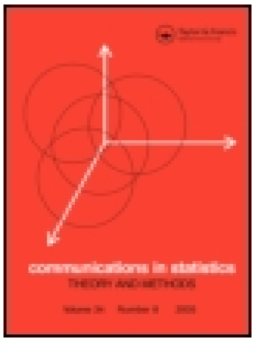

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The Touchard distribution

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ABSTRACT
We present a novel model, which is a two-parameter extension of the Poisson distribution. Its normalizing constant is related to the Touchard polynomials, hence the name of this model. It is a flexible distribution that can account for both under- or overdispersion and concentration of zeros that are frequently found in non-Poisson count data. In contrast to some other generalizations, the Hessian matrix for maximum likelihood estimation of the Touchard parameters has a simple form. We exemplify with three data sets, showing that our suggested model is a competitive candidate for fitting non-Poisson counts.

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1. Introduction

The Poisson model describes counts of events that occur in a number of independent trials under well-known assumptions. If these assumptions are not satisfied, the resulting counting process may exhibit overdispersion, underdispersion or an excess of zeros relative to the Poisson model. Such situations are found in many practical applications, inspiring a number of generalizations of the Poisson distribution in order to describe non-Poisson data (*e.g.*, Satterthwaite 1942; Dandekar 1955; Bardwell and Crow 1964; Sankaran 1970; Consul and Jain 1973; Shmueli et al. 2005; Chakraborty 2010; Kumar and Shibu 2011; Chandra, Roy and Ghosh 2013; Kumar and Nair 2014; Bhati, Sastry, and Maha Qadri 2015).

These models consist of an extra parameter, providing more flexible modeling of a wide range of overdispersion and underdispersion and a better fit to discrete data. However, inflation of zeros is a more difficult issue to deal with when using only a single model. Excess zeros frequently occur because the modeled process is mixed with or contaminated by a different process (Shmueli et al. 2005). In these cases, although models with three parameters can fit the data, they do not model the mixture directly. Thus, mixed-distribution models, such as the zero-inflated Poisson (ZIP), zero-inflated negative binomial (ZINB), and the hurdle model are often used to fit such data (Mullahy 1986; Lambert 1992; Gurmu 1998; Ridout, Hinde, and Demetrio 1998).

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Whereas these approaches allow researchers to model data with an inflated number of zeros relative to the Poisson distribution, we prefer to pursue an approach that does not require mixing distributions.

In this paper, we present a generalization of the Poisson model to a two-parameter model which allows not only overdispersion or underdispersion, but excess zeros as well. Our proposed model was inspired by the moments of the Poisson distribution, whose normalization constant relates to the Touchard polynomials (Touchard 1939; Chrysaphinou 1985).

We find this generalization to be particularly useful due to its simplicity and feasibility. Among its properties, the Touchard is a member of the exponential family, belonging also to the family of power series distributions. For illustrative purposes, we present three example applications using actual data. First, we apply the Touchard model to a data set from the Premier League (the top English football division) which contains the number of goals per match scored by home and away teams during the 2013/14 season. Second, we use our model in a truncated form to fit the dental caries data (Böhning et al. 1996). Finally, we model epileptic seizure count data (Chakraborty 2010; Bhati, Sastry, and Maha Qadri 2015). In all cases, our proposed model is compared with other competing distributions, showing its practical applicability.

The rest of this article is organized as follows. Section 2 introduces the Touchard distribution. Some of its moment characteristics are presented in Section 3. Section 4 deals with maximum likelihood estimation and useful recursive formulas are proposed in Section 5. The practical use of the Touchard distribution is illustrated in Sections 6 and 7 concludes.

2. The Touchard distribution

Let X be a random variable taking nonnegative integer values, $k \in \mathbb{N}$, whose probability distribution is defined as

$$p_k = \mathrm{P}[X = k] = \frac{\lambda^k (k+1)^\delta}{k!\tau(\lambda, \delta)}, \tag{1}$$

where $\lambda > 0$ and $\delta \in \mathbb{R}$ are the distribution parameters, and the function

$$\tau(\lambda, \delta) = \sum_{j \in \mathbb{N}} \frac{\lambda^j (j+1)^\delta}{j!}, \tag{2}$$

which normalizes the previous expression, is related to the Touchard polynomials (Rota 1964; Chrysaphinou 1985) and to the moment of order δ of a shifted Poisson distribution. Thus, we suggest $X \sim \text{Touchard}(\lambda, \delta)$, defined in (1), as a generalization of the Poisson distribution since for $\delta = 0$, $X \sim \text{Poisson}(\lambda)$. Recursively, Eq. (1) may be written as

$$p_{k+1} = \frac{\lambda}{k+1} \left(\frac{k+2}{k+1} \right)^\delta p_k. \tag{3}$$

