CH08-320201

Algorithms and Data Structures

ADS

Lecture 14

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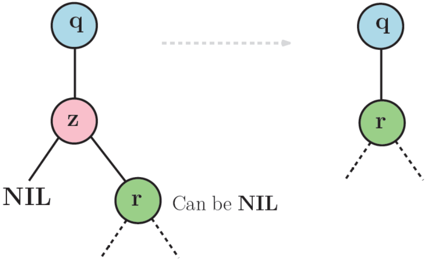
Spring 2019

Modify Operation: Deletion (1)

Case 1:

Deleted node z has no or only right child.



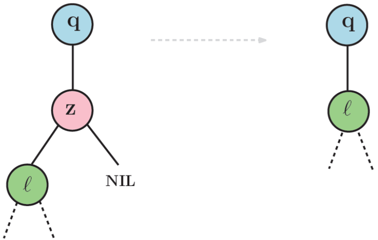
if $z.left$ == NIL

    TRANSPLANT$(T, z, z.right)$

Modify Operation: Deletion (2)

Case 2:

Deleted node z has only left child.



3 **elseif** $z.right$ == NIL

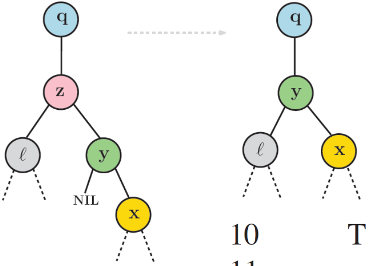
4 TRANSPLANT$(T, z, z.left)$

Remark: For both cases, it does not matter whether z is $q.left$ or $q.right$.

Modify Operation: Deletion (3)

Case 3a:

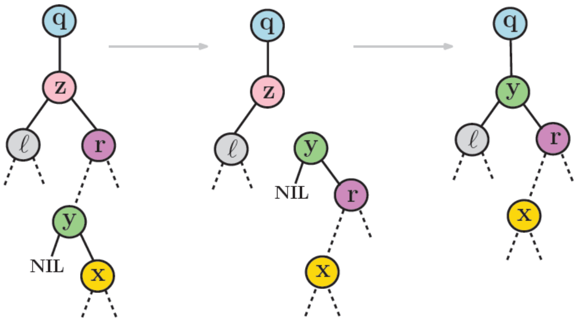
Deleted node z has both children and $Successor(z) = z.right$.




```
TRANSPLANT(T, z, y)
y.left = z.left
y.left.p = y
```

Modify Operation: Deletion (4)

Case 3b:
Deleted node z has both children and $Successor(z) = y \neq z.right$.



Modify Operation: Deletion

```
TREE-DELETE(T, z)
if z.left == NIL
    TRANSPLANT(T, z, z.right)
elseif z.right == NIL
    TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    if y.p ≠ z
        TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
```

Time complexity: $O(h)$

Binary Search Tree: Summary

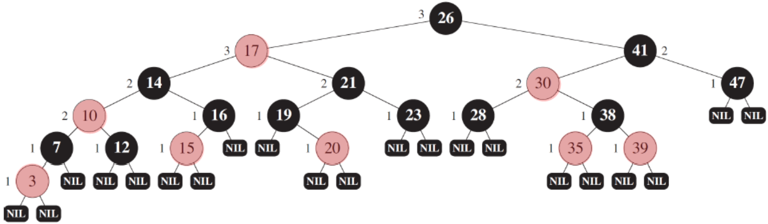
- BST provides all basic dynamic set operations in $O(h)$ running time, including:
  - Search
  - Minimum
  - Maximum
  - Predecessor
  - Successor
  - Insert
  - Delete
- Hence, BST operations are fast if h is small, i.e., if the tree is balanced. Then, $O(h) = O(\lg n)$.

Red-Black Trees: Definition

- A red-black tree is a BST that besides the attributes about parent, left child, right child, and key holds the attribute of a color (**red** or **black**), which is encoded in one additional bit.
- Special convention: All leaves have NIL as key.
- The node colors are used to impose constraints on the nodes such that no path from the root to a leaf is more than twice as long as any other path.
- Hence, the tree is approximately balanced.

