Received: 23 July 2019 | Revised: 17 May 2020 | Accepted: 17 May 2020

DOI: 10.1002/asmb.2556

RESEARCH ARTICLE

WILEY

Birnbaum-Saunders quantile regression and its diagnostics with application to economic data

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Abstract

The Birnbaum-Saunders (BS) distribution is a model that frequently appears in the statistical literature and has proved to be very versatile and efficient across a wide range of applications. However, despite the growing interest in the study of the BS distribution, quantile regression modeling has not been considered for this distribution. To fill this gap, we introduce a class of quantile regression models based on the BS distribution, which allows us to describe positive and asymmetric data when a quantile must be predicted using covariates. We use an approach based on a quantile parameterization to generate the model, permitting us to consider a similar framework to generalized linear models, providing wide flexibility. The methodology proposed includes a thorough study of theoretical properties and practical issues, such as maximum likelihood parameter estimation and diagnostic analytics based on local influence and residuals. The performance of the residuals is evaluated by simulations, whereas an illustrative example of income data is conducted using the methodology to show its potential for applications. The numerical results report an adequate performance of the approach to quantile regression, indicating that the BS distribution is a good modeling choice when dealing with data that have both positive support and asymmetry. The economic implications of our investigation are discussed in the final section. Hence, it can be a valuable addition to the tool kit of applied statisticians and econometricians.

KEYWORDS

data analytics, GLM, local influence, maximum likelihood method, median regression, R software, residuals

1 | BIBLIOGRAPHICAL REVIEW AND MOTIVATING EXAMPLE

1.1 | Introduction

The Birnbaum-Saunders (BS) distribution constantly arises in the applied statistical literature. In the last decades, it has been shown to be versatile and efficient in several fields of science, being widely studied due to its theoretical justification, its good properties, and its close relation with the normal distribution. The BS model is unimodal, with asymmetry to the right and support defined on the positive real numbers, indexed by two parameters that control its shape and scale. This model is often considered as a life distribution due to its origins describing fatigue of materials subject to stress. Hence, the BS distribution assumes a prominent role in the areas of reliability and survival analysis, being a good alternative to standard distributions. For more details, see Birnbaum and Saunders,[1] Johnson et al[2(pp. 651-663)], and Leiva.[3] Although the BS distribution has its origins in physics and engineering, it has also received interest in several other areas, including, but no limited to, business, earth sciences, industry and medicine; for details, see References 4-20. The study of the BS distribution has received growing interest and a considerable amount of work is available; see the recent publications by Leiva,[3] Balakrishnan and Kundu,[21] and references therein, which summarize most of the works to the date. However, no quantile regression models based on the BS distribution have been derived.

Diagnostic analytics plays a relevant aspect in statistical modeling, which can be divided in global and local techniques. Residuals are well known and often used as measures of global influence and for detecting the model adequacy,[22,23] whereas the local influence technique is currently very popular. This technique allows us to evaluate the local effect of perturbations on the estimates of parameters and then to detect potentially influential cases in different models; see, for example, Santana et al[24] and Tapia et al.[25]

Following on from the above, the main objective of this work is to formulate quantile regression models based on the BS distribution and its diagnostics. We employ a quantile parameterization to generate the new model, which allows us to consider a framework as in generalized linear models, providing wide flexibility; see also Mitnik and Baek[26] and Noufaily and Jones[27] for similar parameterizations but not identical. Note that it is not possible to make comparison between our model and that proposed by Noufaily and Jones[27] because these models are postulated in different contexts. In any case, future research about this issue is mentioned in the final section.

1.2 | Motivating example in economy

In economic scenarios, as the behavior of household income, the data can follow a skew distribution and (as it is known) the mean is not a good central tendency measure to summarize the data, but, for example, the median is. Then, we are interested in studying a quantile of the distribution of the data.

A motivation to consider a BS quantile regression model comes from a real data set corresponding to Chilean household income in the year 2016, collected by the National Institute of Statistics, Chile, which are available at: http://www.ine.cl/estadisticas/ingresos-y-gastos/esi/base-de-datos.

For an illustrative purpose in order to show potential applications of our model, we focus on a data subset which considers $n = 100$ cases randomly selected from the full data set. In this subset of the data, the response variable (T) is the household income, whereas the covariates to be considered in our analysis are: the total number of persons in the home work force (X_1), the total income due to salaries (X_2), the total income due to independent work (X_3), the total income due to retirements (X_4), the total income due to pensions (X_5), and the total income due to public subsidy (X_6). These covariates were selected from the full data set (which contains 107 variables including T,X_1,… ,X_6) based on e economic and statistical criteria in relation to the response variable. All incomes are expressed in thousands of Chilean pesos; see http://www.bancocentral.cl for their equivalence in American dollars.

Table 1 provides a descriptive summary of the household income that includes sample median, mean, standard deviation (SD), coefficients of variation (CV), skewness (CS), and kurtosis (CK), as well as minimum ($t_{(1)}$) and maximum ($t_{(n)}$) values. Figure 1 shows the histogram as well as usual and adjusted box plots of the household income; for details of the adjusted box plot for asymmetric data, see Rousseeuw et al.[28] Also, Figure 2 displays scatter plots of household income and each one of the covariates.

Based on Table 1 and Figures 1 and 2, observe the following aspects. First, from the histogram shown in Figure 1A, note that the values of the household income have an empirical distribution that is unimodal and positively skewed, justifying the use of an asymmetric distribution for the response variable. Second, the box plots shown in Figure 1B show some atypical cases for the household income, which we analyze in Section 4. Third, from the scatter plots and correlations shown in Figure 2, linear relationships between some variables are detected as well as some evidence of nonconstant variance for T. Note that X_5 and X_6 have a low correlation with T, reason why we discard them in our

TABLE 1 Descriptive statistics for income data (in thousands of Chilean pesos)

Median	Mean	SD	CV	CS	CK	$t_{(1)}$	$t_{(n)}$	n
698.80	938.10	837.52	0.89	2.45	11.03	70	5369.90	100

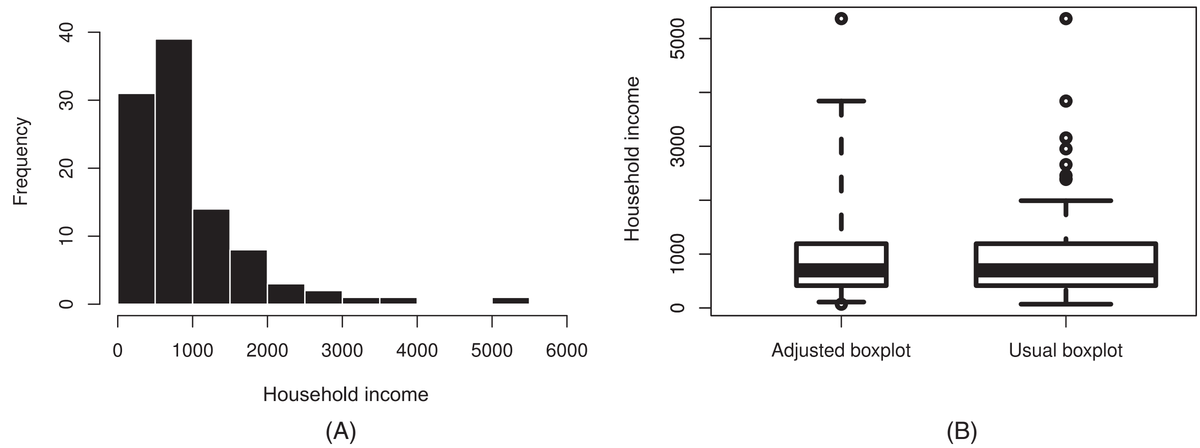


FIGURE 1 Histogram (A) and box plots (B) for income data

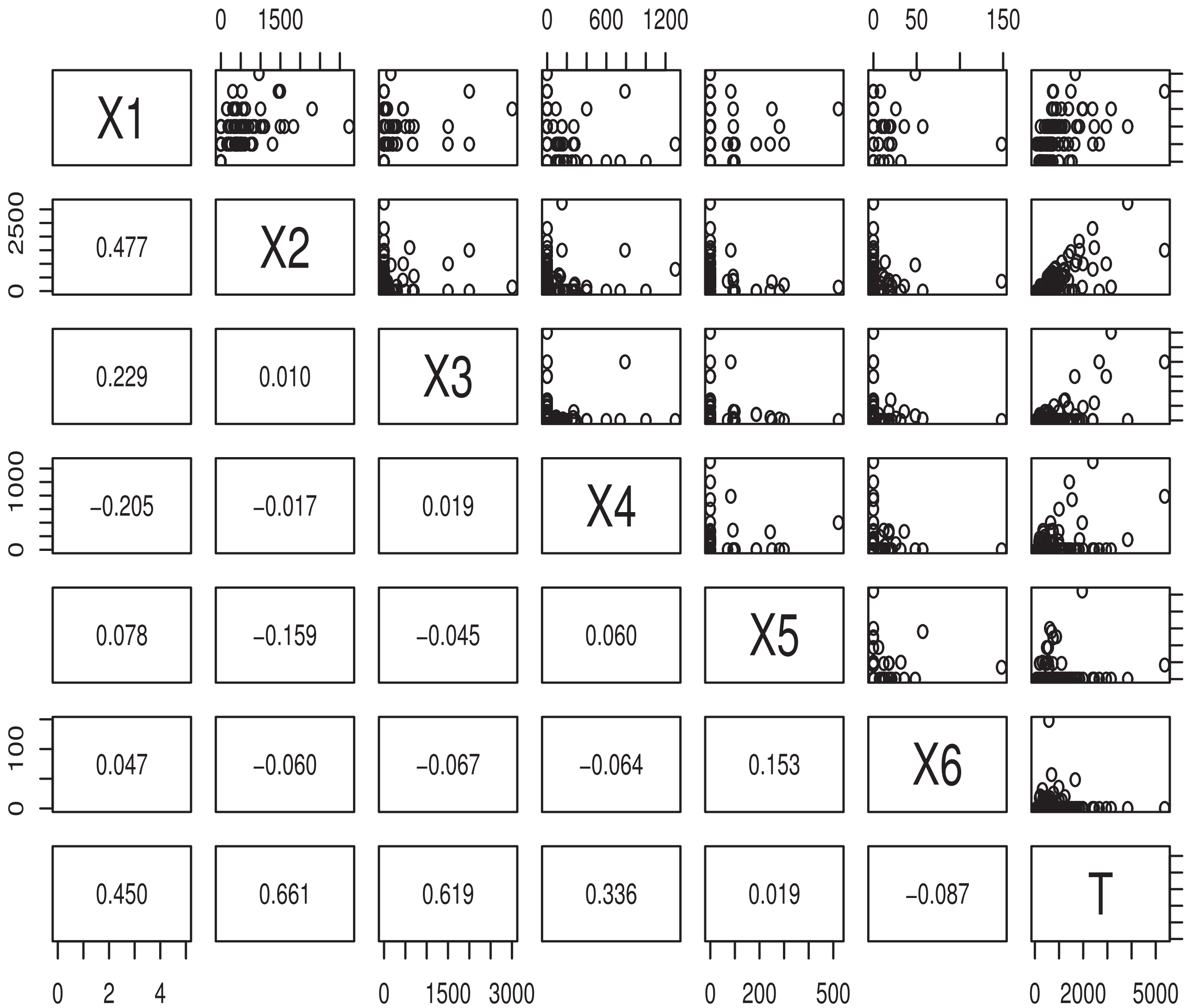


FIGURE 2 Scatter plots and correlations between each pair of variables for income data

illustrative data analysis. In addition, we detect a relatively high correlation between X_1 and X_2 so that we discard X_1 as well due to possible collinearity problems and also because it has less correlation with T than X_2. Therefore, we propose to employ X_2, X_3, and X_4 for illustrating the BS quantile regression model, which has characteristics and properties suitable for describing the median of the data, the nonconstant variance, and the asymmetry detected in these data.

