Integrable PDEs with nonzero (symmetric and asymmetric) boundary conditions: large-time and transient asymptotics

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Outline of the talk

- Part I - An introduction about today's topic
  - Some concepts
  - Some history
- Part II - Three integrable models (defocusing mKdV equation, non-local mKdV equation and mCH equation) with nonzero boundary conditions
  - Relation to earlier work
  - Main results
  - Remarks on the results and the proof
- Part III - Discussable issues

Part I: Introduction

Cauchy problem with nonzero boundary conditions

Cauchy problem

Consider an integrable PDE which satisfies the following initial-value condition

$$F(u, u_x, u_t, u_{xt}, \cdots) = 0,$$

$$u_0(x) := u(x,0) \to \begin{cases} c_l, & x \to -\infty, \\ c_r, & x \to +\infty. \end{cases}$$

- The integrable PDE admits a Lax pair.
- Symmetric case: $|c_l| = |c_r| \neq 0$. (Finite density initial data: under the change of variable u, x, t, one has $|c_l| = |c_r| \equiv 1$)
- Asymmetric case: $|c_l| \neq |c_r|$. (So-called "step-like initial data").
- Using "$=$" to replace "$\to$", we call it "pure step initial data".

Main topics we are concerned with

- The functional space that $u_0(x)$ belongs? $\Rightarrow$ Existence of the global solution? (Not easy).
- The large-time asymptotics and the (possibly) transient asymptotics?

Main topics we are concerned with

- The functional space that $u_0(x)$ belongs? $\Rightarrow$ Existence of the global solution? (Difficult).
- The large-time asymptotics and the (possibly) transient asymptotics?

Large-time and transient asymptotics

"Large-time asymptotics"

Asymptotics with velocity $\xi := \xi(x/t)$ varies along the line (or say in "conventional asymptotic zones").

"Transient asymptotics"

Large-time asymptotics between two different conventional asymptotic zones where phase transitions occur.

Some history on symmetric case (finite density initial data)

- Vartanian computed both the leading and first correction terms in the asymptotic expansion of the solution of the defocusing NLS equation inside/outside "light cone". (RH problem)

  [Vartanian, '02, '03]

- Cuccagna, Jenkins verified the soliton resolution conjecture inside the soliton "light cone". (Uniformization technique, $\bar{\partial} - RH$ problem)

  [Cuccagna-Jenkins, '16]

- The method developed by Cuccagna and Jenkins are extended to compute the asymptotic expansion in soliton-less and transient zones.

  [Wang-Fan, '22, '23]

- Biondini and Mantzavinos study the large-time asymptotics of focusing nonlinear Schrödinger equation with nonzero boundary conditions. (Uniformization technique is not used)

  [Biondini-Mantzavinos, '17]

Some history on symmetric case (finite density initial data)

- Biondini, Li and Mantzavinos study the large-time asymptotics of focusing nonlinear Schrödinger equation with nonzero boundary conditions in the presence of discrete spectrum.

  [Biondini-Li-Mantzavinos, '21]

- ……

Some history on asymmetric case (step-like initial data)

- Gurevich, Pitaevskii and Khruslov are the first who considered such a problem for the KdV equation.

  [Gurevich-Pitaevskii, '73], [Khruslov, '76]

- Asymptotic behavior of solutions of the step-like initial-value problems in a neighborhood of leading edge was done by Khruslov and Kotlyarov by using inverse scattering transform in the form of Marchenko integral equations and the so-called asymptotic solitons.

  [Khruslov, Kotlyarov, et.al, '86, '89, '90, '94, '01, '03]

- A further development of this approach by KK was done by Egorova and Teschl and their co-authors. (Rigorous results for KdV and Toda equation)

  [Egorova-Grunert-Teschl, '09], [Egorova-Michor-Teschl, '09]

Some history on asymmetric case (step-like initial data)

- Minakov and his co-authors studied the large-time asymptotics of the focusing mKdV equation with step-like initial data. (non-self-adjoint Lax operator)

  [Kotlyarov-Minakov, '10, '11, '12], [Grava-Minakov, '20]

- Bertola and Minakov obtain transient asymptotics with the help of parametrices constructed out of Laguerre polynomials in the corresponding RH problem. $(q_0(-\infty) \to c > 0,\ q_0(+\infty) \to 0)$

  [Bertola-Minakov, '18]

- Focusing NLS equation with step-like boundary conditions.

  [Boutet de Monvel-Kotlyarov-Shepelsky, '11]
  [Boutet de Monvel-Lenells-Shepelsky, '21, '22]

- Semi-classical defocusing NLS equation with shock initial data, Defocusing NLS equation with step-like initial data (self-adjoint Lax operator)

  [Jenkins, '15], [Fromm-Lenells-Quirchmayr, '21]

Some history on asymmetric case (step-like initial data)

- Nonlocal NLS equation with step-like boundary conditions. (include transient asymptotics)

  [Rybalko-Shepelsky, '20, '21$\times$4]

- ......

