Poisson Processes

Outline

- Introduction to Poisson Processes
  - Definition of arrival process
  - Definition of renewal process
  - Definition of Poisson process
- Properties of Poisson processes
  - Inter-arrival time distribution
  - Waiting time distribution
  - Superposition and decomposition
- Non-homogeneous Poisson processes (relaxing *stationary*)
- Compound Poisson processes (relaxing *single arrival*)
- Modulated Poisson processes (relaxing *independent*)

Stochastic Process

- A stochastic process $\boldsymbol{X} = \{X(t), t \in T\}$ is a collection of random variables. That is, for each $t \in T$, $X(t)$ is a random variable.

- The index t is often interpreted as "time" and, as a result, we refer to $X(t)$ as the "state" of the process at time t.

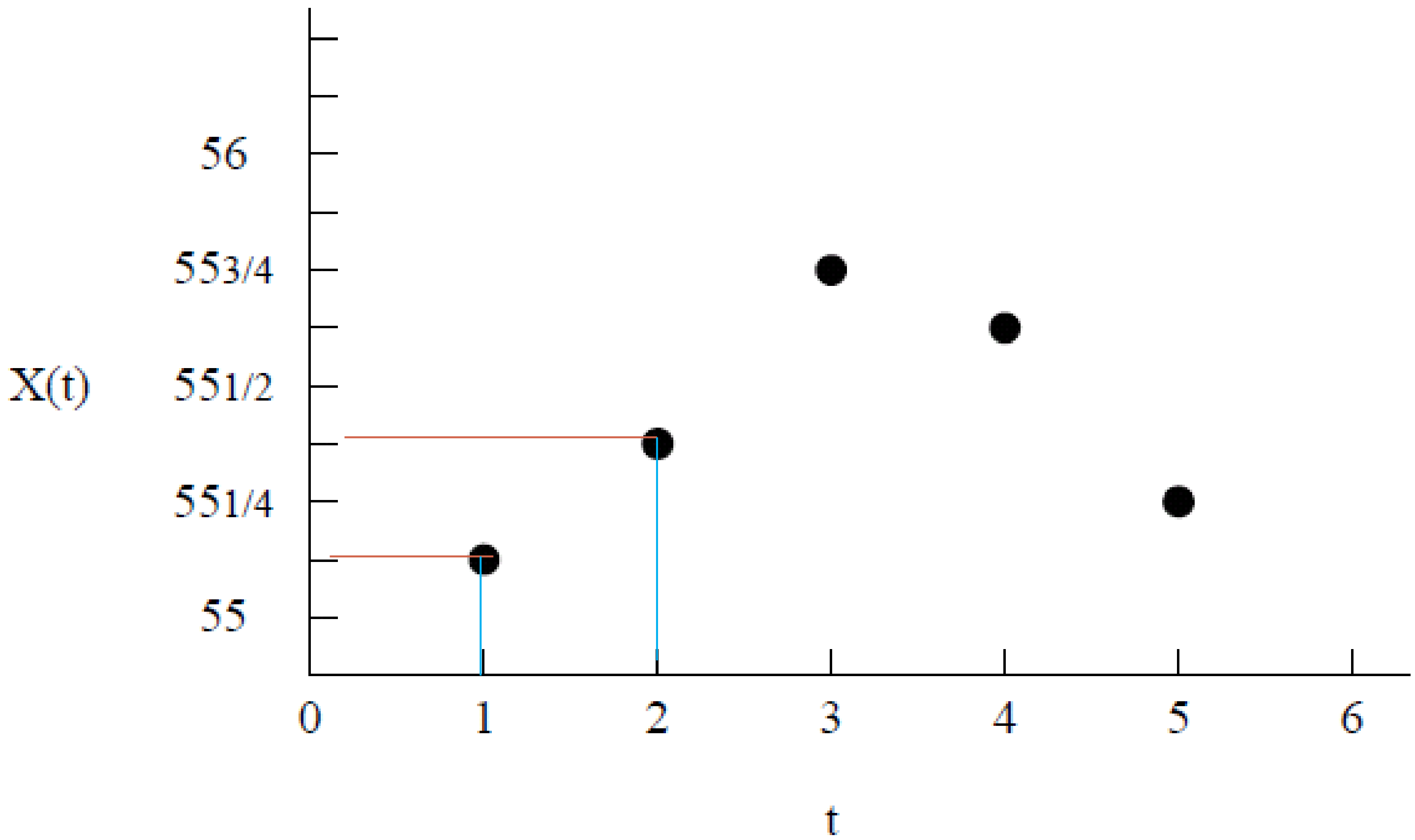
- When index set T of process X is a countable set
  => X is a discrete-time stochastic process

- When index set T of process X is a continuum (uncountable)
  => X is a continuous-time stochastic process

Stochastic Process (con’t)

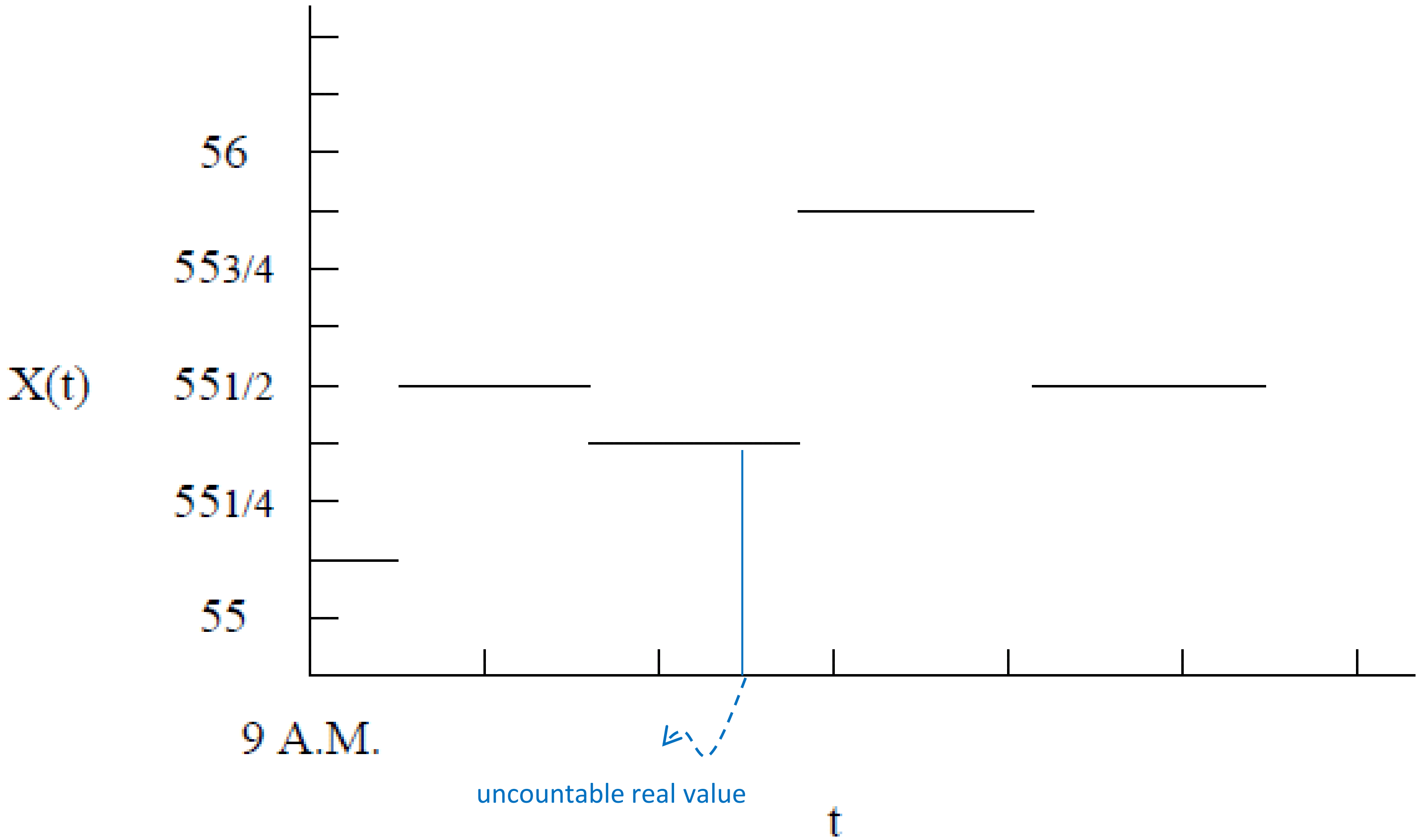
- Four types of stochastic processes
  - Discrete Time with Discrete State Space
  - Continuous Time with Discrete State Space
  - Discrete Time with Continuous State Space
  - Continuous Time with Continuous State Space

