Confluent hypergeometric kernel determinant on multiple large intervals

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@BMS RMT Seminar

Outline

Introduction

Unitary ensembles with Fisher–Hartwig (FH) singularity

The joint probability density function (jpdf) of eigenvalues for GUE:

$$f_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{\mathcal{Z}_n} \prod_{k=1}^{n} e^{-\lambda_k^2} \prod_{1 \leqslant k < j \leqslant n} (\lambda_k - \lambda_j)^2.$$

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- $\chi_\beta(\lambda)$ has the jump-type singularity with $\beta \in i\mathbb{R}$:

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- jump-type singularity & root-type singularity $\rightsquigarrow$ **Fisher-Hartwig** singularity (at $z = 0$).

Unitary ensembles with Fisher–Hartwig (FH) singularity

The jpdf of Dyson circular unitary ensemble with FH singularity:

$$f_n(\theta_1, \dots, \theta_n) = \frac{1}{\mathcal{Z}_n'} \prod_{k=1}^{n} |1 - e^{i\theta_k}|^{2\alpha} e^{i\beta(\theta_k - \pi)} \prod_{1 \leqslant k < j \leqslant n} |e^{i\theta_k} - e^{i\theta_j}|^2,$$

where

- $\mathcal{Z}_n'$: normalization constant.
- $\theta_k \in [0, 2\pi)$, $k = 1, \dots, n$.
- FH singularity at $z = 1$, (i.e., $\theta = 0$).

The correlation kernel

The above eigenvalues form a **determinant point process** (DPP), which is characterized by a correlation kernel $K_n(x, y; \alpha, \beta)$, that is,

$$\rho(\lambda_1, \dots, \lambda_n) = \det\left[K_n(\lambda_i, \lambda_j; \alpha, \beta)\right]_{i,j=1}^{n}.$$

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  $p_j(x)$ are orthonormal polynomials with the weight function $w(x)$ over $\mathbb{R}$, i.e.,

$$\int_{\mathbb{R}} p_i(x) p_j(x) w(x) \, \mathrm{d}x = \delta_{ij}, \quad w(x) = e^{-x^2} |x|^{2\alpha} \chi_\beta(x).$$

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- CUE: $K_n(e^{i\theta}, e^{i\phi}; \alpha, \beta) = \sqrt{w(e^{i\theta})}\sqrt{w(e^{i\phi})}\sum_{j=0}^{n-1} p_j(e^{i\theta})\overline{p_j(e^{i\phi})}$.

$$\int_0^{2\pi} p_i(e^{i\theta})\overline{p_j(e^{i\theta})}w(e^{i\theta})\,\mathrm{d}\theta = \delta_{jk}, \quad w(z) = z^n|z-1|^{2\alpha}z^\beta e^{-i\pi\beta},\ z = e^{i\theta}.$$

Large-n limiting kernel

In the bulk (near the FH singular point), the correlation kernel $K_n(x, y; \alpha, \beta)$ converges to **confluent hypergeometric kernel** $K^{(\alpha,\beta)}(x, y)$ under a suitable scaling.

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- Unitary random matrix ensembles generated by a concrete weight function on the unit circle.

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- it describes the local statistics of eigenvalues in the bulk of the spectrum near a FH singular point for a broad class of unitary ensemble of random matrices.

The confluent hypergeometric kernel

The confluent hypergeometric (CH) kernel with two parameters $\alpha > -1/2$ and $\beta \in i\mathbb{R}$ is defined by

$$K^{(\alpha,\beta)}(x,y) = \frac{1}{2\pi i}\frac{\Gamma(1+\alpha+\beta)\Gamma(1+\alpha-\beta)}{\Gamma(1+2\alpha)^2}\frac{A(x)B(y)-A(y)B(x)}{x-y},$$

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- $\Gamma(z)$ denotes the usual Gamma function. $A(x)$ and $B(x)$ are defined by

$$A(x) := \chi_\beta(x)^{\frac{1}{2}}|2x|^\alpha e^{-ix}\phi(1+\alpha+\beta, 1+2\alpha; 2ix),$$
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- The confluent hypergeometric function is defined by

$$\phi(a,b;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \qquad b \neq 0, -1, -2, \dots,$$

where $(z)_n := z(z+1)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}$ is the Pochhammer symbol.

Universal features of the confluent hypergeometric kernel

The confluent hypergeometric kernel arises in several different, but related areas

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The confluent hypergeometric kernel arises in several different, but related areas

- Infinite random matrices and Hua-Pickrell measure.

  [Borodin-Olshanski, '01]

- Representation theory.

  [Borodin-Deift, '01]

- Circular unitary ensemble.

