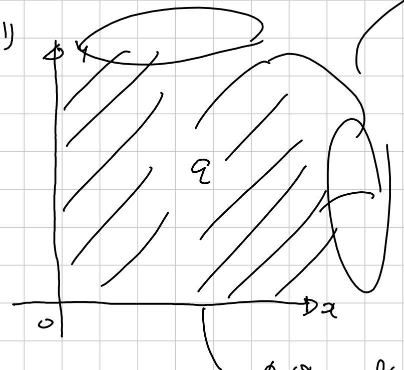
Green function.

例1)



$\phi(\infty, y) = \phi(x, \infty) = 0.$

$\Delta\phi = -\frac{\rho}{\varepsilon}$

$\rho(x,y)$の分布の時のポテンシャル $\phi(x,y)$

$\phi(x,0) = f(x)$, $\frac{\partial\phi}{\partial y}(x,y) = g(y)$ である

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = -\frac{\rho}{\varepsilon}$$

(i) Flourier sine 展開

$$\begin{cases} \hat\phi(k,l) = \frac{2}{\pi}\int_0^\infty \cos kx \sin ly\, \phi(x,y)\, dx dy. \\ \hat\rho(k,l) = \quad \vdots \\ \hat f(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos kx\, f(x) dx, \quad \hat g(l) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin ly\, g(y) dy \end{cases}$$

(1.2.1) を変換

$$\underbrace{\frac{2}{\pi}\int_0^\infty \cos kx \sin ly \cdot \Delta\phi\, dx dy}_{\shortparallel} = -\frac{\hat\rho}{\varepsilon}$$

$$\frac{2}{\pi}\sin ly \underbrace{\int_0^\infty \left(\cos kx \cdot \frac{\partial^2\phi}{\partial x^2} dx\right)}_{\shortparallel} + \frac{2}{\pi}\cos kx \underbrace{\int_0^\infty \left(\sin ly \cdot \frac{\partial^2}{\partial y^2}\phi\, dy\right)}_{\shortparallel}$$

$$\left(-g(y) - k^2\int_0^\infty \cos kx\, \phi(x,y) dx\right) \qquad l f(x) - l^2\int_0^\infty \sin ly\, \phi(x,y) dy$$

$$\Rightarrow \quad -(k^2 + l^2)\hat\phi(k,l) - \sqrt{\frac{2}{\pi}}\hat g(l) + \sqrt{\frac{2}{\pi}}\, l \hat f(k) = -\frac{\hat\rho}{\varepsilon}(k,l)$$

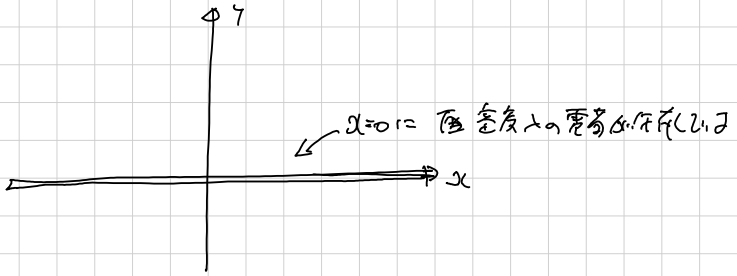
$$\therefore \quad \hat\phi(k,l) = \frac{1}{k^2+l^2}\left\{\sqrt{\frac{2}{\pi}}\, l \hat f(k) - \sqrt{\frac{2}{\pi}}\hat g(l) + \frac{\hat\rho}{\varepsilon}\right\}$$

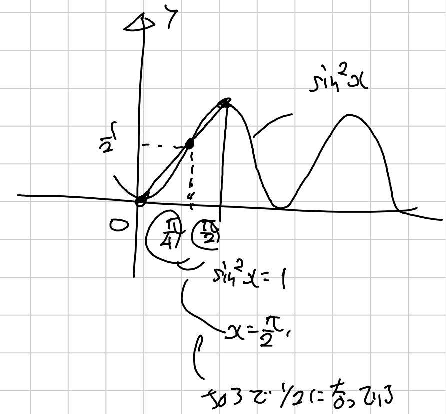
$$\boxed{\phi = \frac{2}{\pi}\int_0^\infty \cos kx \sin ly \cdot \hat\phi\, dk dl}$$

(Fourier 変換の例1)

$\Rightarrow \boxed{\Delta G = -\delta(x-x')\delta(y-y')}$, $x=0$ では 微分で与えている

$\boxed{\text{の解である.}}$





Green 関数の性質

① 相反性 : $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$

② 非正則性

§ 波動方程式の Green function

$$\left(\Delta - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} + \frac{1}{k^2} \cdot \frac{\partial}{\partial t} - \mu^2 \right) \phi(\mathbf{r}, t) = -\rho(\mathbf{r}, t)$$

