Congestion Games

Game Theory I (Prof. Yokoo; Wed. 8:40 am) – 5. Advanced Topic

Spring Quarter, Grad. School of ISEE, Kyushu University

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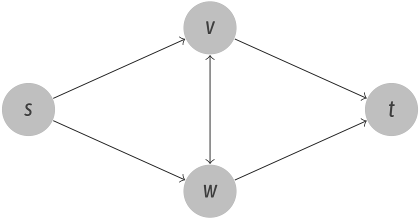
Congestion Games

- A model of game theory representing, e.g., a path-selecting situation in a traffic jam
  - Player i's cost is non-decreasing for #players choosing the same option with i
  - E.g., choosing one from multiple cashiers in a supermarket
- Analysing society and economies based on the idea of Nash equilibrium
  - A Nash equilibrium is not necessarily socially optimal
  - Ad hoc solutions may cause a huge inefficiency to the society
  - Quantitative analysis from the viewpoint of game theory
- A lot of contributions to this field of congestion games from computer scientists

Outline

1 Preliminaries

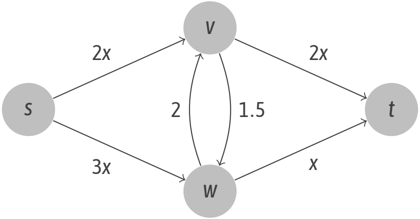
Model and Notations (1/2)



- $G(V, E)$: a given connected directed multigraph
- $s, t \in V$: given source (starting) vertex and terminal (destination) vertex;

  An amount 1 of splittable flow is going to move from s to t.
- P: The set of all s-t paths
- $p := \{(s, v_1), (v_1, v_2), \cdots, (v_{k-1}, v_k), (v_k, t)\}$: an s-t path consists of edges.
  - There are four paths on the above graph, namely svt, swt, svwt, and swvt.

Model and Notations (2/2)



- $c_e : [0,1] \to \mathbb{R}_{\geq 0}$: the cost function associated with edge $e \in E$; assumed to be non-negative, continuous, and non-decreasing
  - Given amount of flow x over edge e, each (infinitely-small) amount of flow over e incurs the cost of $c_e(x)$
- $f := (f_p)_{p \in P}$: a **flow**, indicating the amount over each path
  - $\mathcal{F}$ denotes the set of *feasible* flows satisfying $0 \leq f_p \leq 1 \wedge \sum_{p \in P} f_p = 1$
- $c_p(f_p) := \sum_{e \in p} c_e(f_e)$: the cost function associated with path p
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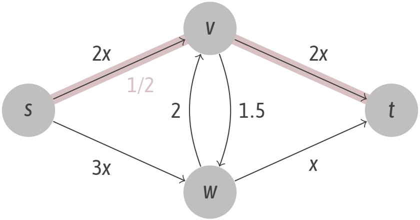

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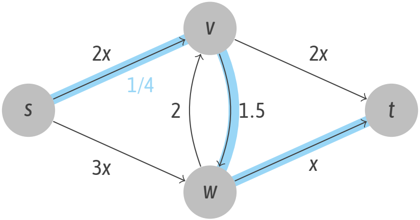

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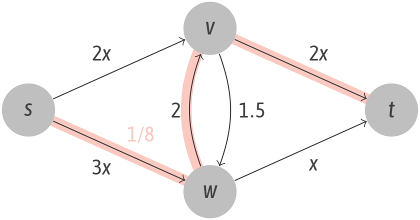

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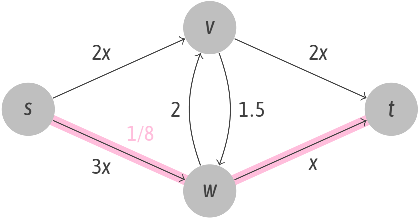

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Intuitive Explanation

The congestion game is not formally defined as a *game* in game-theoretic sense, since the set of players, the set of actions, and the payoff matrix are not explicitly defined.

However, due to its specific structure, we can consider congestion game as a game.

