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EXERCISE 2.1

By the basic properties of probability, $P(\Omega) = 1$

And by the definition of expectation, $\int_\Omega \xi \, dP = E(\xi)$

Now, by the definition of conditional expectation:

$$E(\xi \mid A) = \frac{1}{P(A)} \cdot \int_A \xi \, dP$$

If $A = \Omega$, then:

$$E(\xi \mid \Omega) = \frac{1}{1} \cdot \int_\Omega \xi \, dP$$

$$E(\xi \mid \Omega) = E(\xi)$$

EXERCISE 2.2

$$E(1_A \mid B) = \frac{1}{P(B)} \int_B 1_A \, dP$$

Using the hint, we write: $\int_B 1_A \, dP = P(A \cap B)$

$$E(1_A \mid B) = \frac{1}{P(B)} \cdot P(A \cap B)$$

$$E(1_A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$E(1_A \mid B) = P(A \mid B)$$

EXERCISE 2.3

Using the book's hint that $\{\eta = c\}$ must be $\emptyset$ or Ω for any $c \in \mathbb{R}$

It's possible to say that there is only one $c \in \mathbb{R}$ where $\{\eta = c\} = \Omega$

Then,

$$E(\xi|\eta) = E(\xi|\{\eta = c\})$$
$$= E(\xi|\Omega)$$

By Exercise 2.1

$$E(\xi|\Omega) = E(\xi)$$

Then,

$$E(\xi|\eta) = E(\xi)$$

EXERCISE 2.4

Notice that

$$\{1_B = 1\} = B$$
$$\{1_B = 0\} = \Omega - B \text{ or } B^c$$

Then if $w \in B$ i.e. $\{1_B = 1\}$

$$E(1_A|\{1_B = 1\}) = E(1_A|B)$$

By Exercise 2.2

$$E(1_A|B) = P(A|B)$$

Similarly, if $w \in B^c$ i.e. $\{1_B = 0\}$

$$E(1_A|\{1_B = 0\}) = E(1_A|B^c)$$
$$= P(A|B^c)$$
$$= P(A|\Omega - B)$$

EXERCISE 2.5

Using the hint, we know that

$$\int_B E(\xi|\mathcal{B})\, dP = \int_B \xi\, dP$$

Then,

$$E(E(\xi|\eta)) = \int_\Omega E(\xi|\eta)\, dP \qquad (*)$$

We then suppose that $\{\eta = y_i\} \neq \{\eta = y_j\}$ are pairwise disjoint for any $i \neq j$.
Because $E(\xi|\eta)$ is constant for all $\{\eta = y_n\}$ we have that:

$$\int_{\{\eta = y_n\}} E(\xi|\eta)\, dP = \int_{\{\eta = y_n\}} E(\xi|\{\eta = y_n\})\, dP$$

$$= \int_{\{\eta = y_n\}} \xi\, dP$$

Now we can use that on $(*)$ and take the union of all $\{\eta = y_n\}$ to cover for Ω:

$$E(E(\xi|\eta)) = \sum_n \int_{\{\eta = y_n\}} \xi\, dP$$

$$= \int_\Omega \xi\, dP$$

$$= E(\xi)$$

EXERCISE 2.6

OBJECTIVE: Find $E(\xi|\eta)$

From the book's hint, η is symmetric about $x = 1/2$. In fact:

$$\begin{cases} \text{if } x \in [0, 1/2], & \eta = 2x \text{ (increasing)} \\ \text{if } x \in [1/2, 1], & \eta = 2(1-x) \text{ (decreasing)} \end{cases}$$

That means $\eta(x)$ describes basically how far x is from $1/2$.

The σ-field $\mathcal{F}_\eta$ is generated by η so it contains sets symmetric around $x = 1/2$.

Then any $A \in \mathcal{F}_\eta$ should also be symmetric.

Because $\eta(x)$ is symmetric $E(\xi|\eta)$ should also be (around $x = 1/2$)

in order to be $\mathcal{F}_\eta$-measurable.

Then, $E(\xi|\eta)(x)$ for $x \in [0, 1/2]$ $= E(\xi|\eta)(x-1)$ for $x \in [1/2, 1]$

Now, for any $A \in \mathcal{F}_\eta$:

$$E(\xi|\eta)(x) = \int_A 2x^2\,dx$$ (A' is reflection of A about $x = 1/2$)

$$= \int_A x^2\,dx + \int_{A'} x^2\,dx$$

$$= \int_A x^2\,dx + \int_{1-A} (1-x)^2\,dx$$

$$= \int_A (x^2 + (1-x)^2)\,dx$$

EXERCISE 2.8

OBJECTIVE: Find $E(\xi|\eta)$

① $f_\eta(y) = \int_0^1 f_{\xi,\eta}(x,y)\,dx$

$$= \int_0^1 3/2\,(x^2+y^2)\,dx$$

$$= 1/2 + 3/2\,y^2$$

② $f_{\xi|\eta}(x|y) = \dfrac{f_{\xi,\eta}(x,y)}{f_\eta(y)} = \dfrac{3(x^2+y^2)}{1+3y^2}$

③ $E(\xi|\eta = y) = \int_0^1 x f_{\xi|\eta}(x|y)\,dx$

$$= \int_0^1 x \frac{3(x^2+y^2)}{1+3y^2}\,dx$$

$$= \frac{3}{1+3y^2}\int_0^1 x(x^2+y^2)\,dx$$

$$= \frac{3+6y^2}{4+12y^2}$$

EXERCISE 2.10

By definition,

$\rightarrow \int_\Omega \xi \, dP = E(\xi) = \int_\Omega E(\xi) \, dP$

$\rightarrow \int_\emptyset \xi \, dP = 0 = \int_\emptyset E(\xi) \, dP$

And following the book's hint, we know that if $\mathcal{G} = \{\emptyset, \Omega\}$ then any constant random variable should be $\mathcal{G}$-measurable. So for any value of $\mathcal{G}$:

$E(\xi | \mathcal{G}) = E(\xi)$

EXERCISE 2.11

Knowing that ξ is $\mathcal{G}$-measurable, for any $A \in \mathcal{G}$, by definition

$$\int_A E(\xi|\mathcal{G}) \, dP = \int_A \xi \, dP$$

Then a.s. $E(\xi|\mathcal{G}) = \xi$

EXERCISE 2.12

By definition,

$$E(E(\xi|\mathcal{G})|B) = \frac{1}{P(B)} \cdot \int_B E(\xi|\mathcal{G}) \, dP$$

Also, by definition, if $B \in \mathcal{G}$ then the integral can be written as:

$$= \frac{1}{P(B)} \cdot \int_B \xi \, dP$$

$$= E(\xi | B)$$