Exercise Sheet – Advanced Calculus III

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Exercise 1. Let $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.

(1) Show that f is continuous at $(0,0)$.

(2) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0,0)$.

(3) Determine whether f is differentiable at $(0,0)$.

(4) Discuss the relationship between continuity, existence of partial derivatives, and differentiability for f at $(0,0)$.

Solution 1. (1) For any $(x,y) \to (0,0)$,

$$|f(x,y)| = \left|\frac{x^2y}{x^2+y^2}\right| \leq |y|.$$

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$. Thus, f is continuous at $(0,0)$.

(2) By definition,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

Similarly,

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

(3) f is differentiable at $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}} = 0.$$

Here, $f(0,0)=0$, $f_x(0,0)=0$, $f_y(0,0)=0$, so

$$\frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{x^2y}{(x^2+y^2)^{3/2}}.$$

Along $x=t$, $y=t$,

$$\frac{t^2t}{(t^2+t^2)^{3/2}} = \frac{t^3}{(2t^2)^{3/2}} = \frac{t^3}{2^{3/2}t^3} = \frac{1}{2^{3/2}}.$$

The limit is not 0 along this path, so f is not differentiable at $(0,0)$.

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(4) For f at $(0,0)$, we see:

- f is continuous at $(0,0)$.
- The partial derivatives exist at $(0,0)$.
- f is not differentiable at $(0,0)$.

This example shows that continuity and existence of partial derivatives at a point do not guarantee differentiability at that point.

Exercise 2 (Partial derivatives of homogeneous functions)**.** Complete the following exercises.

(1) (Warm up) Compute the following partial derivatives ($\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$):

$$f(x,y,z) = (x-2y+3z)^2; \quad f(x,y,z) = \frac{x}{\sqrt{x^2+y^2+z^2}}; \quad f(x,y,z) = \left(\frac{x}{y}\right)^{\frac{y}{z}}.$$

(2) A function $f(x,y,z)$ is called a homogeneous function of degree n, if for any $\rho > 0$, we have $f(\rho x, \rho y, \rho z) = \rho^n f(x,y,z)$. Now verify that the above functions are homogeneous and find their degrees n.

(3) (Euler's theorem) Show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x,y,z)$.
(Hint: Differentiate the equation $f(\rho x, \rho y, \rho z) = \rho^n f(x,y,z)$ with respect to ρ and then set $\rho = 1$)

(4) Conversely, show that if $f(x,y,z)$ satisfies the above equation, then $f(x,y,z)$ is a homogeneous function of degree n.

(5) Show that $f_x(x,y,z)$, $f_y(x,y,z)$ and $f_z(x,y,z)$ are homogeneous functions of degree $n-1$.

(6) Prove that $(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})^2 f = n^2 f$.

(7) Examples:

$$\Delta(x_1, x_2, \ldots, x_n) = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1\le i<j\le n} (x_j - x_i).$$

Prove that $\sum_{k=1}^n x_k \frac{\partial \Delta}{\partial x_k} = \frac{n(n-1)}{2}\Delta$ and $\sum_{k=1}^n \frac{\partial \Delta}{\partial x_k} = 0$.

Solution 2. (1) By direct computation, we have

- $f(x,y,z) = (x-2y+3z)^2$:

$$\frac{\partial f}{\partial x} = 2(x-2y+3z), \quad \frac{\partial f}{\partial y} = -4(x-2y+3z), \quad \frac{\partial f}{\partial z} = 6(x-2y+3z).$$

- $f(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$:

$$\frac{\partial f}{\partial x} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial f}{\partial y} = \frac{-xy}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial f}{\partial z} = \frac{-xz}{(x^2+y^2+z^2)^{3/2}}.$$

- $f(x,y,z) = \left(\frac{x}{y}\right)^{\frac{y}{z}}$:

$$\frac{\partial f}{\partial x} = \frac{y}{z}\left(\frac{x}{y}\right)^{\frac{y}{z}-1} \cdot \frac{1}{y} = \frac{y}{zx}\left(\frac{x}{y}\right)^{\frac{y}{z}}.$$

$$\frac{\partial f}{\partial y} = \left(\frac{x}{y}\right)^{\frac{y}{z}}\left[\frac{1}{z}\ln\left(\frac{x}{y}\right) - \frac{y}{z}\frac{1}{y}\right] = \frac{f(x,y,z)}{z}\ln\left(\frac{x}{y}\right) - \frac{f(x,y,z)}{z}.$$

$$\frac{\partial f}{\partial z} = -\frac{y}{z^2}\left(\frac{x}{y}\right)^{\frac{y}{z}}\ln\left(\frac{x}{y}\right) = -\frac{y}{z^2}f(x,y,z)\ln\left(\frac{x}{y}\right).$$