  [Deift-Krasovsky-Vasilevska, '11]

Relations to other kernels

Confluent hypergeometric kernel can reduce to other kernels

- If $\beta = 0$, $\alpha \neq 0$, following from the relation

$$\phi(\alpha, 2\alpha; 2ix) = \Gamma\left(\alpha + \frac{1}{2}\right) e^{ix} \left(\frac{x}{2}\right)^{-\alpha+\frac{1}{2}} J_{\alpha-\frac{1}{2}}(x),$$

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  then we have type-I Bessel kernel

$$K^{(\alpha,0)}(x,y) \equiv K^{(\mathrm{Bessel1})}(x,y) = \left(\frac{|x|}{x}\right)^{\alpha}\left(\frac{|y|}{y}\right)^{\alpha} \frac{\sqrt{xy}}{2} \frac{J_{\alpha+\frac{1}{2}}(x)J_{\alpha-\frac{1}{2}}(y) - J_{\alpha+\frac{1}{2}}(y)J_{\alpha-\frac{1}{2}}(x)}{x-y}.$$

[Akemann-Damgaard-Magnea-Nishigaki, '97] [Kuijlaars-Vanlessen, '03]

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  An episode: Type-II Bessel kernel:

$$K^{(\mathrm{Bessel2})}(x,y) := \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}.$$

  [Forrester, '93]

Relations to other kernels

- If $\alpha = 0$, $\beta \neq 0$, we have a degenerated confluent hypergeometric kernel.

  [Tibboel, '10] [Moreno- Martínez-Finkelshtein -Sousa, '11]

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- If $\alpha = 0$, $\beta \neq 0$, we have a degenerated confluent hypergeometric kernel.

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- If $\alpha = \beta = 0$, we obtain the sine kernel

$$K^{(0,0)}(x,y) \equiv K^{(\mathrm{sine})}(x,y) = \frac{\sin(x-y)}{\pi(x-y)}.$$

Gap probability for DPP

- The **Gap probability**: finding no eigenvalues on a specific interval Σ.

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- **Fredholm determinant representation**:

$$\mathbb{P}(\text{no points lie on } \Sigma) = \mathcal{F}(\Sigma) := \det\left(1 - \mathcal{K}|_{\Sigma}\right),$$

where $\mathcal{K}$ is an integration operator acting on $L^2(\Sigma)$ with integrable kernel $K(x, y)$.

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- Our interest: **large gap asymptotics**

$$\mathcal{F}(s\Sigma) := \det\left(1 - \mathcal{K}|_{s\Sigma}\right) = ?, \quad \text{as} \quad s \to +\infty,$$

especially the case that Σ is **a union of disjoint intervals**.

History: sine kernel determinant

$$\log\det\left(1 - \mathcal{K}^{(\mathrm{sine})}|_{s\Sigma}\right) = C_1 s^2 + C_2 \log s + C_3 \log\theta(V(s)) + C_4 + \mathcal{O}\left(s^{-1}\right), \quad s \to +\infty.$$

[1] Combined the earlier work [Widom '71] on Toeplitz determinant.

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<table>
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<tr><td>$(-1,1)$</td><td>[Dyson, '62]
[Cloizeaux-Mehta, '73]
[Widom, '94]
$C_1=-1/2$</td><td colspan="2">[Deift-Its-Zhou, '97]
$C_2=-1/4,\ C_3=0$</td><td>[Dyson, '76][1]
[Krasovsky, '04]
[Ehrhardt, '04]
[Deift-Its-Krasovsky, '07]
$C_4=(\log 2)/12+3\zeta'(-1)$</td></tr>
<tr><td>$(-1,v_1)\cup(v_2,1)$</td><td colspan="4">[Fahs-Krasovsky, '22]</td></tr>
<tr><td>$\cup_{j=0}^{n}(a_j,b_j)$</td><td colspan="3">[Deift-Its-Zhou, '97]
C_2: integral expression</td><td>?</td></tr>
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Table: Large gap asymptotics for the sine kernel determinant.

[1] Combined the earlier work [Widom '71] on Toeplitz determinant.

History: Airy kernel determinant

$$\log\det\left(1-\mathcal{K}^{(\mathrm{Ai})}|_{s\Sigma}\right) = C_1 s^3 + C_2\log s + C_3\log\theta(V(s)) + C_4 + e(s), \quad s\to+\infty.$$

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Σ	C_1	C_2	C_3	$e(s)$	C_4
$(-1, +\infty)$	[Tracy-Widom, '94] $C_1 = -1/12,\ C_2 = -1/8,$ $C_3 = 0,\ e(s) = \mathcal{O}(s^{-3/2})$				[Deift-Its-Krasovsky, '08] [Baik-Buckingham-DiFranco, '08] $C_4 = (\log 2)/24 + \zeta'(-1)$
$(x_2, x_1) \cup (x_0, +\infty)$ $x_2 < x_1 < x_0 < 0$	[Blackstone-Charlier-Lenells, '22] $\rightsquigarrow e(s) = \mathcal{O}(s^{-1})$ [Krasovsky-Maroudas, '24]				[Krasovsky-Maroudas, '24]
$\bigcup_{j=0}^{g}(x_{2j}, x_{2j-1})$ $x_{-1} = +\infty$	?				
(x_2, x_1)	[Blackstone-Charlier-Lenells, '21] $e(s) = \mathcal{O}(s^{-3/2})$				?
$\bigcup_{j=1}^{g}(x_{2j}, x_{2j-1})$	?				

Table: Large gap asymptotics for the Airy kernel determinant.

History: type-II Bessel kernel determinant

$$\log\det(1-\mathcal{K}^{(\mathrm{Bes2})}|_{s\Sigma}) = C_1 s + C_2 s^{1/2} + C_3 \log s + C_4 \log\theta(V(s)) + C_5 + \mathcal{O}(s^{-1/2}), \quad s\to+\infty.$$

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$C_1=-x_1/4,\ C_2=\alpha\sqrt{x_1},$
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[Ehrhardt, '10] $\rightsquigarrow \alpha\in(-1,1)$
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C_3: exact value (under the ergodic condition)</td><td>?</td></tr>
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Table: Large gap asymptotics for the type-II Bessel kernel determinant.