With the help of this formula, a random number generator can be obtained by performing a sequential search in order to generate a Touchard deviate (Ahrens and Dieter 1982).

It can be seen that $p_{k+1}/p_k \downarrow 0$ as $k \uparrow +\infty$. Furthermore, the Touchard distribution naturally allows zero-inflated counts relative to the Poisson when λ and δ are chosen such that $p_{k^*} < p_0$ and $p_{k^*} < p_{k^*+1}$, for some fixed $k^* \geq 1$; that is,

$$\frac{p_{k^*}}{p_0} = \frac{\lambda^{k^*} (k^* + 1)^\delta}{k^*!} < 1,$$

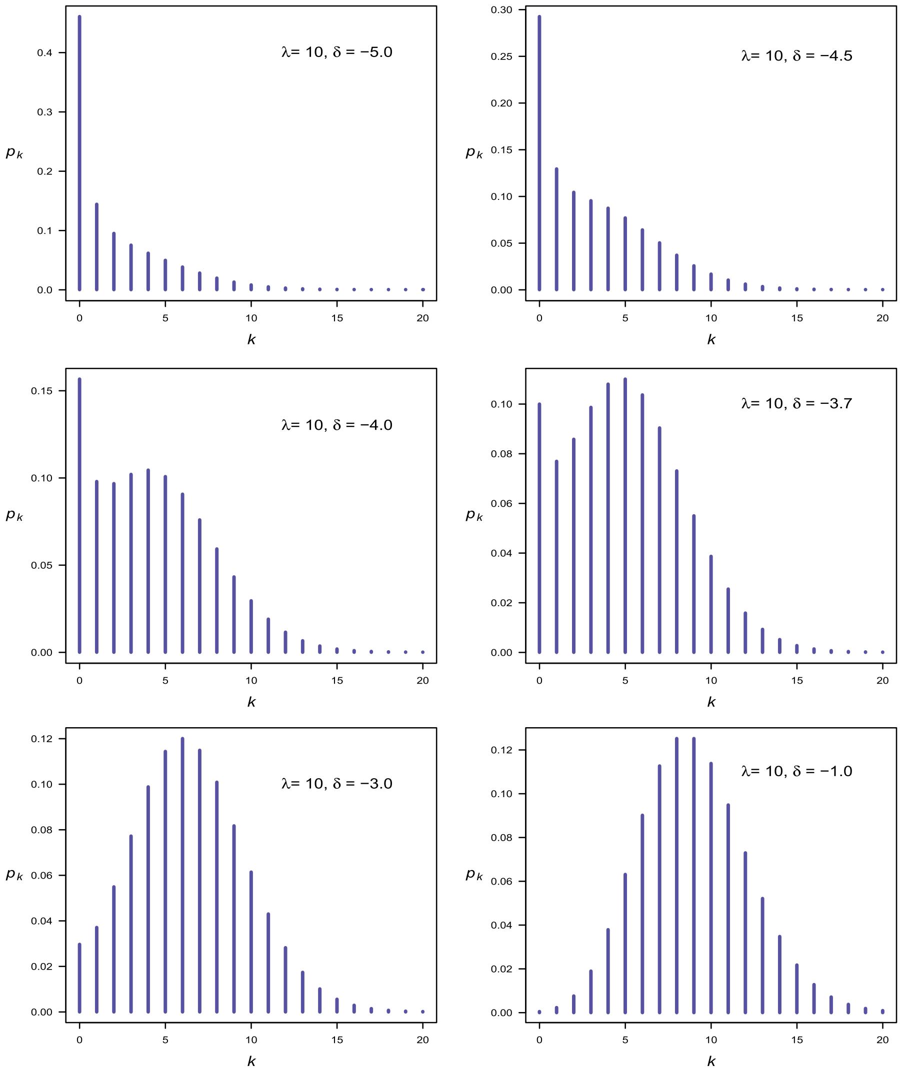

Figure 1. Examples of Touchard distributions, with $\lambda = 10$ and δ ranging from -5.0 to -1.0. Excess zeros appear when $\delta = -4$ and when $\delta = -3.7$.