Part II: Three integrable models with nonzero boundary conditions: Large-time asymptotics

Model I: Defocusing mKdV equation with nonzero boundary conditions

Cauchy problem of the defocusing mKdV equation with finite density initial data

$$
\begin{aligned}
&q_t(x,t) - 6q^2(x,t)q_x(x,t) + q_{xxx}(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\
&q(x,0) = q_0(x), \quad \lim_{x \to \pm\infty} q_0(x) = \pm 1.
\end{aligned}
$$

Cauchy problem of the defocusing mKdV equation with step-like initial data

$$
\begin{aligned}
&q_t(x,t) - 6q^2(x,t)q_x(x,t) + q_{xxx}(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\
&q(x,0) = q_0(x) \to \begin{cases} c_l, & x \to -\infty, \\ c_r, & x \to +\infty, \end{cases}
\end{aligned}
$$

where $c_l > c_r > 0$.

Model I: Relation to earlier work

- Earliest work

  [Ablowitz-Segur, '81]

- Remark work on defocusing mKdV equation (self-adjoint Lax operator, zero bdrys)

  [Deift-Zhou, '93]

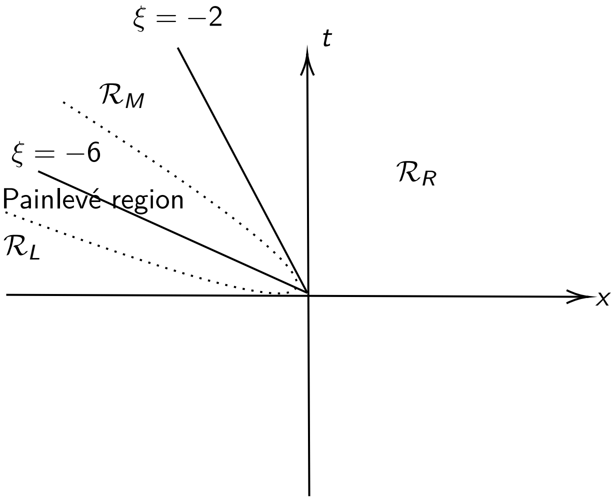
- A nonlinear steepest descent theorem for RH problems with Carleson jump contours (zero bdrys)

  [Lenells, '17]

- Long-time asymptotics of the modified KdV equation in weighted Sobolev spaces (zero bdrys, no soliton)

  [Chen-Liu, '22]

Model I: Statement of results (symmetric bdrys)



Model I: Statement of results (symmetric bdrys)

$$\mathcal{R}_L = \{(x,t) : \xi < -6, |\xi| = \mathcal{O}(1)\}, \quad \mathcal{R}_M = \{(x,t) : -6 < \xi < 6\},$$
$$\mathcal{R}_R = \{(x,t) : \xi > 6, |\xi| = \mathcal{O}(1)\}, \quad \xi := x/t.$$

Theorem (Fan-X.-Zhang)

Let $q(x,t)$ be the solution for the Cauchy problem (the one with finite density initial data) with generic data $q_0(x) - \tanh(x) \in H^{4,4}(\mathbb{R})$ associated to scattering data $\left\{r(z), \{\eta_n, c_n\}_{n=1}^{2N}\right\}$. As $t \to +\infty$, the following three asymptotics are shown.

(a) *For $(x,t) \in \mathcal{R}_L$ (left field),*

$$q(x,t) = -1 + t^{-\frac{1}{2}} f(\xi) + \mathcal{O}(t^{-\frac{3}{4}}).$$

where

$$f(\xi) := \sum_{j=1}^{4} \epsilon_j \left(2\epsilon_j \theta''(\xi_j)\right)^{-\frac{1}{2}} \left(1 - \xi_j^{-2}\right)^{-1} \cdot \left(\beta_{12}^{(\xi_j)} - \frac{1}{\xi_j^2} \beta_{21}^{(\xi_j)}\right),$$

Model I: Statement of results (symmetric bdrys)

Theorem (Fan-X.-Zhang)

(b) *For* $(x,t) \in \mathcal{R}_M$

$$q(x,t) = -1 + \sum_{j=0}^{N} [sol(z_j, x - x_j, t) + 1] + \mathcal{O}(t^{-1}).$$

(soliton resolution, asymptotic stability)

(c) *For* $(x,t) \in \mathcal{R}_R$ *(right field),*

$$q(x,t) = 1 + \mathcal{O}(t^{-1}),$$

Model I: Statement of results (symmetric bdrys)

Theorem (Fan-Wang-X.)

Under the same condition, the long-time asymptotics of the solution to the Cauchy problem (the one with the finite density initial data) for the defocusing mKdV equation in the transition region $\left|\frac{x}{t}+6\right| t^{2/3} < C$ *with* $C > 0$ *is given by the following formula:*

$$q(x,t) = -1 + (3t)^{-\frac{1}{3}} u(s) \cos\varphi_0 + \mathcal{O}\left(t^{-\frac{1}{3}-\epsilon}\right),$$

where ϵ *is a constant with* $0 < \epsilon < \frac{1}{9}$*,*

$$s = \frac{1}{3}(3t)^{\frac{2}{3}}\left(\frac{x}{t}+6\right),\ \varphi_0 = \arg\tilde{r}(1),$$

and $u(s)$ *is a unique solution of the Painlevé II equation* $u''(s) = 2u^3(s) + su(s)$ *with characterization* $u(s) \sim -|r(1)|Ai(s)$*,* $s \to +\infty$*.*

