Data Structures and Algorithms

Recursion

The Recursion Pattern

- **Recursion**: when a method calls itself
- Classic example--the factorial function:
  - $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & \textit{else} \end{cases}$$

The Recursion Pattern

- As a Java method:

```
// recursive factorial function
public static int  recursiveFactorial(int n) {
    if  (n  ==  0)  return  1;        // basis case
    else return  n  *  recursiveFactorial(n- 1); // recursive case
}
```

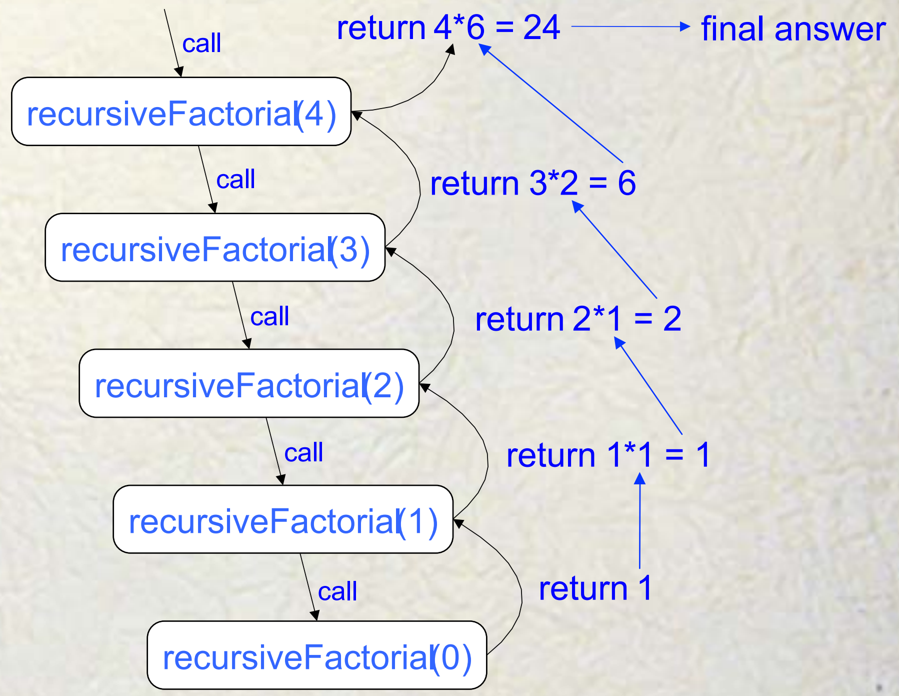
Content of a Recursive Method

- **Base case(s).**
  - Values of the input variables for which we perform no recursive calls are called **base cases** (there should be at least one base case).
  - Every possible chain of recursive calls **must** eventually reach a base case.
- ***Recursive calls.***
  - Calls to the current method.
  - Each recursive call should be defined so that it makes progress towards a base case.

Visualizing Recursion

- Recursion trace
- A box for each recursive call
- An arrow from each caller to callee
- An arrow from each callee to caller showing return value

Example recursion trace:



Example – English Rulers

- Define a recursive way to print the ticks and numbers like an English ruler:

```
---- 0
-
--
-
---
-
--
-
---- 1
-
--
-
---
-
--
-
---- 2
```

```
----- 0
-
--
-
---
-
--
-
----
-
--
-
---
-
--
-
----- 1
```