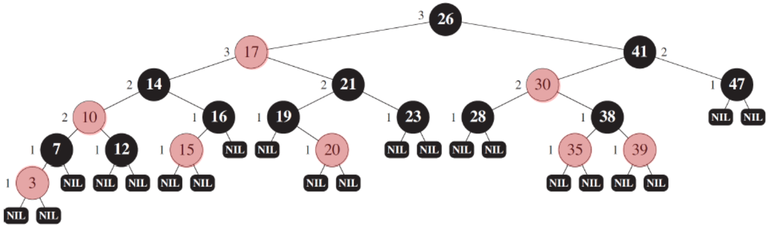
Property 1 (Duh Property)

Every node is either red or black.



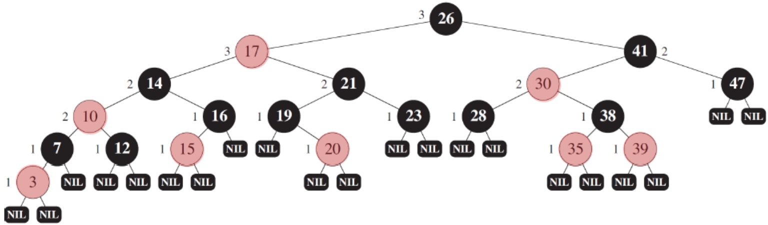
Property 2 (RooB Property)

The root is black.



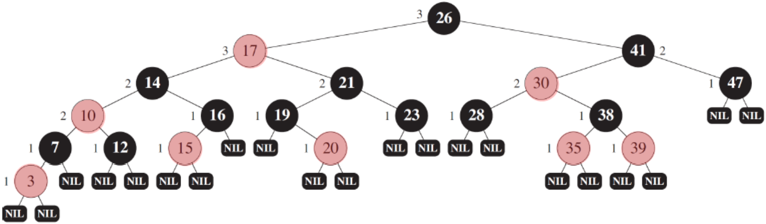
Property 3 (LeaB Property)

All leaves (NIL) are black.



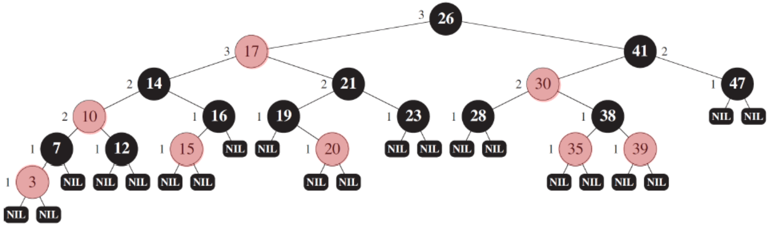
Property 4 (BredB Property)

If a node is red, then both children are black.



Property 5 (BH Property)

For each node all paths from the node to a leaf have the same number of black nodes.

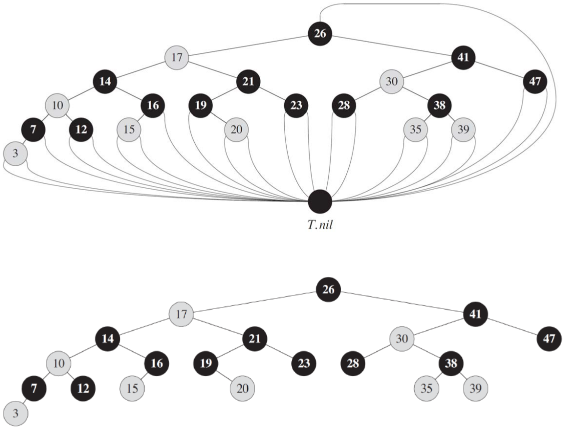


For each node x, we can define a unique black height $bh(x)$.