and

$$\frac{p_{k^*+1}}{p_{k^*}} = \frac{\lambda}{k^*+1}\left(\frac{k^*+2}{k^*+1}\right)^{\delta} > 1.$$

Figure 1 depicts examples with $\lambda = 10$ and δ ranging from -5.0 to -1.0, where excess zeros emerge when $\delta = -4$ and when $\delta = -3.7$.

3. Moments

The rth moment of a Touchard random variable is a polynomial series of binomial type given by

$$\mathrm{E}[X^r] = \sum_{j=0}^{r} \binom{r}{j} \frac{(-1)^{r-j}\tau(\lambda, \delta + j)}{\tau(\lambda, \delta)}, \tag{4}$$

and its moment generation function is

$$\mathrm{M}_X(q) = \mathrm{E}[\mathrm{e}^{qX}] = \frac{\tau(\lambda \mathrm{e}^q, \delta)}{\tau(\lambda, \delta)},$$

where $q \in \mathbb{R}$. Therefore, the Touchard distribution belongs to the family of two-parameter power series distributions (Johnson, Kemp, and Kotz 2005, p. 79). The mean of X can be expressed by

$$\mu = \mathrm{E}[X] = \frac{\tau(\lambda, \delta+1)}{\tau(\lambda, \delta)} - 1 \tag{5}$$

$$= \lambda \cdot \mathrm{E}\left[\left(\frac{X+2}{X+1}\right)^{\delta}\right], \tag{6}$$

and its variance can be written as

$$\sigma^2 = \mathrm{Var}[X] = \frac{\tau(\lambda, \delta+2)}{\tau(\lambda, \delta)} - \left[\frac{\tau(\lambda, \delta+1)}{\tau(\lambda, \delta)}\right]^2$$

$$= \lambda\, \mathrm{E}\left[(X+1)\left(\frac{X+2}{X+1}\right)^{\delta}\right] - \mu^2.$$

From Eq. (6), we observe that $\mu > \lambda$, if $\delta > 0$; and $\mu < \lambda$, if $\delta < 0$. To assess the dispersion, consider the ratio $r = \sigma^2/\mu$, which can be expressed as

$$r = \frac{\mathrm{E}\left[(X+1)\left(\frac{X+2}{X+1}\right)^{\delta}\right]}{\mathrm{E}\left[\left(\frac{X+2}{X+1}\right)^{\delta}\right]} - \mu.$$

In the Poisson case ($\delta = 0$), we have $r = 1$. For $\delta > 0$, as $X + 1$ and $\{[X+2]/[X+1]\}^{\delta}$ are inversely (negatively) correlated, we have $r < 1$. Conversely, if $\delta < 0$, then $r > 1$. For example, with $\lambda = 30$ and $\delta = -5$ we have overdispersion ($r > 1$). But, if $\lambda = 30$ and $\delta = 5$, the distribution is underdispersed ($r < 1$). Figure 2 shows the behavior of the ratio r for $\lambda = 0.05, 0.5, 1, 5, 20, 30$ and 50, and $-5 \leq \delta \leq 5$.

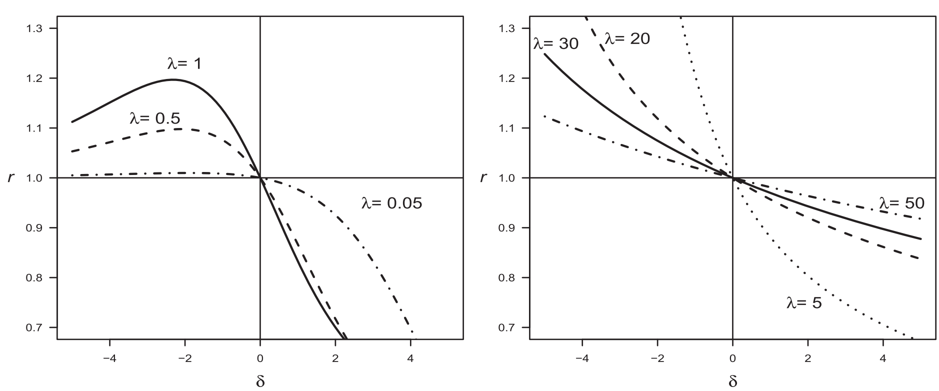

Figure 2. Behaviour of the ratio $r = \sigma^2/\mu$: overdispersion ($r > 1$) and underdispersion ($r < 1$).