Discrete Time with Discrete State Space



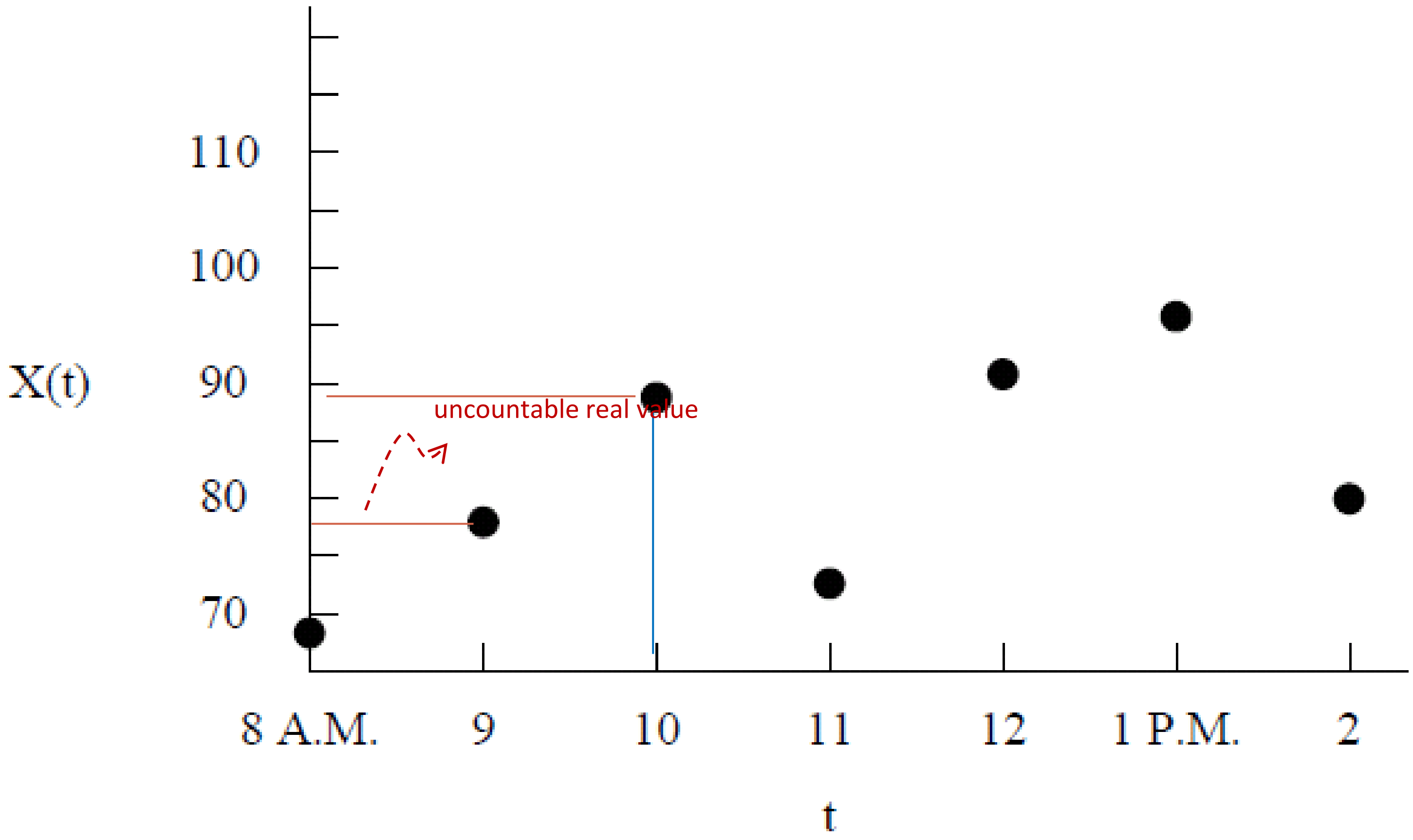
X(t) = closing price of an IBM stock on day t

Continuous Time with Discrete State Space



X(t) = price of an IBM stock at time t on a given day

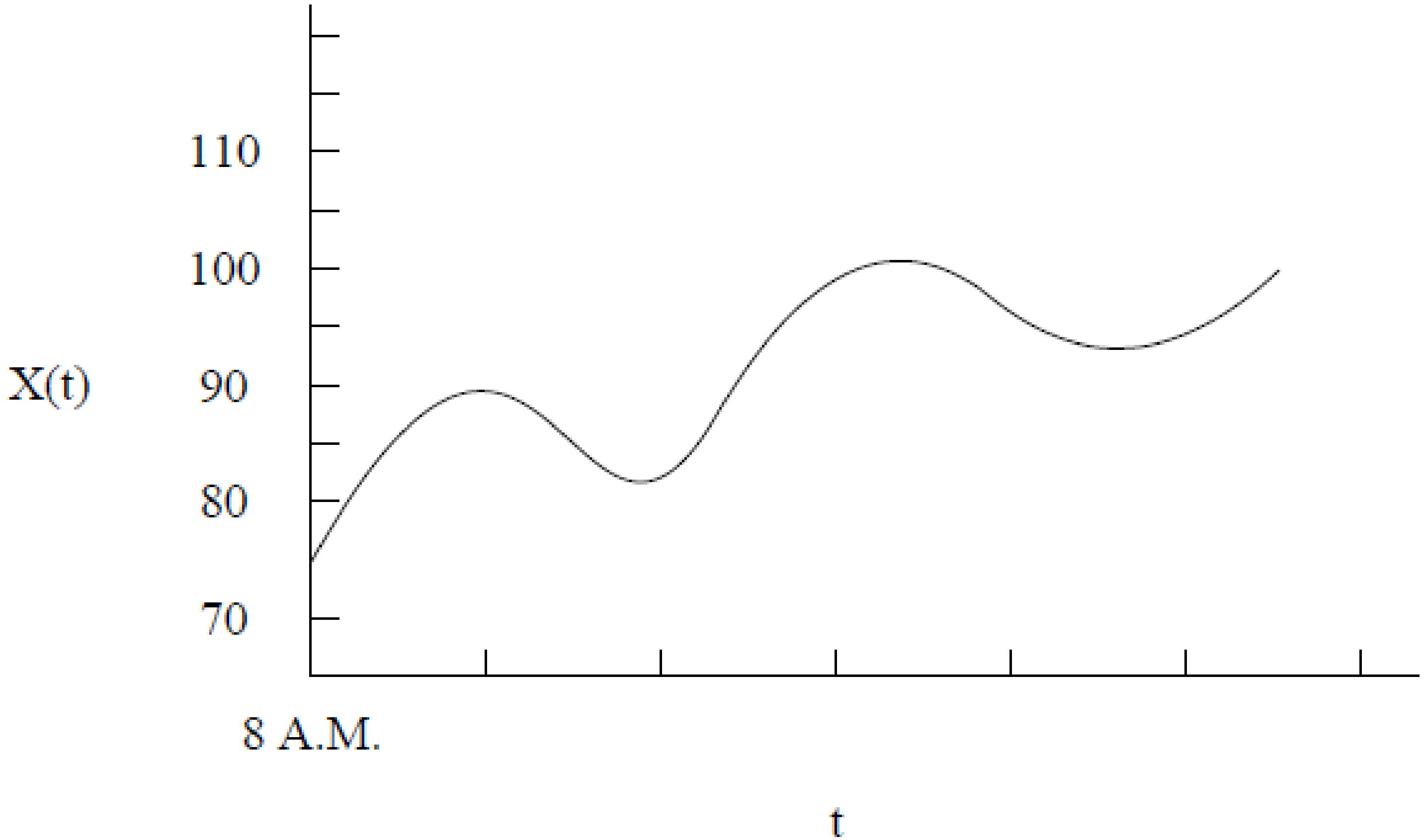
Discrete Time with Continuous State Space