Properties

1. Every node is either red or black (Duh)
2. The root is black (RooB)
3. All leaves are black (LeaB)
4. If a node is red, then both children are black (BredB)
5. For each node all paths from the node to a leaf have the same number of black nodes (BH)

NIL Sentinel



Number of Nodes vs. Black-Height

Lemma 1:
Let $n(x)$ be the number of non-leaf nodes of a red-black subtree rooted at x. Then, $n(x) \geq 2^{bh(x)} - 1$.
Proof (by induction on height $h(x)$ of node x):

- $h(x) = 0$: x is a leaf. $bh(x) = 0$. $2^{bh(x)} - 1 = 0$. $n(x) \geq 0$. True.
- $h(x) > 0$: x is a non-leaf node. It has two children c_1 and c_2. If c_i is red, then $bh(c_i) = bh(x)$, else $bh(c_i) = bh(x) - 1$. Use assumption, since $h(c_i) < h(x)$,
  $n(c_i) \geq 2^{bh(c_i)} - 1 \geq 2^{bh(x)-1} - 1$.
  Thus,
  $n(x) = n(c_1) + n(c_2) + 1 \geq 2(2^{bh(x)-1} - 1) + 1 = 2^{bh(x)} - 1$.

Height vs. Black-Height

Lemma 2:
Let h be the height of a red-black tree with root r. Then, $bh(r) \geq h/2$.
Proof:

- Let $r, v_1, v_2, ..., v_h$ be the longest path in the tree.
- The number of black nodes in the path is $bh(r)$.
- Thus, the number of red nodes is $h - bh(r)$.
- Since v_h is black (LeaB property) and every red node in the path must be followed by a black one (BredB property), we have $h - bh(r) \leq bh(r)$.
- Hence, $bh(r) \geq h/2$.

Height of a Red-Black Tree

Theorem:
A red-black tree with n non-leaf nodes has height $h \leq 2\lg(n+1)$.
Proof:

- Lemma 1: $n \geq 2^{bh(r)} - 1$ (r being the root).
- Lemma 2: $bh(r) \geq h/2$.
- Thus, $n \geq 2^{h/2} - 1$.
- So, $h \leq 2\lg(n+1)$.

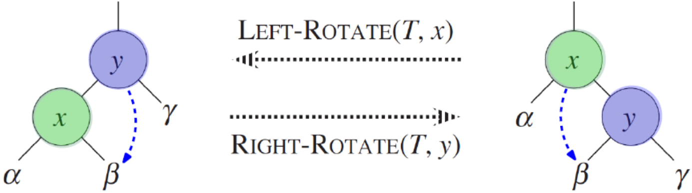
Corollary:
The height of a red-black tree is $O(\lg n)$.
All dynamic set operations can be performed in $O(\lg n)$, if we maintain the red-black tree properties.

Operations

- Querying
  - Search/Minimum & Maximum/Successor & Predecessor
  - Just as in normal BST
  - $O(\lg n)$
- Modifying
  - Tree-Insert/Tree-Delete $\rightarrow O(\lg n)$
  - But, need to guarantee red-black tree properties:
    - must change color of some nodes
    - change pointer structure through rotation

Rotations (1)

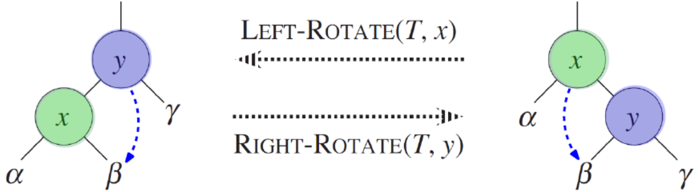
- *Right-Rotate*(T, y):
  - node y becomes right child of its left child x.
  - new left child of y is former right child of x.
- *Left-Rotate*(T, x):
  - node x becomes left child of its right child y.
  - new right child of x is former left child of y.



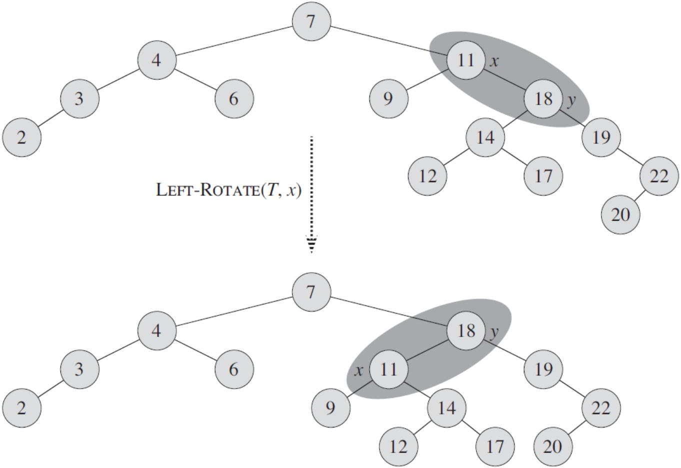
Rotations (2)

