No-hair theorem and Spontaneous scalarization of BHs

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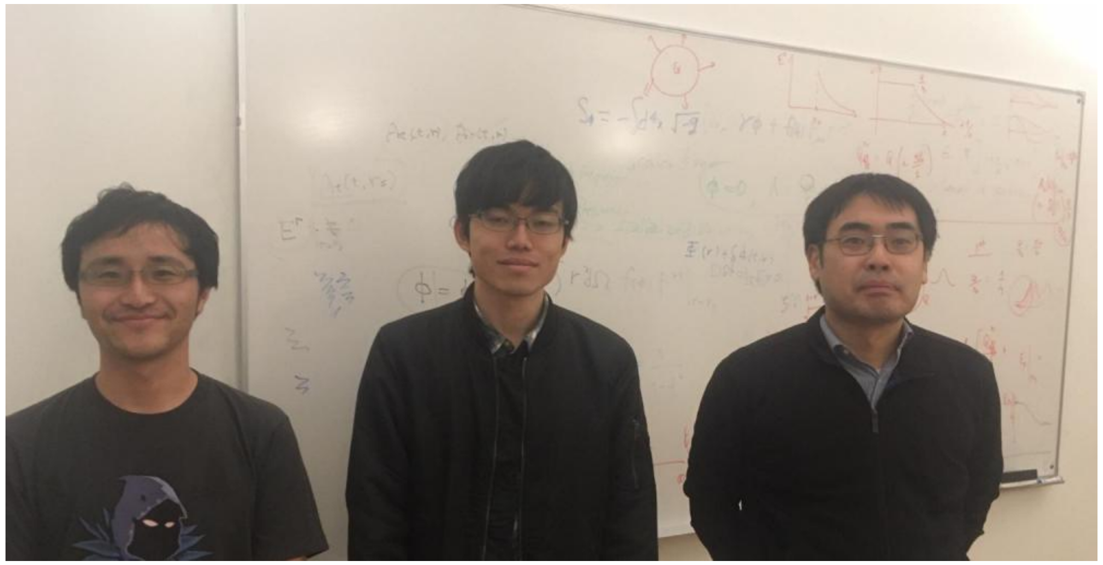
My collaborators



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Outline of my talk

1. Introduction
2. No-hair theorem for ST theory
3. Spontaneous scalarization of BH
    1. Scalar-Gauss-Bonnet theory
    2. Other models
    3. Dynamical behavior of the scalarizaion
4. Summary

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Modification of gravity

- General Relativity (GR)

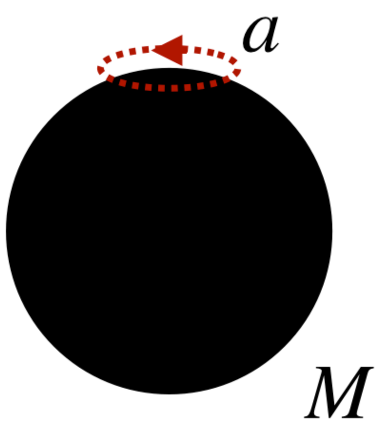
$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi} R - 2\Lambda + \mathscr{L}_m \right)$$

  - GR is quite successful gravitational theory.
  - It can explain cosmology, gravitational waves, and so on …
- Why modify gravity ?
  - Dark Energy ?, Dark Matter ?, Quantum correction ?
  - Test of GR
  - How special is GR ?
  - How universal are several GR's properties and theorems ?

➡ Let us modify gravity, and compare with GR.

No-hair theorem of BH

- BH is the simplest compact object in GR.
  - vacuum solution of GR
  - No-hair theorem of BH
  - parameters : M, a, (Q)



- How universal is no-hair theorem ?

<u>no-hair</u>

- Real scalar field (Bekenstein 1972)
- Non-canonical real scalar (A.Graham et al 2014)
- Horndeski theory with some conditions (L.Hui et al 2013)

....

<u>solution with scalar hair</u>

- Shift symmetric Scalar Gauss Bonnet gravity (T.P.Sotiriou et al 2014)
- GR with a conformally coupled scalar field (N.M.Bocharova et al.1970)

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Horndeski theory

$$X := -\frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi$$
$$\Phi_\mu := \nabla_\mu \Phi$$

- Scalar-Tensor theory (with 2nd order EoM, 2+1 DoF)
  - Physical degrees of freedom : $(g_{\mu\nu}, \Phi)$
  - action

$$S[g, \Phi, A] = \int d^4x \sqrt{-g} \sum_{i=2}^{5} \mathscr{L}^i_{\rm ST}$$

$$\mathscr{L}^2_{\rm ST} = G_2(\Phi, X)$$

$$\mathscr{L}^3_{\rm ST} = G_3(\Phi, X) \Phi_\mu{}^\mu$$

$$\mathscr{L}^4_{\rm ST} = G_4(\Phi, X) R + G_{4,X}(\Phi, X) \left((\Phi_\mu{}^\mu)^2 - \Phi_\mu{}^\nu \Phi_\nu{}^\mu \right)$$

$$\mathscr{L}^5_{\rm ST} = G_5(\Phi, X) G_{\mu\nu} \Phi^{\mu\nu} - \frac{G_{5,X}(\Phi, X)}{6} \left((\Phi_\mu{}^\mu)^3 - 3(\Phi_\mu{}^\mu)(\Phi_\nu{}^\rho \Phi_\rho{}^\nu) + 2 \Phi_\mu{}^\nu \Phi_\nu{}^\rho \Phi_\rho{}^\mu \right)$$

* $G_i(\Phi, X)$ $(i = 2,3,4,5)$ are arbitrary functions.

  - If $G_i = G_i(X)$, the theory has shift symmetry.

$$\Phi \to \Phi + c \quad \Longrightarrow \quad J_\Phi$$: Noether curent

No-hair theorem

- No- hair theorem for Shift-symmetric Horndeski theory

  (L.Hui et. al. (2013))

  - Assumptions

    1. Static, spherically symmetric, asymptotically flat spacetime
    2. Scalar field is also static and spherical symmetry.
    3. $\nabla_\mu \Phi \to 0$ at $r \to \infty$
    4. $J_{\Phi\mu} J^\mu_\Phi < \infty$ on and outside the BH horizon
    5. Action has canonical kinetic term of scalar field. : $X \subset \mathscr{L}$
    6. $G_{2,3,4,5}(X)$ are analytic at $X = 0$.

  - Statement

    - Under these assumptions, BHs can not have a nontrivial scalar hair.

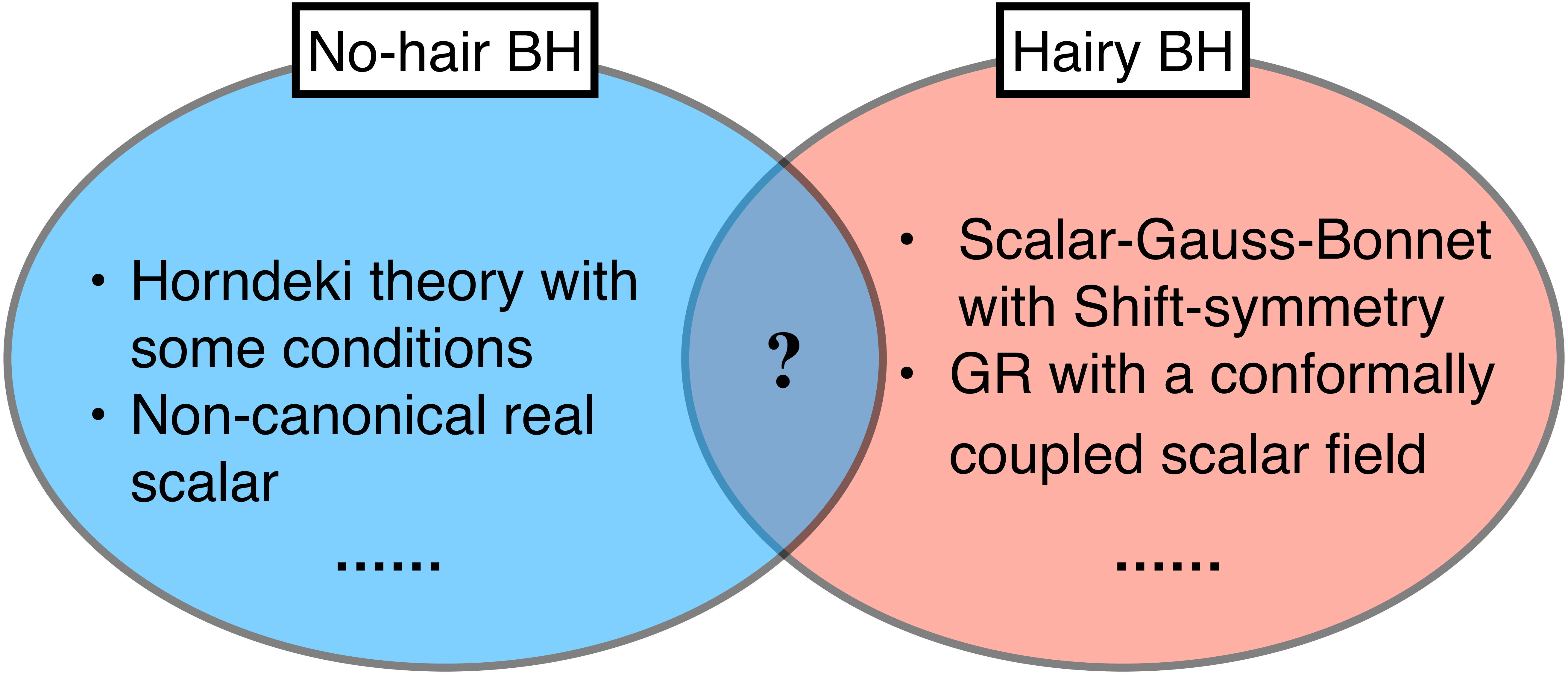
No-hair theorem

- Comment 1 : Extension
  - We can naturally extend no-hair theorem to Scalar-Vector-Tensor theory with additional conditions.

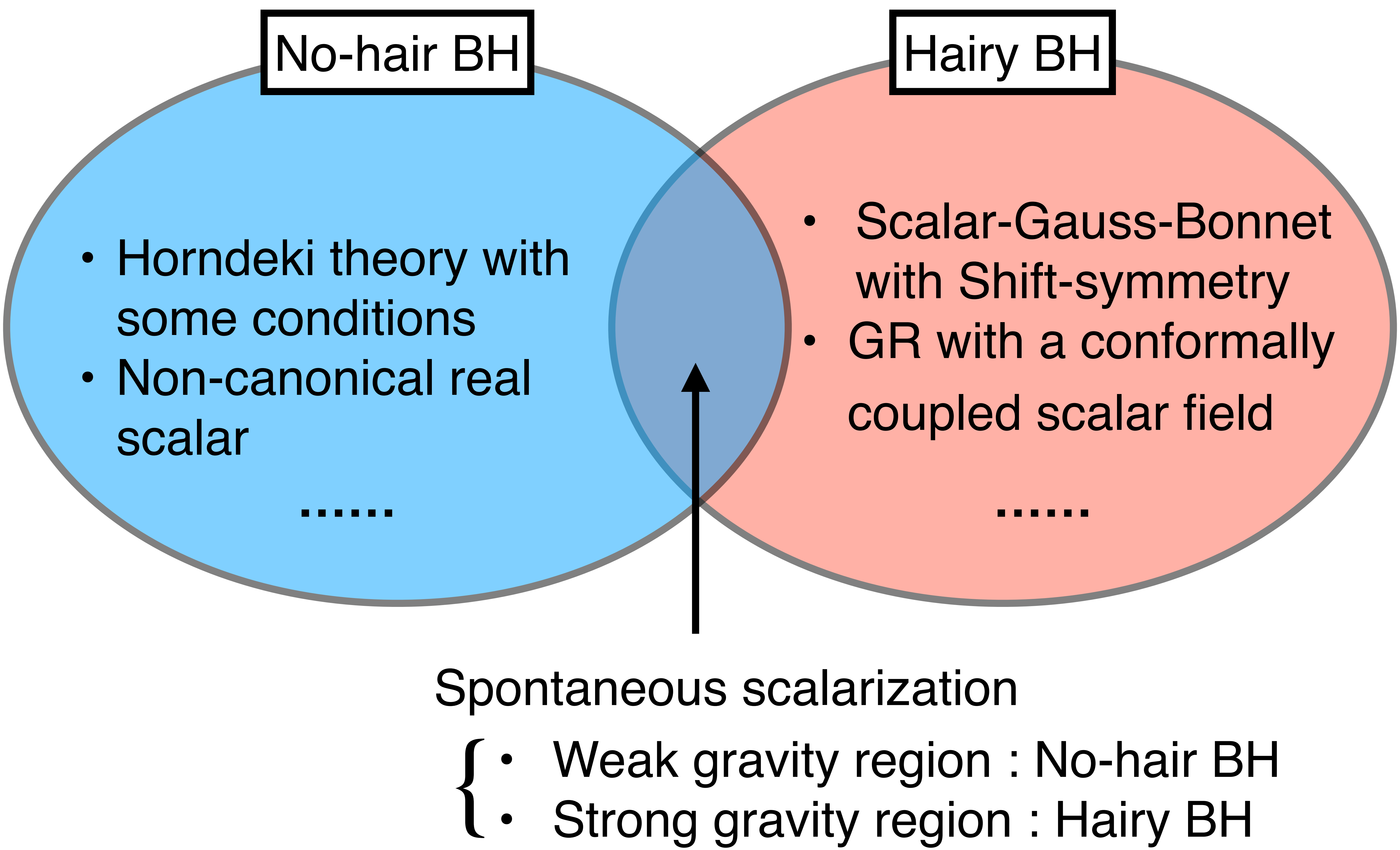
(T.Ikeda et al (2019))

