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Assignment 3

Exercise 3.1

$\xi_1, \xi_2, \cdots$ coin tosses

Events A, B, C, D

[Objective: find the smallest n such that each event belongs to $\mathcal{F}_n$ (σ-field $\xi_1, \cdots, \xi_n$)]

Event A: smallest $n = 11$ → A belongs to $\mathcal{F}_{11}$ but not to $\mathcal{F}_{10}$

Event B: smallest n $\nexists$ → There no n such that $B \in \mathcal{F}_n$

Event C: smallest $n = 100$ → C belongs to $\mathcal{F}_{99}$ but not to $\mathcal{F}_{100}$

Event D: smallest $n = 1$ → $D = \emptyset$, by definition it belongs to all $\mathcal{F}_n$

Exercise 3.3

By definition,

$$\xi_n = E(\xi | \mathcal{F}_n)$$

If we look at $E(\xi_{n+1} | \mathcal{F}_n)$ we could take the expectation at both sides:

$$* \quad E(\xi_n) = E(E(\xi_{n+1} | \mathcal{F}_n))$$

ξ_n is $\mathcal{F}_n$-measurable, which implies that

$$E(E(\xi_{n+1} | \mathcal{F}_n)) = E(\xi_{n+1})$$

Then, going back to $(*)$, we see that

$$E(\xi_n) = E(\xi_{n+1}) \quad \text{for each} \quad n = \{1, 2, \cdots\}$$

Exercise 3.4

We know that:

I. $\mathcal{G}_n \subset \mathcal{F}_n$

II. $\mathcal{G}_n$ is a σ-field generated by $\xi_1, \cdots, \xi_n$

III. ξ_n is integrable since ξ_n is a martingale with respect to $\mathcal{F}_n$

And because of that we can say that: → ξ_n is $\mathcal{G}_n$-measurable

$$\xi_n = E(\xi_n | \mathcal{G}_n) = E(E(\xi_{n+1} | \mathcal{F}_n) | \mathcal{G}_n) = E(\xi_{n+1} | \mathcal{G}_n)$$

So ξ_n is martigale with respect to $\mathcal{G}_n$.

EXERCISE 3.5

<u>objective:</u> show that $\xi_n^2 - n$ is a martingale with respect to $\mathcal{F}_n = \sigma(\eta_1, \cdots, \eta_n)$

$\xi_n^2 - n$ can be written as $(\eta_1 + \cdots + \eta_n)^2 - n$. And because $\eta_1, \cdots, \eta_n$ is $\mathcal{F}_n$-measurable $\xi_n^2 - n$ is adapted to $\mathcal{F}_n$.

Now, by Example 3.3

$$|\xi_n| = |\eta_1 + \cdots + \eta_n| \leq |\eta_1| + \cdots + |\eta_n| = n$$

Then, ξ_n^2 is integrable $\forall n$:

$$E(|\xi_n^2 - n|) \leq E(\xi_n^2) + n \leq n^2 + n < \infty$$

Now, analysing ξ_{n+1}^2 we notice that

$$\xi_{n+1}^2 = \eta_{n+1}^2 + 2\eta_{n+1}\xi_n + \xi_n^2$$

Then,

$\boxed{\mathcal{F}_n \text{ independent}}$ $\boxed{\mathcal{F}_n\text{-measurable}}$

$$E(\xi_{n+1}^2 | \mathcal{F}_n) = E(\eta_{n+1}^2 | \mathcal{F}_n) + 2E(\eta_{n+1}\xi_n | \mathcal{F}_n) + E(\xi_n^2 | \mathcal{F}_n)$$

$$= E(\eta_{n+1}^2) + 2\xi_n E(\eta_{n+1}) + \xi_n^2$$

Now, since $P(\eta_n = 1) = P(\eta_n = -1) = 1/2$:

$$E(\eta_{n+1}) = 1 \cdot 1/2 + (-1) \cdot 1/2 = 0$$

And, since $\eta_{n+1} \in \{-1, 1\}$:

$$E(\eta_{n+1}^2) = 1$$

Then, we are left with

$$E(\xi_{n+1}^2 | \mathcal{F}_n) = 1 + \xi_n^2$$

Now, $E(\xi_{n+1}^2 - n - 1 | \mathcal{F}_n) = \xi_n^2 - n$

So ξ_n is martingale with respect to $\mathcal{F}_n$.

EXERCISE 3.6

<u>objective:</u> prove that $\zeta_n = (-1)^n \cos(\pi \xi_n)$ is martingale

ζ_{n+1} can be expressed as:

$$\zeta_{n+1} = (-1)^{n+1}(\cos(\pi\xi_n) \cdot \cos(\pi\eta_{n+1}) - \sin(\pi\xi_n) \cdot \sin(\pi\eta_{n+1}))$$

$\hookrightarrow \cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

Since $\eta \in \{-1, 1\}$:

- $\cos(\pi\eta_{n+1}) = \cos(\pi) = \cos(-\pi) = -1$
- $\sin(\pi\eta_{n+1}) = \sin(\pi) = \sin(-\pi) = 0$

Then,

$$\varsigma_{n+1} = (-1)^{n+1}(-1)\cos(\pi\xi_n) = (-1)^n\cos(\pi\xi_n)$$

Now, ς_n is F_n-measurable, so:

$$E(\varsigma_{n+1} \mid F_n) = E((-1)^n\cos(\pi\xi_n) \mid F_n)$$
$$= (-1)^n\cos(\pi\xi_n)$$
$$= \varsigma_n$$

And we conclude that ς_n is martingale with respect to F_n.