History: Confluent hypergeometric kernel determinant

As $s \to +\infty$,

$$\log\det(1 - \mathcal{K}^{(\alpha,\beta)}|_{s\Sigma}) = C_1 s^2 + C_2 s + (\beta^2 - \alpha^2 + C_3)\log s + C_4 \log\theta(V(s)) + C_5 + \mathcal{O}(s^{-1}).$$

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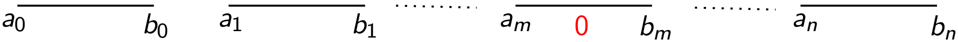
$\cup_{j=0}^{n}(a_j,b_j)$ $0\in(a_m,b_m)$ for some m: $1\leqslant m\leqslant n$	?				

Table: Large gap asymptotics for the confluent hypergeometric kernel determinant.

Today's topic

Aim: establish large gap asymptotics for the confluent hypergeometric kernel determinant $\det(1 - \mathcal{K}^{(\alpha,\beta)})$ on multiple large intervals.



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a_0 ——— b_0 a_1 ——— b_1 $\cdots\cdots$ a_m — 0 — b_m $\cdots\cdots$ a_n ——— b_n

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$\cup_{j=0}^{n}(a_j,b_j)$ $0\in(a_m,b_m)$ for some m: $1\leqslant m\leqslant n$	[Xu-Zhang-Zhao, '25]				?

Table: Large gap asymptotics for the confluent hypergeometric kernel determinant.

Main results

The Riemann surface

We will encounter a hyperelliptic Riemann surface $\mathcal{W}$ associated to the algebraic equation

$$\sqrt{\mathcal{R}(z)} := \sqrt{\prod_{j=0}^{n} (z - a_j)\,(z - b_j)}.$$

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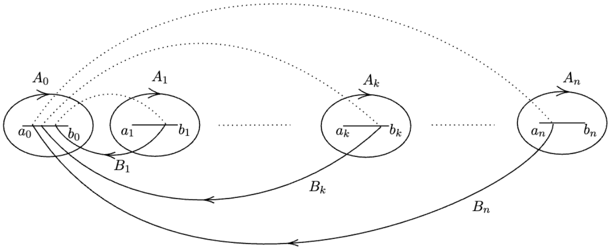

Figure: The canonical homology basis $\{A_j, B_j\}_{j=1}^{n}$ for the Riemann surface $\mathcal{W}$.

- $\sqrt{\mathcal{R}(z)} \sim \pm z^{n+1}$, as $z \to \infty$ on the first (second) sheet.
- Canonical homology basis $\{A_j, B_j\}$.

Preliminaries – The $\mathbb{A}$-matrix

The $\mathbb{A}$-matrix:

$$\mathbb{A} := (a_{k,l})_{0\leqslant k\leqslant n, 0\leqslant l\leqslant n} = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n} \end{pmatrix},$$

$$\tilde{\mathbb{A}} := (a_{k,l})_{k=1,\dots,n}^{l=0,\dots,n-1} = \begin{pmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ a_{2,0} & a_{2,1} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n-1} \end{pmatrix},$$

$$\vec{a} = (a_{0,n+1}, a_{1,n+1}, \dots, a_{n,n+1})^{\mathrm{T}},$$

where $a_{k,l} := \oint_{A_k} \frac{z^l}{\sqrt{\mathcal{R}(z)}} \,\mathrm{d}z = 2i(-1)^{n-k+1} \int_{a_k}^{b_k} \frac{z^l}{|\mathcal{R}(z)|^{\frac{1}{2}}} \,\mathrm{d}z.$

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where $a_{k,l} := \oint_{A_k} \frac{z^l}{\sqrt{\mathcal{R}(z)}}\,\mathrm{d}z = 2i(-1)^{n-k+1}\int_{a_k}^{b_k} \frac{z^l}{|\mathcal{R}(z)|^{\frac{1}{2}}}\,\mathrm{d}z.$

$\rightsquigarrow$ $\mathbb{A}$ and $\tilde{\mathbb{A}}$ are invertible.

[Farkas-Kra, '92, *Riemann Surfaces* 2nd ed]

Preliminaries – The basis of one-form

Introduce the basis of holomorphic one-forms:

$$\vec{\omega} := (\omega_1, \omega_2, \ldots, \omega_n) = \frac{\mathrm{d}z}{\sqrt{\mathcal{R}(z)}}(1, z, \ldots, z^{n-1})\tilde{\mathbb{A}}^{-1},$$

such that

$$\oint_{A_k} \omega_j = \delta_{jk}, \qquad j, k = 1, \ldots, n.$$

Meanwhile, we have the Riemann matrix of B_j periods:

$$\tau := (\tau_{ij})_{i,j=1}^{n} = \left(\oint_{B_j} \omega_i\right)_{i,j=1}^{n}.$$

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$\rightsquigarrow$ τ is symmetric and has a positively definite imaginary part, i.e, $-i\tau$ is positive definite.