1.3 | Organization of the article

The remainder of the article is organized as follows. Section 2 presents the BS distribution in its original parameterization and a new parameterization of it which allows us to model a quantile. In Section 3, we formulate the model and provide estimation based on the maximum likelihood (ML) method. In this section, we also derive diagnostic analytics using the local influence technique. Section 4 proposes four types of residuals for the BS quantile regression model and then we evaluate their performance by using Monte Carlo simulations. In this section, we also apply the obtained results to household income data, including formulation, estimation, inference, and diagnostic analytics to illustrate the potential of the new model. Section 5 discusses economic implications, concluding remarks, and future research.

2 | PRELIMINARIES

2.1 | The standard parametrization of the BS distribution

If $Z \sim \mathrm{N}(0,1)$, then the random variable T, defined as

$$T = \frac{\beta}{4}\left(\alpha Z + \sqrt{\alpha^2 Z^2 + 4}\right)^2, \tag{1}$$

follows a BS distribution with parameters of shape $\alpha > 0$ and scale $\beta > 0$, which is denoted by $T \sim \mathrm{BS}(\alpha,\beta)$. The random variable T has positive support and the transformation given in (1) is one-to-one, which allows us to state that

$$Z = \frac{1}{\alpha}\left(\sqrt{T/\beta} - \sqrt{\beta/T}\right) \sim \mathrm{N}(0,1).$$

The probability density and cumulative distribution functions of T are expressed as

$$f_T(t) = \frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\sqrt{\beta/t} + \sqrt{(\beta/t)^3}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right], \quad t > 0,$$
$$F_T(t) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{t/\beta} - \sqrt{\beta/t}\right)\right], \quad t > 0,$$

respectively, where Φ is the standard normal cumulative distribution function. Given $q \in (0,1)$ and based on (1), note that the $q \times 100$th quantile of the BS distribution is represented as

$$Q = t_q = \frac{\beta}{4}\left(\alpha z_q + \sqrt{\alpha^2 z_q^2 + 4}\right)^2, \quad q \in (0,1), \tag{2}$$

where z_q is the $q \times 100$th quantile of the standard normal distribution and $Q > 0$.

2.2 | A BS distribution parametrized by its quantiles

Let $q \in (0,1)$ be a fixed number and the transformation $(\alpha,\beta) \mapsto (\alpha,Q)$ be one-to-one, where Q is defined in (2). Then, we can state a parameterization of the BS model based on Q so that the associated cumulative distribution and probability density functions can be written, respectively, as

$$F(t) = \Phi\left[\frac{1}{\alpha\gamma_\alpha}\sqrt{\frac{4Q}{t}}\left(\frac{t\gamma_\alpha^2}{4Q} - 1\right)\right], \quad t > 0,$$

$$f_T(t) = \frac{1}{\alpha\gamma_\alpha\sqrt{8\pi Q t}}\left(\frac{\gamma_\alpha^2}{2} + \frac{2Q}{t}\right)\exp\left[-\frac{2Q}{\alpha^2\gamma_\alpha^2 t}\left(\frac{t\gamma_\alpha^2}{4Q} - 1\right)^2\right], \quad t > 0, \tag{3}$$

where $\gamma_\alpha = \alpha z_q + \sqrt{\alpha^2 z_q^2 + 4}$. Hence, under this parameterization, we write $T \sim \mathrm{BS}(\alpha,Q)$. The mean and variance of T are, respectively, given by

$$\mathrm{E}[T] = \frac{4Q}{\gamma_\alpha^2}\left(1 + \frac{\alpha^2}{2}\right), \quad \mathrm{Var}[T] = \frac{16Q^2\alpha^2}{\gamma_\alpha^4}\left(1 + \frac{5}{4}\alpha^2\right).$$

2.3 | Shape analysis

Figure 3 shows some shapes of the probability density function of $T \sim \mathrm{BS}(\alpha,Q)$ defined in (3). From Figure 3A,D,G, observe that the parameter α modifies the skewness and kurtosis of the model, as expected since it is a shape parameter. From

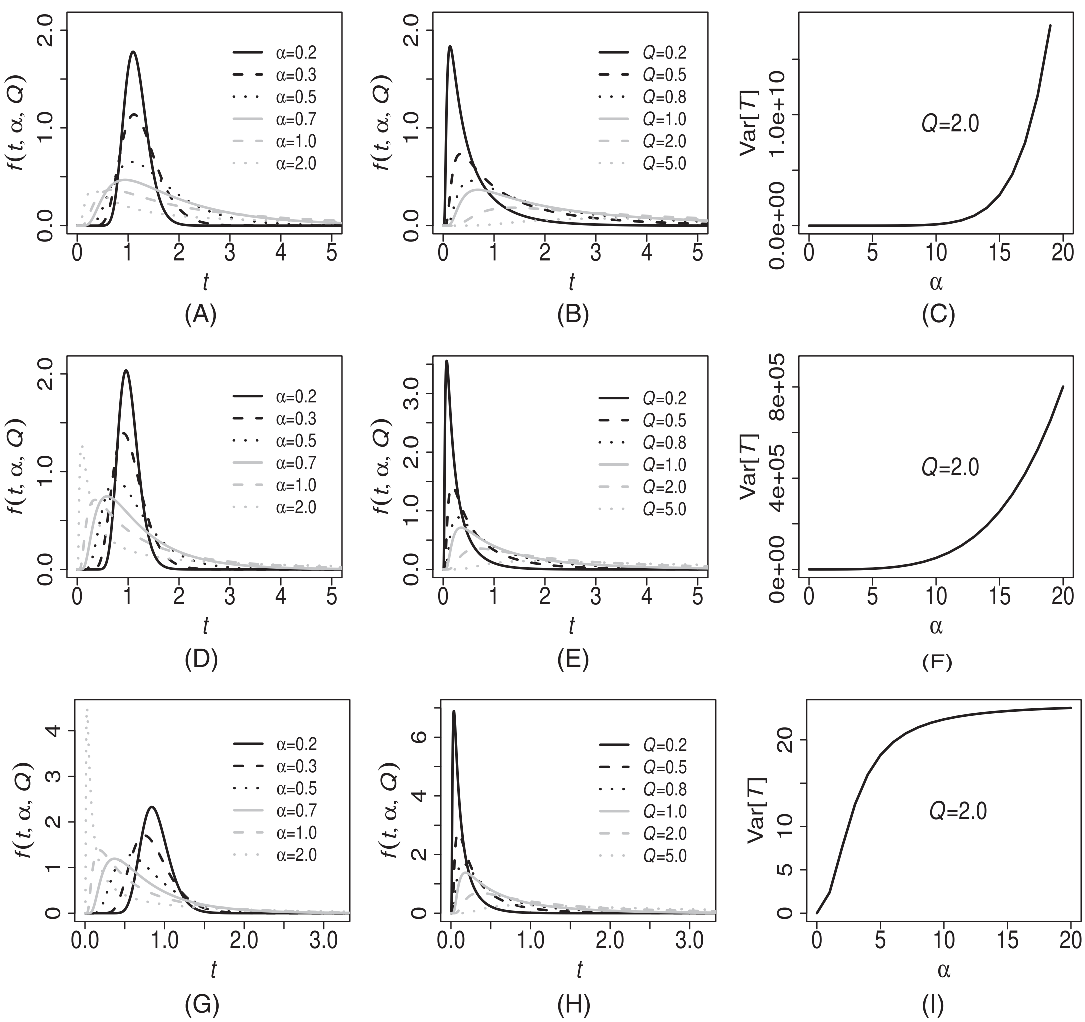


FIGURE 3 Plots of the BS probability density function for $Q = 1.0$ (left) and $\alpha = 1.0$ (center), with $q = 0.25$ (A) (B), $q = 0.50$ (D), (E), and $q = 0.75$ (G), (H); and of the BS variance against α for $Q = 2.0$ (right) with $q = 0.25$ (C), $q = 0.50$ (F), and $q = 0.75$ (I)

Figure 3B,E,H, note that, as Q increases, the kurtosis decreases, also as expected because it is a quantile parameter so that, as it increases, less kurtosis is detected. Also there is more concentration around the quantile as α decreases and therefore the variability decreases. Furthermore, notice that, when α increases, the variance increases exponentially for the first and second quartiles (see Figure 3C, F), whereas it increases in a controlled way for the third quartile (see Figure 3I).

3 | A BS QUANTILE REGRESSION MODEL

3.1 | Formulation and estimation

Let $T_1,\ldots,T_n$ be independent random variables, where $T_i \sim \text{BS}(\alpha,Q_i)$, for $i=1,\ldots,n$, and $\boldsymbol{t}=(t_1,\ldots,t_n)^\top$ be their associated observations. Then, we define a statistical model based on (3) by the systematic component

$$h(Q_i) = \eta_i = \boldsymbol{x}_i^\top \boldsymbol{\beta}, \quad i = 1, \ldots, n, \tag{4}$$

such that $Q_i = h^{-1}(\boldsymbol{x}_i^\top \boldsymbol{\beta})$, where $\boldsymbol{\beta}=(\beta_0,\beta_1,\ldots,\beta_{p-1})^\top$, for $p<n$, is a vector of unknown regression parameters to be estimated, and $\boldsymbol{x}_i^\top = (1, x_{i1}, \ldots, x_{i(p-1)})$ represents the values of p covariates. In the model defined from (4), the link function h is invertible, at least twice differentiable and it has positive support. Examples of link functions are $h(u) = \log_k(u)$ and $h(u) = \sqrt[a]{u}$, with a,k being positive integer numbers. Also, the link function $h(u)=u$ can be considered, that is, the identity function, with $\mathbb{R}^+$ as domain of h.

The log-likelihood function of the model given in (4) for $\boldsymbol{\theta}=(\boldsymbol{\beta}^\top,\alpha)^\top$ is $\ell(\boldsymbol{\theta};\boldsymbol{t}) = \ell(\boldsymbol{\theta}) = \sum_{i=1}^n \ell_i(Q_i,\alpha;t_i)$, where

$$\ell_i(Q_i,\alpha;t_i) = \ell_i(Q_i,\alpha) = -\frac{1}{2}\log(8\pi t_i) - \log(\alpha\gamma_\alpha) - \frac{1}{2}\log(Q_i) + \log\left(\frac{\gamma_\alpha^2}{2} + \frac{2Q_i}{t_i}\right) - \frac{2Q_i}{\alpha^2\gamma_\alpha^2 t_i}\left(\frac{t_i\gamma_\alpha^2}{4Q_i} - 1\right)^2. \tag{5}$$

The score functions for β_j, with $j=0,1,\ldots,p-1$, and α are, respectively, expressed as

$$\begin{aligned}
\dot{\ell}_{\beta_j} &= \frac{\partial \ell(\boldsymbol{\theta})}{\partial \beta_j} = \sum_{i=1}^n \underbrace{\left(-\frac{1}{2Q_i} - \frac{2}{\alpha^2\gamma_\alpha^2 t_i} + \frac{\gamma_\alpha^2 t_i}{8\alpha^2 Q_i^2} + \frac{4}{t_i\gamma_\alpha^2 + 4Q_i}\right)}_{z_i}\underbrace{\frac{1}{h'(Q_i)}}_{a_i} x_{ij}, \\
\dot{\ell}_\alpha &= \frac{\partial \ell(\boldsymbol{\theta})}{\partial \alpha} = \sum_{i=1}^n \underbrace{\left[-\frac{(\gamma_\alpha + \alpha\gamma_\alpha')}{\alpha\gamma_\alpha} + \frac{2t_i\gamma_\alpha\gamma_\alpha'}{t_i\gamma_\alpha^2 + 4Q_i} - \frac{(\gamma_\alpha\gamma_\alpha'\alpha - \gamma_\alpha^2)t_i}{4Q_i\alpha^3} - \frac{2}{\alpha^3} + \frac{4Q_i(\gamma_\alpha + \alpha\gamma_\alpha')}{(\alpha\gamma_\alpha)^3 t_i}\right]}_{b_i},
\end{aligned} \tag{6}$$

where h' is the derivative of h and γ_α' is the derivative of γ_α. Thus, we can write (6) in matrix form as

$$\dot{\boldsymbol{\ell}}_{\boldsymbol{\beta}} = (\dot{\ell}_{\beta_j}) = \boldsymbol{X}^\top \boldsymbol{D}(\boldsymbol{a})\boldsymbol{z}, \quad \dot{\ell}_\alpha = \text{trace}(\boldsymbol{D}(\boldsymbol{b})),$$

where $\boldsymbol{X}^\top = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$, with $\boldsymbol{x}_i$ being defined in (4), for $i=1,\ldots,n$, $\boldsymbol{z}=(z_1,\ldots,z_n)^\top$ and $\boldsymbol{D}$ denoting the diagonalization operator of a vector, such that $\boldsymbol{D}(\cdot)=\text{diag}(\cdot)$, with $\boldsymbol{a}=(a_1,\ldots,a_n)^\top$ and $\boldsymbol{b}=(b_1,\ldots,b_n)^\top$. Then, the score vector is $\dot{\boldsymbol{\ell}}_{\boldsymbol{\theta}} = (\dot{\boldsymbol{\ell}}_{\boldsymbol{\beta}}, \dot{\ell}_\alpha)^\top$.

