Phần V

Quan hệ RELATIONS

Relations

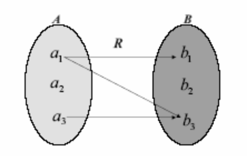
1. Định nghĩa và tính chất
2.Biểu diễn quan hệ
3.Quan hệ tương đương. Đồng dư. Phép toán số học trên$\mathbf{Z}_n$
4.Quan hệ thứ tự. Hasse Diagram

1. Definitions

Definition. A quan hệ hai ngôi từ tập *A đến tập B* là tập con của tích Descartess $R \subseteq A$ x B.

Chúng ta sẽ viết *a R b* thay cho $(a, b) \in R$

Quan hệ từ *A* đến chính nóđược gọi là quan hệ trên *A*

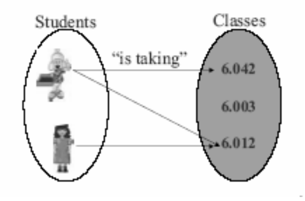


$R = \{ (a_1, b_1), (a_1, b_3), (a_3, b_3) \}$

1. Definitions

Example. A = students; B = courses.

$R = \{(a, b) \mid$ student a is enrolled in class $b\}$



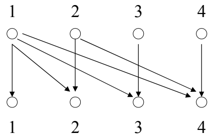
1. Definitions

Example. Let $A = \{1, 2, 3, 4\}$, and

$$R = \{(a, b) \mid a \text{ divides } b\}$$

Then R consists of the pairs:

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4,4)\}$



2. Properties of Relations

Definition. A relation R on a set A is ***reflexive(phản xạ)*** if:

$$(a, a) \in R \text{ for all } a \in A$$

Example. On the set $A = \{1, 2, 3, 4\}$, the relation:

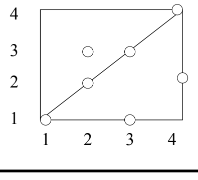
- $R_1 = \{(1,1), (1,2), (2,1), (2, 2), (3, 4), (4, 1), (4, 4)\}$ is not reflexive since $(3, 3) \notin R_1$
- $R_2 = \{(1,1), (1,2), (1,4), (2, 2), (3, 3), (4, 1), (4, 4)\}$ is reflexive since $(1,1), (2, 2), (3, 3), (4, 4) \in R_2$

- The relation $\leq$ on $\boldsymbol{Z}$ is reflexive since $a \leq a$ for all $a \in \boldsymbol{Z}$
- The relation $>$ on $\boldsymbol{Z}$ is not reflexive since $1 \not> 1$

- The relation " | " ("divides") on $\boldsymbol{Z}^+$ is reflexive since any integer a divides itself

Note. A relation R on a set A is reflexive iff it contains the diagonal of $A \times A$:

$$\Delta = \{(a, a); a \in A\}$$



2. Properties of Relations

Definition. A relation R on a set A is ***symmetric(đối xứng)*** if:

$$\forall a \in A\ \forall b \in A\ (a\ R\ b) \rightarrow (b\ R\ a)$$

The relation R is said to be ***antisymmetric(Phản xứng)*** if:

$$\forall a \in A\ \forall b \in A\ (a\ R\ b) \wedge (b\ R\ a) \rightarrow (a = b)$$

Example.

- The relation $R_1 = \{(1,1), (1,2), (2,1)\}$ on the set $A = \{1, 2, 3, 4\}$ is symmetric
- The relation $\leq$ on $\mathbf{Z}$ is not symmetric.
- However it is antisymmetric since

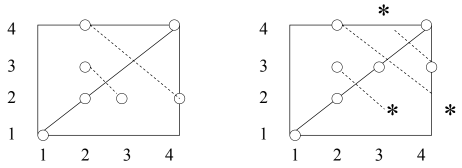
$$(a \leq b) \wedge (b \leq a) \rightarrow (a = b)$$

- The relation “ | ” (“divides”) on $\boldsymbol{Z}^+$ is not symmetric. However it is antisymmetric since

$$(a \mid b) \wedge (b \mid a) \rightarrow (a = b)$$

Note. A relation R on a set A is symmetric iff it is self symmetric with respect to the diagonal Δ of $A \times A$.

The relation R is antisymmetric iff the only self symmetric parts lie on the diagonal Δ of $A \times A$.



2. Properties of Relations

Definition. A relation R on a set A is ***transitive(bắc cầu, truyền)*** if:

$$\forall a \in A\ \forall b \in A\ \forall c \in A\ (a\ R\ b) \wedge (b\ R\ c) \rightarrow (a\ R\ c)$$

Example.

- The relation R = {(1,1), (1,2), (2,1), (2, 2), (1, 3), (2, 3)} on the set A = {1, 2, 3, 4} is transitive
- The relations ≤ and “|”on $\boldsymbol{Z}$ are transitive

$$(a \leq b) \wedge (b \leq c) \rightarrow (a \leq c)$$

$$(a \mid b) \wedge (b \mid c) \rightarrow (a \mid c)$$

3. Representing Relations

Introduction
Matrices
Representing Relations

Introduction

Let R be a relation from A = {1,2,3,4} to B = {u,v,w}:

R = {(1,u),(1,v),(2,w),(3,w),(4,u)}.

