(\omega_2)}tr(\omega_2\omega_1)$, we have $tr(a\wedge a\wedge a\wedge a) = 0$ and $da\wedge da, da\wedge a\wedge a, a\wedge a\wedge da$ are in fact the same.

$$= \frac{1}{8\pi^2}dtr(a\wedge da + \frac{2}{3}a\wedge a\wedge a)$$

$$\int_X \frac{1}{8\pi^2}tr(F_a\wedge F_a)$$

$$= \int_{|y|\leq 1}\frac{1}{8\pi^2}dtr(a\wedge da + \frac{2}{3}a\wedge a\wedge a)$$

So by stokes theorem,

$$= \int_{|y|=1}\frac{1}{8\pi^2}tr(a\wedge da + \frac{2}{3}a\wedge a\wedge a)$$

replace $a = g^{-1}dg$, it will be

$$= \int_{|y|=1}\frac{1}{8\pi^2}tr(\frac{-1}{3}g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg)$$

Replace $g\to gh$ or $g\to hg$,$\int_{|y|=1}\frac{1}{8\pi^2}tr(\frac{-1}{3}g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg)$ won't change, which imply its a volume form of $SU(2)$,so for the coefficient we only need to consider $|y|^{-1}$ $\begin{pmatrix} y_4+iy_3 & iy_1-y_2 \\ iy_1+y_2 & y_4-y_3 \end{pmatrix}$ at the original point,i.e.$y=(0,0,0,1)$, which is $12\times Vol(S^3)$, so the second Chern class(evaluate at fundamental class) $=-1$. □

Remark: In the past day, people trend to consider SD connection so the calculate the dimension in the case intersection form is positive definite, but topologically ASD case and SD case only differ by the choosing of orientation, however people always consider ASD connection now since the complex surface admits a natural orientation(define by its complex structure) and ASD connection is relative to the holomorphic structure.

Theorem 6. *Kuranishi model of ASD moduli space*

At the beginning we recall some definition, consider the complex(A is irreducible ASD connection):

$$0 \longrightarrow \Omega^0(adP) \xrightarrow{d_A} \Omega^1(adP) \xrightarrow{P^+d_A} \Omega^2_+(adP) \longrightarrow 0$$

And let $\triangle^1_A = d_Ad^*_A + d^*_Ad_A; \triangle^+_A = P^+d_AP^+{d_A}^*$ and G_A the Green operator for $\triangle^+_A = P^+d_AP^+{d_A}^*$. Suppose H_A the projection $\Omega^1(adP)\to H^1_A$, or by abusing the notation, the projection $\Omega^2_+(adP)\to H^2_A$.

Theorem 7. *If $H_A^2 = 0$, then the map $K_A : \Omega^1(adP) \to \Omega^1(adP) : K_A(a) = a + P^+ {d_A}^*(G_A(a \wedge a)^+)$ give a diffeomorphism from $T_{A,\epsilon}^+$ to H_A^1, here $T_{A,\epsilon}^+$ means a small set of $\Omega^1(adP)$ such that any $a \in T_{A,\epsilon}^+$ satisfies $F_{A+a}^+ = 0; d_A^* a = 0$ with norm $< \epsilon$. Which give a local model of ASD moduli space.*

We quote a lemma frist:

lemma 4. *$a \in T_{A,\epsilon}^+ \Leftrightarrow K_A(a) \in H_A^1$ and $H_A((a \wedge a)^+) = 0$*

Proof. We only need the lemma above for the case $H_A^2 = 0$, Since G_A commutes with $P^+ {d_A}^*$,

$$P^+ d_A(K_A(a)) = P^+ d_A a + G_A \triangle_A^+ (a \wedge a)^+$$

Since $F_{A+a}^+ = 0$

$$= P^+ d_A a - G_A \triangle_A^+ (d_A^+ a)$$

Since $G_A \triangle_A^+ - H_A = Id$ and $H_A^2 = 0$, we have $G_A \triangle_A^+ = Id$

$$= P^+ d_A a - P^+ d_A a = 0$$

The proof of another side is similar. □

Proof. First we identify H_A^1 as the linear subspace of $\Omega^1(adP)$ satisfies $\triangle_A^1 = 0$,Note that $\frac{dK_A}{da}(\phi) = \frac{d}{dt} K_A(t\phi)|_{t=0} = \phi$, by inverse function theorem K_A give a local diffeomorphism from $\Omega^1(adP)$ to itself, we restrict it to the H_A^1 to obtain a diffeomorphism K_A^{-1} from H_A^1 to $\Omega^1(adP)$ such that $K_A^{-1}(H_A^1)$ diffeomorphic to its(K_A^{-1}) image, for the case $H_A^2 = 0$, which is $T_{A,\epsilon}^+$. Since H_A^1 a finite dimension space(the dimension is same for all irreducible A and $H_A^2 = 0$), so we indeed give a local model of ASD moduli space. □

