# Exercise Sheet – Advanced Calculus III

Taiyang Xu*

18/09/2025, Week 2

**Exercise 1** (Warm up: Directional derivatives and gradient)**.** Complete the following exercises.

(1) Let $f(x, y, z) = x^2y + yz^3$. Compute the gradient $\nabla f$ at the point $(1, 2, 1)$.

(2) For $f(x, y, z)$ as above, compute the directional derivative of $f$ at $(1, 2, 1)$ in the direction of the vector $\vec{v} = (2, -1, 2)$.

(3) Let $f(x, y) = x^3 - 3xy^2$. Find all points $(x, y)$ where the gradient $\nabla f$ is parallel to the vector $(1, 1)$.

Application in ML:

(4) (Linear regression) A common loss function is Mean Squared Error (MSE). For a single data point $(x, y)$, the loss is defined as $L(m, b) = (y - (mx + b))^2$, where $m$ is the slope and $b$ is the $y$-intercept of the regression line. Compute the gradient $\nabla L(m, b)$ and interpret its components in terms of how they influence the loss.

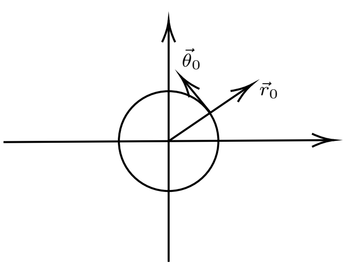
(5) (Gradient descent step) Suppose you are using gradient descent to minimize the function $J(\theta_0, \theta_1) = \theta_0^2 + 2\theta_1^2$. Calculate the gradient $\nabla J(\theta_0, \theta_1)$ firstly, and write down the update rule for $\theta_0$ and $\theta_1$ using a learning rate $\alpha$.

**Exercise 2.** Complete the following exercises

(1) Let $u = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$. Prove that

$$\nabla u = \frac{\partial f}{\partial r}\vec{r}_0 + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{\theta}_0,$$

where $\vec{r}_0 = (\cos\theta, \sin\theta)$ and $\vec{\theta}_0 = (\cos(\theta + \frac{\pi}{2}), \cos\theta)$.




(2) We have the following equation:

$$\frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u} = 1.$$

*School of Mathematical Sciences, Fudan University, Shanghai 200433, China. Email: tyxu19@fudan.edu.cn

Prove that

$$(\nabla u)^2 = 2 < \vec{A}, \nabla u >,$$

where $\vec{A} := (x, y, z)$.
(Hint: see $u = u(x, y, z)$)

**Exercise 3.** Suppose the function $f(x, y)$ has a nonzero directional derivative at some point $(x_0, y_0)$, and the directional derivatives along three different (non-colinear) directions at $(x_0, y_0)$ are equal. Prove that $f(x, y)$ is not differentiable at $(x_0, y_0)$.