BST property is preserved:

- (left): $key(\alpha) \leq x.key \leq key(\beta) \leq y.key \leq key(\gamma)$
- (right): $key(\alpha) \leq x.key \leq key(\beta) \leq y.key \leq key(\gamma)$



Rotation: Example



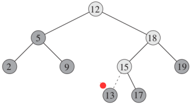
Rotation Pseudocode

```
LEFT-ROTATE(T, x)
  y = x.right                  // set y
  x.right = y.left             // turn y's left subtree into x's right subtree
  if y.left ≠ T.nil
      y.left.p = x
  y.p = x.p                    // link x's parent to y
  if x.p == T.nil
      T.root = y
  elseif x == x.p.left
      x.p.left = y
  else x.p.right = y
  y.left = x                   // put x on y's left
  x.p = y
```

Time complexity: $O(1)$

Insertion

```
TREE-INSERT(T, z)
y = NIL
x = T.root
while x ≠ NIL
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == NIL
    T.root = z
elseif z.key < y.key
    y.left = z
else y.right = z
```




```
RB-INSERT(T, z)
y = T.nil
x = T.root
while x ≠ T.nil
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == T.nil
    T.root = z
elseif z.key < y.key
    y.left = z
else y.right = z
z.left = T.nil
z.right = T.nil
z.color = RED
RB-INSERT-FIXUP(T, z)
```

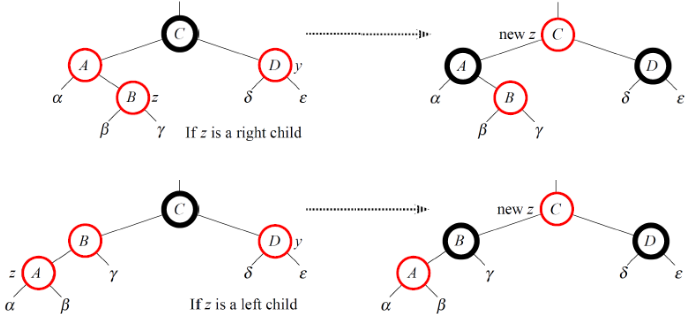
Fixing Red-Black Tree Properties

- We are inserting a **red** node to a valid red-black tree.
- Which properties may be violated?
  1. Duh: Cannot be violated. ✓
  2. RooB: Violated if inserted node is root. ✗
  3. LeaB: Inserted node is not a leaf, i.e., no violation. ✓
  4. BredB: Violated if parent of inserted node is red. ✗
  5. BH: Not affected by red nodes, i.e., no violation. ✓

Fixing BredB

- BredB for node z is violated, if $z.p$ is red.
- Then, $z.p.p$ is black. (BredB property)
- We need to consider different cases depending on the uncle y of z, i.e., the child of $z.p.p$ that is not $z.p$.
- There are 6 cases:
  - $z.p$ is left child of $z.p.p$
    - y is red (Case 1)
    - y is black
      - z is right child of $z.p$ (Case 2)
      - z is left child of $z.p$ (Case 3)
  - $z.p$ is right child of $z.p.p$
    - y is red (symmetric to Case 1)
    - y is black
      - z is right child of $z.p$ (symmetric to Case 2)
      - z is left child of $z.p$ (symmetric to Case 3)

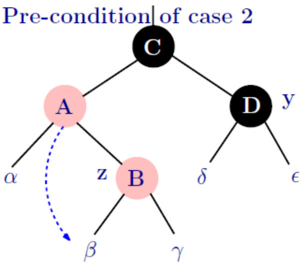
Case 1 (Red Uncle)

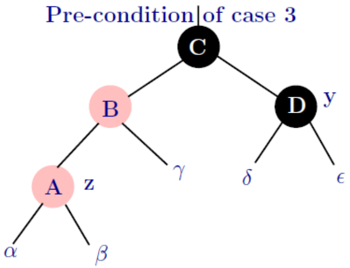



```
if z.p == z.p.p.left
    y = z.p.p.right
    if y.color == RED
        z.p.color = BLACK
        y.color = BLACK          Case 1
        z.p.p.color = RED
        z = z.p.p
```

There is a new problem, if *z.p.p.p* is red.
Algorithm needs to continue with *z.p.p*.