Remark: For the case $H_A^2 \neq 0$, we consider $\xi : a \to H_A(a \wedge a)^+$, then the ASD moduli space can be locally described as $K_A^{-1} \cdot \xi^{-1}(0)$.

Remark: For the case A is reducible, the local model is given by H_A^1/Γ_A, here Γ_A the isotopy group of A.

Note the harmonic space H_A^2 is the orthogonal complement of Imd_A^+, let's give a more universal description. We review a fundamental result in infinite dimension at first:

lemma 5. *Suppose F a (fredholm-)smooth map between two Banach space U and V, then we can spilt $U \cong U_1 \oplus U_2$ and $V \cong V_1 \oplus V_2$ such that any point $p \in U$ and for some neighborhood of p, F behaviors as a linear isomorphism (up to a diffeomorphism the derivative of F ar p) from U_1 to V_1 and a non-inear map from U_2 to V_2 with its derivative vanishing at p, meanwhile $U_2 \cong kerF$ and $V_2 \cong cokerF$.*

Proof. this lemma is known as the inverse function theorem when both $kerF$ and $cokerF$ vanish, thing won't change too much in this case, for detail one can read section 4.2 in Donaldson's Geometry of four manifolds. □

Proof. As we seen in the previous theorem, P^+d_A is a fredholm map when we restrict it in $kerd_A^*$. And $kerP^+d_A = H_A^1; cokerP^+d_A = H_A^2$. So we just get a map that represent the tangent space of moduli space as a zero set of a smooth map. Note that the map P^+d_A is differential of $\phi : a \to (d_Aa + a \wedge a)^+$, so the local model is given by $\phi^{-1}(0)$, which under a local diffeomorphism will be identify with P^+d_A since $kerd_A^*|_{\overline{T_{A,\epsilon}}}$ a Banach space.

Remark: In the general case if we require the moduli space is a manifold, we need 0 the regular value of ϕ, which require $cokerP^+d_A = H_A^2 = 0$.

□

4 Cone over $\mathbb{CP}^2$

In this section we study the behavior of moduli space in the reducible connection,we assume $H_A^2 = 0$ as before. By the classification of SU(2)− bundle, the number of the topological splitting of bundle is one-one correspondent to the $\frac{1}{2}$ number of $a \in H^2(X, \mathbb{Z}|a^2 = -1)$. (recall that $Prin_{SU(2)}(X) \cong [X, BSU(2)] \cong [X, HP^\infty] \cong [X, K(\mathbb{Z}, 4)] \cong H^4(X, \mathbb{Z})$).

Now we are curious about the question: how many (splitting-)connection can be equipped to a specific splitting line bundle?

lemma 6. *For the case the based $4-$ manifold X with negative definite intersection form and $\pi_1(X) = 0$, every topological splitting bundle admits only one splitting connection up to gauge transformation.*

Proof. We prove the existence first,since every splitting $SU(2)-$ bundle has the from $P = Q \times_{U(1)} SU(2)$, the transition function of P takes the form $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ and Q a $U(1)-$ bundle . fix any connection A_0 on Q, suppose their exist $a \in i\Omega(X)$ such that $F_{A_0+a}^+ = 0$, then we have

$$P^+da = -F_{A_0}^+$$

The condition $b^+ = 0$ implies $H_+^2(X, \mathbb{Z})$ vanishing, so the map $\Omega^1(adP) \to \Omega_+^2(adP)$ is surjective. Which prove the existence of solution of the equation above, thus prove the existence of ASD connection.

