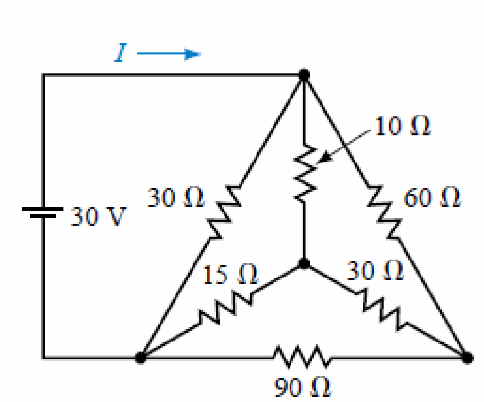
Delta-Wye Transformation

Delta-Wye (Pi-Tee) Conversion:

- Previously we have examined resistive circuits involving series, parallel, and series-parallel combinations.
- There are some networks which cannot be placed into any of the above categories. For example, consider the circuit shown below:



- The above circuit may be analyzed using techniques discussed earlier, e.g. mesh analysis, nodal analysis etc, but they are very time-consuming and prone to error. For example, mesh analysis would involve solving four simultaneous linear equations, since there are four separate loops in the circuit. If we were to use nodal analysis, the solution would require determining three node voltages, since there are three nodes in addition to a reference node.
- Rather than solving the above circuit using the techniques mentioned, it is occasionally easier to examine the circuit after it has been converted to some equivalent form.
- Two such widely used forms of circuit are:
    - delta (or pi) network
    - Wye (or Tee or Star) network

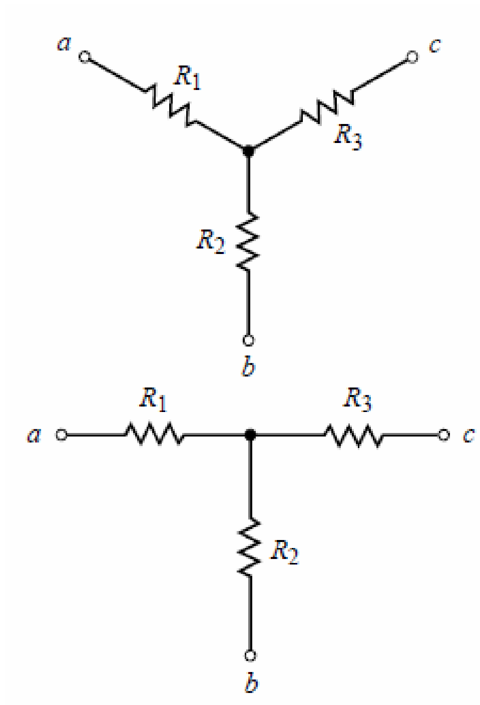


Figure A: Wye (Y) or Tee (T) network

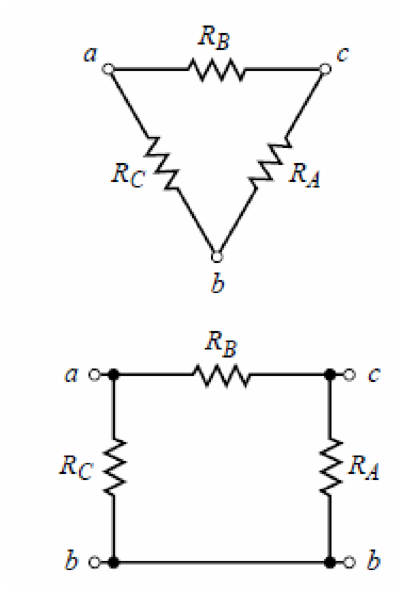


Figure B: Delta (Δ) or Pi (π) network

Converting a Delta Network to its Equivalent Wye Network (Δ to Y):

