Lecture-07: Delta-Wye Transformation

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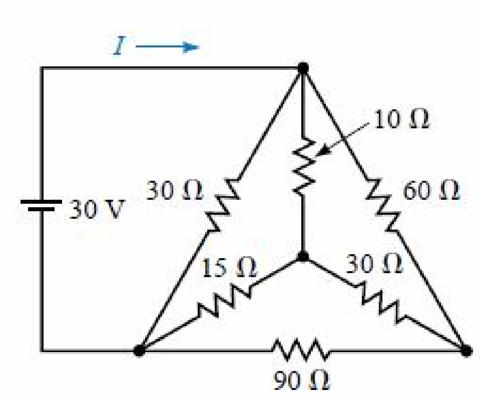
Delta-Wye (Pi-Tee) Conversion:

- Previously we have examined resistive circuits involving series, parallel, and series-parallel combinations.
- There are some networks which cannot be placed into any of the above categories. For example, consider the circuit shown below:



- The above circuit may be analyzed using techniques such as mesh analysis, nodal analysis etc, but they are very time-consuming and prone to error. For example, mesh analysis would involve solving four simultaneous linear equations, since there are four separate loops in the circuit. If we were to use nodal analysis, the solution would require determining three node voltages, since there are three nodes in addition to a reference node.
- Rather than solving the above circuit using the techniques mentioned, it is occasionally easier to examine the circuit after it has been converted to some equivalent form.
- Two such widely used forms of circuit are:
    - Delta (or pi) network
    - Wye (or Tee or Star) network

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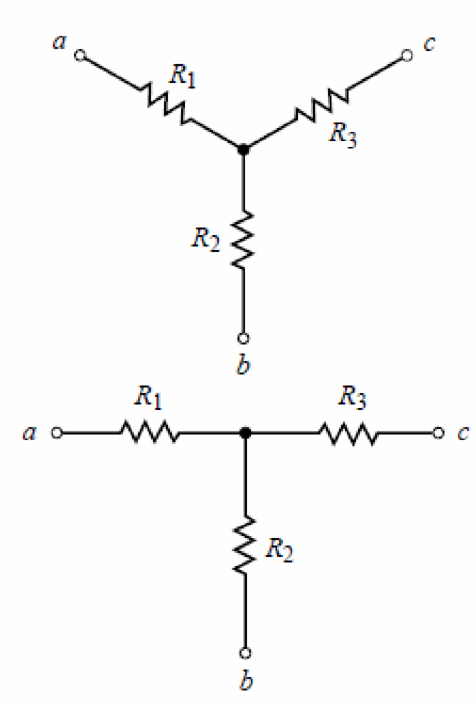

Figure A: Wye (Y) or Tee (T) network

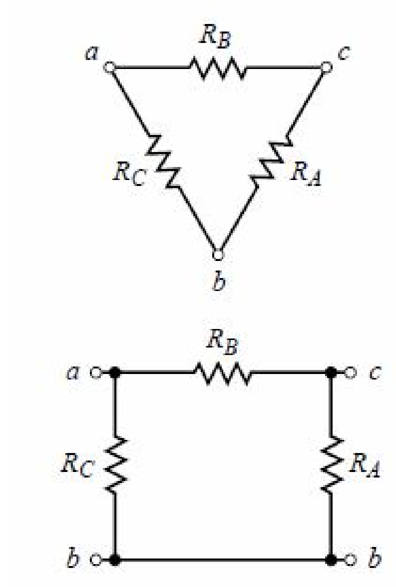

Figure B: Delta (Δ) or Pi (п) network

Converting a Delta Network to its Equivalent Wye Network (Δ to Y):

- Consider the Delta and Wye circuits shown in the figure above.
- We start by making the assumption that the network shown in figure A is equivalent to that shown in figure B.
- Then, using this assumption, we will determine the mathematical relationships between the various resistors in the equivalent circuits.
- In figure A, three resistances R_1, R_2 and R_3 are connected in Y or star fashion between terminals a, b and c.
- In figure B, three resistances R_A, R_B and R_C are connected in delta fashion between the same terminals.
- The circuit of figure A can be equivalent to the circuit of figure B only if the resistance "seen" between any two terminals is exactly the same.
- At first, consider the Y connection. If we were to connect a source between terminals ***a*** and ***b*** of the "Y," the resistance between the terminals would be:

$$R_{ab} = R_1 + R_2 \qquad (1)$$

- Now, consider the delta connection. The resistance between terminals ***a*** and ***b*** of the "Δ" is

$$R_{ab} = R_C \parallel (R_A + R_B) \qquad (2)$$

- As terminal resistances in both connections have to be the same, combining equations (1) and (2), we get

$$R_1 + R_2 = \frac{R_C(R_A + R_B)}{R_A + R_B + R_C}$$

$$\boxed{\Rightarrow \quad R_1 + R_2 = \frac{R_A R_C + R_B R_C}{R_A + R_B + R_C} \qquad (3)}$$