(2)
- For $f(x,y,z) = (x-2y+3z)^2$:

$$f(\rho x, \rho y, \rho z) = (\rho x - 2\rho y + 3\rho z)^2 = \rho^2(x-2y+3z)^2 = \rho^2 f(x,y,z).$$

  So, degree $n = 2$.

- For $f(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$:

$$f(\rho x, \rho y, \rho z) = \frac{\rho x}{\sqrt{(\rho x)^2 + (\rho y)^2 + (\rho z)^2}} = \frac{\rho x}{\rho\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}.$$

  So, degree $n = 0$.

- For $f(x,y,z) = \left(\frac{x}{y}\right)^{\frac{y}{z}}$:

$$f(\rho x, \rho y, \rho z) = \left(\frac{\rho x}{\rho y}\right)^{\frac{\rho y}{\rho z}} = \left(\frac{x}{y}\right)^{\frac{y}{z}}.$$

  So, degree $n = 0$.

(3) Differentiate $f(\rho x, \rho y, \rho z) = \rho^n f(x,y,z)$ with respect to ρ:

$$\frac{d}{d\rho}f(\rho x, \rho y, \rho z) = n\rho^{n-1}f(x,y,z).$$

By the chain rule,

$$\frac{d}{d\rho}f(\rho x, \rho y, \rho z) = xf_1'(\rho x, \rho y, \rho z) + yf_2'(\rho x, \rho y, \rho z) + zf_3'(\rho x, \rho y, \rho z).$$

Thus,

$$\rho xf_1'(\rho x, \rho y, \rho z) + \rho yf_2'(\rho x, \rho y, \rho z) + \rho zf_3'(\rho x, \rho y, \rho z) = n\rho f(x,y,z).$$

Setting $\rho = 1$, we arrive at the result.

(4) Define

$$g(\rho) = \frac{f(\rho x_0, \rho y_0, \rho z_0)}{\rho^n}.$$

Then

$$g'(\rho) = \frac{\rho x_0 f_1'(\rho x_0, \rho y_0, \rho z_0) + \rho y_0 f_2'(\rho x_0, \rho y_0, \rho z_0) + \rho z_0 f_3'(\rho x_0, \rho y_0, \rho z_0)}{\rho^n \cdot \rho} - \frac{nf(\rho x_0, \rho y_0, \rho z_0)}{\rho^{n+1}}.$$

Noticing that $xf_x + yf_y + zf_z = nf(x,y,z)$, then the numerator equals $nf(\rho x_0, \rho y_0, \rho z_0)$, so

$$g'(\rho) = 0.$$

For any $\rho > 0$, $g(\rho)$ is a constant. Recalling that $g(1) = f(x_0, y_0, z_0)$, we have

$$g(\rho) = f(x_0, y_0, z_0),$$

which implies the desired result.

(5) Let f be homogeneous of degree n. Then

$$f(\rho x, \rho y, \rho z) = \rho^n f(x, y, z).$$

Differentiate both sides with respect to x:

$$\rho f_1'(\rho x, \rho y, \rho z) = \rho^n f_1'(x, y, z).$$

So,

$$f_1'(\rho x, \rho y, \rho z) = \rho^{n-1} f_1'(x, y, z).$$

Thus, f_x, f_y, f_z are homogeneous of degree $n-1$.