[Farkas-Kra, '92, *Riemann Surfaces* 2nd ed]

Preliminaries – Multi-dimensional Riemann-θ function and Abel's map

The multi-dimensional Riemann-θ function is defined by

$$\theta\left(\vec{z}\right) = \sum_{\vec{m}\in\mathbb{Z}^n} e^{2\pi i\vec{m}^{\mathrm{T}}\vec{z}+i\pi\vec{m}^{\mathrm{T}}\tau\vec{m}}, \qquad \vec{z} = \left(z_1, \ldots, z_n\right)^{\mathrm{T}} \in \mathbb{C}^n \text{ modulo } \mathbb{Z}^n.$$

- Converging absolutely and uniformly on compact sets of the $\mathbb{C}^n$.
- Even ($\theta(\vec{z}) = \theta(-\vec{z})$), entire function for $\vec{z} \in \mathbb{C}^n$.
- Periodic properties: $\theta(\vec{z} + \vec{e}_j) = \theta(\vec{z})$ and $\theta(\vec{z} \pm \vec{\tau}_j) = e^{\mp 2\pi i z_j - \pi i \tau_{jj}}\theta(\vec{z})$, where $\vec{e}_j = (0, \ldots, 1, \ldots, 0)^{\mathrm{T}}$ with 1 in the j-th position and $\vec{\tau} := \tau\vec{e}_j$.
- Vanishing at each odd half-period: $\theta(\vec{z}) = 0$ if $\vec{z} = \frac{\vec{m}}{2} + \frac{\tau\vec{n}}{2}$ with $\vec{m}, \vec{n} \in \mathbb{Z}^n$ and $\vec{m}^{\mathrm{T}}\vec{n}$ is odd.

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Abel's map:

$$\vec{\mathcal{A}}(z) := \int_{a_0}^{z} \vec{\omega}^{\mathrm{T}}.$$

Preliminaries – The polynomial, linear vector and frequencies

The following polynomial of degree $n+1$ plays an important role in our analysis:

$$\mathrm{p}(z) = z^{n+1} + \sum_{j=0}^{n} p_j z^j.$$

$\oint_{A_k} \frac{\mathrm{p}(s)}{\sqrt{\mathcal{R}(s)}}\,\mathrm{d}s = 0,\ k = 0, 1, \ldots, n. \implies (p_0, \ldots, p_n)^{\mathrm{T}} = -\mathbb{A}^{-1}\vec{a}.$

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A column vector with linear components:

$$\vec{V}(s) := (V_1(s), \ldots, V_n(s))^{\mathrm{T}}, \quad V_j(s) := \frac{s}{2\pi}\Omega_j + \frac{1}{2\pi}\operatorname{Im}(\zeta_j) \in \mathbb{R}.$$

- Frequencies $\Omega_j := 2\sum_{k=0}^{j-1}(-1)^{n-k}\int_{b_k}^{a_{k+1}} \frac{\mathrm{p}(s)}{|\mathcal{R}(s)|^{\frac{1}{2}}}\,\mathrm{d}s > 0.$
- $\operatorname{Im}\zeta_j \in \mathbb{R}$ is dependent of α and β.

A function $\mathcal{L} : \mathbb{C} \times \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{C}$ defined by

$$\mathcal{L}(z, \vec{\mu}) = \frac{h(z)}{\mathrm{p}(z)}\eta(z, \vec{\mu}).$$

where $h(z) = \prod_{k=0}^{n}(z - a_k) + \prod_{k=0}^{n}(z - b_k)$, and $\eta\,(z, \vec{\mu}) = \frac{\theta(\vec{0})^2\theta(\vec{\mathcal{A}}(z)+\vec{\mu}+\vec{d})\theta(\vec{\mathcal{A}}(z)-\vec{\mu}+\vec{d})}{\theta(\vec{\mu})^2\theta(\vec{\mathcal{A}}(z)+\vec{d})^2}$.

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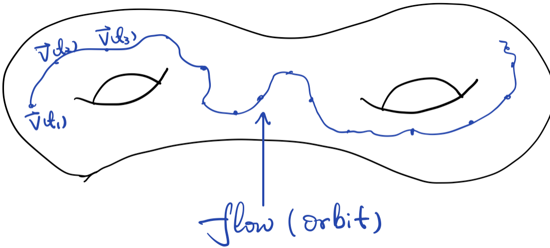
- $\mathcal{L}(p, \vec{V}(t))$ is real-valued and analytic.
- $\mathcal{L}(p, \cdot)$ is periodic on $\mathbb{R}^n/\mathbb{Z}^n$.

$\implies$ $\vec{V}(t)$ is a **linear flow (orbit)** on the torus $\mathbb{R}^n/\mathbb{Z}^n$.