- Consider the Delta and Wye circuits shown in the figure above.
- We start by making the assumption that the network shown in figure A is equivalent to that shown in figure B.
- Then, using this assumption, we will determine the mathematical relationships between the various resistors in the equivalent circuits.
- In figure A, three resistances R_1, R_2 and R_3 are connected in Y or star fashion between terminals a, b and c.
- In figure B, three resistances R_A, R_B and R_C are connected in delta fashion between the same terminals.
- The circuit of figure A can be equivalent to the circuit of figure B only if the resistance "seen" between any two terminals is exactly the same.
- At first, consider the Y connection. If we were to connect a source between terminals ***a*** and ***b*** of the "Y," the resistance between the terminals would be:

$$R_{ab} = R_1 + R_2 \qquad (1)$$

- Now, consider the delta connection. The resistance between terminals ***a*** and ***b*** of the "Δ" is

$$R_{ab} = R_C \parallel (R_A + R_B) \qquad (2)$$

- As terminal resistances in both connections have to be the same, combining equations (1) and (2), we get

$$R_1 + R_2 = \frac{R_C (R_A + R_B)}{R_A + R_B + R_C}$$

$$\Rightarrow \quad R_1 + R_2 = \frac{R_A R_C + R_B R_C}{R_A + R_B + R_C} \qquad (3)$$

- Similarly, considering the resistances between terminal b and c of both network, we get

$$R_2 + R_3 = \frac{R_A R_B + R_A R_C}{R_A + R_B + R_C} \qquad (4)$$

- And, in the same way, considering the resistances between terminal c and a of both network, we get

$$R_1 + R_3 = \frac{R_A R_B + R_B R_C}{R_A + R_B + R_C} \qquad (5)$$

- Subtract equation (4) from equation (3):

$$R_1 + R_2 = \frac{R_A R_C + R_B R_C}{R_A + R_B + R_C} \qquad (3)$$

$$R_2 + R_3 = \frac{R_A R_B + R_A R_C}{R_A + R_B + R_C} \qquad (4)$$

$$R_1 - R_3 = \frac{R_A R_C + R_B R_C}{R_A + R_B + R_C} - \frac{R_A R_B + R_A R_C}{R_A + R_B + R_C}$$

$$= \frac{R_A R_C + R_B R_C - R_A R_B - R_A R_C}{R_A + R_B + R_C}$$

$$= \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C}$$

$$R_1 - R_3 = \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C} \qquad (6)$$

- Add equation (5) and (6):

$$R_1 + R_3 = \frac{R_A R_B + R_B R_C}{R_A + R_B + R_C} \qquad (5)$$

$$R_1 - R_3 = \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C} \qquad (6)$$

$$2R_1 = \frac{R_A R_B + R_B R_C}{R_A + R_B + R_C} + \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C}$$

$$= \frac{R_A R_B + R_B R_C + R_B R_C - R_A R_B}{R_A + R_B + R_C}$$

$$= \frac{2 R_B R_C}{R_A + R_B + R_C}$$

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \qquad (7)$$

- Using similar approach we obtain,

$$R_2 = \frac{R_C R_A}{R_A + R_B + R_C} \qquad (8)$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} \qquad (9)$$

Rewriting equations (7), (8) and (9) we get

$$\boxed{R_1 = \frac{R_B R_C}{R_A + R_B + R_C}}$$

$$\boxed{R_2 = \frac{R_C R_A}{R_A + R_B + R_C}}$$

$$\boxed{R_3 = \frac{R_A R_B}{R_A + R_B + R_C}}$$

- Notice that any resistor connected to a point of the "Y" is obtained by finding the product of the resistors connected to the same point in the "Δ" and then dividing by the sum of all the "Δ" resistances.
- If all the resistors in a Δ circuit have the same value, R_Δ, then the resulting resistors in the equivalent "Y" network will also be equal and have a value given as:

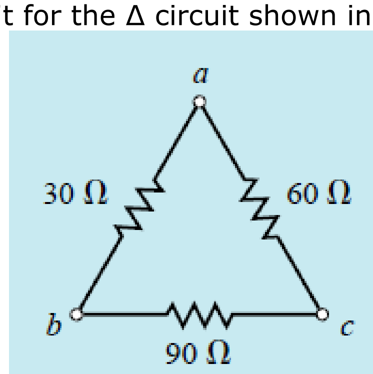
$$R_Y = \frac{R_\Delta R_\Delta}{R_\Delta + R_\Delta + R_\Delta}$$

$$\Rightarrow \quad = \frac{R_\Delta R_\Delta}{3R_\Delta}$$

$$\boxed{R_Y = \frac{R_\Delta}{3}}$$

Example-1:

Find the equivalent Y circuit for the Δ circuit shown in the figure below.



