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ON NEW PARAMETERIZATIONS OF THE BIRNBAUM-SAUNDERS DISTRIBUTION

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ABSTRACT

The Birnbaum-Saunders (BS) model is a two-parameter, unimodal distribution with positive skewness and non-negative support. This model has attracted considerable attention in the statistical literature during the past decade. The interest for the BS model is due to its physical theoretical arguments, its attractive properties and its relationship with the normal model. However, no much attention has been payed in parameterizations of the BS distribution distinct to that proposed originally. In this paper, we study different parameterizations of the BS distribution. Specifically, we analyze the properties of these parameterizations, use the maximum likelihood method for estimating the corresponding parameters and carry out a simulation study for detecting their performance.

KEYWORDS

Confidence intervals; Coverage probabilities, Delta and ML methods; Monte Carlo simulations

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1 INTRODUCTION

A distribution with non-negative support that has attracted considerable interest is the Birnbaum-Saunders model. This distribution is unimodal, positively skewed and has two parameters, which correspond to the shape and scale of the model. For more details about the BS distribution, see Birnbaum and Saunders (1969a) and Johnson et al. (1995, pp. 651-663). This interest for the BS distribution is due to its physical theoretical arguments, its attractive properties and its relationship with the normal distribution. Although the BS distribution has its genesis from engineering, this distribution has been applied to several other fields and topics; see Galea et al. (2004), Díaz-García and Leiva (2005), Vilca and Leiva (2006), Balakrishnan et al. (2007, 2011), Leiva et al. (2007, 2008a, b, c, b, 2009, 2010a, b, 2011a, b), Barros et al. (2008), Sanhueza et al. (2008), Guiraud et al. (2009), Ahmed et al. (2010), Vilca et al. (2010, 2011), and Paula et al. (2011). Different estimation aspects related to the BS distribution have been studied; see Birnbaum and Saunders (1969b), Engelhardt et al. (1981), Chang and Tang (1994), Dupuis and Mills (1998), Ng et al. (2003), From and Li (2006), Lemonte et al. (2007), Cysneiros et al. (2008), and Balakrishnan et al. (2009b). Implementations in `R` language (`www.R-project.org`) of this distribution and its generalized version have been developed by Leiva et al. (2006) and Barros et al. (2009), called `bs` and `gbs` packages, respectively; see R Development Core Team (2009). However, in spite of all these studies, no much attention has been payed in parameterizations of the two-parameter BS model distinct to that originally proposed by Birnbaum and Saunders (1969a) based on the physics of materials. Some few works in this line are accredited to Ahmed et al. (2008), Fox et al. (2008), and Lio et al. (2010).

Our main motivation for studying several parameterizations of the BS distribution is based on the search of (i) parameter estimators that are unbiased and consistent, (ii) coverage probabilities of the respective confidence intervals that are close to the nominal level, (iii) orthogonality of the parameters, and (iv) the possibility of directly describing the mean of a BS random variable (r.v.) by a regression model without the need of transforming the response variable. In order to reach this search, we consider the following ideas for proposing parameterizations. The BS distribution is a mixture of the inverse Gaussian (IG) distribution and its complementary reciprocal; see Jörgensen et al. (1991), Balakrishnan et al. (2009a) and references therein. Tweedie (1957) proposed several parameterizations of the IG distribution, which can be considered for the BS distribution. In addition, an interesting parameterization proposed by Ferrari and Cribari-Neto (2004) applied to regression models following a similar idea to that of generalized linear models (GLM) may be also considered for the BS model.

The aim of this paper is to study several parameterizations of the two-parameter BS distribution. The article is organized as follows. Section 2 contains a preliminary notion of the BS model, including structural properties, parameter estimation, and parameterizations proposed for this model until now. Section 3 introduces the new proposed parameterizations, investigates what type of statistical parameters they are and determines their properties of scale, reciprocity and orthogonality. Section 4 studies the performance of the proposed parameterizations through Monte Carlo simulations using the maximum likelihood (ML) and delta methods. Finally, some conclusions are sketched in the final section.

2 BACKGROUND

In this section, we present some properties of the BS model, the ML estimation of its parameters and the parameterizations proposed for this model.

2.1 The first BS parameterization

The genesis of the BS distribution considers a material exposed to repetitive loads produced under stress in a specific cyclic; see Mann et al. (1974, pp. 150-152). The imposition of m loads under stress in the jth cyclic, say l_{ij}, for $i = 1,2,\ldots,m$, and $m \in \mathbb{N}$, provokes fatigue in the material inducting a crack or wear-out that is extended by a random amount X_{ij}. The total size of the crack due to fatigue for the jth cycle is the r.v. $Y_j = \sum_{i=1}^{m} X_{ij}$, which accumulates aging and wear-out during this cycle. Here, it assumed that Y_j follows a distribution of mean ν and variance η^2. Then, the failure or rupture of the material occurs when the total size of the crack exceeds a level of resistance (threshold) ρ. In summary, the BS model looks for the distribution of the smallest n, say n^*, such that $S_n = \sum_{j=1}^{n} Y_j$ exceeds the threshold ρ, i.e., $n^* = \inf\{n \in \mathbb{N}\colon S_n = \sum_{j=1}^{n} Y_j > \rho\}$. The BS distribution is derived by assuming that the Y_j's are iid r.v.'s and applying the central limit theorem, such that $S_n \dot{\sim} \mathrm{N}(n\nu, n\eta^2)$. Hence, if N is the number of cycles until the failure, we have that

$$\mathbb{P}(N \leq n) = \mathbb{P}(S_n > \rho) \approx \mathbb{P}\left(\frac{S_n - n\nu}{\sqrt{n}\eta} > \frac{\rho - n\nu}{\sqrt{n}\eta}\right) = \Phi\left(\frac{\sqrt{\rho\nu}}{\eta}\left[\sqrt{\frac{n}{\rho/\nu}} - \sqrt{\frac{\rho/\nu}{n}}\right]\right), \qquad (2.1)$$

where Φ is the $\mathrm{N}(0,1)$ distribution function. Based on (2.1), Birnbaum and Saunders (1969a) developed a distribution regarding the discrete r.v. N as a continue r.v. T and defined the parameterization (P1) $\alpha = \eta/\sqrt{\rho\nu}$ and $\beta = \rho/\nu$. This allows us to obtain an r.v. T with BS distribution of shape and scale parameters, $\alpha > 0$ and $\beta > 0$, respectively. This is denoted by $T \sim \mathrm{BS}(\alpha,\beta)$. The parameterization (P1) is the most one known and studied. Thus, if $T = \beta[\alpha Z/2 + \sqrt{\{\alpha Z/2\}^2 + 1}]^2 \sim \mathrm{BS}(\alpha,\beta)$, then $Z = [\sqrt{T/\beta} - \sqrt{\beta/T}]/\alpha \sim \mathrm{N}(0,1)$ and $W = [T/\beta + \beta/T - 2]/\alpha^2 \sim \chi^2(1)$. The probability density (p.d.f.) of T is

$$f(t;\alpha,\beta) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2\alpha^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)\frac{[t+\beta]}{2\alpha\sqrt{\beta t^3}},\ t > 0.$$

In addition, the BS distribution holds the following properties: (A1) $cT \sim \mathrm{BS}(\alpha, c\beta)$, with $c > 0$, and (A2) $1/T \sim \mathrm{BS}(\alpha, 1/\beta)$. The rth moment of T is

$$\mathrm{E}[T^r] = \beta^r \sum_{j=0}^{r}\binom{2r}{2j}\sum_{i=0}^{j}\binom{j}{i}\frac{(2r-2j+2i)!}{2^{r-j+i}(r-j+i)!}\left[\frac{\alpha}{2}\right]^{2r-2j+2i},$$

and so $\mathrm{E}[T] = \beta[1+\alpha^2/2]$ and $\mathrm{Var}[T] = [\beta\alpha]^2[1+5\alpha^2/4]$. The quantile function of T is $t(q) = [\beta/4][\alpha z(q) + \sqrt{\alpha^2 z(q)^2 + 4}]^2$, where $z(p)$ is the qth quantile of $Z \sim \mathrm{N}(0,1)$. Notice that $t(0.5) = \beta$, i.e., β is the median of the distribution of T.

From Figure 1(a), we observe that, as α increases, the BS distribution is turned to be more asymmetrical and its variability and kurtosis increase. From Figure 1(b), we detect that, as β increases, the distribution is dislocated and its variability increases too. This confirms that α modifies the shape of the distribution and β modifies its position and its scale.

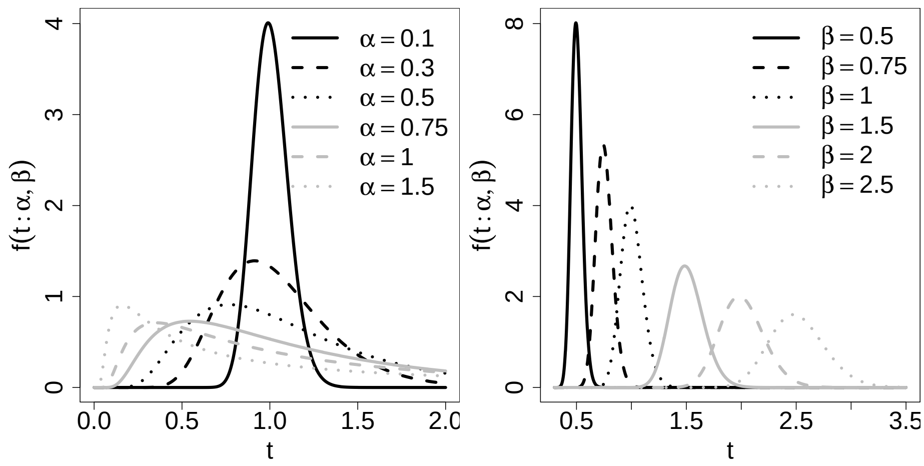


Figure 1: plots of BS p.d.f. for $\beta = 1.0$ (a) and $\alpha = 0.1$ (b).