Large gap asymptotics: general case

Theorem (X.-Zhang-Zhao, '25)

Let $\mathcal{F}(s\Sigma) := \det(1 - \mathcal{K}^{(\alpha,\beta)}|_{s\Sigma})$ *and* $\Sigma := \cup_{j=0}^{n}(a_j, b_j)$ *be such that* $a_1 < b_1 < \cdots < a_m < 0 < b_m < \ldots < a_n < b_n$ *for some* $0 \leqslant m \leqslant n$*. For* $\alpha > -1/2$ *and* $\beta \in i\mathbb{R}$*, we have, as* $s \to +\infty$*,*

$$\log \mathcal{F}(s\Sigma) = -\gamma_0 s^2 - 2i\mathcal{D}_{\infty,1} s + \log\theta\left(\vec{V}(s)\right) + (\beta^2 - \alpha^2)\log s - \frac{1}{16}\sum_{j=0}^{n}\int_{\hat{s}}^{s}\left(\mathcal{L}\left(a_j, \vec{V}(t)\right) + \mathcal{L}\left(b_j, \vec{V}(t)\right)\right)\frac{\mathrm{d}t}{t} + \check{C}_1 + \mathcal{O}(s^{-1}),$$

where

$$\gamma_0 = -\frac{1}{\pi i}\sum_{j=0}^{n}\int_{a_j}^{b_j}\frac{z\mathrm{p}(z)}{\sqrt{\mathcal{R}(z)}_+}\,\mathrm{d}z \in \mathbb{R},$$

$\mathcal{D}_{\infty,1}$ *is purely imaginary and depends on the parameters* α *and* β*,* $\vec{V}(s) \in \mathbb{R}^n$ *is defined above,* $\hat{s} > 0$ *is a sufficiently large number independent of* s*,* $\mathcal{L}(p, \vec{V}(t))$ *is real for* $p \in \mathcal{I}_e := \{a_j, b_j\}_{j=0}^{n}$ *and* $\check{C}_1$ *is an undetermined constant independent of* s*. Moreover, for* $p \in \mathcal{I}_e$*, as* $s \to +\infty$*, we have*

$$\int_{\hat{s}}^{s}\mathcal{L}\left(p, \vec{V}(t)\right)\frac{\mathrm{d}t}{t} = \hat{\mathcal{L}}_p \log s + o(\log s), \quad \hat{\mathcal{L}}_p := \lim_{T\to+\infty}\frac{1}{T}\int_0^T \mathcal{L}\left(p, \vec{V}(t)\right)\mathrm{d}t.$$

Remarks on the general case

- If $n = 0$, $\Sigma = (-1, 1)$, we have

$$\mathrm{p}(z) = z, \quad \gamma_0 = \frac{1}{2}, \quad \mathcal{D}_{\infty,1} = i\alpha, \quad \mathcal{L}(p, \vec{V}(t)) = \mathcal{L}(z, 0) = 2.$$

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- For general $n > 1$, $\beta = 0$ $\Longrightarrow$ The large gap asymptotics for the type-I Bessel kernel determinant on multiple intervals.

- For general $n > 1$, if $\alpha = \beta = 0$ $\Longrightarrow$ The results is consistent with the Eq. (1.34) in [Deift-Its-Zhou, '97].

Large gap asymptotics: "good Diophantine property" case

Question: Could we improve the asymptotics

$$\int_{\hat{s}}^{s} \mathcal{L}\left(p, \vec{V}(t)\right) \frac{\mathrm{d}t}{t} = \hat{\mathcal{L}}_p \log s + o(\log s), \quad \hat{\mathcal{L}}_p := \lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathcal{L}\left(p, \vec{V}(t)\right) \mathrm{d}t.$$

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Aim 1: $o(\log s) \rightsquigarrow \mathcal{O}(s^{-1})$

Definition (good Diophantine property)

The linear flow

$$(0,+\infty) \ni s \mapsto (V_1(s) \mod 1, \quad V_2(s) \mod 1 \quad \ldots \quad V_n(s) \mod 1), \quad V_j(s) = \frac{s}{2\pi}\Omega_j + \frac{1}{2\pi}\operatorname{Im}\zeta_j,$$

has "**good Diophantine properties**" if there exist $\delta_1, \delta_2 > 0$ such that

$$|\vec{m}^{\mathrm{T}}\vec{\Omega}| \geqslant \delta_1 \|\vec{m}\|_2^{-\delta_2} \quad \text{for all } \vec{m} \in \mathbb{Z}^{n\times 1} \text{ with } \vec{m}^{\mathrm{T}}\vec{\Omega} \neq 0,$$

where $\vec{\Omega} := (\Omega_1, \ldots, \Omega_n)^{\mathrm{T}}$ and $\|\vec{m}\|_2 = |\vec{m}^{\mathrm{T}}\vec{m}|^{\frac{1}{2}}$.

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Let $\Sigma = \cup_{j=0}^{n}(a_j, b_j)$ be fixed such that $a_0 < b_0 < \cdots < a_m < 0 < b_m < \cdots < a_n < b_n$ for some $0 \leqslant m \leqslant n$ and assume that the good diophantine properties holds. As $s \to +\infty$, one has

$$\int_{\hat{s}}^{s} \mathcal{L}\left(p, \vec{V}(t)\right) \frac{\mathrm{d}t}{t} = \hat{\mathcal{L}}_p \log s + C_p + \mathcal{O}(s^{-1}),$$

where $p \in \mathcal{I}_e$ and C_p is independent of s. Thus, we have, as $s \to +\infty$,

$$\log \mathcal{F}(s\Sigma) = -\gamma_0 s^2 - 2i\mathcal{D}_{\infty,1} s + \log \theta\left(\vec{V}(s)\right) + \left[\beta^2 - \alpha^2 - \frac{1}{16}\sum_{j=0}^{n}(\hat{\mathcal{L}}_{a_j} + \hat{\mathcal{L}}_{b_j})\right] \log s + \check{C}_2 + \mathcal{O}(s^{-1}),$$

where $\check{C}_2 = \check{C}_1 - \frac{1}{16}\sum_{j=0}^{n}(C_{a_j} + C_{b_j})$ is a constant independent of s with $\check{C}_1$ as in the asymptotics of the general case.