Model I: Remarks on the proof and results (symmetric bdrys)

- An RH characterization of the corresponding Cauchy problem (Uniformization technique)

  [Faddeev, Takhtajan, '87]
- Contour deformation to a "regular" RH problem ($\bar{\partial}$ nonlinear descent appoach)

  [McLaughlin-Miller, '06; Dieng-McLaughlin-Miller, '19]
- Pure RH problem
  - Global parametrix (A model RH problem with singularity at $z = 0 \to$ leading term)
  - Local parametrix (Prabolic cylinder parametrix / Painlevé II parametrix $\to$ sub-leading term)
  - Small norm RH problem
- $\bar{\partial}$ problem (Main obstacle: How to balance the singularity at $z = \pm 1$?)
- Good: No use of g-function;

  Bad: Singularities from the uniformization technique.

Model I: Statement of results (asymmetric bdrys)

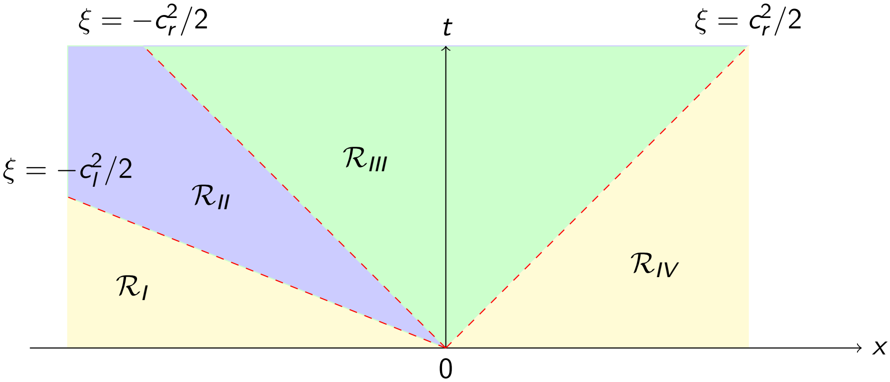


Figure: The different asymptotic regions of the (x, t)-half plane, where $\xi := x/(12t)$.

Model I: Statement of results (asymmetric bdrys)

Definition

For the constants c_l and c_r, we define

- Left field: $\mathcal{R}_I := \left\{(x,t)|\xi < -\frac{c_l^2}{2}\right\}$,
- Oscillatory region: $\mathcal{R}_{II} := \left\{(x,t)| -\frac{c_l^2}{2} < \xi < -\frac{c_r^2}{2}\right\}$,
- Central field: $\mathcal{R}_{III} := \left\{(x,t)| -\frac{c_r^2}{2} < \xi < \frac{c_r^2}{2}\right\}$,
- Right field: $\mathcal{R}_{IV} := \left\{(x,t)|\xi > \frac{c_r^2}{2}\right\}$,

where recall that $\xi = x/(12t)$.

Model I: Statement results (asymmetric bdrys)

Assumption

(a) The initial data $q_0(x)$ admits a compact perturbation of the shock initial data

$$q_{0c_j}(x) = \begin{cases} c_l, & x < 0, \\ c_r, & x > 0, \end{cases}$$

i.e., $q_0(x) - q_{0c_j}(x) = 0$ for $|x| > N$ with some positive N.

(b) The initial data $q_0(x)$ is assumed to be locally a function of bounded variation $BV_{loc}(\mathbb{R})$ and satisfying the following conditions

$$\int_{\mathbb{R}} |x|^2 |\,\mathrm{d}q_0(x)| < \infty,$$

and

$$\int_{\mathbb{R}^+} e^{2\sigma x} |q_0(x) - c_r| \,\mathrm{d}x < \infty, \quad \int_{\mathbb{R}-} e^{2\sigma x} |q_0(x) - c_l| \,\mathrm{d}x < \infty$$

where $\sigma > \sqrt{c_l^2 - c_r^2} > 0$, and $\mathrm{d}q_0(x)$ is the corresponding signed measure.

Model I: Statement of results (asymmetric bdrys)

Theorem (Fan-X.)

Let $q(x,t)$ be the global solution of the Cauchy problem (the one with the step-like initial data) for the defocusing mKdV equation over the real line under the Assumption, and denote by $r(k)$ the reflection coefficient. As $t \to +\infty$, we have the following asymptotics of $q(x,t)$ in the regions $\mathcal{R}_I - \mathcal{R}_{IV}$.

(a) *For $\xi \in \mathcal{R}_I$, we have*

$$q(x,t) = D_{I,\infty}^{-2}(\xi)\left(c_I + t^{-1/2} f_I(\xi)\right) + \mathcal{O}(t^{-1}),$$

(b) *For $\xi \in \mathcal{R}_{II}$, we have*

$$q(x,t) = D_{II,\infty}^{-2}(\xi)\left(\sqrt{-\frac{x}{6t}} + t^{-1} f_{II}(\xi)\right) + \mathcal{O}(t^{-2}),$$

Model I: Statement results (asymmetric bdrys)

Theorem (Fan-X.)

