

Exercise Sheet – Advanced Calculus III

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Exercise 1. Let $f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$.

- (1) Show that f is continuous at $(0, 0)$.
- (2) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0, 0)$.
- (3) Determine whether f is differentiable at $(0, 0)$.
- (4) Discuss the relationship between continuity, existence of partial derivatives, and differentiability for f at $(0, 0)$.

Solution 2. (1) For any $(x, y) \rightarrow (0, 0)$,

$$|f(x, y)| = \left| \frac{x^2 y}{x^2 + y^2} \right| \leq |y|.$$

So $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$. Thus, f is continuous at $(0, 0)$.

(2) By definition,

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0.$$

Similarly,

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0.$$

(3) f is differentiable at $(0, 0)$ if

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - f(0, 0) - f_x(0, 0)x - f_y(0, 0)y}{\sqrt{x^2 + y^2}} = 0.$$

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Here, $f(0, 0) = 0$, $f_x(0, 0) = 0$, $f_y(0, 0) = 0$, so

$$\frac{f(x, y)}{\sqrt{x^2 + y^2}} = \frac{x^2 y}{(x^2 + y^2)^{3/2}}.$$

Along $x = t$, $y = t$,

$$\frac{t^2 t}{(t^2 + t^2)^{3/2}} = \frac{t^3}{(2t^2)^{3/2}} = \frac{t^3}{2^{3/2} t^3} = \frac{1}{2^{3/2}}.$$

The limit is not 0 along this path, so f is not differentiable at $(0, 0)$.

(4) For f at $(0, 0)$, we see:

- f is continuous at $(0, 0)$.
- The partial derivatives exist at $(0, 0)$.
- f is not differentiable at $(0, 0)$.

This example shows that continuity and existence of partial derivatives at a point do not guarantee differentiability at that point.

Exercise 3 (Partial derivatives of homogeneous functions). Complete the following exercises.

(1) (Warm up) Compute the following partial derivatives $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$:

$$f(x, y, z) = (x - 2y + 3z)^2; \quad f(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}; \quad f(x, y, z) = \left(\frac{x}{y}\right)^{\frac{y}{z}}.$$

(2) A function $f(x, y, z)$ is called a homogeneous function of degree n , if for any $\rho > 0$, we have $f(\rho x, \rho y, \rho z) = \rho^n f(x, y, z)$. Now verify that the above functions are homogeneous and find their degrees n .

(3) (Euler's theorem) Show that $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = n f(x, y, z)$.

(Hint: Differentiate the equation $f(\rho x, \rho y, \rho z) = \rho^n f(x, y, z)$ with respect to ρ and then set $\rho = 1$)

(4) Conversely, show that if $f(x, y, z)$ satisfies the above equation, then $f(x, y, z)$ is a homogeneous function of degree n .

(5) Show that $f_x(x, y, z)$, $f_y(x, y, z)$ and $f_z(x, y, z)$ are homogeneous functions of degree $n - 1$.

(6) Prove that $(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z})^2 f = n^2 f$.

(7) Examples:

$$\Delta(x_1, x_2, \dots, x_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Prove that $\sum_{k=1}^n x_k \frac{\partial \Delta}{\partial x_k} = \frac{n(n-1)}{2} \Delta$ and $\sum_{k=1}^n \frac{\partial \Delta}{\partial x_k} = 0$.

Solution 4. (1) By direct computation, we have

- $f(x, y, z) = (x - 2y + 3z)^2$:

$$\frac{\partial f}{\partial x} = 2(x - 2y + 3z), \quad \frac{\partial f}{\partial y} = -4(x - 2y + 3z), \quad \frac{\partial f}{\partial z} = 6(x - 2y + 3z).$$

- $f(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$:

$$\frac{\partial f}{\partial x} = \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{\partial f}{\partial y} = \frac{-xy}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{\partial f}{\partial z} = \frac{-xz}{(x^2 + y^2 + z^2)^{3/2}}.$$

