INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY

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Solutions to Tutorial 1
PART A

1. **If A is an $m \times n$ matrix and B an $n \times p$ matrix, then show that the i^{th} row of the product AB is equal to the product of the i^{th} row vector A_i of A with B. Give a similar description of the columns of AB.**

   Solution: Routine verification. The j-th column of AB is the product of A and the j-th column vector of B.

2. **Prove that matrix multiplication is distributive over matrix addition. That is, for any three matrices A, B, C $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$. (In all problems involving sums and products of matrices, it is to be assumed that the matrices meet all the size requirements needed for these operations to be defined.)**

   Solution: Routine verification.

3. **Prove that for square matrices A and B, in general $(A + B)^2 \neq A^2 + 2AB + B^2$. Prove however, that if A and B commute with each other (i.e. $AB = BA$) then this holds and so does the binomial theorem for any power of $A + B$.**

   Solution: In general, using the distributive law, we only have $(A + B)^2 = A^2 + AB + BA + B^2$. If A and B commute then the middle two terms are equal. In fact, in this case any product of the form $X_1X_2 \dots X_n$ where some of the $X_i's$ are A and the remaining are B, equals A^rB^{n-r} where r is the number of factors that equal A (with the understanding that A^0 and B^0 equal the identity matrix). So the proof of the binomial theorem for real numbers goes through.

4. (a) **For any two matrices A and B and any numbers α and β, prove that (i) $(\alpha A+\beta B)^t = \alpha A^t + \beta B^t$ and (ii) $(AB)^t = B^tA^t$.**

   (b) **State and prove the corresponding results about the Hermitian adjoint.**

Solution: (a) Routine verification. (b) This is routine too, except that (i) becomes $(\alpha A+\beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*$.

5. **A matrix A is called skew-symmetric (skew-Hermitian) if $A^t = -A$ (if $A^* = -A$). Prove that every square matrix can be expressed uniquely as a sum of a symmetric and a skew-symmetric matrix (as a sum of a Hermitian and a skew Hermitian matrix).**

   Solution: Note that $A + A^t$ is always symmetric and $A - A^t$ is always skew-symmetric. Write A as $\dfrac{A+A^t}{2} + \dfrac{A-A^t}{2}$. Then it is a sum of a symmetric and a skew-symmetric matrix. For uniqueness, suppose $S_1 + T_1 = S_2 + T_2$ where S_1, S_2 are symmetric and T_1, T_2 are skew-symmetric. Let $B = S_1 - S_2 = T_2 - T_1$ is both symmetric and skew-symmetric. So $B = O$. The proofs for the assertian about Hermitian and skew Hermitian are obtained by replacing transposes with conjugate transposes.

6. **Suppose $A = (A_{ij})$ is an $r \times s$ partitioned matrix and $B = (B_{ij})$ is an $s \times t$ partitioned matrix. Then A and B are said to be conformably partitioned for multiplication if for every $i = 1, 2, \ldots, r$, every $j = 1, 2, \ldots, s$ and every $k = 1, 2, \ldots, t$, the column size of A_{ij} equals the row size of B_{jk}. As a result, A_{ij} and B_{jk} can be multiplied as matrices. Prove that this gives an $r \times t$ partition of the matrix AB. When can two partitioned matrice be added? When they can, what is their sum?**

   Solution: Routine verification. (Best seen from a diagram.) In order to add two partitioned matrices, the sizes of the corresponding blocks must match. When this happens, by adding these blocks, we get a partition of the sum matrix.

7. **The diagonal of a square amtrix $A = (a_{ij})$ is defined as the set of all elements of the form a_{ii}, $i = 1, 2, \ldots, n$. A matrix in which all non-diagonal enries vanish is called a diagonal matrix. A matrix in which all entries 'below' the diagonal vanish is called upper triangular while a matrix in which all entries 'above' the diagonal vanish is called lower triangular. Prove that the sum and the product of diagonal matrices (of**

equal sizes) is again a diagonal matrix and similar results for upper/lower triangular matrices. (Occasionally, we talk of the upper and the lower subdiagonals of a square matrix defined in the obvious way. All these definitions appear natural when a matrix is drawn as an array. Otherwise they appear clumsy.)

Solution: Routine verifications. (Best seen from diagrams.)