- The amount 1 of the total flow is considered as the set of players
- A flow from s to t denotes a profile of actions by the set of players
- A player corresponds to a infinitely-small amount of flow on a s-t path
  - Each player's action is to choose one specific s-t path
- Given flow f, the cost that a player incurs is the sum of the flows passing over each edge in the s-t path that she chooses
  - As same as the traditional game theory, each (infinitely small) player is willing to choose a path that incurs her less cost

Equilibrium Flow

Definition (Equilibrium Flow)

A flow f is said to be an *equilibrium flow* if, for any s-t path p such that $f_p > 0$ and for any other s-t path $\tilde{p}$, it holds that

$$c_p(f_p) \leq c_{\tilde{p}}(f_{\tilde{p}}).$$

- For any s-t path p, $f_p > 0$ indicates the fact that there are some players choosing the path p
- If $c_p(f_p) > c_{\tilde{p}}(f_{\tilde{p}})$ holds for another path $\tilde{p}$, we can see that some small amount of players on path p can decrease their costs by moving to path $\tilde{p}$.

Optimal Flow

- In this lecture we focus on *minimizing the sum of the costs in the whole society*, or **social cost** in short, as an objective.
- Given flow f, the social cost is defined as follows:

$$\sum_{p \in P} c_p(f_p) * f_p$$

Definition (Optimal Flow)

A flow f is said to be an *optimal flow* if, for any other feasible flow $\tilde{f} \neq f$, the following holds:

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$$\sum_{p \in P} \underbrace{\boxed{c_p(f_p) * f_p}}_{\text{Total cost from path } p}$$

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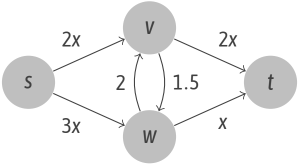
Computing Equilibrium Flow



- The cost $3x + 1.5$ of path svwt exceeds the one of svt, $4x$, for any feasible flow $f \in \mathcal{F}$
  - Note that $f_p \leq 1$ for any feasible flow f and any path p
- Analogously, the cost $5x + 2$ of swvt exceeds that one of swt, $4x$
- Thus, in any equilibrium flow f, $f_p > 0$ holds for $p \in \{\text{svt}, \text{swt}\}$
- From the definition of equlibrium flow, the amount of flow on each path is 1/2, resulting in the social cost

$$\frac{1}{2} * 4 * \frac{1}{2} + \frac{1}{2} * 4 * \frac{1}{2} = 2$$

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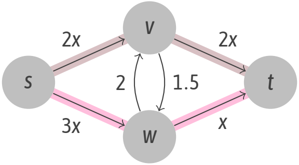

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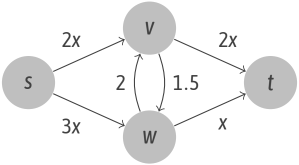
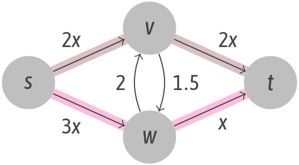


- As observed in the previous page, both paths svwt and swvt are dominated by other paths, and therefore not used in the optimal flow
- Lett x be the amount of flow under the optimal flow f. The social cost is therefore given by the following, which is minimized by choosing $x = 1/2$.

$$x * 4x + (1 - x) * 4(1 - x) = 8\big((x - \frac{1}{2})^2 + \frac{1}{4}\big)$$

Computing Optimal Flow



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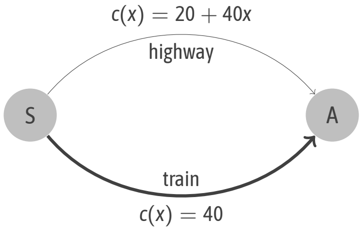
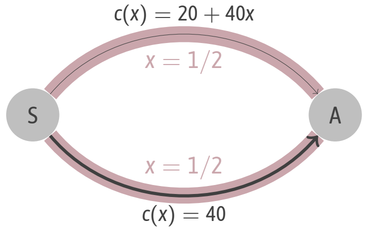
2 Pigou's Paradox



- You are travelling from Kyudai-Gakkentoshi Station (S) to Fukuoka Airport (A)
  - When travelling by car taking highway, time required depends on the traffic amount. Just 20mins. without traffic jam, 60mins. in the worst case, and increases linearly on the traffic amount x.
  - Trains bring you to the airport in a constant time, 40mins.
- Which option will you choose, only considering the time required?