2.2 Estimation and inference

Birnbaum and Saunders (1969b) developed ML estimation for the parameterization (P1). Specifically, based on a sample $\mathbf{T} = [T_1, \ldots, T_n]^\top$, where $T_i \sim \mathrm{BS}(\alpha, \beta)$, for $i = 1, \ldots, n$, with observations $\mathbf{t} = [t_1, \ldots, t_n]^\top$, the log-likelihood function for $[\alpha, \beta]^\top$ is

$$\ell(\alpha, \beta; \mathbf{t}) = k - n\log(\alpha) - \frac{n}{2}\log(\beta) + \sum_{i=1}^{n}\log(t_i + \beta) - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right], \tag{2.2}$$

where k is a constant that does not depend on α and β. In order to obtain the ML estimators of α and β, we need to compute the maximum of $\ell(\alpha, \beta; \mathbf{T})$ given in (2.2). This maximization conducts to

$$\widehat{\alpha} = \left[\frac{S}{\widehat{\beta}} + \frac{\widehat{\beta}}{R} - 2\right]^{1/2}, \tag{2.3}$$

where $S = [\sum_{i=1}^{n} T_i]/n$ and $R = n/[\sum_{i=1}^{n}\{1/T_i\}]$ are the arithmetic and harmonic means, respectively. However, it is not possible to find an analytic expression for the ML estimator of β. To solve this, Birnbaum and Saunders (1969b) proposed the expression $g(\beta) = \beta^2 - \beta[2R + K(\beta)] + R[S + K(\beta)]$, where $K(\beta) = 1/\left[\sum_{i=1}^{n}[\beta + T_i]^{-1}/n\right]$, for $\beta > 0$. The function $g(\cdot)$ must be equated to zero and the ML estimate of β must be numerically obtained. Once we have the ML estimate of β, we replace it in (2.3) and obtain the ML estimate of α. Engelhardt et al. (1981) derived asymptotic inference for α and β by

$$\begin{pmatrix}\widehat{\alpha}\\ \widehat{\beta}\end{pmatrix} \quad \dot{\sim} \quad \mathrm{N}\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\frac{\alpha^2}{2n} & 0\\ 0 & \frac{\beta^2}{n[1/4 + 1/\alpha^2 + I(\alpha)]}\end{pmatrix}\right), \tag{2.4}$$

where

$$I(\alpha) = 2\int_0^\infty \left[\frac{1}{1+h(\alpha x)} - \frac{1}{2}\right]^2 d\Phi(x) \quad \text{and} \quad h(y) = 1 + \frac{y^2}{2} + y\left[1 + \frac{y^2}{4}\right]^{1/2}.$$

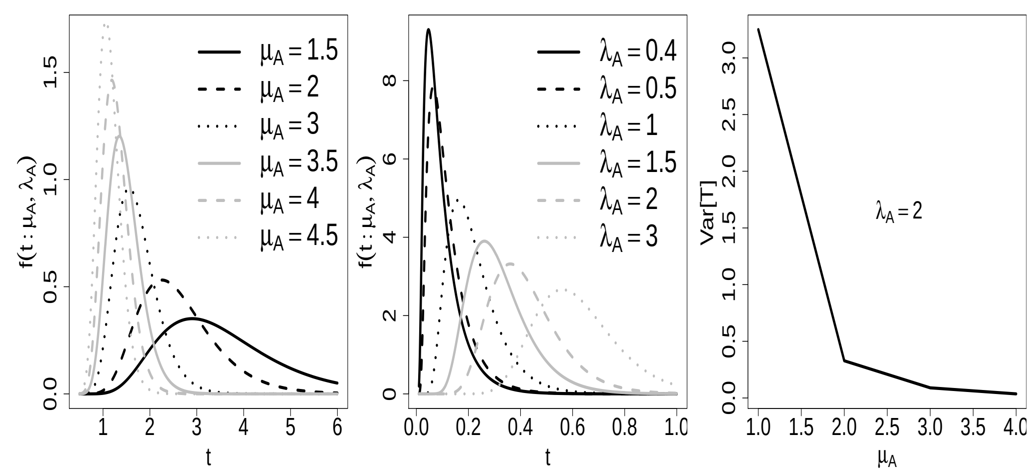
Note the expected value of the second derivative of the likelihood function given in (2.2) with respect to α and β is zero, i.e., $\mathrm{E}[\partial^2 \ell(\alpha,\beta;\mathbf{T})/\partial\alpha\partial\beta] = 0$. Therefore, α and β are orthogonal parameters.

2.3 A second BS parameterization

An interesting parameterization of the BS distribution proposed by Ahmed et al. (2008) was also based on physical aspects. Specifically, this parameterization is given by (P2) $\alpha = 1/\sqrt{\mu_A \lambda_A}$ and $\beta = \lambda_A/\mu_A$, where $\lambda_A > 0$ and $\mu_A > 0$ correspond, according to Ahmed et al. (2008)'s proposal, to the amplitude and nominal load of the treatment in the sample, respectively. Thus, under this parameterization, the p.d.f. of $T \sim \mathrm{BS}(\mu_A, \lambda_A)$ is

$$f(t;\mu_A,\lambda_A) = \frac{1}{2\sqrt{2\pi}}\left[\frac{\lambda_A}{t\sqrt{t}} + \frac{\mu_A}{\sqrt{t}}\right]\exp\left(-\frac{1}{2}\left[\frac{\lambda_A}{\sqrt{t}} - \mu_A\sqrt{t}\right]^2\right), \quad t > 0.$$

In this case, $cT \sim \mathrm{BS}(\mu_A/\sqrt{c}, \sqrt{c}\lambda_A)$, with $c > 0$, and $1/T \sim \mathrm{BS}(\lambda_A, \mu_A)$. In addition, the mean and variance are $\mathrm{E}[T] = [\lambda_A \mu_A + 1/2]/\mu_A^2$ and $\mathrm{Var}[T] = [\lambda_A \mu_A + 5/4]/\mu_A^4$. For this parameterization, from Figure 2(a), we observe that, as μ_A increases, the variability of the distribution decreases. From Figure 2(b), we detect that, as λ_A increases, the distribution goes having more light tails. Based on Figure 2(c), it is possible to note that μ_A is a precision parameter.



(a) (b) (c)

Figure 2: plots of BS p.d.f. for $\lambda_A = 5$ (a) and $\mu_A = 5$ (b) and of its variance vs. μ_A (c).

3 NEW PARAMETERIZATIONS

In this section, we present several parameterizations that can be useful in diverse aspects related to the BS distribution. All of these parameterizations are related to the parameterization (P1). Here, we also comment about the type of statistical parameters they are. Property (A1) is confirmed in each one of the new parameterizations, but the reciprocal (A2) and orthogonality properties only hold in the case of a parameterization proposed by Tweedie (1957) that we call (P10).

3.1 Parameterization 3 (based on GLM)

A new first parameterization of the BS distribution that we propose, in addition to the two already existing (P1) and (P2), is (P3) given by $\alpha = \sqrt{2/\delta}$ and $\beta = \delta\mu/[\delta+1]$, where $\delta > 0$ and $\mu > 0$ are shape and mean parameters, respectively. Thus, under this parameterization, the p.d.f. of $T \sim \mathrm{BS}(\delta,\mu)$ is

$$f(t;\delta,\mu) = \frac{\exp(\delta/2)\sqrt{\delta+1}}{4\sqrt{\pi\mu}\,t^{3/2}}\left[t+\frac{\delta\mu}{\delta+1}\right]\exp\left(-\frac{\delta}{4}\left[\frac{t\{\delta+1\}}{\delta\mu}+\frac{\delta\mu}{t\{\delta+1\}}\right]\right), \quad t>0.$$

In this case, $cT \sim \mathrm{BS}(\delta, c\mu)$, with $c > 0$. The mean and variance are $\mathrm{E}[T] = \mu$ and $\mathrm{Var}[T] = g(\mu)/h(\delta)$, where $g(\mu) = \mu^2$ and $h(\delta) = [2\delta+5]/[\delta+1]^2$, which follows a similar idea to that used in GLM. For this parameterization, from Figure 3(a), we observe that, as δ increases, the shape of the distribution changes turning to be more symmetrical. From Figure 3(b), we detect that, as μ increases, the variability and kurtosis of the distribution increase too. Note from Figure 3(c) that δ can also be interpreted as a precision parameter, as expected, since, the larger value of δ corresponds to the smaller value of the variance.

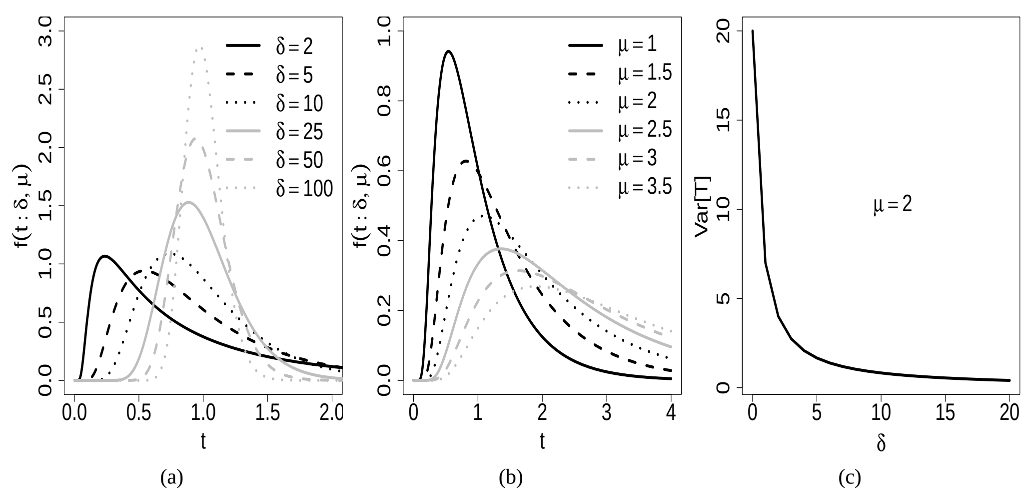


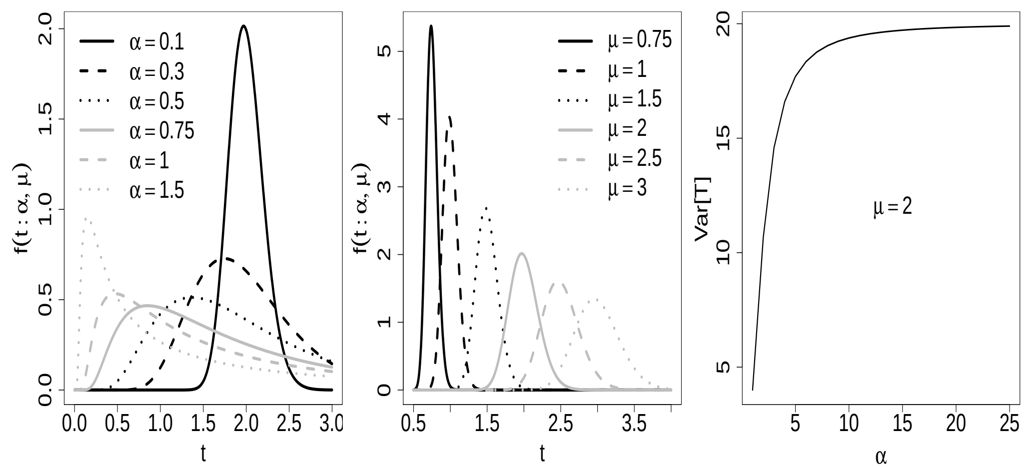
Figure 3: plots of BS p.d.f. for $\mu = 1.0$ (a) and $\delta = 5.0$ (b) and of its variance vs. δ (c).

3.2 Parameterization 4 (based on the mean)

A new second parameterization of the BS distribution that we propose is (P4) given by α and $\beta = 2\mu/[2+\alpha^2]$, where $\alpha > 0$ and $\mu > 0$ are shape and mean parameters, respectively. Thus, under this parameterization, the p.d.f. of $T \sim \mathrm{BS}(\alpha,\mu)$ is

$$f(t;\alpha,\mu) = \frac{\exp(1/\alpha^2)\sqrt{2+\alpha^2}}{4\alpha\sqrt{\pi\mu}\,t^{3/2}}\left[t+\frac{2\mu}{2+\alpha^2}\right]\exp\left(-\frac{1}{2\alpha^2}\left[\frac{\{2+\alpha^2\}t}{2\mu}+\frac{2\mu}{\{2+\alpha^2\}t}\right]\right), t>0.$$

In this case, $cT \sim \mathrm{BS}(\alpha, c\mu)$, with $c > 0$. In addition, the mean and variance are $\mathrm{E}[T] = \mu$ and $\mathrm{Var}[T] = [\mu\alpha]^2[4+5\alpha^2]/[2+\alpha^2]^2$. For this parameterization, from Figure 4(a), we detect that, as α increases, the distribution is turned to be more asymmetrical and its variability and kurtosis increase, as expected. From Figure 4(b), we observe that, as μ increases, the distribution is dislocated and its variability increases too. Based on Figure 4(c), it is possible to note that α is also a dispersion parameter.