Solution:

Given that

R_A=Resistance between terminal b and c of Δ circuit=90Ω

R_B=Resistance between terminal c and a of Δ circuit=60Ω

R_C=Resistance between terminal a and b of Δ circuit=30Ω

Equivalent Y resistance can be calculated using the following relations:

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C}$$

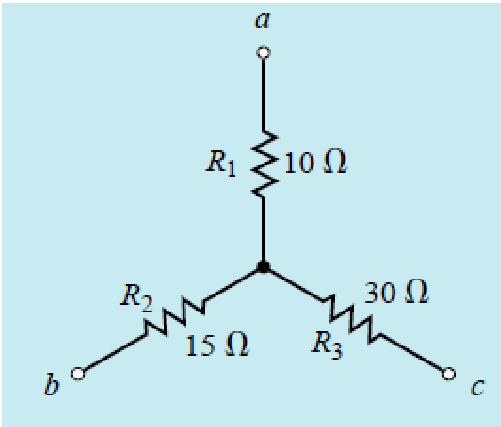
$$R_2 = \frac{R_C R_A}{R_A + R_B + R_C}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$

$$\begin{aligned} R_1 &= \frac{R_B R_C}{R_A + R_B + R_C} \\ &= \frac{(60\Omega)(30\Omega)}{90\Omega + 60\Omega + 30\Omega} \\ &= \frac{1800}{180}\Omega \\ &= 10\Omega \end{aligned}$$

Similarly, R_2=15Ω, R_3=30Ω

Therefore, the resulting Y circuit is:



Converting a Wye Network to its Equivalent Delta Network (Y to Δ):

- By using equations (7), (8) and (9) above, it is possible to derive another set of equations which allow the conversion from a "Y" into an equivalent "Δ."

Re-writing equations (7), (8) and (9) we get

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \qquad (7)$$

$$R_2 = \frac{R_C R_A}{R_A + R_B + R_C} \qquad (8)$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} \qquad (9)$$

From equation (7) we get

$$R_A + R_B + R_C = \frac{R_B R_C}{R_1} \qquad (10)$$

Similarly, from equation (8) we get

$$R_A + R_B + R_C = \frac{R_C R_A}{R_2} \qquad (11)$$

And, in the same way, from equation (9) we get

$$R_A + R_B + R_C = \frac{R_A R_B}{R_3} \qquad (12)$$

From equations (10), (11) and (12) we get

$$\frac{R_B R_C}{R_1} = \frac{R_C R_A}{R_2} = \frac{R_A R_B}{R_3} \qquad (13)$$

Using equation (13) we get

$$\frac{R_B R_C}{R_1} = \frac{R_C R_A}{R_2} \qquad (14)$$

$$\frac{R_B R_C}{R_1} = \frac{R_A R_B}{R_3} \qquad (15)$$

From equation (14) we get

$$\frac{R_B R_C}{R_1} = \frac{R_C R_A}{R_2} \qquad (14)$$

$$\Rightarrow \quad \frac{R_B}{R_1} = \frac{R_A}{R_2}$$

$$\Rightarrow \quad R_B = \frac{R_A R_1}{R_2} \qquad (16)$$

Similarly, from equation (15) we get

$$\frac{R_B R_C}{R_1} = \frac{R_A R_B}{R_3} \qquad (15)$$

$$\Rightarrow \quad \frac{R_C}{R_1} = \frac{R_A}{R_3}$$

$$\Rightarrow \quad R_C = \frac{R_A R_1}{R_3} \qquad (17)$$

Putting the value of R_B and R_C and from equations (16) and (17) to equation (7) we get:

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \qquad (7)$$

$$\Rightarrow \qquad = \frac{\left(\frac{R_A R_1}{R_2}\right)\left(\frac{R_A R_1}{R_3}\right)}{R_A + \frac{R_A R_1}{R_2} + \frac{R_A R_1}{R_3}}$$

$$\Rightarrow \qquad = \frac{R_A\left[\left(\frac{R_A R_1 R_1}{R_2 R_3}\right)\right]}{R_A\left[1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right]}$$

$$\Rightarrow \qquad = \frac{\frac{R_A R_1 R_1}{R_2 R_3}}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}}$$

$$\Rightarrow \qquad = \frac{\frac{R_A R_1 R_1}{R_2 R_3}}{\frac{R_2 R_3 + R_1 R_3 + R_1 R_2}{R_2 R_3}}$$

$$\Rightarrow \qquad R_1 = \frac{R_A R_1 R_1}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

$$\Rightarrow \qquad 1 = \frac{R_A R_1}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

$$\Rightarrow \quad R_A R_1 = R_1 R_2 + R_2 R_3 + R_3 R_1$$

$$\Rightarrow \qquad R_A = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1} \qquad (18)$$

Similarly, putting the value of R_B and R_C from equations (16) and (17) to equation (8) we get:

$$\Rightarrow \qquad R_B = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} \qquad (19)$$

Similarly, putting the value of R_B and R_C from equations (16) and (17) to equation (9) we get:

$$\Rightarrow \qquad R_C = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} \qquad (20)$$

Re-writing equations (18), (19) and (20), we get:

$$\boxed{\begin{aligned} R_A &= \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \\ R_B &= \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} \\ R_C &= \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} \end{aligned}}$$

- In general, we see that the resistor in any side of a "Δ" is found by taking the sum of all two-product combinations of "Y" resistor values and then dividing by the resistance in the "Y" which is located directly opposite to the resistor being calculated.

- If the resistors in a Y network are all equal, then the resultant resistors in the equivalent Δ circuit will also be equal and given as:

$$R_C = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

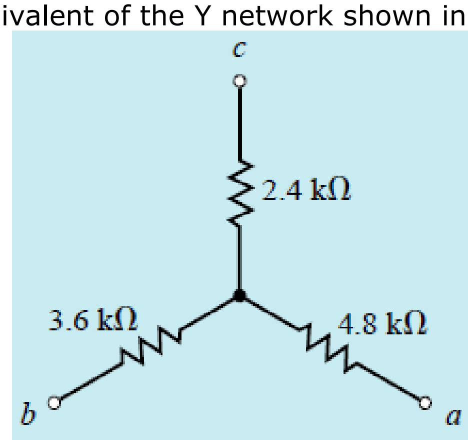
$$\Rightarrow R_\Delta = \frac{R_YR_Y + R_YR_Y + R_YR_Y}{R_Y}$$

$$\Rightarrow \quad = \frac{3R_YR_Y}{R_Y}$$

$$\Rightarrow \quad R_\Delta = 3R_Y$$

Example-2:

Find the Δ network equivalent of the Y network shown in the figure below.



Solution:

Given that

R_1=Resistance at terminal ***a*** of Y circuit=4.8kΩ

R_2= Resistance at terminal ***b*** of Y circuit =3.6kΩ

R_3= Resistance at terminal ***c*** of Y circuit =2.4kΩ

Equivalent Δ resistance can be calculated using the following relations:

$$R_A = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$
$$R_B = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$
$$R_C = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

$$\begin{aligned} R_A &= \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \\ &= \frac{(4.8\Omega)(3.6\Omega) + (3.6\Omega)(2.4\Omega) + (2.4\Omega)(4.8\Omega)}{4.8\Omega} \\ &= 7.8k\Omega \end{aligned}$$

