Exercise Sheet – Advanced Calculus III

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Exercise 1 (Warm up: Directional derivatives and gradient)**.** Complete the following exercises.

(1) Let $f(x, y, z) = x^2y + yz^3$. Compute the gradient ∇f at the point $(1, 2, 1)$.

(2) For $f(x, y, z)$ as above, compute the directional derivative of f at $(1, 2, 1)$ in the direction of the vector $\vec{v} = (2, -1, 2)$.

(3) Let $f(x, y) = x^3 - 3xy^2$. Find all points (x, y) where the gradient ∇f is parallel to the vector $(1, 1)$.

Application in ML:

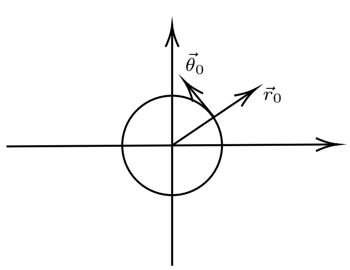
(4) (Linear regression) A common loss function is Mean Squared Error (MSE). For a single data point (x, y), the loss is defined as $L(m, b) = (y - (mx + b))^2$, where m is the slope and b is the y-intercept of the regression line. Compute the gradient $\nabla L(m, b)$ and interpret its components in terms of how they influence the loss.

(5) (Gradient descent step) Suppose you are using gradient descent to minimize the function $J(\theta_0, \theta_1) = \theta_0^2 + 2\theta_1^2$. Calculate the gradient $\nabla J(\theta_0, \theta_1)$ firstly, and write down the update rule for θ_0 and θ_1 using a learning rate α.

Exercise 2. Let $u = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$. Prove that

$$\nabla u = \frac{\partial f}{\partial r}\vec{r}_0 + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{\theta}_0,$$

where $\vec{r}_0 = (\cos\theta, \sin\theta)$ and $\vec{\theta}_0 = (\cos(\theta + \frac{\pi}{2}), \cos\theta)$.



Exercise 3. Suppose the function $f(x, y)$ has a nonzero directional derivative at some point (x_0, y_0), and the directional derivatives along three different (non-colinear) directions at (x_0, y_0) are equal. Prove that $f(x, y)$ is not differentiable at (x_0, y_0).

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