- Comment 2 : How wide theories are included ?
  - Scalar-Gauss-Bonnet gravity is “not” included.

$$f(\Phi)\mathcal{G}_{\mathrm{GB}} \Longleftrightarrow \begin{aligned} G_2 &= 8f^{(4)}(\Phi)X^2\left(3-\ln X\right) \\ G_3 &= 4f^{(3)}(\Phi)X\left(7-3\ln X\right) \\ G_4 &= 4f^{(2)}(\Phi)X\left(2-\ln X\right) \\ G_5 &= -4f'(\Phi)\ln X \end{aligned}$$



No-hair BH
Hairy BH
• Horndeki theory with some conditions
• Non-canonical real scalar
......
?
• Scalar-Gauss-Bonnet with Shift-symmetry
• GR with a conformally coupled scalar field
......


No-hair BH
Hairy BH
• Horndeki theory with some conditions
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......
• Scalar-Gauss-Bonnet with Shift-symmetry
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......
Spontaneous scalarization
• Weak gravity region : No-hair BH
• Strong gravity region : Hairy BH

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Spontaneous scalarization

- Spontaneous scalarization is one of the mechanism to produce a scalar hair in extreme situation.

$$\Box\Phi = \underline{\mu_{\rm eff}^2(g_{\mu\nu}, T)}\Phi$$

cf:T. Damour et al (1993)
T.Harada (1997)

$\mu_{\rm eff}^2 < 0$ around $\begin{cases} \text{dense matter region} \\ \quad \Rightarrow \text{Neutron star scalarizarion} \\ \boxed{\text{strong gravity region} \quad \Rightarrow \text{Black hole scalarizarion}} \end{cases}$

➡ The trivial scalar profile becomes unstable, and scalar hair appears.

- The theory can be distinguish from GR around extreme situation.

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Scalar-Gauss-Bonnet theory

- Action

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\nabla_\mu\Phi\,\nabla^\mu\Phi + f(\Phi)\mathscr{G}_{\rm GB}\right)$$

$$\mathscr{G}_{\rm GB} := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

- EOM

$$\Box\,\Phi = -f'(\Phi)\mathscr{G}$$

- We assume $f'(\Phi_0) = 0$, then Schwarzschild BH is solution with $\Phi = \Phi_0$.
- Around the Schwarzschild BH, $\Phi = \Phi_0 + \delta\Phi$

$$\Box\,\delta\Phi = \underbrace{-f''(\Phi_0)\mathscr{G}_{\rm GB}}_{\text{effective mass}}\delta\Phi$$

Scalar-Gauss-Bonnet theory

- We can proof following theorem.

cf: $\Box\,\delta\Phi = -f''(\Phi_0)\mathscr{G}_{\rm GB}\delta\Phi$

  - Assumption :
    - stationary (Killing vector :$\vec{\xi}$) and asymptotic flat BH spacetime
    - Event horizon is a Killing horizon associated with $\vec{\xi}$
    - $\mathscr{L}_{\xi}\Phi = 0$
    - $-f''(\Phi)\mathscr{G}_{\rm GB} > 0$ in the whole region
  - Then :
    - Scalar field is constant
- This theorem means the scalar field may be able to have nontrivial configuration when $-f''(\Phi)\mathscr{G}_{\rm GB} < 0$ at some regions.

negative mass square

Spontaneous scalarization

- Spontaneous scalarization of BH in quadratic Scalar-Gauss-Bonnet theory

(H.O.Silva et al.(2018), G.Antoniou et al. (2018), D.D.Doneva et al. (2018))

$$S = \int d^4x \sqrt{-g} \left(R - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi + f(\Phi) \mathscr{G}_{\mathrm{GB}} \right)$$

with $f(\Phi) = \dfrac{\eta \Phi^2}{8}$

- EOM : $\Box \Phi = \mu_{\mathrm{eff}}^2 \Phi$
- effective mass : $\mu_{\mathrm{eff}}^2(r) = -(\eta/4) \mathscr{G}_{GB}$
- Around Schwarzschild BH : $\mathscr{G}_{\mathrm{GB}}(r) \Big|_{\mathrm{Sch}} = 48 M^2 / r^6 > 0$

$$\mu_{\mathrm{eff}}^2(r) < 0 \quad (\eta > 0)$$

➡ Scalar field can have non-trivial profile.

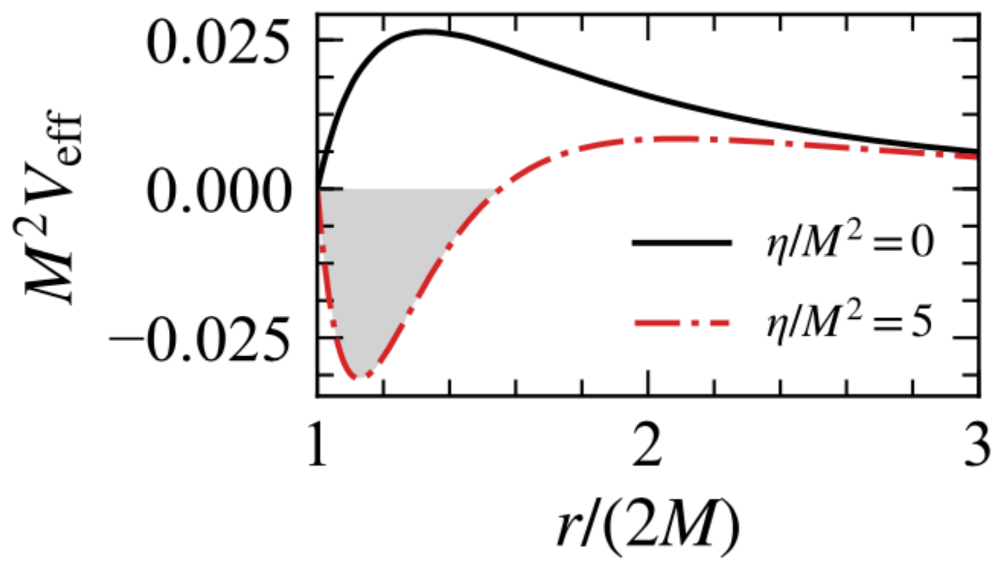
Spontaneous scalarization

$$\Phi(t,r,\theta,\phi) = \sum_{l,m} \frac{e^{i\omega t}\tilde{\sigma}_{lm}(r)}{r} Y_{lm}(\theta,\phi) + \Phi_0$$

- Perturbation around Schwarzschild BH

$$-\partial_{r_*}^2 \tilde{\sigma}_{lm\omega} + V_{\rm eff}(r)\tilde{\sigma}_{lm\omega} = \omega^2 \tilde{\sigma}_{lm\omega}$$

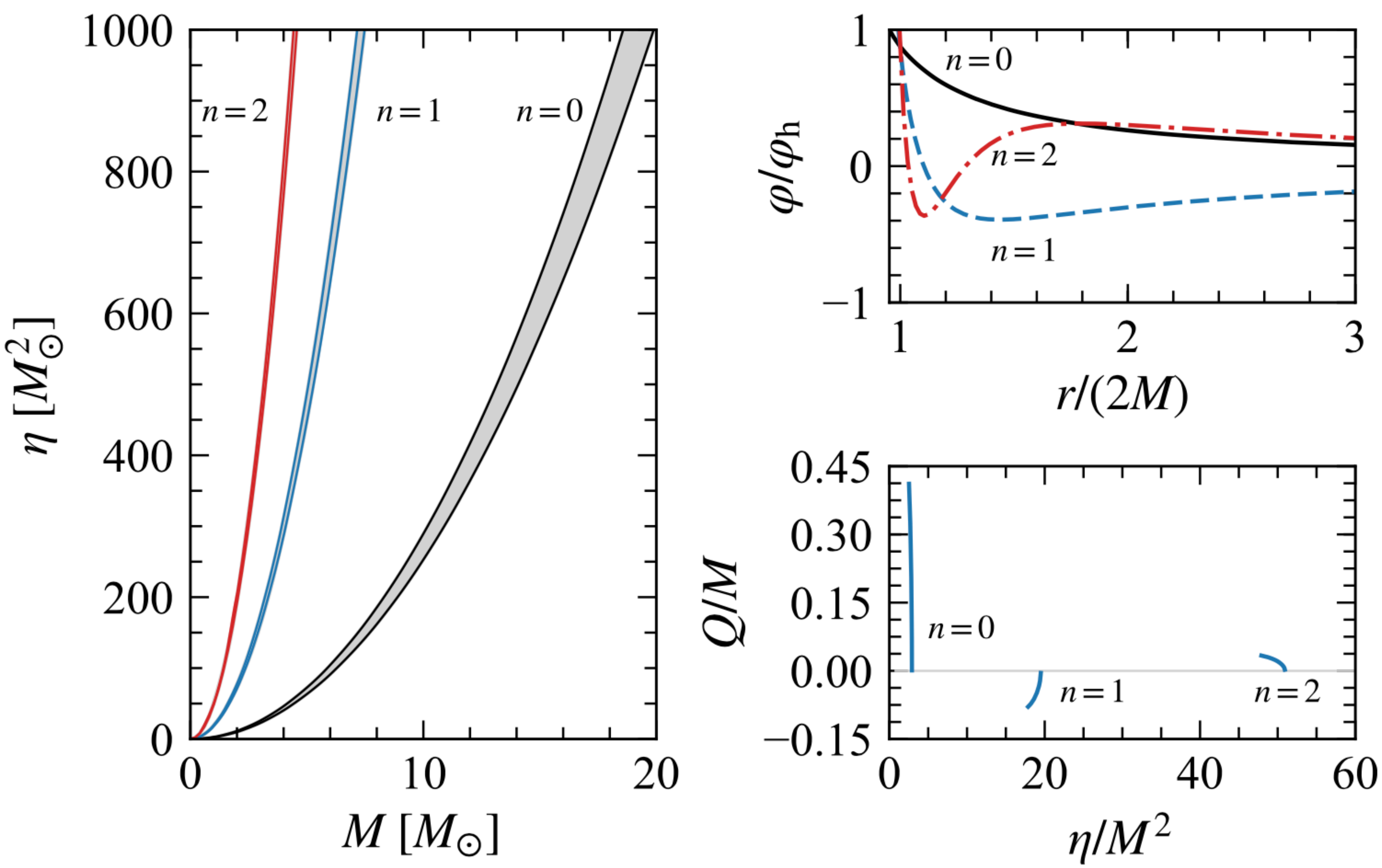
$$V_{\rm eff}(r) = (r-2M)\left(\frac{2M}{r^7}\left(r^3 - 6M\eta\right) + \frac{l(l+1)}{r^3}\right)$$



- Schwarzschild BH is unstable when $\frac{10}{3}M^2 < \eta$
- New scalarized BH appears.

Spontaneous scalarization

- Scalarized BH
  - Boundary condition : regular on $r_{\rm H}$, $\Phi \to 0\ (r \to \infty)$
  - BHs are labeled by nodes and $\Phi(r_{\rm H})$.