(a) (b) (c)

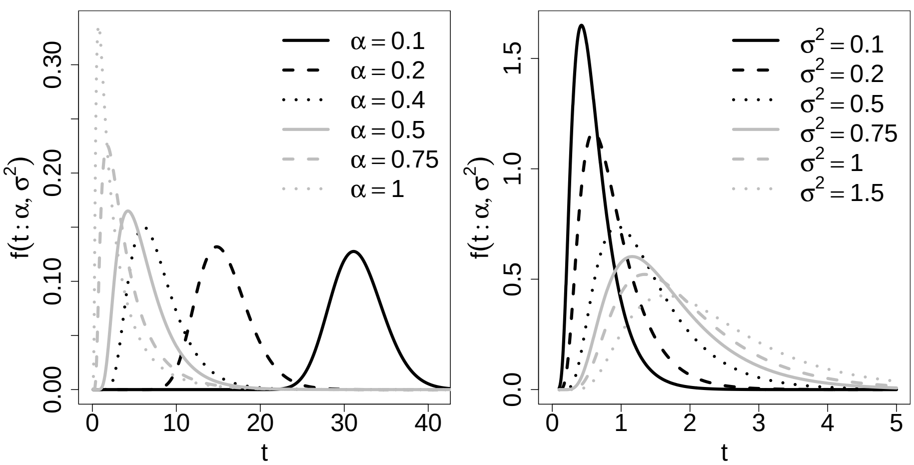
Figure 4: plots of BS p.d.f. for $\mu = 2.0$ (a) and $\alpha = 0.1$ (b) and of its variance vs. α (c).

3.3 Parameterization 5 (based on the variance 1)

A new third parameterization of the BS distribution that we propose is (P5) given by α and $\beta = [2\sqrt{\sigma^2}]/[\alpha\sqrt{4+5\alpha^2}]$, where $\alpha > 0$ and $\sigma^2 > 0$ are shape and variance parameters, respectively. Thus, under this parameterization, the p.d.f. of $T \sim \mathrm{BS}(\alpha,\sigma^2)$ is

$$\begin{aligned} f(t;\alpha,\sigma^2) &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2\alpha^2}\left[\frac{\alpha\sqrt{4+5\alpha^2}}{2\sqrt{\sigma^2}t^{-1}}+\frac{2\sqrt{\sigma^2}\{t\alpha\}^{-1}}{\sqrt{4+5\alpha^2}}-2\right]\right) \\ &\times\left[\frac{\{t\alpha\}^{-1/2}\{4+5\alpha^2\}^{1/4}}{2^{3/2}\sigma^{1/2}}+\frac{\sigma^{1/2}}{\{t\alpha\}^{3/2}\sqrt{2}\{4+5\alpha^2\}^{1/4}}\right]. \end{aligned}$$

In this case, $cT \sim \mathrm{BS}(\alpha, c^2\sigma^2)$, with $c > 0$. In addition, the mean and variance are $\mathrm{E}[T] = [2 + \alpha^2]\sqrt{\sigma^2}/[\alpha\sqrt{4+5\alpha^2}]$ and $\mathrm{Var}[T] = \sigma^2$. For this parameterization, from Figure 5(a), we observe that, as α increases, the distribution is turned to be more asymmetrical and its kurtosis decreases. From Figure 5(b), as σ^2 increases, the variability of the distribution increases too, which is obvious since σ^2 is its variance.



(a) (b)

Figure 5: plots of BS p.d.f. for $\sigma^2 = 10$ (a) and $\alpha = 0.1$ (b).

3.4 Parameterization 6 (based on the variance 2)

A fourth parameterization of the BS distribution that we propose is (P6) given by $\alpha = 2\sqrt{\gamma - 1}/\sqrt{5}$ and $\beta = \sqrt{5\sigma^2}/[2\sqrt{\gamma\{\gamma-1\}}]$, where $\gamma > 1$ and $\sigma^2 > 0$ are shape and variance parameters, respectively. Here, γ must be greater than one to warrant α and β are positive reals. Thus, under this parameterization, the p.d.f. of $T \sim \mathrm{BS}(\gamma, \sigma^2)$ is

$$\begin{aligned} f(t;\gamma,\sigma^2) &= \frac{\sqrt{\gamma}}{2\sqrt{2\pi\sigma^2}} \left[\left\{ \frac{1}{2t}\sqrt{\frac{5\sigma^2}{\gamma(\gamma-1)}} \right\}^{1/2} + \left\{ \frac{1}{2t}\sqrt{\frac{5\sigma^2}{\gamma(\gamma-1)}} \right\}^{3/2} \right] \\ &\quad \times \exp\left(-\frac{5}{8[\gamma-1]} \left[\frac{2t\sqrt{\gamma\{\gamma-1\}}}{\sqrt{5\sigma^2}} + \frac{\sqrt{5\sigma^2}}{2t\sqrt{\gamma\{\gamma-1\}}} - 2 \right] \right), \quad t > 0. \end{aligned}$$

In this case, $cT \sim \mathrm{BS}(\gamma, c^2\sigma^2)$, with $c > 0$. In addition, the mean and variance are $\mathrm{E}[T] = [2\gamma + 3]\sqrt{\sigma^2}/\sqrt{20\gamma[\gamma-1]}$ and $\mathrm{Var}[T] = \sigma^2$. For this parameterization, from Figure 6(a), we observe that, as γ increases, the variability of the distribution decreases and its shape changes. From Figure 6(b), we detect that, as σ^2 increases, the variability of the distribution increases too, since σ^2 is once again its variance.

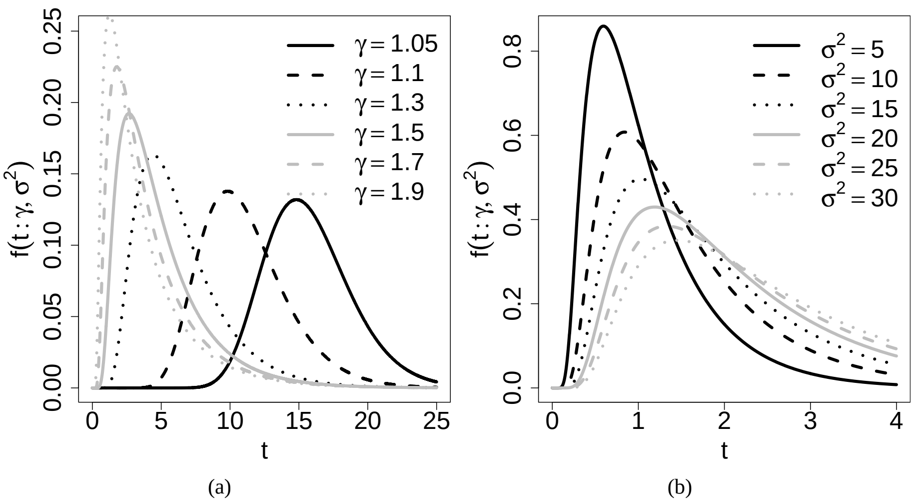


Figure 6: plots of BS p.d.f. for $\sigma^2 = 10$ (a) and $\gamma = 1.5$ (b).

3.5 Parameterization 7 (based on the mean and a bounded variance)

A new fifth parameterization of the BS distribution that we propose is (P7) given by $\alpha = \sqrt{2\,[\phi-1]}$ and $\beta = \mu/\phi$, where $\phi > 1$ is a parameter that allows us to obtain a bounded variance following a similar idea to that used in GLM, since $\mathrm{Var}[T] = g(\mu)/h(\phi)$, with $g(\mu)$ and $h(\phi)$ being functions of μ and ϕ, respectively. Here, $\mu > 0$ is the mean and $\phi > 1$ to warrant α is positive real. Thus, under this parameterization, the p.d.f. of $T \sim \mathrm{BS}(\phi,\mu)$ is

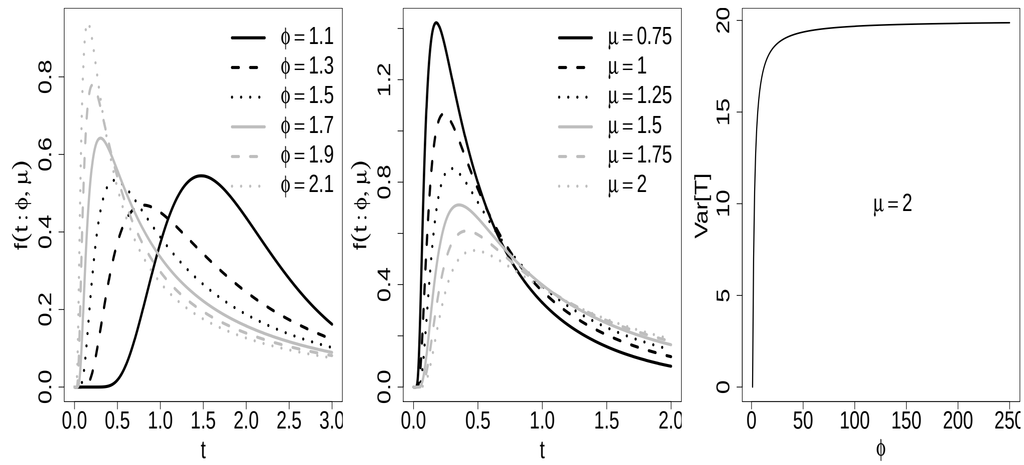
$$f(t;\phi,\mu) = \frac{\exp\left(1/[2\{\phi-1\}]\right)[t\,\phi+\mu]}{4\sqrt{\pi\phi\,[\phi-1]\mu}\,t^{3/2}}\exp\left(-\frac{1}{4[\phi-1]}\left[\frac{t\phi}{\mu}+\frac{\mu}{t\phi}\right]\right), \quad t > 0.$$

In this case, $c\,T \sim \mathrm{BS}(\phi, c\,\mu)$, with $c > 0$. In addition, the mean and variance are $\mathrm{E}[T] = \mu$ and $\mathrm{Var}[T] = \mu^2[\phi-1][5\phi-3]/\phi^2$. For this parameterization, from Figure 7(a), we detect that, as ϕ increases, the variability of the distribution decreases. This parameter has an interesting property since it limits the variability of the distribution, i.e., when ϕ approaches to ∞, its variance is turned to be constant (for fixed μ, $\mathrm{Var}[T] \to 5\mu^2$), such as we can see in Figure 7(c). From Figure 7(b), we observe that, as μ increases, the variability of the distribution increases too.

