Exercise Sheet – Advanced Calculus III

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Exercise 1. Let $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.

(1) Show that f is continuous at $(0,0)$.

(2) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0,0)$.

(3) Determine whether f is differentiable at $(0,0)$.

(4) Discuss the relationship between continuity, existence of partial derivatives, and differentiability for f at $(0,0)$.

Solution 1. (1) For $(x,y) \neq (0,0)$, $f(x,y) = \frac{x^2y}{x^2+y^2}$. We check $\lim_{(x,y)\to(0,0)} f(x,y)$. Along any path $y = kx$,

$$f(x,kx) = \frac{x^2(kx)}{x^2+(kx)^2} = \frac{kx^3}{x^2(1+k^2)} = \frac{kx}{1+k^2} \to 0 \text{ as } x \to 0.$$

Along $x = 0$, $f(0,y) = 0$. Along $y = 0$, $f(x,0) = 0$. For any $(x,y) \to (0,0)$,

$$|f(x,y)| = \left|\frac{x^2y}{x^2+y^2}\right| \leq |y|.$$

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$. Thus, f is continuous at $(0,0)$.

(2) By definition,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Similarly,

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

(3) f is differentiable at $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2+y^2}} = 0.$$

Here, $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 0$, so

$$\frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{x^2y}{(x^2+y^2)^{3/2}}.$$

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Along $x = t$, $y = t$,

$$\frac{t^2 t}{(t^2+t^2)^{3/2}} = \frac{t^3}{(2t^2)^{3/2}} = \frac{t^3}{2^{3/2}t^3} = \frac{1}{2^{3/2}}.$$

The limit is not 0 along this path, so f is not differentiable at $(0,0)$.

(4) For f at $(0,0)$, we see:

- f is continuous at $(0,0)$.
- The partial derivatives exist at $(0,0)$.
- f is not differentiable at $(0,0)$.

This example shows that continuity and existence of partial derivatives at a point do not guarantee differentiability at that point.

Exercise 2 (Partial derivatives of homogeneous functions)**.** Complete the following exercises.

(1) (Warm up) Compute the following partial derivatives ($\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$):

$$f(x,y,z) = (x-2y+3z)^2; \quad f(x,y,z) = \frac{x}{\sqrt{x^2+y^2+z^2}}; \quad f(x,y,z) = \left(\frac{x}{y}\right)^{\frac{y}{z}}.$$

(2) A function $f(x,y,z)$ is called a homogeneous function of degree n, if for any $\rho > 0$, we have $f(\rho x, \rho y, \rho z) = \rho^n f(x,y,z)$. Now verify that the above functions are homogeneous and find their degrees n.

(3) (Euler's theorem) Show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x,y,z)$.
(Hint: Differentiate the equation $f(\rho x, \rho y, \rho z) = \rho^n f(x,y,z)$ with respect to ρ and then set $\rho = 1$)

(4) Conversely, show that if $f(x,y,z)$ satisfies the above equation, then $f(x,y,z)$ is a homogeneous function of degree n.

(5) Show that $f_x(x,y,z)$, $f_y(x,y,z)$ and $f_z(x,y,z)$ are homogeneous functions of degree $n-1$.

(6) Prove that $(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z})^2 f = n(n-1)f$.
(Hint: Use the results of (3) and (5))

(7) Examples:

$$\Delta(x_1, x_2, \ldots, x_n) = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1\le i<j\le n}(x_j - x_i).$$

Prove that $\sum_{k=1}^n x_k \frac{\partial \Delta}{\partial x_k} = \frac{n(n-1)}{2}\Delta$ and $\sum_{k=1}^n \frac{\partial \Delta}{\partial x_k} = 0$.

Solution 2. (1) By direct computation, we have

- $f(x,y,z) = (x-2y+3z)^2$:

$$\frac{\partial f}{\partial x} = 2(x-2y+3z), \quad \frac{\partial f}{\partial y} = -4(x-2y+3z), \quad \frac{\partial f}{\partial z} = 6(x-2y+3z).$$

- $f(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$:

$$\frac{\partial f}{\partial x} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial f}{\partial y} = \frac{-xy}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial f}{\partial z} = \frac{-xz}{(x^2+y^2+z^2)^{3/2}}.$$