(6) If you see $D := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. By Euler's theorem, we have

$$D(Df) = D(nf) = n^2 f.$$

This also could be directly verified by chain rule:

$$\begin{aligned}
&\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) \\
&= x\left(\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}\right) + xy\frac{\partial^2 f}{\partial x \partial y} + xy\frac{\partial^2 f}{\partial y \partial x} + y\left(\frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2}\right) \\
&= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + x\left[x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right] + y\left[x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right] \\
&= nf + x\cdot(n-1)\frac{\partial f}{\partial x} + y\cdot(n-1)\frac{\partial f}{\partial y} \\
&= nf + (n-1)nf \\
&= n^2 f.
\end{aligned}$$

Remark (The other understanding): we see that $(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})^2$ equals to $x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x \partial y} + y^2\frac{\partial^2}{\partial y^2}$. Then we have

$$\begin{aligned}
&x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} \\
&= x\left[x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right] + y\left[x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right] \\
&= x\cdot(n-1)\frac{\partial f}{\partial x} + y\cdot(n-1)\frac{\partial f}{\partial y} \\
&= n(n-1)f.
\end{aligned}$$

For the second understanding, try to prove that

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^m f = n(n-1)\cdots(n-m+1)f. \tag{1}$$

For $m = 1$, the equation is valid.
For $m = k$, we suppose that the equation is valid.
What we need to do is to prove that it holds for $m = k + 1$. As it holds for $m = k$, we have

$$\sum_{i=0}^{k} \left(C_k^i x^i y^{k-i} \frac{\partial^k f}{\partial x^i \partial y^{k-i}} \right) \Bigg|_{(tx,ty)} = n(n-1)\cdots(n-k+1) f(tx,ty) = n(n-1)\cdots(n-k+1) t^n f(x,y).$$

Differentiate t^k in both sides:

$$\sum_{i=0}^{k} C_k^i x^i y^{k-i} \left(\frac{\partial^k f}{\partial x^i \partial y^{k-i}} \right) \Bigg|_{(tx,ty)} = n(n-1)\cdots(n-k+1) t^{n-k} f(x,y). \tag{2}$$

Now we differentiate the both side of the above equation with respect to t. Noting the chain rule

$$\frac{d}{dt} \left(\frac{\partial^k f}{\partial x^i \partial y^{k-i}} \Bigg|_{(tx,ty)} \right) = x \left(\frac{\partial^{k+1} f}{\partial x^{i+1} \partial y^{k-i}} \Bigg|_{(tx,ty)} \right) + y \left(\frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Bigg|_{(tx,ty)} \right).$$

Thus the t-derivative of the l.h.s of (2) equals to

$$\begin{aligned}
&\sum_{i=0}^{k} \left[C_k^i x^{i+1} y^{k-i} \frac{\partial^{k+1} f}{\partial x^{i+1} \partial y^{k-i}} \Bigg|_{(tx,ty)} + C_k^i x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Bigg|_{(tx,ty)} \right] \\
&= \sum_{i=1}^{k+1} C_k^{i-1} x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k-i+1}} \Bigg|_{(tx,ty)} + \sum_{k=0}^{i} C_k^i x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Bigg|_{(tx,ty)} \\
&= \sum_{i=0}^{k+1} C_{k+1}^i x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Bigg|_{(tx,ty)},
\end{aligned}$$

where we used $C_k^i + C_k^{i-1} = C_{k+1}^i$ for the last "=".
Thus

$$\sum_{i=0}^{k+1} C_{k+1}^i x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Bigg|_{(tx,ty)} = n(n-1)\cdots(n-k+1)(n-k) t^{n-k-1} f(x,y).$$

Taking $t = 1$ in the above equation, we reach that the equation (1) holds for $m = k + 1$.

(7) First, recall that Δ is a homogeneous polynomial of degree $d = \frac{n(n-1)}{2}$ in the variables $x_1, \ldots, x_n$ (since there are $n(n-1)/2$ factors, each linear in x_k). By Euler's theorem for homogeneous functions,

$$\sum_{k=1}^{n} x_k \frac{\partial \Delta}{\partial x_k} = \frac{n(n-1)}{2} \Delta.$$

Secondly, notice that Δ admits the translation invariance, i.e., $\Delta(x_1, \ldots, x_n) = \Delta(x_1 + t, \ldots, x_n + t)$ for any $t \in \mathbb{R}$.

$$0 = \frac{\partial \Delta}{\partial t}(x_1, \ldots, x_n) \overset{u_k = x_k + t}{=} \sum_{k=1}^{n} \frac{\partial \Delta}{\partial u_k}(u_1, \ldots, u_n) \cdot \frac{du_k}{dt}\Big|_{t=0} = \sum_{k=1}^{n} \frac{\partial \Delta}{\partial x_k}(x_1, \ldots, x_n).$$