- Similarly, considering the resistances between terminal b and c of both network, we get

$$\boxed{R_2 + R_3 = \frac{R_A R_B + R_A R_C}{R_A + R_B + R_C} \qquad (4)}$$

- And, in the same way, considering the resistances between terminal c and a of both network, we get

$$\boxed{R_1 + R_3 = \frac{R_A R_B + R_B R_C}{R_A + R_B + R_C} \qquad (5)}$$

- Subtract equation (4) from equation (3):

$$R_1 + R_2 = \frac{R_A R_C + R_B R_C}{R_A + R_B + R_C} \qquad (3)$$

$$R_2 + R_3 = \frac{R_A R_B + R_A R_C}{R_A + R_B + R_C} \qquad (4)$$

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$$R_1 - R_3 = \frac{R_A R_C + R_B R_C}{R_A + R_B + R_C} - \frac{R_A R_B + R_A R_C}{R_A + R_B + R_C}$$
$$= \frac{R_A R_C + R_B R_C - R_A R_B - R_A R_C}{R_A + R_B + R_C}$$

$$= \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C}$$

$$R_1 - R_3 = \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C} \qquad (6)$$

- Add equation (5) and (6):

$$R_1 + R_3 = \frac{R_A R_B + R_B R_C}{R_A + R_B + R_C} \qquad (5)$$

$$R_1 - R_3 = \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C} \qquad (6)$$

$$2R_1 = \frac{R_A R_B + R_B R_C}{R_A + R_B + R_C} + \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C}$$
$$= \frac{R_A R_B + R_B R_C + R_B R_C - R_A R_B}{R_A + R_B + R_C}$$
$$= \frac{2 R_B R_C}{R_A + R_B + R_C}$$

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \qquad (7)$$

- Using similar approach we obtain,

$$R_2 = \frac{R_C R_A}{R_A + R_B + R_C} \qquad (8)$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} \qquad (9)$$

Rewriting equations (7), (8) and (9) we get

$$\boxed{R_1 = \frac{R_B R_C}{R_A + R_B + R_C}}$$

$$\boxed{R_2 = \frac{R_C R_A}{R_A + R_B + R_C}}$$

$$\boxed{R_3 = \frac{R_A R_B}{R_A + R_B + R_C}}$$

- Notice that any resistor connected to a point of the "Y" is obtained by finding the product of the resistors connected to the same point in the "Δ" and then dividing by the sum of all the "Δ" resistances.
- If all the resistors in a Δ circuit have the same value, R_Δ, then the resulting resistors in the equivalent "Y" network will also be equal and have a value given as:

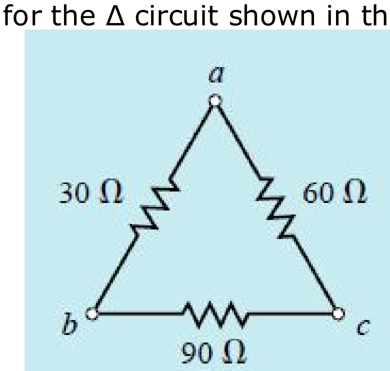
$$R_Y = \frac{R_\Delta R_\Delta}{R_\Delta + R_\Delta + R_\Delta}$$

$$\Rightarrow \quad = \frac{R_\Delta R_\Delta}{3R_\Delta}$$

$$\boxed{R_Y = \frac{R_\Delta}{3}}$$

Example-1:

Find the equivalent Y circuit for the Δ circuit shown in the figure below.