4. Sufficient statistics and maximum likelihood equations

The Touchard distribution is a member of the exponential family since Eq. (1) can be rearranged as

$$p_k = \frac{1}{k!}\exp\{k\ln(\lambda) + \delta\ln(k+1) - \ln(\tau(\lambda,\delta))\}.$$

Given a sample of n independent realizations of this distribution, $x_1, \ldots, x_n$, the resulting likelihood can be written as

$$L(\lambda,\delta|\{x_i\}) = \left(\prod_i x_i!\right)^{-1} \lambda^{S_1} e^{\delta S_2}\left[\tau(\lambda,\delta)\right]^{-n}, \tag{7}$$

with $S_1 = \sum_i x_i$ and $S_2 = \sum_i \ln(x_i+1)$ being sufficient statistics by the factorization theorem. To maximize the log-likelihood function $l(\lambda,\delta) = \ln L(\lambda,\delta|\{x_i\})$, we need the first two derivatives of $\tau(\lambda,\delta)$ with respect to λ and δ, which are

$$\begin{aligned}\frac{\partial\tau(\lambda,\delta)}{\partial\lambda} &= \frac{\tau(\lambda,\delta)}{\lambda}\cdot\mu;\\ \frac{\partial\tau(\lambda,\delta)}{\partial\delta} &= \tau(\lambda,\delta)\,\mathrm{E}\{\ln[X+1]\}.\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\tau(\lambda,\delta)}{\partial\lambda^2} &= \tau(\lambda,\delta)\cdot\frac{\mathrm{E}[X^2]-\mu}{\lambda^2};\\ \frac{\partial^2\tau(\lambda,\delta)}{\partial\delta^2} &= \tau(\lambda,\delta)\,\mathrm{E}\{\ln^2[X+1]\};\\ \frac{\partial^2\tau(\lambda,\delta)}{\partial\delta\partial\lambda} &= \frac{\tau(\lambda,\delta)}{\lambda}\,\mathrm{E}\{X\ln(X+1)\}.\end{aligned}$$

Thus, the maximum likelihood equations are

$$\begin{cases} S_1 - n\mu = 0, \\ S_2 - n\,\mathrm{E}\{\ln[X+1]\} = 0. \end{cases} \tag{8}$$

Indeed, from the factorization theorem, as the likelihood in (7) can be written as $L(\lambda,\delta|\{x_i\}) = \eta(\lambda,\delta|S_1,S_2)\cdot\Psi(\{x_i\})$, maximizing $\ln L(\lambda,\delta|\{x_i\})$ with respect to λ and δ is equivalent to maximizing $\eta(\lambda,\delta|S_1,S_2)$ with respect to λ and δ. Therefore, the moments estimates of λ and δ, which satisfy (8), coincide with their corresponding maximum likelihood estimates. For the examples in Section 6, rather than solving this nonlinear system, we directly maximize the log-likelihood function by using a Newton-Raphson solver. For this purpose, the Hessian matrix is

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} H_{11} &= \frac{\partial^2 l(\lambda,\delta)}{\partial\lambda^2} = -\frac{n[\sigma^2 + (\bar{x}-\mu)]}{\lambda^2};\\ H_{22} &= \frac{\partial^2 l(\lambda,\delta)}{\partial\delta^2} = -n\,\mathrm{Var}[\ln(X+1)];\\ H_{12} &= \frac{\partial^2 l(\lambda,\delta)}{\partial\delta\partial\lambda} = -\frac{n\,\mathrm{Cov}[X,\ln(X+1)]}{\lambda}. \end{aligned}$$

In these examples, when optimizing the Touchard likelihood function, the sample mean was used as the initial value for λ, and 0 for δ (that is, we start as a Poisson model).