(c) *For $\xi \in \mathcal{R}_{III}$, we have*

$$q(x,t) = c_r + t^{-1} f_{III}(\xi) + \mathcal{O}(t^{-2}),$$

(d) *For $\xi \in \mathcal{R}_{IV}$, we have*

$$q(x,t) = c_r + \mathcal{O}(t^{-1/2} e^{-16t\xi^{3/2}}),$$

where the qutities $D_{I,\infty}$, $D_{II,\infty}$, $f_I(\xi)$, $f_{II}(\xi)$ and $f_{III}(\xi)$ admit the concrete forms (They are not listed here for limiting the length of the slides).

Model I: Remarks on the proof and results (asymmetric bdrys)

- An RH characterization of the corresponding Cauchy problem (Branch cut stays)
- Contour deformation to a "regular" RH problem (How to construct the g-function case by case?)
- RH analysis
  - Global parametrix (A model RH problem with singularity at branch points and a cut $\to$ leading term)
  - Local parametrix (Prabolic cylinder parametrix / Airy parametrix $\to$ sub-leading term)
  - Small norm RH problem
- Good: Less singular points; Bad: newly generated g-function.
- Comparision to the focusing cases with the same step-like initial data
  - Location of the branch cut.
  - No elliptic wave.

Model II: Nonlocal mKdV equation with step-like initial data

Cauchy problem for the nonlocal mKdV equation with step-like initial data

$$\begin{aligned} &u_t(x,t)+6u(x,t)u(-x,-t)u_x(x,t)+u_{xxx}(x,t)=0, \\ &u_0(x)=u(x,t=0), \qquad\qquad x\in\mathbb{R},\ t\in\mathbb{R}. \end{aligned}$$

with

$$u(x,t)=\begin{cases} A+o(1), & x\to+\infty, \\ o(1), & x\to-\infty, \end{cases}$$

Model II: Relation to earlier work

- Long-time asymptotics of the nonlocal mKdV equation with zero bdrys.

  [He-Xu-Fan, '19]

- Long-time asymptotics of the nonlocal mKdV equation with symmetric bdrys.

  [Zhou-Fan, '22, '23]

Model II: Statement of results

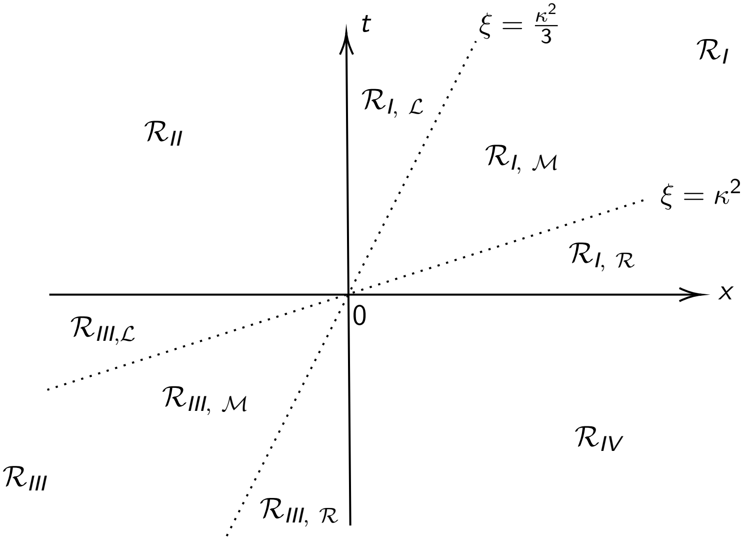


Figure: The space-time cones with $\xi := \frac{x}{12t}$

Model II: Statement of results

Theorem (Fan-X.)

Consider the initial-valued problem, where the initial data $u_0(x)$ is a compact perturbation of the pure step initial function: $u_0(x) - u_{0A}(x) = 0$ for $|x| > N$ with some $N > 0$. Assume that the spectral functions $a_j(k)$, $j = 1, 2$ and $b(k)$ associated to $u_0(x)$ satisfy

(i) *$a_1(k)$ has a single, simple zero in $\overline{\mathbb{C}_+}$ located $k = i\kappa$, and $a_2(k)$ either has no zeros in $\overline{\mathbb{C}_-}$ (generic case) or has a single, simple zero at $k = 0$ (non-generic case);*

(ii) $\operatorname{Im} \nu(-k_0(\xi)) = -\frac{1}{2\pi} \int_{-\infty}^{-k_0(\xi)} d \arg(1 + r_1(s) r_2(s)) \in (-\frac{1}{2}, \frac{1}{2})$, *where* $r_1(k) = \frac{b(k)}{a_1(k)}$, $r_2(k) = \frac{b(k)}{a_2(k)}$.

Assuming that the solution $u(x,t)$ of Cauchy problem exists, the long-time asymptotics of $u(x,t)$ along any line $\xi := \frac{x}{12t} < 0$, $|\xi| = O(1)$ can be described as follows:

Model II: Statement of results

Theorem (Fan-X.)