Similarly, R_B=10.4Ω, R_C=15.6Ω

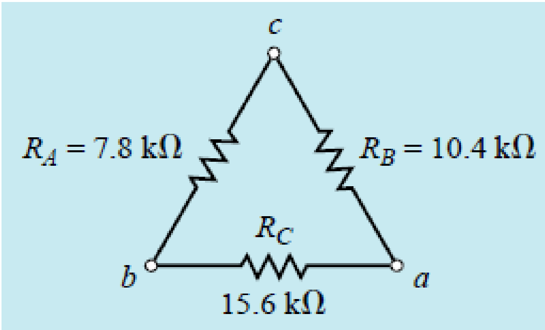
Therefore,

R_A=Resistance between terminal b and c of Δ circuit=7.8kΩ

R_B=Resistance between terminal c and a of Δ circuit=10.4kΩ

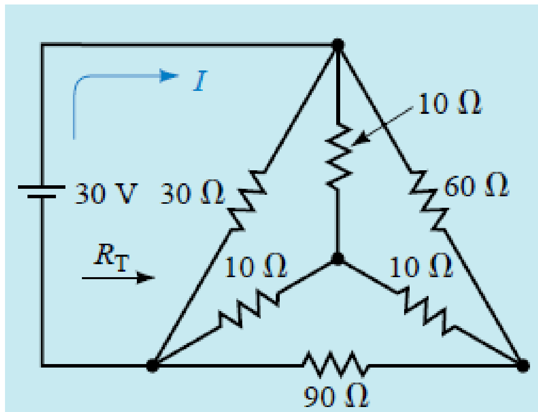
R_C=Resistance between terminal a and b of Δ circuit=15kΩ

Therefore, the resulting Δ circuit is:



Example-3:

Find the total resistance, R_T, and the total current, I for the circuit shown in the figure below.



Solution:

The given circuit may be solved in one of two ways:

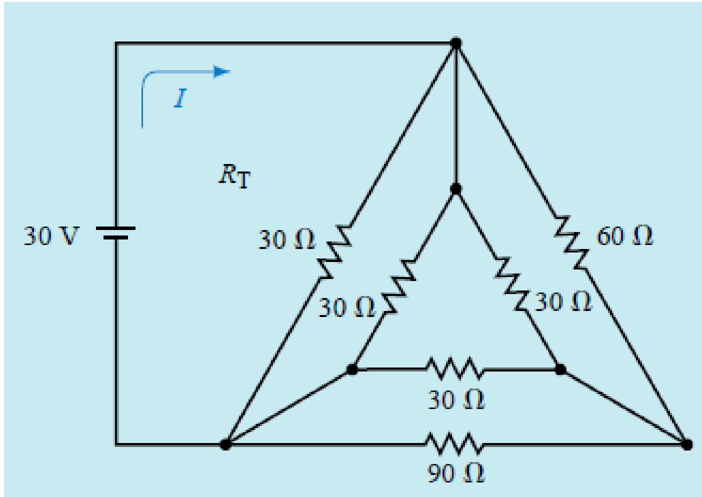
- We may convert the "Δ" into its equivalent "Y," and solve the circuit by placing the resultant branches in parallel, or
- We may convert the "Y" into its equivalent "Δ."

We choose to use the conversion from "Y" to "Δ" since the resistors in the "Y" have the same value, each of 10Ω.

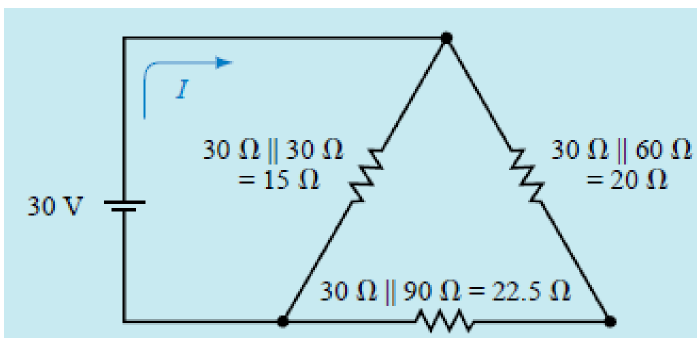
Then the resultant resistors in the equivalent Δ circuit will also be equal and given as:

$$\Rightarrow \quad R_\Delta = 3R_Y \\ = 3 \times 10\Omega \\ = 30\Omega$$

The resulting circuit is shown in the figure below.



