Exercise Sheet – Mathematical Analysis III

Taiyang Xu*

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定义 1. 设 X 是复线性空间, 如果对任意 $x, y \in X$ 有一复数 $\langle x, y\rangle$ 与之对应, 且满足以下条件:

(1) $\langle x, x\rangle \geq 0$, 且当且仅当 $x = 0$ 时取等号;

(2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(3) 对任意复数 α, β 有 $\langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$;

则称 $\langle \cdot, \cdot\rangle$ 为 X 上的一个**内积**, 称配备内积的复线性空间为**内积空间**.

设 X 是内积空间, 令

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad \forall x \in X.$$

可以验证 $\|x\|$ 满足范数的三条性质 (正定性, 齐次性, 三角不等式), 我们称 $\|\cdot\|$ 为 X 上的**由内积诱导的范数**. 可以看到内积空间也是一种特殊的赋范空间. 若 X 按照由内积诱导的范数完备, 则称 X 为**希尔伯特空间**.

练习 1. 设 X 是内积空间, 证明柯西-施瓦茨不等式: 对任意 $x, y \in X$ 有

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|,$$

其中 $\|x\| = \sqrt{\langle x, x\rangle}$.

解答 1. 若 $y = 0$, 则不等式显然成立. 现设 $y \neq 0$, 则对任意复数 λ 有

$$0 \leq \langle x - \lambda y, x - \lambda y\rangle = \langle x, x\rangle - \lambda\langle y, x\rangle - \overline{\lambda}\langle x, y\rangle + |\lambda|^2\langle y, y\rangle.$$

取 $\lambda = \frac{\langle x, y\rangle}{\langle y, y\rangle}$ 可得

$$0 \leq \langle x, x\rangle - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle},$$

即

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle.$$

因此

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$$

*School of Mathematical Sciences, Fudan University, Shanghai 200433, China. Email: tyxu19@fudan.edu.cn

练习 2. 内积是关于两个变元的连续函数.

解答 2. 设 $\{x_n\}$ 和 $\{y_n\}$ 分别是内积空间 X 中收敛于 x 和 y 的序列, 则

$$\begin{aligned}|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle| \\ &\le |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\le \|x_n\| \cdot \|y_n - y\| + \|x_n - x\| \cdot \|y\|.\end{aligned}$$

由于 $\{x_n\}$ 收敛, 因此 $\{\|x_n\|\}$ 有界, 设存在常数 $M > 0$ 使得 $\|x_n\| \le M$ 对任意 n 成立. 由于 $\{y_n\}$ 收敛于 y, 因此 $\|y_n - y\| \to 0$ $(n \to \infty)$. 同理, $\|x_n - x\| \to 0$ $(n \to \infty)$. 因此

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| \le M \cdot \|y_n - y\| + \|x_n - x\| \cdot \|y\| \to 0 \quad (n \to \infty).$$

即

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

练习 3. 考虑平方可积函数空间 $L^2(a, b)$, 定义内积为

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx, \quad \forall f, g \in L^2(a, b),$$

由该内积诱导的范数为

$$\|f\|_2 = \left(\int_a^b |f(x)|^2\, dx\right)^{1/2}, \quad \forall f \in L^2(a, b).$$

事实上, $L^2(a, b)$ 是一个希尔伯特空间.

解答 3. 这个定理只需证明它的完备性, 但是为此还需要很多准备, 因此不是证明它的好时机.

定义 2. 设 X 是内积空间, x, y 是 X 中的元素, 如果 $\langle x, y\rangle = 0$, 则称 x 与 y **正交**, 记作 $x \perp y$. 如果 X 的子集 A 中每个向量都与子集 B 中的每个向量正交, 则称 A 与 B 正交.

定义 3 (正交系)**.** 设 M 是内积空间 X 的一个不含 0 元的子集, 如果 M 中任意两个不同的向量都正交, 则称 M 为 X 的一个**正交系**. 进一步地, 若 M 中的每个向量范数都为 1, 则称 M 为 X 的一个**规范正交系**.

例子 1. 在 $L^2(-\pi, \pi)$ 中, 定义内集为

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx, \quad \forall f, g \in L^2(-\pi, \pi).$$

那么集合

$$\left\{\frac{1}{\sqrt{2}},\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots\right\}$$

是 $L^2(-\pi,\pi)$ 的一个规范正交系.