X(t) = temperature at the airport at time t

(every hour on the hour)

Continuous Time with Continuous State Space



X(t) = temperature at the airport at time t

Two Structural Properties of Stochastic Processes

- **Independent increment** : if for all $t_0 < t_1 < t_2 < \ldots < t_n$ in the process $\boldsymbol{X} = \{X(t), t \geq 0\}$, random variables $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$, $\ldots$, $X(t_n) - X(t_{n-1})$ are independent

  ⇒ the magnitudes of state change over non-overlapping time intervals are mutually independent

- **Stationary increment** : if the random variable $X(t + s) - X(t)$ has the same probability distribution for all t and any $s > 0$,

  ⇒ the probability distribution governing the magnitude of state change depends only on the difference in the lengths of the time indices and is independent of the time origin used for the indexing variable

  $(t + s) - t$

  t

Counting Processes

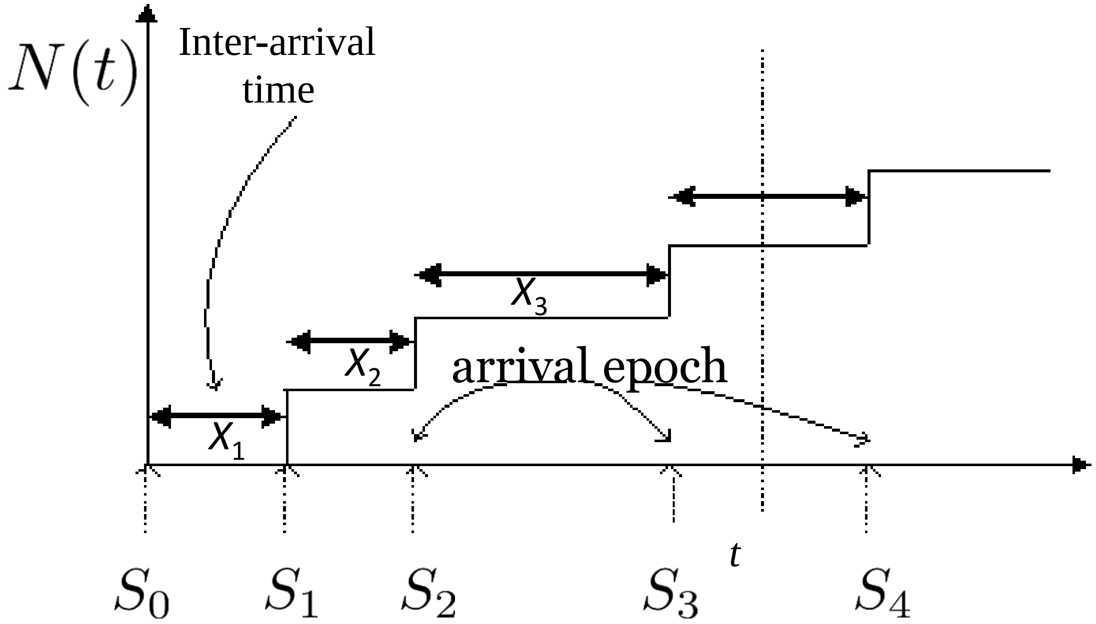
- A stochastic process $\{N(t), t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of "events" that have occurred up to time *t*.

- From the definition we see that for a counting process $N(t)$ must satisfy:

  1. $N(t) \geq 0$.
  2. $N(t)$ is integer valued.
  3. If $s < t$, then $N(s) \leq N(t)$
  4. For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval $(s, t]$.

Ex. 1 number of cars passing by
Ex. 2 number of home runs hit by a baseball player

Counting Processes



(i) n_{th} arrival epoch S_n is

$$S_n = X_1 + X_2 + ... + X_n = \sum_{i=1}^{n} X_i$$

(ii) Number of arrival at time t is : $N(t)$. Notice that :

$$\{N(t) \geq n\} \overset{iff}{\iff} \{S_n \leq t\}, \{N(t) = n\} \overset{iff}{\iff} \{S_n \leq t \text{ and } S_{n+1} > t\}$$

Three Types of Counting Processes

Arrival Process: $\{X_i, i = 1, 2, ...\}$; $X_i's$ can be any

$\{S_i, i = 0, 1, 2, ...\}$; $S_i's$ can be any

$\{N(t), t \geq 0\}$; $\rightarrow$ called arrival process

Renewal Process: $\{X_i, i = 1, 2, ...\}$; $X_i's$ are i.i.d

$\{S_i, i = 0, 1, 2, ...\}$; $S_i's$ are general distributed

$\{N(t), t \geq 0\}$; $\rightarrow$ called renewal process

Poisson Process: $\{X_i, i = 1, 2, ...\}$; $X_i's$ are iid exponential distributed

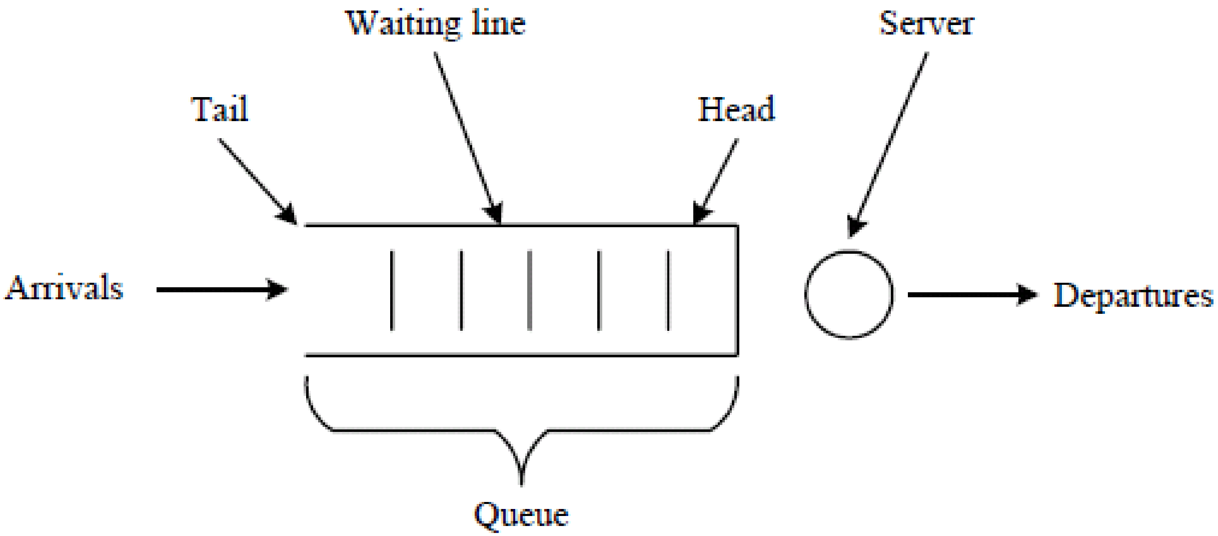
$\{S_i, i = 0, 1, 2, ...\}$; $S_i's$ are Erlang distributed

$\{N(t), t \geq 0\}$; $\rightarrow$ called Poisson process

Siméon Denis Poisson

- Born: 6/21/1781-Pithiviers, France
- Died: 4/25/1840-Sceaux, France
- “Life is good for only two things:
  - discovering mathematics and
  - teaching mathematics.”

A Single Server Queue



- Arrivals: Poisson process, renewal process, etc.
- Queue length: Markov process, semi-Markov process, etc.
- . . .