On the other hand, if there exist $b \in i\Omega(X)$ such that

$$P^+ da + P^+ db = -F^+_{A_0}$$

Which implies $d^+ b = 0$, by Stokes formula

$$0 = \int_X d(a \wedge da)$$

$$= \int_X da \wedge da = -\int_X |P^+ da|^2 dvol + \int_X |P^- da|^2 dvol$$

$$= \int_X |P^+ da|^2 dvol$$

we have $db = 0$, and we can get $b = df$ for some $f \in i\Omega^0(X)$ since $\pi_1(X, \mathbb{Z}) = 0$.

Recall that we have $g \cdot d_{A_0} = d_{A_0} + g^{-1} d_{A_0} g = d_{A_0} + g^{-1} dg = A + d\log g$(since $U(1)$ is commutative and $[X, S^1] = 0$ so we can take log and exp pointwise without changing the smoothness), so gauge transformation differ by elements in the form $a = df, f \in i\Omega^0(X)$. Hence we prove that ASD connection is unique up to gauge transformation. $\square$

Theorem 8. *In a neighborhood of reducible connection, the moduli space is a cone of* $\mathbb{CP}^2$.

Proof. To see what the neighborhood look like, it's sufficient to describe the shape of H^1_A. In the case P is splitting(with splitting connection),$adP = (Q \times_{U(1)} SU(2)) \times_{ad} \mathfrak{su}(2) = Q \times_{ad(U(1))} \mathfrak{su}(2)$. The transition function is give by

$$\begin{pmatrix} it & z \\ -\bar{z} & it \end{pmatrix} \to \begin{pmatrix} it & ze^{2i\theta} \\ -\bar{z}e^{-2i\theta} & it \end{pmatrix}$$

So adP splits into $i\mathbb{R} \oplus L^{\otimes 2}_{\mathbb{R}}$. It's sufficient to consider two sequence:

$$0 \longrightarrow i\Omega^0(X, i\mathbb{R}) \xrightarrow{d} i\Omega^1(X, \mathbb{R}) \xrightarrow{P^+ d} i\Omega^2(X, \mathbb{R}) \longrightarrow 0$$

$$0 \longrightarrow \Omega^0(X, L^{\otimes 2})_{\mathbb{R}} \xrightarrow{d_B} \Omega^1(X, L^{\otimes 2})_{\mathbb{R}} \xrightarrow{P^+ d_B} \Omega^2(X, L^{\otimes 2})_{\mathbb{R}} \longrightarrow 0$$

Here d_B is the induced connection, d is the ordinary differential operator since $d_{A_0} a = da + [A_0, a]$ and $[A_0, a]$ vanishing.

Remember that the Euler characteristic of sequence:

$$0 \longrightarrow \Omega^0(adP) \xrightarrow{d_A} \Omega^1(adP) \xrightarrow{P^+ d_A} \Omega^2_+(adP) \longrightarrow 0$$

is -5 by Atiyah-Singer index theorem. And the Euler characteristic of the first sequence above is 1, so the Euler characteristic of the second sequence above is -6. Since $\dim ker d_A = 1$, so $\dim ker d_B = 0$ hence $H^1_B = 0$, using the fact that $H^2_+(X) = 0$ and $H^2_A = 0$ we have $\dim H^1_B = 6$, therefore $\dim H^1_A = 6$(the real dimension). Fixed a complex structure on $L_{\mathbb{R}}^{\otimes 2}$ we have $H^1_A \cong \mathbb{C}^3$.

What we want is H^1_A/Γ_A, in this case $\mathbb{C}^3/S^1$. Since S^1 acts on $\mathbb{C}^3$ as

$$\begin{pmatrix} e^{2i\theta} & 0 & 0 \\ 0 & e^{2i\theta} & 0 \\ 0 & 0 & e^{2i\theta} \end{pmatrix}$$

Let's describe it in a geometric way. Recall that $\mathbb{C}^3/\mathbb{C} \cong \mathbb{CP}^2$. And consider sphere in $\mathbb{C}^3$ and $\mathbb{C}$. We have $\mathbb{C}^3/\mathbb{C} \cong S^5/S^1$, we views $\mathbb{C}^3 - 0$ as $S^5 \times (0,1)$, since 0 is fixed under S^1 action, which represent the singularity. So we can view $\mathbb{C}^3/S^1$ as $\mathbb{CP}^2 \times (0,1)$ unions a singularity, which is a cone of $\mathbb{CP}^2$. □

5 Orientation of ASD moduli space

I'd like to introduce an alternative approach to translate the orientation problem of moduli space to the calculation of fundamental group of configuration space $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G} = \mathcal{A}^*/(\mathcal{G}/Z(G))$. Roughly speaking, we construct a determinant line bundle of configuration space by pulling back the universal one.