To test $H_0 : X \sim \text{Poisson}(\lambda_0)$ against $H_1 : X \sim \text{Touchard}(\lambda, \delta)$ for large samples, we may consider the likelihood ratio and the Wald statistics, which are, respectively, $D = -2\ln[L(\hat{\lambda}_0, 0)/L(\hat{\lambda}, \hat{\delta})]$ and $W = \hat{\delta}^2/I_{22}$, where $\hat{\lambda}_0$, $\hat{\lambda}$ and $\hat{\delta}$ are the corresponding ML estimates, I_{22} is the element(2,2) of $-H^{-1}$ and, under the null, both are approximately distributed as a $\chi^2_{(1)}$.

5. More recursive formulas

Calculations with and analyses of Touchard random variables require an accurate and efficient implementation of the Touchard polynomials in Eq. (2). To obtain a numerical value for this sum, it must be truncated (except when $\delta = 0$, yielding $\tau(\lambda, 0) = e^{\lambda}$). We suggest an alternative form of the summands, where the factorial term is avoided and the relative size of each term can be more clearly observed:

$$\tau(\lambda, \delta) = \sum_{j\in\mathbb{N}} A_j, \tag{10}$$

with

$$A_{j+1} = \frac{\lambda}{j+1}\left(\frac{j+2}{j+1}\right)^{\delta} A_j,$$

and $A_0 = 1$. In this way, $\tau(\lambda, \delta)$ can be calculated given a prespecified relative precision. Similarly, the numerical evaluation of the Hessian can be performed recursively by rewriting Eq. (4) as

$$\mathrm{E}[X^r] = \frac{1}{\tau(\lambda, \delta)} \sum_{j=1}^{\infty} A_{r,j},$$

for $r \geq 1$, where $A_{r,1} = \lambda 2^{\delta}$ and

$$A_{r,j+1} = \frac{\lambda}{j+1}\left(\frac{j+1}{j}\right)^{r}\left(\frac{j+2}{j+1}\right)^{\delta} A_{r,j}.$$

The rth moment of $\ln(X+1)$, $r \geq 1$, can be expressed as

$$\mathrm{E}[\ln^r(X+1)] = \frac{1}{\tau(\lambda, \delta)} \sum_{j=1}^{\infty} B_{r,j},$$

where $B_{r,1} = \lambda 2^{\delta}(\ln 2)^r$ and

$$B_{r,j+1} = \frac{\lambda}{j+1}\left(\frac{\ln(j+2)}{\ln(j+1)}\right)^{r}\left(\frac{j+2}{j+1}\right)^{\delta} B_{r,j}.$$

Finally, the expectation of the product $X\ln(X+1)$, may be written as

$$\mathrm{E}[X\ln(X+1)] = \frac{1}{\tau(\lambda, \delta)} \sum_{j=1}^{\infty} C_j,$$

Table 1. Number of iterations for the τ function calculation in (10) with different combinations of (λ, δ) and a relative error of 10^{-15}.

		δ						
		−10	−5	−1	0	1	5	10
λ	0.1	6	7	9	10	10	12	14
	0.5	8	11	14	14	15	17	20
	1	9	13	17	18	18	21	24
	5	17	26	31	32	33	36	40
	10	27	38	44	45	46	49	53
	20	47	60	64	65	66	70	74

where $C_1 = \lambda 2^{\delta} \ln 2$ and

$$C_{j+1} = \frac{\lambda \ln(j+2)}{j \ln(j+1)} \left(\frac{j+2}{j+1} \right)^{\delta} C_j.$$

These recursive formulas are linear homogeneous recurrence equations in the forward direction. As an illustration, Table 1 shows the number of iterations needed to reach the prespecified relative precision of 10^{-15} for the τ function in (10) with some combinations of (λ, δ). One can observe that the number of iterations grows slowly as we move southeast [↘] (large λ, large δ) inside Table 1. This phenomenon is well known and better schemes taking into consideration the parameter values can be studied further (Ahrens and Dieter 1982; Fishman 2003). A detailed discussion on the stability of linear recurrence equations in the forward direction can be found in Panjer and Wang (1993).