Spontaneous scalarization

- Test field analysis

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi + f(\Phi) \mathcal{G}_{\rm GB} \right)$$

- Let us consider $G \to 0$ with fixed $GM = \text{const}$.
  - $\mathcal{O}(G^{-1})$: Schwarzschild BH
  - $\mathcal{O}(G^0)$: $\Box \Phi = - f'(\Phi) \mathcal{G}$
- In this limit,

Scalarized BH $\Rightarrow$ Schwarzschild BH + (test) non-trivial scalar profile

Spontaneous scalarization

- Stability analysis of test field

$$f(\Phi) = \frac{\eta \Phi^2}{8}$$

$$\Phi(t, r, \theta, \phi) = \sum_{l,m} \frac{e^{i\omega t}\tilde{\sigma}_{lm}(r)}{r} Y_{lm}(\theta, \phi) + \Phi_0(r)$$

non-trivial profile around Schwarzschild BH

- In quadratic coupling, the scalar field equation is linear.

➡ Effective potential is same as one of constant scalar field.

$$V_{\text{eff}}(r) = (r - 2M)\left(\frac{2M}{r^7}\left(r^3 - 6M\eta\right) + \frac{l(l+1)}{r^3}\right)$$

➡ Test non-trivial scalar field is unstable.

- All scalarized BHs of quadric coupling are unstable.

(J.L.Blazquez-Salcedo et al (2018))

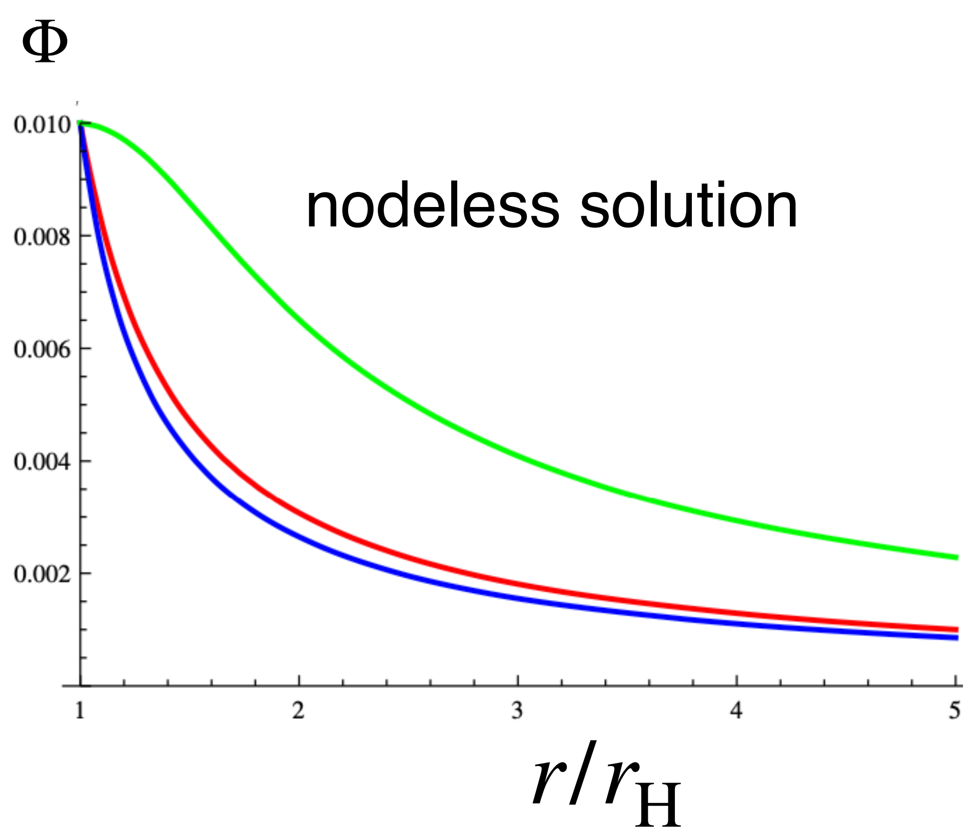
Spontaneous scalarization

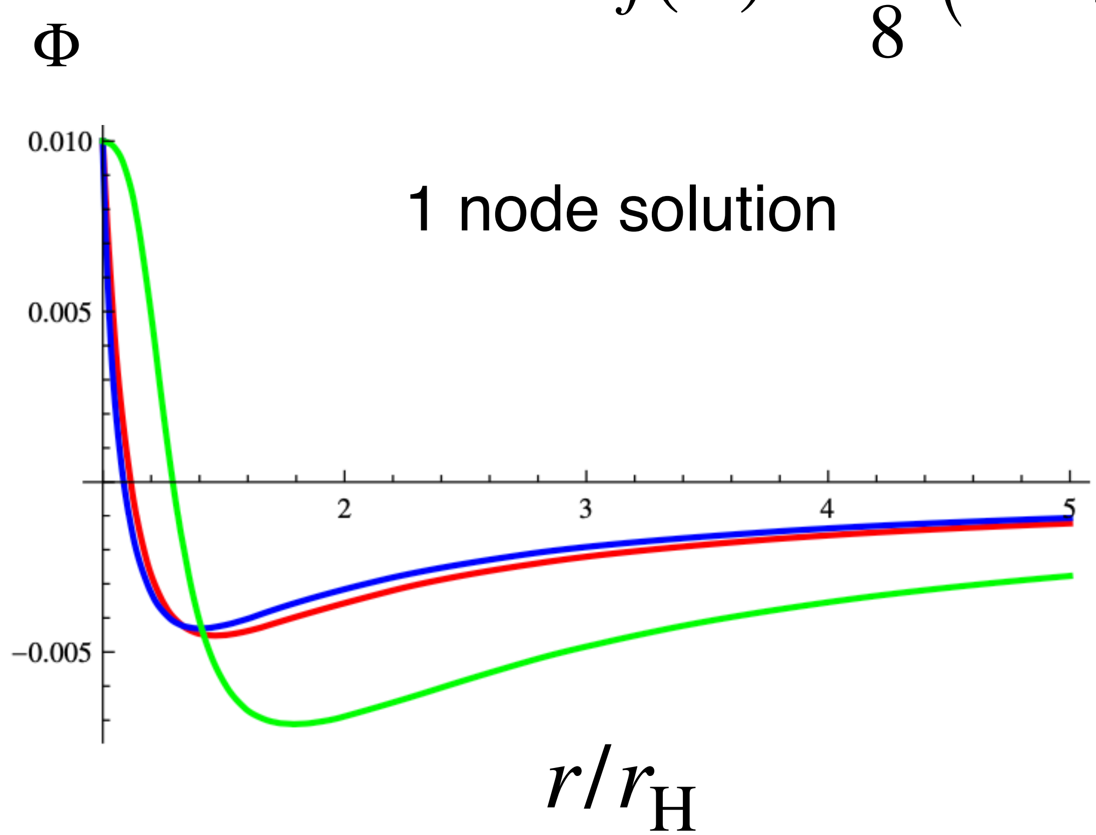
- Higher order coupling can stabilize the scalarized BH.
  - Quartic coupling

M.Minamitsuji, T.Ikeda (2019), H.O.Silva (2019)

$$S = \int d^4x \sqrt{-g} \left(R - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi + f(\Phi) \mathscr{G}_{\rm GB} \right)$$

$$f(\Phi) = \frac{\eta}{8} \left(\Phi^2 + \alpha \Phi^4 \right)$$





$(\eta/r_{\rm H}^2, \alpha) = (0.725, 0), (0.338, 10000), (7.31, -4990)$ $\quad (4.87, 0), (3.09, 10000), (20.2, -4990)$

Spontaneous scalarization

- Stability analysis of test field $G \to 0,\ GM : \mathrm{fix}$

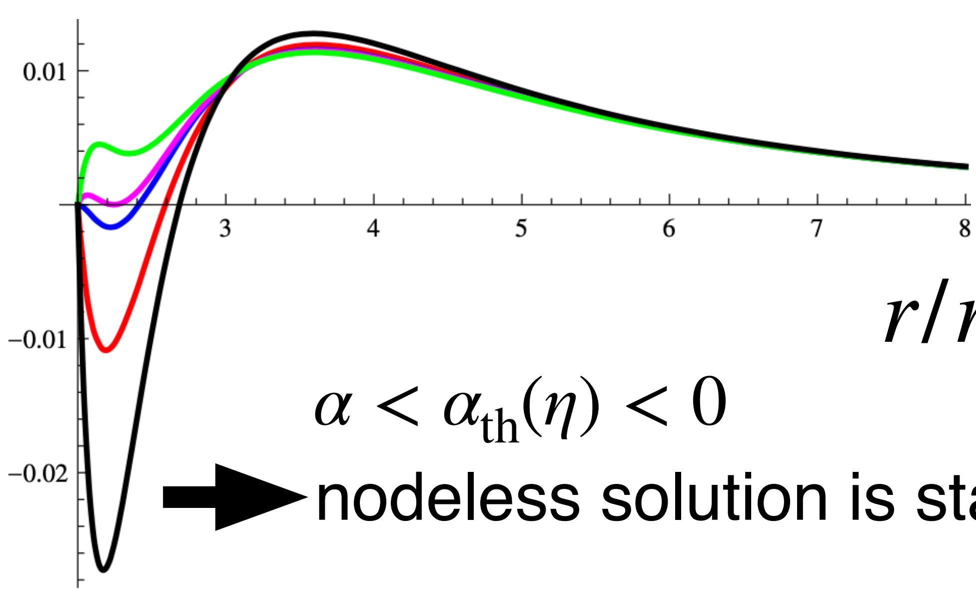
$$f(\Phi) = \frac{\eta}{8}\left(\Phi^2 + \alpha\Phi^4\right)$$

$$\Box\Phi = -\frac{\eta}{4}\left(\Phi + 2\alpha\Phi^3\right)\mathscr{G}_{\mathrm{GB}}$$

$$V_{\mathrm{eff}}(r) = (r-2M)\left(\frac{2M}{r^7}\left(r^3 - 6M\eta(1 + 6\alpha\Phi_0(r)^2)\right) + \frac{l(l+1)}{r^3}\right)$$

if $\alpha < 0$, this term contribute plus.