I. *For $x < 0$, $t > 0$ (corresponding to $\mathcal{R}_{II}$), as $t \to +\infty$*

$$u(x,t) = -4\eta(-\tau)^{-\frac{1}{2}-\operatorname{Im}\nu(-k_0(\xi))}\operatorname{Re}\left(\gamma(\xi)e^{t\varphi(\xi,0)}(-\tau)^{i\operatorname{Re}\nu(-k_0(\xi))}\right) + R_1(\xi,-t).$$

II. *For $x > 0$, $t < 0$ (corresponding to $\mathcal{R}_{IV}$), as $t \to -\infty$, three types of asymptotic forms are possible, depending on the $\operatorname{Im}\nu(-k_0)$, in detail,*

(II.a) *if* $\operatorname{Im}\nu(-k_0(\xi)) \in (-\frac{1}{2}, -\frac{\alpha}{6}]$,

$$u(x,t) = A\delta^2(0,\xi) - \frac{4c_0^2}{k_0^2(\xi)}\eta\tau^{-\frac{1}{2}-\operatorname{Im}\nu(-k_0(\xi))} \cdot \operatorname{Re}\left(i\gamma(\xi)e^{t\varphi(\xi,0)}\tau^{i\operatorname{Re}\nu(-k_0(\xi))}\right) + R_1(\xi,t),$$

Model II: Statement of results

Theorem (Fan-X.)

(II.b) *if* $\operatorname{Im}\nu(-k_0(\xi)) \in (-\frac{\alpha}{6}, \frac{\alpha}{6})$,

$$u(x,t) = A\delta^2(0,\xi) - \frac{4c_0^2}{k_0^2(\xi)}\eta\tau^{-\frac{1}{2}-\operatorname{Im}\nu(-k_0(\xi))} \cdot \operatorname{Re}\left(i\gamma(\xi)e^{t\varphi(\xi,0)}\tau^{i\operatorname{Re}\nu(-k_0(\xi))}\right) + 4\eta\tau^{-\frac{1}{2}+\operatorname{Im}\nu(-k_0(\xi))} \cdot \operatorname{Re}\left(\beta(\xi)e^{-t\varphi(\xi,0)}\tau^{-i\operatorname{Re}\nu(-k_0(\xi))}\right) + R_3(\xi,t),$$

(II.c) *if* $\operatorname{Im}\nu(-k_0(\xi)) \in [\frac{\alpha}{6}, \frac{1}{2})$,

$$u(x,t) = A\delta^2(0,\xi) + 4\eta\tau^{-\frac{1}{2}+\operatorname{Im}\nu(-k_0(\xi))} \cdot \operatorname{Re}\left(\beta(\xi)e^{-t\varphi(\xi,0)}\tau^{-i\operatorname{Re}\nu(-k_0(\xi))}\right) + R_2(\xi,t),$$

Model II: Statement of results

Theorem (Fan-X.)

$$\delta(0,\xi)=\exp\left\{\frac{1}{2\pi i}\int_{(-\infty,-k_0(\xi))\cup(k_0(\xi),+\infty)}\frac{\log\left(1+r_1(s)r_2(s)\right)}{s}ds\right\},$$

$$k_0(\xi):=\sqrt{-\xi},\quad \eta:=\frac{k_0(\xi)}{2},\quad \rho=\eta\sqrt{48k_0(\xi)},\quad \tau:=-t\rho^2=-12tk_0^3(\xi),$$

$$\epsilon:=\min\left\{\eta:=\frac{k_0(\xi)}{2},\quad \frac{1}{2}|i\kappa+k_0(\xi)|\right\},\ \alpha\in\left(\frac{1}{2},1\right)$$

$$\varphi(\xi;\zeta):=2i\theta\left(\xi,-k_0(\xi)+\frac{\eta}{\rho}\right)=16ik_0^3(\xi)-\frac{i}{2}\zeta^2+\frac{i\zeta^3}{12\rho},$$

$$\beta(\xi)=\frac{\sqrt{2\pi}e^{\frac{i\pi}{4}}e^{-\frac{\pi\nu(-k_0(\xi))}{2}}}{q_1(-k_0(\xi))\Gamma(-i\nu(-k_0(\xi)))},\quad q_1(-k_0(\xi))=e^{-2\chi(\xi,-k_0(\xi))}r_1(-k_0(\xi))e^{2i\nu(-k_0(\xi))\log 4},$$

$$\gamma(\xi)=\frac{\sqrt{2\pi}e^{-\frac{i\pi}{4}}e^{-\frac{\pi\nu(-k_0(\xi))}{2}}}{q_2(-k_0(\xi))\Gamma(i\nu(-k_0(\xi)))},\quad q_2(-k_0(\xi))=e^{2\chi(\xi,-k_0(\xi))}r_2(-k_0(\xi))e^{-2i\nu(-k_0(\xi))\log 4}.$$

Model II: Statement of results

Theorem (Fan-X.)

The error estimates are as follows

$$R_1(\xi,t)=\begin{cases}O(\epsilon\tau^{-\frac{1+\alpha}{2}}), & \operatorname{Im}\nu(-k_0(\xi))\geqslant 0\\ O(\epsilon\tau^{-\frac{1+\alpha}{2}+2|\operatorname{Im}\nu(-k_0(\xi))|}), & \operatorname{Im}\nu(-k_0(\xi))<0\end{cases}$$

$$R_2(\xi,t)=\begin{cases}O(\epsilon\tau^{-\frac{1+\alpha}{2}+2|\operatorname{Im}\nu(-k_0(\xi))|}), & \operatorname{Im}\nu(-k_0(\xi))\geqslant 0\\ O(\epsilon\tau^{-\frac{1+\alpha}{2}}), & \operatorname{Im}\nu(-k_0(\xi))<0\end{cases}$$

and

$$R_3(\xi,t):=R_1(\xi,t)+R_2(\xi,t)=O(\epsilon\tau^{-\frac{1+\alpha}{2}+2|\operatorname{Im}\nu(-k_0(\xi))|})$$

Model II: Statement of results

Theorem (Fan-X.)