8. **A square matrix N is called nilpotent if there exists a positive integer r such that $N^r = O$. Prove that every square matrix with all enries on and below the diagonal is nilpotent. Show by examples that if A, B are nilpotent, $A + B$ and AB need not be so. Prove, however, that this is the case if A and B commute with each other.**

   Solution: Suppose $A = (a_{ij})$ is an $n \times n$ matrix in which $a_{ij} = 0$ for $j \leq i$. Then a direct computation shows that the ij-th entry of A^2 is 0 whenever $j \leq i - 1$. Similarly, in A^3 the ij-th entry will vanish whenever $j \leq i-3$. Continuing, all entries in A^n will vanish. (Easier to see this with a diagram. Later a proof based on linear transformations will be given.) For the second part, let $A = E_{12}$ and $B = E_{21}$ be two 2×2 matrices where E_{ij} is a matrix in which only the ij-th entry is 1 and all others are 0. Then A, B are nilpotent, but neither $A + B$ nor AB is so. For the third part, if $A^r = B^s = O$, then $(AB)^r = A^r B^r = O$ while $(A+B)^{r+s} = 0$ as every term in the binomial expansion vanishes.

9. **If A, B are invertible matrices of the same size, prove that AB is invertible and has $B^{-1}A^{-1}$ as its inverse. Is $A+B$ necessrily invertible?**

   Solution: For the first part, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$. Similarly $(B^{-1}A^{-1})(AB) = I$. For the second part a trivial counterexample is to take $B = -A$.

10. **If N is a nilpotent matrix, prove that $I - N$ is invertible where I is the identity matrix of the same size as N. [*Hint:* Expand $\dfrac{1}{I-N}$ as a formal power series in N.]**

    Solution: The infinite power series $I + N + N^2 + N^3 + \ldots$ is terminating since $N^r = O$ for some r. For a valid proof, assume $N^r = O$. Then by a direct computation, $(I-N)(I+N+N^2+\ldots+N^{r-1}) = I-N^r = I$ and similalrly $(I+N+N^2+\ldots+N^{r-1})(I-N) = I$.

11. A matrix A is called idempotent if $A^2 = A$. Show that a diagonal matrix with only 0 and 1 as possible diagonal entries is idempotent. Can you find some others?

Solution: The first part is trivial. For the second, one example is $A = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$. (This can be obtained by trial. Later it will be shaown that A represents the projection onto the line $x+y=0$ in the plane and every projection gives an idempotent transformation.)

12. If A is an idempotent matrix, prove that $(A+I)^n = I+(2^n-1)A$ for every positive integer n.

Solution: Apply the binomial theorem and the fact that $A^k = A$ for all $k \geq 1$.

13. An $m \times n$ matrix all whose entries are 1, is often denoted by $J_{m\times n}$ (or by J_n when $m = n$) or simply by J when the size is understood. If $A = J_n$ and $B = J_{n\times 1}$ prove that $AB = nB$, $A^2B = n^2B$ and in general, for any polynomial $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_rx^r$, $p(A)B = p(n)B$.

Solution: Routine verification.

14. A Markov or stochastic matrix is a square matrix with non-negative entries such that the sum of the entries in each row is 1. Prove that the product of two Markov matrices is again a Markov matrix.

Solution: A direct proof is possible. But, for a slick proof observe that an $n \times n$ matrix A with non-negative entries is a Markov matrix if and only if $AJ = J$ where $J = J_{n\times 1}$. If A, B are two Markov matrices of order n then $(AB)J = A(BJ) = AJ = J$ whence AB is a Markov matrix.

15. For any θ, let $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. For any α, β, prove that $A_{\alpha+\beta} = A_\alpha A_\beta$. Which complex number is represented by A_θ?

Solution: This follows from the trigonometric identities for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$. The matrix A_θ represnts the complex number $\cos\theta + i\sin\theta = e^{i\theta}$. (This correspondence can also be used to give an alternate proof of the first part since $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$.)

16. Find the 100-th power of the following matrices:

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & \omega \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $(\lambda \in \mathbb{R})$

Solution: (i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{bmatrix}$. (The 100-th power will be a diagonal matrix with entries $1^{100}, i^{100}$ and ω^{100}.) (ii) $\begin{bmatrix} -2^{50} & 0 \\ 0 & -2^{50} \end{bmatrix}$. (The matrix, say A, corresponds to the complex number $(1+i) = \sqrt{2}e^{i\pi/4}$. So, A^{100} corresponds to the complex number $(1+i)^{100} = 2^{50}e^{25i\pi} = -2^{50}$.) (iii) $\begin{bmatrix} \lambda^{100} & 100\lambda^{99} \\ 0 & \lambda^{100} \end{bmatrix}$. (Apply induction on n to show that the n-th power equals $\begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$ for all $n \in \mathbb{N}$. Or write the given matrix as $\lambda I + E_{12}$. Apply the binomial theorem and the nilpotency of E_{12}.)