Aim 2: Simplify the $\hat{\mathcal{L}}_p := \lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathcal{L}\left(p, \vec{V}(t)\right) \mathrm{d}t$.

Large gap asymptotics: ergodic case

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is ergodic in the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ if $\{\vec{V}(s) \mod \mathbb{Z}^n\}_{s>0}$ is dense in $\mathbb{R}^n/\mathbb{Z}^n$. Equivalently, the linear flow is ergodic in $\mathbb{R}^n/\mathbb{Z}^n$ if $\{\Omega_j\}_{j=1}^n$ are rationally independent, that is, if there exist $(c_1, c_2, \dots, c_n) \in \mathbb{Z}^n$ such that

$$c_1\Omega_1 + c_2\Omega_2 + \cdots + c_n\Omega_n = 0,$$

then $c_1 = c_2 = \cdots = c_n = 0$.

Large gap asymptotics: ergodic case

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$$c_1\Omega_1 + c_2\Omega_2 + \cdots + c_n\Omega_n = 0,$$

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Theorem (Birkhoff's ergodic theorem)

The time average exists everywhere, and coincides with the space average if f is continuous (or merely Riemann integrable) and the frequencies Ω_j are independent.

Large gap asymptotics: ergodic case

Applying Birkhoff's ergodic theorem to $\hat{\mathcal{L}}_p := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathcal{L}\left(p, \vec{V}(t)\right) \mathrm{d}t$

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$$\int_{\hat{s}}^{s} \mathcal{L}\left(p, \vec{V}(t)\right) \frac{\mathrm{d}t}{t} = \hat{\mathcal{L}}_p \log s + o(\log s), \quad \hat{\mathcal{L}}_p = \frac{h(p)}{\mathrm{p}(p)} \int_{[0,1)^n} \eta(p; u_1, u_2, \ldots u_n)\, \mathrm{d}u_1 \cdots \mathrm{d}u_n.$$

Thus, we have, as $s \to +\infty$,

$$\log \mathcal{F}(s\Sigma) = -\gamma_0 s^2 - 2i\mathcal{D}_{\infty,1} s + \left[\beta^2 - \alpha^2 - \frac{1}{16}\sum_{j=0}^{n}(\hat{\mathcal{L}}_{a_j} + \hat{\mathcal{L}}_{b_j})\right] \log s + o(\log s),$$

where the constant $\hat{\mathcal{L}}_p$, $p \in \mathcal{I}_e$, is explicitly given by the n-fold integral.

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In the case of $n = 1$, the linear flow satisfies both good Diophantine properties and ergodic properties.

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It could be calculated that $\hat{\mathcal{L}}_p = \frac{h(p)}{\mathrm{p}(p)} \int_{[0,1)} \eta(p; u_1) \, \mathrm{d}u_1 = 2$ for $p \in \{a_0, b_0, a_1, b_1\}$.

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where C is an undetermined constant independent of s.

Remarks

Let

$$\mathcal{S}_D := \left\{ \vec{\Omega} : \ \vec{\Omega} \text{ has ``good Diophantine property''} \right\}, \ \mathcal{S}_E := \left\{ \vec{\Omega} : \ \vec{\Omega} \text{ is rationally independent} \right\}.$$

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We have some examples:

- Genus $n = 1$ case: $\mathcal{S}_D = \mathcal{S}_E = (0, +\infty)$;
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- Genus $n = 3$ case: $\vec{\Omega} = (1, C_L, 1) \notin \mathcal{S}_D \cup \mathcal{S}_E$.
- ……

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Let

$$\mathcal{S}_D := \left\{\vec{\Omega}: \ \vec{\Omega} \text{ has ``good Diophantine property''}\right\}, \ \mathcal{S}_E := \left\{\vec{\Omega}: \ \vec{\Omega} \text{ is rationally independent}\right\}.$$

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For $n \geqslant 2$, all of the following cases can and do occur for certain choices of the edges points $\{a_j, b_j\}_{j=0}^{n}$.

$$\vec{\Omega} \notin \mathcal{S}_D \cup \mathcal{S}_E, \quad \vec{\Omega} \in \mathcal{S}_D \setminus \mathcal{S}_E, \quad \vec{\Omega} \in \mathcal{S}_E \setminus \mathcal{S}_D, \quad \vec{\Omega} \in \mathcal{S}_D \cap \mathcal{S}_E.$$

[Deift-Its-Zhou, '97]

Remarks

- For the n-fold integral obtained in the ergodic case

$$\hat{\mathcal{L}}_p = \frac{h(p)}{\mathrm{p}(p)} \int_{[0,1)^n} \eta(p; u_1, u_2, \ldots u_n) \, \mathrm{d}u_1 \cdots \mathrm{d}u_n,$$

(a) It's been proved that $\hat{\mathcal{L}}_p \equiv 2$ for $n = 1$.
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- The multiplicative constant $\rightsquigarrow$ challenge.