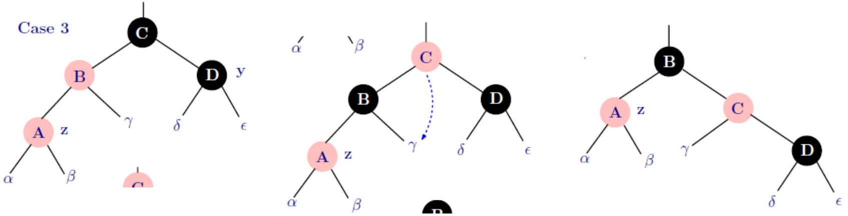
Case 2 (Black Uncle, z Right Child)






```
else if z == z.p.right
    z = z.p                    Case 2
    LEFT-ROTATE(T, z)
```

Case 3 (Black Uncle, z Left Child)



$z.p.color = \text{BLACK}$
$z.p.p.color = \text{RED}$
RIGHT-ROTATE$(T, z.p.p)$

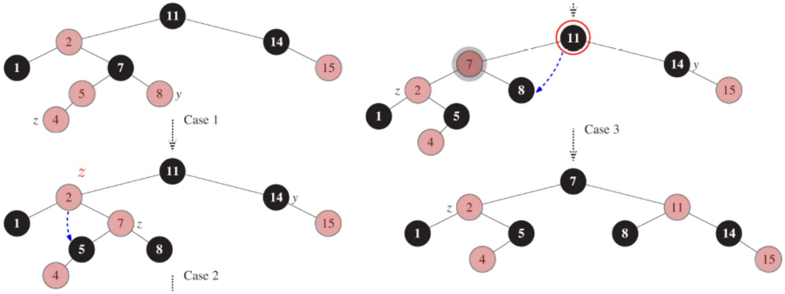
Case 3

Putting It All Together

- We need to put the 3 cases (and the 3 symmetric cases) together.
- Moreover, we need to propagate the considerations upwards (see Case 1).
- Finally, we have to fix RooB.

```
RB-INSERT-FIXUP(T, z)
while z.p.color == RED
    if z.p == z.p.p.left
        y = z.p.p.right
        if y.color == RED
            z.p.color = BLACK           Case 1
            y.color = BLACK             Case 1
            z.p.p.color = RED           Case 1
            z = z.p.p                   Case 1
        else if z == z.p.right
                z = z.p                 Case 2
                LEFT-ROTATE(T, z)       Case 2
            z.p.color = BLACK           Case 3
            z.p.p.color = RED           Case 3
            RIGHT-ROTATE(T, z.p.p)      Case 3
    else (same as then clause
            with "right" and "left" exchanged)
T.root.color = BLACK
```

Insert Example

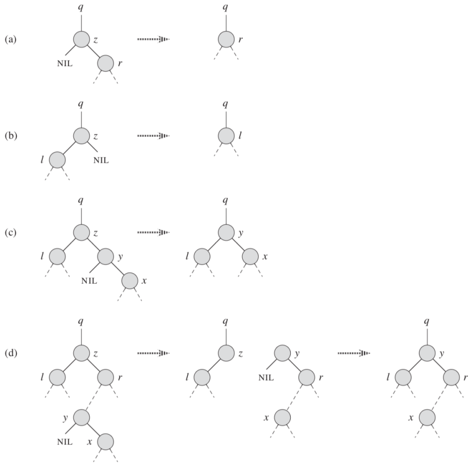


Time Complexity

- In worst case, we have to go all the way from the leaf to the root along the longest path within the tree.
- Hence, running time is $O(h) = O(\lg n)$ for the fixing of the red-black tree properties.
- Overall, running time for insertion is $O(h) = O(\lg n)$.
- Example for building up a red-black tree by iterated node insertion:
  http://www.youtube.com/watch?v=vDHFF4wjWYU

Deletion (Remember BST)

```
TREE-DELETE(T, z)
if z.left == NIL
    TRANSPLANT(T, z, z.right)
elseif z.right == NIL
    TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    if y.p ≠ z
        TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
```