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Solution:

Given that,

R_A=Resistance between terminal b and c of Δ circuit=90Ω

R_B=Resistance between terminal c and a of Δ circuit=60Ω

R_C=Resistance between terminal a and b of Δ circuit=30Ω

Equivalent Y resistance can be calculated using the following relations:

$$\boxed{R_1 = \frac{R_B R_C}{R_A + R_B + R_C}}$$

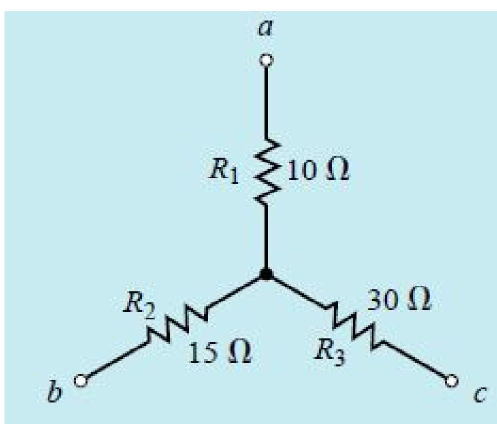
$$\boxed{R_2 = \frac{R_C R_A}{R_A + R_B + R_C}}$$

$$\boxed{R_3 = \frac{R_A R_B}{R_A + R_B + R_C}}$$

$$\begin{aligned} R_1 &= \frac{R_B R_C}{R_A + R_B + R_C} \\ &= \frac{(60\Omega)(30\Omega)}{90\Omega + 60\Omega + 30\Omega} \\ &= \frac{1800}{180}\Omega \\ &= 10\Omega \end{aligned}$$

Similarly, R_2=15Ω, R_3=30Ω

Therefore, the resulting Y circuit is:



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Converting a Wye Network to its Equivalent Delta Network (Y to Δ):

- By using equations (7), (8) and (9) above, it is possible to derive another set of equations which allow the conversion from a "Y" into an equivalent "Δ."

Re-writing equations (7), (8) and (9) we get

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \qquad (7)$$

$$R_2 = \frac{R_C R_A}{R_A + R_B + R_C} \qquad (8)$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} \qquad (9)$$

From equation (7) we get

$$\boxed{R_A + R_B + R_C = \frac{R_B R_C}{R_1} \qquad (10)}$$

Similarly, from equation (8) we get

$$\boxed{R_A + R_B + R_C = \frac{R_C R_A}{R_2} \qquad (11)}$$

And, in the same way, from equation (9) we get

$$\boxed{R_A + R_B + R_C = \frac{R_A R_B}{R_3} \qquad (12)}$$

From equations (10), (11) and (12) we get

$$\frac{R_B R_C}{R_1} = \frac{R_C R_A}{R_2} = \frac{R_A R_B}{R_3} \qquad (13)$$

Using equation (13) we get

$$\frac{R_B R_C}{R_1} = \frac{R_C R_A}{R_2} \qquad (14)$$

$$\frac{R_B R_C}{R_1} = \frac{R_A R_B}{R_3} \qquad (15)$$

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From equation (14) we get

$$\frac{R_B R_C}{R_1} = \frac{R_C R_A}{R_2} \qquad (14)$$

$$\Rightarrow \quad \frac{R_B}{R_1} = \frac{R_A}{R_2}$$

$$\Rightarrow \quad R_B = \frac{R_A R_1}{R_2} \qquad (16)$$

Similarly, from equation (15) we get

$$\frac{R_B R_C}{R_1} = \frac{R_A R_B}{R_3} \qquad (15)$$

$$\Rightarrow \quad \frac{R_C}{R_1} = \frac{R_A}{R_3}$$

$$\Rightarrow \quad R_C = \frac{R_A R_1}{R_3} \qquad (17)$$

Putting the value of R_B and R_C from equations (16) and (17) to equation (7) we get:

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \qquad (7)$$

$$\Rightarrow \quad = \frac{\left(\frac{R_A R_1}{R_2}\right)\left(\frac{R_A R_1}{R_3}\right)}{R_A + \frac{R_A R_1}{R_2} + \frac{R_A R_1}{R_3}}$$

$$\Rightarrow \quad = \frac{R_A\left[\left(\frac{R_A R_1 R_1}{R_2 R_3}\right)\right]}{R_A\left[1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right]}$$

$$\Rightarrow \quad = \frac{\frac{R_A R_1 R_1}{R_2 R_3}}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}}$$

$$\Rightarrow \quad = \frac{\frac{R_A R_1 R_1}{R_2 R_3}}{\frac{R_2 R_3 + R_1 R_3 + R_1 R_2}{R_2 R_3}}$$

$$\Rightarrow \quad R_1 = \frac{R_A R_1 R_1}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

$$\Rightarrow \quad 1 = \frac{R_A R_1}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

$$\Rightarrow R_A R_1 = R_1 R_2 + R_2 R_3 + R_3 R_1$$

$$\boxed{\Rightarrow \quad R_A = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1} \qquad (18)}$$