Then we can represent R as:

	u	v	w
1	1	1	0
2	0	0	1
3	0	0	1
4	1	0	0

The labels on the outside are for clarity.

It’s really the matrix in the middle that’s important.

This is a 4×3-matrix whose entries indicate membership in R

Representing Relations

Definition. Let R be a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then the ***representing matrix*** of R is the $m \times n$ zero-one matrix $\mathbf{M}_R = [m_{ij}]$ defined by

$$m_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \\ 1 & \text{if } (a_i, b_j) \in R \end{cases}$$

Example. Let R be the relation from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ such that a R b if $a > b$.

Then the representing matrix of R is

	1	2
1	0	0
2	1	0
3	1	1

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Example. Let R be the relation from $A = \{a_1, a_2, a_3\}$ to $B = \{b_1, b_2, b_3, b_4, b_5\}$ represented by the matrix

b_1 b_2 b_3 b_4 b_5

a_1
a_2
a_3

Then R consists of the pairs:

$\{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$

Representing Relations

- Let R be a relation on a set A, then the matrix $\mathbf{M}_R$ that represents R is a ***square matrix***
- R is ***reflexive*** if and only if all ***diagonal entries*** of $\mathbf{M}_R$ are equal to 1: $m_{ii} = 1$ for all i

	u	v	w
u	1	1	0
v	0	1	1
w	0	0	1

Representing Relations

- Let R be a relation on a set A, then the matrix $\mathbf{M}_R$ that represents R is a ***square matrix***
- R is symmetric if and only if $\mathbf{M}_R$ is ***symmetric***

$$m_{ij} = m_{ji} \qquad \text{for all } i, j$$

	u	v	w
u	1	0	1
v	0	0	1
w	1	1	0

Representing Relations

- Let R be a relation on a set A, then the matrix $\mathbf{M}_R$ that represents R is a ***square matrix***
- R is ***antisymmetric*** if and only if $\mathbf{M}_R$ satisfies:

$$m_{ij} = 0 \text{ or } m_{ji} = 0 \quad \text{if } i \neq j$$

$$\begin{array}{c|ccc} & u & v & w \\ \hline u & 1 & 0 & 1 \\ v & 0 & 0 & 0 \\ w & 0 & 1 & 1 \end{array}$$

4.Equivalence Relations

Introduction
Equivalence Relations
Representation of Integers
Equivalence Classes
Linear Congruences.

Introduction

- Example:

Let S = {people in this classroom}, and let

R = {(a,b): a's last name starts with the same letter as b's last name }

- Quiz time:

Is R reflexive? Yes

Is R symmetric? Yes

Is R transitive? Yes

Everyone whose last name starts with the same letter as yours belongs to your assignment group.

Equivalence Relations
Quan hệ tương đương

Definition. A relation R on a set A is an ***equivalence relation*** if it is reflexive, symmetric and transitive:

Example. Let R be the relation on the set of strings of English letters such that aRb if and only if a and b have the same length, then R is an equivalence relation

Example. Let R be the relation on $\mathbf{R}$ such that aRb if and only if $a - b$ is an integer, then R is an equivalence relation

Recall that if a and b are integers, then a is said to be divisible by b, or a is a multiple of b, or b is a divisor of a if there exists an integer k such that $\boldsymbol{a = kb}$

Example. Let m be a positive integer and R the relation on **Z** such that aRb if and only if $a – b$ is divisible by m, then R is an equivalence relation

■The relation is clearly reflexive and symmetric.

■Let a, b, c be integers such that $a – b$ and $b – c$ are both divisible by m, then $a – c = a – b + b – c$ is also divisible by m. Therefore R is transitive

■This relation is called the ***congruence modulo m*** and we write

$$a \equiv b \pmod{m}$$

instead of aRb

Equivalence Classes
Lớp tương đương

Definition. Let R be an equivalence relation on a set A , and $a \in A$. The ***equivalence class of a*** denoted by $[a]_R$ or simply $[a]$ is the subset

$$[a]_R = \{b \in A, b\ R\ a\}$$

Equivalence Classes

Example. What are the equivalence classes modulo 8 of 0 and 1?

Solution. The equivalence class modulo 8 of 0 contains all integer a with the same remainder mod 8 as 0, i.e. a is a multiple of 8. Therefore

$$[0]_8 = \{ \dots, – 16, – 8, 0, 8, 16, \dots \}$$

Similarly

$$[1]_8 = \{a, a \text{ has remainder 1 mod 8}\}$$
$$= \{ \dots, – 15, – 7, 1, 9, 17, \dots \}$$

Note. In the last example, the equivalence classes $[0]_8$ and $[1]_8$ are disjoint.