Definition 1: Poisson Processes

The counting process $\{N(t), t \geq 0\}$ is Possion process with rate λ $(\lambda > 0)$, if:

1. $N(0) = 0$
2. Independent increments
3. The number of events in any interval of length *t* is Poisson distributed with mean λt. That is, for all *s, t* ≥ 0

$$P\{\underbrace{N(t+s) - N(s) = n}_{\text{dependent on t, independent on s}}\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \; n = 0, 1, \ldots$$

dependent on t, independent on s ← stationary increments property

Definition 2: Poisson Processes

The counting process $\{N(t), t \geq 0\}$ is Possion process with rate λ $(\lambda > 0)$, if:

1. $N(0) = 0$
2. Independent increments relaxed ⇒ Modulated Poisson Process

   For any $0 \leq s \leq t \leq u \leq v$, $N(t) - N(s)$ is independent of $N(v) - N(u)$

3. Stationary increments relaxed ⇒ Non-homogeneous Poisson Process

$$P\{N(t+s) - N(t) = k\} = P\{N(l+s) - N(l) = k\}$$

4. Single arrival relaxed ⇒ Compound Poisson Process

$$\begin{aligned} P\{N(h) = 1\} &= \lambda \mathrm{h} + o(h) \\ P\{N(h) \geq 2\} &= o(h) \end{aligned}$$

f is said to be $o(\mathrm{h})$ if

$$\lim_{\mathrm{h} \to 0} \frac{f(\mathrm{h})}{\mathrm{h}} = 0$$

Theorem: Definition 1 and Definition 2 are equivalent

Proof: We show that **Definition 1** implies **Definition 2**

$$P\{N(h)=0\} = P\{N(h)-N(0)=0\} = e^{-\lambda h}\frac{(\lambda h)^0}{0!} = e^{-\lambda h}$$

The Taylor series for the exponential function e^x at $a = 0$ is

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$P\{N(h)=0\} = e^{-\lambda h} = 1 - \lambda h + \frac{(\lambda h)^2}{2} + \ldots$$
$$= 1 - \lambda h + o(h)$$

f is said to be $o(\mathrm{h})$ if

$$\lim_{\mathrm{h}\to 0}\frac{f(\mathrm{h})}{\mathrm{h}} = 0$$

$$P\{N(h)=1\} = e^{-\lambda h}\lambda h = \lambda h(1-\lambda h + o(h))$$
$$= \lambda h + o(h)$$

$$P\{N(h)\geq 2\} = 1 - P\{N(h)=0\} - P\{N(h)=1\} = o(h)$$

From condition (3) of Definition 1, the number of events in any interval of length t is Poisson distributed with mean λt.

=> We know that $N(t)$ also has stationary increments . $\#$

Theorem: Definitions 1 and 2 are equivalent (con't)

Proof: We show that Definition 2 implies Definition 1

Let $P_n(t) = P\{N(t) = n\}$

$$
\begin{aligned}
P_n(t+h) = P\{N(t+h) = n\} &= P\{N(t) = n, N(t+h) - N(t) = 0\} \\
&+ P\{N(t) = n-1, N(t+h) - N(t) = 1\} \\
&+ P\{N(t) = n-2, N(t+h) - N(t) = 2\} + \ldots \\
&+ P\{N(t) = 0, N(t+h) - N(t) = n\}
\end{aligned}
$$

Independent

Stationary

$$
\begin{aligned}
&= P\{N(t) = n\}P\{N(t+h) - N(t) = 0\} \\
&+ P\{N(t) = n-1\}P\{N(t+h) - N(t) = 1\} \\
&+ P\{N(t) = n-2\}P\{N(t+h) - N(t) = 2\} + \ldots \\
&+ P\{N(t) = 0\}P\{N(t+h) - N(t) = n\}
\end{aligned}
$$

$P\{N(h) = 0\}$

$P\{N(h) = 1\}$

$$
\begin{aligned}
P_n(t+h) &= P_n(t)P_0(h) \\
&\quad + P_{n-1}(t)P_1(h) \\
&\quad + P_{n-2}(t)P_2(h) + \dots \\
&\quad + P_0(t)P_n(h) \\
&= P_n(t)(1-\lambda h + o(h)) \\
&\quad + P_{n-1}(t)(\lambda h + o(h)) \\
&\quad + P_{n-2}(t)o(h) + \dots \\
&\quad + P_0(t)o(h) \\
&= P_n(t) - \lambda h P_n(t) + \lambda h P_{n-1}(t) + o(h)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + o(h)
$$

$$
\text{As } h \to 0 \quad \begin{cases} P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) & \text{As n=1, 2, ...} \\ P_0'(t) = -\lambda P_0(t) & \text{As n=0} \end{cases}
$$

$$P_0'(t) = -\lambda P_0(t)$$

$$\text{Integral } \frac{P_0'(t)}{P_0(t)} = -\lambda$$

$$\Rightarrow \log P_0(t) = -\lambda t + \text{const.} \Rightarrow P_0(t) = ce^{-\lambda t}$$

$$\because \; P_0(0) = P\{N(0) = 0\} = 1 \qquad \Rightarrow c = 1$$

$$\Rightarrow P_0(t) = e^{-\lambda t}$$

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)$$
$$\Rightarrow e^{\lambda t}[P_n'(t) + \lambda P_n(t)] = \lambda e^{\lambda t} P_{n-1}(t)$$
$$\Rightarrow \frac{d}{dt} e^{\lambda t} P_n(t) = \lambda e^{\lambda t} P_{n-1}(t)$$

To show that $P_n(t) = e^{-\lambda t}(\lambda t)^n/n!$, we use mathematical induction and assume $P_{n-1}(t) = e^{-\lambda t}(\lambda t)^{n-1}/(n-1)!$ for $n-1$, i.e.,

$$\frac{d}{dt} e^{\lambda t} P_n(t) = \lambda e^{\lambda t} P_{n-1}(t) = \cancel{\lambda} e^{\lambda t}\, \cancel{e}^{-\lambda t} (\lambda t)^{n-1}/(n-1)! = \frac{\lambda(\lambda t)^{n-1}}{(n-1)!}$$

$$\Rightarrow e^{\lambda t} P_n(t) = \frac{(\lambda t)^n}{n!} + c$$ integral

$$\because\ P_n(0) = P\{N(0) = n\} = 0 \Rightarrow c = 0$$

$$\Rightarrow P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$ #

The Inter-Arrival Time Distribution

Theorem

Poisson Processes have exponential inter-arrival time distribution, i.e., $\{X_n, n = 1, 2, \ldots\}$ are i.i.d. and exponentially distributed with parameter λ (i.e., mean inter-arrival time = $1/\lambda$).

Proof:

$$X_1 : P\{X_1 > t\} = P\{N(t) = 0\} = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda \mathrm{t}}$$

$$\therefore \; X_1 \; \sim \; e(t; \lambda)$$

$$X_2 : P\{X_2 > t | X_1 = s\}$$

$$= P\{0 \text{ arrivals in } (s, s+t] | X_1 = s\}$$

by independent increment

$$= P\{0 \text{ arrivals in } (s, s+t]\}$$

by stationary increment

$$= P\{0 \text{ arrivals in } (0, t]\}$$

$$= e^{-\lambda \mathrm{t}} \quad \therefore \; X_2 \text{ is independent of } X_1, \text{ and } X_2 \; \sim exp(t; \lambda).$$

(The procedure repeats for the rest of Xi's.)

The Arrival Time Distribution of the n_{th} Event

Theorem

The arrival time of the n_{th} event, S_n (also called the waiting time until the n_{th} event), is *Erlang* distributed with parameter (*n*, *λ*).