Similarly, putting the value of R_B and R_C from equations (16) and (17) to equation (8) we get:

$$\boxed{\Rightarrow \quad R_B = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} \qquad (19)}$$

Similarly, putting the value of R_B and R_C from equations (16) and (17) to equation (9) we get:

$$\boxed{\Rightarrow \quad R_C = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} \qquad (20)}$$

Re-writing equations (18), (19) and (20), we get:

$$\boxed{\begin{aligned} R_A &= \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1} \\ R_B &= \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} \\ R_C &= \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} \end{aligned}}$$

- In general, we see that the resistor in any side of a "Δ" is found by taking the sum of all two-product combinations of "Y" resistor values and then dividing by the resistance in the "Y" which is located directly opposite to the resistor being calculated. For example, when calculating R_A, which is between terminal ***a*** and

c in Delta network, resistor R_1 is opposite to terminal ***a*** and ***c*** in the Wye network. So, we divide by R_1.

- If the resistors in a Y network are all equal, then the resultant resistors in the equivalent Δ circuit will also be equal and given as:

$$R_C = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

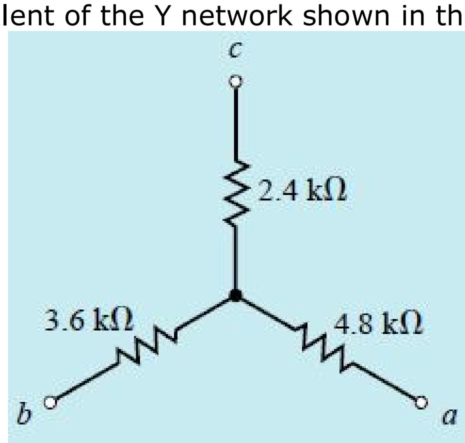
$$\Rightarrow R_\Delta = \frac{R_YR_Y + R_YR_Y + R_YR_Y}{R_Y}$$

$$\Rightarrow \quad = \frac{3R_YR_Y}{R_Y}$$

$$\boxed{\Rightarrow \quad R_\Delta = 3R_Y}$$

Example-2:

Find the Δ network equivalent of the Y network shown in the figure below.



Solution:

Given that

R_1=Resistance at terminal ***a*** of Y circuit=4.8kΩ

R_2= Resistance at terminal ***b*** of Y circuit =3.6kΩ

R_3= Resistance at terminal ***c*** of Y circuit =2.4kΩ

Equivalent Δ resistance can be calculated using the following relations:

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$$R_A = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$
$$R_B = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$
$$R_C = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

$$\begin{aligned} R_A &= \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \\ &= \frac{(4.8\Omega)(3.6\Omega) + (3.6\Omega)(2.4\Omega) + (2.4\Omega)(4.8\Omega)}{4.8\Omega} \\ &= 7.8k\Omega \end{aligned}$$

Similarly, R_B=10.4Ω, R_C=15.6Ω

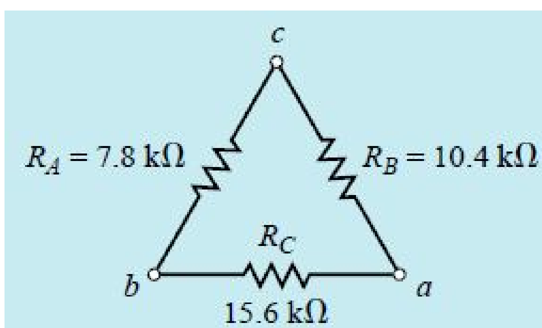
Therefore,

R_A=Resistance between terminal b and c of Δ circuit=7.8kΩ

R_B=Resistance between terminal c and a of Δ circuit=10.4kΩ

R_C=Resistance between terminal a and b of Δ circuit=15kΩ

Therefore, the resulting Δ circuit is:



Example-3:

Find the total resistance, R_T, and the total current, I for the circuit shown in the figure below.