Pigou's Paradox



- Under an **equilibrium flow**, the costs/required time on both paths must coincide, namely 40mins. Therefore, 1/2 of the total population chooses to use the highway.
- Under an **optimal flow**, 1/4 chooses the highway, and the remaining 1/3 uses train.
  - Let H be the fraction using the highway. The social cost then becomes the following, which is minimized for $H = 1/4$;

$$H * (20 + 40H) + (1 - H) * 40 = 40\left((H - \frac{1}{4})^2 + \frac{15}{16}\right)$$

  - The time required on the highway is $20 + 40 * \frac{1}{4} =$ 30mins, meaning that some amount of population on the train wants to move to highway.

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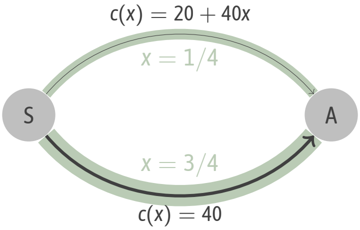

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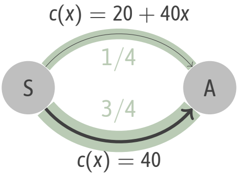
Why is the optimal flow optimal?



- Obviously, the optimal flow is not an equilibrium flow.
- One might think that, by moving some amount of the optimal flow from train to highway, the social cost slightly reduces.
- The costs that the moved amount Δx of players incurs certainly reduces;

  $40 \rightarrow \left(20 + 40(\frac{1}{4} + \Delta x)\right)$
- However, at the same time, the costs that the players already on the highway incur increases, which exceeds the above reduced amount;

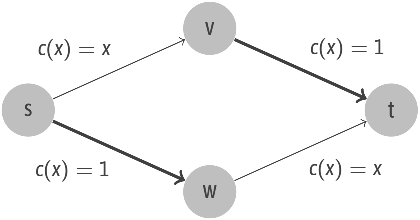
$$\underbrace{\frac{1}{4} * 40\Delta x}_{\text{Increased Costs of Highway Players}} + \underbrace{\Delta x * \left(20 + 40(\frac{1}{4} + \Delta x)\right)}_{\text{Newly Incurred Cost of Moved Players}} = \underbrace{40\Delta x}_{\text{Reduced Cost of Moved Players}} + \underbrace{40(\Delta x)^2}_{\text{Total Increase}}$$

Lessons from Pigou's Paradox

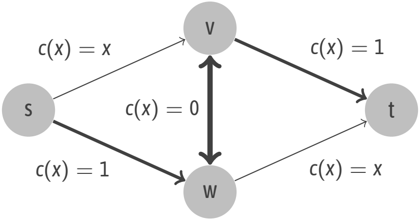
- The socially optimal solution does not necessarily coincide with the ideal options of each player
  - Quite similar structure with the Prisoner's Dilemma
  - The optimal flow is *Pareto optimal* when we assume that utility/costs are transferrable
    - I.e., the solution minimizing the social cost is Pareto optimal
- In general, Nash equilibrium is not socially optimal/efficient
  - This is quite natural since, in an equilibrium, players are *anarchically* behaving
  - Can we design a game in which (some) equilibrium state is socially optimal?

  $\Rightarrow$ **Mechanism Design**, which you will learn in GTII Course in the Summer Quarter

3 Braess' Paradox



- There are narrow edges ($c(x) = x$), and therefore both routes tend to get jammed.
- To clear the traffic jam, consider building a bypass road between v and w
  - The new bypass road is wide enough, so that traffic jam never occurs on it
- How much the social cost could be reduced?