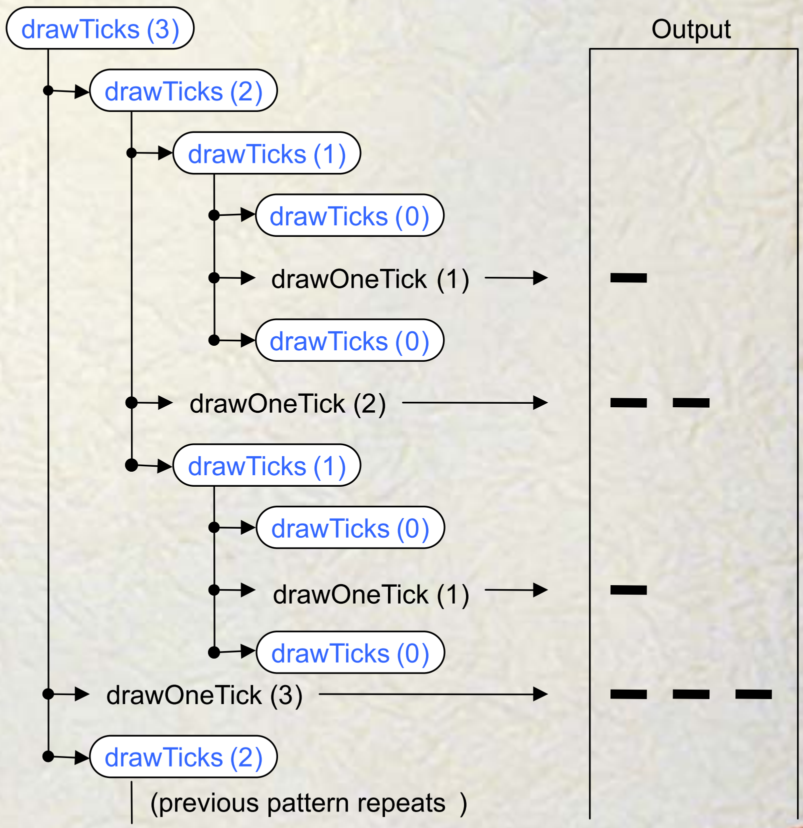
```
--- 0
-
--
-
--- 1
-
--
-
--- 2
-
--
-
--- 3
```

A Recursive Method for Drawing Ticks on an English Ruler

```
    // draw a tick with no label
public static void  drawOneTick(int  tickLength) {  drawOneTick(tickLength,  - 1);  }
    // draw one tick
public static void  drawOneTick(int  tickLength,  int  tickLabel) {
   for  (int i =  0; i <  tickLength;  i++)
       System.out.print("-");
   if  (tickLabel  >=  0)  System.out.print(" "  +  tickLabel);
   System.out.print("\n");
}
public static void  drawTicks(int  tickLength)  {  // draw ticks of given length
   if  (tickLength  >  0)  {                    // stop when length drops to 0
       drawTicks(tickLength- 1);               // recursively draw left ticks
       drawOneTick(tickLength);                // draw center tick
       drawTicks(tickLength- 1);               // recursively draw right ticks
  }
}
public static void  drawRuler(int  nInches,  int  majorLength)  {  // draw ruler
   drawOneTick(majorLength,  0);              // draw tick 0 and its label
   for  (int i =  1; i <=  nInches;  i++)        {
       drawTicks(majorLength- 1);             // draw ticks for this inch
       drawOneTick(majorLength,  i);          // draw tick i and its label
  }
}
```

Visualizing the DrawTicks Method

- An interval with a central tick length $L \geq 1$ is composed of the following:
  - an interval with a central tick length L-1,
  - a single tick of length L,
  - an interval with a central tick length L-1.



Recall the Recursion Pattern

- **Recursion**: when a method calls itself
- Classic example--the factorial function:
  - $n! = 1\cdot 2\cdot 3\cdot \cdots \cdot (n-1)\cdot n$
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

- As a Java method:

```
// recursive factorial function
public static int  recursiveFactorial(int n) {
    if  (n  ==  0)  return  1;      // basis case
    else return  n  *  recursiveFactorial(n- 1); // recursive case
}
```

Linear Recursion

- **Test for base cases.**
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls **must** eventually reach a base case, and the handling of each base case should not use recursion.
- ***Recur once.***
  - Perform a single recursive call. (This recursive step may involve a test that decides which of several possible recursive calls to make, but it should ultimately choose to make just one of these calls each time we perform this step.)
  - Define each possible recursive call so that it makes progress towards a base case.