More generally, we have

Theorem. Let R be an equivalence relation on a set A and a, $b \in A$, then

(i) a R b if and only if $[a]_R = [b]_R$

(ii) $[a]_R \neq [b]_R$ if and only if $[a]_R \cap [b]_R = \varnothing$

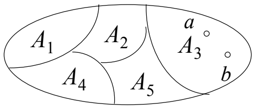
Note. The equivalence classes form a partition of the set A in the sense that it divides A into disjoint subsets.

Note. Let $\{A_1, A_2, \dots\}$ be a partition of A into disjoint nonempty subsets then there is a unique equivalence relation R on A such that the given sets A_i are precisely the equivalence classes.

Let indeed $a, b \in A$, then we define $a\ R\ b$ if and only if there is a subset A_i such that $a, b \in A_i$

We can prove that R is an equivalence relation on A and

$$[a]_R = A_i \text{ if and only if } a \in A_i$$



Example. Let m be a positive integer, then there are m different congruence classes $[0]_m, [1]_m, \dots, [m-1]_m$.

They form a partition of $\mathbf{Z}$ into disjoint subsets.

- Note that

$$[0]_m = [m]_m = [2m]_m = \dots$$
$$[1]_m = [m+1]_m = [2m+1]_m = \dots$$
$$[2]_m = [m+2]_m = [2m+2]_m = \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$[m-1]_m = [2m-1]_m = [3m-1]_m = \dots$$

- They are called the ***integers modulo m***
- The set of all integers modulo m is denoted by $\mathbf{Z}_m$

$$\mathbf{Z}_m = \{[0]_m, [1]_m, \dots, [m-1]_m\}$$

5 Linear Congruences

Example. Let m be a positive integer, then we define the two operations " + " and " × " on $\mathbf{Z}_m$ as follows

$$[a\ ]_m + [b]_m = [a+b]_m$$
$$[a\ ]_m\,[b]_m = [a\,b]_m$$

Theorem. The foregoing operations are well defined, i.e. If $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$, then

$$a + b \equiv c + d \pmod{m} \quad\text{and}\quad a\,b \equiv c\,d \pmod{m}$$

Example. $7 \equiv 2 \pmod 5$ and $11 \equiv 1 \pmod 5$ so that

$$7 + 11 \equiv 2 + 1 = 3 \pmod 5$$
$$7 \times 11 \equiv 2 \times 1 = 2 \pmod 5$$

Note. The operations " + " and " × " on $\mathbf{Z}_m$ satisfy the same property as the similar operations on $\mathbf{Z}$

$$[a\ ]_m + [b]_m = [b]_m + [a]_m$$
$$[a\ ]_m + ([b]_m + [c\ ]_m) = ([a]_m + [b]_m) + [c]_m$$
$$[a\ ]_m + [0]_m = [a]_m$$
$$[a\ ]_m + [m-a]_m = [0]_m\ ,$$

we also write $-[a]_m = [m-a]_m$

$$[a\ ]_m\,[b]_m = [b]_m\,[a\ ]_m$$
$$[a\ ]_m\,([b]_m\,[c\ ]_m) = ([a]_m\,[b]_m)\,[c]_m$$
$$[a\ ]_m\,[1]_m = [a]_m$$
$$[a\ ]_m\,([b]_m + [c\ ]_m) = [a]_m\,[b]_m + [a]_m\,[c]_m$$

Example. The " linear equation" on $\mathbf{Z}_m$

$$[x]_m + [a]_m = [b]_m$$

where $[a]_m$ and $[b]_m$ are given, has a unique solution:

$$[x]_m = [b\,]_m - [a]_m = [b - a]_m$$

Let $m\ = 26$ so that the equation $[x]_{26} + [3]_{26}\ = [b]_{26}$ has a unique solution for any $[b]_{26}$ in $\mathbf{Z}_{26}$.
It follows that the function $[x]_{26} \to [x]_{26} +\ [3]_{26}$ is a bijection of $\mathbf{Z}_{26}$ to itself.
We can use this to define the Caesar's ***encryption***: the English letters are represented in a natural way by the elements of $\mathbf{Z}_{26}$: $A \to [0]_{26}$, $B \to [1]_{26}$, …, $Z \to [25]_{26}$
For simplicity, we write: $A \to 0$, $B \to 1$, …, $Z \to 25$

- These letters are encrypted so that A is *encrypted* by the letters represented by $[0]_{26} + [3]_{26} = [3]_{26}$, i.e. D.

- Similarly B is encrypted by the letters represented by $[1]_{26} + [3]_{26} = [4]_{26}$, i.e. E, … and finally Z is encrypted by $[25]_{26} + [3]_{26} = [2]_{26}$, i.e. C.