$M^2V_{\mathrm{eff}}(r)$ nodeless solution

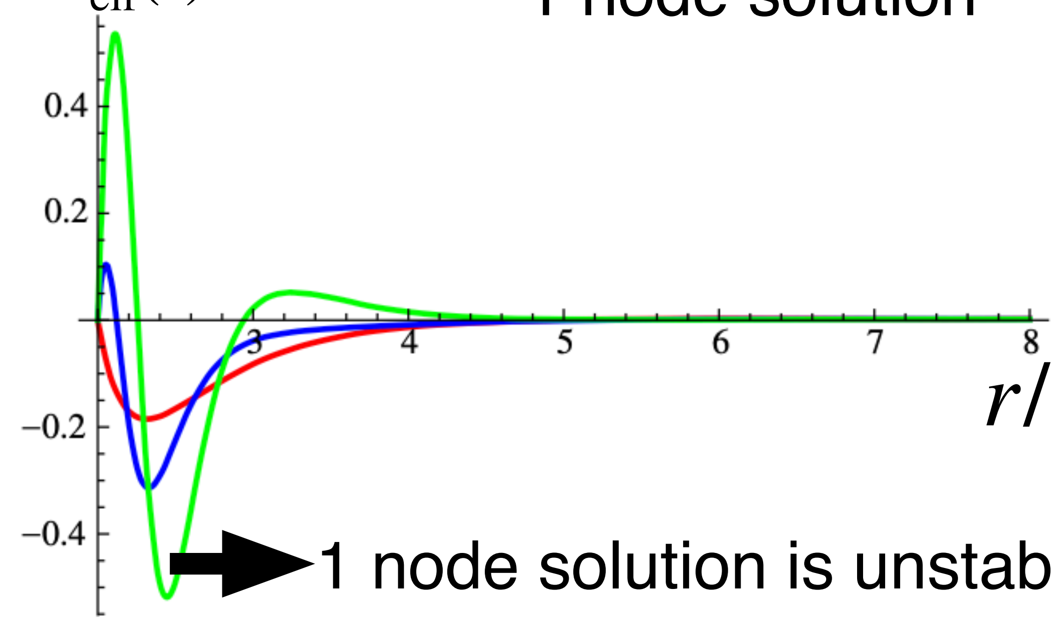


r/r_{H}

$\alpha < \alpha_{\mathrm{th}}(\eta) < 0$

→ nodeless solution is stable

$M^2V_{\mathrm{eff}}(r)$ 1 node solution



r/r_{H}

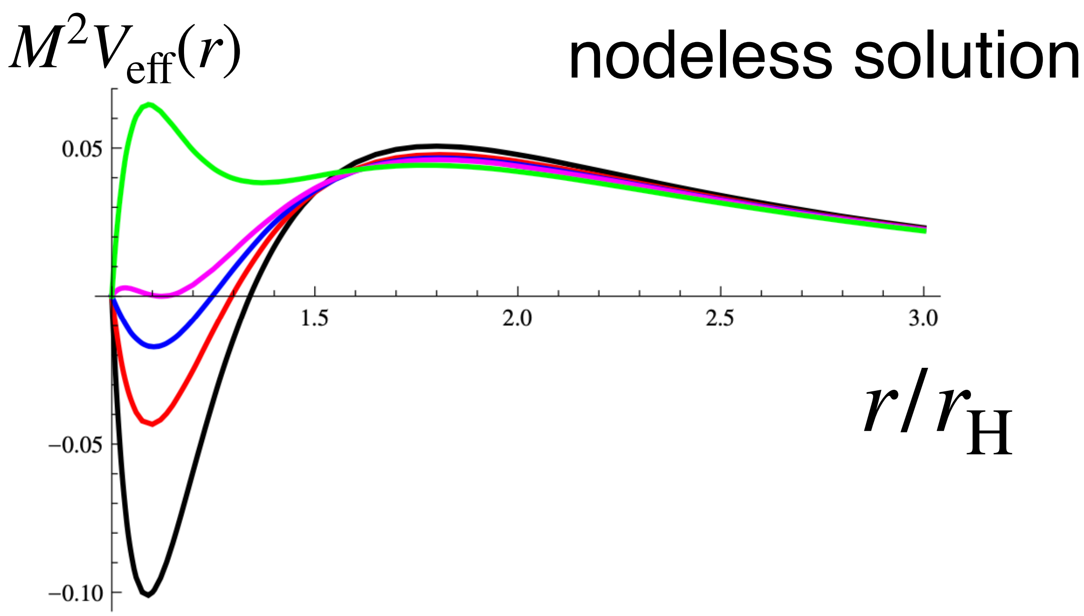
→ 1 node solution is unstable

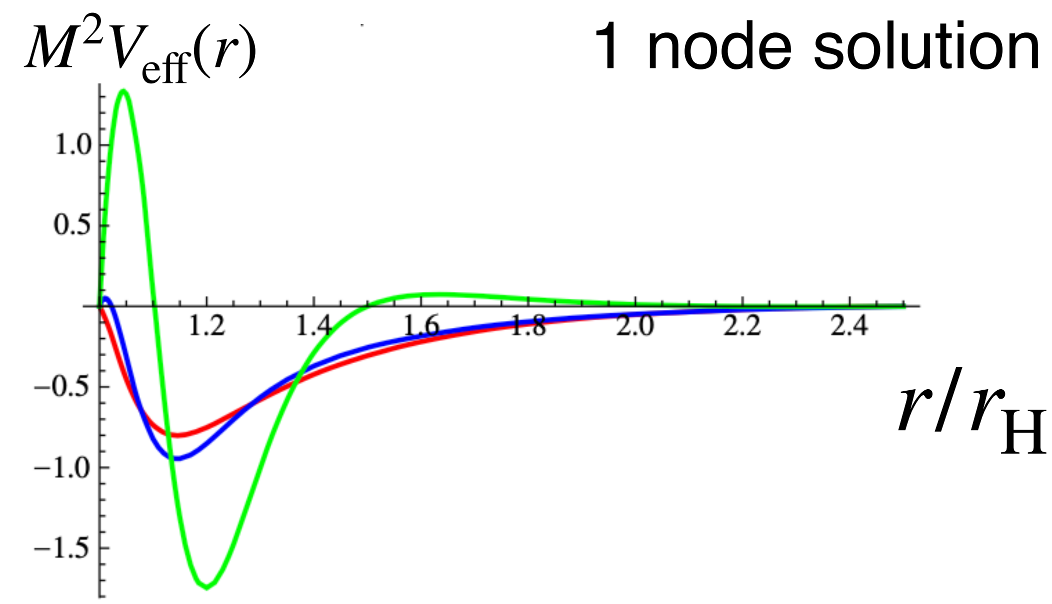
$(\eta/M^2, \alpha) = (2.14, 0.3), (2.9, 0), (3.32, -0.1), (3.4, -0.115), (3.59, -0.15)$

Spontaneous scalarization

- Stability of the scalarized BH

$$f(\Phi) = \frac{\eta}{8}\left(\Phi^2 + \alpha\Phi^4\right)$$





- Nodeless scalarized BH solution can stabilize by non-linear term.
- Scalarized BH with one or more nodes are unstable.

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Spontaneous scalarization

- Other model of the spontaneous scalarizarion
  - Einstein-Maxwell-Scalar model C.A.R.Herdeiro et al (2018)

$$S = \int d^4x \sqrt{-g} \left(R + \frac{1}{2} (\nabla \Phi)^2 - \frac{1}{4} f(\Phi) F^{\mu\nu} F_{\mu\nu} \right)$$

  - Scalar-Vector-Tensor with double-dual Riemann T.I et al (2019)

$$S = \int d^4x \sqrt{-g} \left(R + \frac{1}{2} (\nabla \Phi)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + H(\Phi) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right)$$

$$L^{\mu\nu\alpha\beta} = \frac{1}{4} \epsilon^{mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} R_{\rho\sigma\rho\sigma}$$

Reissner-Nordström BH + constant scalar field ➡ Charged BH + non-trivial scalar field

  - Which class of ST theory can realize scalarization ?

➡ Only theory close to SGB gravity M.Minamitsuji, T.I, (2019)

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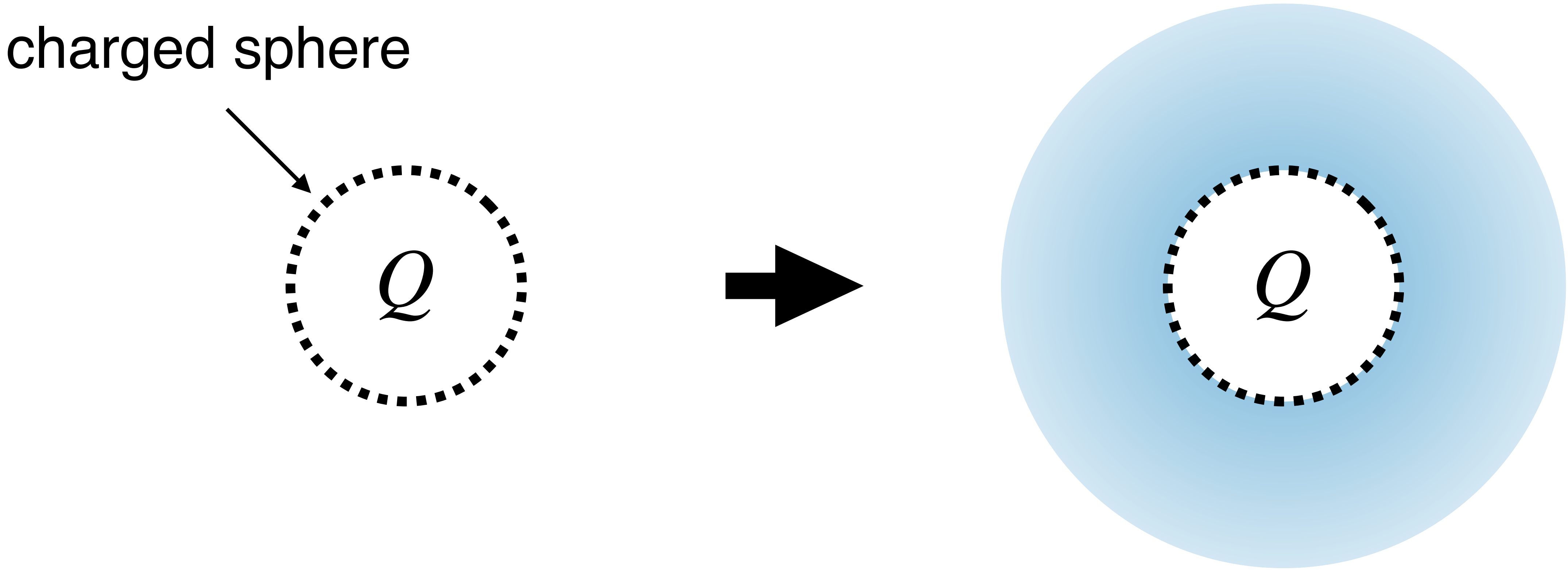
Scalarizaion on charged sphere

- Spontaneous scalarization is dynamical process.
  - To understand the process, let us consider simple toy model.

C.A.R.Herdeiro, TI, et al (2018)

- Scalar-Maxwell theory on Minkowski spacetime with charged sphere.

$$S = -\int d^4x \left(2\partial^\mu \Phi \partial_\mu \Phi + f(\Phi) F^{\mu\nu} F_{\mu\nu} \right)$$



Scalarizaion on charged sphere

- Equation of motion

$$\begin{cases} \Box\Phi - \frac{1}{4}f'(\Phi)F^{\mu\nu}F_{\mu\nu} = 0 \\ D_\mu\left(f(\Phi)F^{\mu\nu}\right) = 0 \end{cases}$$

- First, we construct spherically symmetric scalarized solution.

$$\Phi = \Phi_0(r) \qquad A_\mu = (A_0(r),0,0,0)$$

$$\begin{cases} \Phi_0''(r) + \frac{2}{r}\Phi_0'(r) + \frac{1}{2}f'(\Phi_0)A_0'(r)^2 = 0 \\ A_0''(r) + \frac{2}{r}A_0'(r) + \frac{f'(\Phi_0)}{f(\Phi_0)}\Phi_0'(r)A_0'(r) = 0 \end{cases} \Rightarrow A_0'(r) = \frac{Q}{r^2 f(\Phi_0)}$$

$$\Rightarrow \Phi_0''(r) + \frac{2}{r}\Phi_0'(r) + \frac{Q^2}{2r^4}\frac{f'(\Phi_0)}{f(\Phi)^2} = 0$$

Scalarizaion on charged sphere

- Boundary conditions
  - Asymptotic region : $\Phi_0 \to 0 \quad (r \to \infty)$
  - On the charged sphere
    - Dirichlet B.C. (D): $\Phi_0 |_{r_s} = 0$
    - Neumann B.C.(N) : $\partial_r \left(r^{-1} \Phi_0 \right) |_{r_s} = 0$
    - Radiative B.C.(R) : $(\partial_r \Phi_\omega + i\omega\Phi_\omega) |_{r_s} = 0$

$\Phi_0 |_{r_s} = \Phi_{\mathrm{min}}$

- Coupling function
  - inverse quartic model

$$f(\Phi)^{-1} = 1 - a\Phi^2 + \frac{k^2 a^2}{4}\Phi^4$$

  - inverse cosine model

$$f(\Phi)^{-1} = \cos(\sqrt{2a}\Phi)$$

We can solve these models analytically.