- $f(x,y,z) = \left(\frac{x}{y}\right)^{\frac{y}{z}}$:

$$\frac{\partial f}{\partial x} = \frac{y}{z}\left(\frac{x}{y}\right)^{\frac{y}{z}-1} \cdot \frac{1}{y} = \frac{y}{zx}\left(\frac{x}{y}\right)^{\frac{y}{z}}.$$

$$\frac{\partial f}{\partial y} = \left(\frac{x}{y}\right)^{\frac{y}{z}}\left[\frac{1}{z}\ln\left(\frac{x}{y}\right) - \frac{y}{z}\frac{1}{y}\right] = \frac{f(x,y,z)}{z}\ln\left(\frac{x}{y}\right) - \frac{f(x,y,z)}{z}.$$

$$\frac{\partial f}{\partial z} = -\frac{y}{z^2}\left(\frac{x}{y}\right)^{\frac{y}{z}}\ln\left(\frac{x}{y}\right) = -\frac{y}{z^2}f(x,y,z)\ln\left(\frac{x}{y}\right).$$

(2)
- For $f(x,y,z) = (x-2y+3z)^2$:

$$f(\rho x, \rho y, \rho z) = (\rho x - 2\rho y + 3\rho z)^2 = \rho^2(x-2y+3z)^2 = \rho^2 f(x,y,z).$$

  So, degree $n = 2$.
- For $f(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$:

$$f(\rho x, \rho y, \rho z) = \frac{\rho x}{\sqrt{(\rho x)^2 + (\rho y)^2 + (\rho z)^2}} = \frac{\rho x}{\rho\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}.$$

  So, degree $n = 0$.
- For $f(x,y,z) = \left(\frac{x}{y}\right)^{\frac{y}{z}}$:

$$f(\rho x, \rho y, \rho z) = \left(\frac{\rho x}{\rho y}\right)^{\frac{\rho y}{\rho z}} = \left(\frac{x}{y}\right)^{\frac{y}{z}}.$$

  So, degree $n = 0$.

(3) Differentiate $f(\rho x, \rho y, \rho z) = \rho^n f(x,y,z)$ with respect to ρ:

$$\frac{d}{d\rho} f(\rho x, \rho y, \rho z) = n\rho^{n-1} f(x,y,z).$$

By chain rule:

$$\frac{\partial f}{\partial x}(\rho x, \rho y, \rho z) \cdot x + \frac{\partial f}{\partial y}(\rho x, \rho y, \rho z) \cdot y + \frac{\partial f}{\partial z}(\rho x, \rho y, \rho z) \cdot z = n\rho^{n-1} f(x,y,z).$$

Set $\rho = 1$:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x,y,z).$$

(4) Define

$$g(\rho) = \frac{f(\rho x_0, \rho y_0, \rho z_0)}{\rho^n}.$$

Then

$$g'(\rho) = \frac{\rho x_0 f_x(\rho x_0, \rho y_0, \rho z_0) + \rho y_0 f_y(\rho x_0, \rho y_0, \rho z_0) + \rho z_0 f_z(\rho x_0, \rho y_0, \rho z_0)}{\rho^n \cdot \rho} - \frac{n f(\rho x_0, \rho y_0, \rho z_0)}{\rho^{n+1}}.$$

Noticing that $x f_x + y f_y + z f_z = n f(x, y, z)$, then the numerator equals $n f(\rho x_0, \rho y_0, \rho z_0)$, so

$$g'(\rho) = 0.$$

For any $\rho > 0$, $g(\rho)$ is a constant. Recalling that $g(1) = f(x_0, y_0, z_0)$, we have

$$g(\rho) = f(x_0, y_0, z_0),$$

which implies the desired result.

(5) Let f be homogeneous of degree n. Then

$$f(\rho x, \rho y, \rho z) = \rho^n f(x, y, z).$$

Differentiate both sides with respect to x:

$$\frac{\partial}{\partial x} f(\rho x, \rho y, \rho z) = \rho \frac{\partial f}{\partial x}(\rho x, \rho y, \rho z) = \rho^n \frac{\partial f}{\partial x}(x, y, z).$$

So,

$$\frac{\partial f}{\partial x}(\rho x, \rho y, \rho z) = \rho^{n-1} \frac{\partial f}{\partial x}(x, y, z).$$

Thus, f_x, f_y, f_z are homogeneous of degree $n-1$.