- $f(x, y, z) = \left(\frac{x}{y}\right)^{\frac{y}{z}}$:

$$\frac{\partial f}{\partial x} = \frac{y}{z} \left(\frac{x}{y}\right)^{\frac{y}{z}-1} \cdot \frac{1}{y} = \frac{y}{zx} \left(\frac{x}{y}\right)^{\frac{y}{z}}.$$

$$\frac{\partial f}{\partial y} = \left(\frac{x}{y}\right)^{\frac{y}{z}} \left[\frac{1}{z} \ln \left(\frac{x}{y}\right) - \frac{y}{z} \frac{1}{y} \right] = \frac{f(x, y, z)}{z} \ln \left(\frac{x}{y}\right) - \frac{f(x, y, z)}{z}.$$

$$\frac{\partial f}{\partial z} = -\frac{y}{z^2} \left(\frac{x}{y}\right)^{\frac{y}{z}} \ln \left(\frac{x}{y}\right) = -\frac{y}{z^2} f(x, y, z) \ln \left(\frac{x}{y}\right).$$

- (2) • For $f(x, y, z) = (x - 2y + 3z)^2$:

$$f(\rho x, \rho y, \rho z) = (\rho x - 2\rho y + 3\rho z)^2 = \rho^2(x - 2y + 3z)^2 = \rho^2 f(x, y, z).$$

So, degree $n = 2$.

- For $f(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$:

$$f(\rho x, \rho y, \rho z) = \frac{\rho x}{\sqrt{(\rho x)^2 + (\rho y)^2 + (\rho z)^2}} = \frac{\rho x}{\rho \sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}.$$

So, degree $n = 0$.

- For $f(x, y, z) = \left(\frac{x}{y}\right)^{\frac{y}{z}}$:

$$f(\rho x, \rho y, \rho z) = \left(\frac{\rho x}{\rho y}\right)^{\frac{\rho y}{\rho z}} = \left(\frac{x}{y}\right)^{\frac{y}{z}}.$$

So, degree $n = 0$.

- (3) Differentiate $f(\rho x, \rho y, \rho z) = \rho^n f(x, y, z)$ with respect to ρ :

$$\frac{d}{d\rho} f(\rho x, \rho y, \rho z) = n\rho^{n-1} f(x, y, z).$$

By the chain rule,

$$\frac{d}{d\rho} f(\rho x, \rho y, \rho z) = x f'_1(\rho x, \rho y, \rho z) + y f'_2(\rho x, \rho y, \rho z) + z f'_3(\rho x, \rho y, \rho z).$$

Thus,

$$\rho x f'_1(\rho x, \rho y, \rho z) + \rho y f'_2(\rho x, \rho y, \rho z) + \rho z f'_3(\rho x, \rho y, \rho z) = n\rho f(x, y, z).$$

Setting $\rho = 1$, we arrive at the result.

- (4) Define

$$g(\rho) = \frac{f(\rho x_0, \rho y_0, \rho z_0)}{\rho^n}.$$

Then

$$g'(\rho) = \frac{\rho x_0 f'_1(\rho x_0, \rho y_0, \rho z_0) + \rho y_0 f'_2(\rho x_0, \rho y_0, \rho z_0) + \rho z_0 f'_3(\rho x_0, \rho y_0, \rho z_0)}{\rho^n \cdot \rho} - \frac{n f(\rho x_0, \rho y_0, \rho z_0)}{\rho^{n+1}}.$$

Noticing that $x f_x + y f_y + z f_z = n f(x, y, z)$, then the numerator equals $n f(\rho x_0, \rho y_0, \rho z_0)$, so

$$g'(\rho) = 0.$$

For any $\rho > 0$, $g(\rho)$ is a constant. Recalling that $g(1) = f(x_0, y_0, z_0)$, we have

$$g(\rho) = f(x_0, y_0, z_0),$$

which implies the desired result.

- (5) Let f be homogeneous of degree n . Then

$$f(\rho x, \rho y, \rho z) = \rho^n f(x, y, z).$$

Differentiate both sides with respect to x :

$$\rho f'_1(\rho x, \rho y, \rho z) = \rho^n f'_1(x, y, z).$$

So,

$$f'_1(\rho x, \rho y, \rho z) = \rho^{n-1} f'_1(x, y, z).$$

Thus, f_x, f_y, f_z are homogeneous of degree $n - 1$.