- In this way the message "MEET YOU IN THE PARK" is encrypted as

M E E T	Y O U	I N	T H E	P A R K
12 4 4 19	24 14 20	8 13	19 7 4	15 0 17 10
↓ ↓ ↓ ↓				
15 7 7 22	1 17 23	11 16	22 10 7	18 3 20 13
P H H W	B R X	L Q	W K H	S D U N

- To *decrypt* a message, we use the inverse function:

$$[x]_{26} \to [x]_{26} -\ [3]_{26} = [x - 3]_{26}$$

P H H W is represented by	15 7 7 22
	↓ ↓ ↓ ↓
And hence decrypted by	12 4 4 19
The corresponding decrypted message is	M E E T

However this simple encryption method is easily detected.

- We can improve the encryption using the function

$$f: [x]_{26} \to [ax + b]_{26}$$

where a and b are constants chosen so that this function is a bijection

First we choose an *invertible* element a in $\mathbf{Z}_{26}$ i.e. there exists a' in $\mathbf{Z}_{26}$ such that

$$[a]_{26}\,[a'\,]_{26} = [a\,a'\,]_{26}\ = [1]_{26}$$

We write $[a'\,]_{26}\ = [a]_{26}^{-1}$ if it exists.
The solution of the equation

$$[a]_{26}\,[x]_{26} = [c]_{26}$$

is $\quad [x]_{26} = [a]_{26}^{-1}\ [c]_{26} = [a'c]_{26}$

We also say that the solution of the linear congruence

$$a\,x \equiv c \pmod{26}$$

is $\quad x \equiv a'c \pmod{26}$

Now the inverse function of *f* is given by

$$[x]_{26} \rightarrow [a'(x - b)]_{26}$$

Example. Let $a = 7$ and $b = 3$, then the inverse of $[7]_{26}$ is $[15]_{26}$ since $[7]_{26}[15]_{26} = [105]_{26} = [1]_{26}$

Now the letter M is encrypted as

$$[12]_{26} \rightarrow [7 \cdot 12 + 3]_{26} = [87]_{26} = [9]_{26}$$

which corresponds to I. Conversely I is decrypted as

$$[9]_{26} \rightarrow [15 \cdot (9 - 3)]_{26} = [90]_{26} = [12]_{26}$$

which corresponds to M.

To obtain more secure encryption method, more sophisticated modular functions can be used

6. Partial Orderings

Introduction
Lexicographic Order
Hasse Diagrams
Maximal and Minimal Elements
Upper Bounds and Lower Bounds
Topological Sorting

Introduction

Example Let R be the relation on the real numbers:

$$a\,R\,b \text{ if and only if } a \le b$$

Quiz time:

- ■Is *R* reflexive? Yes
- ■Is *R* transitive? Yes
- ■Is *R* symmetric? No
- ■Is *R* antisymmetric? Yes

Introduction

Definition. A relation *R* on a set *A* is a ***partial order(quan hệ thứ tự, thứ tự)*** if it is ***reflexive***, ***antisymmetric*** and ***transitive***.

We often denote a partial order by

The pair (*A*,) is called a ***partially ordered set(tập sắp thứ tự)*** or a ***poset***

Reflexive: $a \quad a$

Antisymmetric: $(a \quad b) \wedge (b \quad a) \rightarrow (a = b)$

Transitive: $(a \quad b) \wedge (b \quad c) \rightarrow (a \quad c)$

Introduction

Definition. A relation *R* on a set *A* is a ***partial order*** if it is reflexive, antisymmetric and transitive.

Example. The divisibility relation " | "on the set of positive integers is a partial ordering, i.e. (**Z**$^+$, |) is a poset

Reflexive?

Yes, *x* | *x* since *x* = 1 · *x*

Transitive?

Yes?

a | *b* means *b* = *ka*, *b* | *c* means *c* = *jb*.
Then *c* = *j*(*ka*) = *jka*: *a* | *c*

Example. The divisibility relation " | "on the set of positive integers is a partial ordering, i.e. (**Z**$^+$, |) is a poset

Antisymmetric?

Yes?

a | *b* means *b* = *ka*, *b* | *a* means *a* = *jb*.
Then *a* = *jka*
It follows that *j* = *k* = 1, i.e. *a* = *b*

Example. Is (**Z**, |) a poset?

Not a poset.

Antisymmetric?

No

3|-3, and -3|3,

but 3 ≠ -3.

Ex. Is (2^S, ⊆), where 2^S the set of all subsets of S, a poset?

Yes, A poset.

Reflexive?

Yes, A ⊆ A, ∀A∈ 2^S

Transitive?

A ⊆ B, B ⊆ C. Does that mean A ⊆ C?

Yes

Antisymmetric?

A ⊆ B, B ⊆ A. Does that mean A =B?

Yes

Definition. The elements *a* and *b* of a poset (*S*,   ) are ***comparable*** if either *a*   *b* or *b*   *a* .
Otherwise, they are said to be ***incomparable(không so sánh được)***.