3.2 | Information matrix and inference

The elements of the associated Hessian matrix are expressed as

$$\ddot{\ell}_{\beta_l\beta_j} = \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta_l \partial \beta_j} = \sum_{i=1}^n \underbrace{\left[\frac{\partial^2 \ell_i(Q_i,\alpha)}{\partial Q_i^2}\left(\frac{dQ_i}{d\eta_i}\right)^2 + \frac{\partial \ell_i(Q_i,\alpha)}{\partial Q_i}\left(\frac{\partial}{\partial Q_i}\frac{dQ_i}{d\eta_i}\right)\frac{dQ_i}{d\eta_i}\right]}_{c_i} x_{ij}x_{il},$$

where

$$\frac{\partial \ell_i(Q_i,\alpha)}{\partial Q_i} = z_i,\ \frac{\partial^2 \ell_i(Q_i,\alpha)}{\partial Q_i^2} = \frac{1}{2Q_i^2} - \frac{16}{(t_i\gamma_\alpha^2 + 4Q_i)^2} - \frac{\gamma_\alpha^2 t_i}{4\alpha^2 Q_i^3},\ \frac{\mathrm{d}Q_i}{\mathrm{d}\eta_i} = a_i,\ \frac{\partial}{\partial Q_i}\left(\frac{\mathrm{d}Q_i}{\mathrm{d}\eta_i}\right) = -\frac{h''(Q_i)}{(h'(Q_i))^2},$$

with h'' being the second derivative of h. Hence, we can group the expressions obtained in matrix form as $\ddot{\boldsymbol{\ell}}_{\beta\beta} = \boldsymbol{X}^\top \boldsymbol{D}(\boldsymbol{c})\boldsymbol{X}$, where $\boldsymbol{c} = (c_1,\ldots,c_n)^\top$. Furthermore, we have

$$\ddot{\ell}_{\beta_j\alpha} = \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta_j \partial \alpha} = \sum_{i=1}^{n} \underbrace{\left[-\frac{8t_i\gamma_\alpha\gamma_\alpha'}{(t_i\gamma_\alpha^2 + 4Q_i)^2} + \frac{(\gamma_\alpha\gamma_\alpha'\alpha - \gamma_\alpha^2)t_i}{4\alpha^3 Q_i^2} + \frac{4(\gamma_\alpha + \alpha\gamma_\alpha')}{(\alpha\gamma_\alpha)^3 t_i}\right]}_{m_i} a_i x_{ij},$$

which can be represented in matrix form as $\ddot{\boldsymbol{\ell}}_{\beta\alpha} = \boldsymbol{X}^\top \boldsymbol{D}(\boldsymbol{a})\,\boldsymbol{m}$, where $\boldsymbol{m} = (m_1,\ldots,m_n)$. In addition, we get

$$\ddot{\ell}_{\alpha\alpha} = \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha^2} = \sum_{i=1}^{n} \underbrace{\left\{\mathcal{B}_i \frac{t_i}{(t_i\gamma_\alpha^2 + 4Q_i)^2} + 2[\gamma_\alpha^3\gamma_\alpha'' - \gamma_\alpha^2(\gamma_\alpha')^2]\left(\frac{t_i}{t_i\gamma_\alpha^2 + 4Q_i}\right)^2 - \mathcal{C}_i t_i + \frac{6}{\alpha^4} + \mathcal{D}_i\frac{1}{t_i} - \mathcal{A}\right\}}_{d_i},$$

where γ_α'' is the second derivative of γ_α and

$$\begin{aligned}
\mathcal{A} &= \frac{(2\gamma_\alpha' + \alpha\gamma_\alpha'')(\alpha\gamma_\alpha) - (\gamma_\alpha + \alpha\gamma_\alpha')^2}{(\alpha\gamma_\alpha)^2},\\
\mathcal{B}_i &= 8Q_i[(\gamma_\alpha')^2 + \gamma_\alpha\gamma_\alpha''],\\
\mathcal{C}_i &= \frac{1}{4Q_i}\left[\frac{\alpha^2(\gamma_\alpha')^2 + \alpha^2\gamma_\alpha\gamma_\alpha'' - \alpha\gamma_\alpha\gamma_\alpha' - 3\gamma_\alpha\gamma_\alpha'\alpha + 3\gamma_\alpha^2}{\alpha^4}\right],\\
\mathcal{D}_i &= 4Q_i\left[\frac{(2\gamma_\alpha' + \alpha\gamma_\alpha'')(\alpha\gamma_\alpha) - 3(\gamma_\alpha + \alpha\gamma_\alpha')^2}{(\alpha\gamma_\alpha)^4}\right].
\end{aligned}$$

In matrix notation, we can write $\ddot{\ell}_{\alpha\alpha} = \mathrm{trace}(\boldsymbol{D}(\boldsymbol{d}))$, where $\boldsymbol{d} = (d_1,\ldots,d_n)^\top$.

The associated expected Fisher information $\boldsymbol{K}_{\theta\theta} = \mathrm{E}[-\ddot{\boldsymbol{\ell}}_{\theta\theta}]$ can be expressed in form matrix as

$$\boldsymbol{K}_{\theta\theta} = \begin{pmatrix} \boldsymbol{K}_{\beta\beta} & \boldsymbol{K}_{\beta\alpha} \\ \boldsymbol{K}_{\alpha\beta} & K_{\alpha\alpha} \end{pmatrix},$$

where $\boldsymbol{K}_{\beta\beta} = \boldsymbol{X}^\top \boldsymbol{D}(\boldsymbol{v})\,\boldsymbol{X}$, with $\boldsymbol{v} = (v_1,\ldots,v_n)^\top$, whose elements are

$$v_i = \left[16V_i(\boldsymbol{\theta}) - \frac{1}{2Q_i^2} + \frac{1}{\alpha^2 Q_i^2}\left(1 + \frac{\alpha^2}{2}\right)\right]\frac{1}{[h'(Q_i)]^2} - \left[\frac{1}{2Q_i} + \frac{1}{2\alpha^2 Q_i}\left(1 + \frac{\alpha^2}{2}\right) - 4W_i(\boldsymbol{\theta})\right]\frac{h''(Q_i)}{[h(Q_i)]^3}; \tag{7}$$

$\boldsymbol{K}_{\beta\alpha} = \boldsymbol{K}_{\alpha\beta}^\top = \boldsymbol{X}^\top \boldsymbol{D}(\boldsymbol{a})\,\boldsymbol{s}$, with $\boldsymbol{s} = (s_1,\ldots,s_n)$, whose elements are

$$s_i = 8\gamma_\alpha\gamma_\alpha' U_i(\boldsymbol{\theta}) - \frac{(\gamma_\alpha\gamma_\alpha'\alpha - \gamma_\alpha^2)}{\alpha^3\gamma_\alpha^2 Q_i^2}\left(1 + \frac{\alpha^2}{2}\right) - \frac{(\gamma_\alpha + \alpha\gamma_\alpha')}{\alpha^3\gamma_\alpha Q_i}\left(1 + \frac{\alpha^2}{2}\right);$$

and $K_{\alpha\alpha} = \mathrm{trace}(\boldsymbol{D}(\boldsymbol{u}))$, for $\boldsymbol{u} = (u_1,\ldots,u_n)^\top$, whose elements are

$$u_i = \mathcal{A} - \mathcal{B}_i U_i(\boldsymbol{\theta}) - 2[\gamma_\alpha^3\gamma_\alpha'' - \gamma_\alpha^2(\gamma_\alpha')^2]S_i(\boldsymbol{\theta}) + \mathcal{C}_i\frac{4Q_i}{\gamma_\alpha^2}\left(1 + \frac{\alpha^2}{2}\right) - \frac{6}{\alpha^4} - \mathcal{D}_i\frac{\gamma_\alpha^2}{4Q_i}\left(1 + \frac{\alpha^2}{2}\right),$$

with

$$S_i(\boldsymbol{\theta}) = \int_0^\infty \left(\frac{t}{t\gamma_\alpha^2 + 4Q_i}\right)^2 f_{T_i}(t)\,\mathrm{d}t,$$
$$U_i(\boldsymbol{\theta}) = \int_0^\infty \frac{t}{(t\gamma_\alpha^2 + 4Q_i)^2} f_{T_i}(t)\,\mathrm{d}t,$$
$$V_i(\boldsymbol{\theta}) = \int_0^\infty \frac{1}{(t\gamma_\alpha^2 + 4Q_i)^2} f_{T_i}(t)\,\mathrm{d}t,$$
$$W_i(\boldsymbol{\theta}) = \int_0^\infty \frac{1}{t\gamma_\alpha^2 + 4Q_i} f_{T_i}(t)\,\mathrm{d}t.$$

To estimate the model parameters by the ML method, we solve the equations $\dot{\boldsymbol{\ell}} = \mathbf{0}$. However, no closed-form expressions for the ML estimates are available. Following the definitions in Leiva et al[5] and Santos-Neto et al,[29] we can write the iterative algorithm

$$\boldsymbol{\theta}^{(m+1)} = (\tilde{\boldsymbol{X}}^\top \tilde{\boldsymbol{W}}^{(m)} \tilde{\boldsymbol{X}})^{-1} \tilde{\boldsymbol{X}}^\top \tilde{\boldsymbol{W}}^{(m)} \boldsymbol{z}^{*(m)}, \quad m = 0, 1, 2, \dots,$$

which uses the Fisher scoring method, where

$$\tilde{\boldsymbol{X}} = \begin{pmatrix} \boldsymbol{X} & 0 \\ 0 & 1 \end{pmatrix},$$
$$\tilde{\boldsymbol{W}} = \begin{pmatrix} \boldsymbol{D}(\boldsymbol{v}) & \boldsymbol{D}(\boldsymbol{a})\boldsymbol{s} \\ \boldsymbol{s}^\top \boldsymbol{D}(\boldsymbol{a}) & \mathrm{trace}(\boldsymbol{D}(\boldsymbol{u})) \end{pmatrix},$$
$$\boldsymbol{z}^{*(m)} = \tilde{\boldsymbol{X}}\boldsymbol{\theta}^{(m)} + (\tilde{\boldsymbol{W}}^{(m)})^{-1} \begin{pmatrix} \boldsymbol{D}(\boldsymbol{a})^{(m)} & 0 \\ 0 & \mathrm{trace}(\boldsymbol{D}(\boldsymbol{b}))^{(m)} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}^{(m)} \\ 1 \end{pmatrix}.$$