A Simple Example of Linear Recursion

Algorithm LinearSum(*A, n*):

Input:

An integer array *A* and an integer *n* >= 1, such that *A* has at least *n* elements

Output:

The sum of the first *n* integers in *A*

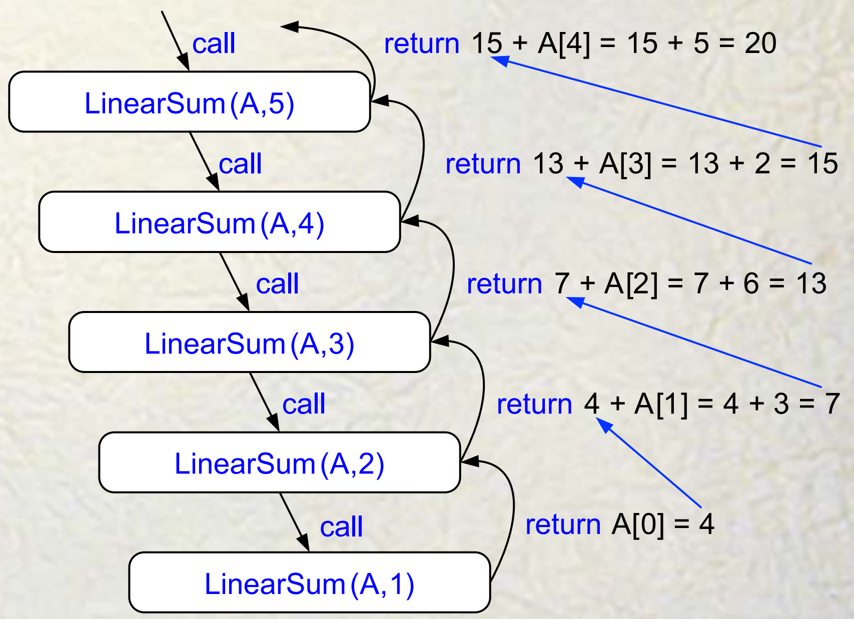
if *n* = 1 **then**

return *A*[0]

else

return LinearSum(*A, n* - 1) + *A*[*n* - 1]

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

Swap $A[i]$ and $A[j]$

ReverseArray(A, $i + 1$, $j - 1$)

return

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional paramaters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(*A, i, j*), not ReverseArray(*A*).

Computing Powers

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to a power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$

$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$

$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$

$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$

A Recursive Squaring Method

Algorithm Power(x, n):

Input: A number x and integer $n = 0$

Output: The value x^n

if $n = 0$ **then**

return 1

if n is odd **then**

$y = \text{Power}(x, (n - 1)/2)$

return $x \cdot y \cdot y$

else

$y = \text{Power}(x, n/2)$

return $y \cdot y$

Analyzing the Recursive Squaring Method

Algorithm Power(x, n):

Input: A number x and integer $n = 0$

Output: The value x^n

if $n = 0$ **then**

  return 1

if n is odd **then**

  $y = \text{Power}(x, (n - 1)/2)$

  return $x \cdot y \cdot y$

else

  $y = \text{Power}(x, n/2)$

  return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we used a variable twice here rather than calling the method twice.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

Algorithm IterativeReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

while $i < j$ **do**

Swap $A[i]$ and $A[j]$

$i = i + 1$

$j = j - 1$

return

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example: the DrawTicks method for drawing ticks on an English ruler.

```
---- 0      ----- 0     --- 0
-           -           -
--          --          --
-           -           -
---         ---         --- 1
-           -           -
--          --          --
-           -           -
---- 1      ----        --- 2
-           -           -
--          --          --
-           -           -
---         ---         --- 3
-           -
--          --
-           -
---- 2      ----- 1
```

A Binary Recursive Method for Drawing Ticks

```
    // draw a tick with no label
public static void drawOneTick(int tickLength) { drawOneTick(tickLength, - 1); }
    // draw one tick
public static void drawOneTick(int tickLength, int tickLabel) {
  for (int i = 0; i < tickLength; i++)
      System.out.print("-");
  if (tickLabel >= 0) System.out.print(" " + tickLabel);
  System.out.print("\n");
}
public static void drawTicks(int tickLength) { // draw ticks of given length
  if (tickLength > 0) {                    // stop when length drops to 0
      drawTicks(tickLength- 1);             // recursively draw left ticks
      drawOneTick(tickLength);              // draw center tick
      drawTicks(tickLength- 1);             // recursively draw right ticks
  }
}
public static void drawRuler(int nInches, int majorLength) { // draw ruler
  drawOneTick(majorLength, 0);             // draw tick 0 and its label
  for (int i = 1; i <= nInches; i++)       {
      drawTicks(majorLength- 1);            // draw ticks for this inch
      drawOneTick(majorLength, i);          // draw tick i and its label
  }
}
```

