IMO 2023 Problem 1

64th International Mathematical Olympiad

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Problem 1

Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with

$$1 = d_1 < d_2 < \cdots < d_k = n,$$

then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leq i \leq k - 2$.

Key Insight

Main Observation: The divisor structure of prime powers p^a forms a geometric progression $\{1, p, p^2, \ldots, p^a\}$, making the divisibility condition automatic:

$$d_i = p^{i-1} \mid p^i + p^{i+1} = p^i(1 + p) = d_{i+1}(1 + p)$$

For numbers with ≥ 2 distinct primes, there's always a "transition point" where the condition fails.

Answer: $\boxed{n = p^a \text{ where } p \text{ is prime and } a \geq 2}$
All composite integers satisfying the condition are exactly the **prime powers** with exponent at least 2.

Solution

Step 1: Exploration with Small Cases

n	Prime Power?	Divisors	Works?
$4 = 2^2$	Yes	$1, 2, 4$	✓
$6 = 2 \cdot 3$	No	$1, 2, 3, 6$	$\times$ $(2 \nmid 9)$
$8 = 2^3$	Yes	$1, 2, 4, 8$	✓
$9 = 3^2$	Yes	$1, 3, 9$	✓
$10 = 2 \cdot 5$	No	$1, 2, 5, 10$	$\times$ $(2 \nmid 15)$
$12 = 2^2 \cdot 3$	No	$1, 2, 3, 4, 6, 12$	$\times$ $(2 \nmid 7)$
$16 = 2^4$	Yes	$1, 2, 4, 8, 16$	✓
$25 = 5^2$	Yes	$1, 5, 25$	✓
$27 = 3^3$	Yes	$1, 3, 9, 27$	✓

Conjecture: The answer is exactly the prime powers $n = p^a$ with $a \geq 2$.

Step 2: Prime Powers Satisfy the Condition

Theorem 1. *For any prime p and integer $a \geq 2$, the number $n = p^a$ satisfies the given property.*

Chứng minh. The divisors of $n = p^a$ are exactly $\{1, p, p^2, \ldots, p^a\}$, so $d_j = p^{j-1}$ for $j = 1, 2, \ldots, a+1$.

For any $1 \leq i \leq a-1$, we verify $d_i \mid d_{i+1} + d_{i+2}$:

$$d_{i+1} + d_{i+2} = p^i + p^{i+1} = p^i(1+p)$$

Since $d_i = p^{i-1}$ divides $p^i = p \cdot p^{i-1}$, we have:

$$d_i = p^{i-1} \mid p^i(1+p) = d_{i+1} + d_{i+2}$$

Therefore the condition holds for all valid i. □

Step 3: Non-Prime-Powers Fail the Condition

Theorem 2. *If n has at least two distinct prime divisors, then n fails the condition.*

Chứng minh. Let $p < q$ be the two smallest distinct prime divisors of n. The divisor sequence begins:

$$d_1 = 1, \quad d_2 = p$$

The key is analyzing what happens at d_3 and d_4.

Case 1: $q < p^2$ (so $d_3 = q$)

Then $d_4 \geq \min(p^2, pq)$.

If $d_4 = p^2$: We need $p \mid d_3 + d_4 = q + p^2$, which requires $p \mid q$. But q is prime and $q > p$, so $p \nmid q$. **Contradiction.**

If $d_4 = pq$: We need $p \mid q + pq = q(1+p)$. Since $\gcd(p, q) = 1$ and $p \nmid (1+p)$, this fails. **Contradiction.**

Case 2: $p^2 \leq q$ (so $d_3 = p^2$ if $p^2 \mid n$)

Eventually, q must appear in the divisor list. Let $d_j = p^m$ and $d_{j+1} = q$ for some m.

At $i = j-1$: $d_i = p^{m-1}$, and we need $p^{m-1} \mid p^m + q$.

Since $p^{m-1} \mid p^m$, we need $p^{m-1} \mid q$. But q is prime and $q \neq p$. **Contradiction.**

In all cases, non-prime-powers fail the condition. □

Conclusion

The complete set of composite integers satisfying the property is:

$$\boxed{n = p^a \text{ where } p \text{ is a prime and } a \geq 2}$$

Examples: $4, 8, 9, 16, 25, 27, 32, 49, 64, 81, 121, 125, \ldots$

Commentary

Techniques Used

- **Small cases exploration** - Testing $n = 4, 6, 8, \ldots$ to identify pattern
- **Divisibility chain analysis** - Studying the structure $d_i \mid d_{i+1} + d_{i+2}$
- **Proof by contradiction** - Showing non-prime-powers fail
- **Prime factorization structure** - Key insight about geometric vs non-geometric divisor sequences

Why This is an IMO P1

- Accessible entry point (small cases)
- Clean answer (prime powers)
- Requires careful case analysis but not deep theory
- Tests mathematical maturity more than specialized knowledge

Generalization

Question: For which n does $d_i \mid d_{i+1} + d_{i+2} + \cdots + d_{i+m}$ hold for all valid i?

Conjecture: The answer is still prime powers, since the geometric structure of $\{1, p, p^2, \ldots\}$ makes any sum divisible by earlier terms.