(6) From (5), we have $\frac{\partial f}{\partial x}(x, y, z)$, $\frac{\partial f}{\partial y}(x, y, z)$ and $\frac{\partial f}{\partial z}(x, y, z)$ are homogeneous functions of degree $n-1$. Thus,

$$(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z})\partial\frac{\partial f}{\partial x} = (n-1)\frac{\partial f}{\partial x},$$

$$(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z})\partial\frac{\partial f}{\partial y} = (n-1)\frac{\partial f}{\partial y},$$

$$(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z})\partial\frac{\partial f}{\partial z} = (n-1)\frac{\partial f}{\partial z},$$

Multiplying the first equation by x, the second by y and the third by z, then adding them up, we get

$$(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z})^2 f = (n-1)(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z})f = n(n-1)f = n(n-1)f.$$

(7) First, recall that Δ is a homogeneous polynomial of degree $d = \frac{n(n-1)}{2}$ in the variables $x_1, \ldots, x_n$ (since there are $n(n-1)/2$ factors, each linear in x_k). By Euler's theorem for homogeneous functions,

$$\sum_{k=1}^{n} x_k \frac{\partial \Delta}{\partial x_k} = \frac{n(n-1)}{2}\Delta.$$

Secondly, notice that Δ admits the translation invariance, i.e., $\Delta(x_1, \ldots, x_n) = \Delta(x_1 + t, \ldots, x_n + t)$ for any $t \in \mathbb{R}$.

$$0 = \frac{\partial \Delta}{\partial t}(x_1, \ldots, x_n) \overset{u_k = x_k + t}{=} \sum_{k=1}^{n} \frac{\partial \Delta}{\partial u_k}(u_1, \ldots, u_n) \cdot \frac{du_k}{dt}\Big|_{t=0} = \sum_{k=1}^{n} \frac{\partial \Delta}{\partial x_k}(x_1, \ldots, x_n).$$

Exercise 3 (Directional derivatives and gradient)**.** Complete the following exercises.

(1) Let $f(x, y, z) = x^2y + yz^3$. Compute the gradient ∇f at the point $(1, 2, 1)$.

(2) For $f(x, y, z)$ as above, compute the directional derivative of f at $(1, 2, 1)$ in the direction of the vector $\vec{v} = (2, -1, 2)$.

(3) Let $f(x, y) = x^3 - 3xy^2$. Find all points (x, y) where the gradient ∇f is parallel to the vector $(1, 1)$.

Application in ML:

(4) (Linear regression) A common loss function is Mean Squared Error (MSE). For a single data point (x, y), the loss is defined as $L(m, b) = (y - (mx + b))^2$, where m is the slope and b is the y-intercept of the regression line. Compute the gradient $\nabla L(m, b)$ and interpret its components in terms of how they influence the loss.

(5) (Gradient descent step) Suppose you are using gradient descent to minimize the function $J(\theta_0, \theta_1) = \theta_0^2 + 2\theta_1^2$. Calculate the gradient $\nabla J(\theta_0, \theta_1)$ firstly, and write down the update rule for θ_0 and θ_1 using a learning rate α.

Solution 3. (1) The gradient is

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Compute:

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + z^3, \quad \frac{\partial f}{\partial z} = 3yz^2.$$

At $(1, 2, 1)$:

$$\frac{\partial f}{\partial x} = 2 \cdot 1 \cdot 2 = 4, \quad \frac{\partial f}{\partial y} = 1^2 + 1^3 = 2, \quad \frac{\partial f}{\partial z} = 3 \cdot 2 \cdot 1^2 = 6.$$

So,

$$\nabla f(1, 2, 1) = (4,\ 2,\ 6).$$

(2) The directional derivative in direction $\vec{v} = (2, -1, 2)$ is

$$D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|}.$$

Compute $|\vec{v}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$. So, unit vector $\vec{u} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$. Then,

$$D_{\vec{v}} f = 4 \cdot \frac{2}{3} + 2 \cdot \left(-\frac{1}{3}\right) + 6 \cdot \frac{2}{3} = \frac{8}{3} - \frac{2}{3} + \frac{12}{3} = \frac{18}{3} = 6.$$

(3) Compute $\nabla f(x, y) = (3x^2 - 3y^2,\ -6xy)$. We want $\nabla f(x, y)$ parallel to $(1, 1)$, i.e.,

$$(3x^2 - 3y^2,\ -6xy) = \lambda(1, 1).$$

So,

$$3x^2 - 3y^2 = \lambda, \quad -6xy = \lambda.$$

Equate:

$$x^2 - y^2 + 2xy = 0 \implies (x + y)^2 - 2y^2 = 0.$$

So,

$$x + y = \pm\sqrt{2}y.$$

Thus, all points (x, y) with $x = (\sqrt{2} - 1)y$ or $x = (-\sqrt{2} - 1)y$.