Deletion (RB) (1)

```
TREE-DELETE(T, z)
if z.left == NIL
    TRANSPLANT(T, z, z.right)
elseif z.right == NIL
    TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    if y.p ≠ z
        TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
```

```
RB-DELETE(T, z)
y = z
y-original-color = y.color
if z.left == T.nil
    x = z.right
    RB-TRANSPLANT(T, z, z.right)
elseif z.right == T.nil
    x = z.left
    RB-TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    y-original-color = y.color
    x = y.right
    if y.p == z
        x.p = y
    else RB-TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    RB-TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
    y.color = z.color
if y-original-color == BLACK
    RB-DELETE-FIXUP(T, x)
```

Deletion (RB) (2)

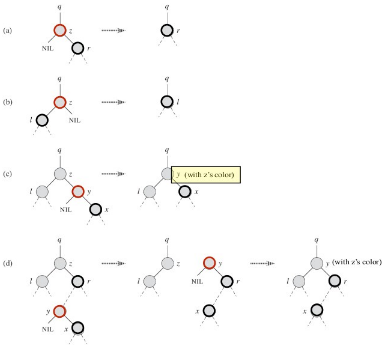
- node y
  - either removed (a/b)
  - or moved in the tree (c/d)
  - y-original-color
- node x
  - the node that moves into y's original position
  - x.p points to y's original parent (since it moves into y's position, note special case in 12/13)

RB-DELETE(T, z)

```
1  y = z
2  y-original-color = y.color
3  if z.left == T.nil
4      x = z.right
5      RB-TRANSPLANT(T, z, z.right)
6  elseif z.right == T.nil
7      x = z.left
8      RB-TRANSPLANT(T, z, z.left)
9  else y = TREE-MINIMUM(z.right)
10     y-original-color = y.color
11     x = y.right
12     if y.p == z
13         x.p = y
14     else RB-TRANSPLANT(T, y, y.right)
15         y.right = z.right
16         y.right.p = y
17     RB-TRANSPLANT(T, z, y)
18     y.left = z.left
19     y.left.p = y
20     y.color = z.color
21 if y-original-color == BLACK
22     RB-DELETE-FIXUP(T, x)
```

Deletion (RB) (3)

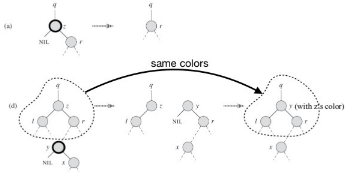
- y-original-color == red




```
RB-DELETE(T, z)
y = z
y-original-color = y.color
if z.left == T.nil
    x = z.right
    RB-TRANSPLANT(T, z, z.right)
elseif z.right == T.nil
    x = z.left
    RB-TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    y-original-color = y.color
    x = y.right
    if y.p == z
        x.p = y
    else RB-TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    RB-TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
    y.color = z.color
if y-original-color == BLACK
    RB-DELETE-FIXUP(T, x)
```

Deletion (RB) (4)

- y-original-color == red
  - no problem
- y-original-color == black
  - violations might occur (2,4,5)
  - main idea to fix
    - x gets an “**extra black**” & needs to get rid of it
  - 4 cases

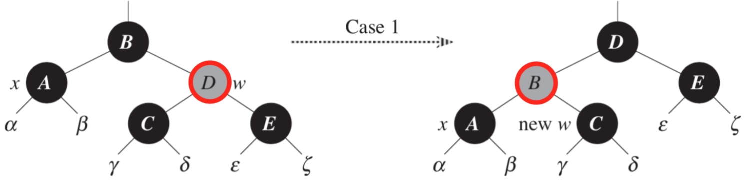



```
RB-DELETE(T, z)
y = z
y-original-color = y.color
if z.left == T.nil
    x = z.right
    RB-TRANSPLANT(T, z, z.right)
elseif z.right == T.nil
    x = z.left
    RB-TRANSPLANT(T, z, z.left)
else y = TREE-MINIMUM(z.right)
    y-original-color = y.color
    x = y.right
    if y.p == z
        x.p = y
    else RB-TRANSPLANT(T, y, y.right)
        y.right = z.right
        y.right.p = y
    RB-TRANSPLANT(T, z, y)
    y.left = z.left
    y.left.p = y
    y.color = z.color
if y-original-color == BLACK
    RB-DELETE-FIXUP(T, x)
```

Fixing Red-Black Tree Properties (1)

Case 1: x's sibling w is red.