About the proofs

An RH characterization of the $\mathcal{F}(s\Sigma)$

Rely on an integrable structure of the confluent hypergeometric kernel $K^{(\alpha,\beta)}(x,y)$

$$K^{(\alpha,\beta)}(x,y) = \frac{\vec{f}(x)^{\mathsf{T}}\vec{h}(y)}{x-y} = \frac{\sum_{k=1}^{2} f_k(x)h_k(y)}{x-y}.$$

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- $Y(z) := I - \int_{s\Sigma} \frac{\vec{F}(x)\vec{h}(x)^{\mathsf{T}}}{x-z}\,\mathrm{d}x$ satisfies an 2×2 RH problem, where $\vec{F}(z) = (1-\mathcal{K}^{(\alpha,\beta)}|_{s\Sigma})^{-1}\vec{f}(z)$.

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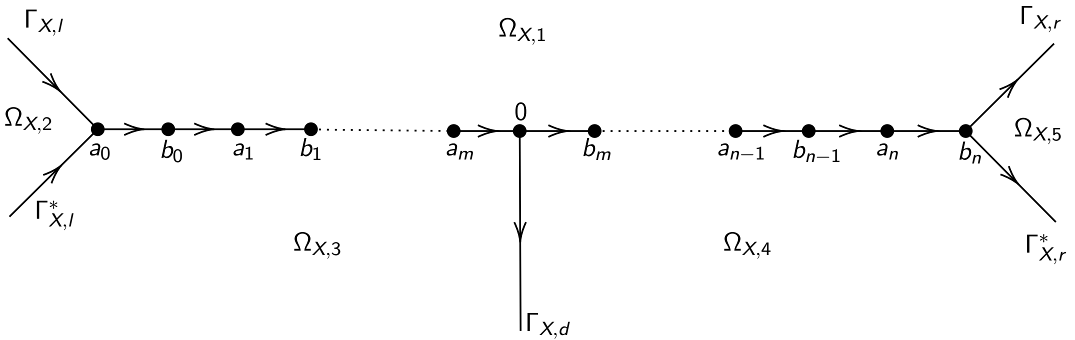
Undressing transform $\rightsquigarrow$ an RH problem for X with constant jumps.

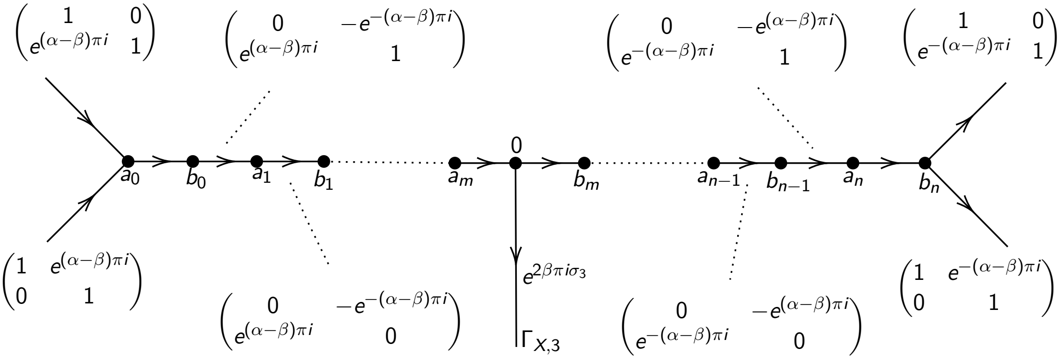
RH problem for X

(a) $X(z)$ is holomorphic for $z \in \mathbb{C} \setminus \Gamma_X$



RH problem for X

(b) For $z \in \Gamma_X$, we have $X_+(z) = X_-(z) J_X(z)$.



(c) As $z \to \infty$, we have

$$X(z) = \left(I + \frac{X_1(s)}{z} + \mathcal{O}\left(z^{-2}\right) \right) z^{-\beta\sigma_3} e^{-isz\sigma_3},$$

where

$$X_1(s) = (2s)^{\beta\sigma_3} \begin{pmatrix} \frac{1}{2s}\left(\Phi_{\mathrm{CH},1}\right)_{11} + \frac{1}{s}(Y_1(s))_{11} & \frac{1}{2s}(\Phi_{\mathrm{CH},1})_{12} + \frac{1}{s}(Y_1(s))_{12} e^{-\pi i\beta} \\ \frac{1}{2s}(\Phi_{\mathrm{CH},1})_{21} + \frac{1}{s}(Y_1(s))_{21} e^{\pi i\beta} & \frac{1}{2s}(\Phi_{\mathrm{CH},1})_{22} + \frac{1}{s}(Y_1(s))_{22} \end{pmatrix} (2s)^{-\beta\sigma_3}$$

with $\Phi_{\mathrm{CH},1} := \lim_{z\to\infty} z(\Phi_{\mathrm{CH}}(z) z^{\beta\sigma_3} e^{\frac{i}{2}z\sigma_3} - I)$.

(d) As $z \to p$ from $\Omega_{X,1}$, $p \in \mathcal{I}_e$, we have $X(z) = \mathcal{O}(\log(z-p))$.