定义 4. 设 X 是赋范线性空间, $x_1,x_2,\cdots$ 是 X 中的一列向量, $\alpha_1,\alpha_2,\cdots$ 是一列数, 作形式级数

$$\sum_{i=1}^{\infty}\alpha_i x_i,$$

称 $S_n=\sum_{i=1}^{n}\alpha_i x_i$ 是该级数的部分和, 如果部分和列 $\{S_n\}$ 收敛于 $x\in X$, 则称该级数在 X 中**收敛**, 并称 x 为该级数的和, 记作

$$x=\sum_{i=1}^{\infty}\alpha_i x_i.$$

练习 4. 若 M 为 X 中的规范正交系, $e_1,e_2,\cdots$ 是 M 中的有限个或可列个向量, 且 $x=\sum_{i=1}^{\infty}\alpha_i e_i$, 则

$$x=\sum_{j=1}^{\infty}\langle x,e_j\rangle e_j.$$

解答 4. 由于 M 为规范正交系, 因此对任意 $i\neq j$ 有 $\langle e_i,e_j\rangle=0$, 且 $\|e_i\|=1$. 因此对任意 $n\in\mathbb{N}^+$ 有

$$\left\langle x,e_j\right\rangle=\left\langle\sum_{i=1}^{\infty}\alpha_i e_i,e_j\right\rangle=\sum_{i=1}^{\infty}\alpha_i\langle e_i,e_j\rangle=\alpha_j.$$

因此

$$\sum_{j=1}^{\infty}\langle x,e_j\rangle e_j=\sum_{j=1}^{\infty}\alpha_j e_j=x.$$

定义 5. 设 M 是内积空间 X 的规范正交系, $x\in X$, 称数集

$$\{\langle x,e\rangle|e\in M\}$$

为 x 关于规范正交系 M 的**傅里叶系数集**, $\langle x,e\rangle$ 称为 x 关于 e 的**傅里叶系数**.

练习 5. 设 M 是内积空间 X 的规范正交系, 任取 M 中的有限个向量 $e_1,\ldots,e_n$, 证明

(1) 对任意 $x\in X$ 有

$$\left\|x-\sum_{i=1}^{n}\langle x,e_i\rangle e_i\right\|^2=\|x\|^2-\sum_{i=1}^{n}|\langle x,e_i\rangle|^2\geq 0.$$

(2) 对任意的 $x\in X$ 有

$$\left\|x-\sum_{i=1}^{n}\alpha_i e_i\right\|\geq\left\|x-\sum_{i=1}^{n}\langle x,e_i\rangle e_i\right\|,\quad \forall\alpha_1,\alpha_2,\ldots,\alpha_n\in\mathbb{C}.$$

解答 5. (1) 由于 M 为规范正交系, 因此对任意 $i \neq j$ 有 $\langle e_i, e_j\rangle = 0$, 且 $\|e_i\| = 1$. 因此对任意 $n \in \mathbb{N}^+$ 有

$$\begin{aligned}
&\left\|x-\sum_{i=1}^{n}\langle x,e_i\rangle e_i\right\|^2\\
=&\left\langle x-\sum_{i=1}^{n}\langle x,e_i\rangle e_i, x-\sum_{j=1}^{n}\langle x,e_j\rangle e_j\right\rangle\\
=&\langle x,x\rangle-\sum_{j=1}^{n}\overline{\langle x,e_j\rangle}\langle x,e_j\rangle-\sum_{i=1}^{n}\langle x,e_i\rangle\overline{\langle x,e_i\rangle}+\sum_{i=1}^{n}|\langle x,e_i\rangle|^2\\
=&\|x\|^2-\sum_{i=1}^{n}|\langle x,e_i\rangle|^2.
\end{aligned}$$

由于范数非负, 因此

$$\left\|x-\sum_{i=1}^{n}\langle x,e_i\rangle e_i\right\|^2=\|x\|^2-\sum_{i=1}^{n}|\langle x,e_i\rangle|^2\geq 0.$$