EXERCISE 3.7

Recalling the work done on 3.5, we saw that

$$\xi_{n+1}^2 = \eta_{n+1}^2 + 2\eta_{n+1}\xi_n + \xi_n^2$$

Then, $E(\xi_{n+1}^2 \mid F_n) = E(\eta_{n+1}^2 \mid F_n) + 2E(\eta_{n+1}\xi_n \mid F_n) + E(\xi_n^2 \mid F_n)$

$$= E(\eta_{n+1}^2) + 2\xi_n E(\eta_{n+1}) + \xi_n^2$$

Now, since $P(\eta_n = 1) = P(\eta_n = -1) = 1/2$:

$$E(\eta_{n+1}) = 1\cdot 1/2 + (-1)\cdot 1/2 = 0$$

And, since $\eta_{n+1} \in \{-1, 1\}$:

$$E(\eta_{n+1}^2) = 1$$

Then, we are left with

$$E(\xi_{n+1}^2 \mid F_n) = 1 + \xi_n^2$$

Which means $E(\xi_{n+1}^2 \mid F_n) \geq \xi_n^2$ which is the definition of submartingale, in this case, in respect to F_n.

EXERCISE 3.8

By the definition of stopping times, knowing wether $\tau \leq n$ means also knowing wether $\tau = n$, as we have full knowledge of F_n.

- $1 \Rightarrow 2$: $\{\tau = n\} = \{\tau \leq n\} \setminus \{\tau \leq n-1\}$

  Since both $\{\tau \leq n\}$ and $\{\tau \leq n-1\}$ belong to F_n, then $\{\tau = n\} \in F_n$.

- $2 \Rightarrow 1$: $\{\tau \leq n\} = \{\tau = 1\} \cup \cdots \cup \{\tau = n\} \in F_n$

  Since each $\{\tau = k\} \in F_k \subset F_n$, then $\{\tau \leq n\} \in F_n$ proving

EXERCISE 3.9

To show that $\{\tau \leq n\} \in \mathcal{F}_n$, we must start with the fact that there should be some $k \leq n$ such that $\xi_k \in B$:

$$\{\tau \leq n\} = \bigcup_{k=1}^{n} \{\xi_k \in B\}$$

Since each $\{\xi_k \in B\} \in \mathcal{F}_k \subset \mathcal{F}_n$, the union is also in $\mathcal{F}_n$.
Then $\tau = \min\{n : \xi_n \in B\}$ is a stopping time.

EXERCISE 3.10

Using the hint, we can write:

$$\{\xi_{\tau \wedge n} \in B\} = [\{\xi_n \in B\} \cap \{\tau > n\} \in \mathcal{F}_n] \cup [\bigcup_{k=1}^{n} \{\xi_k \in B, \tau = k\}]$$

where $B \subset \mathbb{R}$ is a Borel set.

Then, for $k = 1, \dots, n$

$$\{\xi_k \in B, \tau = k\} = \{\xi_k \in B\} \cap \{\tau = k\} \in \mathcal{F}_k \subset \mathcal{F}_n$$

Then,

$$\{\xi_{\tau \wedge n} \in B\} \in \mathcal{F}_n \quad \forall n$$

EXERCISE 3.11

Following the hint, the probability of the game terminating at step n is $P(\tau = n) = \frac{1}{2^n}$

Then, by the definition of expectation:

$$E(\xi_{n-1}) = \sum_{n=1}^{\infty} \xi_{n-1} \cdot P(\tau = n)$$

$$= \sum_{n=1}^{\infty} (-1 - 2 - \dots - 2^{n-2}) \cdot 1/2^n$$

$$= -\sum_{1}^{\infty} \frac{2^{n-1} - 1}{2^n} \longrightarrow -\infty$$

Then, as asked, $E(\xi_{n-1}) = -\infty$

EXERCISE 3.12

condition 1 : $\tau < \infty$ a.s. or $P(\tau = \infty) = 0$

By example 3.7, we take

$$P(\tau < 2Kn) \leq \left(1 - \frac{1}{2^{2K}}\right)^n \to 0 \quad \text{as} \quad n \to \infty$$

Because $\{\tau < 2Kn\} \supset \{\tau > 2K(n+1)\} \supset \cdots$, then:

$$P(\tau < \infty) = P\left(\bigcap_{n=1}^{\infty} (\tau > 2Kn)\right) = \lim_{n\to\infty} P(\tau < 2Kn) = 0$$

condition 2 : ξ_τ is integrable

$|G_\tau| \leq 1$, so $E(|G_\tau|) \leq 1 < \infty$

condition 3 : $E(\xi_n 1_{\{\tau > n\}}) \to 0$ as $n \to \infty$

$$|E(G_n 1_{\{\tau>n\}})| \leq E(|G_n| 1_{\{\tau>n\}})$$
$$\leq E(1_{\{\tau>n\}}) \to \text{because } |G_n| \leq 1 \ \forall n$$
$$= P(\tau = n) \qquad \to 0 \text{ by condition 1}$$
$$\overset{n\to\infty}{=} P(\tau > n) \searrow P(\tau = \infty)$$

$|E(G_n 1_{\{\tau>n\}})| \to 0$ as $n \to \infty$

Now, because $\xi_\tau \in \{-K, K\}$, $\cos(\pi) = \cos(-\pi) = -1$

$$G_\tau = (-1)^\tau \cos(\pi(K + \xi_\tau)) = (-1)^\tau$$

Then, $E(G_\tau) = E((-1)^\tau) = (-1)^K$