For $\nabla f(x, y)$ to be parallel to $(1, 1)$, there must exist $\lambda \in \mathbb{R}$ such that

$$(3x^2 - 3y^2,\ -6xy) = \lambda(1, 1).$$

So,

$$3x^2 - 3y^2 = \lambda, \quad -6xy = \lambda.$$

Equate the two expressions for λ:

$$3x^2 - 3y^2 = -6xy \implies x^2 + 2xy - y^2 = 0.$$

This factors as

$$(x + y)^2 - 2y^2 = 0 \implies (x + y)^2 = 2y^2 \implies x + y = \pm\sqrt{2}\,y.$$

Thus,

$$x = (\sqrt{2} - 1)y \quad \text{or} \quad x = (-\sqrt{2} - 1)y.$$

So all points (x, y) with $x = (\sqrt{2} - 1)y$ or $x = (-\sqrt{2} - 1)y$ have ∇f parallel to $(1, 1)$.

(4)

$$\frac{\partial L}{\partial m} = -2x(y - (mx + b))$$
$$\frac{\partial L}{\partial b} = -2(y - (mx + b))$$

So,

$$\nabla L(m, b) = \big(-2x(y - (mx + b)),\ -2(y - (mx + b))\big)$$

Interpretation: the gradient points in the direction of steepest increase of the loss. The m-component shows how changing the slope affects the loss (scaled by x), and the b-component shows how changing the intercept affects the loss. Both are proportional to the residual $y - (mx + b)$.

(5) The gradient is

$$\nabla J(\theta_0, \theta_1) = \left(\frac{\partial J}{\partial \theta_0}, \frac{\partial J}{\partial \theta_1}\right) = (2\theta_0,\ 4\theta_1).$$

The gradient descent update rule is:

$$\theta_0^{\text{new}} = \theta_0 - \alpha \cdot 2\theta_0 = \theta_0(1 - 2\alpha)$$

$$\theta_1^{\text{new}} = \theta_1 - \alpha \cdot 4\theta_1 = \theta_1(1 - 4\alpha)$$

where α is the learning rate.

Exercise 4 (Riemann θ function)**.** Define the Riemann θ function as

$$\theta(\vec{z}) := \sum_{\vec{m} \in \mathbb{Z}^n} e^{2\pi i \vec{m}^{\mathrm{T}} \vec{z} + i\pi \vec{m}^{\mathrm{T}} \tau \vec{m}}, \quad \vec{z} = (z_1, \ldots, z_n)^{\mathrm{T}} \in \mathbb{C}^n,$$

where τ is a complex $n \times n$ matrix with positive definite imaginary part.

(1) Prove that the series converges absolutely and uniformly on any compact subset of $\mathbb{C}^n$.

(2) Prove that θ function is an even entire function. (It's not a proper time to prove it is entire)

(3) Prove that the periodicity of θ function:

$$\theta(\vec{z}+\vec{e}_j)=\theta(\vec{z}), \quad \theta(\vec{z}\pm\tau_j)=e^{\mp 2\pi i z_j-\pi i\tau_{jj}}\theta(\vec{z}), \quad j=1,2,\ldots,n,$$

where $\vec{e}_j$ is the j-th standard basis vector of $\mathbb{C}^n$ and τ_j is the j-th column of τ.

(4) Define a ratio

$$\mathcal{G}(\vec{z};\vec{\nu}):=\frac{\theta(\vec{z}+\vec{\nu})}{\theta(\vec{z})}.$$

Now prove that $\mathcal{G}(\vec{z})$ satisfies the following equation:

$$\frac{\partial_j\mathcal{G}(\vec{z})}{\mathcal{G}(\vec{z})}=\frac{\partial_j\theta(\vec{z}+\vec{\nu})}{\theta(\vec{z}+\vec{\nu})}-\frac{\partial_j\theta(\vec{z})}{\theta(\vec{z})}.$$

Furthermore, show that $\frac{\partial_j\mathcal{G}(\vec{0})}{\mathcal{G}(\vec{0})}=\frac{\partial_j\theta(\vec{\nu})}{\theta(\vec{\nu})}$.