Under the same conditions, the long-time asymptotics of $u(x,t)$ along any line $\xi := \frac{x}{12t} > 0$, can be described as follows:

I. *For $x > 0$, $t > 0$ (corresponding to $\mathcal{R}_I$), as $t \to +\infty$, three asymptotic forms are presented for different ξ as follows*

(I.a) *if $\xi \in (0, \frac{\kappa^2}{3})$ (corresponding to solitonic region $\mathcal{R}_{I,\mathcal{L}}$)*

$$u(x,t) = \frac{A}{1 - C_1(\kappa)e^{-2\kappa x + 8\kappa^3 t}} + O\left(t^{-\frac{1}{2}} e^{-16t\xi^{3/2}}\right),$$

(I.b) *if $\xi \in (\frac{\kappa^2}{3}, \kappa^2)$ (corresponding to region $\mathcal{R}_{I,\mathcal{M}}$)*

$$u(x,t) = A + O\left(t^{-\frac{1}{2}} e^{-16t\xi^{3/2}}\right),$$

(I.c) *if $\xi \in (\kappa^2, +\infty)$ (corresponding to region $\mathcal{R}_{I,\mathcal{R}}$)*

$$u(x,t) = A + O\left(t^{-\frac{1}{2}} e^{-8t\kappa_\delta(3\xi - \kappa_\delta^2)}\right).$$

Model II: Statement of results

Theorem (Fan-X.)

II. *For $x < 0$, $t < 0$ (corresponding to $\mathcal{R}_{III}$), as $t \to -\infty$, three asymptotic forms are presented for different ξ as follows*

(II.a) *if $\xi \in (0, \frac{\kappa^2}{3})$ (corresponding to solitonic region $\mathcal{R}_{III,\mathcal{R}}$)*

$$u(x,t) = \frac{4}{A\kappa^{-2} - C_2(\kappa)e^{-2\kappa x + 8\kappa^3 t}} + O\left((-t)^{-\frac{1}{2}} e^{16t\xi^{3/2}}\right),$$

(II.b) *if $\xi \in (\frac{\kappa^2}{3}, \kappa^2)$ (corresponding to region $\mathcal{R}_{III,\mathcal{M}}$)*

$$u(x,t) = O\left((-t)^{-\frac{1}{2}} e^{16t\xi^{3/2}}\right),$$

(II.c) *if $\xi \in (\kappa^2, +\infty)$ (corresponding to region $\mathcal{R}_{III,\mathcal{L}}$)*

$$u(x,t) = O\left((-t)^{-\frac{1}{2}} e^{8t\kappa_\delta(3\xi - \kappa_\delta^2)}\right).$$

Model II: Remarks on the proof and results

- An RH characterization of the corresponding Cauchy problem (Both focusing and defocusing cases; Main obstacle: singularity, generic case / non-generic case $\to$ residue condition)
- Contour deformation to a "regular" RH problem (δ function is considered when we discuss large negative t $\to$ avoiding high oscillation; nonlocal properties $u(-x,-t)$)
- Regular RH problem
  - Global parametrix (Blaschke-Potapov factors method $\to$ leading term)
  - Local parametrix
  - Small norm RH problem
- Comparision to the local mKdV equation with the same step-like initial data

[Kotlyarov-Minakov, '10]

  - The Riemann-Hilbert problem formalism. Branch cut v.s. Singular point $k=0$ and $i\kappa \in i\mathbb{R}^+$.
  - The large-time asymptotics analysis technique. g-function v.s. singular condition $\to$ residue condition
  - The asymptotic results. $\operatorname{Im}\nu(-k_0)$.

Model III: mCH equation with nonzero boundary conditions

(The part “Model III” of this slide is made by L. Zhang, and revised by me)

Cauchy problem for the mCH equation with nonzero boundary conditions

$$\begin{aligned} & m_t + \left(m\left(u^2 - u_x^2\right)\right)_x = 0, \quad m = u - u_{xx}, && x \in \mathbb{R},\ t > 0, \\ & u(x,0) = u_0(x), && x \in \mathbb{R}, \end{aligned}$$

with nonzero boundary condition

$$u_0(x) \to 1, \qquad |x| \to \infty.$$

Modell III: Relation to earlier work

- A Lax pair

  [Schiff, '96; Qiao, '06]

- A Riemann-Hilbert (RH) formalism of the Cauchy problem has recently been developed.

  [Boutet de Monvel-Karpenko-Shepelsky, '20]

- Large-time asymptotics: initial value in the Schwartz space.