Proof:

Method 1

$$N(t) \geq n \iff S_n \leq t$$

$$\because \quad F_{S_n}(t) = P\{S_n \leq t\} = P\{N(t) \geq n\} = \sum_{j=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

$$\therefore \quad f_{S_n}(t) = -\sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} + \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!}$$

$$= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

The Arrival Time Distribution of the n-th Event (con't)

Method 2:

?

$$
\begin{aligned}
&P\{t < S_n < t + dt\} \\
&= P\{n - 1 \text{arrivals in } (0, t] \text{ and } 1 \text{ arrival in } (t, t + dt)\} + o(dt) \\
&= P\{N(t) = n - 1 \text{ and } 1 \text{ arrival in } (t, t + dt)\} + o(dt) \\
&= P\{N(t) = n - 1\}P\{1 \text{ arrival in } (t, t + dt)\} + o(dt) \\
&= \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\lambda dt + o(dt)
\end{aligned}
$$

independent

$$
\therefore \quad \lim_{dt \to 0} \frac{P\{t < S_n < t + dt\}}{dt} = f_{S_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}
$$

$$
\begin{aligned}
&P\{t < S_n < t + dt\} \\
&= P\{\text{n-1 arrivals in (0,t] and 1 arrival in (t,t+dt)}\} \\
&+ P\{\text{n-2 arrivals in (0,t] and 2 arrival in (t,t+dt)}\} + \ldots \\
&+ P\{\text{0 arrivals in (0,t] and n arrival in (t,t+dt)}\} \\
&= P\{N(t) = \text{n-1 and 1 arrival in (t,t+dt)}\} \\
&+ P\{N(t) = \text{n-2 and 2 arrival in (t,t+dt)}\} + \ldots \\
&+ P\{N(t) = \text{0 and n arrival in (t,t+dt)}\} \\
&= P\{N(t) = n-1\}P\{\text{1 arrival in (t,t+dt)}\} \\
&+ P\{N(t) = n-2\}P\{\text{2 arrival in (t,t+dt)}\} + \ldots \\
&+ P\{N(t) = 0\}P\{\text{n arrival in (t,t+dt)}\} \\
&= \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\lambda dt + o(dt)
\end{aligned}
$$

$$P\{N(h) = 1\} \quad = \quad \lambda \text{h} + o(h)$$

$$P[N(h) \geq 2] \quad = \quad o(h)$$

$$\therefore \quad \lim_{dt \to 0} \frac{P\{t < S_n < t + dt\}}{dt} = f_{S_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

Erlang

Order Statistics

Let $Y_1, ..., Y_n$ be n random variables defined on a common probability space.

We say that $Y_{(1)}, Y_{(2)}, ..., Y_{(n)}$ are the **order statistics** of these n random variables if $Y_{(k)}$ is the kth smallest value among $Y_{(1)}, Y_{(2)}, ..., Y_{(n)}$.

$$Y_{(1)} = min(Y_1, Y_2, ...Y_n)$$

$$Y_{(n)} = max(Y_1, Y_2, ...Y_n)$$

Conditional Distribution of the Arrival Times

Theorem

Given that $N(t) = n$, the n arrival times $S_1, S_2, \ldots, S_n$ have the same distribution as the order statistics corresponding to ***n* i.i.d. uniformly** distributed random variables from (0, t).

Order Statistics:

Let $Y_1, Y_2, \ldots, Y_n$ be n i.i.d. continuous RVs having common pdf f.
Define $Y_{(k)}$ as the k_{th} smallest value among all $Y_i' s$, i.e.,
$Y_{(1)} \le Y_{(2)} \le Y_{(3)} \le \ldots \le Y_{(n)}$ then $Y_{(1)}, \ldots, Y_{(n)}$ are order statistics corresponding to random variables $Y_1, \ldots, Y_n$

Joint pdf of $Y_{(1)}, Y_{(2)} \ldots, Y_{(n)}$ is

$$f(y_1, y_2, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i),$$

where $y_1 < y_2 < \ldots < y_n.$

Conditional Distribution of the Arrival Times

Proof.

Let $0 < t_1 < t_2 < \ldots < t_{n+1} = t$ and let h_i be small enough so that $t_i + h_i < t_{i+1}, i = 1, \ldots, n$.

$$\because \quad P\{t_i < S_i < t_i + h_i, i = 1, \ldots, n | N(t) = n\}$$

$$= \frac{P\left\{\substack{\text{exactly one arrival in each}[t_i, t_i + h_i] \\ i=1,2,\ldots,n,\text{and no arrival elsewhere in } [0,\mathrm{t}]}\right\}}{P\{N(t) = n\}}$$

$$= \frac{\lambda h_1 e^{-\lambda h_1} \ldots \lambda h_n e^{-\lambda h_n} e^{-\lambda(t - h_1 - h_2 \ldots - h_n)}}{e^{-\lambda t}(\lambda t)^n / n!}$$

$$= \frac{n!(h_1 \cdot h_2 \cdots h_n)}{t^n}$$

$$\therefore \frac{P\{t_i < S_i < t_i + h_i, i = 1, \ldots, n | N(t) = n\}}{h_1 \cdot h_2 \cdots h_n} = \frac{n!}{t^n}$$

Stationary

P{exact one in $[t_i, t_i + h_i]$}
= P{exact one in$[0, h_i]$}

Independent

P{exact one in $[t_1, t_1 + h_1]$, exact one in $[t_2, t_2 + h_2]$}
=P{exact one in $[t_1, t_1 + h_1]$}P{exact one in $[t_2, t_2 + h_2]$}

Conditional Distribution of the Arrival Times

Taking $\lim_{h_i \to 0, i=1,\ldots,n} (\quad)$, then

$$f(t_1, \ldots t_n) = \boxed{\frac{n!}{t^n}}, \quad 0 < t_1 < t_2 < \ldots < t_n.$$

$$\boxed{f(y_1, y_2, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i)}$$

$\Rightarrow$ f = 1/t (a uniform distribution)

Proposition 1

Given that $N(t) = 1$, the arrival time of $S_1 = X_1$ has a uniform distribution on $[0, t]$.

Proof.

For $s \in [0, t]$, we condition the arrival time S_1 on $N(t) = 1$

$$
\begin{aligned}
P\{S_1 < s | N(t) = 1\} &= \frac{P\{N(s) = 1, N(t) - N(s) = 0\}}{P\{N(t) = 1\}} \\
&= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\
&= \frac{s}{t}
\end{aligned}
$$

Conditional Distribution of the Arrival Times

Theorem

Given that $N(t) = n$, the arrival times $S_1, ..., S_n$ have the same conditional joint probability density as the order statistics $Y_{(1)}, Y_{(2)}, ..., Y_{(n)}$ of n independent random variables $Y_1, ..., Y_n$ uniformly distributed on the interval $[0, t]$.

Proof.

Our goal is to compute the conditional density of $S_1, ..., S_n$ given that $N(t) = n$.

Observing that for every $0 < t_1 < ... < t_n \leq t$ the event

$$P\{S_1 = t_1, S_2 = t_2, ..., S_n = t_2, N(t) = n\}$$
$$= P\{X_1 = t_1, X_2 = t_2 - t_1, ..., X_n = t_n - t_{n-1}, X_{n+1} > t - t_n\}$$

$$f_{(S_1,S_2,\dots,S_n)|N(t)=n}(t_1,t_2,\dots,t_n)$$

$$=\frac{P\{S_1=t_1,S_2=t_2,\dots,S_n=t_2,N(t)=n\}}{P\{N(t)=n\}}$$

$$=\frac{P\{X_1=t_1,X_2=t_2-t_1,\dots,X_n=t_n-t_{n-1},X_{n+1}>t-t_n\}}{P\{N(t)=n\}}$$