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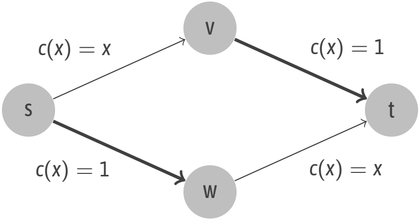
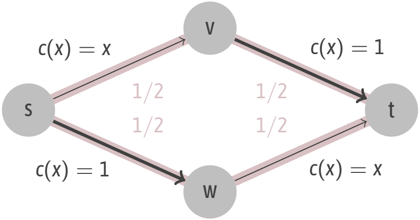


- **Without the bypass road**, the total amount is equally splitted for both route, in both equilibrium and optimal flows. The total cost is then $1/2 * (1 + 1/2) * 2 = 3/2$.
- **With the bypass road**:
  - In an equlibrium flow, the whole amount of 1 takes the path svwt. The social cost is then $1 * (1 + 0 + 1) = 2$, which is **worse than the above!**
  - The optimal flow is the same for the case without bypass, which has the social cost of 3/2.



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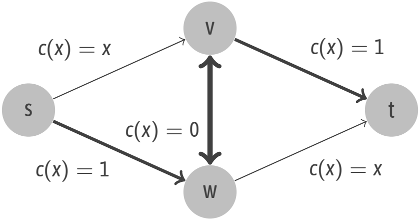

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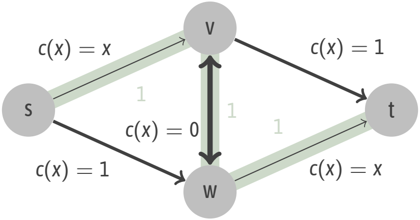

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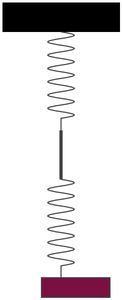
A Similar Physical Phenomenon

- Connects two identical springs with a short string/ joint, hang it from the ceil, and hang a weight on the bottom of it.
- Connects two additional strings, with a slight bend, as follows:
  - From the top of the lower spring to the ceil
  - From the bottom of the upper spring to the weight
- Removing the short joint moves the weight upward.



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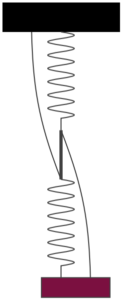

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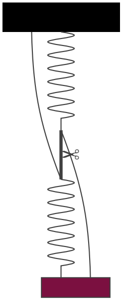

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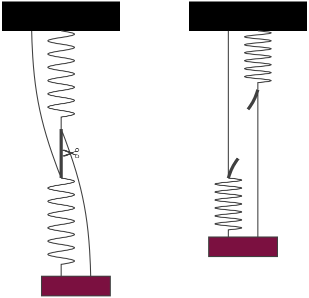


Lessons from Braess' Paradox

- In the system where individuals behaves selfishly and strategically, an ad-hoc change of the system may cause inefficiency.
  - A way of thinking like “mean well” is sometimes dangerous.
  - A detailed analysis of the change of behaviors is necessary.
- We observed a quite counter-intuitive bahavior in the corresponding physical phenomenon.

4 Nash Equilibrium and Price of Anarchy

- If you cannot modify the components of a game, such as the payoff matrix, it is almost impossible to fall into inefficient equilibrium
  - Players do not follow your instructrions; they choose their actions based on the payoff matrix
  - Cf. Cases where you can modify the payoff matrix; the *mechanism design* problem
- Given the possibility of inefficiency, now we *quantify* the inefficiency in the equilibrium
  - Saying something like "In the worst case, the social welfare is 0.*** times (or the social cost is *** times) the social optimal solution. "
  - Motivated from the *approximation/competitive analysis* in theoretical computer science

Price of Anarchy

- Based on the approximation analysis, we give the upper bound of the ratio of the social cost under an equilibrium to the optimal social cost.