3.6 Parameterization 8 (based on the first Tweedie parameterization)

A new sixth parameterization of the BS distribution that we propose is (P8) given by $\alpha = \tau/\sqrt{2\omega}$ and $\beta = \tau^2/2$, where $\tau > 0$ and $\omega > 0$ are both of them scale parameters, but ω is also a shape parameter. Thus, under this parameterization, the p.d.f of $T \sim \mathrm{BS}(\tau,\omega)$ is

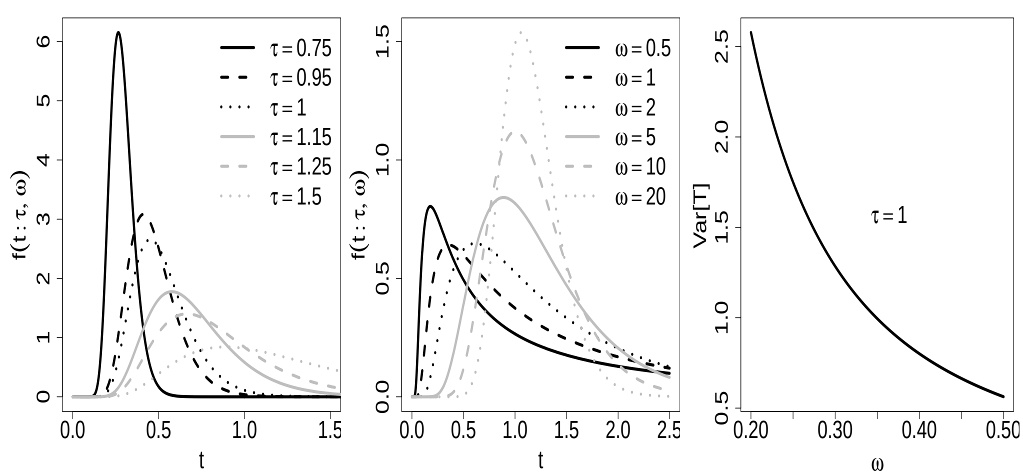
$$f(t;\tau,\omega) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\omega}{\tau^2}\left[\frac{2t}{\tau^2}+\frac{\tau^2}{2t}-2\right]\right)\frac{[2t+\tau^2]\sqrt{\omega}}{2\tau^2\sqrt{t^3}}, \quad t > 0.$$



(a) (b) (c)

Figure 7: plots of BS p.d.f. for $\mu = 2.0$ (a) and $\phi = 1.5$ (b) and of its variance vs ϕ(c).

In this case, $cT \sim \mathrm{BS}(\sqrt{c}\tau, c\omega)$, with $c > 0$. In addition, that mean and variance are $\mathrm{E}[T] = \tau^2/2 + \tau^4/[8\omega]$ and $\mathrm{Var}[T] = \tau^6/[8\omega] + 5\tau^8/[64\omega^2]$. For this parameterization, from Figure 8(a), we observe that, as τ increases, the distribution is turned to be more asymmetrical and its variability and kurtosis increase too. From 8(b)-(c), we detect that, as ω increases, the distribution is turned to be more symmetrical and the variability and kurtosis decrease.



(a) (b) (c)

Figure 8: plots of BS p.d.f. for $\omega = 5.0$ (a) and $\tau = 1.5$ (b) and of its variance vs. ω (c).

3.7 Parameterization 9 (based on the second Tweedie parameterization)

A new seventh parameterization of the BS distribution that we propose is (P9) given by $\alpha = \sqrt{\beta/\omega}$ and β, where $\omega > 0$ and $\beta > 0$ are both scale parameters. Thus, under this parameterization, the p.d.f of $T \sim \mathrm{BS}(\omega, \beta)$ is

$$f(t;\beta,\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\omega}{2\beta}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right) \frac{[t+\beta]\sqrt{\omega}}{2\beta\sqrt{t^3}}, \quad t > 0.$$

In this case, $c\,T \sim \mathrm{BS}(c\,\omega, c\,\beta)$, with $c > 0$. In addition, the mean and variance are $\mathrm{E}[T] = \beta + \beta^2/[2\omega]$ and $\mathrm{Var}[T] = \beta^3/\omega + 5\beta^4/[4\omega^2]$. For this parameterization, from Figure 9(a), we observe that, as ω increases, the distribution is turned to be more symmetrical, so that ω modifies its shape. From Figure 9(b), we detect that β is a scale parameter, as expected From Figure 9(c), we note that the variability of the distribution is inversely proportional to ω.

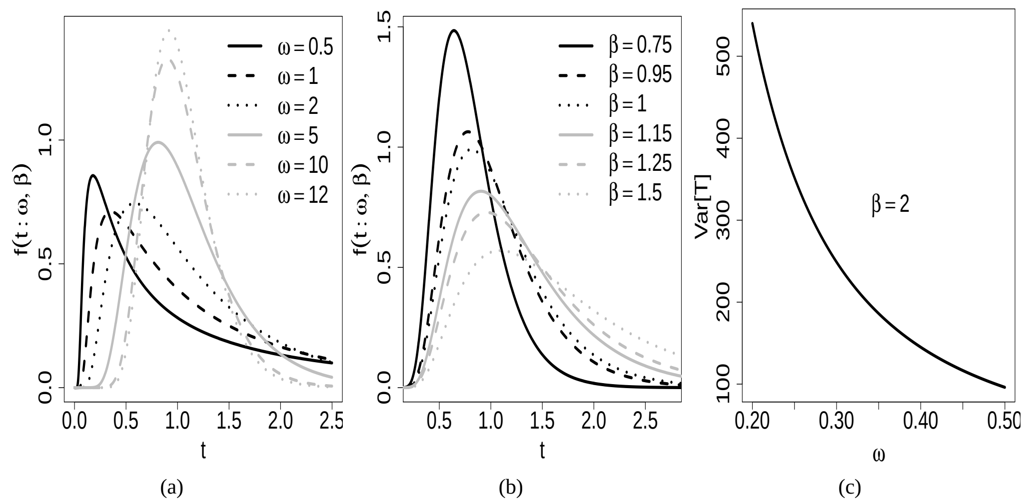


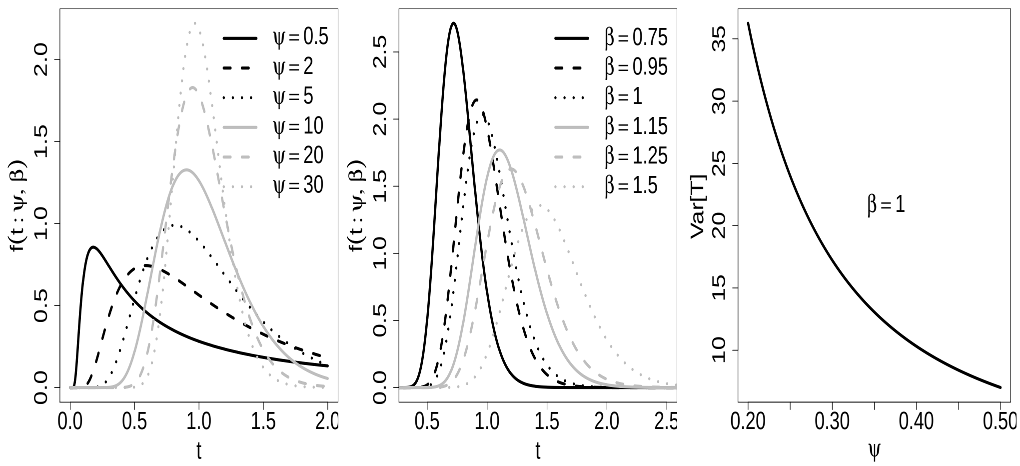
Figure 9: plot of BS p.d.f. for $\beta = 1.0$ (a) and $\omega = 5.0$ (b) and of its variance vs. ω (c).

3.8 Parameterization 10 (based on the third Tweedie parameterization)

A new eighth parameterization of the BS distribution that we propose is (P10) given by $\alpha = 1/\sqrt{\psi}$ and β, where $\psi > 0$ and $\beta > 0$ are shape and scale parameters. Thus, under this parameterization, the p.d.f of $T \sim \mathrm{BS}(\psi, \beta)$ is

$$f(t;\beta,\psi) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\psi}{2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right) \frac{[t+\beta]\sqrt{\psi}}{2\sqrt{\beta t^3}}, \quad t > 0.$$

In this case, $c\,T \sim \mathrm{BS}(\psi, c\,\beta)$, with $c > 0$ and $1/T \sim \mathrm{BS}(\psi, 1/\beta)$. In addition, the mean and variance are $\mathrm{E}[T] = \beta + \beta/[2\psi]$ and $\mathrm{Var}[T] = \beta^2/\psi + 5\beta^2/[4\psi^2]$. For this parameterization, first, we highlight that ψ and β are orthogonal parameters, since the expected value of the second derivative of the likelihood function with respect to ψ and β is zero, i.e., $\mathrm{E}[\partial^2 \ell(\psi,\beta;\mathbf{T})/\partial\psi\partial\beta] = 0$; see details in Appendix. Second, from Figure 10(a), we observe that, as ψ increases, the distribution is turned to be more symmetrical confirming that ψ is a shape parameter. From 10(b), we detect that β modifies the position and the scale, as expected. From Figure 10(c), we note that the variability of the distribution is inversely proportional to ψ.



(a) (b) (c)

Figure 10: plot of BS p.d.f. for $\beta = 1.0$ (a) and $\psi = 25$ (b) and of its variance vs. ψ (c).

3.9 Parameterization 11 (based on the fourth Tweedie parameterization)

A new ninth parameterization of the BS distribution that we propose is (P11) given by $\alpha = 1/\sqrt{\psi}$ and $\beta = \omega/\psi$, where $\psi > 0$ and $\omega > 0$ are shape and scale parameters. Thus, under this parameterization, the p.d.f of $T \sim \mathrm{BS}(\psi,\omega)$ is

$$f(t;\psi,\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\psi}{2}\left[\frac{t\psi}{\omega} + \frac{\omega}{t\psi} - 2\right]\right) \frac{[t\,\psi + \omega]}{2\sqrt{\omega t^3}}, \quad t > 0.$$

In this case, $cT \sim \mathrm{BS}(\psi, c\,\omega)$, with $c > 0$. In addition, the mean and variance are $\mathrm{E}[T] = \omega/\psi + \omega/[2\psi^2]$ and $\mathrm{Var}[T] = \omega^2/\psi^3 + 5\omega^2/[4\psi^4]$. For this parameterization, from Figure 11(a), we observe that, as ψ increases, the variability and kurtosis of the distribution decrease and its shape changes turning to be more symmetrical. From Figure 11(b), we detect that, as ω increases, the variability of the distribution increases too. From Figure 11(c), we note that the variability decreases when ψ increases.

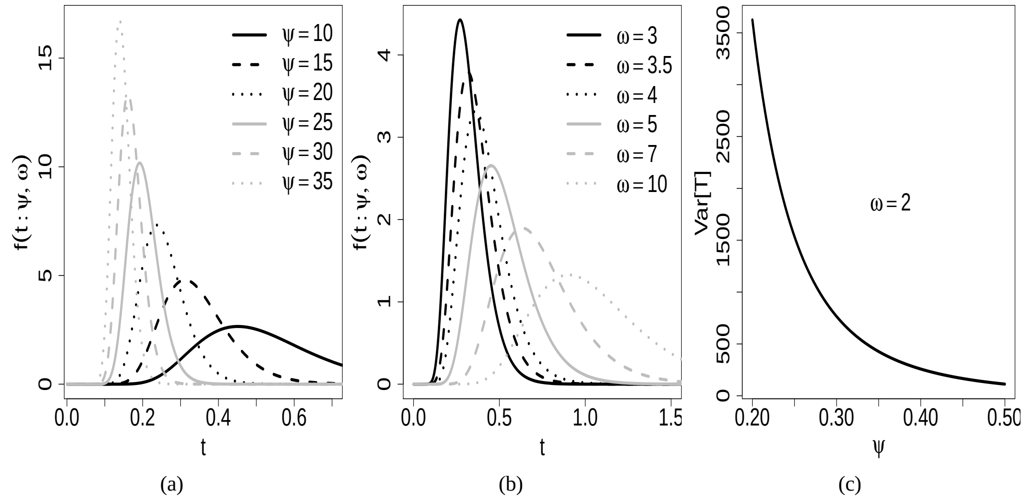


Figure 11: plot of BS p.d.f. for $\omega = 5.0$ (a) and $\psi = 10$ (b) and of its variance vs. ψ (c).

4 NUMERICAL EVALUATION

In this section, we show and discuss the results of a Monte Carlo study that allows us to evaluate the performance of the different parameterizations analyzed in this article.