To proof ASD moduli space $\mathcal{M}$ is oriented, it suffices to show the tangent bundle $T\mathcal{M}$ is oriented as bundle, i.e. the determinant line bundle $\bigwedge^{max} T\mathcal{M}$ is trivial.

Now we write $P^+d_A + d_A^*$ as D for short, First note that $\bigwedge^{max} kerD \otimes \bigwedge^{max} (cokerD)^*$

(which can be simplify to $\bigwedge^{max} kerP^+d_A \cong \bigwedge^{max} T\mathcal{M}$ for some nice metric) give the "determinant line bundle" of ASD moduli space pointwisely and formally(because $P^+d_A + d_A^*$ indeed change under gauge transformation), hence extend to a "line bundle" of configuration space. Nevertheless $dim(kerD)$ and $dim(cokerD)$ may vary even under a sufficiently slight deformation, hence we fail to construct the line bundle in this direct way(we don't have the trivialization property, which request the local homeomorphism, but we only get the pointwise and not well-defined one).

However it is possible to construct a universal line bundle, and represent the one we need by pulling back.

Theorem 9. *For X and Y two Banach space, we denote the space of fredholm operator from X to Y as $Fred[X,Y]$. Then for any $D \in Fred[X,Y]$, we construct a line bundle $\bigwedge^{max} kerD \otimes \bigwedge^{max}(cokerD)^*$ (or $\bigwedge^{max} kerD \otimes \bigwedge^{max}(kerD^*)$ for short) pointwisely, this construction indeed give a real line bundle λ over $Fred[X,Y]$*

Proof. We consider the case D a surjection first, by open mapping theorem, we have

$$|x|_{X/kerD} \leq c|D(x)|_Y$$

Which means that for a sufficient small P, $D+P$ is still surjective, and $ker(D+P) \cong kerD$ giving by $x \to x + TPx$, here T the right inverse of D. This isomorphism is continuously depending on P, then we have the local trivialization.

If D is not surjective, consider $F : V \to Y$ such that $D \oplus F : X \oplus V \to Y$ is surjective, here V is some finite dimensional vector space. We have the short exact sequence

$$0 \longrightarrow kerD \longrightarrow ker(D+F) \longrightarrow F^{-1}(ImD) \longrightarrow 0$$

In elements level:

$$0 \longrightarrow x \longrightarrow (x,0)$$

$$(x,\xi) \longrightarrow \xi \longrightarrow 0$$

Note $\bigwedge^{max}(V \oplus W) \cong \bigwedge^{max} V \otimes \bigwedge^{max} W$. We have

$$\bigwedge^{max} ker(D \oplus F) \cong \bigwedge^{max} kerD \otimes \bigwedge^{max} (F^{-1}(ImD))$$

Since

$$\bigwedge^{max} W \otimes \bigwedge^{max} (V/W) \cong \bigwedge^{max} V \cong \mathbb{R}$$

here we use the fact that $\bigwedge^{max} V \cong \bigwedge^{max} \mathbb{R}^n \cong \mathbb{R}$,hence

$$\bigwedge max(F^{-1}(ImD)) \cong \bigwedge(V/F^{-1}(ImD))^* \cong \bigwedge^{max} (cokerD)^*$$

Finally we obtain

$$\bigwedge^{max} (ker(D \oplus F)) \cong \bigwedge^{max} kerD \otimes \bigwedge max(cokerD)^*$$

Replace D to $D+P$ for any sufficient small P, it is easy to see the continuously depending. Thus we prove the local trivialization. $\square$

Now we consider $X = \Omega^1(adP)$ and $Y = \Omega^0(adP) \oplus \Omega^2_+(adP)$, and $\mathcal{A}^* \to Fred[X, Y]$ by $f : A \to P^+ d_A + d_A^*$, then $f^*\lambda$ give a determinant line bundle, which is very closed to what we want.