(2) 同样地, 对任意 $n \in \mathbb{N}^+$ 有

$$\begin{aligned}
&\left\|x-\sum_{i=1}^{n}\alpha_i e_i\right\|^2-\left\|x-\sum_{i=1}^{n}\langle x,e_i\rangle e_i\right\|^2\\
=&\left\langle x-\sum_{i=1}^{n}\alpha_i e_i, x-\sum_{j=1}^{n}\alpha_j e_j\right\rangle-\left\langle x-\sum_{i=1}^{n}\langle x,e_i\rangle e_i, x-\sum_{j=1}^{n}\langle x,e_j\rangle e_j\right\rangle\\
=&\|x\|^2-\sum_{j=1}^{n}\overline{\alpha_j}\langle x,e_j\rangle-\sum_{i=1}^{n}\alpha_i\overline{\langle x,e_i\rangle}+\sum_{i=1}^{n}|\alpha_i|^2-\|x\|^2+\sum_{j=1}^{n}|\langle x,e_j\rangle|^2\\
=&\sum_{i=1}^{n}|\alpha_i|^2-\sum_{i=1}^{n}\alpha_i\overline{\langle x,e_i\rangle}-\sum_{i=1}^{n}\overline{\alpha_i}\langle x,e_i\rangle+\sum_{i=1}^{n}|\langle x,e_i\rangle|^2\\
=&\sum_{i=1}^{n}|\alpha_i-\langle x,e_i\rangle|^2\geq 0.
\end{aligned}$$

因此

$$\left\|x-\sum_{i=1}^{n}\alpha_i e_i\right\|\geq\left\|x-\sum_{i=1}^{n}\langle x,e_i\rangle e_i\right\|.$$

从证明中我们可以看出, 当且仅当 $\alpha_i = \langle x, e_i\rangle$ 时等号成立. 并且若用 $e_1, \ldots, e_n$ 的线性组合去逼近 x, 则最佳的选择就是取傅里叶系数.

练习 6. 证明如下的 Bessel 不等式: 设 M 是内积空间 X 的规范正交系, 则对任意 $x \in X$ 有

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2.$$

解答 6. 由上一个练习的结论可知, 对任意 $n \in \mathbb{N}^+$ 有

$$\left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - \sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \geq 0.$$

因此

$$\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2.$$

由于上式对任意 n 都成立, 因此取极限可得

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2.$$

练习 7. 设 $\{e_k\}$ 为希尔伯特空间 X 中可数规范正交系, 那么

(1) 级数 $\sum_{i=1}^{\infty} \alpha_i e_i$ 收敛的充要条件是 $\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty$.

(2) 若 $x = \sum_{i=1}^{\infty} \alpha_i e_i$，则 $\alpha_i = \langle x, e_i \rangle$, $\forall i \in \mathbb{N}^+$; 故

$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i.$$

(3) 对任意 $x \in X$, $\sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ 收敛.

解答 7. (1) 设 $S_n = \sum_{i=1}^{n} \alpha_i e_i$, $\sigma_n = \sum_{i=1}^{n} |\alpha_i|^2$, 由于 $\{e_i\}$ 是规范正交系, 因此对任意正整数 $n > m$ 有

$$\|S_n - S_m\|^2 = \left\| \sum_{i=m+1}^{n} \alpha_i e_i \right\|^2 = \sum_{i=m+1}^{n} |\alpha_i|^2 = \sigma_n - \sigma_m.$$

所以 $\{S_n\}$ 是 X 中 Cauchy 列的充要条件是 $\{\sigma_n\}$ 是 $\mathbb{R}$ 中 Cauchy (数) 列, 由 X 和 R 的完备性可以知道 $\{S_n\}$ 收敛的充要条件是 $\{\sigma_n\}$ 收敛, 即 $\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty$.

(2) 证明与之前习题类似.

(3) 由 Bessel 不等式可知, 对任意 $x \in X$ 有

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2 < \infty.$$

由 (1) 和 (2) 可知, 级数 $\sum_{i=1}^{\infty}\langle x,e_i\rangle e_i$ 收敛.

我们很容易得到以下的推论: 设 $\{e_k\}$ 为希尔伯特空间 X 中可数规范正交系, 则对任意 $x\in X$,

$$\lim_{n\to\infty}\langle x,e_n\rangle=0.$$

(级数收敛的必要条件)

定义 6. 设 M 是希尔伯特空间 X 的规范正交系, 如果

$$\overline{\mathrm{span}M}=X,$$

则称 M 为 X 的**完全规范正交系**.

