Lecture-06: DC Network Theorem: Maximum Power Transfer & Reciprocity Theorems

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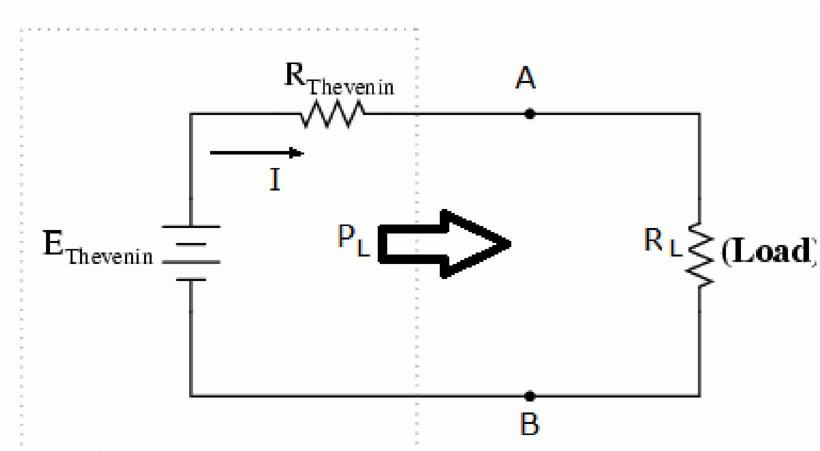
Introduction:

- In today's lecture, we will explain two theorems:
  5. Maximum Power Transfer theorem
  6. Reciprocity theorem

5. Maximum Power Transfer Theorem:

- When designing a circuit, it is often important to be able to answer one of the following questions:
  - What load should be applied to a system to ensure that the load is receiving maximum power from the system? Or,
  - For a particular load, what conditions should be imposed on the source to ensure that it will deliver the maximum power available?
- At times it is desired to obtain the maximum power transfer from an active network to an external load resistor R_L.
- Assume that the network is linear. So, it can be reduced to a Thevenin's equivalent circuit with load resistor connected across terminals *A* and *B* to this network as shown in the figure below:



- Power absorbed by the load will vary according to the resistance R_{Th} of the network. Then, for the load resistance, to absorb the maximum power possible, it has to be "Matched" to the resistance of the network. This forms the basis of **Maximum Power Transfer**.

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- The process of finding the load that will receive maximum power from a particular system is quite straightforward due to the maximum power transfer theorem.
- As applied to DC networks, this theorem may be stated as follows:
  - A resistive load will receive maximum power from a network when the load resistance is equal to the Thevenin resistance of the network applied to the load. That is, when

$$\boxed{R_L = R_{Th}}$$

  - But, if the load resistance is lower or higher in value than the Thevenin resistance of the network, its dissipated power will be less than the maximum.
- With $R_L = R_{Th}$, the maximum power delivered to the load can be determined as follows:

Circuit current, $I_L = \dfrac{E_{Th}}{R_{Th} + R_L}$

Power consumed by the load, $P_L = I_L^2 R_L = \left(\dfrac{E_{Th}}{R_{Th} + R_L}\right)^2 (R_L)$

$$P_L = \frac{E_{Th}^2 R_L}{(R_{Th} + R_L)^2} \qquad \text{........(1)}$$

- For P_L to be maximum, $\dfrac{dP_L}{dR_L}$ must be equal to zero, that is,

$$\boxed{\frac{dP_L}{dR_L} = 0}$$

- Therefore, differentiating equation (1), we have,

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$$\frac{dP_L}{dR_L} = E_{Th}^2\left[\frac{1}{(R_{Th}+R_L)^2} + R_L\left(\frac{-2}{(R_{Th}+R_L)^3}\right)\right] = E_{Th}^2\left[\frac{1}{(R_{Th}+R_L)^2} - \frac{2R_L}{(R_{Th}+R_L)^3}\right]$$