  [Boutet de Monvel-Karpenko-Shepelsky, '22]

- Step-like bdry conditions

  [Karpenko-Shepelsky-Teschl, '23; Yang-Li-Fan, '22]

Model III: Relation to earlier work

Large-time asymptotics: initial value in the Schwartz space.
[Boutet de Monvel-Karpenko-Shepelsky, '22]

- a soliton region: $\{(x,t) : \xi > 2\}$,
- a fast decay region: $\{(x,t) : \xi < -1/4\}$,
- two oscillatory regions: $\{(x,t) : 0 < \xi < 2\} \cup \{(x,t) : -1/4 < \xi < 0\}$,

Here, $\xi := x/t$.

A left problem in [Boutet de Monvel-Karpenko-Shepelsky, '22]: what are the transient asymptotics?

Model III: Statement of results

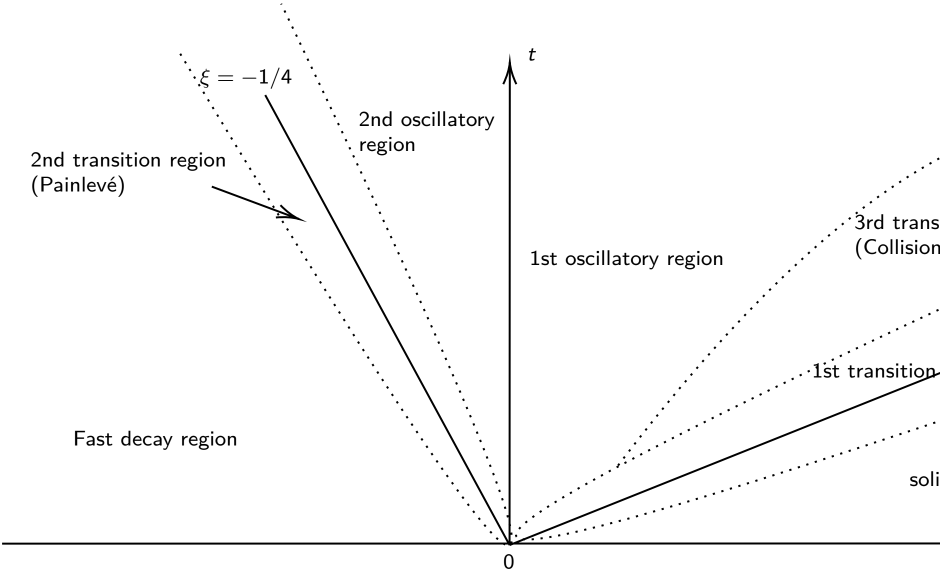
Three transient zones:

(a) the first transition region (Painlevé) $\mathcal{R}_I := \{(x,t) : 0 \leqslant |\xi - 2| t^{2/3} \leqslant C\}$,

(b) the second transition region (Painlevé) $\mathcal{R}_{II} := \{(x,t) : 0 \leqslant |\xi + 1/4| t^{2/3} \leqslant C\}$,

(c) the third transition region (collisionless shock) $\mathcal{R}_{III} := \{(x,t) : 2 \cdot 3^{1/3} (\log t)^{2/3} < (2-\xi) t^{2/3} < C(\log t)^{2/3},\ C > 2 \cdot 3^{1/3}\}$,

Model III: Statement of results



Model III: Statement of results

Assumption

- $m_0(x) := m(x,0) > 0$ for $x \in \mathbb{R}$.
- $m_0(x) - 1 \in H^{2,1}(\mathbb{R}) \cap H^{1,2}(\mathbb{R})$, where

$$H^{k,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \langle \cdot \rangle^s \partial_x^j f \in L^2(\mathbb{R}), j = 0, 1, \dots, k \right\}, \quad s \geqslant 0,$$

with $\langle x \rangle := (1+x^2)^{1/2}$ is the weighted Sobolev space.

Global well-posedness of the Cauchy problem.

[Yang-Fan-Liu, '22]

Model III: Statement of results

Theorem (X.-Yang-Zhang)

(a) *For $\xi \in \mathcal{R}_I$,*

$$u(x,t) = 1 + 36^{-2/3} t^{-2/3} \left(v(s) \int_s^{+\infty} v^2(\zeta)\, \mathrm{d}\zeta - v^2(s) \right) + \mathcal{O}(t^{-\min\{1-2\delta_1, 1/3+3\delta_1\}}),$$

where δ_1 is any real number belonging to $(1/9, 1/6)$,

$$s = 6^{-2/3} \left(\frac{x}{t} - 2 \right) t^{2/3},$$

and $v(s)$ is the unique solution of the Painlevé II equation

$$v''(s) = s v(s) + 2 v^3(s)$$

characterized by $v(s) \sim r(1)\mathrm{Ai}(s)$, $s \to +\infty$ with Ai *being the classical Airy function and $r(1) \in [-1, 1]$.*

Model III: Statement of results

Theorem (X.-Yang-Zhang)