练习 8. 设 M 是希尔伯特空间 X 的完全规范正交系, 那么 M 完全的充要条件是 $M^{\perp}=\{0\}$.

解答 8. 我们事实上只需要证明: 若 M 是希尔伯特空间 X 中的非空子集, 则 M 的线性包 $\mathrm{span}M$ 在 X 中稠密的充要条件是 $M^{\perp}=\{0\}$.

设 $x\in M^{\perp}$, 若 $\mathrm{span}M$ 在 X 中稠密, 则 $x\in\overline{\mathrm{span}M}$, 因此存在 $\{x_n\}\in\mathrm{span}M$, 使得 $x_n\to x$ $(n\to\infty)$. 由内积的连续性, $\langle x,x\rangle=0$, 因此 $x=0$. 所以 $M^{\perp}=\{0\}$.

反之, 若 $M^{\perp}=\{0\}$, 如果 $x\perp\mathrm{span}M$, 则 $x\perp M$, 因此 $x\in M^{\perp}$. 由 $M^{\perp}=\{0\}$ 可知 $x=0$. 因此 $(\overline{\mathrm{span}M})^{\perp}=\{0\}$. 但 $(\overline{\mathrm{span}M})^{\perp}=(\mathrm{span}M)^{\perp}$, 利用投影定理可知 $\overline{\mathrm{span}M}=X$. 事实上, 令 $Y:=\overline{\mathrm{span}\,M}$. 由假设有 $Y^{\perp}=\{0\}$. 对任意 $x\in X$, 由投影定理可得唯一分解

$$x=y+z,\qquad y\in Y,\ z\in Y^{\perp}.$$

由于 $z\in Y^{\perp}=\{0\}$, 所以 $z=0$, 从而 $x=y\in Y$. 因 x 任取, 故 $Y=X$, 即 $\overline{\mathrm{span}\,M}=X$.

注意: 这里的投影定理是希尔伯特空间中的投影定理, 它的证明需要更多的努力. 现在不是直接证明它的好时机.

练习 9. M 是希尔伯特空间 X 的完全规范正交系的充要条件是: 对任意 $x\in X$ 有如下的 Parseval 等式成立:

$$\|x\|^2=\sum_{e\in M}|\langle x,e\rangle|^2.$$

解答 9. 充分性. 设 Parseval 等式对任意 $x\in X$ 成立. 若 M 不是完全规范正交系, 由以上练习, 则存在非零的 $x_0\in X$, 且 $x_0\perp M$. 所以对任何的 $e\in M$, 有 $\langle x_0,e\rangle=0$. 但由 Parseval 等式可知

$$\|x_0\|^2=\sum_{e\in M}|\langle x_0,e\rangle|^2=0,$$

因此 $x_0 = 0$, 与 $x_0 \neq 0$ 矛盾. 因此 M 是完全规范正交系.

必要性. 设 M 是完全规范正交系, 则对任意 $x \in X$, 设其非零的傅里叶系数为 $\langle x, e_1 \rangle$, $\langle x, e_2 \rangle$, $\dots$, 由之前的练习可以知道, $y = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i < \infty$. 对任意 $i \in \mathbb{N}^+$ 有

$$\langle x - y, e_i \rangle = \langle x, e_i \rangle - \langle y, e_i \rangle = \langle x, e_i \rangle - \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle e_j, e_i \rangle = \langle x, e_i \rangle - \langle x, e_i \rangle = 0.$$

又对 M 中一切使得 $\langle x, e \rangle = 0$ 的 e 也有

$$\langle x - y, e \rangle = \langle x, e \rangle - \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle e_j, e \rangle = 0.$$

因此 $x - y \in M^{\perp}$. 由 M 的完全性可知 $M^{\perp} = \{0\}$, 因此 $x - y = 0$, 即 $x = y$.

练习 10. 例子 1 中的三角函数系是 $L^2(-\pi, \pi)$ 的完全规范正交系.

解答 10. 这个证明现在不是介绍它的最好时机. 略.

练习 11. Consider the Legendre polynomials defined on $[-1, 1]$ by

$$L_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n, \quad n = 0, 1, 2, \dots.$$

(a) Show that if f is indefinitely differentiable on $[-1, 1]$, then

$$\int_{-1}^{1} L_n(x) f(x)\, dx = (-1)^n \int_{-1}^{1} (x^2 - 1)^n f^{(n)}(x)\, dx.$$

In particular, show that L_n is orthogonal to x^m whenever $m < n$. Hence $\{L_n\}_{n=0}^{\infty}$ is an orthogonal family.