Under usual regularity conditions, the asymptotic distribution of $\hat{\boldsymbol{\theta}}$ is $\hat{\boldsymbol{\theta}} \dot{\sim} \mathrm{N}_{p+1}(\boldsymbol{\theta}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})$, where $\dot{\sim}$ means "approximately distributed" and $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ is the asymptotic variance-covariance matrix of $\hat{\boldsymbol{\theta}}$; see Cox and Hinkley.[30] This matrix can be obtained by using the inverse expected Fisher information matrix $\boldsymbol{K}_{\boldsymbol{\theta}\boldsymbol{\theta}}^{-1}$ and estimated by replacing $\hat{\boldsymbol{\theta}}$ with $\boldsymbol{\theta}$. Thus, an approximate $100\times(1-\xi)\%$ confidence region for $\boldsymbol{\theta}$ is defined by $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^\top \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}^{-1} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \leq \chi^2_{1-\xi}(p+1)$, for $\boldsymbol{\theta}$ in $\mathbb{R}^{p+1}$, where $\chi^2_{1-\xi}(p+1)$ is the $(1-\xi)\times 100$th quantile of the chi-squared distribution with p+1 degrees of freedom and $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}$ is a consistent estimator of $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$. Therefore, it is possible to construct asymptotic $100\times(1-\xi)\%$ confidence bands for the linear predictor $Q(\boldsymbol{x}_{\mathrm{pred}}) = h^{-1}(\boldsymbol{x}_{\mathrm{pred}}^\top \boldsymbol{\beta})$, $\forall\, \boldsymbol{x}_{\mathrm{pred}} \in \mathbb{R}^p$, where $\boldsymbol{x}_{\mathrm{pred}}$ is an arbitrary $p\times 1$ vector. Note that the asymptotic distribution of $\hat{\boldsymbol{\beta}}$ is given by $\hat{\boldsymbol{\beta}} \dot{\sim} \mathrm{N}_p(\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\boldsymbol{\beta}})$, where $\boldsymbol{\Sigma}_{\boldsymbol{\beta}}$ is the asymptotic variance-covariance matrix of $\hat{\boldsymbol{\beta}}$, which can be obtained appropriately from $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$. Then, an approximate $100\times(1-\xi)\%$ confidence region for $Q(\boldsymbol{x}_{\mathrm{pred}})$ is expressed as

$$\left\{ h^{-1}\left[\boldsymbol{x}_{\mathrm{pred}}^\top \hat{\boldsymbol{\beta}} - \sqrt{\chi^2_{1-\xi}(p)}\left(\boldsymbol{x}_{\mathrm{pred}}^\top \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}}\, \boldsymbol{x}_{\mathrm{pred}}\right)^{1/2}\right],\quad h^{-1}\left[\boldsymbol{x}_{\mathrm{pred}}^\top \hat{\boldsymbol{\beta}} + \sqrt{\chi^2_{1-\xi}(p)}\left(\boldsymbol{x}_{\mathrm{pred}}^\top \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}}\, \boldsymbol{x}_{\mathrm{pred}}\right)^{1/2}\right] \right\},$$

where $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}} = (\boldsymbol{X}^\top \hat{\boldsymbol{V}} \boldsymbol{X})^{-1}$, with $\hat{\boldsymbol{V}} = \boldsymbol{D}(\hat{\boldsymbol{v}}) - \boldsymbol{D}(\hat{\boldsymbol{a}})\hat{\boldsymbol{s}}[\mathrm{trace}(\boldsymbol{D}(\hat{\boldsymbol{u}}))]^{-1}\hat{\boldsymbol{s}}^\top \boldsymbol{D}(\hat{\boldsymbol{a}})$, for $\boldsymbol{x}_{\mathrm{pred}} \in \mathbb{R}^p$ and $0 < \xi < 1$.

3.3 | Local influence analytics

The likelihood distance (LD) is defined by $\mathrm{LD}(\boldsymbol{\omega}) = 2[\ell(\hat{\boldsymbol{\theta}}) - \ell(\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}})]$, where $\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}}$ is the ML estimate of $\boldsymbol{\theta}$ for a perturbed model and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)^\top$ is a perturbation vector. Cook[31] studied the local behavior of $\mathrm{LD}(\boldsymbol{\omega})$ around the nonperturbed vector $\boldsymbol{\omega}_0$, such that $\mathrm{LD}(\boldsymbol{\omega}_0) = 0$. The normal curvature for $\hat{\boldsymbol{\theta}}$ at the direction $\boldsymbol{l}$, with $\|\boldsymbol{l}\| = 1$, is defined as $C_{\boldsymbol{l}}(\hat{\boldsymbol{\theta}}) = 2|\boldsymbol{l}^\top \boldsymbol{\Delta}^\top \ddot{\boldsymbol{\ell}}_{\hat{\theta}\hat{\theta}}^{-1} \boldsymbol{\Delta} \boldsymbol{l}|$, where $\ddot{\boldsymbol{\ell}}_{\hat{\theta}\hat{\theta}}$ is the Hessian matrix of $\boldsymbol{\ell}(\boldsymbol{\theta})$ evaluated at $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\Delta}$ is a $(p+1)\times n$ perturbation matrix also evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}, \boldsymbol{\omega} = \boldsymbol{\omega}_0$. Hence, its elements are given by

$$\Delta_{ij} = \left.\frac{\partial^2 \ell_{\boldsymbol{\omega}}(\boldsymbol{\theta})}{\partial\theta_i \partial\omega_j}\right|_{\theta=\hat{\theta},\ \omega=\omega_0}, \quad i = 0, 1, \dots, p, \quad j = 1, \dots, n,$$

with $\ell_{\boldsymbol{\omega}}(\boldsymbol{\theta})$ being the log-likelihood function associated with the model perturbed by $\boldsymbol{\omega}$. For the model defined in (4), the elements of $\ddot{\boldsymbol{\ell}}_{\theta\theta}$ are $\ddot{\boldsymbol{\ell}}_{\beta\beta} = \boldsymbol{X}^\top \boldsymbol{D}(\boldsymbol{c})\boldsymbol{X}$, $\ddot{\boldsymbol{\ell}}_{\beta\alpha} = \ddot{\boldsymbol{\ell}}_{\alpha\beta} = \boldsymbol{X}^\top \boldsymbol{D}(\boldsymbol{a})\boldsymbol{m}$, and $\ddot{\ell}_{\alpha\alpha} = \text{trace}(\boldsymbol{D}(\boldsymbol{d}))$. We consider the direction $\boldsymbol{l}_{max}$ as the eigenvector associated with the largest eigenvalue of the matrix

$$\ddot{\boldsymbol{F}} = -\boldsymbol{\Delta}^\top \ddot{\boldsymbol{\ell}}_{\hat{\theta}\hat{\theta}}^{-1} \boldsymbol{\Delta}. \tag{8}$$

The index plot of $\boldsymbol{l}_{max}$ may be considered to detect cases that are potentially influential on $\hat{\boldsymbol{\theta}}$. If our interest is only on the vector $\hat{\boldsymbol{\beta}}$, then the normal curvature at the direction $\boldsymbol{l}$ is given by $C_{\boldsymbol{l}}(\hat{\boldsymbol{\beta}}) = 2|\boldsymbol{l}^\top \boldsymbol{\Delta}^\top [\ddot{\boldsymbol{\ell}}_{\hat{\theta}\hat{\theta}}^{-1} - \ddot{\boldsymbol{\ell}}_1]\boldsymbol{\Delta}\boldsymbol{l}|$, where the $(p+1)\times(p+1)$ matrix $\ddot{\boldsymbol{\ell}}_1$ is given by

$$\ddot{\boldsymbol{\ell}}_1 = \begin{pmatrix} \boldsymbol{0} & 0 \\ \boldsymbol{0} & \ddot{\ell}_{\hat{\alpha}\hat{\alpha}}^{-1} \end{pmatrix}.$$

To study local influence on $\hat{\alpha}$, the normal curvature in the direction of $\boldsymbol{l}$ is defined by $C_{\boldsymbol{l}}(\hat{\alpha}) = 2|\boldsymbol{l}^\top \boldsymbol{\Delta}^\top (\ddot{\boldsymbol{\ell}}_{\hat{\theta}\hat{\theta}}^{-1} - \ddot{\boldsymbol{\ell}}_2)\boldsymbol{\Delta}\boldsymbol{l}|$, where the $(p+1)\times(p+1)$ matrix $\ddot{\boldsymbol{\ell}}_2$ is expressed as

$$\ddot{\boldsymbol{\ell}}_2 = \begin{pmatrix} \ddot{\boldsymbol{\ell}}_{\hat{\beta}\hat{\beta}}^{-1} & 0 \\ \boldsymbol{0} & 0 \end{pmatrix}.$$

The vector $\boldsymbol{l} = \boldsymbol{e}_{in}$, where $\boldsymbol{e}_{in}$ is an $n \times 1$ vector of zeros, with one at the ith position, is other relevant direction. In that case, the normal curvature, called total local influence of the case i, is calculated by $C_i = 2|\boldsymbol{e}_{in}\ddot{\boldsymbol{F}}\boldsymbol{e}_{in}| = 2|\ddot{F}_{ii}|$, where $\ddot{F}_{ii}$ is the ith diagonal element of $\ddot{\boldsymbol{F}}$ defined in (8). Lesaffre and Verbeke[32] proposed to pay attention to those cases with $C_i > 2\,\overline{C}$, where $\overline{C} = \sum_{i=1}^n C_i/n$. Next, we specify the matrix $\boldsymbol{\Delta}$ for usual perturbation schemes.

3.3.1 | Case-weight perturbation

Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)^\top$ be a weight vector. In this case, the perturbed log-likelihood function is defined by $\ell_{\boldsymbol{\omega}}(\boldsymbol{\theta}) = \sum_{i=1}^n \omega_i \ell_i(Q_i, \alpha)$, where $\ell_i(Q_i, \alpha)$ is given in (5), with $0 \le \omega_i \le 1$, for $i = 1, \ldots, n$. Hence, the perturbation matrix is expressed as $\boldsymbol{\Delta} = [\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_n]$, where for a_i, b_i, and z_i given in Subsection 3.1, with

$$\boldsymbol{\delta}_i = \begin{pmatrix} \boldsymbol{x}_i a_i z_i \\ b_i \end{pmatrix}, \quad i = 1, \ldots, n.$$

Here, $\boldsymbol{\Delta}$ must be evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0 = (1, \ldots, 1)^\top$.