$$\Rightarrow \quad 0 = E_{Th}^2\left[\frac{1}{(R_{Th}+R_L)^2} - \frac{2R_L}{(R_{Th}+R_L)^3}\right]$$

$$\Rightarrow \quad E_{Th}^2\left[\frac{1}{(R_{Th}+R_L)^2} - \frac{2R_L}{(R_{Th}+R_L)^3}\right] = 0$$

$$\Rightarrow \quad \frac{1}{(R_{Th}+R_L)^2} - \frac{2R_L}{(R_{Th}+R_L)^3} = 0$$

$$\Rightarrow \quad \frac{1}{(R_{Th}+R_L)^2} = \frac{2R_L}{(R_{Th}+R_L)^3}$$

$$\Rightarrow 1 = \frac{2R_L}{(R_{Th}+R_L)}$$

$$\Rightarrow 2R_L = R_{Th} + R_L$$

$$\Rightarrow R_L = R_{Th} \qquad (\text{Proved})$$

- Now, using $R_{Th}=R_L$ in equation (1), we have the equation of maximum power:

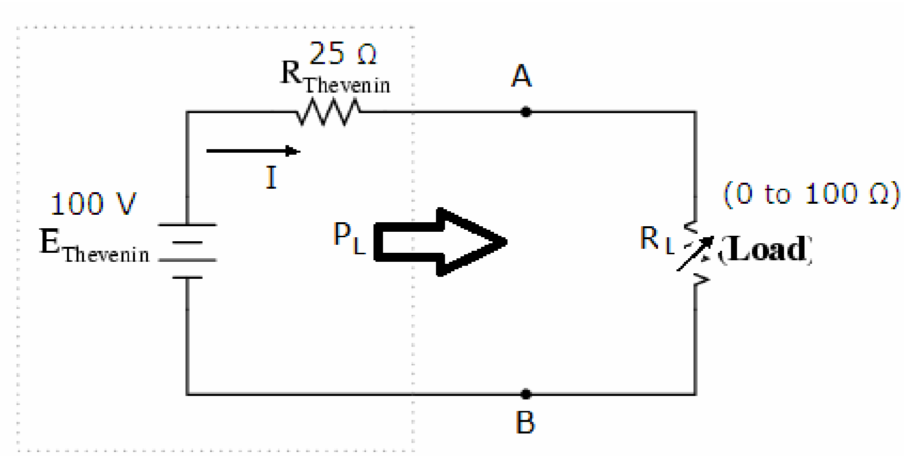
$$P_L = \frac{E_{Th}^2 R_L}{(R_{Th}+R_L)^2} = \frac{E_{Th}^2 R_{Th}}{(R_{Th}+R_{Th})^2} = \frac{E_{Th}^2 R_{Th}}{(2R_{Th})^2} = \frac{E_{Th}^2 R_{Th}}{4{R_{Th}}^2} = \frac{E_{Th}^2}{4R_{Th}}$$

$$\boxed{P_{L_{\max}} = \frac{E_{Th}^2}{4R_{Th}}}$$

- Therefore, **Maximum Power Transfer Theorem** is a useful analysis method to ensure that the maximum amount of power will be dissipated in the load resistance when the value of the load resistance is exactly equal to the resistance of the network.

Example-1:

- Consider the following Thevenin equivalent circuit with E_{Th}=100 V and R_{Th}=25 Ω. Determine the circuit current and power dissipated by the load for different values of load resistance (e.g. 0 to 100 Ω).
    - Demonstrate that *maximum power transfer occurs in the load* when the load resistance, R_L is equal in value to the Thevenin equivalent resistance, R_{Th}.
    - And also prove that *maximum power transfer occurs* when the load voltage and circuit current are one-half of their maximum possible values.