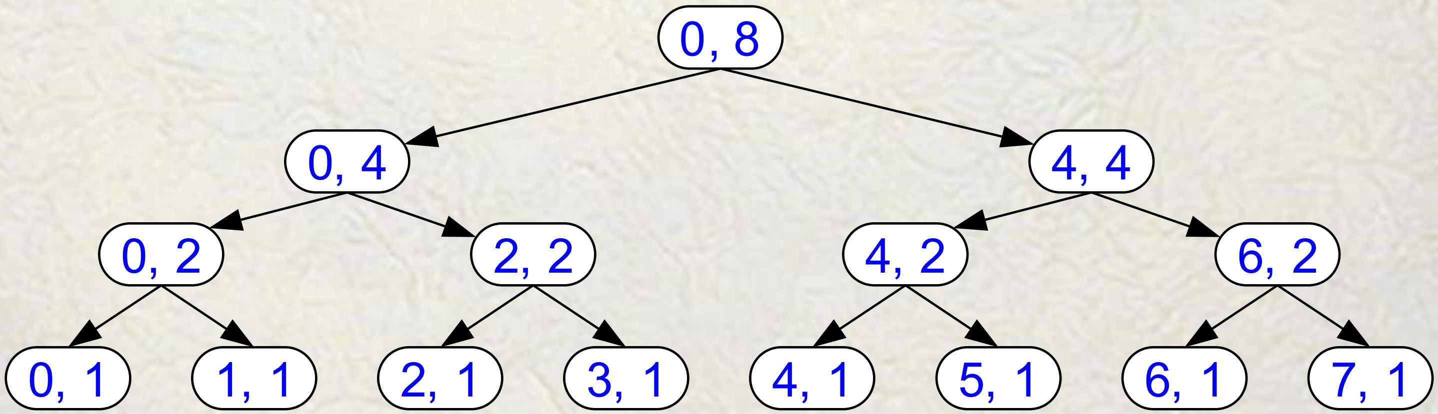
Note the two recursive calls

Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

Algorithm BinarySum(A, i, n):
Input: An array A and integers i and n
Output: The sum of the n integers in A starting at index i
if $n = 1$ **then**
  return $A[i]$
return BinarySum($A, i, n/2$) + BinarySum($A, i + n/2, n/2$)

- **Example trace:**



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

  $F_0 = 0$
  $F_1 = 1$
  $F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

- As a recursive algorithm (first attempt):

  Algorithm BinaryFib(k):
  Input: Nonnegative integer k
  Output: The kth Fibonacci number F_k
  if $k <= 1$ **then**
  return k
  else
  return BinaryFib(k - 1) + BinaryFib(k - 2)

Analyzing the Binary Recursion Fibonacci Algorithm

- Let n_k denote number of recursive calls made by BinaryFib(k). Then
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$.
- Note that the value at least doubles for every other value of n_k. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead:

Algorithm LinearFibonacci(k):

Input: A nonnegative integer k

Output: Pair of Fibonacci numbers (F_k, F_{k-1})

if $k = 1$ **then**

  return (k, 0)

else

  (i, j) = LinearFibonacci(k - 1)

  return ($i + j$, i)

- Runs in $O(k)$ time.

Multiple Recursion

- Motivating example: summation puzzles
  - *pot + pan = bib*
  - *dog + cat = pig*
  - *boy + girl = baby*

- Multiple recursion: makes potentially many recursive calls (not just one or two).
- Find all subset of a certain length.

Algorithm for Multiple Recursion

Algorithm PuzzleSolve(k,S,U):

Input: An integer k, sequence S, and set U (the universe of elements to test)

Output: An enumeration of all k-length extensions to S using elements in U without repetitions

```
if k = 0 then
    Test whether S is a configuration that solves the puzzle
    if S solves the puzzle then
        return "Solution found: " S
else
    for all e in U do
        Remove e from U    {e is now being used}
        Add e to the end of S
        PuzzleSolve(k - 1, S,U)
        Add e back to U    {e is now unused}
        Remove e from the end of S
```

Visualizing PuzzleSolve