Scalarizaion on charged sphere

- Scalarized solution with $\Phi(r \to \infty) \to 0$
  - inverse quartic model

$$\bar{\Phi}(r) = \sqrt{\frac{ak^2}{2}}\Phi_0 = \sqrt{1-\sqrt{1-C_0}}\,\mathrm{sn}\left(\frac{Q}{\sqrt{2}r}\sqrt{1+\sqrt{1-C_0}}, \frac{1-\sqrt{1-C_0}}{1+\sqrt{1-C_0}}\right)$$

Jacobi elliptic sine function

  - inverse cosine model

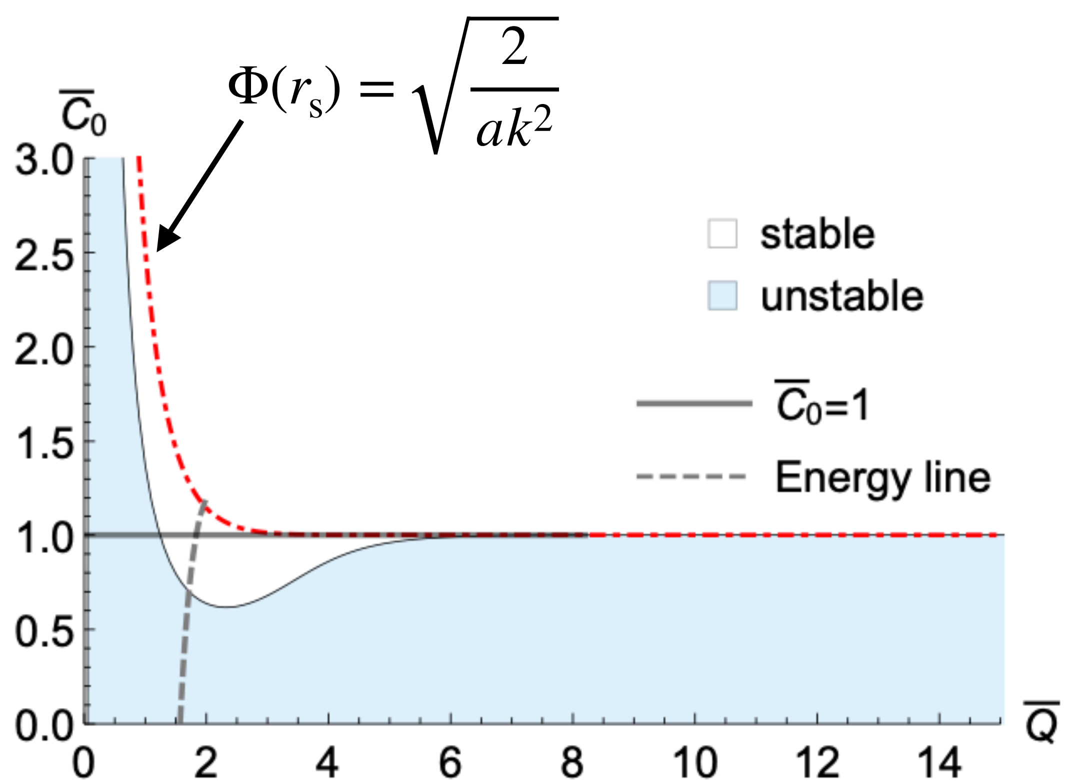
$$\bar{\Phi} = \sqrt{\frac{a}{2}}\Phi_0 = \mathrm{am}\left(\sqrt{\frac{C_1}{2}}\frac{Q}{r}, \frac{2}{C_1}\right)$$

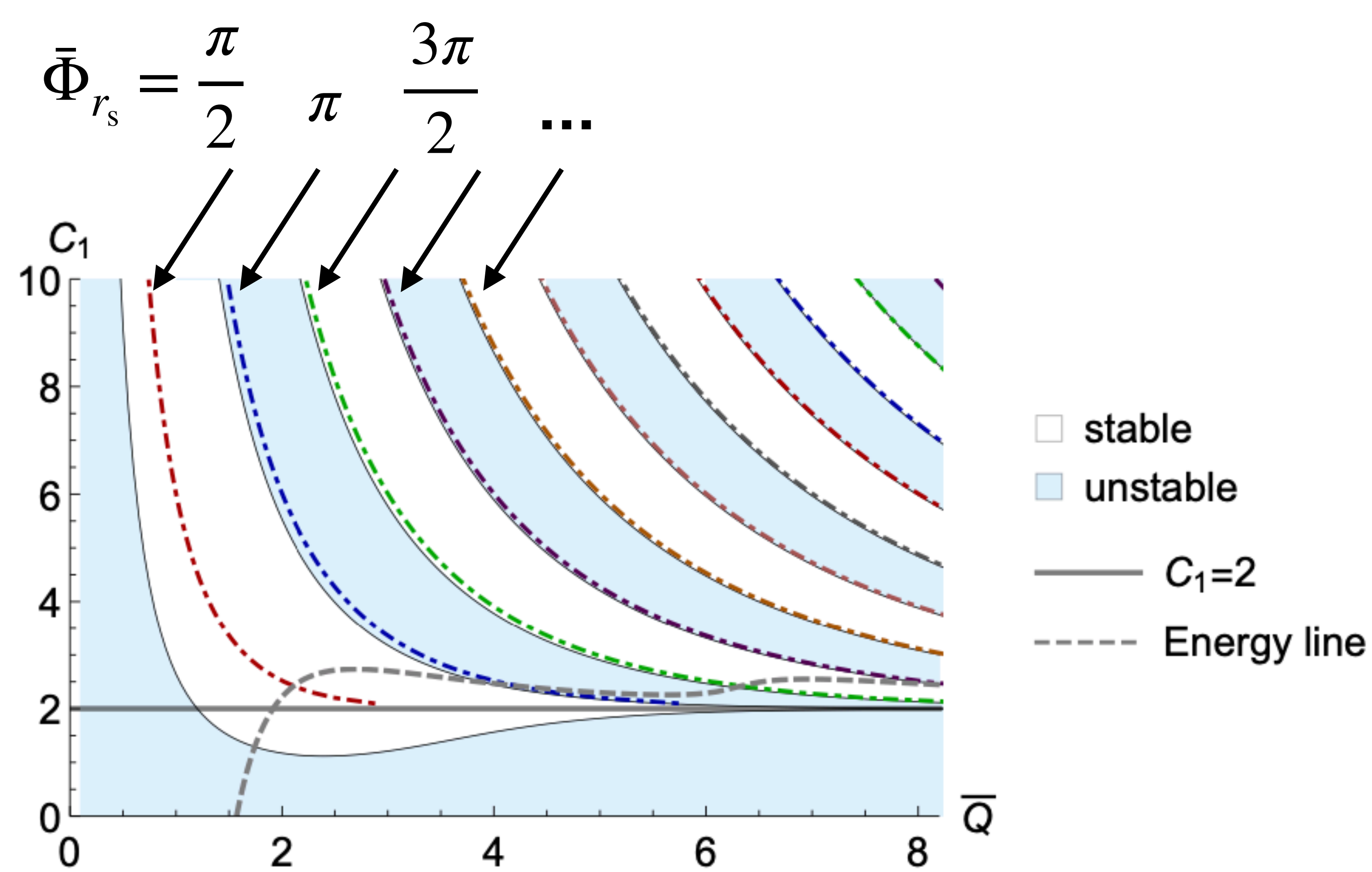
Jacobi amplitude

  - Constant C_0, C_1 are determined from boundary condition on the sphere.
- We can check the stability of each solutions with each B.C.

Scalarizaion on charged sphere

- (Un)stable region of the scalarized solutions.
  - Radiative boundary condition





Inverse quartic

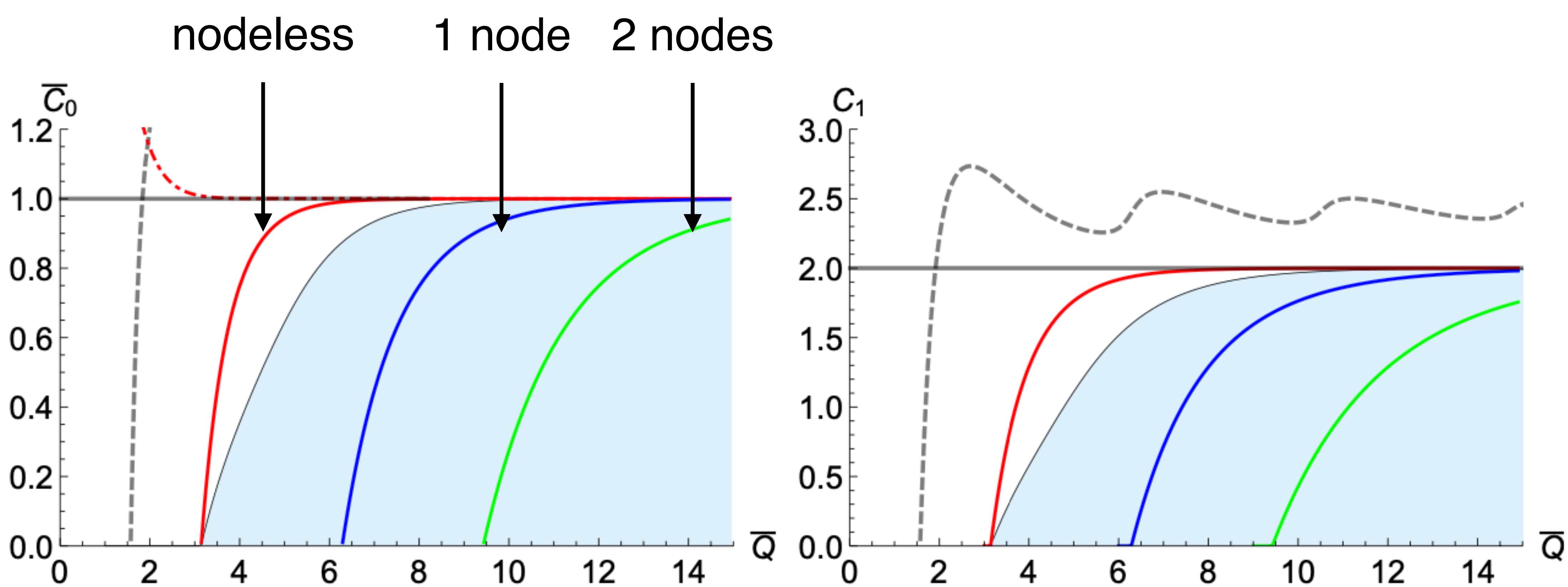
Inverse cosine

Scalarized solution is stable.

Scalarized solution with $\Phi(r_s) = \frac{2n+1}{2}\pi$ is stable.

Scalarizaion on charged sphere

- (Un)stable region of the scalarized solutions.
  - Dirichlet Boundary condition