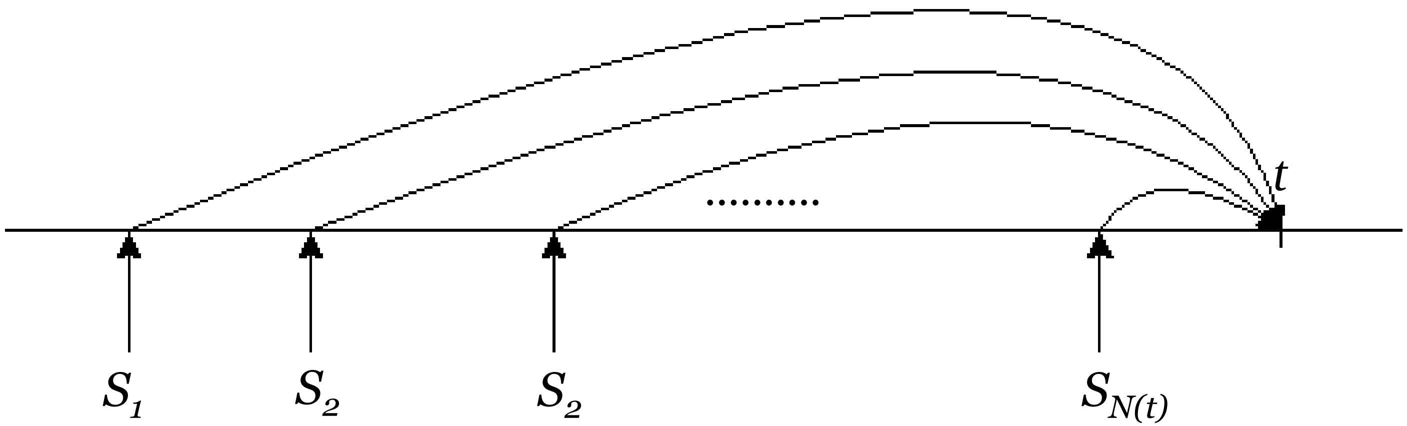
$$=\frac{\lambda e^{-\lambda t_1}\lambda e^{-\lambda(t_2-t_1)}\dots\lambda e^{-\lambda(t_n-t_{n-1})}e^{-\lambda(t-t_n)}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$=\frac{n!}{t^n}$$

Example (Ex. 2.3(A) p.68 [Ross])

Suppose that travelers arrive at a train depot in accordance with a Poisson process with rate λ. If the train departs at time t, what is the expected sum of the waiting times of travelers arriving in $(0,t)$?

$$\text{That is, } E\left[\sum_{i=1}^{N(t)} (t - S_i)\right] = ?$$



Solution

Conditioning $E[\sum_{i=1}^{N(t)}(t-S_i)]$ on $N(t)$ yields

$$E[\sum_{i=1}^{n}(t-S_i)|N(t)=n]$$

$$=nt-E[\sum_{i=1}^{n}S_i|N(t)=n]$$

Let $U_1, U_2, \ldots, U_n$ denote a set of n independent uniform (0, t) RV, then

$$\because E[\sum_{i=1}^{n}S_i|N(t)=n]=E[\sum_{i=1}^{n}U_{(i)}]$$

$$=E[\sum_{i=1}^{n}U_i]$$

$$=nt/2$$

$$\therefore E[\sum_{i=1}^{n}(t-S_i)|N(t)=n] = nt - \frac{nt}{2} = \frac{nt}{2}$$

$$\begin{aligned}
\Rightarrow E[\sum_{i=1}^{N(t)}(t-S_i)] &= \sum_{n=0}^{\infty} E[\sum_{i=1}^{n}(t-S_i)|N(t)=n]\,P\{N(t)=n\} \\
&= \sum_{n=0}^{\infty} \frac{nt}{2} P\{N(t)=n\} \\
&= \frac{t}{2} E[N(t)] \\
&= \frac{\lambda t^2}{2}
\end{aligned}$$

#

9. Suppose that people arrive at a bus stop according to a Poisson process with rate λ. The bus leaves at time t. Let X denote the total amount of waiting time of all those who get on at time t. Let $N(t)$ denote the number of arrivals by time t.

(a) What is $E[X|N(t)]$?

(b) Show that $\mathrm{Var}(X|N(t)) = N(t)t^2/12$

(c) What is $\mathrm{Var}(X)$?

solution

9. (a) Each arrival distributed uniformly on $(0,t)$, so

$$E[X|N(t)] = N(t)\int_0^t (t-s)\frac{ds}{t} = N(t)t/2.$$

(b) Let $(U_i)_{i=1}^n$ be i.i.d. uniformly distributed on $(0,t)$. Then

$$\mathrm{Var}(X|N(t)=n) = n\mathrm{Var}(U_i) = \frac{nt^2}{12}.$$

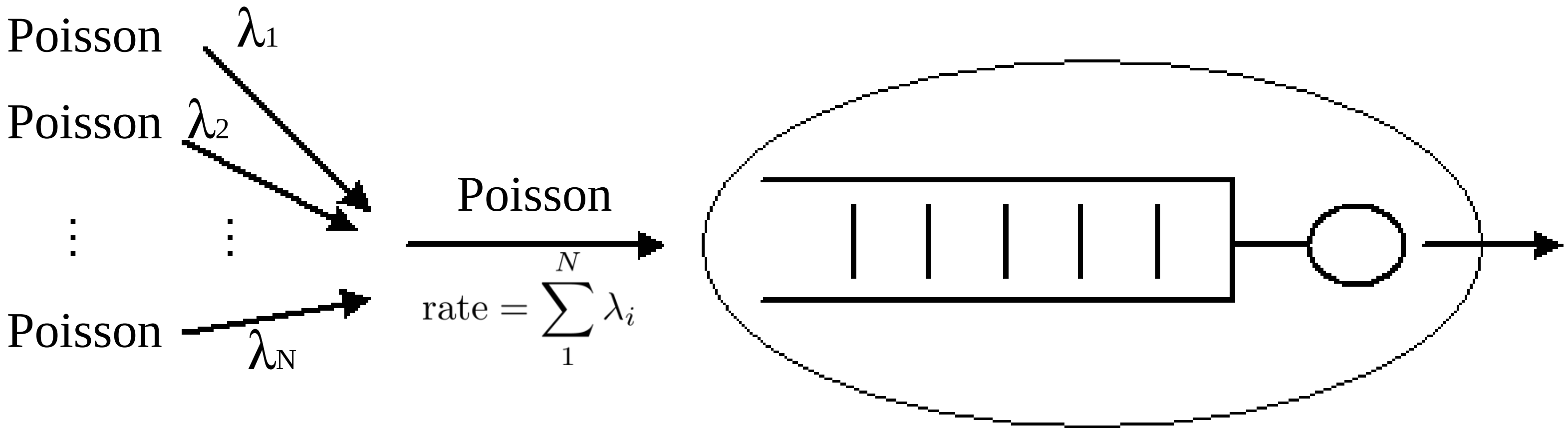
(c)

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - E[X]^2 \\
&= E[E[X^2|N(t)]] - E[X]^2 \\
&= E[E[(X - E[X|N(t)])^2|N(t)]] + E[(E[X|N(t)]^2] - E[X^2] \\
&= E[\mathrm{Var}(X|N(t))] + E[(E[X|N(t)])^2] - E[X^2] \\
&= \frac{t^2}{12}E[N(t)] + \frac{t^2}{4}E[N(t)^2] - \frac{t^2}{4}E[N(t)]^2 \\
&= \frac{\lambda t^3}{12} + \frac{\lambda t^3}{4} \\
&= \frac{\lambda t^3}{3}.
\end{aligned}$$

Superposition of Independent Poisson Processes

Theorem. Superposition of independent Poisson Processes

(λ_i, $i = 1, \ldots, N$), is also a Poisson process with rate $\sum_1^N \lambda_i$



<Homework> Prove the superposition theorem.