4.1 Methodological details

We carry out Monte Carlo simulations to compare the performance of the ML estimators for the proposed parameterizations under diverse scenarios. The considered sample sizes are $n = 10, 25, 100$ (small, moderate and large sizes, respectively) and the run size for the simulations is $M = 10000$ in each case. The true values of the parameters are $\alpha = 0.2, 0.5, 1.0$ (low, moderate and high asymmetry, respectively) and, without loss of generality, we assume $\beta = 1.0$. In each of 10000 replicates, the ML estimates of the parameterizations (P1)-(P11) are calculated. The function `rbs2` of the `bs` package produced in `R` language, and the relationship established among the parameters given in (P2)-(P11), are used for the generation of random numbers. We compute the empirical mean, standard error (SE), bias and mean square error (MSE) of the parameter estimators. Based on these results, we study the statistical properties of the corresponding ML estimators. All the results of this simulation study are developed upon the platform `R` (version 2.11.1). Table 1 presents the true values of the parameters by using the corresponding relationships among them indicated in (P2)-(P11).

Table 1: true values of the parameters.

α	β	μ_A	λ_A	δ	μ	σ^2	γ	ϕ	τ	ω	ψ
0.20	1.00	5.00	5.00	50.00	1.02	0.04	1.05	1.02	1.41	25.00	25.00
0.50	1.00	2.00	2.00	8.00	1.12	0.33	1.31	1.12	1.41	4.00	4.00
1.00	1.00	1.00	1.00	2.00	1.50	2.25	2.25	1.50	1.41	1.00	1.00

4.2 Coverage probabilities and confidence intervals

From (2.4), asymptotic confidence intervals (CI's) of $[1-\xi]\times 100\%$ level for the parameters α and β are

$$\widehat{\alpha}\pm z(1-\xi/2)\sqrt{\widehat{\alpha}^2/2n} \quad \text{and} \quad \widehat{\beta}\pm z(1-\xi/2)\sqrt{\frac{\widehat{\beta}^2}{n[1/4+1/\widehat{\alpha}^2+I(\widehat{\alpha})]}},$$

where $z(1-\xi/2)$ is the $(1-\xi/2)$th N(0,1) quantile.

In order to obtain asymptotic CI's for the other parameters, we can approximate the variances of their ML estimators using the delta method; see Rao (1965, pp. 387-388). Specifically, let $\widehat{\theta}=h(\widehat{\alpha},\widehat{\beta})$, for $\theta=\mu_A$, λ_A, δ, μ, σ^2, γ, ϕ, τ, ω, or ψ. Then,

$$\mathrm{Var}[\widehat{\theta}]\approx\left[\frac{\partial\theta}{\partial\alpha}\right]^2\mathrm{Var}[\widehat{\alpha}]+\left[\frac{\partial\theta}{\partial\beta}\right]^2\mathrm{Var}[\widehat{\beta}],$$

where, as mentioned, α and β are orthogonal parameters and their corresponding asymptotic variances are as given in (2.4) by $\mathrm{Var}[\widehat{\alpha}]\approx\alpha^2/[2n]$ and $\mathrm{Var}[\widehat{\beta}]\approx\beta^2/[n[\{1/4+1/\alpha^2+I(\alpha)\}]$. Hence, the asymptotic variance of the estimators of the other parameters are

$$\begin{aligned}
\mathrm{Var}[\widehat{\mu}_A] &\approx \frac{1}{2n\alpha^2\beta}+\frac{1}{4n\alpha^2\beta[1/4+1/\alpha^2+I(\alpha)]},\\
\mathrm{Var}[\widehat{\lambda}_A] &\approx \frac{\beta}{2n\alpha^2}+\frac{\beta}{4n\alpha^2[1/4+1/\alpha^2+I(\alpha)]},\\
\mathrm{Var}[\widehat{\delta}] &\approx \frac{8}{n\alpha^4},\\
\mathrm{Var}[\widehat{\mu}] &\approx \frac{\alpha^4\beta^2}{2n}+\frac{\beta^2[1+\alpha^2/2]^2}{n[1/4+1/\alpha^2+I(\alpha)]},\\
\mathrm{Var}[\widehat{\sigma^2}] &\approx \frac{\left[2\alpha^5\beta^2+5\beta^2\alpha^7\right]^2}{2n}+\frac{\left[4\alpha^4\beta^6+20\alpha^6\beta^2\beta^6/4\right]^2}{n\,[1/4+1/\alpha^2+I(\alpha)]},\\
\mathrm{Var}[\widehat{\gamma}] &\approx \frac{25\alpha^4}{8n},\\
\mathrm{Var}[\widehat{\phi}] &\approx \frac{\alpha^4}{2n},\\
\mathrm{Var}[\widehat{\tau}] &\approx \frac{\beta}{2n[1/4+1/\alpha^2+I(\alpha)]},\\
\mathrm{Var}[\widehat{\omega}] &\approx \frac{2\beta^2}{n\alpha^4}+\frac{\beta^2}{n[1/4+1/\alpha^2+I(\alpha)]\alpha^4} \quad \text{and}\\
\mathrm{Var}[\widehat{\psi}] &\approx \frac{2}{n\alpha^4}.
\end{aligned}$$

where $I(\alpha)$ is as given in (2.4). Tables 3-5 provide $[1-\xi]\times 100\%$ asymptotic CI's and their corresponding coverage probabilities for the indicated parameters.

4.3 Results and discussion

The ML estimates under the eleven (11) parameterizations analyzed in this work are presented in Table 2. Notice that the empirical bias, SE and MSE of all the parameter estimators decrease when the sample size (n) increases, as expected, since the asymptotic variance is a decreasing function on n. We also verify that, as α increases, the SE's of the ML estimators increase, excepting the case of the parameters $\mu_A, \lambda_A, \delta, \omega$ and ψ. This behavior can occur due to the fact that these parameters are an inverse function of α. Such a behavior is also observed for the corresponding MSE. The ML estimators of the parameters μ_A, λ_A, δ, ω and ψ also present the largest biases and dispersions among all the analyzed parameters. We can note that most of the estimators of the studied parameters tend to be unbiased and consistent, as expected. Since the values of the empirical SE's of the ML estimators are close to the true values of the parameters, this indicates a stability in the values of the variances of the ML estimators for the different sample sizes considered. Furthermore, it worthwhile to observe that the ML estimators of α, γ and ϕ underestimate their respective true values, while the estimators of μ_A, λ_A, δ, ω and ψ overestimate these values. From Table 2, we conclude that, as n increases, the bias approaches zero, as expected. Biases of $\widehat{\delta}$, $\widehat{\omega}$ and $\widehat{\psi}$ show superior values to those of the other parameters. In general, kurtosis values of the ML estimators (omitted here) decrease as n increases being close to a value equal to three. In addition, as n increases, the variability and MSE of the estimators decrease. Empirical distributions of $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\mu}$ seem to be symmetrical, but those of the others estimators are asymmetric (omitted here); see Figures 12-14.

Next, we present the simulation results regarding the construction of the asymptotic CI's for α, β, μ_A, λ_A, δ, μ, σ^2, γ, ϕ, τ, ω and ψ. A total of 324 intervals with nominal levels of coverage equal to $1-\xi = 0.90, 0.95, 0.99$, for the sample sizes $n = 10, 25, 100$, are evaluated. All of the asymptotic CI's are constructed in such a way that (i) these contain the true value of the parameter with probability $1-\xi$, (ii) the lower limit (LL) is lesser than this true value with probability $\xi/2$, and (iii) the upper limit (UL) is greater than the true value of the parameter also with probability $\xi/2$, for $0 < \xi/2 < 0.5$. For each one of the calculated asymptotic CI's, the coverage probabilities are estimated by considering the frequency with that the confidence LL exceeds the true value of the parameter and the confidence UL does not exceed this value, in both of the cases for the 10000 asymptotic CI's. These coverage probabilities that the confidence LL is greater and the confidence LC is lesser than the true values of the parameters are shown in Tables 3-5 by lower (LC) and upper (UC) coverages, respectively. We observe that, in general, for the three nominal levels of confidence and the different sample sizes considered, the asymptotic CI's for the parameters μ_A, λ_A, δ, μ, ω and ψ present the best coverage probabilities. In addition, as expected, the coverage probabilities increase as n increases.

Figures 12, 13 and 14 display empirical distributions of $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\mu}$, respectively, by histograms constructed with the 10000 ML estimates of the parameters for several values of ξ and n. Histograms for other estimators can be obtained from the authors upon request. In these histograms, the straight line segments represent each asymptotic CI computed, where stippled vertical line indicates the true value of the parameter. In general, such histograms are approximately symmetrical, except in the case of δ, σ^2, ω and ψ (omitted here). Furthermore, it is important to highlight that the asymptotic CI's for β, μ_A, λ_A, μ, ω and ψ are approximately balanced, i.e., the observed lower and upper coverage probabilities coincide approximately at the tails of the distribution of the ML estimator of the corresponding parameter, for the different confidence levels and sample sizes. We also observe that, as n increases, the symmetry of the histograms and the balancing of the asymptotic CI's increase, but when α increases, the balancing deteriorates.

5 CONCLUDING REMARKS

In this paper, we have analyzed eleven parameterizations (P1)-(P11) of the two-parameter Birnbaum-Saunders distribution. We have examined the structural properties of these parameterizations. We have considered the maximum likelihood method for estimating the corresponding parameters. We have conducted a Monte Carlo simulations study for analyzing the performance of these parameterizations. By means of this study, we can highlight the parameterizations (P1), (P2), (P3), (P10) and (P11). As expected, the mentioned simulation study has confirmed that (i) the maximum likelihood estimators of these parameters are asymptotically unbiased and consistent and (ii) the coverage probabilities of the respective confidence intervals are close to the nominal level. In addition to the orthogonality of α and β in the original parameterization (P1), ψ and β in the parameterization (P10) are also orthogonal. Another aspect to be highlighted is the possibility of directly describing the mean of a Birnbaum-Saunders random variable by a regression model without the need of transforming the response variable when the parameterization (P3) is used, which has been not proposed until now, but follows a similar idea to that of generalized linear models. Other aspect that can be analyzed in further studies is the performance of moment estimators for these parameterizations.

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Table 2: ML estimates, bias, standard error and mean square error of the indicated parameters ($\beta = 1.0$).