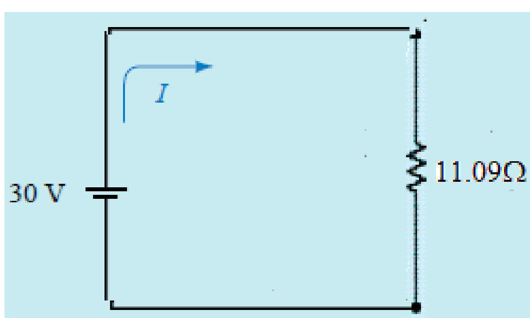
From the above circuit, we see that the sides of the resulting "Δ" are in parallel, which allows us to simplify the circuit even further as shown in the figure below.



The total resistance of the circuit is now easily determined as

$$\begin{aligned} R_T &= 15\Omega \,||\, (20\Omega + 22.5\Omega) \\ &= 15\Omega \,||\, 42.5\Omega \\ &= 11.09\Omega \end{aligned}$$

Now the above circuit becomes

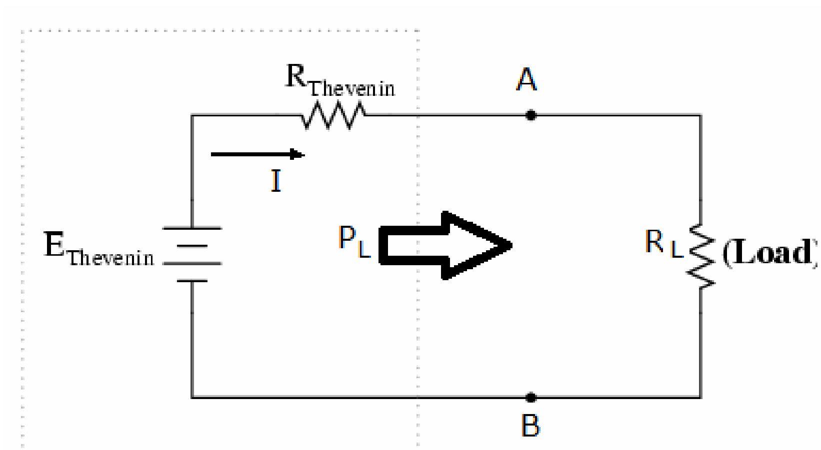


This results in a circuit current of

$$
\begin{aligned}
I &= \frac{V}{R_T} \\
&= \frac{30V}{11.09\Omega} \\
&= 2.706A
\end{aligned}
$$

Maximum Power Transfer Theorem

- When designing a circuit, it is often important to be able to answer one of the following questions:
    - What load should be applied to a system to ensure that the load is receiving maximum power from the system? Or,
    - For a particular load, what conditions should be imposed on the source to ensure that it will deliver the maximum power available?
- At times it is desired to obtain the maximum power transfer from an active network to an external load resistor R_L.
- Assume that the network is linear. So, it can be reduced to a Thevenin's equivalent circuit with load resistor connected across terminals *A* and *B* to this network as shown in the figure below:



- Power absorbed by the load will vary according to the resistance R_{Th} of the network. Then, for the load resistance, to absorb the maximum power possible, it has to be "Matched" to the resistance of the network. This forms the basis of **Maximum Power Transfer**.
- The process of finding the load that will receive maximum power from a particular system is quite straightforward due to the maximum power transfer theorem. As applied to DC networks, this theorem may be stated as follows:
    - A resistive load will receive maximum power from a network when the load resistance is equal to the Thevenin resistance of the network applied to the load. That is, when

$$\boxed{R_L = R_{Th}}$$

- But, if the load resistance is lower or higher in value than the Thevenin resistance of the network, its dissipated power will be less than the maximum.

