TAILINE NONATO
DC427182

EXERCISE 1.1

Since the sequence $A_1 \subset A_2 \subset \cdots$ is expanding, the union can be written as:

$$A = \bigcup_{n=1}^{\infty} A_n$$ where each A_n event grows larger

Notice that $P(A_n) = P(A_1 \cup A_2 \cup \cdots)$ is increasing because $A_1 \subset A_2 \subset \cdots$
Then $P(A_1) \leq P(A_2) \leq \cdots$ and $\lim_{n \to \infty} P(A_n)$ exists.

By the definition of the probability measure, the idea of continuity of probability can be seen:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n)$$

Similarly, since the sequence $A_1 \supset A_2 \supset \cdots$ is contracting the union can be written as

$$B = \bigcap_{n=1}^{\infty} A_n$$ where each A_n event becomes smaller

If one event contains the other, its probability can not increase, then $P(A_n) = P(A_1 \cap A_2 \cap \cdots)$ and
$P(A_1) \geqslant P(A_2) \geqslant \cdots$ and $\lim_{n \to \infty} P(A_n)$ exists.

Then by the definition of probability measure

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n)$$

EXERCISE 1.2

Given that B_n is a contracting sequence $(B_1 \supset B_2 \supset \cdots)$, the results from Exercise 1.1 show that

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$$
$$= \lim_{n\to\infty} P(A_n \cup A_{n+1} \cup \cdots)$$

By the Boole inequality

$$P(B_n) \leq P(A_n) + P(A_{n+1}) + \cdots$$

Assume that $\sum_{n=1}^{\infty} P(A_n) < \infty$

Notice that when $n \to \infty$, the terms in $\sum_{n=1}^{\infty} P(A_n)$ get progressively smaller.

Therefore $P(B_n) \to 0$ when $n \to \infty$

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) = 0$$

EXERCISE 1.3

X is a r.v. $\sigma(X)$-measurable, by that for any Borel set A
$\{X \in A\} = X^{-1}(A) \in \mathcal{F}$

Considering $\{f(X) \in B\}$ where B is a Borel set
$\{f(X) \in B\} = X^{-1}(f^{-1}(B))$

Notice that $X^{-1}(f^{-1}(B)) \in \sigma(X)$ because f is Borel.
And $\sigma(X)$ conteins all events that depend on X including $\{X \in A\}$ where A is a Borel set.
And that implies that $f(X)$ is $\sigma(X)$-measurable.

EXERCISE 1.4

(I) Assume $x_1 < x_2$, then the event $\{X \le x_1\} \supset \{X \le x_2\}$
meaning $P(X \le x_1) \le P(X \le x_2)$
which implies $F_\xi(x_1) \le F_\xi(x_2)$
Thus $F_\xi(x)$ is decreasing.

(II) By the definition, $F_\xi(x_n) = P(\xi \le x_n)$
And by Exercise 1.1
$F_\xi(x_n) = P(\xi \le x_n) = \lim_{n\to\infty} P(\xi \le x_n) = \lim_{n\to\infty} F_\xi(x_n)$

Proving that F_ξ is right continuous.

(III) As $x \to -\infty$ the probability of $X \le x$ goes to 0
Therefore $\lim_{n\to\infty} F_\xi(x) = P(\emptyset) = 0$

Also, by Exercise 1.1
$\lim_{n\to\infty} F_\xi(n) = \lim_{n\to\infty} P(\xi \le n) = P(\Omega) = 1$

EXERCISE 1.5

By definition,
$$F_\xi(x) = P(\xi \le x) = \int_{-\infty}^{x} f_\xi(x)dx$$

Assuming that $f_\xi(x)$ is defined.
Now, assuming $f_\xi(x)$'s continuity and that F_ξ is differentiable:
$$\frac{d}{dx} F_\xi(x) = f_\xi(x)$$

EXERCISE 1.6

By definition: $F_\xi(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i)$

As $(s,t]$ is an interval that does not contain $x_1, x_2, \cdots$, the probability of X taking any value is 0.
Therefore, for any $s < x < t$:

$$F_\xi(t) - F_\xi(s) = P(s < \xi \leq t) = 0$$

Now, the size of the jump is given by:

$$F_\xi(x_i) - \underbrace{F_\xi(x_i^-)}_{\text{the value before } x_i} = P(\xi \leq x_i) - P(\xi < x_i) = P(\xi = x_i)$$

EXERCISE 1.8

The Schwarz inequality is given by

$$[E(\xi\eta)]^2 \leq E(\xi^2) \cdot E(\eta^2)$$

The objective is to prove that

$E(\xi^2)$ exists

Then if $\eta = 1$ is choosen:

$$[E(\xi \cdot 1)]^2 \leq E(\xi^2) \cdot E(1^2) = E(\xi^2) < \infty$$

EXERCISE 1.9

$E(\eta^2)$ can be written as:

$$E(\eta^2) = \int_0^\infty t^2 \, dF_\eta(t) \quad \text{where} \quad F_\eta(t) = P(\eta \leq t)$$

Integrating by parts:

$u = t^2$, $du = 2t\,dt$
$dv = dF_\eta(t)$, $v = 1 - F_\eta(t)$

$$\int_0^\infty t^2 dF_\eta(t) = \left[t^2(1-F_\eta(t))\right]_0^\infty - \int_0^\infty 2t(1-F_\eta(t))\,dt$$

Then, $E(\eta^2) = 2\int_0^\infty t(1-F_\eta(t))\,dt$

Since $F_\eta(t) \to 1$ as $t \to \infty$, the integrable is finite and it is proven that $E(\eta^2)$ is square integrable.

EXERCISE 1.10

Given that $B_1, B_2, \cdots$ is a sequence or disjoint sets, A can be written as:

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots$$

By the property of additivity:

$$P(A) = P((A \cap B_1) \cup (A \cap B_2) \cup \cdots)$$
$$= \sum_{n=1}^{\infty} P(A \cap B_n)$$
$$= \sum_{n=1}^{\infty} P(A \mid B_n) \cdot P(B_n)$$

EXERCISE 1.11

If A and B are independent, then $P(A|B) = P(A)$ and vice versa. Therefore, if $P(B) \neq 0$, A and B are independent if and only if $P(A|B) = P(A)$

Since $P(B) \neq 0$ we can multiply by $P(B)$ on both sides:

$$P(A) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A) \cdot P(B) = \frac{P(A \cap B) \cdot P(B)}{P(B)}$$

and find $P(A \cap B) = P(A) \cdot P(B)$