(b) *For* $\xi \in \mathcal{R}_{II}$,

$$u(x,t) = 1 + 3^{-2/3} t^{-1/3} f_{II}(s) v_{II}(s) + \mathcal{O}\left(t^{\max\{-2/3+2\delta_2, -1/3-\delta_2\}}\right),$$

where $\delta_2 \in (0, 1/6)$, $s = -\left(\frac{8}{9}\right)^{1/3}\left(\frac{x}{t} + \frac{1}{4}\right) t^{2/3}$,

$$\begin{aligned} f_{II}(s) = {} & 2\sqrt{2+\sqrt{3}}\left(\sin\psi_a(s,t)\cos\gamma_a - \frac{iT_1}{T(i)}\cos\psi_a(s,t)\sin\gamma_a\right) \\ & + 2\sqrt{2-\sqrt{3}}\left(\sin\psi_b(s,t)\cos\gamma_b - \frac{iT_1}{T(i)}\cos\psi_b(s,t)\sin\gamma_b\right) \\ & + \sqrt{3}\cos\left(\frac{\Lambda_a+\Lambda_b}{2}\right)\sin\left(\frac{\Lambda_a+\Lambda_b}{2}\right) \end{aligned}$$

Model III: Statement of results

Theorem (X.-Yang-Zhang)

(c) *For $\xi \in \mathcal{R}_{III}$, if $|r(\pm 1)| = 1$, $r \in H^s$ with $s > 5/2$, we have*

$$u(x,t) = 1 + \sqrt{\frac{2-\xi}{48}}(q/p)^{1/2}\,(a-b)\left(\frac{\Theta\left(-A(\infty)+\frac{\varkappa}{4}\right)\Theta\left(A(\infty)+\frac{\varkappa}{4}+\frac{\phi}{\pi}\right)}{\Theta\left(A(\infty)+\frac{\varkappa}{4}\right)\Theta\left(-A(\infty)+\frac{\varkappa}{4}+\frac{\phi}{\pi}\right)}e^{i\phi}\right.$$
$$\left.+\frac{\Theta\left(A(\infty)-\frac{\varkappa}{4}\right)\Theta\left(-A(\infty)-\frac{\varkappa}{4}+\frac{\phi}{\pi}\right)}{\Theta\left(-A(\infty)-\frac{\varkappa}{4}\right)\Theta\left(A(\infty)-\frac{\varkappa}{4}+\frac{\phi}{\pi}\right)}e^{-i\phi}\right)(1+o(1)),$$

where p and q are any two fixed positive constants, $\Theta(z)$ is the Jacobi theta function defined by

$$\Theta(z) := \Theta(s) := \sum_{n\in\mathbb{Z}} e^{2\pi ins+\varkappa\pi in^2},$$

and

$$A(\hat{k}) = \left(2\int_b^a \frac{1}{w_+(\zeta)}\,\mathrm{d}\zeta\right)^{-1}\int_b^{\hat{k}}\frac{1}{w(\zeta)}\,\mathrm{d}\zeta, \qquad \hat{k} \in \mathbb{C}\setminus[-b,b],$$

with $w^2(\hat{k}) = (\hat{k}^2-a^2)(\hat{k}^2-b^2)$ is the Abel integral.

Model III: Remarks on the proof and results

- An RH characterization of the corresponding Cauchy problem
  [Boutet de Monvel-Karpenko-Shepelsky, '22; Yang-Fan-Liu, '22]
- Contour deformation to a "regular" RH problem ($\bar{\partial}$ nonlinear descent apporach)
  [McLaughlin-Miller, '06; Dieng-McLaughlin-Miller, '19]
- Pure RH problem and $\bar{\partial}$ problem.
- Painlevé regions: PII parametrix. (Shrinking skill is very important !)
- Collisionless shock region: g function $\to$ a model RH problem solvable in terms of the Jacobi theta function.

Part III: Discussable issues

Discussable issues

- Nonlocal mKdV equation with step-like initial data, of which two sides are nonzero.
- Transient asymptotics of the nonlocal mKdV equation with step-like initial data $(A, 0)$.
- Transient asymptotics of the mCH equation with $\kappa \neq 0$. (Just modify the global parametrix).
- $n \times n$, $n \geqslant 3$ spectral problem under the nonzero background ?
- ... ...

References

T.-Y Xu, Z.-C Zhang, E.-G Fan

On the Cauchy problem of defocusing mKdV equation with finite density initial data: long-time asymptotics in soliton-less regions.

J. Differential Equations 372 (2023), 55-122.

Z.-C Zhang, T.-Y Xu, E.-G Fan

Soliton resolution and asymptotic stability of N-soliton solutions for the defocusing mKdV equation with finite density type initial data.

Preprint, arXiv:2108.03650.

Z.-Y Wang, T.-Y Xu, E.-G Fan

Painlevé transcendents in the defocusing mKdV equation with non-zero boundary conditions.

Preprint, arXiv:2306.07073.

T.-Y Xu, E.-G Fan

On the large-time asymptotics of the defocusing mKdV equation with step-like initial data.

Preprint, arXiv:2204.01299.

References

T.-Y Xu, E.-G Fan

Large-time asymptotics to the focusing nonlocal modified Kortweg-de Vries equation with step-like boundary conditions.

Stud. Appl. Math, 150 (2023), 1217-1273.

T.-Y Xu, Y.-L Yang, L Zhang

Transient asymptotics of the modified Camassa-Holm equation.

Preprint, arXiv:2308.06950..

- My personal webpage: taiyangyxu.github.io/tyxumaths/

Thanks for your attention !

Any comments and questions are welcome !