A poset (*S*,   ) such that every two elements are comparable is called a ***totally ordered set(tập sắp thứ tự toàn phần)***
We also say that   is a ***total order(thứ tự toàn phần)*** or a ***linear order(thứ tư tuyến tính)*** on *S*

Example. The relation "≤ " on the set of positive integers is a total order.

Example. The divisibility relation " | "on the set of positive integers is not a total order, since the elements 5 and 7 are not comparable

Lexicographic Order

Thứ tự tự điển

Ex. A straight forward partial order on bit strings of length n, is defined as:

$$a_1a_2\ldots a_n \le b_1b_2\ldots b_n$$

if and only if $a_i \le b_i$, $\forall$ i.

With respect to this order, 0110 and 1000 are "incomparable" …
We can't tell which is "bigger."

For many applications in computer, it is convenient to have a total order on bit strings, or more generally on strings of characters:
This is the lexicographic order

Lexicographic Order

Let $(A, \le)$ and $(B, \le')$ be two totally ordered sets. We define a partial order   on $A \times B$ as follows:

$(a_1 , b_1) \quad (a_2, b_2)$ if and only if
$a_1 < a_2$ or ($a_1 = a_2$ and $b_1 \le$ ' b_2)

Now we can verify that this is a total order on $A \times B$ called the **lexicographic order**

Note that if A and B are well ordered by ≤ and ≤ ' respectively, then $A \times B$ is also well ordered by

Note also that this definition can be extended to the cartesian product of a finite number of totally ordered sets

Lexicographic Order

Recall that if Σ is a finite set called an alphabet, then the set of strings on Σ, denoted by Σ* is defined by:

- λ ∈ Σ*, where λ denotes the null or empty string.
- If $x \in \Sigma$, and $w \in \Sigma^*$, then $wx \in \Sigma^*$, where wx is the concatenation of string w with symbol x.

Example. Let Σ = {a, b, c}. Then
Σ* = {λ, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab,…}

Lexicographic Order

Now assume that ≤ is a total order on Σ, then we can define a total order   on Σ* as follows.

Let $s = a_1\, a_2\, \ldots\, a_m$ and $t = b_1\, b_2\, \ldots\, b_n$ be two strings in Σ*

Then $s \quad t$ if and only if

- either $a_i = b_i$ for $1 \le i \le m$ so that
  $t = a_1\, a_2\, \ldots\, a_m\, b_{m+1}\, b_{m+2}\, \ldots\, b_n$
- or there exists $k < m$ such that
  ✓ $a_i = b_i$ for $1 \le i \le k$ and
  ✓ $a_{k+1} < b_{k+1}$ so that
  $s = a_1\, a_2\, \ldots\, a_k\, a_{k+1}\, a_{k+2}\, \ldots\, a_m$
  $t = a_1\, a_2\, \ldots\, a_k\, b_{k+1}\, b_{k+2}\, \ldots\, b_n$

- ■ We can prove again that is a total order on the set Σ* called the ***lexicographic order*** on Σ*

Example. If Σ is the English alphabet with the usual order on the characters: a < b < … < z, then the lexicographic order is precisely the order of the words in a dictionary

For example

✓ *discreet* *discrete*

discreet
↓↓↓↓↓↓ ↓
discrete

e ≠ *t*

✓*discreet* *discreetness*

discreet
↓↓↓↓↓↓↓↓
discreetness

is a total order called the ***lexicographic order*** on Σ*

Example. If Σ = {0, 1} with the usual order 0 < 1, then Σ* is the set of all bit strings.

We have

✓ 0110 10

✓ 0110 01100

Hasse Diagrams

A poset can be represented visually using a special kind of graphs called the ***Hasse diagram***

To define the Hasse diagram we need the concept of direct upper bound.

Definition. An element *b* in a poset (*S*,) is said to be an ***upper bound*** of an element *a* in *S* if *a* *b*

We also say that *a* is a ***lower bound*** of *b*

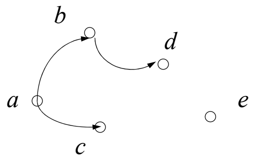
b is said to be a ***direct upper bound*** of *a* if *b* is an upper bound of *a*, and there is no upper bound *c* such that

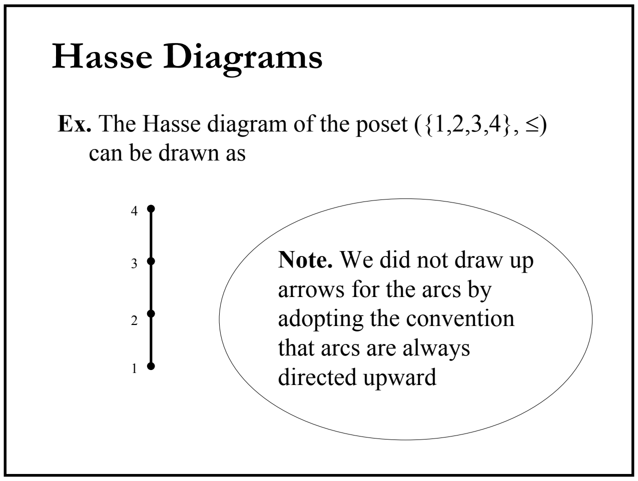
Hasse Diagrams

- Now the ***Hasse diagram*** of a finite poset (*S*,) is the graph:

✓whose vertices are points in the plane in one-to-one correspondence with *S*,

✓two vertices *a*, *b* are joined by an arc directed from *a* to *b* if *b* is a direct upper bound of *a*

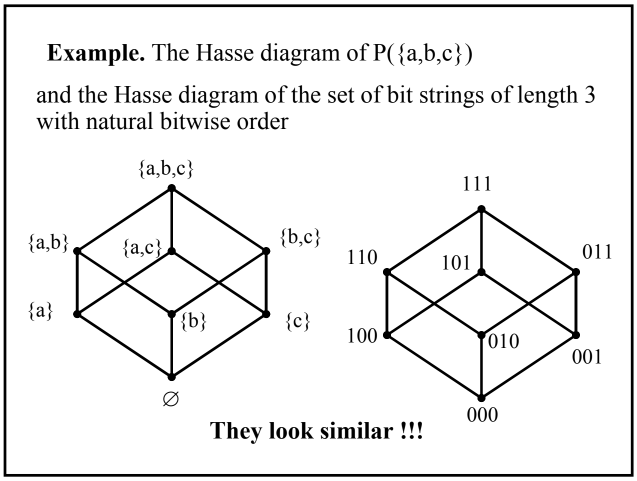




Hasse Diagrams

Ex. The Hasse diagram of the poset ({1,2,3,4}, ≤) can be drawn as

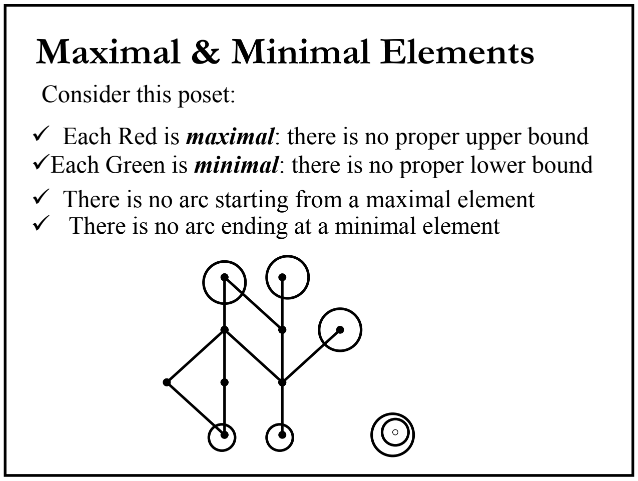
Note. We did not draw up arrows for the arcs by adopting the convention that arcs are always directed upward



Example. The Hasse diagram of P({a,b,c})

and the Hasse diagram of the set of bit strings of length 3 with natural bitwise order

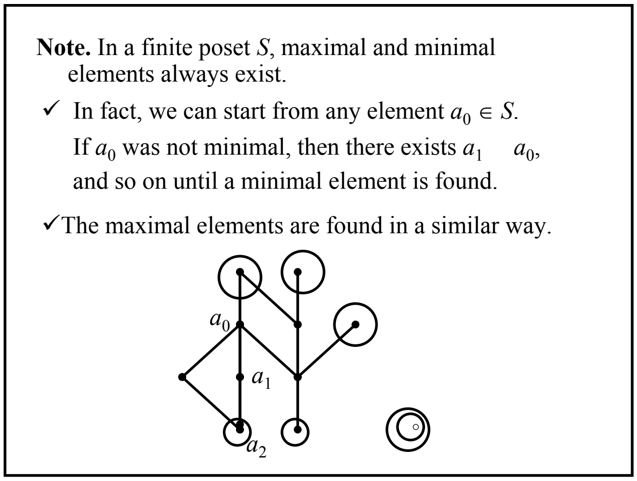
They look similar !!!



Maximal & Minimal Elements

Consider this poset:

- ✓ Each Red is ***maximal***: there is no proper upper bound
- ✓Each Green is ***minimal***: there is no proper lower bound
- ✓ There is no arc starting from a maximal element
- ✓ There is no arc ending at a minimal element



Note. In a finite poset *S*, maximal and minimal elements always exist.

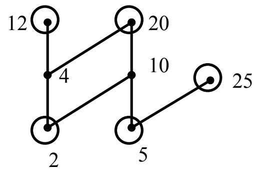
- ✓ In fact, we can start from any element $a_0 \in S$. If a_0 was not minimal, then there exists a_1 a_0, and so on until a minimal element is found.
- ✓The maximal elements are found in a similar way.

Example. What are the maximal and minimal elements of the poset ({2, 4, 5, 10, 12, 20, 25}, |) ?