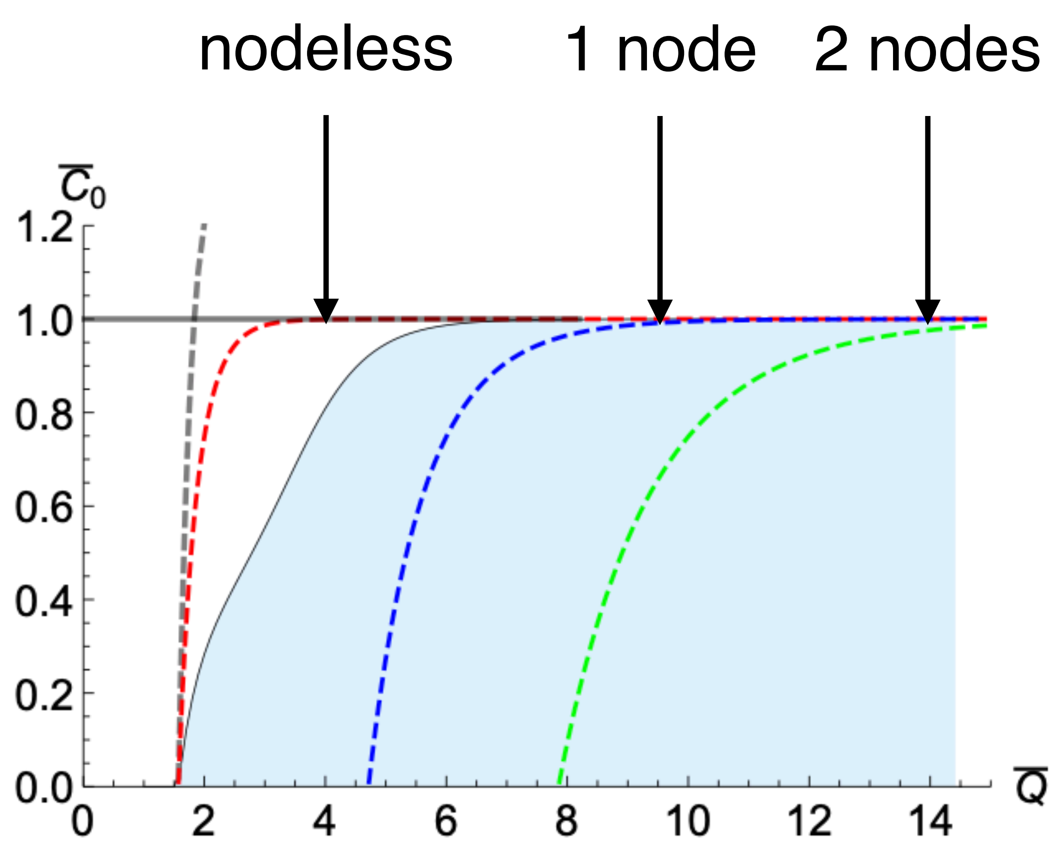


Inverse quartic | Inverse cosine

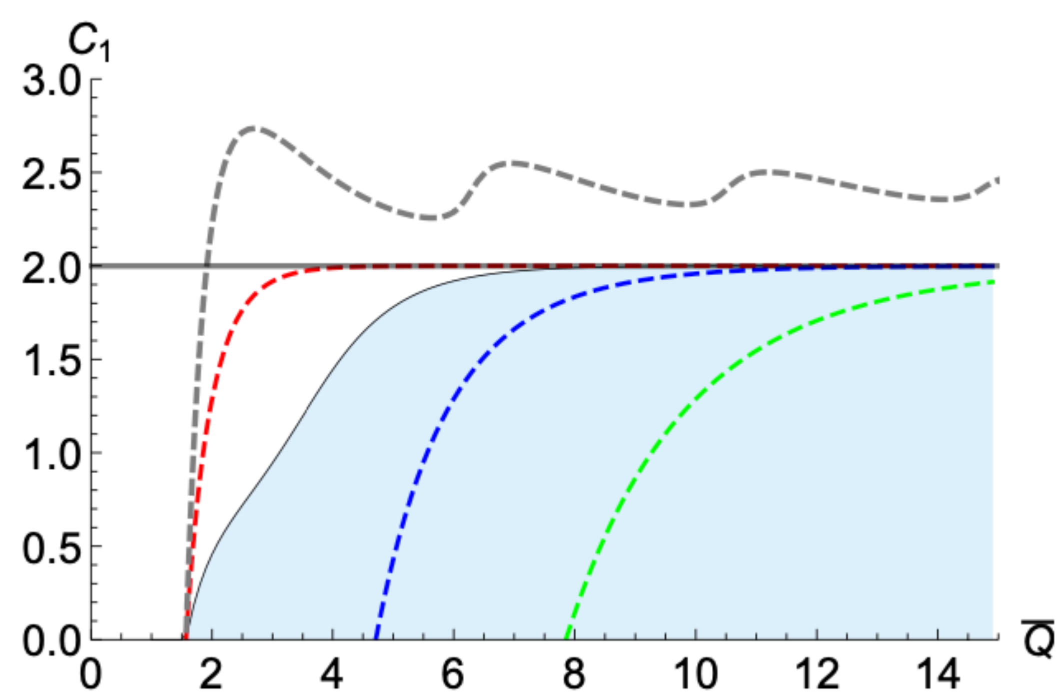
Nodeless scalarized solution is stable.

Scalarizaion on charged sphere

- (Un)stable region of the scalarized solutions.
  - Neumann Boundary condition



Inverse polynomial



Inverse cosine

Nodeless scalarized solution is stable.

Time evolution

- We can solve evolution equation of this system.
- Time evolution

$$-\ddot{\Phi} + \Phi'' + \frac{2}{r}\Phi' = -\frac{Q^2}{r^4}\frac{f'(\Phi)}{f(\Phi)^2}$$

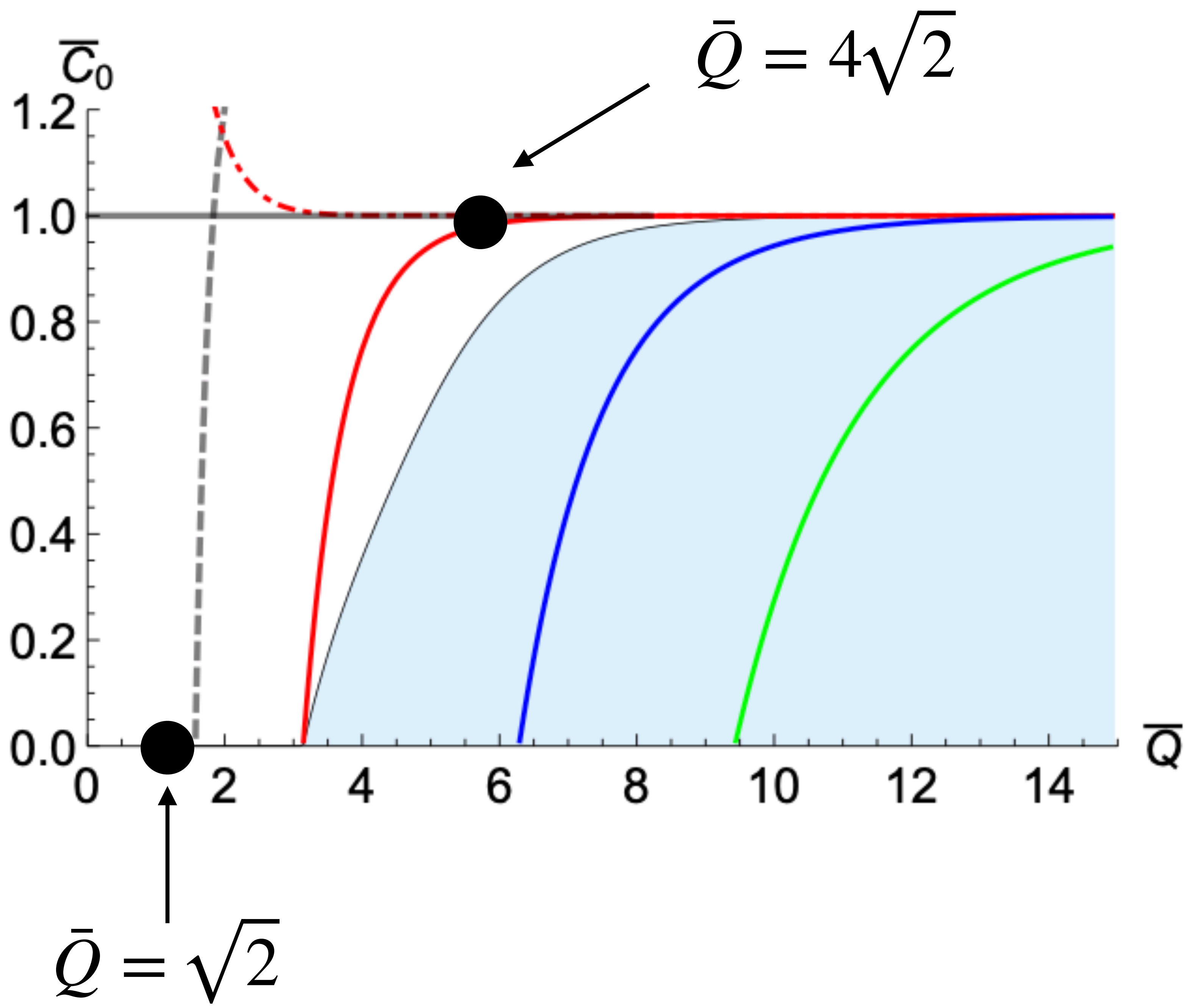
- Initial data

$$\Phi = Ae^{-\left(\frac{r-r_0}{w}\right)^2} \qquad \dot{\Phi} = 0 \qquad (t=0)$$

- Boundary condition
  - Infinity : out-going boundary condition
  - On the sphere
    - Dirichlet B.C. : $\Phi|_{r_s} = 0$
    - Neumann B.C. : $\partial_r\left(r^{-1}\Phi\right)|_{r_s} = 0$
    - Radiative B.C. : $(\partial_r\Phi_\omega + i\omega\Phi_\omega)|_{r_s} = 0$

Time evolution

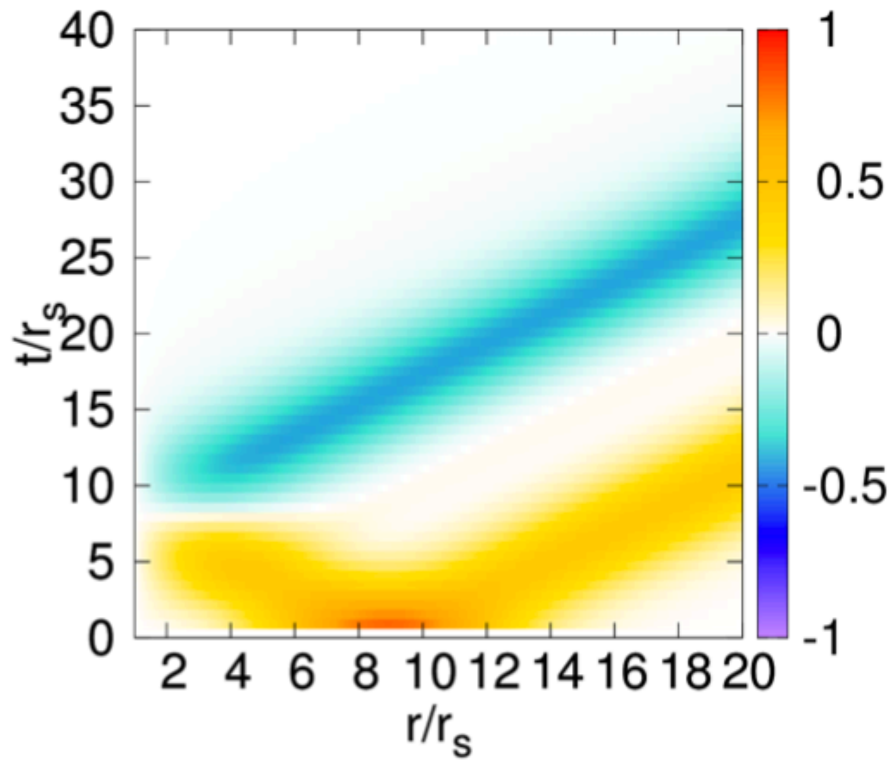
- Inverse quartic, Dirichlet B.C.



Time evolution

- Inverse quartic, Dirichlet B.C.
  - Initial parameters: $(A, w, r_0) = (0.1\Phi_{\rm min}, 4r_s, 8r_s)$
  - charge : $\bar{Q} = Q\sqrt{a}/r_s = \sqrt{2}$

$r\Phi(t, r)$

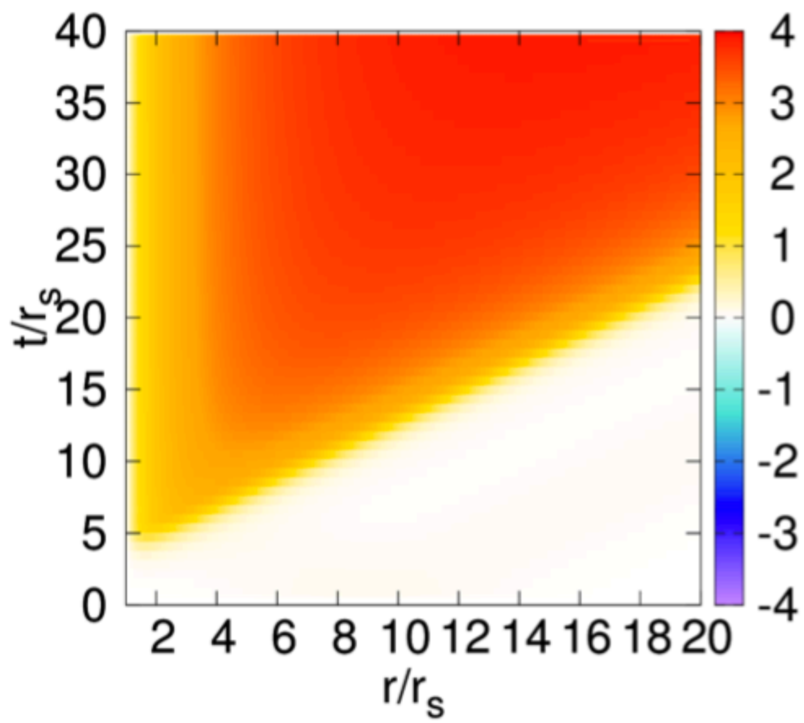


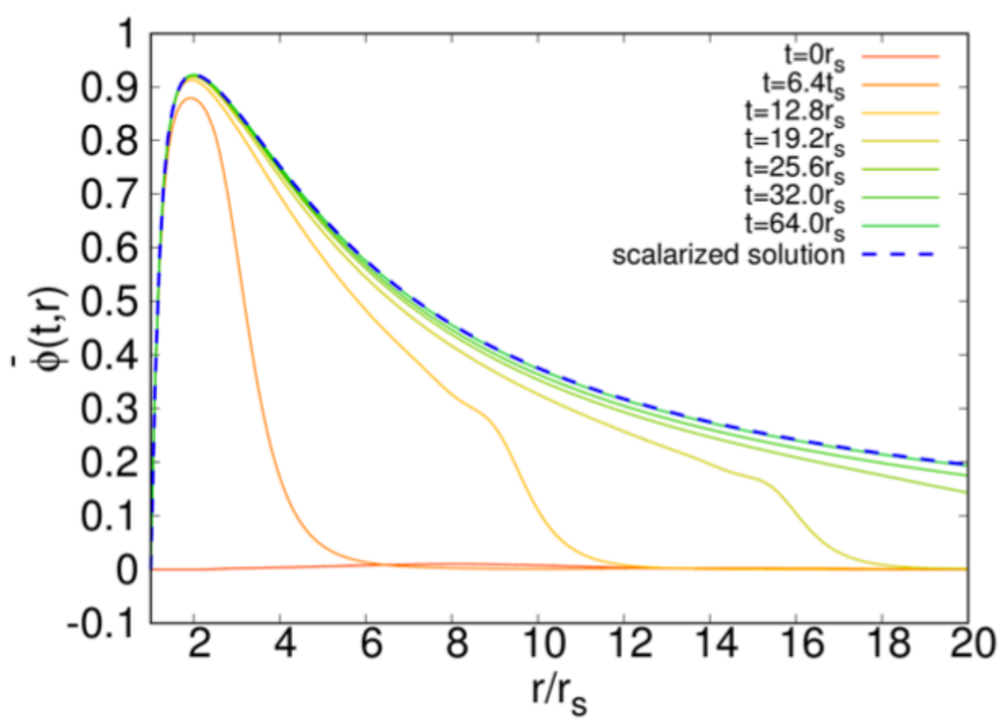
- In this parameter, the charged sphere does not scalarize.

