東大物理工学科 2017

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第一問

[1.1]

エネルギー保存則より

$$mgR = mgR\cos\theta + \frac{1}{2}mv^2, \quad \therefore v = \sqrt{2gR(1-\cos\theta)} \tag{1}$$

[1.2]

回転系で見ると遠心力がかかるため

$$m\frac{v^2}{R} + N = mg\cos\theta, \quad \therefore N = mg\cos\theta - m\frac{v^2}{R} = mg(3\cos\theta - 2) = 0 \tag{2}$$

の時に質点が離れるため

$$\cos\theta = \frac{2}{3} \tag{3}$$

よって，速度は

$$v_{c1} = \sqrt{\frac{2}{3}gR} \tag{4}$$

[2.1]

$$\begin{aligned} I &= \int dmr'^2 = \frac{3m}{4\pi r^3}\int r'^3 dr'd\theta dz, \quad 0 \le r' \le \sqrt{r^2 - z^2} && (5) \\ &= \frac{3m}{4\pi r^3}\cdot 2\pi \int_{-r}^{r} \frac{1}{4}(r^2 - z^2)^2 dz && (6) \\ &= \frac{2}{5}mr^2 && (7) \end{aligned}$$

[2.2]

図より

$$R\theta = r(\phi - \theta), \quad \therefore (R+r)\dot{\theta} = r\dot{\phi} \tag{8}$$

よって

$$v = \frac{\dot{\theta}}{R+r}, \omega = \dot{\phi} \tag{9}$$

より，

$$(R+r)^2 v = r\omega \tag{10}$$

[2.3]

束縛条件は

$$f_r(r') = r' - (r+R) = 0, \quad f_{\theta,\phi}(\theta,\phi) = (R+r)\theta - r\phi = 0 \tag{11}$$

であり，束縛条件がない時のラグランジアンは

$$L_0 = \frac{1}{2}m(\dot{r}'^2 + r'^2\dot{\theta}^2) + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\dot{\phi}^2 - mgr'\cos\theta \tag{12}$$

よって，束縛条件がある場合のラグランジュ方程式は

$$m\ddot{r}' - (mr'\dot{\theta}^2 - mg\cos\theta) = \lambda_r\frac{\partial f_r}{\partial r} = \lambda_r \tag{13}$$

$$mr'^2\ddot{\theta} - mgr'\sin\theta = \lambda_{\theta,\phi}(R+r) \tag{14}$$

$$\frac{1}{5}mr'^2\ddot{\phi} = \lambda_{\theta,\phi}(-r) \tag{15}$$

よって，$\lambda_{\theta,\phi}$ を消去して，また，r の束縛条件より

$$mr'^2\ddot{\theta} - mgr'\sin\theta = -\frac{m}{5}(R+r)^2\ddot{\theta}, \quad \therefore \frac{6}{5}mr'\ddot{\theta} - mg\sin\theta = 0 \tag{16}$$

ここで，

$$\frac{6}{5}mr'\dot{\theta}\ddot{\theta} - \dot{\theta}mg\sin\theta = 0 \quad \therefore \frac{3}{5}mr'\dot{\theta}^2 + mg\cos\theta = 0 \tag{17}$$

よって，束縛条件より $\dot{r}', \ddot{r}' = 0$ であるため，

$$-mr'\frac{5}{3m}g(1-\cos\theta) + mg\cos\theta = \lambda_r, \quad \therefore \cos\theta = \frac{2}{5} \tag{18}$$

また，

$$\dot{\theta} = \sqrt{\frac{5}{3(R+r)}\frac{3}{5}} = \frac{\sqrt{g}}{\sqrt{R+r}} \tag{19}$$

より，

$$v = \frac{\sqrt{g}}{(R+r)^{3/2}} \tag{20}$$

[2.4]

回転により寄与がかかるから．