Solution. From the Hasse diagram, we see that 12, 20, 25 are maximal elements

and 2, 5 are minimal elements

Thus the maximal and minimal elements of a poset are not necessarily unique



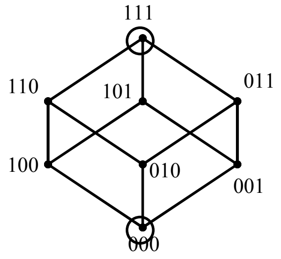
Example. What are the maximal and minimal elements of the poset consisting of bit strings of length 3?

Solution. From the Hasse diagram, we see that 111 is the unique maximal element and 000 is the unique minimal element

111 is also the ***greatest element*** and
000 is the ***least element***
in the sense:

000 *abc* 111

for all string *abc*



In fact we have

Theorem. In a finite poset, if the maximal element is unique, then it is the greatest element .
Similarly for the least element.

Proof. Let *g* be the unique maximal element.

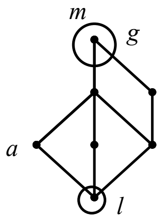
Let *a* be an arbitrary element, then there is a maximal element *m* such that

a *m*

Since *g* is unique we must have *m* = *g* , i.e. *a* *g*

Therefore *g* is the greatest element.

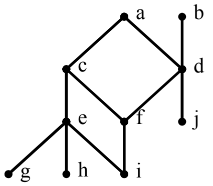
Similar proof for the existence of the least element *l*



Upper and Lower Bounds

Definition. Let (*S*,) be a partial order. If *A* ⊆ *S*, then an ***upper bound*** for *A* is an element *x* ∈ *S* (perhaps in A also) such that ∀ *a* ∈ *A*, *a* *x*.

A ***lower bound*** for *A* is an *x* ∈ *S* such that ∀ *a* ∈ *A*, *x* *a*



Ex. The upper bound of {*g*,*j*} is *a*.

Why not *b*?

Upper and Lower Bounds

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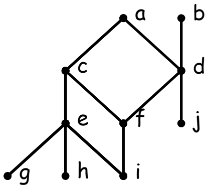

Ex. The upper bounds of {*g*,*i*} are

(A) *e*

(B) *c*

(C) *e*, *c*, and *a*

{*a*, *b*} has no UB.

Upper and Lower Bounds

Definition. Let (*S*,) be a partial order. If *A* ⊆ *S*, then an ***upper bound*** for *A* is an element *x* ∈ *S* (perhaps in A also) such that ∀ *a* ∈ *A*, *a* *x*.

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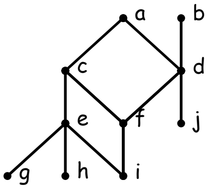


Ex. The lower bounds of {*c*,*d*} are

(A) *f*

(B) *f, i*

(C) *e*

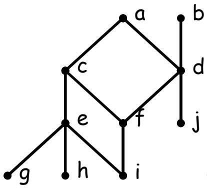
NO! *e*, *d* are not comparable

{*g*, *h*} has no LB.

Definition. Let (*S*,) be a partial order. If *A* ⊆ *S*, then the ***least upper bound*** for *A* is an upper bound *x* such that for any upper bound *y* of *A*, *y* *x*

The ***greatest lower bound*** for *A* is a lower bound *x* such that for any lower bound *y* of *A*, *y* *x*

Ex. The LUB of {*i*,*j*} is *d*



Ex. The GLB of {*g*,*j*} is

(A) *e, c, a*

NO! they are upper bounds

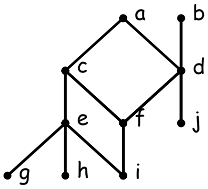
(B) *d, a*

NO! they are upper bounds

{*g*, *j*} has no lower bounds, and hence no GLB

If the least upper bound of *A* = {*a*, *b*} exists, then we denote it by $a \vee b$

Similarly if the greatest lower bound of *A* = {*a*, *b*} exists, then we denote it by $a \wedge b$



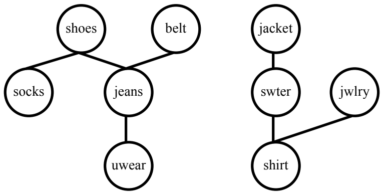
Ex. $i \vee j = d$

Ex. $b \wedge c = f$

Topological Sorting

Consider the problem of getting dressed.

Precedence constraints are modeled by a poset in which *a* b if and only if you must put on *a* before *b*.

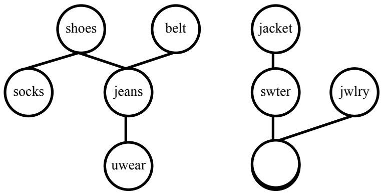


In what order will you get dressed while respecting constraints?

In other words, we will find a new total order so that *a* is a lower bound of *b* if *a* *b*

Topological Sorting

Recall that every finite non-empty poset has at least one minimal element a_1.