- With $R_L=R_{Th}$, the maximum power delivered to the load can be determined as follows:

Circuit current, $I_L = \dfrac{E_{Th}}{R_{Th}+R_L}$

Power consumed by the load, $P_L = I_L^2 R_L = \left(\dfrac{E_{Th}}{R_{Th}+R_L}\right)^2 (R_L)$

$$P_L = \frac{E_{Th}^2 R_L}{(R_{Th}+R_L)^2} \qquad \text{........(1)}$$

- For P_L to be maximum, $\dfrac{dP_L}{dR_L}$ must be equal to zero, that is,

$$\boxed{\frac{dP_L}{dR_L}=0}$$

- Therefore, differentiating equation (1), we have,

$$\frac{dP_L}{dR_L} = E_{Th}^2\left[\frac{1}{(R_{Th}+R_L)^2} + R_L\left(\frac{-2}{(R_{Th}+R_L)^3}\right)\right] = E_{Th}^2\left[\frac{1}{(R_{Th}+R_L)^2} - \frac{2R_L}{(R_{Th}+R_L)^3}\right]$$

$$\Rightarrow \quad 0 = E_{Th}^2\left[\frac{1}{(R_{Th}+R_L)^2} - \frac{2R_L}{(R_{Th}+R_L)^3}\right]$$

$$\Rightarrow \quad E_{Th}^2\left[\frac{1}{(R_{Th}+R_L)^2} - \frac{2R_L}{(R_{Th}+R_L)^3}\right] = 0$$

$$\Rightarrow \quad \frac{1}{(R_{Th}+R_L)^2} - \frac{2R_L}{(R_{Th}+R_L)^3} = 0$$

$$\Rightarrow \quad \frac{1}{(R_{Th}+R_L)^2} = \frac{2R_L}{(R_{Th}+R_L)^3}$$

$$\Rightarrow 1 = \frac{2R_L}{(R_{Th}+R_L)}$$

$$\Rightarrow 2R_L = R_{Th}+R_L$$

$$\Rightarrow R_L = R_{Th} \qquad (\text{Proved})$$

- Now, using $R_{Th}=R_L$ in equation (1), we have the equation of maximum power:

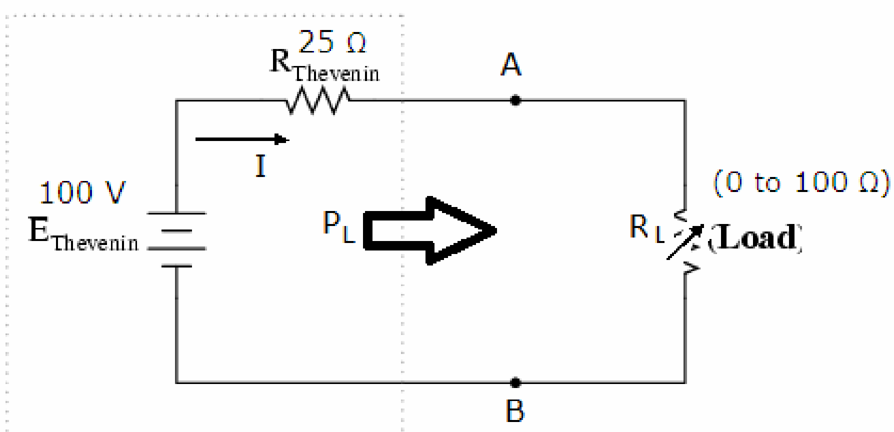
$$P_L = \frac{E_{Th}^2 R_L}{(R_{Th}+R_L)^2} = \frac{E_{Th}^2 R_{Th}}{(R_{Th}+R_{Th})^2} = \frac{E_{Th}^2 R_{Th}}{(2R_{Th})^2} = \frac{E_{Th}^2 R_{Th}}{4R_{Th}^{\ 2}} = \frac{E_{Th}^2}{4R_{Th}}$$

$$\boxed{P_{L_{\max}} = \frac{E_{Th}^2}{4R_{Th}}}$$

- The **Maximum Power Transfer Theorem** is a useful analysis method to ensure that the maximum amount of power will be dissipated in the load resistance when the value of the load resistance is exactly equal to the resistance of the network.