(note that a Poisson process must satisfy Definitions 1 or 2)

Decomposition of a Poisson Process

Theorem

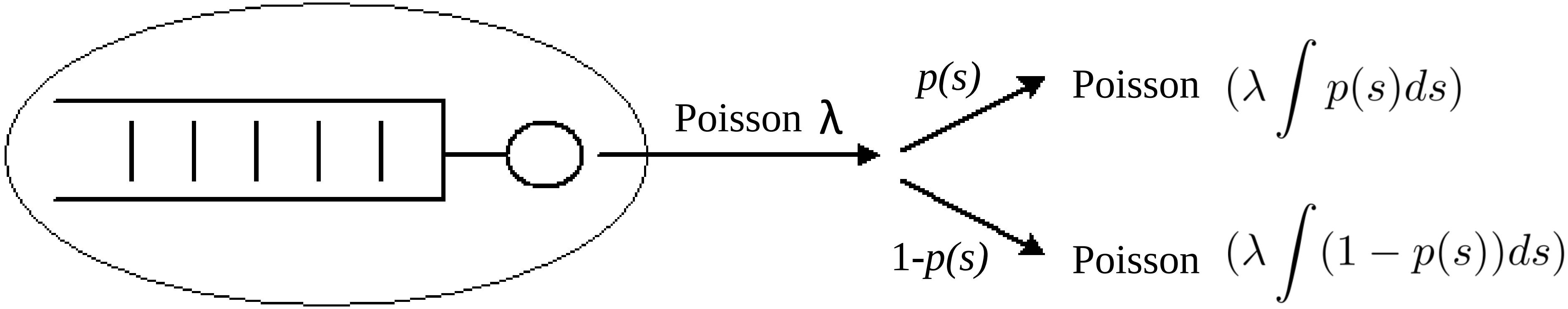
$P(\lambda t)$

- Given a Poisson process $\{N(t), t \geq 0\}$;
- If $N_i(t)$ represents the number of type-i events that occur by time t, $i = 1, 2$;
- Arrival occurring at time s is a **type-1** arrival with probability $p(s)$, and **type-2** arrival with probability $1 - p(s)$

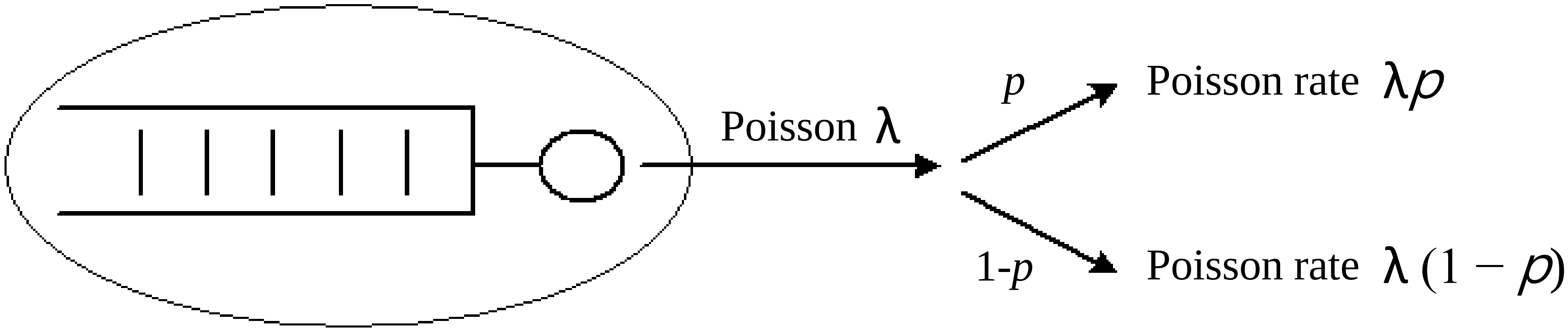
$\Downarrow$ then

- $N_1,\ N_2$ are independent,
- $N_1(t) \sim P(\lambda t p),$ and
- $N_2(t) \sim P(\lambda t(1-p))$, where $p = \dfrac{1}{t}\displaystyle\int_0^t p(s)ds$

Decomposition of a Poisson Process



special case: If $p(s) = p$ is constant, then



Proof

$$P\{N_1(t) = n, N_2(t) = m\}$$
$$= P\{N_1(t) = n, N_2(t) = m | N(t) = n + m\} P\{N(t) = n + m\}$$

Let $f_S(s)$ denote the pdf of arrival time

$$f_S(s|N(t) = 1) = \frac{1}{t}, 0 < s < t$$

(proof is n next slide)

$$P_r\{\text{type} - 1 | N(t) = 1\} = \int_0^t P_r\{\text{type} - 1 | S = s, N(t) = 1\} f_S(s|N(t) = 1) ds$$
$$= \int_0^t p(s) \frac{1}{t} ds$$

Since the arrival time of the n+m events are independent (each is uniform (0,t])

$\therefore \; P\{N_1(t) = n, N_2(t) = m | N(t) = n + m\}$ just equal to the probability

of n successes and m failures independent tials with success probability p

$$\Rightarrow P\{N_1(t)=n, N_2(t)=m | N(t)=n+m\} = \binom{n+m}{m} p^n (1-p)^m$$

$$\begin{aligned}
\Rightarrow & P\{N_1(t)=n, N_2(t)=m\} \\
& = P\{N_1(t)=n, N_2(t)=m | N(t)=n+m\} P\{N(t)=n+m\} \\
& = \frac{(n+m)!}{n!m!} p^n (1-p)^m \, e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!} \\
& = \boxed{e^{-\lambda t p} \frac{(\lambda t p)^n}{n!}} \; \boxed{e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^m}{m!}}
\end{aligned}$$

$N_1(t) \sim P(\lambda t p)$ $\qquad$ $N_2(t) \sim P(\lambda t(1-p))$

$$= P\{N_1(t)=n\} P\{N_2(t)=m\}$$

$\Rightarrow$ $N_1(t)$ and $N_2(t)$ are independent Poisson processes!

#

Proof of $f_S(s|N(t)=1) = \frac{1}{t}, 0 < s < t$

Let $f_S(s)$ denote the pdf of arrival time.

$$
\begin{aligned}
P\{S_1 < s|N(t)=1\} &= \frac{P\{X1 < s, N(t)=1\}}{P\{N(t)=1\}} \\
&= \frac{\text{one event in } [0,s], \text{ 0 event in } [s,t)}{P\{N(t)=1\}} \\
&= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{P\{N(t)=1\}} \\
&= \frac{s}{t}
\end{aligned}
$$

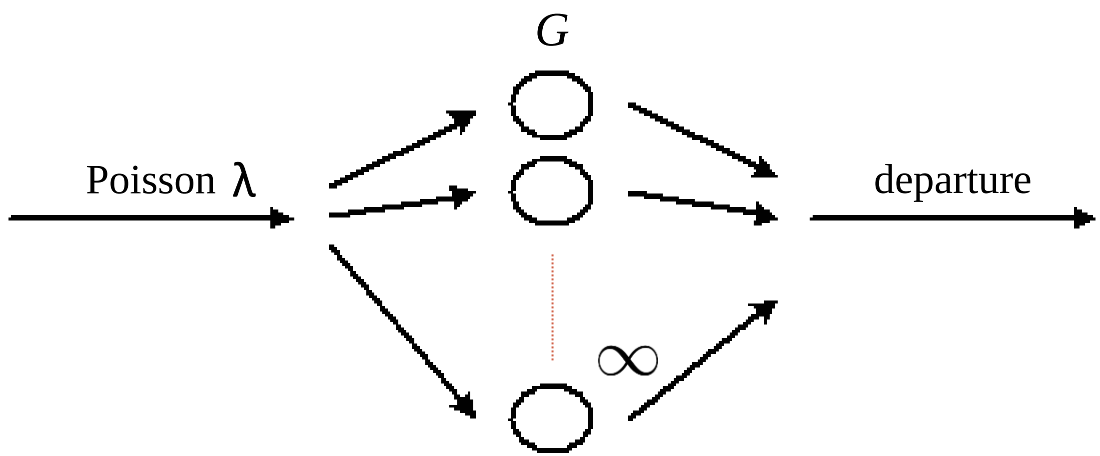
$$
\Rightarrow f_S(s|N(t)=1) = \frac{1}{t}
$$

Example (Infinite Server Queue, textbook [Ross])

Suppose that customers arrive at a service station in accordance with a Poisson process with rate λ. Upon arrival the customer is immediately served by one of an infinite number of possible servers, and the service times are assumed to be independent with a common distribution *G*.

type-1 customer: if it is complete its service by time *t*

type-2 customer: if it does not complete its service by time *t*



- $G_s(t) = P\{S \leq t\}$,where S is the service time
- $G_s(t)$ is independent of each other and of the arrival process
- $N_1(t)$ denotes the number of customers that have left before t
- $N_2(t)$ denotes the number of customers which are still in the system at time t

Question 1: What is $P\{N_1(t) = j\}$?