α	n	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\mu}_A$	$\widehat{\lambda}_A$	$\widehat{\delta}$	$\widehat{\mu}$	$\widehat{\sigma}^2$	$\widehat{\gamma}$	$\widehat{\phi}$	$\widehat{\tau}$	$\widehat{\omega}$	$\widehat{\psi}$
							MLE						
0.2	10	0.1847	1.0021	5.7726	5.7734	68.5140	1.0202	0.0390	1.0459	1.0184	1.4150	35.8888	35.8082
	25	0.1942	1.0012	5.2638	5.2663	56.6998	1.0205	0.0407	1.0482	1.0193	1.4148	28.3994	28.3605
	100	0.1986	1.0000	5.0619	5.0607	51.4959	1.0198	0.0417	1.0496	1.0198	1.4141	25.7467	25.7474
0.5	10	0.4606	1.0121	2.3201	2.3209	11.5003	1.1256	0.3154	1.2804	1.1122	1.4185	5.8316	5.7576
	25	0.4851	1.0056	2.1089	2.1112	9.0924	1.1264	0.3248	1.3004	1.1202	1.4165	4.5741	4.5462
	100	0.4964	1.0006	2.0261	2.0249	8.2430	1.1245	0.3270	1.3096	1.1238	1.4142	4.1240	4.1215
1.0	10	0.9159	1.0410	1.1762	1.1769	2.9196	1.5016	2.4150	2.1182	1.4473	1.4286	1.5271	1.4595
	25	0.9681	1.0173	1.0595	1.0615	2.2841	1.5034	2.3352	2.1966	1.4786	1.4208	1.1635	1.1420
	100	0.9922	1.0027	1.0146	1.0134	2.0630	1.4988	2.2676	2.2370	1.4948	1.4147	1.0344	1.0315
							Bias						
0.2	10	−0.0153	0.0021	0.7726	0.7734	18.5140	0.0002	−0.0030	−0.0041	−0.0016	0.0008	10.8888	10.8082
	25	−0.0058	0.0012	0.2638	0.2663	6.6998	0.0005	−0.0013	−0.0018	−0.0007	0.0005	3.3994	3.3605
	100	−0.0014	−0.0000	0.0619	0.0607	1.4959	−0.0002	−0.0003	−0.0004	−0.0002	−0.0001	0.7467	0.7474
0.5	10	−0.0394	0.0121	0.3201	0.3209	3.5003	0.0006	−0.0127	−0.0321	−0.0128	0.0043	1.8316	1.7576
	25	−0.0149	0.0056	0.1089	0.1112	1.0924	0.0014	−0.0033	−0.0121	−0.0048	0.0023	0.5741	0.5462
	100	−0.0036	0.0006	0.0261	0.0249	0.2430	−0.0005	−0.0011	−0.0029	−0.0012	0.0000	0.1240	0.1215
1.0	10	−0.0841	0.0410	0.1762	0.1769	0.9196	0.0016	0.1650	−0.1318	−0.0527	0.0143	0.5271	0.4595
	25	−0.0319	0.0173	0.0595	0.0615	0.2841	0.0034	0.0852	−0.0534	−0.0214	0.0065	0.1635	0.1420
	100	−0.0078	0.0027	0.0146	0.0134	0.0630	−0.0012	0.0176	−0.0130	−0.0052	0.0005	0.0344	0.0315
							SE						
0.2	10	0.0442	0.0633	1.5965	1.5990	41.3684	0.0650	0.0192	0.0205	0.0082	0.0447	22.9310	22.6773
	25	0.0284	0.0399	0.8120	0.8159	18.1333	0.0410	0.0128	0.0140	0.0056	0.0282	9.2006	9.0912
	100	0.0142	0.0199	0.3679	0.3683	7.4853	0.0205	0.0065	0.0071	0.0028	0.0141	3.7815	3.7421
0.5	10	0.1104	0.1566	0.6642	0.6671	7.1761	0.1818	0.2161	0.1316	0.0526	0.1092	3.8817	3.6495
	25	0.0709	0.0979	0.3379	0.3418	2.9154	0.1145	0.1342	0.0873	0.0349	0.0688	1.5481	1.4578
	100	0.0354	0.0485	0.1539	0.1542	1.1981	0.0572	0.0662	0.0442	0.0177	0.0342	0.6341	0.5990
1.0	10	0.2209	0.3007	0.3727	0.3770	1.8634	0.4751	2.3909	0.9056	0.3623	0.2032	1.1657	0.9317
	25	0.1420	0.1823	0.1883	0.1918	0.7345	0.2992	1.3664	0.3489	0.1396	0.1264	0.4419	0.3672
	100	0.0709	0.0885	0.0858	0.0860	0.3000	0.1497	0.6478	0.1766	0.0706	0.0624	0.1776	0.1500
							MSE						
0.2	10	0.0022	0.0040	3.1457	3.1550	2054.1154	0.0042	0.0004	0.0004	0.0001	0.0020	644.3970	631.0783
	25	0.0008	0.0016	0.7289	0.7366	373.7037	0.0017	0.0002	0.0002	0.0000	0.0008	96.2076	93.9422
	100	0.0002	0.0004	0.1392	0.1394	58.2668	0.0004	0.0000	0.0001	0.0000	0.0002	14.8573	14.5617
0.5	10	0.0137	0.0247	0.5436	0.5480	63.7478	0.0331	0.0469	0.0184	0.0029	0.0120	18.4228	16.4079
	25	0.0053	0.0096	0.1260	0.1292	9.6931	0.0131	0.0180	0.0078	0.0012	0.0047	2.7263	2.4234
	100	0.0013	0.0023	0.0244	0.0244	1.4945	0.0033	0.0044	0.0020	0.0003	0.0012	0.4175	0.3736
1.0	10	0.0559	0.0921	0.1699	0.1734	4.3179	0.2257	5.7434	0.8375	0.1340	0.0415	1.6368	1.0792
	25	0.0212	0.0335	0.0390	0.0406	0.6201	0.0895	1.8744	0.1246	0.0199	0.0160	0.2220	0.1550
	100	0.0051	0.0078	0.0076	0.0076	0.0940	0.0224	0.4200	0.0314	0.0050	0.0039	0.0327	0.0235

Table 3: $[1-\xi]\times 100\%$ asymptotic CI's and coverage probabilities for the indicated parameters ($\alpha = 0.2, \beta = 1.0$).

		LL			UL			LC(%)			UC(%)			Coverage(%)		
ξ	θ	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$
0.01	α	0.0783	0.1235	0.1624	0.2910	0.2650	0.2348	0.00	0.01	0.06	8.98	4.39	1.66	91.02	95.60	98.28
	β	0.8529	0.9020	0.9493	1.1513	1.1003	1.0506	1.55	0.66	0.53	2.49	1.22	0.85	95.96	98.12	98.62
	μ_A	2.4218	3.3290	4.1311	9.1234	7.1985	5.9927	0.59	0.56	0.55	0.23	0.44	0.40	99.18	99.00	99.05
	λ_A	2.4221	3.3306	4.1302	9.1247	7.2019	5.9913	0.63	0.64	0.61	0.30	0.42	0.45	99.07	98.94	98.94
	δ	0.0000	15.3755	32.7375	151.0122	98.0242	70.2543	0.00	0.05	0.17	0.92	0.92	0.75	99.08	99.03	99.08
	μ	0.8608	0.9145	0.9657	1.1795	1.1264	1.0739	0.81	0.41	0.30	2.89	1.34	0.74	96.30	98.25	98.96
	σ^2	0.0000	0.0084	0.0252	0.0873	0.0729	0.0581	0.00	0.00	0.01	12.67	6.69	2.53	87.33	93.31	97.46
	γ	1.0002	1.0131	1.0315	1.0978	1.0832	1.0676	0.00	0.00	0.01	12.13	6.57	2.51	87.87	93.43	97.48
	ϕ	1.0001	1.0052	1.0126	1.0391	1.0333	1.0270	0.00	0.00	0.01	12.13	6.57	2.51	87.87	93.43	97.48
	τ	1.3096	1.3447	1.3783	1.5204	1.4848	1.4499	1.68	0.76	0.56	2.25	1.08	0.77	96.07	98.16	98.67
	ω	0.0000	7.5308	16.2789	77.5130	49.2679	35.2146	0.00	0.02	0.16	0.90	1.01	0.74	99.10	98.97	99.10
	ψ	0.0000	7.6983	16.3682	77.0573	49.0226	35.1266	0.00	0.03	0.16	0.92	0.92	0.75	99.08	99.05	99.09
0.05	α	0.1037	0.1404	0.1711	0.2656	0.2481	0.2261	0.01	0.32	0.96	15.21	8.75	5.16	84.78	90.93	93.88
	β	0.8886	0.9257	0.9614	1.1156	1.0766	1.0385	4.08	2.86	2.36	5.75	3.77	3.10	90.17	93.37	94.54
	μ_A	3.2229	3.7916	4.3537	8.3222	6.7359	5.7702	4.66	3.68	2.88	1.18	1.79	2.27	94.16	94.53	94.85
	λ_A	3.2234	3.7934	4.3527	8.3234	6.7391	5.7688	4.60	3.76	2.96	1.29	1.82	2.18	94.11	94.42	94.86
	δ	5.7407	25.2559	37.2225	131.2874	88.1438	65.7693	0.00	1.35	1.97	2.63	2.79	3.03	97.37	95.86	95.00
	μ	0.8989	0.9398	0.9786	1.1414	1.1011	1.0610	2.76	2.13	1.80	6.24	3.97	2.87	91.00	93.90	95.33
	σ^2	0.0023	0.0161	0.0291	0.0757	0.0652	0.0542	0.00	0.00	0.46	17.88	11.86	6.40	82.12	88.14	93.14
	γ	1.0064	1.0215	1.0358	1.0854	1.0749	1.0633	0.00	0.03	0.56	17.32	11.11	6.14	82.68	88.86	96.30
	ϕ	1.0026	1.0086	1.0143	1.0342	1.0299	1.0253	0.00	0.03	0.56	17.32	11.11	6.14	82.68	88.86	93.30
	τ	1.3348	1.3614	1.3869	1.4952	1.4680	1.4414	4.49	3.12	2.51	5.44	3.60	3.04	90.07	93.28	94.45
	ω	4.2167	12.5200	18.5426	67.5609	44.2784	32.9509	0.12	1.34	2.03	2.66	2.79	2.95	97.22	95.87	95.02
	ψ	4.4215	12.6385	18.6107	67.1949	44.0824	32.8841	0.12	1.42	1.97	4.25	2.80	3.03	97.24	95.78	95.00
0.10	α	0.1167	0.1490	0.1755	0.2526	0.2394	0.2217	0.22	1.13	2.91	19.80	12.52	8.24	79.98	86.35	88.85
	β	0.9068	0.9378	0.9676	1.0974	1.0645	1.0323	7.05	5.83	4.85	8.53	6.46	5.52	84.42	87.71	89.63
	μ_A	3.6328	4.0283	4.4675	7.9123	6.4992	5.6563	8.96	7.14	6.02	2.67	3.55	4.80	88.37	89.31	89.18
	λ_A	3.6334	4.0302	4.4665	7.9134	6.5023	5.6550	9.05	7.28	5.96	2.69	3.55	4.52	88.26	89.17	89.52
	δ	15.8330	30.3113	39.5173	121.1951	83.0884	63.4745	1.80	4.17	4.82	4.25	4.88	5.49	93.95	90.95	89.69
	μ	0.9184	0.9528	0.9852	1.1219	1.0881	1.0543	5.18	4.24	4.00	8.78	6.46	5.25	86.04	89.30	90.75
	σ^2	0.0082	0.0201	0.0311	0.0698	0.0613	0.0522	0.01	0.14	1.79	21.57	15.74	9.87	78.42	84.12	88.34
	γ	1.0127	1.0258	1.0380	1.0790	1.0706	1.0611	0.00	0.39	2.05	20.96	15.16	9.42	79.04	84.45	88.53
	ϕ	1.0051	1.0103	1.0152	1.0316	1.0282	1.0244	0.00	0.39	2.05	20.96	15.17	9.42	79.04	84.44	88.53
	τ	1.3477	1.3700	1.3912	1.4823	1.4595	1.4370	7.51	6.15	5.05	8.16	6.20	5.39	84.33	87.65	89.56
	ω	9.3087	15.0733	19.7009	62.4688	41.7255	31.7926	2.77	4.36	4.69	4.22	4.96	5.39	93.01	90.68	89.92
	ψ	9.4677	15.1662	19.7581	62.1487	41.5548	31.7367	2.67	4.24	4.82	4.25	4.86	5.49	93.08	90.90	89.69

Table 4: $[1-\xi]\times 100\%$ asymptotic CI's and coverage probabilities for the indicated parameters ($\alpha = 0.5, \beta = 1.0$).