Solution:

Circuit current ***I***, power absorbed by the load ***P_L*** and load voltage ***V_L*** can be determined using the following formulas:

$$\text{Circuit current, } I = \frac{E_{Th}}{R_{Th} + R_L}$$

$$\text{Power consumed by the load, } P_L = I^2 R_L$$

$$\text{Load Voltage, } V_L = IR_L$$

- Using the above three equations, the circuit current, power absorbed by the load and load voltage are calculated for different values of load resistance, which is tabulated as below:

R_L	I	P_L	V_L
0	4	0	0
5	3.3	55	16.5
10	2.8	78	28
15	2.5	93	37.5
20	2.2	97	44
25 (R_{Th})	2.0 ($I_{max}/2$)	100 (Maximum)	50 ($E_{Th}/2$)
30	1.8	97	54
40	1.5	94	60
60	1.2	83	72
100	0.8	64	80

- From the above table we see that maximum power transfer occurs when the load voltage is one-half (50 V) of Thevenin's voltage (100 V) and circuit current is one-half (2 amp) of its maximum possible value (4 amp).
- Using the data from the table above, we can plot a graph of load resistance, R_L against power, P_L for different values of load resistance.

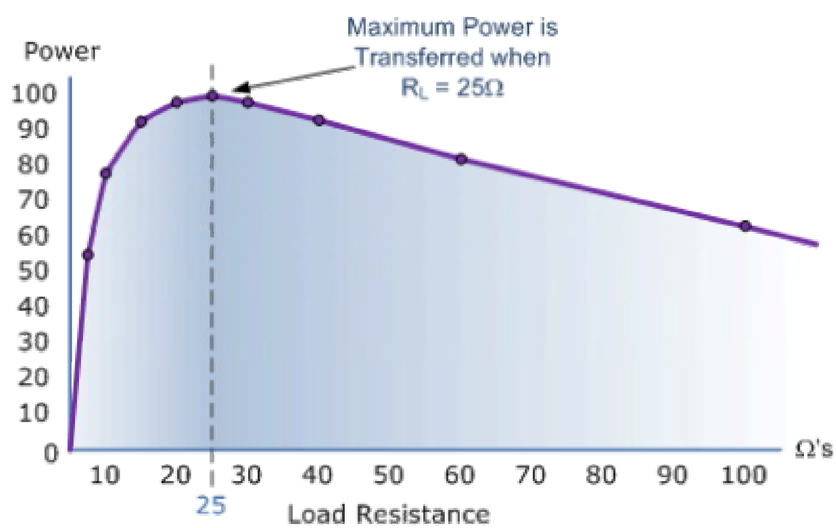
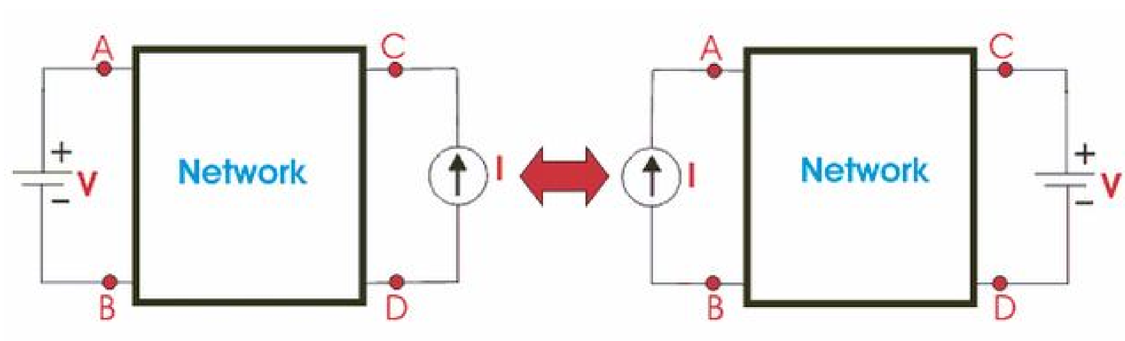


Figure: Graph of Power against Load Resistance

- From the above table and graph we can see that the **Maximum Power Transfer** occurs in the load when the load resistance, R_L is equal in value to the Thevenin's equivalent resistance, R_{Th}, so then: $R_{Th} = R_L = 25\Omega$. This is called a "matched condition" and as a general rule, maximum power is transferred from an active device such as a power supply or battery to an external device occurs when the impedance of the external device matches that of the source. Improper impedance matching can lead to excessive power use and dissipation.