Question 2: What is $E[N_1(t)]$ and $E[N_2(t)]$?

Solution

Any customer arrives at time s, $s \leq t$, then it is a type-1 customer if its service time is less than $t - s$ and the probability is

$$p(s) = G(t-s), s \leq t.$$

The probability that a customer has left before time t equals

$$\begin{aligned}
p &= P\{\text{type-1} | N(t) = 1\} \\
&= \int_0^t P\{\text{type-1} | S = s, N(t) = 1\} \underbrace{f_s(s | N(t) = 1)}_{} ds \\
&= \int_0^t G(t-s) \frac{1}{t} ds \\
&= \frac{1}{t} \int_0^t G(s) ds
\end{aligned}$$

Theorem: Conditional Distribution of the Arrival Times => Uniform , iid

Theorem: Conditional Distribution of the Arrival Times

Again, condition on $N(t)=n$, the probability of a customer that has left before t is independent of the ohers. Therefore,

$$P\{N_1(t)=j|N(t)=n\}=\begin{cases}\binom{n}{j}p^j(1-p)^{n-j}, & j=1,2,\ldots n\\ 0, & j>n\end{cases}$$

Finally, for $j=0,1,2\ldots$

$$P\{N_1(t)=j\}=\sum_{n=j}^{\infty}P\{N_1(t)=j|N(t)=n\}p\{N(t)=n\}$$

$$=\sum_{n=j}^{\infty}\binom{n}{j}p^j(1-p)^{n-j}e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

$$=e^{-\lambda t}\frac{(\lambda tp)^j}{j!}\sum_{n=j}^{\infty}\frac{(\lambda t(1-p))^{n-j}}{(n-j)!}$$

$$=e^{-\lambda t}\frac{(\lambda tp)^j}{j!}e^{-\lambda t(1-p)}$$

$$=e^{-\lambda tp}\frac{(\lambda tp)^j}{j!}$$

Poisson(λtp)

$$\Rightarrow \quad E[N_1(t)] = \lambda t \int_0^t G(t-s)\frac{1}{t}ds \quad = \lambda \int_0^t G(t-s)ds$$

$$\Rightarrow \quad E[N_2(t)] = \lambda t \int_0^t (1-G(t-s))\frac{1}{t}ds \quad = \lambda \int_0^t (1-G(t-s))ds$$

#

Non-homogeneous Poisson Processes (Relaxing Stationary Increment)

- The counting process $N(t), t \geq 0$ is said to be a non-stationary or non-homogeneous Poisson Process with time-varying intensity function $\lambda(t), t \geq 0$, if

  1. $N(0) = 0$
  2. $N(t), t \geq 0$ has independent increments
  3. $P\{N(t+h) - N(t) \geq 2\} = o(h)$
  4. $P\{N(t+h) - N(t) = 1\} = \lambda(t) \cdot h + o(h)$

- Define "integrated intensity function" as $m(t) = \int_0^t \lambda(s) ds.$

Theorem.

$$P\{N(t+s) - N(t) = n\} = \frac{e^{-[m(t+s)-m(t)]}[m(t+s) - m(t)]^n}{n!}$$

Proof. < Homework >.

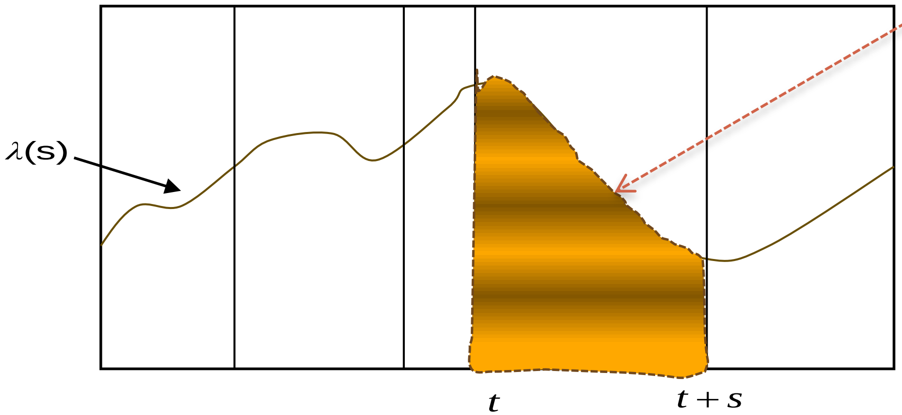
Non-homogeneous Poisson Processes

Poisson

$$e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

- $N(t+s)$ - $N(t)$ is Poisson with mean $m(t+s)$ - $m(t)$

$$P\{N(t+s)-N(t)=n\}=\frac{e^{-[m(t+s)-m(t)]}[m(t+s)-m(t)]^n}{n!}$$



$$m(t)=\int_0^t \lambda(s)ds.$$

- Non- homogeneous Poisson process allows for the arrival rate to be a function of time λ(t) instead of a constant λ.

- It is useful when the rate of events varies.
  - Ex: when observing customers entering a restaurant, the numbers will much greater during meal times than during off hours.

Example (Infinite Server Queue 2.4

Poisson Arrival
General service distribution
Infinite server queue

The "output process" of the $M/G/\infty$ queue is a non-homogeneous Poisson process having intensity function $\lambda(t) = \lambda G(t)$, where G is the service distribution.

Hint: Let $D(s, s+t)$ denote the number of service completions (departures) in the interval (*s*, *s* +*t*).

To prove the above statement, we shall first show

(1) $D(s, s+t)$ follows a Poisson distribution with mean = $\lambda \int_s^{s+t} G(y)dy$

(2) The numbers of service completions (departures) in disjoint intervals are independent, i.e., let $s < s+t < s' < s'+t$, $D(s, s+t)$ is independent of $D(s', s'+t')$

Proof (1)

Let an arrival event is type-1 if is departs in interval (*s*, *s*+*t*),
An arrival at *y* will be type-1 with probability

$$P(y) = \begin{cases} G(s+t-y) - G(s-y), & \text{if } y < s \\ G(s+t-y), & \text{if } s < y < s+t \\ 0, & \text{if } y > s+t \end{cases}$$

According to the theorem of **Decomposition of a Poisson Process**, the arrival rate of type-1 event is still Poisson with $N_1(t) \sim P(\lambda t p)$, where

$$\begin{aligned} p &= P\{\text{type-1} | N(t) = 1\} \\ &= \int_0^t P\{\text{type-1} | S = y, N(t) = 1\} f_s(y | N(t) = 1) dy \\ &= \int_0^5 P(y) f_s(y | N(t) = 1) dy \\ &= \frac{1}{t} \int_0^t P(y) dy \end{aligned}$$

$$
\begin{aligned}
E[D(s, s+t)] = \lambda t p &= \lambda \int_0^{\infty} P(y)dy \\
&= \lambda \int_0^{s} (G(s+t-y) - G(s-y)dy + \\
&\quad \lambda \int_s^{s+t} G(s+t-y)dy \\
&= \lambda \int_s^{s+t} G(y)dy
\end{aligned}
$$

Therefore, the "output process" of the $M/G/\infty$ queue is Poisson distribution with parameter $\lambda \int_s^{s+t} G(y)dy$, where G is the service distribution.

Proof (2)

Let I_1, I_2 denote disjoint time intervals.

An arrival is **type-1** if it departs at I_1
An arrival is **type-2** if it departs in I_2
Otherwise, it is **type-3**

According to the theorem of "**Decomposition of a Poisson Process**", type-1 and type-2 are independent Poisson random variables.

Note : The output process of the $M/G/\infty$ queue is not a general Poisson since it doesn't have the stationay property.

It is "Non-homogeneous Poisson Processes."

Example

Suppose cars enter a one-way infinite highway at a Poisson rate λ. The ith car to enter chooses a velocity V_i and travels at this velocity. Assume that the V_is are independent positive random variables having a common distribution F. Find the distribution of the number of cars that are located in the interval (a, b) at time t. Assume that no time is lost when one car overtakes another car.