3.3.2 | Response perturbation

We consider now an additive perturbation on the response i by making $t_i(\omega_i) = t_i + \omega_i s_{T_i}$, where $\omega_i \in \mathbb{R}$ and s_{T_i} is a scale factor often represented by the sample SD of T, for $i = 1, \ldots, n$. Then, under the scheme of response perturbation, the log-likelihood function is given by $\ell_{\boldsymbol{\omega}}(\boldsymbol{\theta}) = \sum_{i=1}^n \ell_{\omega_i}(Q_i, \alpha)$, where

$$\begin{aligned} \ell_{\omega_i}(Q_i, \alpha) = & -\frac{1}{2}\log[8\pi t_i(\omega_i)] - \log(\alpha\gamma_\alpha) - \frac{1}{2}\log(Q_i) \\ & + \log\left[\frac{\gamma_\alpha^2}{2} + \frac{2Q_i}{t_i(\omega_i)}\right] - \frac{2Q_i}{\alpha^2\gamma_\alpha^2 t_i(\omega_i)}\left[\frac{t_i(\omega_i)\gamma_\alpha^2}{4Q_i} - 1\right]^2. \end{aligned}$$

Thus, the column vectors of the matrix $\boldsymbol{\Delta}$ are now expressed as

$$\boldsymbol{\delta}_i = \begin{pmatrix} \boldsymbol{x}_i a_i \psi_i \vartheta_i \\ \tau_i \vartheta_i \end{pmatrix}, \quad i = 1, \ldots, n,$$

with $\mathbf{\Delta}$ being evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0 = (0,\ldots,0)^\top$ so that

$$\hat{\psi}_i = \frac{2}{\hat{\alpha}^2\gamma_{\hat{\alpha}}^2 t_i^2} + \frac{\gamma_{\hat{\alpha}}^2}{8\hat{\alpha}^2\hat{Q}_i^2} - \frac{4\gamma_{\hat{\alpha}}^2}{(t_i\gamma_{\hat{\alpha}}^2 + 4\hat{Q}_i)^2},$$

$$\hat{\tau}_i = 2\gamma_{\hat{\alpha}}\gamma'_{\hat{\alpha}}\left[\frac{t_i\gamma_{\hat{\alpha}}^2 + 4\hat{Q}_i - \gamma_{\hat{\alpha}}^2 t_i}{(t_i\gamma_{\hat{\alpha}}^2 + 4\hat{Q}_i)^2}\right] - \frac{\gamma_{\hat{\alpha}}\gamma'_{\hat{\alpha}}\hat{\alpha} - \gamma_{\hat{\alpha}}^2}{4\hat{Q}_i\hat{\alpha}^3} - \frac{4\hat{Q}_i(\gamma_{\hat{\alpha}} + \hat{\alpha}\gamma'_{\hat{\alpha}})}{(\hat{\alpha}\gamma_{\hat{\alpha}})^3 t_i^2},$$

$$\hat{\vartheta}_i = s_{T_i},$$

where $\gamma_{\hat{\alpha}}$ is γ_α evaluated at $\hat{\alpha}$.

3.3.3 | Perturbation of a continuous covariate

Consider now an additive perturbation on a particular continuous covariate, namely, x_t, for $t = 1,\ldots,p-1$, by making $x_{ti}(\omega_i) = x_{ti} + \omega_i s_{X_t}$, where s_{X_t} is a scale factor, which can be the sample SD of X_t, and $\omega_i \in \mathbb{R}$, for $i = 1,\ldots,n$. Then, under the scheme of covariate perturbation, the log-likelihood function is given by $\ell_{\boldsymbol{\omega}}(\boldsymbol{\theta}) = \sum_{i=1}^n \ell_{\omega_i}(Q_i, \alpha)$, where

$$\ell_{\omega_i}(Q_i, \alpha) = -\frac{1}{2}\log(8\pi t_i) - \log(\alpha\gamma_\alpha) - \frac{1}{2}\log[Q_i(\omega_i)] + \log\left[\frac{\gamma_\alpha^2}{2} + \frac{2Q_i(\omega_i)}{t_i}\right] - \frac{2Q_i(\omega_i)}{\alpha^2\gamma_\alpha^2 t_i}\left[\frac{t_i\gamma_\alpha^2}{4Q_i(\omega_i)} - 1\right]^2,$$

with $Q_i(\omega_i) = h^{-1}[\boldsymbol{x}_i^\top(\omega_i)\boldsymbol{\beta}]$ and $\boldsymbol{x}_i^\top(\omega_i) = [1, x_{i1}, \ldots, x_{ti}(\omega_i), \ldots, x_{i(p-1)}]^\top$. Hence, the perturbation matrix assumes the form

$$\boldsymbol{\Delta} = \begin{pmatrix} \boldsymbol{\Delta}_\beta \\ \boldsymbol{\Delta}_\alpha \end{pmatrix},$$

where $\boldsymbol{\Delta} = (\Delta_{\beta_{ij}})$ is a $p \times n$ matrix with elements, when $j \neq t$, defined as

$$\Delta_{\beta_{ij}} = s_X\beta_t a'_i x_{ij} q_i + s_X\beta_t x_{ij} a_i^2\left[\frac{1}{2Q_i^2} - \frac{16}{(t_i\gamma_\alpha^2 + 4Q_i)^2} - \frac{\gamma_\alpha^2 t_i}{4\alpha^2 Q_i^3}\right],$$

with $q_i = z_i$, whereas, when $j = t$, it is given by

$$\Delta_{\beta_{it}} = s_X a_i q_i + s_X\beta_t a'_i x_{it} q_i + s_X\beta_t x_{it} a_i^2\left[\frac{1}{2Q_i^2} - \frac{16}{(t_i\gamma_\alpha^2 + 4Q_i)^2} - \frac{\gamma_\alpha^2 t_i}{4\alpha^2 Q_i^3}\right],$$

where a'_i is the derivative of a_i defined in (6). In addition, $\mathbf{\Delta}_\alpha = (\zeta_1,\ldots,\zeta_n)$, with $\zeta_i = s_t\beta_t a_i m_i$. Note that here $\mathbf{\Delta}$ must be evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0 = (0,\ldots,0)^\top$.

3.3.4 | Perturbation of the parameter α

Here α is perturbed as $\alpha_i = \alpha/\omega_i$, with $\omega_i > 0$ and then the perturbed log-likelihood function is $\ell_{\boldsymbol{\omega}}(\boldsymbol{\theta}) = \sum_{i=1}^n \ell_{\omega_i}(Q_i, \alpha_i)$, where

$$\ell_{\omega_i}(Q_i, \alpha_i) = -\frac{1}{2}\log(8\pi t_i) - \log(\alpha_i\gamma_{\alpha_i}) - \frac{1}{2}\log(Q_i) + \log\left(\frac{\gamma_{\alpha_i}^2}{2} + \frac{2Q_i}{t_i}\right) - \frac{2Q_i}{\alpha_i^2\gamma_{\alpha_i}^2 t_i}\left(\frac{t_i\gamma_{\alpha_i}^2}{4Q_i} - 1\right)^2, \quad i = 1,\ldots,n.$$

Hence, the column vectors of $\boldsymbol{\Delta}$ are expressed as

$$\boldsymbol{\delta}_i = \begin{pmatrix} \boldsymbol{x}_i a_i \varpi_i \\ \varphi_i \end{pmatrix},$$

where $\boldsymbol{\varpi} = (\varpi_1, \dots, \varpi_n)^\top$ and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)^\top$, with $\varpi_i = -\alpha\, m_i$ and $\varphi_i = -d_i\, \alpha - b_i$, for $i = 1, \dots, n$, with $\boldsymbol{\Delta}$ being now evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0 = (1, \dots, 1)^\top$.

4 | EMPIRICAL ILLUSTRATION

4.1 | Computational framework

For purposes of simulation and estimation based on the proposed model, we use the `R` software, which allows us to conduct statistical analyses and graphs, with its open source code. This software can be downloaded from http://www.r-project.org. Our methodology was implemented in `R` and its codes are available upon request. Some `R` packages employed in this implementation are:

(i) `maxLik` package: This offers a set of functions for ML estimation and nonlinear optimization, as well as related tools. We use the `maxBFGS` function of this `R` package to maximize the log-likelihood function and to obtain the estimates of parameters.
(ii) `VGAM` package: This contains the functions `dbisa` and `rbisa`, which are useful for calculating Pearson, deviance, likelihood, and quantile residuals as well as for generating BS random numbers, respectively.

The time spent to obtain the descriptive statistics of the four residuals for each link function and values of α is close to 9 minutes in a personal computer Lenovo Ideapad 300-14ISK, Intel core i7-6500U CPU, 2.50 GHz, RAM 3.7 GB. The routines implemented worked satisfactorily.

4.2 | Simulation study

Note that an important aspect to be considered in all regression is the use of residuals to evaluate the model adequacy. Upon nonnormality, the identification of a suitable residual is not an easy task. We consider one of the possible simulation studies for our model based on the residuals. Comments on other possible simulation studies regarding our model are mentioned in the final section about "Limitations and future studies." Also, note that we are assessing the adequacy of these residuals to our BS quantile regression model and not in a general context. Therefore, this study is relevant and necessary to our model and not to others. Thus, in order to evaluate the fit of our model to a data set, we consider the four following types of residuals proposed in the literature.

4.2.1 | Pearson-type residual

First, we utilize a modification of the standardized Pearson residual given by

$$r_i^{(1)} = \frac{t_i - \hat{Q}_i}{\sqrt{(1-h_i)\widehat{\mathrm{Var}}[T_i]}} = \frac{\hat{\gamma}_\alpha^2 (t_i - \hat{Q}_i)}{4\hat{Q}_i \hat{\alpha} \sqrt{(1-h_i)(1+5\hat{\alpha}^2/4)}}, \quad i = 1, \dots, n, \tag{9}$$

where α, Q_i are defined in (3), (4), respectively, and h_i is the ith element of the matrix $\boldsymbol{H} = \boldsymbol{D}(\hat{\boldsymbol{v}})^{1/2}\boldsymbol{X}[\boldsymbol{X}^\top \boldsymbol{D}(\hat{\boldsymbol{v}})\boldsymbol{X}]^{-1}\boldsymbol{X}^\top \boldsymbol{D}(\hat{\boldsymbol{v}})^{1/2}$, which is equivalent to the hat matrix of regression but for generalized linear models.

4.2.2 | Deviance-type residual

Second, we consider a deviance type residual, replacing the mean by the quantile, expressed as

$$r_i^{(2)} = \frac{\mathrm{sgn}(t_i - \hat{Q}_i)\sqrt{D_i}}{\sqrt{1-h_i}}, \quad i = 1, \ldots, n, \tag{10}$$

where $D_i = 2[\ell_i(\tilde{\boldsymbol{\theta}}) - \ell_i(\hat{\boldsymbol{\theta}})]$, with $\tilde{\boldsymbol{\theta}}$ being the ML estimate of $\boldsymbol{\theta}$ under the saturated model (with n parameters), $\hat{\theta}$ is the ML estimate of $\boldsymbol{\theta}$ under the model of interest (with p parameters), and $\mathrm{sgn}(z)$ denotes the sign of z.

4.2.3 | Likelihood residual

Third, we derive a likelihood residual, which is a combination of the two previous residuals, given by

$$r_i^{(3)} = \mathrm{sgn}(t_i - \hat{Q}_i)\left\{ h_i[r_i^{(1)}]^2 + (1-h_i)[r_i^{(2)}]^2 \right\}^{1/2}, \quad i = 1, \ldots, n, \tag{11}$$

with its elements previously defined.

4.2.4 | Randomized quantile residual

Fourth, we employ also the randomized quantile residual proposed by Dunn and Smyth.[33] In our case, this residual is given by

$$r_i^{(4)} = \Phi^{-1}(F(t_i; \hat{\alpha}, \hat{Q}_i)), \quad i = 1, \ldots, n, \tag{12}$$

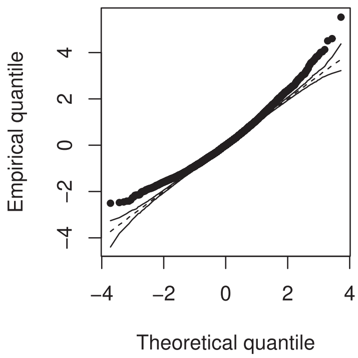
where F is the BS cumulative distribution function defined in (3). This residual follows approximately a normal standard distribution.