6. Examples

6.1. Premier League match-by-match data

The Premier League is the top English football league contested by 20 clubs. We collected match-by-match data from `http://www.premierleague.com`. They refer to the number of goals per match scored by home and away teams during the 2013-2014 season. A season runs from August to May and each team plays 38 matches, totaling 380 matches during a season. The distribution of the number of goals scored in football matches has been investigated by several authors (*e.g.*, Hirotsu and Wright 2003; Skinner and Freeman 2008). Among the models which have been considered, goal scoring is frequently regarded as a Poisson or negative binomial distribution. Here, we suggest the Touchard distribution as an alternative. Table 2 shows the empirical distributions of the number of goals per game scored by home and away teams and compares the fit of some competing models: Touchard, hyper-Poisson, generalized Poisson (GPD) (Chandra, Roy and Ghosh 2013), Conway-Maxwell-Poisson (COM-Poisson), zero-inflated Poisson (ZIP), zero-inflated negative binomial (ZINB), Poisson-Lindley, new generalized Poisson-Lindley (NGPL) (Bhati, Sastry, and Maha Qadri 2015), Poisson and negative binomial. A comparison of the log-likelihood, AIC, BIC and χ^2 statistics indicates that the Touchard distribution is a competitive model. The resulting maximum likelihood estimates of λ and δ are shown in Table 3.

Table 2. Observed and expected number of goals per game in the Premier League (season 2013/2014, England) and comparison of the log-likelihood and χ^2 statistics for competing models (the models are ordered by ascending AIC values).

	# goals (home)									
	0	1	2	3	4	≥ 5	log-like.	AIC	BIC	χ^2
observed	95	113	85	49	28	10	—	—	—	—
Touchard	95.0	112.1	86.9	50.1	23.1	12.8	− 617.2	1,238.4	1,246.3	1.72
hyper-Poisson	95.0	112.0	86.9	50.2	23.1	12.8	− 617.2	1,238.4	1,246.3	1.73
GPD*	95.7	109.0	88.8	51.4	23.1	12.0	− 617.3	1,238,7	1,246.5	1.80
COM-Poisson	93.5	115.0	86.6	49.0	22.6	13.3	− 617.4	1,238.7	1,246.6	2.20
ZIP	95.0	106.7	91.9	52.8	22.8	10.8	− 617.8	1,239.6	1,247.5	2.41
neg. binomial	91.5	118.3	87.0	47.8	21.9	13.5	− 617.8	1,239.6	1,247.5	3.05
ZINB	95.0	111.0	88.6	50.2	22.6	12.6	− 617.3	1,240.7	1,252.5	2.07
Poisson	78.7	123.9	97.5	51.2	20.1	8.6	− 621.3	1,244.6	1,248.5	9.36
NGPL	119.0	104.8	69.2	40.7	22.4	23.9	− 628.0	1,258.2	1,262.1	20.27
Poisson-Lindley	136.5	93.7	60.2	37.1	22.2	30.3	− 642.6	1,287.2	1,291.2	45.74
	# goals (away)									
	0	1	2	3	4	≥ 5	log-like.	AIC	BIC	χ^2
observed	137	114	66	49	10	4	—	—	—	—
ZIP	137.0	110.4	78.1	36.8	13.0	4.8	− 553.3	1,110.6	1,118.5	6.86
GPD*	136.4	113.8	75.2	35.8	13.3	5.5	− 553.4	1,110.9	1,118.8	7.22
Touchard	134.7	118.0	73.1	34.8	13.4	6.0	− 553.9	1,111.8	1,119.7	8.19
hyper-Poisson	134.0	118.8	73.3	34.7	13.3	5.9	− 553.9	1,111.9	1,119.8	8.31
ZINB	137.0	110.4	78.0	36.8	13.0	4.8	− 553.3	1,112.6	1,124.4	6.84
COM-Poisson	132.4	121.9	72.8	33.7	13.0	6.2	− 554.6	1,113.2	1,121.1	9.73
neg. binomial	130.5	125.3	72.4	32.7	12.7	6.4	− 555.5	1,115.0	1,122.9	11.51
Poisson	115.0	137.5	82.1	32.7	9.8	2.9	− 559.5	1,121.0	1,124.9	19.93
NGPL	148.9	111.4	62.5	31.2	14.5	11.5	− 559.6	1,121.2	1,125.1	17.65
Poisson-Lindley	165.0	98.1	55.1	29.8	15.7	16.3	− 568.8	1,139.6	1,143.5	33.21