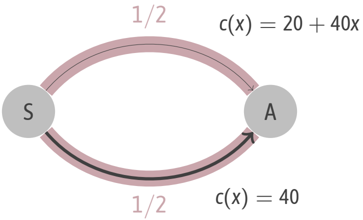
Definition (Price of Anarchy)

Given game with the set O of outcomes and the set $\mathrm{NE} \subseteq O$ of Nash equilibrium, the *price of anarchy* PoA for objective function C is defined by the following:

$$\mathrm{PoA} = \frac{\max_{o \in \mathrm{NE}} C(o)}{\min_{o^* \in O} C(o^*)}$$

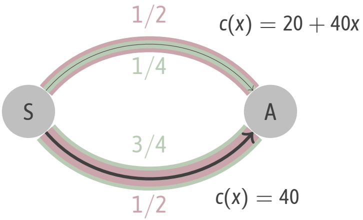
- The numerator is the objective function value under the worst Nash equilibrium e
- The denominator is the objective function value under the best possible outcome o^*

Pigou's Paradox Revisited



- The social cost of the **equilibrium flow** is $\frac{1}{2} * 40 + \frac{1}{2} * 40 = 40$
  - It is known that, under an non-atomic congestion game, the social cost of any equilibrium flow is identical.
- The social cost of the **optimal flow** is $\frac{1}{4} * 30 + \frac{3}{4} * 40 = \frac{150}{4}$
- The price of anarchy of the above game is $\frac{16}{15}$
- Implying that, the social cost could be $16/15$ times larger that the optimal value.

Pigou's Paradox Revisited



- The social cost of the **equilibrium flow** is $\frac{1}{2} * 40 + \frac{1}{2} * 40 = 40$
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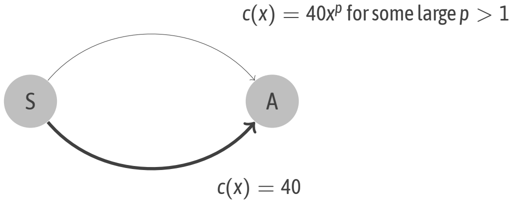
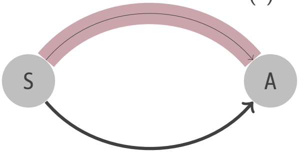
Upper Bound of Price of Anarchy



- The value $16/15$ is not that worse.
  - Of course, from the viewpoint of engineering, it is important to achieve lower PoA
- The above example, a non-linear variant of the Pigou's example, has an unbounded PoA
  - **Equilibrium Flow**: all the flow route the upper path. The social cost is 40.
  - **Optimal Flow**: a small-enough amount $\epsilon = 1 - (p+1)^{-1/p}$ uses the lower path. The social coist is $40\epsilon + 40 * (1-\epsilon)^{p+1}$.
  - PoA = $\lim_{p \to \infty} \frac{40}{40\epsilon + 40 * (1-\epsilon)^{p+1}} = \infty$

Upper Bound of Price of Anarchy



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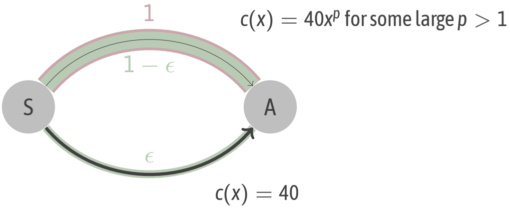

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On the Analysis based on Price of Anarchy

- Game theory and the idea of PoA enables us to quantitatively analyze the performance of the system
  - When players behave rationally and strategically, the PoA guarantees the worst-case performance of the systems
  - The PoA varies accoring to the choice of the objective function
  - Wr focused on minimizing the social cost in this pdf.
- The PoA does not necessarily have an upper bound
  - The fact that there is no upper bound implies that, under an equilibrium, the performance of the system is quite worse than the social optimal (but non-equilibirum) outcome.

5 Summary

Summary

- The model of congestion games
- Pigou's Paradox
  - In general, equilibrium flow (Nash equilibrium) is not necessarily optimal
- Braess' Paradox
  - An ad-hoc solution may cause a loss of the performance.
- Price of Anarchy
  - A tool that enables us to quantitatively analyze the performance of the system

Contact Taiki Todo (todo@inf.kyushu-u.ac.jp) for any questions.

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