		LL			UL			LC(%)			UC(%)			Coverage(%)		
ξ	θ	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$
0.01	α	0.1953	0.3084	0.4060	0.6300	0.5976	0.5541	0.00	0.01	0.06	9.12	4.47	1.66	90.88	95.52	98.28
	β	0.6543	0.7690	0.8801	1.3699	1.2422	1.1210	1.05	0.49	0.46	3.74	1.89	1.18	95.21	97.62	98.36
	μ_A	0.9264	1.3025	1.6376	3.7139	2.9152	2.4146	0.64	3.86	0.55	0.25	0.33	0.50	99.11	99.08	98.95
	λ_A	0.9266	1.3040	1.6366	3.7153	2.9185	2.4132	0.68	0.68	0.61	0.25	0.44	0.48	99.07	98.88	98.91
	δ	0.0000	2.4682	5.2404	24.7650	15.7167	11.2456	0.00	0.03	0.17	0.89	0.92	0.74	99.11	99.05	99.09
	μ	0.6824	0.8325	0.9747	1.5687	1.4203	1.2743	0.27	0.12	0.12	5.00	2.45	1.26	94.73	97.43	98.62
	σ^2	0.0000	0.0000	0.1592	0.8361	0.6604	0.4949	0.00	0.00	0.00	16.87	8.60	3.18	83.13	91.40	96.82
	γ	1.0000	1.0816	1.1968	1.6034	1.5193	1.4224	0.00	0.00	0.01	13.95	6.65	2.50	86.05	93.35	97.49
	ϕ	1.0000	1.0326	1.0787	1.2413	1.2077	1.1689	0.00	0.00	0.01	13.95	6.65	2.51	86.05	93.35	97.48
	τ	1.1679	1.2499	1.3291	1.6691	1.5831	1.4993	1.51	0.69	0.56	2.98	1.54	0.99	95.51	97.77	98.45
	ω	0.0000	1.0825	2.5430	12.8050	8.0657	5.7049	0.00	0.03	0.20	1.09	1.06	0.84	98.91	98.91	98.96
	ψ	0.0000	1.2341	2.6202	12.3899	7.8584	5.6228	0.00	0.03	0.17	0.89	0.92	0.74	99.11	99.05	99.09
0.05	α	0.2587	0.3506	0.4276	0.6624	0.6196	0.5652	0.01	0.32	0.96	15.41	8.86	5.20	84.58	90.82	93.84
	β	0.7398	0.8256	0.9089	1.2843	1.1856	1.0922	3.19	2.41	2.33	7.37	4.87	3.67	89.44	92.72	94.00
	μ_A	1.2596	1.4953	1.7304	3.3806	2.7224	2.3217	4.58	3.86	2.78	1.31	1.80	2.34	94.11	94.34	94.88
	λ_A	1.2599	1.4970	1.7294	3.3819	2.7255	2.3204	4.67	3.94	3.01	1.32	1.80	2.15	94.01	94.26	94.84
	δ	1.4071	4.0520	5.9583	21.5935	14.1329	10.5277	0.06	1.43	2.00	2.53	2.77	3.00	97.41	95.80	95.00
	μ	0.7884	0.9027	1.0105	1.4628	1.3500	1.2385	1.13	0.98	1.27	8.98	5.67	3.74	89.89	93.35	94.99
	σ^2	0.0000	0.0695	0.1993	0.7116	0.5802	0.4547	0.00	0.00	0.15	22.29	13.47	7.25	77.71	86.53	92.60
	γ	1.0346	1.1339	1.2236	1.5262	1.4670	1.3954	0.00	0.03	0.56	19.88	11.26	6.16	80.12	88.71	93.28
	ϕ	1.0138	1.0536	1.0895	1.2105	1.1868	1.1582	0.00	0.03	0.56	19.88	11.26	6.16	80.12	88.71	93.28
	τ	1.2279	1.2897	1.3494	1.6092	1.5433	1.4790	4.16	3.01	2.64	6.45	4.26	3.41	89.39	92.73	93.95
	ω	0.5253	1.9173	2.9210	11.1380	7.2309	5.3269	0.13	1.32	2.12	2.79	2.90	2.84	97.08	95.78	95.04
	ψ	0.7109	2.0260	2.9791	10.8041	7.0664	5.2638	0.12	1.43	2.00	2.53	2.77	3.00	97.35	95.80	95.00
0.10	α	0.2912	0.3723	0.4387	0.7258	0.6618	0.5868	0.21	1.13	2.91	20.08	12.70	8.37	79.71	86.17	88.72
	β	0.7836	0.8545	0.9236	1.2406	1.1567	1.0775	5.91	5.32	4.79	10.16	7.71	6.37	83.93	86.97	88.84
	μ_A	1.4301	1.5940	1.7780	3.2101	2.6238	2.2742	9.21	7.10	5.84	2.67	3.61	4.70	88.12	89.29	89.46
	λ_A	1.4305	1.5957	1.7770	3.2113	2.6267	2.2729	9.17	7.49	6.06	2.73	3.72	4.42	88.10	88.79	89.52
	δ	3.0298	4.8624	6.3256	19.9708	13.3225	10.1604	2.63	4.28	4.83	4.15	4.78	5.47	93.22	90.94	89.70
	μ	0.8426	0.9387	1.0288	1.4086	1.3141	1.2201	2.64	2.67	3.55	12.22	8.39	6.36	85.14	88.94	90.09
	σ^2	0.0000	0.1106	0.2198	0.6479	0.5391	0.4342	0.00	0.00	1.32	26.35	17.08	10.64	73.65	82.92	88.04
	γ	1.0741	1.1607	1.2376	1.4866	1.4402	1.3816	0.00	0.39	2.02	24.40	15.19	9.42	75.60	84.42	88.56
	ϕ	1.0296	1.0643	1.0950	1.1946	1.1761	1.1526	0.00	0.39	2.02	24.40	15.19	9.42	75.60	84.42	88.56
	τ	1.2585	1.3101	1.3598	1.5786	1.5229	1.4686	7.11	6.10	5.13	9.28	7.14	6.06	83.61	86.76	88.81
	ω	1.3784	2.3445	3.1144	10.2848	6.8037	5.1335	2.64	4.42	4.66	4.64	5.20	5.33	92.72	90.38	90.01
	ψ	1.5223	2.4312	3.1628	9.9928	6.6613	5.0802	2.71	4.30	4.84	4.15	4.78	5.47	93.14	90.92	89.69

Table 5: $[1-\xi]\times 100\%$ asymptotic CI's and coverage probabilities for the indicated parameters ($\alpha = 1.0, \beta = 1.0$).

		LL			UL			LC(%)			UC(%)			Coverage(%)		
ξ	θ	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$	$n=10$	$n=25$	$n=100$
0.01	α	0.3884	0.6154	0.8115	1.4435	1.3207	1.1730	0.00	0.01	0.06	9.76	4.53	1.70	90.24	95.46	98.24
	β	0.4070	0.6063	0.7959	1.6751	1.4282	1.2094	0.60	0.34	0.41	7.02	3.79	2.09	92.38	95.87	97.50
	μ_A	0.4178	0.6200	0.8025	1.9346	1.4989	1.2267	0.79	0.62	0.69	0.24	0.31	0.50	98.97	99.07	98.81
	λ_A	0.4161	0.6207	0.8014	1.9377	1.5023	1.2253	0.76	0.75	0.24	0.45	0.58	1.06	98.79	98.67	98.70
	δ	0.0000	0.6200	1.3115	6.2820	3.9482	2.8145	0.03	0.05	0.17	0.80	0.91	0.75	99.17	99.04	99.08
	μ	0.4227	0.7950	1.1401	2.5806	2.2119	1.8574	0.05	0.02	0.04	9.98	5.27	2.74	89.97	94.71	97.22
	σ^2	0.0000	0.0000	0.6702	7.8986	5.6495	3.8648	0.00	0.00	0.00	20.47	11.39	4.61	79.53	88.61	95.39
	γ	1.0089	1.3248	1.7864	3.3964	3.0684	2.6876	0.05	0.00	0.01	14.45	6.84	2.57	85.50	93.16	97.42
	ϕ	1.0036	1.1299	1.3146	1.9586	1.8274	1.6750	0.05	0.00	0.01	14.45	6.84	2.57	85.50	93.16	97.42
	τ	0.9958	1.1345	1.2690	1.8613	1.7070	1.5605	1.22	0.67	0.55	5.49	2.79	1.78	93.29	96.54	97.67
	ω	0.0000	0.2000	0.6020	3.4831	2.1270	1.4667	0.00	0.02	0.24	1.50	1.36	1.06	98.50	98.62	98.70
	ψ	0.0000	0.3100	0.6558	3.1407	1.9741	1.4073	0.00	0.05	0.17	0.81	0.91	0.75	99.19	99.04	99.08
0.05	α	0.5145	0.6997	0.8547	1.3173	1.2364	1.1298	0.01	0.32	0.96	15.98	9.12	5.22	84.01	90.56	93.82
	β	0.5586	0.7046	0.8453	1.5235	1.3300	1.1600	2.32	1.88	2.29	10.97	7.64	5.29	86.71	90.48	92.42
	μ_A	0.5991	0.7251	0.8532	1.7532	1.3939	1.1760	4.95	3.89	3.18	1.66	2.08	2.46	93.39	94.03	94.36
	λ_A	0.5980	0.7261	0.8521	1.7558	1.3969	1.1747	4.91	3.99	2.07	1.79	2.28	3.47	93.30	93.73	94.46
	δ	0.3610	1.0179	1.4912	5.4781	3.5503	2.6349	0.17	1.48	2.05	2.46	2.69	2.97	97.37	95.83	94.98
	μ	0.6807	0.9643	1.2258	2.3226	2.0425	1.7717	0.25	0.31	1.24	14.79	9.57	6.23	84.96	90.12	92.53
	σ^2	0.0000	0.0000	1.0521	6.5875	4.8571	3.4829	0.00	0.00	0.00	25.72	15.99	8.66	74.28	84.01	91.34
	γ	1.1455	1.5333	1.8941	3.0908	2.8600	2.5798	0.06	0.03	0.53	20.66	11.57	6.24	79.28	88.40	93.23
	ϕ	1.0582	1.2133	1.3576	1.8363	1.7440	1.6319	0.06	0.03	0.53	20.66	11.57	6.24	79.28	88.40	93.23
	τ	1.0993	1.2029	1.3038	1.7578	1.6386	1.5256	3.56	2.82	2.89	9.20	6.62	4.94	87.24	90.56	92.17
	ω	0.0483	0.4304	0.7054	3.0154	1.8966	1.3634	0.10	1.24	2.07	3.66	3.76	3.47	96.24	95.00	94.46
	ψ	0.1802	0.5089	0.7456	2.7387	1.7752	1.3174	0.14	1.48	2.05	2.50	2.69	2.97	97.36	95.83	94.98
0.10	α	0.5790	0.7429	0.8768	1.2528	1.1932	1.1076	0.22	1.13	2.90	20.88	13.13	8.50	78.90	85.74	88.60
	β	0.6361	0.7548	0.8706	1.4459	1.2797	1.1347	4.70	4.54	4.93	14.03	10.63	8.66	81.27	84.83	86.41
	μ_A	0.6919	0.7789	0.8791	1.6605	1.3401	1.1501	9.67	7.71	6.14	3.10	3.92	4.84	87.23	88.37	89.02
	λ_A	0.6911	0.7800	0.8780	1.6627	1.3430	1.1487	9.26	7.85	4.64	3.39	4.29	5.84	87.35	87.86	98.70
	δ	0.7724	1.2214	1.5831	5.0667	3.3467	2.5429	2.88	4.39	4.97	4.01	4.71	5.40	93.11	90.90	89.63
	μ	0.8126	1.0510	1.2697	2.1906	1.9558	1.7278	0.75	1.40	3.58	18.52	12.95	9.37	80.73	85.65	87.05
	σ^2	0.0000	0.2188	1.2475	5.9167	4.4516	3.2875	0.02	0.00	0.63	28.98	19.79	12.09	71.00	80.21	87.28
	γ	1.3019	1.6399	1.9492	2.9344	2.7533	2.5247	0.06	0.39	1.95	25.14	15.63	9.49	74.80	83.98	88.56
	ϕ	1.1208	1.2560	1.3797	1.7738	1.7013	1.6099	0.06	0.39	1.95	25.14	15.63	9.49	74.80	83.98	88.56
	τ	1.1522	1.2379	1.3216	1.7049	1.6036	1.5078	6.60	5.91	5.63	12.5	9.63	8.00	80.90	84.46	86.37
	ω	0.2781	0.5482	0.7582	2.7762	1.7788	1.3105	4.64	2.40	4.21	5.84	5.62	5.99	91.98	89.80	89.52
	ψ	0.3859	0.6107	0.7916	2.5331	1.6734	1.2715	2.85	4.39	4.97	4.04	4.71	5.40	93.11	90.90	89.63