Time evolution

- Inverse quartic, Dirichlet B.C.
  - Initial parameters: $(A, w, r_0) = (0.1\Phi_{\rm min}, 4r_{\rm s}, 8r_{\rm s})$
  - charge : $\bar{Q} = Q\sqrt{a}/r_{\rm s} = 4\sqrt{2}$

$r\Phi(t, r)$

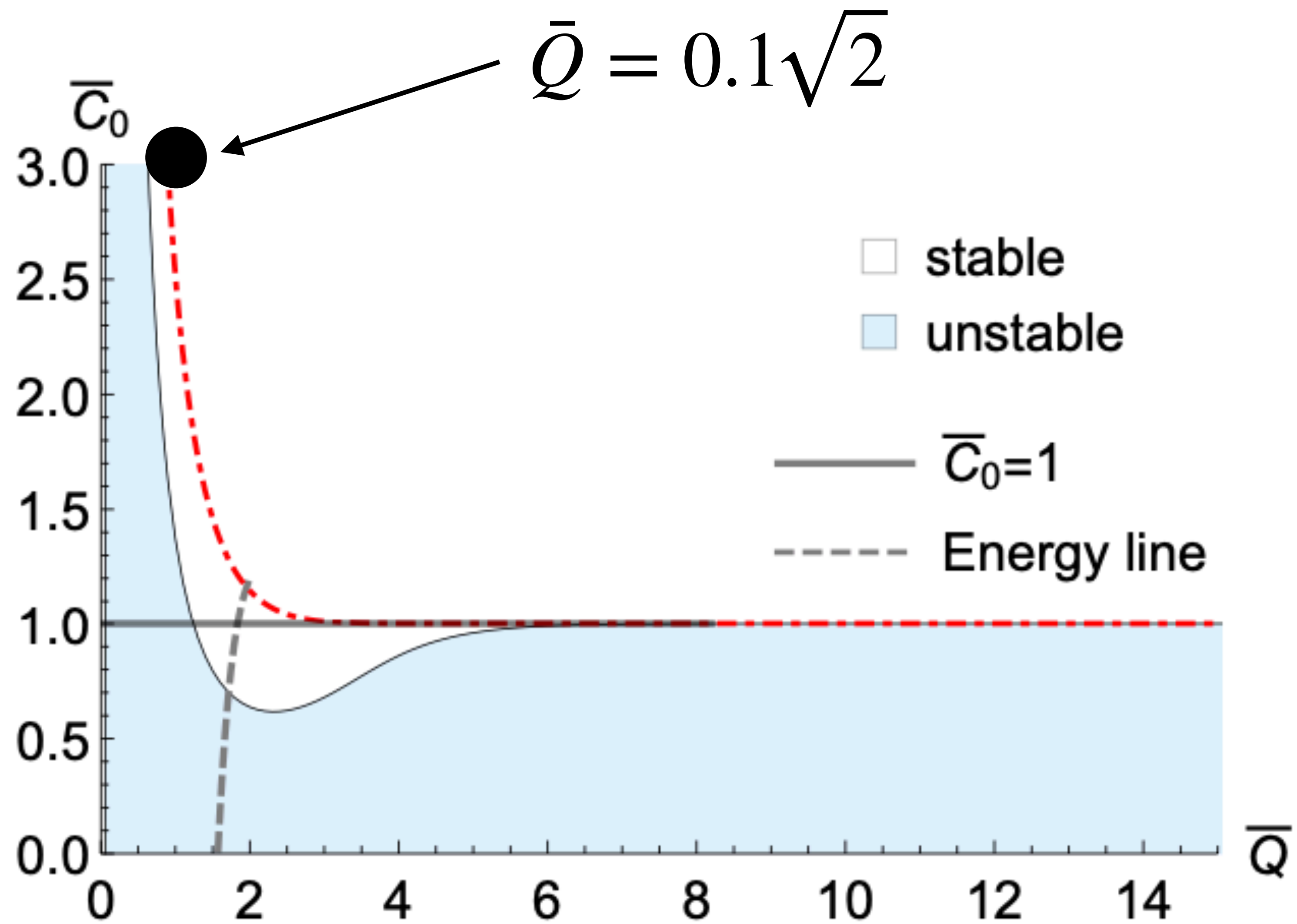




- The scalar field propagates speed of light, and the charged sphere scalarize as expected.

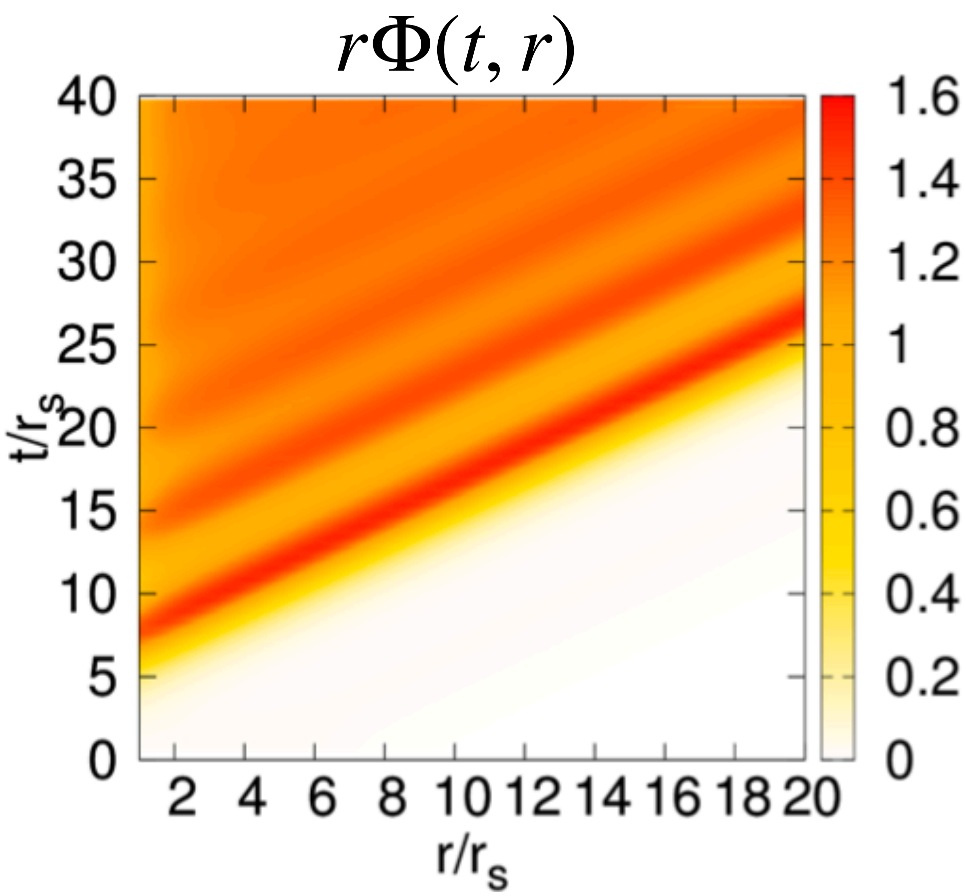
Time evolution

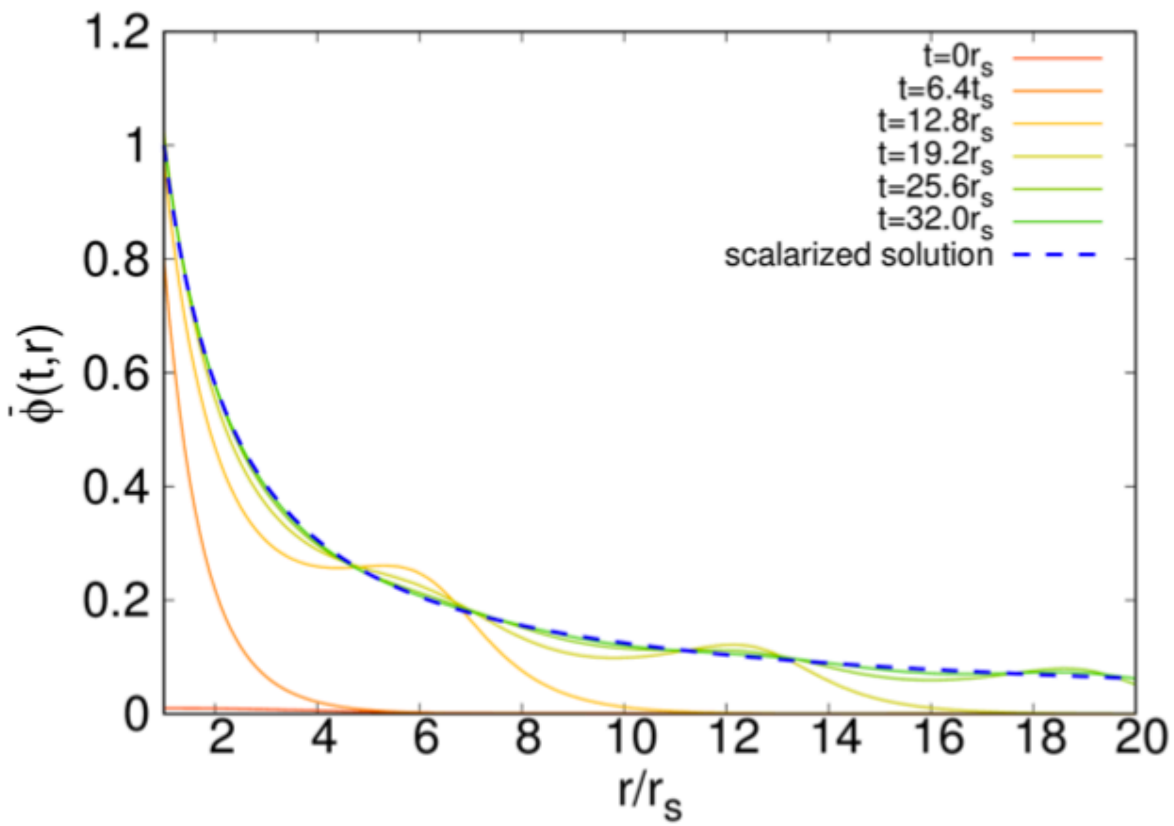
- Inverse quartic, Radiative B.C.