(b) Show that

$$\|L_n\|^2 = \int_{-1}^{1} |L_n(x)|^2\, dx = \frac{(n!)^2 2^{2n+1}}{2n + 1}.$$

(c) Prove that any polynomial of degree n that is orthogonal to $1, x, x^2, \dots, x^{n-1}$ is a constant multiple of L_n.

(d) Let $\tilde{L}_n = L_n / \|L_n\|$, the normalized Legendre polynomials. Prove that $\{\tilde{L}_n\}$ is the family obtained by applying the Gram−Schmidt process to $\{1, x, \dots, x^n, \dots\}$, and conclude that every Riemann integrable function f on $[-1, 1]$ has a Legendre expansion

$$\sum_{n=0}^{\infty} \langle f, \tilde{L}_n \rangle \tilde{L}_n$$

which converges to f in the mean-square sense.

解答 11. (a) We start by integrating by parts n times. Let $u = f(x)$ and $dv = L_n(x)dx = \frac{d^n}{dx^n}(x^2-1)^n dx$. Then

$$\int_{-1}^{1} L_n(x) f(x)\, dx = \left[f(x) \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \right]_{-1}^{1} - \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx.$$

Since $(x^2-1)^n = (x-1)^n(x+1)^n$ vanishes to order n at $x = \pm 1$, all boundary terms vanish after n integrations by parts. After n steps we obtain

$$\int_{-1}^{1} L_n(x) f(x)\, dx = (-1)^n \int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\, dx.$$

Now take $f(x) = x^m$ with $m < n$. Then $f^{(n)}(x) = 0$, so

$$\int_{-1}^{1} L_n(x) x^m\, dx = 0.$$

Thus L_n is orthogonal to all polynomials of degree less than n, and hence the family $\{L_n\}_{n=0}^{\infty}$ is orthogonal.

(b) Using the result from part (a) with $f = L_n$, note that $L_n^{(n)}(x) = \frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)!$, since $(x^2-1)^n$ is a polynomial of degree $2n$. Thus

$$\|L_n\|^2 = \int_{-1}^{1} L_n(x)^2\, dx = (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n\, dx.$$

Let $J_n = \int_{-1}^{1} (x^2-1)^n\, dx$. Write $(x^2-1)^n = (x-1)^n(x+1)^n$ and integrate by parts with $u = (x-1)^n$ and $dv = (x+1)^n dx$. The boundary terms vanish, and we obtain

$$J_n = -\frac{n}{n+1} \int_{-1}^{1} (x-1)^{n-1}(x+1)^{n+1}\, dx.$$

Note that $(x-1)^{n-1}(x+1)^{n+1} = (x^2-1)^{n-1}(x+1)^2$. Expanding,

$$(x+1)^2 = x^2 + 2x + 1,$$

so

$$(x^2-1)^{n-1}(x+1)^2 = (x^2-1)^{n-1}x^2 + 2(x^2-1)^{n-1}x + (x^2-1)^{n-1}.$$

Now,

$$\int_{-1}^{1} (x^2-1)^{n-1}x^2\, dx = \int_{-1}^{1} (x^2-1)^{n-1}(x^2-1+1)\, dx = \int_{-1}^{1} (x^2-1)^n\, dx + \int_{-1}^{1} (x^2-1)^{n-1}\, dx = J_n + J_{n-1}.$$

The term $2\int_{-1}^{1} (x^2-1)^{n-1}x\, dx = 0$ since the integrand is odd. Therefore,

$$\int_{-1}^{1} (x^2-1)^{n-1}(x+1)^2\, dx = (J_n + J_{n-1}) + 0 + J_{n-1} = J_n + 2J_{n-1}.$$

Substituting into the expression for J_n gives

$$J_n = -\frac{n}{n+1}(J_n + 2J_{n-1}) \quad \Rightarrow \quad (2n+1)J_n = -2nJ_{n-1}.$$

Solving the recurrence with $J_0 = 2$ yields

$$J_n = (-1)^n \frac{2^{2n+1}(n!)^2}{(2n+1)!}.$$

Therefore,

$$\|L_n\|^2 = (-1)^n(2n)! \cdot (-1)^n \frac{2^{2n+1}(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{2n+1}.$$

(c) Let $P(x)$ be a polynomial of degree n that is orthogonal to $1, x, x^2, \ldots, x^{n-1}$. We want to show that $P(x) = cL_n(x)$ for some constant c.