6. Reciprocity Theorem:

- In many electrical networks, it is found that if positions of voltage source and ammeter are interchanged, the reading of ammeter remains the same.
- Suppose a voltage source is connected to a passive network and an ammeter is connected to other part of the network to indicate the response. Now suppose that you interchange the positions of ammeter and voltage source that means you connect the voltage source at the part of the network where the ammeter was connected and connect ammeter to that part of the network where the voltage source was connected. The response of the ammeter means current through the ammeter would be same in both cases.



Therefore, Reciprocity theorem can be stated as follows:

- In any linear bilateral network, if a source of e.m.f. ***E*** in any branch of a reciprocal network produces a current ***I*** in any other branch, then if the e.m.f. ***E*** is moved from the first to the second branch, it will produce the same current ***I*** in the first branch, where the e.m.f. has been replaced by a short circuit.
- In other words, it simply means that ***E*** and ***I*** are mutually transferable.

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- It also means that interchange of an ideal voltage source and an ideal ammeter in any network will not change the ammeter reading. Same is the case with the interchange of an ideal current source and an ideal voltmeter.

Note:

- The ratio ***E/I*** is known as **transfer resistance**.
- The voltage source and the ammeter used in this theorem must be ideal. That means the internal resistance of both voltage source and ammeter must be zero.
- The main limitation of this theorem is that it is applicable only to single source networks and not to multi-source networks.
- It is also applicable only to linear bilateral networks, not to non-linear networks.

- The circuit in the figure below is a concrete example of reciprocity theorem.
- In the first figure, if you solve the circuit, you will get the values of the current ***I*** as 0.35294 Amp. In the second figure, ***E*** is moved where I was. If you solve this circuit, then you will get the same value of ***I*** as before.
- If ***E*** is reversed, then the direction of ***I*** is reversed, so the direction does not matter so long as both ***E*** and ***I*** are reversed at the same time.

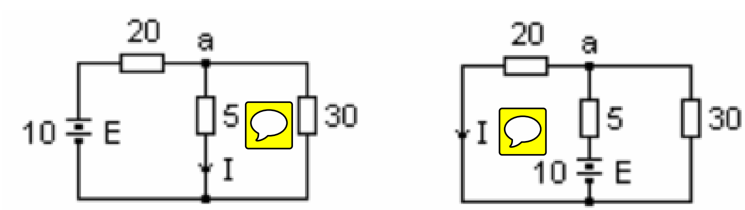

Figure: Illustrating reciprocity

Example: 2.86 (Page-133)

In the networks shown below in figure-A and figure-B, find ammeter current for both networks and prove that $I_1=I_2=1/3$ Amp.

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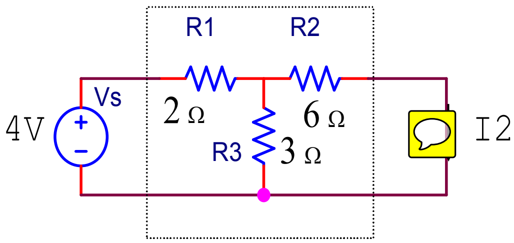

Figure-A

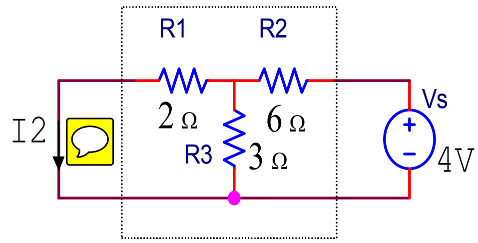

Figure-B

Solution:

DIY (Do it yourself)