$\Downarrow$ Green function

$$\left(\Delta - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} - \frac{1}{k^2} \cdot \frac{\partial}{\partial t} - \mu^2 \right) G(\mathbf{r}, t', \mathbf{r}', t) = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

$A(\mathbf{r}) \cdot \mathbf{n} \cdot \nabla G + B(\mathbf{r}) G = 0$ が成立する

$\Downarrow$ 遅延条件

$G(\mathbf{r}, \mathbf{r}', t, t')$ ($t < t'$)

$\Downarrow$

⇒ 具体的な Green 関数の求め方

$$(\Delta + k^2)\phi(\mathbf{r}) = -\rho(\mathbf{r}).$$

Helmholtz equation: $(\Delta + k^2)\phi(\mathbf{r}) = -\rho(\mathbf{r})$.

$\Downarrow$

Green function は. $(\Delta + k^2) G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$ である.

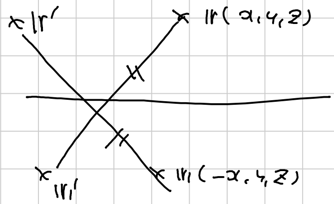
$$\boxed{\begin{aligned} \hat{G}(\mathbf{p}) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} G(\mathbf{r}) e^{-i\mathbf{p} \cdot \mathbf{r}} d\mathbf{r} \\ G(\mathbf{r}) &= \int_{-\infty}^{\infty} \hat{G}(\mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{r}} d\mathbf{r} \end{aligned}}$$

$$\left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} \underbrace{e^{-i\mathbf{p} \cdot \mathbf{r}} (\Delta + k^2) G(\mathbf{r})}_{\shortparallel} = -\frac{1}{(2\pi)^3}$$

$$\underbrace{\int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^3 (\Delta + k^2) \left[e^{-i\mathbf{p} \cdot \mathbf{r}} G(\mathbf{r}) \right]}_{\Downarrow \; 0.} - \underbrace{(-p^2 + k^2) \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} G(\mathbf{r}) e^{-i\mathbf{p} \cdot \mathbf{r}} d\mathbf{r}}_{\shortparallel \; -(-p^2 + k^2) \hat{G}(\mathbf{r})}$$

③ 境界のある場合のGreen function

↳ 鏡像法



$G^D(\mathbf{r},\mathbf{r}') = G^\infty(|\mathbf{r}-\mathbf{r}'|) - G^\infty(|\mathbf{r}-\mathbf{r}_1'|)$ も方程式をみたしている。

◎ 相反性:

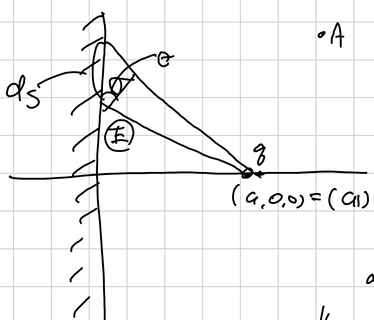
$$G^D(\mathbf{r}',\mathbf{r}) = G^\infty(|\mathbf{r}'-\mathbf{r}|) - G^\infty(|\mathbf{r}'-\mathbf{r}_1|) \quad \text{OK}$$

$G^D(|\mathbf{r}-\mathbf{r}'|)=0$ というDirichletの条件をみたしている

$G^N(\mathbf{r},\mathbf{r}') = G^\infty(|\mathbf{r}'-\mathbf{r}|) + G^\infty(|\mathbf{r}'-\mathbf{r}_1'|)$ — Neumann

(ex) $V = \frac{1}{4\pi\varepsilon}\left(\frac{q}{r} - \frac{q}{r'}\right)$ である

境界面上からのキョリ(a, y, z)は.



$E = \frac{\sigma}{\varepsilon_0}$ より.

$\sigma = \varepsilon_0 E = \varepsilon_0\left(-\frac{\partial V}{\partial x}\right) = -\frac{aq}{2\pi r^3}$ である

$E = -\nabla\phi$ 　E_xのせいぶんのみ

$d\omega = \frac{dS\cos\theta}{r^2}$ 　∴ $\sigma dS = -\frac{q}{2\pi}d\omega$

◎ 総電荷を求める

$\sigma \cdot \underset{d(\pi s^2)}{2\pi s\,ds} = -\frac{aq}{r^3}s\,ds$ 　∴ $-\int_0^\infty \frac{aqs}{(a^2+s^2)^{3/2}}ds = -q$ である.