A simulation is performed to evaluate the distributions of $r^{(1)}$, $r^{(2)}$, $r^{(3)}$, and $r^{(4)}$ formulated in (9), (10), (11), and (12), respectively. In this simulation study, we use the regression model

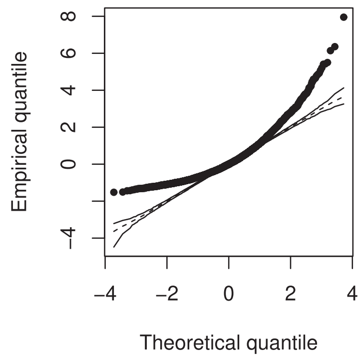
$$h(Q_i) = \beta_0 + \beta_1 x_i, \quad i = 1, \ldots, 100, \tag{13}$$

considering the logarithm (log) and identity link functions for h, with $\beta_0 > 0$ and $\beta_1 > 0$ to guarantee $Q_i > 0$ in the case of the identity link. The true values of the parameters are taken as $\beta_0 = 0.3$ and $\beta_1 = 0.5$, whereas $\alpha \in \{0.2, 0.5, 1.0\}$, which indicates low, moderate and high degrees of skewness. Note that, using the identity link function, we have that $Q_i \in [0.30, 0.80]$, whereas for the log link function, we have that $Q_i \in [1.35, 2.23]$. Other ranges for Q_i need other values for β_0, β_1 and other link functions so that is a limitation of this simulation study. We assume that the values of the covariate X are generated from a uniform distribution in the interval (0,1). The number of Monte Carlo replications is 5000. Using the relation $Q_i = h^{-1}(\beta_0 + \beta_1 x_i)$, we calculate the values of Q_i. In each of the 5000 replications, we obtain the observations $\boldsymbol{t} = (t_1, \ldots, t_{100})^\top$ from the BS distribution with parameters α and Q_i, for $i = 1, \ldots, 100$. Then, the model given in (13) is fitted using the implemented functions in the R software.

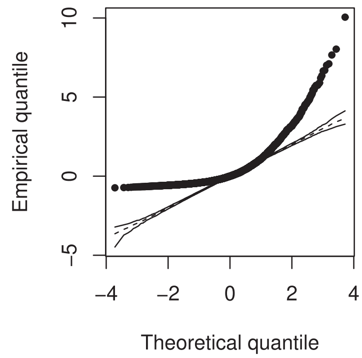
Statistical behavior of $r^{(1)}$, $r^{(2)}$, $r^{(3)}$, and $r^{(4)}$ can be graphically viewed when comparing the empirical distribution of the residuals and the standard normal distribution. We employ a quantile against quantile (QQ) plot with simulated envelope to make this comparison; see Atkinson.[34] Figures 4 and 5 display QQ plots with these envelopes, one for each 5000 residuals $r^{(1)}$, $r^{(2)}$, $r^{(3)}$, and $r^{(4)}$, using log and identity links, with $\alpha \in \{0.2, 0.5, 1.0\}$. We observe that the QQ plot with simulated envelope of $r^{(1)}$ shows that it is further away from the diagonal line and outside of the envelope, which says us that this residual is not suitable for the proposed model. The QQ plots associated with $r^{(2)}$, $r^{(3)}$, and $r^{(4)}$ are adequately over the diagonal and inside of the envelope, indicating that these residuals follow approximately a standard normal distribution, at least when α is in [0.2, 0.5, 1.0], showing their adequacy for the BS quantile regression model. A study when the number of observations is greater than $n = 100$, namely, $n = 200$ and $n = 500$, provides similar results to $n = 100$. Other study



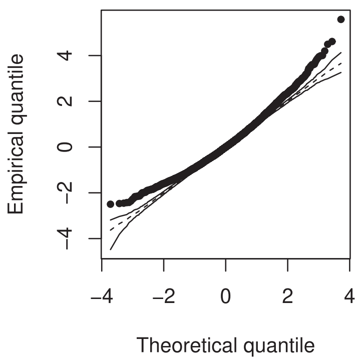
(A) $r^{(1)}$, $h = \log$, $\alpha = 0.2$



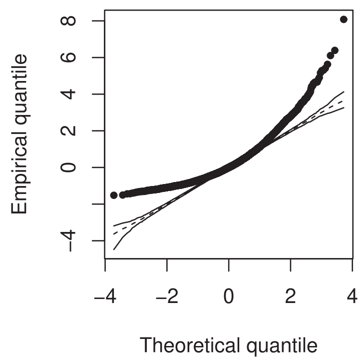
(B) $r^{(1)}$, $h = \log$, $\alpha = 0.5$



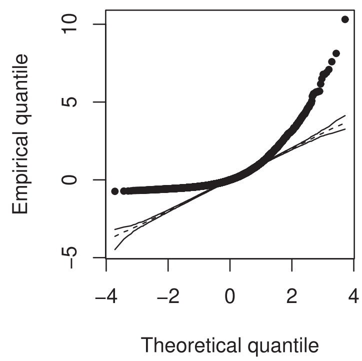
(C) $r^{(1)}$, $h = \log$, $\alpha = 1.0$



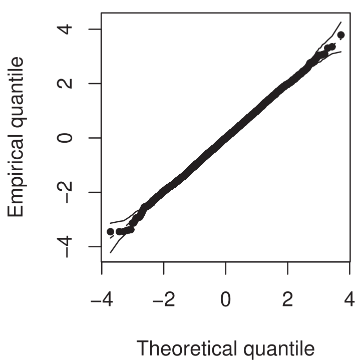
(D) $r^{(1)}$, $h = \mathbf{id}$, $\alpha = 0.2$



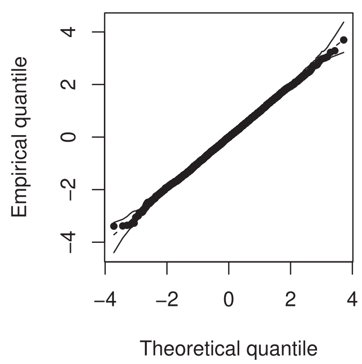
(E) $r^{(1)}$, $h = \mathbf{id}$, $\alpha = 0.5$



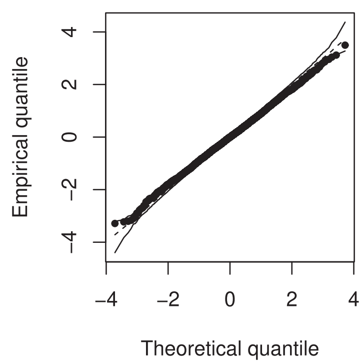
(F) $r^{(1)}$, $h = \mathbf{id}$, $\alpha = 1.0$



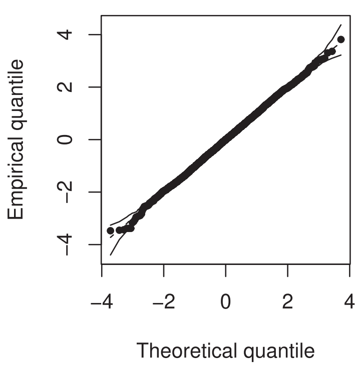
(G) $r^{(2)}$, $h = \log$, $\alpha = 0.2$



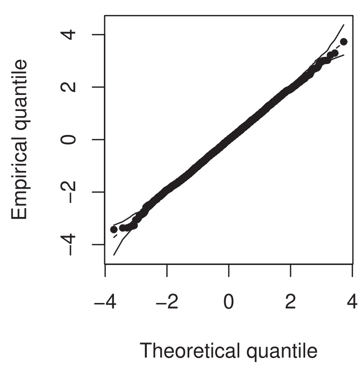
(H) $r^{(2)}$, $h = \log$, $\alpha = 0.5$



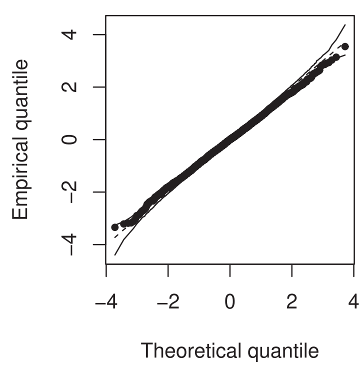
(I) $r^{(2)}$, $h = \log$, $\alpha = 1.0$



(J) $r^{(2)}$, $h = \mathbf{id}$, $\alpha = 0.2$

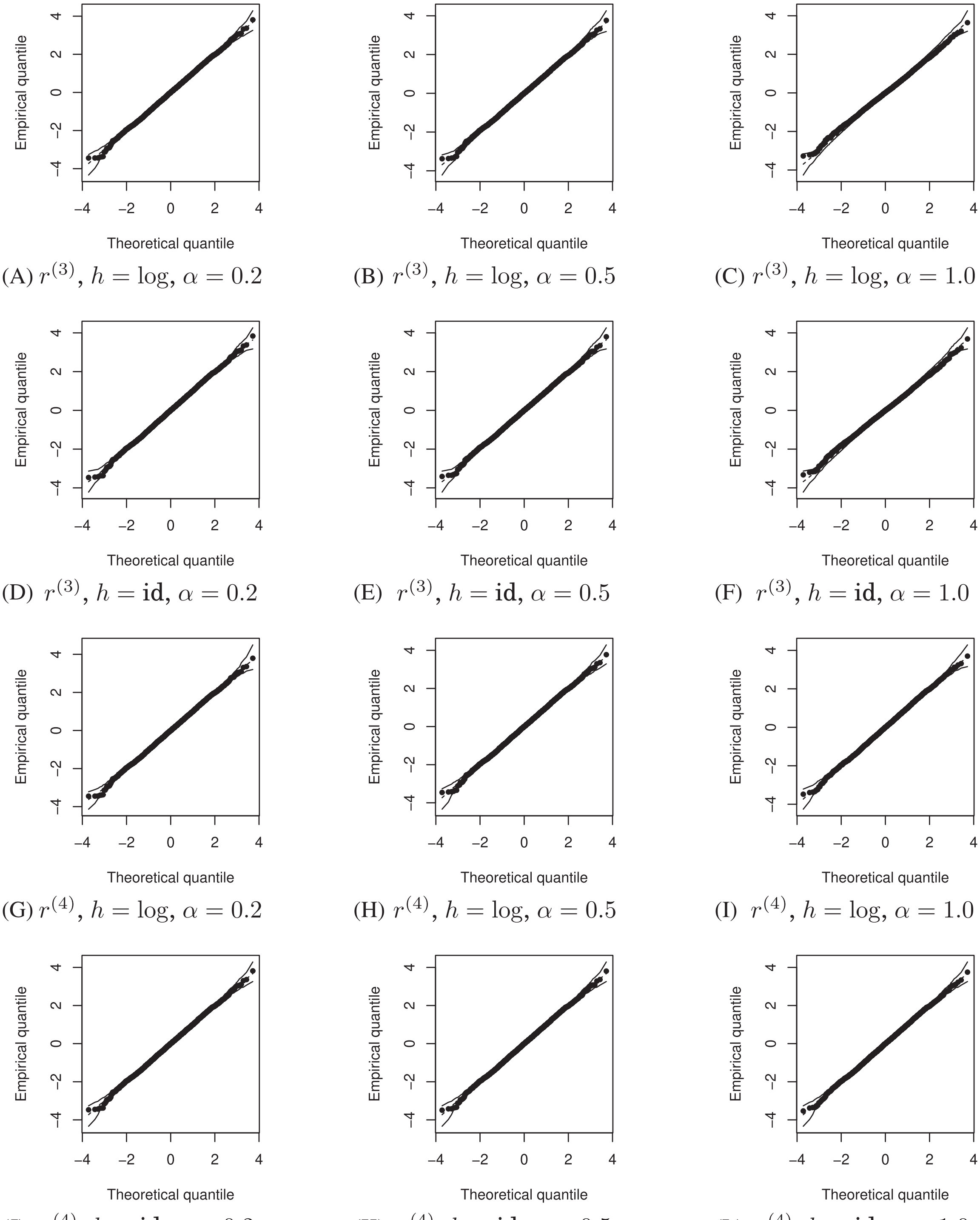


(K) $r^{(2)}$, $h = \mathbf{id}$, $\alpha = 0.5$



(L) $r^{(2)}$, $h = \mathbf{id}$, $\alpha = 1.0$

FIGURE 4 Plots of the indicated residual, link function, and value of α with simulated data



(A) $r^{(3)}$, $h = \log$, $\alpha = 0.2$ (B) $r^{(3)}$, $h = \log$, $\alpha = 0.5$ (C) $r^{(3)}$, $h = \log$, $\alpha = 1.0$

(D) $r^{(3)}$, $h = \textbf{id}$, $\alpha = 0.2$ (E) $r^{(3)}$, $h = \textbf{id}$, $\alpha = 0.5$ (F) $r^{(3)}$, $h = \textbf{id}$, $\alpha = 1.0$

(G) $r^{(4)}$, $h = \log$, $\alpha = 0.2$ (H) $r^{(4)}$, $h = \log$, $\alpha = 0.5$ (I) $r^{(4)}$, $h = \log$, $\alpha = 1.0$

(J) $r^{(4)}$, $h = \textbf{id}$, $\alpha = 0.2$ (K) $r^{(4)}$, $h = \textbf{id}$, $\alpha = 0.5$ (L) $r^{(4)}$, $h = \textbf{id}$, $\alpha = 1.0$

FIGURE 5 Plots of the indicated residual, link function, and value of α with simulated data

for $q = 0.1$ and $q = 0.9$ with $n = 100$ reports that the residuals $r^{(1)}$, $r^{(2)}$, and $r^{(3)}$ do not follow a normal distribution, but $r^{(4)}$ does. Due to reasons of space, the obtained results are omitted here. Then, we recommend the use of $r^{(4)}$ for this model.