Time evolution

- Inverse quartic, Radiative B.C.
  - Initial parameters: $(A, w) = (0.01\Phi_{\rm min}, 4r_s)$
  - charge : $\bar{Q} = Q\sqrt{a}/r_s = 0.1\sqrt{2}$

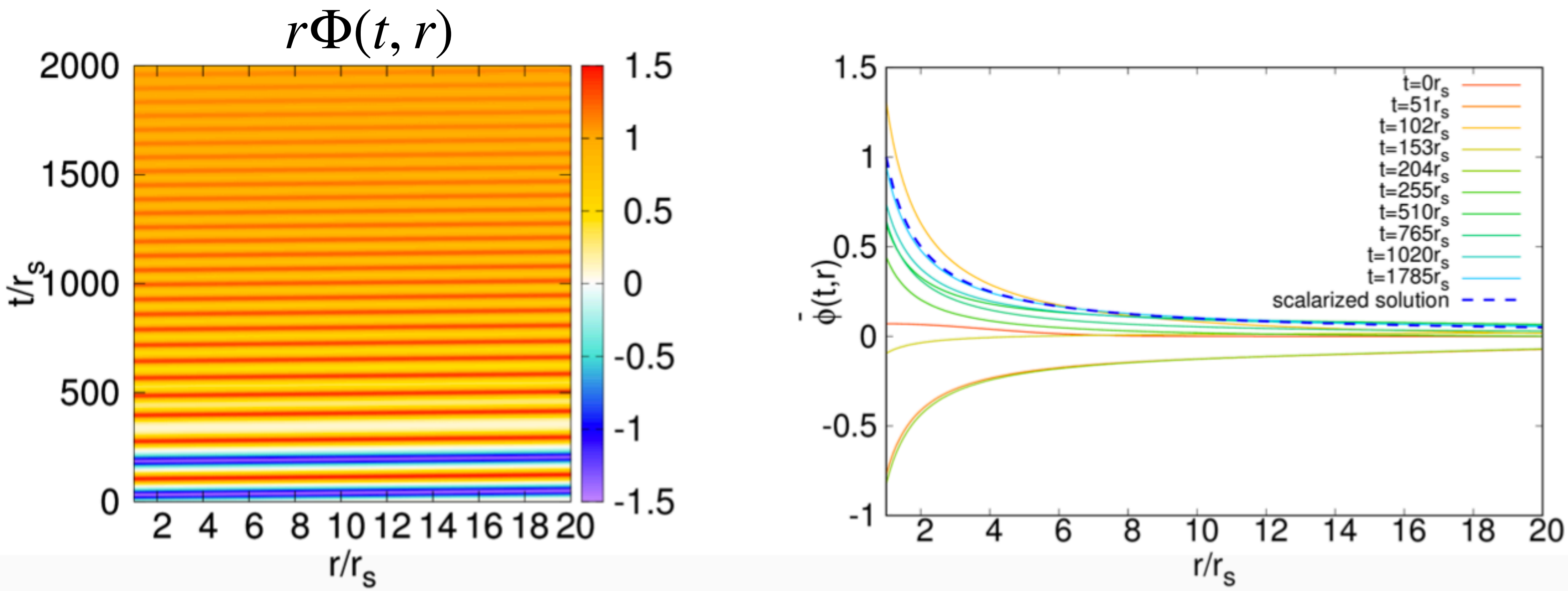




  - Scalar field propagates with speed of light, and reaches the expected scalarized solution, directly.

Time evolution

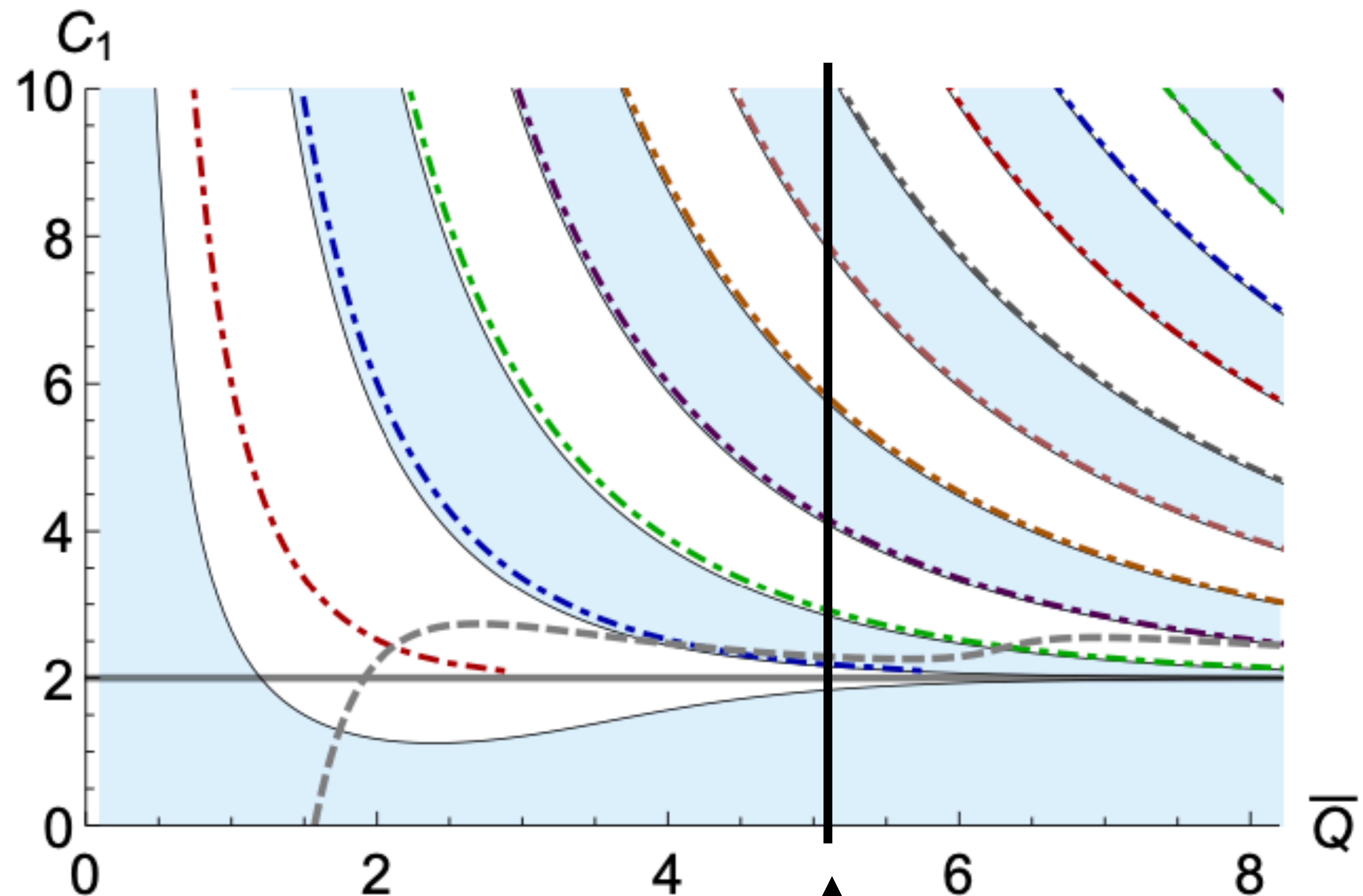
- Inverse quartic, Radiative B.C.
  - Initial parameters: $(A, w) = (0.07\Phi_{\min}, 4r_s)$
  - charge : $\bar{Q} = Q\sqrt{a}/r_s = 0.1\sqrt{2}$



  - Scalar field reaches the expected scalarized solution after many oscillations.

Time evolution

- Inverse cosine, Radiative B.C.



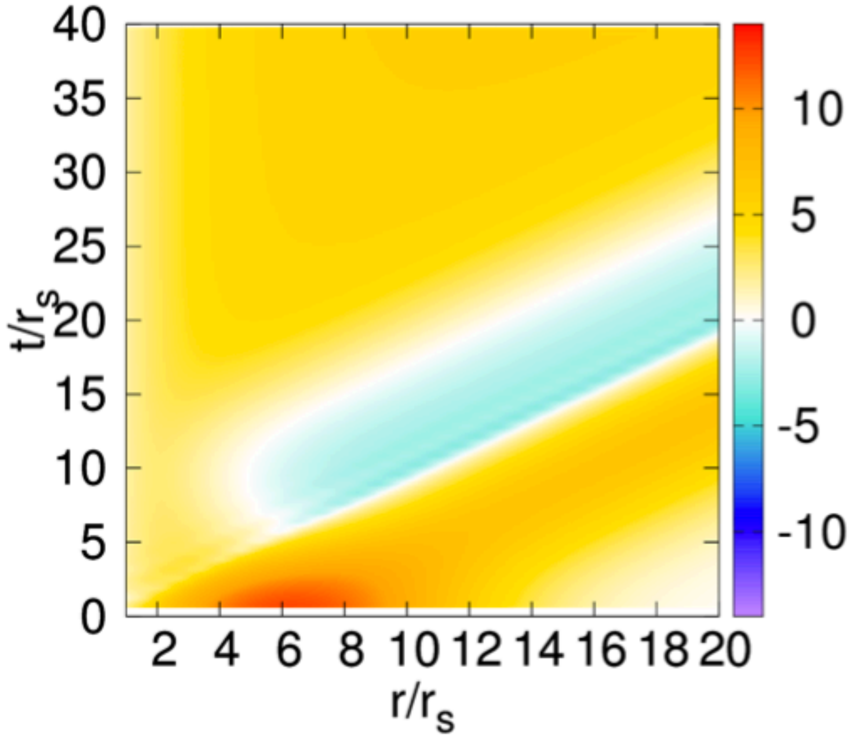
$\bar{Q} = 4\sqrt{2}$

Time evolution

- Inverse cosine, Radiative B.C.
  - charge : $\bar{Q} = Q\sqrt{a}/r_s = 4\sqrt{2}$

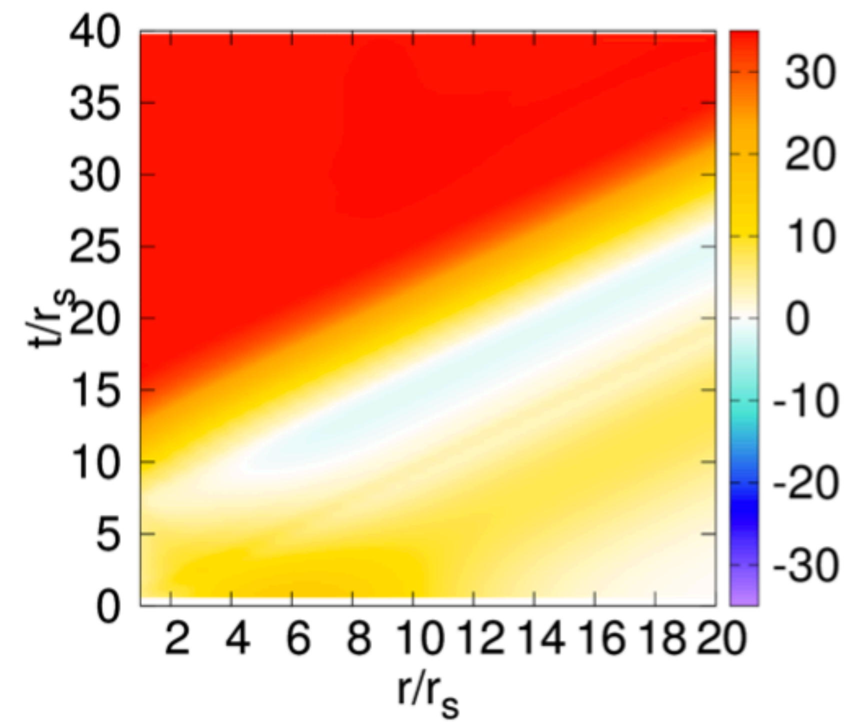
$(A\sqrt{a/2}, w/r_s) = (3.0, 8.0)$

$r\Phi(t, r)$



$(A\sqrt{a/2}, w/r_s) = (3.5, 8.0)$

$r\Phi(t, r)$



- The final scalarized solution depends on the initial data.

Outline of my talk

1. Introduction
2. No-hair theorem for ST theory
3. Spontaneous scalarization of BH
    1. Scalar-Gauss-Bonnet theory
    2. Other models
    3. Dynamical behavior of the scalarizaion
4. Summary

Summary

- Spontaneous scalariation is one of the mechanism to produce a scalar hair in extreme situation.
- Spontaneous scalarization in Scalar-Gauss-Bonnet gravity

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi + f(\Phi)\mathscr{G}_{\rm GB}\right)$$

quadratic : $f(\Phi) = \dfrac{\eta\Phi^2}{8}$ quartic : $f(\Phi) = \dfrac{\eta}{8}\left(\Phi^2 + \alpha\Phi^4\right)$

- Scalarized BHs with quadratic coupling are unstable.
- Non-linear term in quartic coupling can stabilize the scalarized BHs.

- We showed time evolution of the scalarization using the toy model.