導体は.

$$\boxed{\Delta\phi = -\frac{\rho(\mathbf{r})}{\varepsilon_0} = -\frac{e\delta(\mathbf{r}-\mathbf{a})}{\varepsilon_0} \ \wedge\ \phi = 0 \ (x=0) \ \text{Dirichlet 条件}}$$

$G^D(\mathbf{r},\mathbf{r}') \equiv G^\infty(|\mathbf{r}-\mathbf{r}'|) - G^\infty(|\mathbf{r}-\mathbf{r}_1'|)$ をみたす.

$$\Delta G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$$

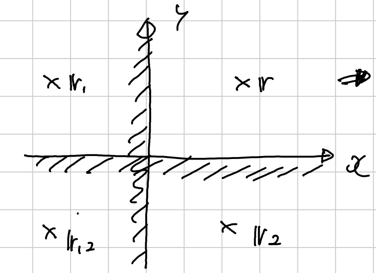
$$\Delta\phi = \int_{-\infty}^{\infty}\Delta G(\mathbf{r},\mathbf{r}')\frac{\rho(\mathbf{r}')}{\varepsilon_0} = \Delta\int_{-\infty}^{\infty}\left(G(\mathbf{r},\mathbf{r}')\frac{\rho(\mathbf{r}')}{\varepsilon_0}\right)$$

$$\therefore \phi = \int_{-\infty}^{\infty} G(\mathbf{r},\mathbf{r}')\cdot\frac{\rho(\mathbf{r}')}{\varepsilon_0} = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\cdot\frac{e\delta(\mathbf{r}'-\mathbf{a})}{\varepsilon_0}d\mathbf{r}'$$

$$= \frac{e}{4\pi|\mathbf{r}-\mathbf{a}|\varepsilon_0}$$

$$G^D(\mathbf{r},\mathbf{r}') = G^\infty(|\mathbf{r}-\mathbf{r}'|) - G^\infty(|\mathbf{r}-\mathbf{r}_1'|)$$

b)



⇒

Dirichlet条件に対応するグリーン関数は

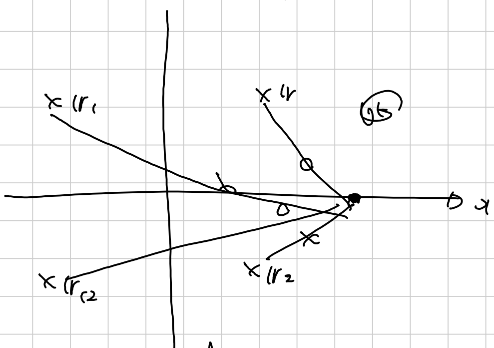
$$\boxed{G^D(r, r') = G^\infty(|r-r'|) - G^\infty(|r-r_1'|) - G^\infty(|r-r_2'|) + G^\infty(|r-r_{12}'|) = 0}$$

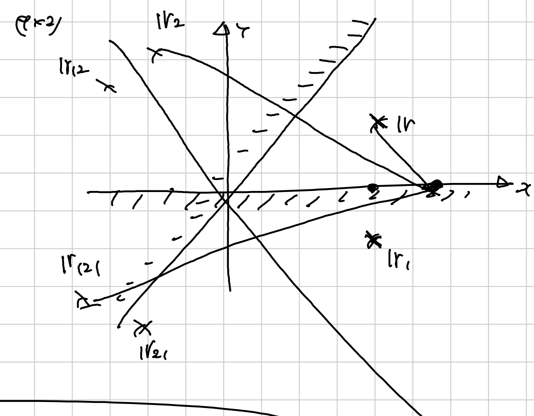
$x=0, y, z$の時 と $y=0, x, z$の時で両方0になる,

⇒

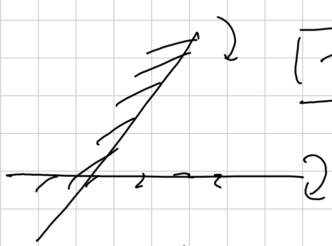
$G^\infty(|r-r'|)\Big|_{a=0} = G^\infty($

