Linear System Theory

Problem Set 6

Controllability, observability, state-feedback controllers and observer design

Issue date: December 21, 2022
Due date: 23:55, January 16, 2023

Exercise 1 (Controllabilty and observability, 15%). Consider the linear time invariant system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} w & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned} \tag{*}$$

where $w \in [1, 2]$ is a constant parameter. Define $A(w) = \begin{bmatrix} w & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C = [1 \quad 0]$.

1. Determine for which values of w, the system in (*) is controllable and observable.
2. Consider the linear observer

$$\begin{aligned} \dot{\hat{x}}(t) &= A(w)\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) &= C\hat{x}(t). \end{aligned}$$

   Find the set of all observer gain matrices $L \in \mathbb{R}^{2\times 1}$ such that, for all possible $w \in [1, 2]$, the observation error $e(t) = x(t) - \hat{x}(t)$ tends to zero asymptotically.

Exercise 2 (Controllable and observable modes, 20%). Given the system

$$\dot{x}(t) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t), \qquad y(t) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} x(t),$$

Discuss controllability, stabilizability, observability and detectability properties of the system.

Exercise 3 (Controllability and observability, 15%). The cart-pole system in Figure 1 consist of two components:

- a cart of mass M moving along the X-axis pushed by the horizontal force F; let p denote the horizontal position of the cart with respect to the origin in X,
- and a pole of length L with a mass m on top; for simplicity we assume that the pole itself is weightless.

The objective is to use the cart force F to balance the pole as an inverted pendulum, so that it stays in a vertical position, i.e., with an angle $\theta \approx 0$.

The linearised dynamics of this cart-pole system are:

$$\begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML} \end{bmatrix} u(t) \tag{1}$$

where $u(t) = F(t)$, $g = 10$, $m = 0.1$, $M = 10$ and $L = 1$.

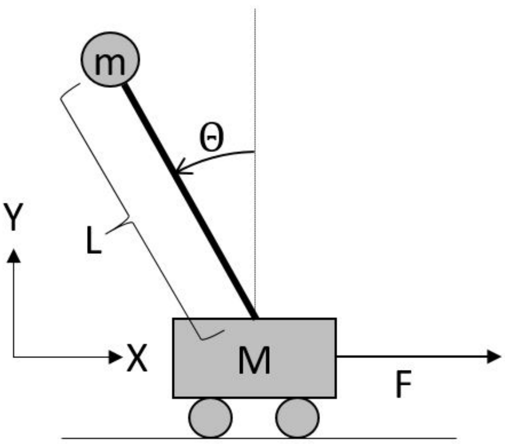

Figure 1: Cart-pole system

1. Is the cart-pole system with the linearised dynamics controllable?
2. Using the state dependencies shown in A, design an output matrix C which uses the smallest number of states to construct an output that leads to an observable system. Explain why your choice of C leads to an observable system.

Exercise 4 (Pole placement, 20%). Consider the system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} u(t) = Ax(t) + Bu(t), \\ y(t) &= \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} x(t) = Cx(t). \end{aligned} \tag{$\square$}$$

1. Is the system observable? Is it controllable? Justify your answer in each case.
2. Design a state feedback $u = Kx$ such that the closed loop system has three poles at $s = -2$.
3. Recall that a state-observer provides an estimate $\hat{x}(t)$ of the state $x(t)$ of ($\square$) by means of the differential equation

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L\big(\hat{y}(t) - y(t)\big), \\ \hat{y}(t) &= C\hat{x}(t). \end{aligned}$$

   Compute the observer gain L such that the dynamics of the error $e := x - \hat{x}$ have three poles at $s = -3$.

Exercise 5 (Controllability & observability, 30%). Consider the controllable linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= \underbrace{\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t), \\ y(t) &= \underbrace{\begin{bmatrix} 0 & 2 \end{bmatrix}}_{C} x(t)\,. \end{aligned} \tag{2}$$

1. Design a gain matrix $K \in \mathbb{R}^{1\times 2}$ such that the closed-loop system under feedback input $u(t) = Kx(t)$ has eigenvalues equal to -3. What is the dominant pole, i.e. the pole closest to the imaginary axis, of the closed-loop dynamics?

2. Is the system observable? If not, find the eigenvalues of the unobservable modes and conclude whether the system is at least detectable. Justify your answer.

3. Consider the linear state observer

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(\hat{y}(t) - y(t)) \\ \hat{y}(t) &= C\hat{x}(t)\,, \end{aligned} \tag{3}$$

   where the matrices A, B and C are the same as in system (2). Is it possible to design L such that the dominant pole of the observer error dynamics $e(t) = x(t) - \hat{x}(t)$ is smaller than -2? Justify your answer.

4. For the linear state observer in Equation (3), find a gain observer matrix $L \in \mathbb{R}^{2\times 1}$ such that the dynamics of the error $e(t) = x(t) - \hat{x}(t)$ have two poles at -2. Comment on the number of solutions you obtain.

5. Consider now the closed-loop system consisting of (2), the state observer dynamics (3), and the observer based output feedback $u(t) = K\hat{x}(t)$ where K was designed in point 1. What is the dominant pole of the closed-loop system? Justify your answer.