Let $\mathcal{P}_n$ denote the vector space of polynomials of degree at most n over $[-1, 1]$ with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

The set $\{1, x, x^2, \ldots, x^{n-1}\}$ spans a subspace $V_n \subset \mathcal{P}_n$ of dimension n. The orthogonal complement of V_n in $\mathcal{P}_n$, denoted $V_n^\perp$, has dimension $\dim(\mathcal{P}_n) - \dim(V_n) = (n+1) - n = 1$.

From part (a), we know that L_n is orthogonal to all polynomials of degree less than n. Specifically, for any $m < n$,

$$\langle L_n, x^m \rangle = 0,$$

so $L_n \in V_n^\perp$. Moreover, L_n is nonzero because it is a polynomial of degree n (its leading coefficient is positive and can be shown to be $(2n)!/n!$).

Now, since P is also in $V_n^\perp$ by assumption, and $V_n^\perp$ is one-dimensional, P must be a scalar multiple of L_n. Formally, there exists a constant c such that $P = cL_n$.

To determine c, consider the difference $Q(x) = P(x) - cL_n(x)$ where we choose c so that the coefficient of x^n in Q is zero (i.e., c is the ratio of the leading coefficients of P and L_n). Then Q is a polynomial of degree at most $n-1$. Moreover, since both P and L_n are orthogonal to V_n, their difference Q is also orthogonal to V_n. In particular, Q is orthogonal to itself:

$$\langle Q, Q \rangle = 0.$$

This implies $Q(x) = 0$ almost everywhere, and since Q is a polynomial, $Q \equiv 0$. Thus $P(x) = cL_n(x)$.

(d) Let $\tilde{L}_n = L_n/\|L_n\|$, so that $\|\tilde{L}_n\| = 1$. We claim that the sequence $\{\tilde{L}_n\}_{n=0}^{\infty}$ is exactly the orthonormal family obtained by applying the Gram–Schmidt process to the monomials $\{1, x, x^2, \ldots\}$ with respect to the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx$.

The Gram-Schmidt process applied to $\{1, x, x^2, \dots\}$ produces an orthonormal sequence $\{p_n\}_{n=0}^{\infty}$ where each p_n is a polynomial of degree n and is orthogonal to all p_m with $m < n$. By construction, p_n is also orthogonal to $1, x, \dots, x^{n-1}$ (since these are linear combinations of $p_0, \dots, p_{n-1}$). By part (c), p_n must be a constant multiple of L_n. That is, $p_n = c_n L_n$ for some nonzero constant c_n. Normalizing gives $p_n = \pm\tilde{L}_n$. We can adjust the sign (if necessary) so that the leading coefficient of p_n is positive, which then ensures $p_n = \tilde{L}_n$. (The standard Legendre polynomials are usually defined to have positive leading coefficient, e.g., $L_n(1) = 1$.)

Thus, the normalized Legendre polynomials $\{\tilde{L}_n\}$ form an orthonormal system obtained via Gram–Schmidt from the monomials.

Now, the Weierstrass approximation theorem states that polynomials are dense in the space of continuous functions on $[-1, 1]$ with respect to the uniform norm. Since the uniform norm dominates the L^2 norm, polynomials are also dense in $L^2[-1, 1]$. Therefore, the orthonormal system $\{\tilde{L}_n\}$ is complete in $L^2[-1, 1]$.

For any Riemann integrable function f on $[-1, 1]$, we have $f \in L^2[-1, 1]$ (since Riemann integrability implies boundedness and hence square integrability). By the general theory of Hilbert spaces, the Fourier series of f with respect to a complete orthonormal system converges to f in the L^2 norm (mean-square sense). That is,

$$\lim_{N\to\infty} \left\| f - \sum_{n=0}^{N} \langle f, \tilde{L}_n \rangle \tilde{L}_n \right\|_{L^2} = 0.$$

Equivalently,

$$f = \sum_{n=0}^{\infty} \langle f, \tilde{L}_n \rangle \tilde{L}_n \quad \text{in the mean-square sense.}$$

This completes the proof.