4.3 | Illustrative example

Based on Subsection 1.2, we assume the response $T_i \sim \text{BS}(\alpha, Q_i)$. We consider the logarithm, square root, and identity link functions for the systematic component of the regression model on the median, which are expressed as: (L1)

$\log(Q_i) = \boldsymbol{x}_i^\top \boldsymbol{\beta}$; (L2) $\sqrt{Q_i} = \boldsymbol{x}_i^\top \boldsymbol{\beta}$; (L3) $Q_i = \boldsymbol{x}_i^\top \boldsymbol{\beta}$, with $\boldsymbol{\beta} > 0$; for $i = 1, \ldots, 100$, where $\boldsymbol{\beta} = (\beta_0, \beta_2, \beta_3, \beta_4)^\top$ is the regression coefficient vector and $\boldsymbol{x}_i^\top = (1, x_{i2}, x_{i3}, x_{i4})$ is the observed value of $\boldsymbol{X}_i$. We emphasize that the median regression is used as it is a robust measure of centrality for right-skewed distributions; see Hao and Naiman,[35(p. 57)] and Davino et al.[36(p. 76)] We fit the BS model by using the command `bsreg.fit()` that we have implemented in the `R` software. The values of the corrected Akaike information criterion (AIC) and of the log-likelihood function (in parenthesis) for the model with indicated link functions are (L1) 1502.116 (−745.739), (L2) 1411.67 (−700.516), and (L3) 1396.313 (−692.837). With these values, we state that the identity link function should be employed in the modeling, as conjectured in Subsection 1.2. The ML estimates for the model parameters with link function (L3), approximate estimated standard errors (SEs), and their significance at 5% are reported in Table 2, where X_2,X_3, and X_4 are identified as significant to explain T. Therefore, we consider the model given by

$$Q_i = \beta_0 + \beta_1 x_{2i} + \beta_2 x_{3i} + \beta_3 x_{4i}, \quad i = 1, \ldots, 100. \tag{14}$$

Additionally, we estimate the model for $q = 0.1$, 0.25, 0.75, and 0.9 obtaining satisfactory fittings such as with the median so that, due to reasons of space, the obtained results are omitted here. Therefore, we decide to continue our illustration with the BS median regression model.

Distributional assumption of the model given in (14) is verified by the QQ plot with envelope for the residual $r^{(4)}$ in Figure 6A. This figure shows that the residuals follow approximately a standard normal distribution, validating that the response variable follows the BS distribution. In addition, no unusual features are detected by the plot of residuals presented in Figure 6B, solving the problem of nonconstant variance detected in Subsection 1.2, but five outlying cases (#13, #27, #32, #80, and #87) are identified, which are analyzed in the diagnostic study presented next. When comparing our model with the normal regression model (where as known the mean is equal to the median), a better performance is detected in favor of the BS quantile regression model, based on the QQ plots with envelope for the residuals displayed in Figure 6A,B. Note that in the case of the normal regression model, we use the usual standardized Pearson residual. Comparison of our model with other similar models based on, for instance, the gamma, lognormal, or Weibull distributions[27] is not possible because this implies to derive models using such distributions with identical parameterizations to that employed in our approach, which are not available in the literature.

Suitability of the identity link function used in the model given in (14) is verified by utilizing $\boldsymbol{z}_2 = \hat{\boldsymbol{\eta}} + \hat{\boldsymbol{v}}^* \odot \hat{\boldsymbol{z}}$, such as in Leiva et al,[5] where $\hat{\boldsymbol{v}}^* = (1/\hat{v}_1, \ldots, 1/\hat{v}_n)^\top$, with v_i being defined in (7) and $\odot$ being the Hadamard product. The plot of $\hat{z}_{2i}$ against $\hat{\eta}_i$ is employ to verify the adequacy of the link function, where a linear tendency is requested. According to Figure 6C, it is possible to note such a linear tendency, and therefore the identity function link is adequate for our model.

Diagnostics based on the local influence technique for the BS quantile regression model given in (4) are provided in Figure 7, which shows index plots of C_i. From there, cases #13, #27, #32, #80, and #87 are detected as potentially influential. Also, we analyze index plots of $\boldsymbol{l}_{\max}$, but the results are similar to those presented for the index plots of C_i, so that we omit these results here. Figure 7 shows plots of C_i against X_{2i}, which indicate that small values of X_2 have a moderate influence on the estimates; see, for example, case #87. Impact on the model inference is analyzed for three cases (#13, #80, and #87) identified as more potentially influential in the diagnostic analytics. Then, we remove the sets of cases {#13}, {#80}, {#87}, {#13, #80}, {#13,#87}, {#80,#87}, {#13,#80,#87} and reestimate the model parameters. Relative changes (RCs) in the parameter estimates and in their associated estimated SEs are calculated as

$$\mathrm{RC}_{\theta_{j(i)}} = \left| \frac{\hat{\theta}_j - \hat{\theta}_{j(i)}}{\hat{\theta}_j} \right| \times 100\%, \quad \mathrm{RC}_{\mathrm{SE}(\hat{\theta}_j)_{(i)}} = \left| \frac{\widehat{\mathrm{SE}}(\hat{\theta}_j) - \widehat{\mathrm{SE}}(\hat{\theta}_j)_{(i)}}{\widehat{\mathrm{SE}}(\hat{\theta}_j)} \right| \times 100\%,$$

where $\hat{\theta}_{j(i)}$ and $\widehat{\mathrm{SE}}(\hat{\theta}_j)_{(i)}$ denote the ML estimates of θ_j and of the estimated SE of the associated estimator, respectively, obtained after removing case i, for $j = 1, \ldots, 5$ and $i = 1, \ldots, 100$, with $\theta_1 = \beta_0$, $\theta_2 = \beta_1$, $\theta_3 = \beta_2$, $\theta_4 = \beta_3$, and $\theta_5 = \alpha$. Table 3

TABLE 2 Estimate, SE, and significance at 5% of the indicated parameter for income data

	$\hat{\beta}_0$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\alpha}$
Estimate	198.0903	1.0440	1.1090	1.0865	0.3646
SE	22.3166	0.0871	0.1502	0.1759	0.0087
Significance	Yes	Yes	Yes	Yes	Yes

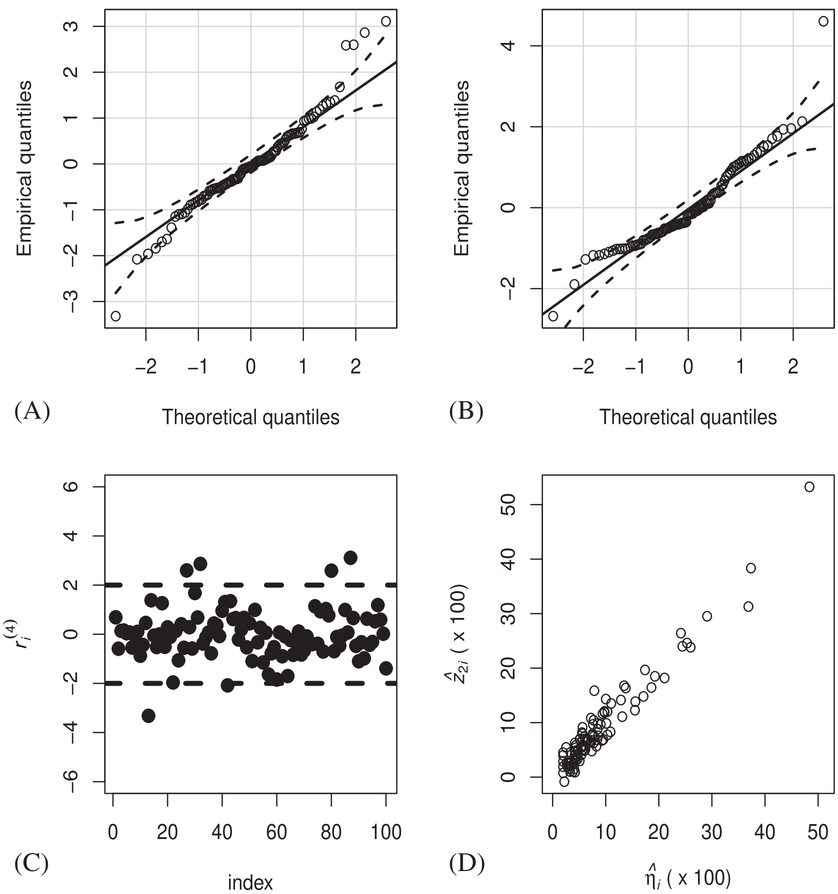


FIGURE 6 QQ plot with envelope of $r^{(4)}$ for the BS median regression (A), and of the standardized residual for the normal regression (B); index plot of $r^{(4)}$ (C), and plot of $\hat{z}_2$ against $\hat{\eta}$ for the model fit with identity link (D), based on income data and BS quantile regression

reports these RCs with income data. From this table, the largest values of RCs are identified when removing simultaneously the cases #80 and #87, which influence importantly on all parameters, with RCs until approximately 21%. However, no inferential changes are found. The results presented in this table show that the diagnostic measures derived in this study identify potentially influential cases, but these do not affect the inference of the model. We can conclude that the diagnostic analytics based on the local influence technique confirm that the BS quantile regression model presented in (14) is nonsensitive to the atypical cases detected and suitable for modeling the income data.

5 | IMPLICATIONS, CONCLUSIONS, AND PROJECTIONS

5.1 | Economic implications

Quantile regression allows us to carry out a deeper analysis of the determinants of household incomes when compared with traditional ordinary least squares (OLS) regression, since such determinants may have different magnitude across income strata. Table 4 reports the estimated coefficients for various BS quantile regression models as well as for the OLS regression. In a quantile regression, the estimated coefficient is interpreted as the change in the $q \times 100$th percentile of the household income corresponding to a unit change in the covariate, whereas, in the OLS regression, the change is in the mean household income; see Hao and Naiman[35(p. 57)].