Now we push down this line bundle to $\mathcal{B}^*$. Consider the principal bundle:

$$\begin{tikzcd} \mathcal{G}/Z(G) \arrow[r] & \mathcal{A}^* \arrow[d] \\ & \mathcal{B}^* \end{tikzcd}$$

And the fibration $f^*\lambda \to \mathcal{A}$ (A is a linear space, hence paracompact),then we obtain the lifting:

$$\begin{tikzcd} & f^*\lambda|_{\mathcal{A}^*} \arrow[d, "projection"] \\ \mathcal{G}/Z(G) \arrow[ur] \arrow[r, "action"'] & \mathcal{A}^* \end{tikzcd} \quad \bullet$$

Since the ation of $\mathcal{G}/Z(G)$ is free, we indeed obtain a real line bundle over $\mathcal{B}^*$:

$$\begin{tikzcd} f^*\lambda/(\mathcal{G}/Z(G)) \arrow[d] \\ \mathcal{B}^* \end{tikzcd}$$

The final bundle we get(when restrict to ASD moduli space) is the desired determinant line bundle, which is classifying by the first Stifel-Witney class, to proof the triviality we only need to show $\pi_1(\mathcal{B}^*) = 0$.

Theorem 10. *The space of irreducible connection $\mathcal{A}^*$ is weak contractible.*

This theorem is easy to image but the way I figure out involved a lot elementary techniques in differential topology, which may make you feel bored.

Proof. First we should give $\mathcal{A} - \mathcal{A}^*$ a clear description. Let's fix a topological splitting of bundle P,i.e. pick a $U(1)-$ bundle Q and inclusion $\rho : U(1) \to SU(2)$ in the form $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ Now $P \cong Q \times_{ad(SU(2))\rho} SU(2)$. Since $\mathfrak{su}(2) \cong \mathfrak{u}(1) \oplus \mathfrak{h}$ for some $\mathfrak{h}$,we have the splitting $adP \cong adQ \oplus V$ for some V, hence the section space split as $\Omega^1(adP) \cong \Omega^1(adQ) \oplus \Omega^1(V)$. Thus for a fix splitting connection(w.r.t. Q), we can identify the space of reducible connection(w.r.t. Q)as $\Omega^1(adQ)$, for short R_Q. Since for different splitting, those set R_Q won't intersect, so we can realize the space $\mathcal{A} - \mathcal{A}^*$ as finite union of some submanifold in $\mathcal{A}$.

The proof I give in following may not correct since I'm not sure these standard differential topological techniques still hold in infinite dimension, I'll update if I comp up with a new proof.

Let's return to the proof, the first observation is that, $\mathcal{A} - \mathcal{A}^*$ has infinite codimension. So now we consider a map $f : S^n \to \mathcal{A}^*$, since $\mathcal{A}$ is affine, we can extend f to a map $f^* : D^{n+1} \to \mathcal{A}$. However we need two more lemma by Witney $\square$

lemma 7. *$f : M_1 \to M_2$ a continuous map(from manifold to manifold), which is smooth in a closed subset A of M_1, then we can find g a smooth map homotopic to f and $f|_A = g|_A$.*

lemma 8. *$f : M_1 \to M_2$ smooth map which is embedding when restrict to close subset A, if $dimM_2 > 2dimM_1 + 1$, then we can find $g : M_1 \to M_2$ such that $g|_A = f|_A$ and g an embedding.*

Combine this two lemma we can find a embedding f_0 to replace f^* such that $f_0|_{S^n} = f^*|_{S^n}$, then we consider the normal bundle of $f_0(D^{n+1})$(which in fact is trivial), since D^{n+1} and $\mathcal{A} - \mathcal{A}^*$ are a close set, we cansider the subet set of normal bundle such that when restrict to S^n the length of fiber is too small to intersect with $\mathcal{A} - \mathcal{A}^*$, since the codimension of $\mathcal{A} - \mathcal{A}^*$ is infinite, by the transversality theorem(we use the version in page 63 differential manifold by kosinski, which can help us to find a transveral section), there exist a section s in the subset such that s do not intersect with $\mathcal{A} - \mathcal{A}^*$, then we can find a homotopy from f_0 to s in $\mathcal{A}^*$, so we prove the theorem.

6 introduction to Taubes theorem

Remark: there is a serious typo in 6.14 page 108, we should replace

$$|P_- F_\lambda(x) - F_\lambda^-(x)| \leq c_4|x|^2$$

to

$$|P_- F_\lambda(x) - F_\lambda^-(x)| \leq c_4|x|^2|F_\lambda(x)|$$