(e) As $z \to 0$ from $\operatorname{Im} z > 0$, we have

$$X(z) = X_0(z) z^{\alpha\sigma_3},$$

where $X_0(z)$ is holomorphic in the neighborhood of 0. The behavior of $X(z)$ as $z \to 0$ is determined by jump conditions.

A "failed" differential identity

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Difficulties: (a) several complex variables; (b) hard to evaluate the coefficients of higher order derivatives. $\rightsquigarrow$ **We need a new one!**

A new differential identity

Proposition (X.-Zhang-Zhao, '25)

We have

$$\partial_s \log \mathcal{F}(s\Sigma) = i\left((X_1(s))_{11} - (X_1(s))_{22}\right) - \frac{\alpha^2 - \beta^2}{s},$$

where X_1 is the residue term of X at infinity.

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Deift-Zhou nonlinear steepest analysis for X

g-function mechanism is taken to normalize the RH problem for X.

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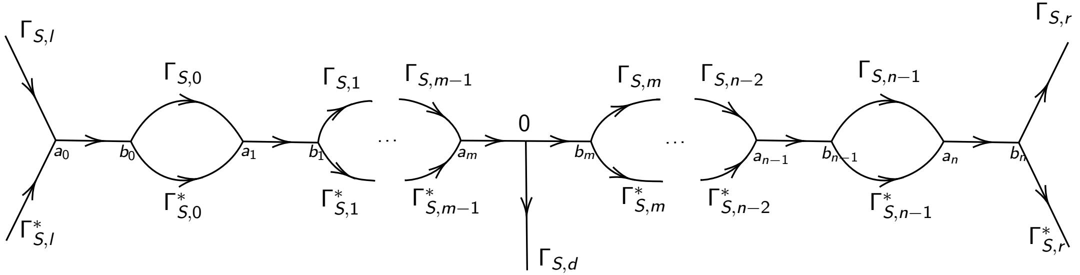
Opening lenses to obtain a solvable RH problem for P



Deift-Zhou nonlinear steepest analysis for X

Global parametrix for $P^{(\infty)}$

- A Szegö function $\mathcal{D}$ to deal with the Fisher-Hartwig singularity.

$$\mathcal{D}(z) := \exp\left\{ \frac{\sqrt{\mathcal{R}(z)}}{2\pi i} \int_{\Sigma} \frac{\mathcal{H}(\xi)}{\xi - z} \, \mathrm{d}\xi \right\},$$

where

$$\mathcal{H}(z) := \frac{\log z^{-2\beta} + \mathrm{sgn}(z)(\alpha - \beta)\pi i + \zeta_j}{\sqrt{\mathcal{R}(z)}_+}, \quad z \in (a_j, b_j), \quad j = 0, 1, \ldots, n.$$

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[Deift-Its-Zhou, '97]

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- Construction of $P^{(\infty)}$: a model RH problem solvable in terms of Riemann-θ function. [Deift-Its-Zhou, '97]
- Some exact formulas related to $P^{(\infty)}$ are required.

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where $*_1$ is the residue term of $*$ at infinity.

How the two properties help?

Go back to the time average integral

$$I = \frac{1}{T}\int_{\hat{s}}^{T} \mathcal{Y}(\vec{V}(t))\,\mathrm{d}t, \quad \vec{V}(t) = t\vec{\Omega} + \vec{\tilde{\zeta}} \in \mathbb{R}^n/\mathbb{Z}^n.$$

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Analytical function $\mathcal{Y}$ on $\mathbb{R}^n/\mathbb{Z}^n$ admits a Fourier expansion

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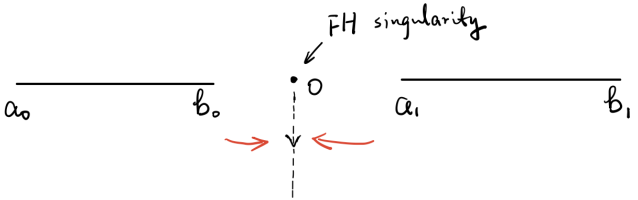
- Ergodic property + Birkhoff's ergodic theorem $\Longrightarrow \sum_{\substack{\vec{m}\in\mathbb{Z}^n \\ \vec{m}^{\mathrm{T}}\vec{\Omega}=0}} \ell_{\vec{m}} e^{i\vec{m}^{\mathrm{T}}\vec{\tilde{\zeta}}} = \ell_{\vec{0}}$.

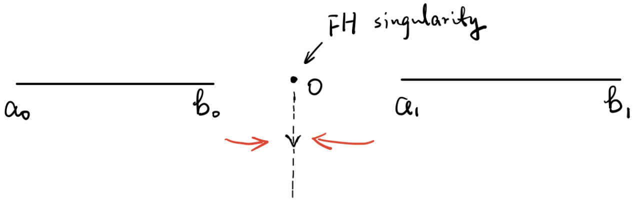
Future work

- Confluent hypergeometric kernel determinant with a FH singularity at the gap and the merging case at the FH singular point.



Future work

- Confluent hypergeometric kernel determinant with a FH singularity at the gap and the merging case at the FH singular point.



- New differential identity $\rightsquigarrow$ Other integrable kernel determinants for general genus $g > 1$ case?
(Progress on the Airy kernel determinant)

Thanks for your attention!