(6) If you see $D := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$. By Euler's theorem, we have

$$D(Df) = D(nf) = n^2 f.$$

This also could be directly verified by chain rule:

$$\begin{aligned} & \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \\ &= x \left(\frac{\partial f}{\partial x} + x \frac{\partial^2 f}{\partial x^2} \right) + xy \frac{\partial^2 f}{\partial x \partial y} + xy \frac{\partial^2 f}{\partial y \partial x} + y \left(\frac{\partial f}{\partial y} + y \frac{\partial^2 f}{\partial y^2} \right) \\ &= x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x \left[x \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) + y \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \right] + y \left[x \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) + y \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \right] \\ &= nf + x \cdot (n-1) \frac{\partial f}{\partial x} + y \cdot (n-1) \frac{\partial f}{\partial y} \\ &= nf + (n-1)nf \\ &= n^2 f. \end{aligned}$$

Remark (The other understanding): we see that $(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})^2$ equals to $x^2 \frac{\partial^2}{\partial x^2} + 2xy \frac{\partial^2}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2}$. Then we have

$$\begin{aligned} & x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} \\ &= x \left[x \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) + y \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \right] + y \left[x \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) + y \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \right] \\ &= x \cdot (n-1) \frac{\partial f}{\partial x} + y \cdot (n-1) \frac{\partial f}{\partial y} \\ &= n(n-1)f. \end{aligned}$$

For the second understanding, try to prove that

$$\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^m f = n(n-1) \cdots (n-m+1)f. \quad (1)$$

For $m = 1$, the equation is valid.

For $m = k$, we suppose that the equation is valid.

What we need to do is to prove that it holds for $m = k + 1$. As it holds for $m = k$, we have

$$\sum_{i=0}^k \left(C_k^i x^i y^{k-i} \frac{\partial^k f}{\partial x^i \partial y^{k-i}} \right) \Big|_{(tx, ty)} = n(n-1) \cdots (n-k+1) f(tx, ty) = n(n-1) \cdots (n-k+1) t^n f(x, y).$$

Differentiate t^k in both sides:

$$\sum_{i=0}^k C_k^i x^i y^{k-i} \left(\frac{\partial^k f}{\partial x^i \partial y^{k-i}} \right) \Big|_{(tx, ty)} = n(n-1) \cdots (n-k+1) t^{n-k} f(x, y). \quad (2)$$

Now we differentiate the both side of the above equation with respect to t . Noting the chain rule

$$\frac{d}{dt} \left(\frac{\partial^k f}{\partial x^i \partial y^{k-i}} \Big|_{(tx, ty)} \right) = x \left(\frac{\partial^{k+1} f}{\partial x^{i+1} \partial y^{k-i}} \Big|_{(tx, ty)} \right) + y \left(\frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Big|_{(tx, ty)} \right).$$

Thus the t -derivative of the l.h.s of (2) equals to

$$\begin{aligned} & \sum_{i=0}^k \left[C_k^i x^{i+1} y^{k-i} \frac{\partial^{k+1} f}{\partial x^{i+1} \partial y^{k-i}} \Big|_{(tx, ty)} + C_k^i x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Big|_{(tx, ty)} \right] \\ &= \sum_{i=1}^{k+1} C_k^{i-1} x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Big|_{(tx, ty)} + \sum_{k=0}^i C_k^i x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Big|_{(tx, ty)} \\ &= \sum_{i=0}^{k+1} C_{k+1}^i x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Big|_{(tx, ty)}, \end{aligned}$$

where we used $C_k^i + C_k^{i-1} = C_{k+1}^i$ for the last “=”.

Thus

$$\sum_{i=0}^{k+1} C_{k+1}^i x^i y^{k+1-i} \frac{\partial^{k+1} f}{\partial x^i \partial y^{k+1-i}} \Big|_{(tx, ty)} = n(n-1) \cdots (n-k+1)(n-k) t^{n-k-1} f(x, y).$$

Taking $t = 1$ in the above equation, we reach that the equation (1) holds for $m = k + 1$.

- (7) First, recall that Δ is a homogeneous polynomial of degree $d = \frac{n(n-1)}{2}$ in the variables x_1, \dots, x_n (since there are $n(n-1)/2$ factors, each linear in x_k). By Euler's theorem for homogeneous functions,

$$\sum_{k=1}^n x_k \frac{\partial \Delta}{\partial x_k} = \frac{n(n-1)}{2} \Delta.$$

Secondly, notice that Δ admits the translation invariance, i.e., $\Delta(x_1, \dots, x_n) = \Delta(x_1 + t, \dots, x_n + t)$ for any $t \in \mathbb{R}$.

$$0 = \frac{\partial \Delta}{\partial t}(x_1, \dots, x_n) \stackrel{u_k = x_k + t}{=} \sum_{k=1}^n \frac{\partial \Delta}{\partial u_k}(u_1, \dots, u_n) \cdot \frac{du_k}{dt} \Big|_{t=0} = \sum_{k=1}^n \frac{\partial \Delta}{\partial x_k}(x_1, \dots, x_n).$$