Example-1:

- Consider the following Thevenin equivalent circuit with E_{Th}=100 V and R_{Th}=25 Ω. Determine the circuit current and power dissipated by the load for different values of load resistance (e.g. 0 to 100 Ω).
    - Demonstrate that maximum power transfer occurs in the load when the load resistance, R_L is equal in value to the Thevenin equivalent resistance, R_{Th}.
    - And also prove that maximum power transfer occurs when the load voltage and circuit current are one-half of their maximum possible values.



Circuit current I, power absorbed by the load P_L and load voltage V_L can be determined using the following formulas:

$$\text{Circuit current, } I = \frac{E_{Th}}{R_{Th} + R_L}$$

$$\text{Power consumed by the load, } P_L = I^2 R_L$$

$$\text{Load Voltage, } V_L = IR_L$$

- Using the above three equations, the circuit current, power absorbed by the load and load voltage are calculated for different values of load resistance, which is tabulated as below:

R_L	I	P_L	V_L
0	4	0	0
5	3.3	55	16.5
10	2.8	78	28
15	2.5	93	37.5
20	2.2	97	44
25 (R_{Th})	2.0 (I_{max}/2)	100 (Maximum)	50 (E_{Th}/2)
30	1.8	97	54
40	1.5	94	60
60	1.2	83	72
100	0.8	64	80

- From the above table we see that maximum power transfer occurs when the load voltage is one-half (50 V) of Thevenin's voltage (100 V) and circuit current is one-half (2 amp) of its maximum possible value (4 amp).
- Using the data from the table above, we can plot a graph of load resistance, R_L against power, P_L for different values of load resistance.

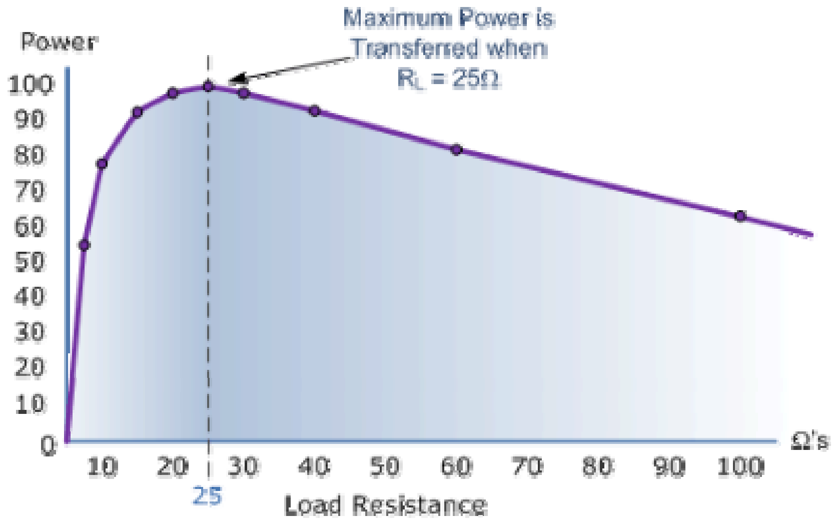


Figure: Graph of Power against Load Resistance

- From the above table and graph we can see that the **Maximum Power Transfer** occurs in the load when the load resistance, R_L is equal in value to the Thevenin's equivalent resistance, R_{Th}, so then: $R_{Th} = R_L = 25\Omega$. This is called a "matched condition" and as a general rule, maximum power is transferred from an active device such as a power supply or battery to an external device occurs when the impedance of the external device matches that of the source. Improper impedance matching can lead to excessive power use and dissipation.