The results of Table 4 report that all the covariates affect positively the household income, as expected. Note that the effects of all the covariates increase with the household income (higher quantiles) in the BS quantile regression model. For example, an increase of 1000 Chilean pesos in salaries (X_2), increases the 10th percentile ($q = 0.10$) of the household income by an amount of $657.1 Chilean pesos. In addition, it increases by $1659.1 Chilean pesos the 90th percentile ($q = 0.90$) of the household income. In other words, the effect of the salaries increases for individuals with higher household income (higher quantiles). As mentioned, see http://www.bancocentral.cl for the equivalence between Chilean pesos and American dollars.

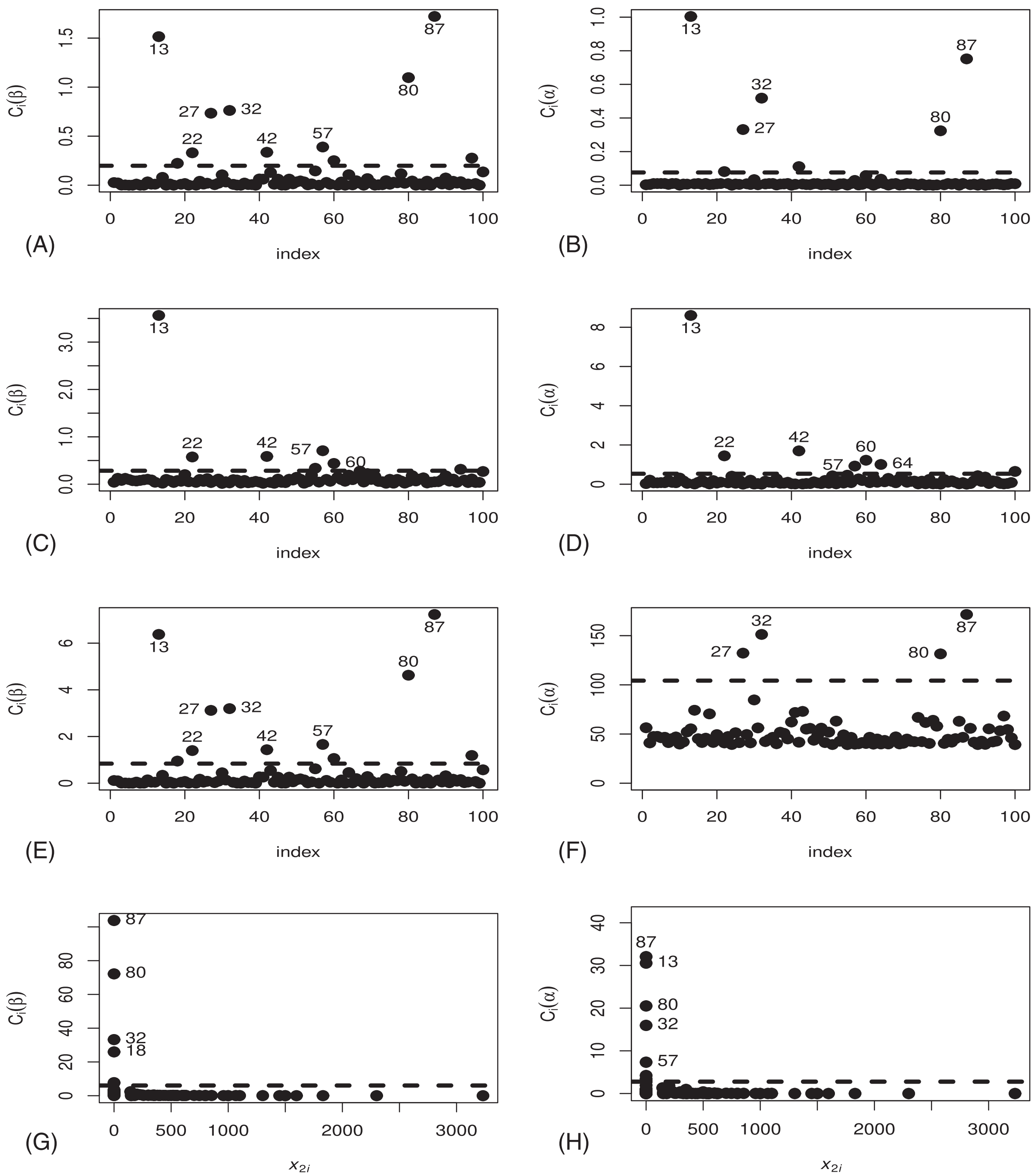


FIGURE 7 Index plots of C_i for $\boldsymbol{\beta}$ (A), and α (B), under case-weight perturbation; for $\boldsymbol{\beta}$ (C), and α (D), under response perturbation; for $\boldsymbol{\beta}$ (E), and α (F), under perturbation of the parameter α; and for $\boldsymbol{\beta}$ (G), and α (H), under covariate perturbation X_2, using income data and BS quantile regression

Note also from Table 4 that the estimated coefficient for the total income due to independent work (X_3) in the BS quantile regression model for $q = 0.25$ is 0.8681, which is less than the estimated coefficient in the mean regression model (OLS), which is 0.9918. This suggests that an increase of 1000 Chilean pesos of income due to pensions provides an average increase of \$991.8 in household income. Observe that the increase would not be substantial for most of the population. Similarly, the estimated coefficient for the total income due to retirements (X_4) in the BS median regression model is 1.0865, which is greater than the corresponding estimated coefficient in the mean regression model.

In general, we can conclude that economic analyses are more informative using quantile regression. In the present example, the BS quantile regression model provided a thorough tool to analyze income data.

Removed cases		β_0	β_1	β_2	β_3	α
None	RC($\hat{\theta}$)	—	—	—	—	—
	RC($\widehat{SE}$)	—	—	—	—	—
	P-value	<.01	<0.01	<.01	<.01	<.01
{13}	RC($\hat{\theta}$)	9.46	2.98	2.41	4.74	5.63
	RC($\widehat{SE}$)	3.41	4.66	4.99	4.84	9.80
	P-value	<.01	<.01	<.01	.01	<.01
{80}	RC($\hat{\theta}$)	7.61	2.4	3.59	3.85	3.24
	RC($\widehat{SE}$)	3.58	2.7	2.04	2.77	5.48
	P-value	<.01	<.01	<.01	<.01	<.01
{87}	RC($\hat{\theta}$)	10.19	3.22	4.84	5.17	5.00
	RC($\widehat{SE}$)	7.01	4.57	3.65	4.64	8.68
	P-value	<.01	<.01	<.01	<.01	<.01
{13,80}	RC($\hat{\theta}$)	1.51	0.53	0.98	0.88	8.37
	RC($\widehat{SE}$)	0.52	6.80	6.64	7.02	14.23
	P-value	<.01	<.01	<.01	<.01	<.01
{13,87}	RC($\hat{\theta}$)	0.50	0.08	1.9	0.14	10.04
	RC($\widehat{SE}$)	2.48	8.57	8.2	8.76	17.14
	P-value	<.01	<.01	<.01	<.01	<.01
{80,87}	RC($\hat{\theta}$)	20.92	6.70	10.31	10.87	9.84
	RC($\widehat{SE}$)	13.54	8.95	7.06	9.10	16.80
	P-value	<.01	<.01	<.01	<.01	<.01
{13,80,87}	RC($\hat{\theta}$)	11.27	3.48	7.00	5.67	14.09
	RC($\widehat{SE}$)	7.65	12.06	10.86	12.24	23.66
	P-value	<.01	<.01	<.01	<.01	<.01

TABLE 3 RCs of ML estimates and of the associated estimated SEs for the indicated case(s), and respective *p*-values using income data and BS quantile regression

	BS quantile regression					
	$q = 0.10$	$q = 0.25$	$q = 0.50$	$q = 0.75$	$q = 0.90$	OLS
$\hat{\beta}_0$	124.7264	155.0139	198.0903	251.4492	314.7382	126.7358
	(<.001)	(<.001)	(<.001)	(<.001)	(<.001)	(<.001)
$\hat{\beta}_2$	0.6571	0.8170	1.0440	1.3369	1.6591	49.9244
	(<.001)	(<.001)	(<.001)	(<.001)	(<.001)	(<.001)
$\hat{\beta}_3$	0.6974	0.8681	1.1090	1.4218	1.7629	0.9918
	(<.001)	(<.001)	(<.001)	(<.001)	(<.001)	(<.001)
$\hat{\beta}_4$	0.6840	0.8498	1.0865	1.3923	1.7259	1.0689
	(<.001)	(<.001)	(<.001)	(<.001)	(<.001)	(<.001)

TABLE 4 OLS regression and BS quantile regression estimated coefficients (*p*-values in parentheses) at five different quantiles with income data

5.2 | Concluding remarks

In this work, we proposed an approach to quantile regression modeling based on the BS distribution. This approach used a quantile parameterization, which allowed us to consider a similar framework to generalized linear models, providing flexibility in the modeling. The ML method was considered for estimating the model parameters and for carrying out inference on these parameters. A diagnostic analytics based on the local influence technique under different perturbation schemes and a residual analysis were also conducted. The performance of four types of residuals was evaluated by simulations, with a randomized quantile residual being identified as the most suitable for our model. A data analysis of the new model was performed with Chilean household income data. This analysis showed an adequate performance of the approach, providing evidence that the BS distribution is a good modeling alternative for dealing with positive data and an asymmetric behavior, as the income data. These results suggested that the BS quantile regression can become a new standard for routine analyses of positive data following an asymmetric distribution in economic sciences and elsewhere. This new methodology can be a valuable addition to the tool kit of applied statisticians and econometricians interested in modeling percentiles, which are often of interest when describing incomes. Our economic analysis showed to be more informative when using quantile regression for household income data than when using OLS regression. Comparison of our model with the normal regression model reported a better performance in favor of the BS quantile regression for the income data used in the empirical illustration. However, comparison of our model with other similar models based on, for instance, the gamma, lognormal, or Weibull distributions[27] is not possible because this implies to derive approaches using such distributions with identical parameterizations to that used in our approach, which are not available in the literature.

5.3 | Limitations and future research

A limitation of our proposal is that covariates can affect simultaneously quantiles and shape parameter. Then, this topic will be studied in a future investigation following the line of the recent work of Ventura et al[37] about joint modeling of two parameters. Regarding how our model performs and the statistical evaluation of the estimation process by means of Monte Carlo simulations are open problems that we are planning to conduct in a future study. As mentioned, comparison of our model with other similar models is not possible because identical parameterizations to that used in our approach are not available in the literature, which opens new lines of work based on these models. Also as part of future research, it is of interest to study how multivariate, spatial, temporal components can be incorporated in the quantile regression modeling based on the BS distribution; see Garcia-Papani et al[9,10] and Martinez et al.[17] In addition, the exploration of the novel quantile regression approach proposed in this work may be considered with other distributions. Furthermore, Cobb-Douglas and tobit-type models can be studied in the context of the present investigation; see Desousa et al,[38] Cysneiros et al,[39] and de la Fuente-Mella et al.[40] The use of censored data can be also of interest to be analyzed; see Villegas et al[41] and Leao et al.[12] The authors are working on these issues and hope to report their finding in future articles.

ACKNOWLEDGEMENT

The authors thank the editors and reviewers for their constructive comments on an earlier version of this manuscript. The research was partially supported by the project grant “FONDECYT 1200525” from the National Commission for Scientific and Technological Research of the Chilean government” as “The research was partially supported by the project grants “FONDECYT 1200525” from the National Commission for Scientific and Technological Research of the Chilean government (V. Leiva) and “Puente 001/2019” from the Research Directorate of the Vice President for Research of the Pontificia Universidad Católica de Chile, Chile (M. Galea).

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How to cite this article: Sánchez L, Leiva V, Galea M, Saulo H. Birnbaum-Saunders quantile regression and its diagnostics with application to economic data. *Appl Stochastic Models Bus Ind*. 2021;37:53–73. https://doi.org/10.1002/asmb.2556