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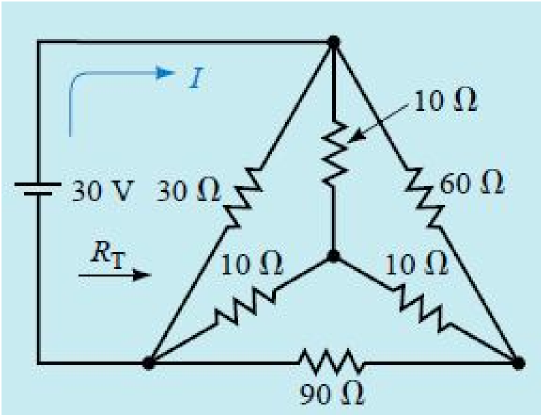


Solution:

The given circuit may be solved in one of two ways:

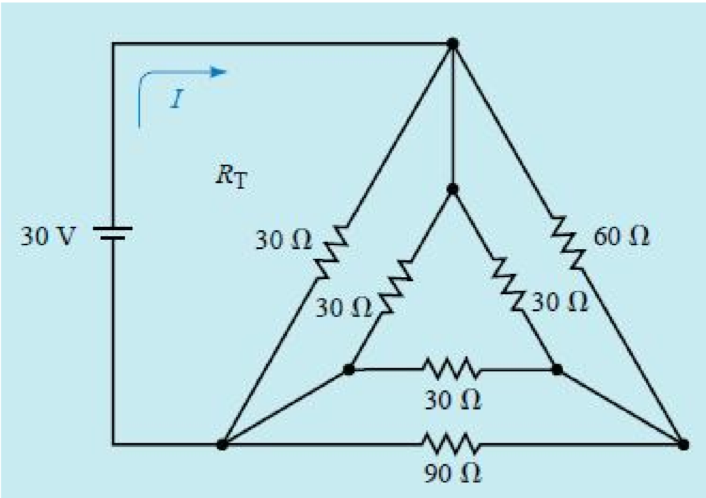
- We may convert the "Δ" into its equivalent "Y," and solve the circuit by placing the resultant branches in parallel, or
- We may convert the "Y" into its equivalent "Δ."

We choose to use the conversion from "Y" to "Δ" since the resistors in the "Y" have the same value, each of 10Ω.

Then the resultant resistors in the equivalent Δ circuit will also be equal and given as:

$$\Rightarrow \quad R_\Delta = 3R_Y$$
$$= 3 \times 10\Omega$$
$$= 30\Omega$$

The resulting circuit is now shown in the figure below.



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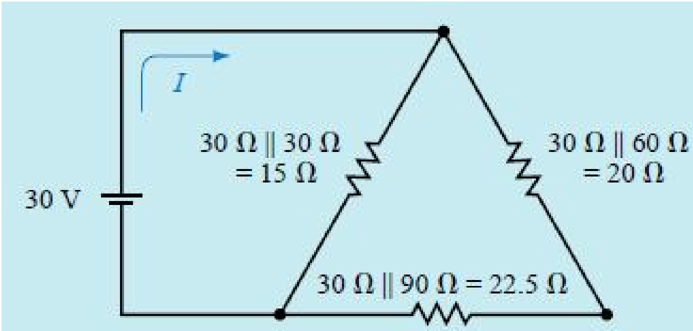
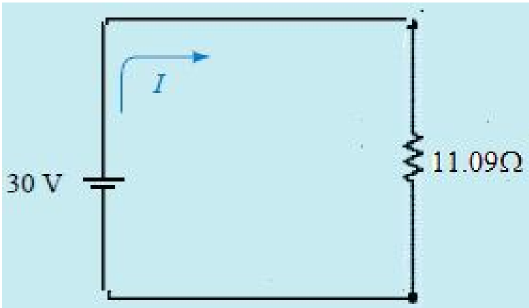
From the above circuit, we see that the sides of the resulting "Δ" are in parallel, which allows us to simplify the circuit even further as shown in the figure below.



The total resistance of the circuit is now easily determined as

$$\begin{aligned} R_T &= 15\Omega \parallel (20\Omega + 22.5\Omega) \\ &= 15\Omega \parallel 42.5\Omega \\ &= 11.09\Omega \end{aligned}$$

Now the above circuit becomes



This results in a circuit current of

$$\begin{aligned} I &= \frac{V}{R_T} \\ &= \frac{30V}{11.09\Omega} \\ &= 2.706A \end{aligned}$$