Transform to Case 2, 3, or 4 by left rotation and changing colors of nodes B and D.



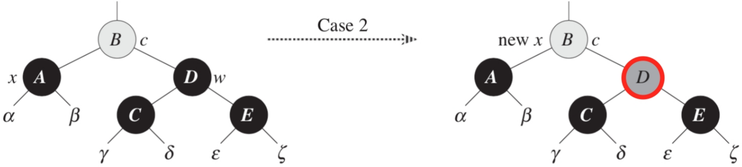
x = node with extra black

w = x's sibling

if $w.color$ == RED
  $w.color$ = BLACK
  $x.p.color$ = RED
  LEFT-ROTATE($T, x.p$)
  $w = x.p.right$

Fixing Red-Black Tree Properties (2)

Case 2: x's sibling w is black and the children of w are black. Set color of w to red and propagate upwards.

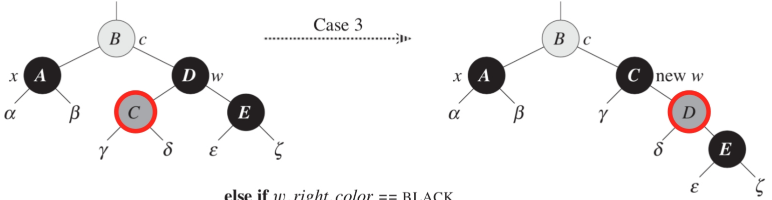


x = node with extra black
w = x's sibling
c = color of the node

if $w.left.color$ == BLACK and $w.right.color$ == BLACK
$w.color$ = RED
$x = x.p$

Fixing Red-Black Tree Properties (3)

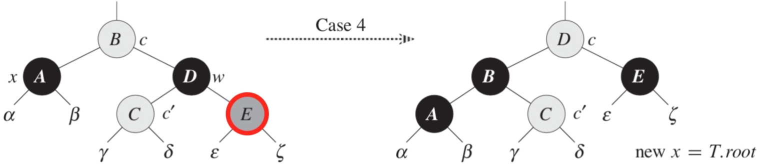
Case 3: x's sibling w is black and the left child of w is red, while the right child of w is black.
Transform to Case 4 by right rotation and changing colors of nodes C and D.



else if $w.right.color$ == BLACK
$w.left.color$ = BLACK
$w.color$ = RED
RIGHT-ROTATE(T, w)
$w = x.p.right$

Fixing Red-Black Tree Properties (4)

Case 4: x's sibling w is black and the right child of w is red. Perform a left-rotate and change colors of B, D, and E. Then, the loop terminates.



$w.color = x.p.color$
$x.p.color = \text{BLACK}$
$w.right.color = \text{BLACK}$
$\text{LEFT-ROTATE}(T, x.p)$

Fixing Red-Black Tree Properties (5)

```
RB-DELETE-FIXUP(T, x)
while x ≠ T.root and x.color == BLACK
    if x == x.p.left
        w = x.p.right
        if w.color == RED
            w.color = BLACK                                  // case 1
            x.p.color = RED                                  // case 1
            LEFT-ROTATE(T, x.p)                              // case 1
            w = x.p.right                                    // case 1
        if w.left.color == BLACK and w.right.color == BLACK
            w.color = RED                                    // case 2
            x = x.p                                          // case 2
        else if w.right.color == BLACK
                w.left.color = BLACK                         // case 3
                w.color = RED                                // case 3
                RIGHT-ROTATE(T, w)                           // case 3
                w = x.p.right                                // case 3
            w.color = x.p.color                              // case 4
            x.p.color = BLACK                                // case 4
            w.right.color = BLACK                            // case 4
            LEFT-ROTATE(T, x.p)                              // case 4
            x = T.root                                       // case 4
    else (same as then clause with "right" and "left" exchanged)
x.color = BLACK
```

Time complexity: $O(h) = O(\lg n)$

Conclusion

Modifying operations on red-black trees can be executed in $O(\lg n)$ time.