*Chandra, Roy and Ghosh (2013)

6.2. Dental caries data

In this example we consider data from a prospective study of school-children conducted in the urban area of Belo Horizonte, Brazil (Böhning et al. 1996). The sample consists of 797 Brazilian school children aged 7 years at the start of the study, in which data were recorded for the eight deciduous molars. As the counts range between 0 and 8, we define the truncated Touchard distribution as

$$p_k = \mathrm{P}[X = k] = \frac{\lambda^k (k+1)^\delta}{k!\mathrm{T}(\lambda, \delta)}, \tag{11}$$

where $0 \leq k \leq 8$, $\lambda > 0$, $\delta \in \mathbb{R}$ and $\mathrm{T}(\lambda, \delta) = \sum_{j=0}^{8} \lambda^j (j+1)^\delta / j!$. In Eq. (11), if $\delta = 0$ we have a truncated Poisson distribution. Figure 3 depicts the observed and predicted counts for truncated Touchard, truncated hyper-Poisson, truncated COM-Poisson, negative binomial and truncated Poisson models. The corresponding log-likelihood and χ^2 statistics are

Table 3. Maximum likelihood estimates of the Touchard distribution parameters, the sample mean and the sample variance for the Premier League data.

team	number of games	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\mu}$	$\hat{\sigma}^2$
away	380	2.02	− 1.205	1.195	1.445
home	380	2.27	− 0.945	1.574	1.897

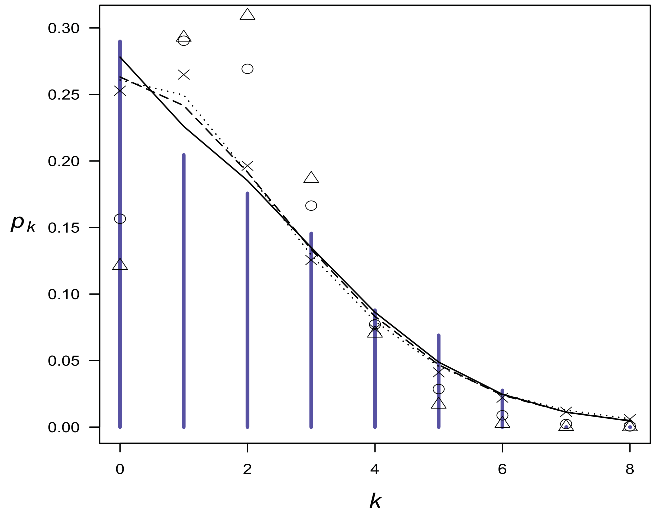

Figure 3. Dental caries data. Comparing observed (vertical bars) *versus* expected counts for $0 \leq k \leq 8$ (solid line = truncated Touchard distribution; dashed line = truncated hyper-Poisson; dotted line = truncated COM-Poisson; $\times$ = negative binomial; $\triangle$ = binomial and $\circ$ = truncated Poisson).

summarized in Table 4. Here, in a truncated form, the Touchard distribution compares favorably to the other suggested models.

6.3. Epileptic seizure counts data

Table 5 shows the epileptic seizure counts data used by Chakraborty (2010) and Bhati, Sastry, and Maha Qadri (2015) to compare some competing two-parameters distributions: negative binomial, weighted generalized Poisson (WGP), generalized Poisson-Lindley (GPL), weighted generalized Poisson-Lindley (WGPL), and new generalized Poisson-Lindley (NGPL). Here, we included hyper-Poisson, COM-Poisson, GPD (Chandra, Roy and Ghosh 2013), ZIP, ZINB and the Touchard distribution (with parameters estimated as $\hat{\lambda} = 4.49$ and $\hat{\delta} = -2.81$). As can be seen from Table 5, a high log-likelihood value in conjunction with a small χ^2 statistic reflect that the Touchard is a viable candidate for modeling this data. In this example, the likelihood ratio and the Wald statistics for Poissonness are respectively $D = 85.9$ and $W = 103.5$. As the p-values are less than 10^{-9}, the Poissonness of the data is rejected in favor of the Touchard model.