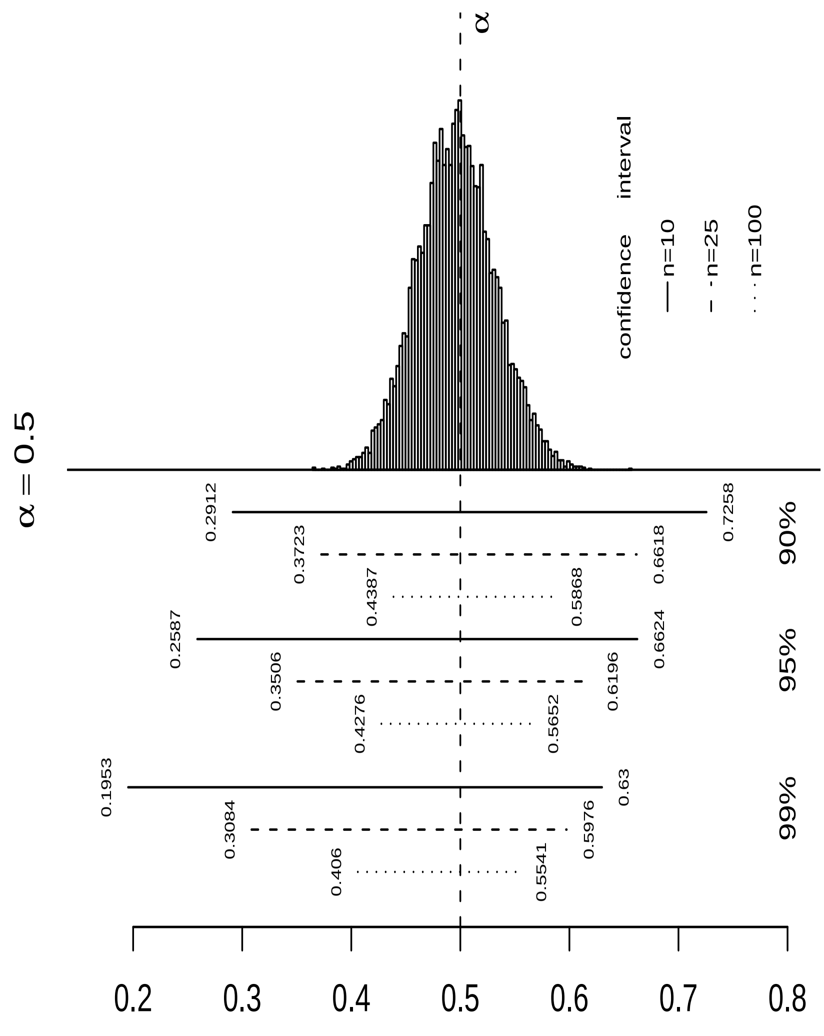

Figure 12: histograms of the ML estimates for $\alpha = 0.5$ and different values of n

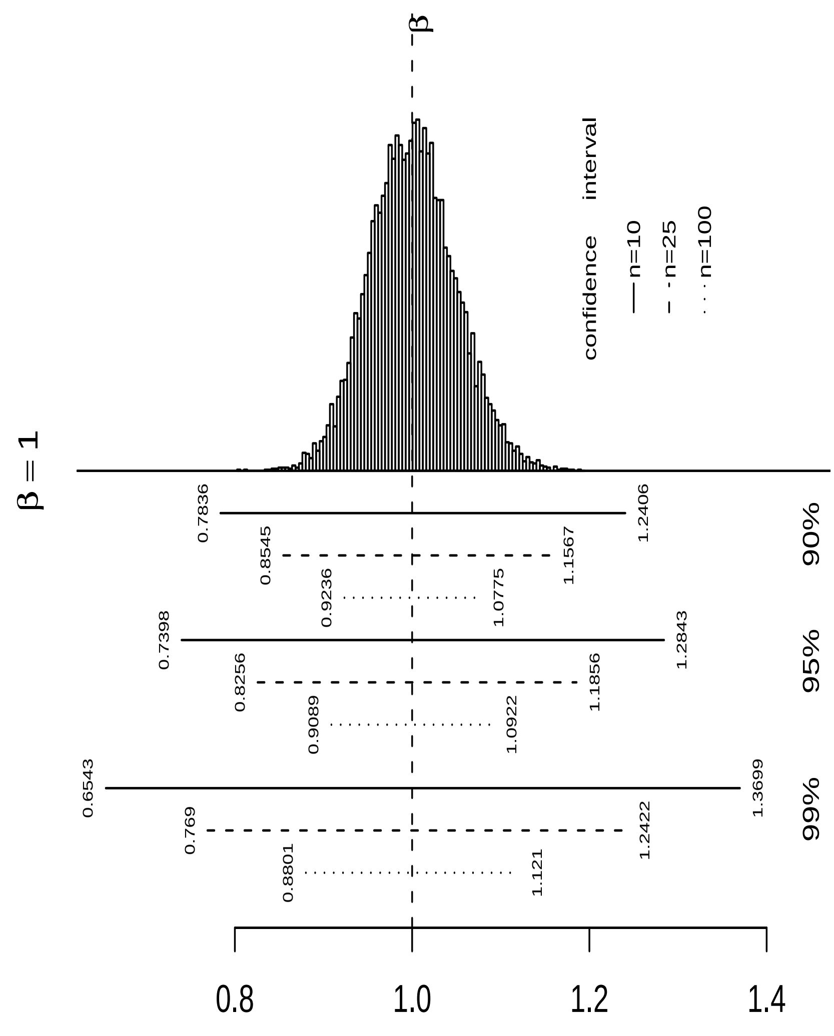

Figure 13: histograms of the ML estimates for $\beta = 1.0$ and different values of n

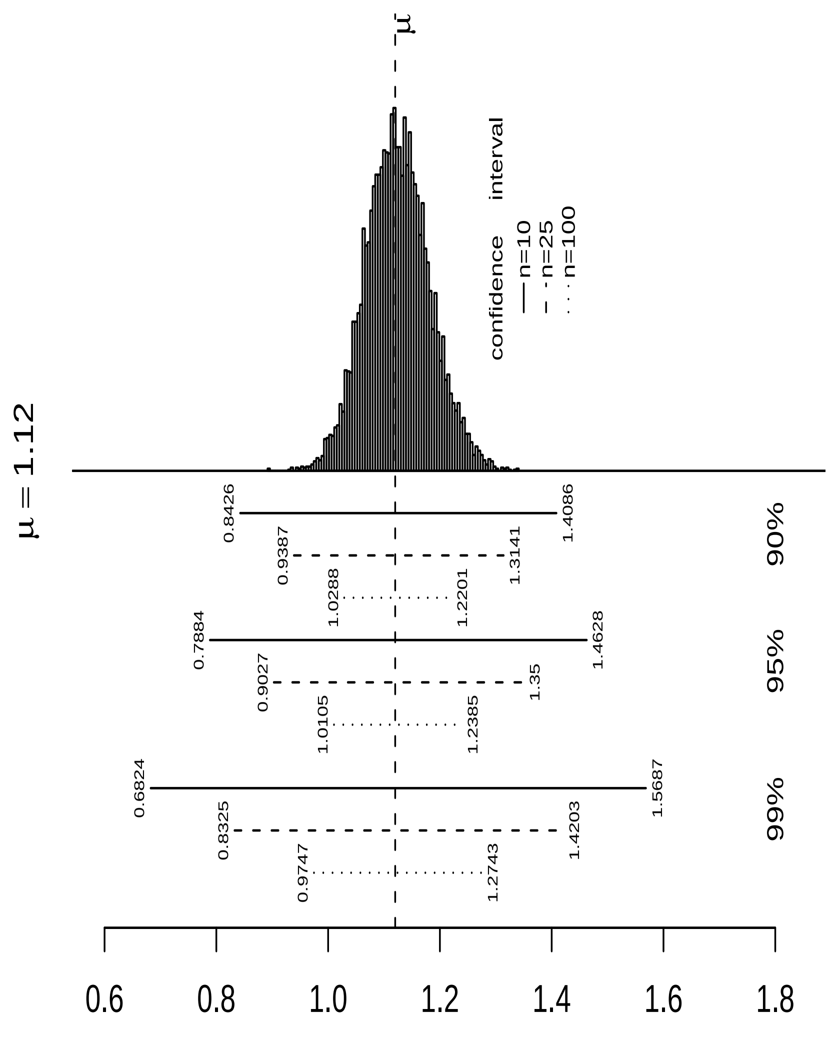

Figure 14: histograms of the ML estimates for $\mu = 1.12$ and different values of n

APPENDIX: ORTHOGONALITY OF THE PARAMETERS OF (P10)

To prove the orthogonality of the parameters ψ and β of (P10), without loss of generality, we can use the log-likelihood function for a single observation given by

$$\ell(\psi,\beta;t) = \log(f(t;\psi,\beta)) = k_1 - \frac{\psi t}{2\beta} - \frac{\psi\beta}{2t} + \psi + \log(t+\beta) + \frac{1}{2}\log(\psi) - \frac{1}{2}\log(\beta),$$

where k_1 is a constant that does not depend on ψ and β. Now, we need the partial derivatives of second order with respect to ψ and β given by

$$\frac{\partial^2\ell(\psi,\beta;t)}{\partial\psi\partial\beta} = \frac{t}{2\beta^2} - \frac{1}{2t}. \tag{5.1}$$

Now, we need to prove that the expected value of (5.1) is zero. Notice that $\mathrm{E}[T] = \beta + \beta/[2\psi]$. Since (P10) has the reciprocal property (A2), then $\mathrm{E}[1/T] = 1/\beta + 1/[2\beta\psi]$. Therefore, we have that

$$\begin{aligned}
\mathrm{E}\left[\frac{\partial^2\ell(\psi,\beta;t)}{\partial\psi\partial\beta}\right] &= \frac{1}{\beta^2}\left[\beta + \frac{\beta}{2\psi}\right] - \frac{1}{2}\left[\frac{1}{\beta} + \frac{1}{2\beta\psi}\right] \\
&= \frac{1}{2\beta} + \frac{1}{4\beta\psi} - \frac{1}{2\beta} - \frac{1}{4\beta\psi} \\
&= 0,
\end{aligned}$$

which proves that the parameters ψ and β are orthogonal.

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