↓





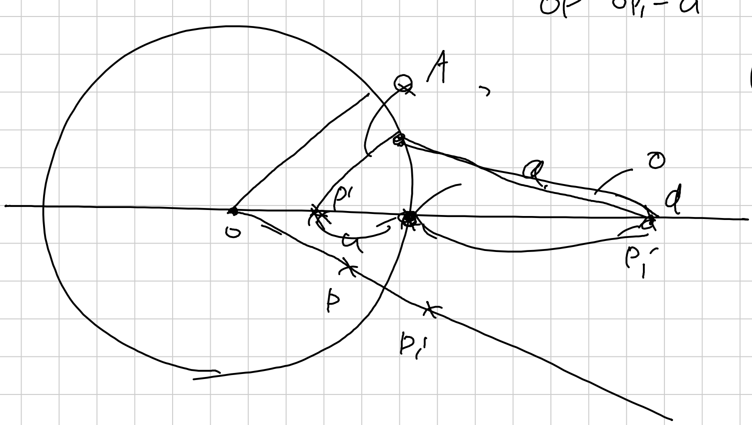
↓



$\boxed{\text{をおりかえしていくことで求める!!}}$

円周上でのLaplace eqにおける

$\overline{OP}\cdot\overline{OP_1}=a^2$



$(a+d)\cdot x = a^2$

$x = \frac{a^2}{d}$の時に

◎ Laplace-equation

$\Delta\phi(\mathbf{r}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{r})$, $r\to\infty$の時. $G(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$ が便利である.

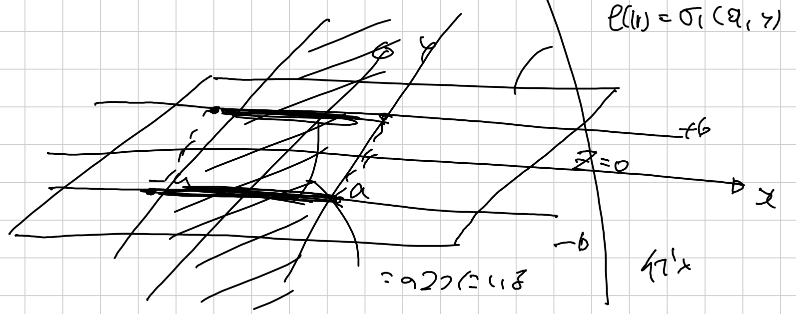
$$\boxed{\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}'}$$ rに展開する

‖

$$\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^{l} P_l(\cos\theta') \quad (\mathbf{r}\cdot\mathbf{r}' = r\cdot r'\cos\theta')$$

b) 平面電荷分布

$$\rho(\mathbf{r}) = \sigma_1(x,y)\,\delta(z) = \sigma_2(\rho,\varphi)\,\delta(z) = \sigma_3(r,\varphi)\,\delta\left(\theta-\frac{\pi}{2}\right)$$

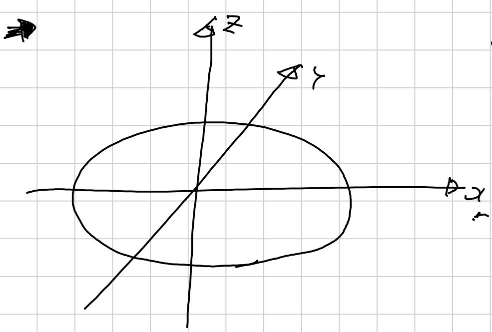


$$\rho(\mathbf{r}) = \sigma_0\Theta(a-|x|)\{\delta(y-b)+\delta(y+b)\}\delta(z)$$ の時を考えてみたい.

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\cdot\sigma\Theta(a-|x|)\}\ \{\delta(z)\,d\mathbf{r}'$$

$$= \frac{\sigma_0}{4\pi\varepsilon_0}\int_{-a}^{a}\left\{\frac{1}{\sqrt{(x-x')^2+(y-b)^2+z^2}}+\frac{1}{\sqrt{(x-x')^2+(y+b)^2+z^2}}\right\}dx'$$

$a\to\infty$ とすると.



$$\boxed{\rho(\mathbf{r}) = \sigma\cdot\Theta(a-\rho)\,\delta(z)} = \sigma_3(r,\varphi)\Theta(a-r)\,\delta(\cos\theta)$$

一図 とばす.

d). 球面電荷分布の時 $\phi(\mathbf{r}) = \frac{a^2\sigma_0}{\varepsilon_0}\left\{\frac{1}{r}\Theta(r-a)+\frac{1}{a}\Theta(a-r)\right\}$

球外では 原点に全電荷 $4\pi a^2\sigma_0$ が集中したのに等しい.

$$\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^{l}P_l(\cos\theta')$$

$|0\rangle|0\rangle \longrightarrow |2\rangle|1\rangle$

$\Downarrow$

高次元を利用して、ZZ相互作用を実現できるのでは??