Table 4. Dental caries data: comparison of the log-likelihood and χ^2 statistics (the results are ordered by ascending AIC values).

	counts										
	0	1	2	3	4	5	6 to 8	log-like.	AIC	BIC	χ^2
observed	231	163	140	116	70	55	22	—	—	—	—
truncated Touchard	221.6	180.1	147.7	107.6	68.8	38.9	32.3	− 1,425.4	2,854.8	2,864.2	13.0
trunc. hyper-Poisson	209.7	192.6	152.7	106.5	66.3	37.3	32.0	− 1,429.4	2,862,8	2,872,2	20.3
trunc. COM-Poisson	207.9	198.8	152.6	103.0	63.4	36.4	34.9	− 1,433.9	2,871,7	2,881,1	26.7
negative binomial	201.5	211.1	156.5	100.1	58.8	32.8	36.2	− 1,444.5	2,893.0	2,902.4	42.3
binomial	96.7	233.3	246.4	148.7	56.1	13.5	2.3	− 1,587.8	3,177.6	3,182.3	570.0
truncated Poisson	124.8	231.4	214.6	132.6	61.5	22.8	9.3	− 2,228.1	4,458.2	4,462.9	202.6

Table 5. Epileptic seizure counts data: log-likelihood and χ^2 statistics (the models are ordered by ascending AIC values).

	counts												
	0	1	2	3	4	5	6	7	8	log-like.	AIC	BIC	χ^2
observed**	126	80	59	42	24	8	5	4	3	—			—
Touchard	126.0	81.0	58.3	39.0	23.4	12.6	6.1	2.7	1.9	− 593.06	1,190.1	1,197.8	3.7
hyper-Poisson	121.6	88.2	59.2	37.0	21.6	11.8	6.1	3.0	2.4	− 593.48	1,190.9	1,198.7	3.8
ZINB	126.0	79.2	61.7	39.3	22.3	11.7	5.8	2.8	2.2	− 592.88	1,191.7	1,203.3	2.5
COM-Poisson	120.4	90.7	59.6	36.2	20.7	11.4	6.0	3.1	3.0	− 594.07	1,192.1	1,199.9	4.4
NGPL*	122.0	91.0	58.7	35.2	20.5	11.2	6.4	3.3	2.5	− 594.48	1,191.0	1,194.8	5.8
GPL*	121.5	92.0	59.0	35.1	20.1	11.2	6.1	3.3	2.7	− 594.61	1,191.2	1,195.1	5.9
neg. binomial*	91.0	86.6	63.4	42.6	27.6	17.6	10.5	6.5	5.0	− 595.22	1,194.4	1,202.2	22.5
WGP**	118.1	95.8	59.9	34.5	19.2	10.6	5.8	3.9	3.2	− 595.83	1,193.7	1,197.5	7.2
GPD***	105.3	91.1	74.1	45.2	22.0	9.0	3.1	1,0	0.3	− 602.23	1,208.4	1,216.2	40.8
ZIP	99.6	89.6	80.6	48.4	21.8	7.8	2.3	0.6	0.2	− 609.11	1,222.2	1,229.9	84.5
Poisson*	74.9	115.7	89.3	46.0	17.8	5.5	1.4	0.3	0.0	− 636.05	1,274.1	1,278.0	256.5

*Bhati, Sastry, and Maha Qadri (2015), **Chakraborty (2010), ***Chandra, Roy and Ghosh (2013)

7. Conclusion

The Touchard model has been demonstrated to be potentially useful to describe non-Poisson counts. As a generalization of the Poisson model, the extra parameter δ controls the dispersion, while λ drives μ. As a result, when compared with the Poisson, the Touchard distribution is able to represent not only over- or underdispersed data, but count data with excess zeros as well (Figure 1). The simplicity of the Hessian matrix for maximum likelihood estimation is remarkable. Our data fitting examples in Section 6 illustrate the model's practical usefulness.

The Touchard distribution has appealing theoretical properties. As a member of the family of generalized power series distributions, recursive expressions for the moments were derived. Also, it belongs to the exponential family. As such, it allows for interesting properties such as sufficiency and the existence of a conjugate family of priors for Bayesian estimation (Kadane et al. 2006).

The numerical computation can be programmed easily in any language. We used the R software environment for our illustrations. Examples of R codes for parameter estimation and generation of Touchard deviates can be found at `https://1drv.ms/f/s!Apx60k7TMXze6U5ApjewoiKtfvre`.

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