1. $X_1 = (13 \cdot -5 + 7) \bmod 12$

$= -5 + 7 = 2$

$\underline{X_1 = 2}$

$X_2 = (13 \cdot 2 + 7) \bmod 12$

$= 2 + 7 = 9$

$X_2 = 9$

$X_2 = (13 \cdot 9 + 7) \bmod 12$

$= (9 + 7) \bmod 12$

$= 16 \bmod 12 = 4$

$\underline{X_2 = 4}$

$X_3 = (13 \cdot 4 + 7) \bmod 12$

$= 4 + 7 = 11$

$\underline{X_3 = 11}$

$X_4 = (13 \cdot 11 + 7) \bmod 12$

$= (11 + 7) \bmod 12$

$= 18 \bmod 12 = 6$

$\underline{X_4 = 6}$

$\boxed{2, 9, 4, 11, 6}$

2. Trailing zeroes occur when $\underline{2 \cdot 5}$

Ex: $10 \times 10 = 100$

2 5 2 5

1

2

$\frac{100}{5^1} = 20$ $\frac{100}{5^2} = 4$

$20 + 4 = \boxed{24}$

3. $(n^5 - 5n^3 + 4n) \bmod 5 = 0$

$$n(n^4 - 5n^2 + 4)$$
$$= n(n^2-1)(n^2-4)$$
$$= n(n+1)(n-1)(n+2)(n-2) \leftarrow \text{5 consecutive numbers}$$

Possible mod 5 values are 0, 1, 2, 3, 4 and with consecutive numbers at least 1 must be 0, meaning it's divisible by 5.

4. $1333^{42} \bmod 11$

$1333 \bmod 11 = 2$

$$= 2^{42} \bmod 11$$
$$= (2^{21})^2 \bmod 11$$
$$= (2 \cdot (2^{10})^2)^2 \bmod 11$$
$$= (2 \cdot ((2^5)^2)^2)^2 \bmod 11$$
$$= (2 \cdot ((32)^2)^2)^2 \bmod 11$$
$$= (2 \cdot (10^2)^2)^2 \bmod 11$$
$$= (2 \cdot (1^2))^2 \bmod 11$$
$$= \boxed{4 \bmod 11}$$

5. $\gcd(309, 112)$

$309 = 112 \cdot 2 + \underline{85}$ or $309 \bmod 112 = \underline{85}$

$\gcd(112, 85)$

$112 = 85 \cdot 1 + \underline{27}$ or $112 \bmod 85 = \underline{27}$

$\gcd(85, 27)$

$85 = 27 \cdot 3 + \underline{4}$ or $85 \bmod 27 = \underline{4}$

$\gcd(27, 4)$

$27 = 4 \cdot 6 + \underline{3}$ or $27 \bmod 4 = 3$

$\gcd(4, 3)$

$4 = 3 \cdot 1 + \underline{1}$ or $4 \bmod 3 = 1$

$\gcd(3, 1)$

$3 = 1 \cdot 3 + \underline{0}$ so $\boxed{1 \text{ is the gcd}}$ 309 and 112 are relatively prime

6. $54x + 16y = \gcd(54, 16)$ $r_0 = 54$ $r_1 = 16$

$\gcd(54,16)$ $54 = 16 \cdot 3 + 6 \longrightarrow 6 = 54 - 16 \cdot 3$ or $r_0 - 3r_1$

$\gcd(16,6)$ $16 = 6 \cdot 2 + 4 \longrightarrow 4 = 16 - 6 \cdot 2$ or $r_1 - 2(\;) = 7r_1 - 2r_0$

$\gcd(6,4)$ $6 = 4 \cdot 1 + 2 \longrightarrow 2 = 6 - 4 \cdot 1$ or $(\;) - (\;) = 3r_0 - 10r_1$

$\gcd(4,2)$ $4 = 2 \cdot 2 + 0$

$\gcd(54,16) = 2 = 3r_0 - 10r_1 = 162 - 160$ ✓

$\boxed{x = 3}$ $\boxed{y = -10}$

7. $x = 33 \bmod 112$ $\qquad r_0 = 112 \quad r_1 = 33$

$\gcd(112, 33) \quad 112 = 33 \cdot 3 + \underline{13} \quad 13 = r_0 - 3 \cdot r_1$

$\gcd(33, 13) \quad 33 = 13 \cdot 2 + \underline{7} \quad 7 = r_1 - 2 \cdot (r_0 - 3 \cdot r_1)$

$\gcd(13, 7) \quad 13 = 7 \cdot 1 + \underline{6} \quad 6 = (r_0 - 3 \cdot r_1) - (r_1 - 2 \cdot (r_0 - 3 \cdot r_1))$

$\gcd(7, 6) \quad 7 = 6 \cdot 1 + \underline{1} \quad 1 = (r_1 - 2 \cdot (r_0 - 3 \cdot r_1)) - ((r_0 - 3 \cdot r_1) - (r_1 - 2 \cdot (r_0 - 3 \cdot r_1)))$

$\gcd(6, 1) \quad 6 = 1 \cdot 6 + \underline{0}$

$1 = 7r_1 + 2r_0 - ((r_0 - 3r_1) - (7r_1 + 2r_0))$

$1 = 7r_1 + 2r_0 - (3r_0 - 10r_1)$

$1 = 17 \cdot r_1 - 5 \cdot r_0$ $\qquad \boxed{17}$