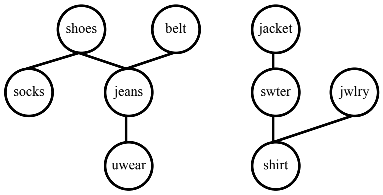


E.g. shirt is a minimal element

✓ Now the new set after we remove a_1 is still a poset.

Topological Sorting

✓ Let a_2 be a minimal of the new poset.



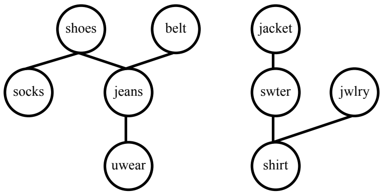
E.g. underwear is a new minimal element

✓ Now every element of this new poset cannot be a proper lower bound of a_1 and a_2 in the original poset

This process continues until all elements are removed

We obtain a new order of the elements satisfying the given constraints:

$a_1, a_2, \ldots, a_m$



The arrangement of the given poset in the new total order $a_1, a_2, \ldots$ compatible with the old order is called the Topological sorting

Bài tập

6. Khảo sát các tính chất của các quan hệ $\mathcal{R}$ sau. Xét xem quan hệ $\mathcal{R}$ nào là quan hệ tương đương. Tìm các lớp tương đương cho các quan hệ tương đương tương ứng.

a) $\forall x, y \in \mathbf{R}$, $x\mathcal{R}y \Leftrightarrow x^2 + 2x = y^2 + 2y$;

b) $\forall x, y \in \mathbf{R}$, $x\mathcal{R}y \Leftrightarrow x^2 + 2x \le y^2 + 2y$;

c) $\forall x, y \in \mathbf{R}$, $x\mathcal{R}y \Leftrightarrow$

$$x^3 - x^2y - 3x = y^3 - xy^2 - 3y;$$

d) $\forall x, y \in \mathbf{R}^+$, $x\mathcal{R}y \Leftrightarrow x^3 - x^2y - x = y^3 - xy^2 - y$.

Bài tập

7. . Khảo sát tính chất của các quan hệ $\mathcal{R}$ sau. Xét xem quan hệ $\mathcal{R}$ nào là quan hệ thứ tự và khảo sát tính toàn phần, tính bộ phận và tìm các phần tử lớn nhất, nhỏ nhất, tối đại, tối tiểu (nếu có) của các quan hệ thứ tự tương ứng.

a) $\forall x, y \in \mathbf{Z}$, $x\mathcal{R}y \Leftrightarrow x|y$;

b) $\forall x, y \in \mathbf{R}$, $x\mathcal{R}y \Leftrightarrow x = y$ hay $x < y + 1$.

c) $\forall x, y \in \mathbf{R}$, $x\mathcal{R}y \Leftrightarrow x = y$ hay $x < y - 1$.

d) $\forall (x, y); (z, t) \in \mathbf{Z}^2$, $(x, y) \le (z, t) \Leftrightarrow x \le z$ hay ($x = z$ và $y \le t$);

e) $\forall (x, y); (z, t) \in \mathbf{Z}^2$, $(x, y) \le (z, t) \Leftrightarrow x < z$ hay ($x = z$ và $y \le t$);

Bài tập

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Bài tập

8. . Xét quan hệ $\mathcal{R}$ trên $\mathbf{Z}$ định bởi:

$\forall x, y \in \mathbf{Z}$, $x\mathcal{R}y \Leftrightarrow \exists n \in \mathbf{Z}$, $x = y2^n$

a) Chứng minh $\mathcal{R}$ là một quan hệ tương đương.

b) Trong số các lớp tương đương $\overline{1}, \overline{2}, \overline{3}, \overline{4}$ có bao nhiêu lớp đôi một phân biệt?

a) Câu hỏi tương tự như câu b) cho các lớp $\overline{6}, \overline{7}, \overline{21}, \overline{24}, \overline{25}, \overline{35}, \overline{42}$ và $\overline{48}$.

Bài tập

9. . Xét tập mẫu tự A = {a, b, c} với a < b < c và các chuỗi kí tự:

s_1 = ccbac

s_2 = abccaa

theo thứ tự tự điển.. Hỏi có bao nhiêu chuỗi kí tự s gồm 6 kí tự thỏa $s_2 \le s \le s_1$?

Bài tập

10. **<u>ĐỀ THI NĂM 20006</u>**

- Xét thứ tự "⊂"trên tập P(S)các tập con của tập S ={1,2,3,4,5}trong đó A⊂B nếu A là tập con của B.
- Tìm một thứ tự toàn phần " ≤ "trên P(S) sao cho với A, B trong P(S), nếu A⊂B thì A≤ B. Tổng quát hoá cho trường hợp S có n phần tử.

Bài tập

11) Đề 2007.Có bao nhiêu dãy bit có độ dài ≤15 sao cho 00001 ≤ s ≤ 011, trong đó "≤ " là thứ tự từ điển.