17. Verify that the companion matrix of the polynomial $p(x) = x^3 + x - 2$ satisfies the matrix equation $p(A) = 0$.

Solution: The companion matrix, say A equals $\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. By a direct calculation, the matrices A^2 and A^3 come out to be $\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ respectively. Adding, we get $A^3 + A = 2I$.

item [**18.**] **Prove that each of the three elementary row operations amounts to premultiplying the matrix by a suitable square matrix. Deduce that all these operations are reversible.**

Solution: Let A be an $m \times n$ matrix. To interchange the i-th and the j-th rows of A, premultiply A by $I_m - E_{ii} - E_{jj} + E_{ij} + E_{ji}$. To multiply the i-th row of A by c, premultiply A by $I_m + (c-1)E_{ii}$. To add c times the i-th row to the j-th row of A, premultiply A by $I_m + cEji$. These three matrices are all invertible and their inverses are of the same type as they. Hence every elementary row operation can be undone by performing a similar row operation on the new matrix. A direct proof is also easy.

PART B

19. Find the locus of the point of intersection of the lines AQ and BP, where $A = (7,0)$, $B = (0,-5)$, P lies on the x-axis, Q lies on the y-axis and PQ is perpendicular to AB. (JEE 1990)

Solution: $x(x-7) + y(y+5) = 0$. (Let $P = (a,0)$ and $Q = (0,b)$. Comparing various slopes gives $\frac{a}{5} = \frac{h}{k+5}, \frac{b}{7} = \frac{k}{7-h}$ and $\frac{b}{a} \times \frac{5}{7} = 1$. Eliminating a, b gives the answer.

20. Can the last problem be done by pure geometry?

Solution: The locus is a circle with the segment joining the origin to the point $(7,-5)$ as a diameter and hence also AB as a diameter. Once we guess this, a pure geometry solution is possible. From the data, P is the orthocentre of the triangle ABQ and hence BP is perpendicular to AQ. So $\angle ARB = 90°$. Hence the locus of R is a circle with AB as a diameter.

21. Suppose at a dance party, there are m boys $b_1, b_2, \ldots, b_m$ and n girls $g_1, g_2, \ldots, g_n$. Let A be the $m \times n$ matrix whose (i,j)-th entry is 1 or 0 according as b_i dances with g_j or not. Interpret the entries of the square matrices AA^t and A^tA.

Solution: The ij-th entry of AA^t is the number of girls with whom both b_i and b_j have danced, that of A^tA gives the number of boys with whom both g_i and g_j have danced.

22. Let A be the incidence matrix of a graph with vertices $v_1, v_2, \ldots, v_m$ and edges $e_1, e_2, \ldots, e_n$. Assume that G has no loops. Interpret the matrices $AJ_{n\times 1}$ and $J_{1\times m}A$.

Solution: The i-th entry in $AJ_{n\times 1}$ is the degree of the vertex v_i, i.e. the number of edges incident on it. (Note that G may have multiple edges, i.e. more than one edges joining the same pair of distinct vertices.) If G has no loops, then every entry in $J_{1\times m}$ is 2.

23. Let A be the adjacency matrix of a graph G with vertices $v_1, v_2, \ldots, v_n$.

(i) Can it happen that some diagonal entry of A is 1?

(ii) What is the graph theoretic interpretation of the matrix $AJ_{n\times 1}$ where $J_{n\times 1}$ is an $n \times 1$ matrix in which every entry is 1 ?

(iii) Assume that none of the diagonal entries of A is 1. What is the graph theoretic interpretation of the ij-th entry of the square matrix A^2 and more generally, of the power A^r for a positive integer r? Compute A^2 and A^3 for the complete graph K_n, i.e. a graph on n vertices in which every two distinct vertices are adjacent.

Solution: (i) This happens if and only if the graph has a loop, i.e. an edge which joins a vertex to itself. (ii) The i-th entry of $AJ_{n\times 1}$ is the **degree** of the vertex v_i, i.e. the number of the vertices it is joined to. (iii) The ij-th entry in A^r is the number of paths of length r between the vertices v_i and v_j. For K_n, the non-diagonal entries in A^2 are all $n-2$ and the diagonal entries are all $n-1$. (iv) A graph G is connected if and only if for some r, A^r has no non-zero entries.

24. A graph is said to be connected if it is impossible to partition its vertex set into two mutually disjoint non-empty subsets in such a way that no vertex in the first subset is adjacent to a vertex in the second subset. Prove that a (finite) graph is connected if and only if every pair, say $\{u, v\}$ of its vertices can be joined by a walk, i.e. by a finite sequence, say $(e_1, e_2, \ldots, e_k)$ of edges in which (i) u lies on e_1, (ii) v lies on e_n and (iii) for every $i = 1, 2, \ldots, k-1$, the edges e_i and e_{i+1} pass through a common vertex of the graph.

Solution: For the direct implication, it suffices to show that a fixed vertex, say v_0 can be joined to every vertex by a path. Take any vertex

v_0. Let A be the subset of those vertices which can be joined to v_0 by a walk and B be the set of those that can't. Then $A \neq \emptyset$. Also no vertex in A can be joined to a vertex in B by an edge. For the converse, suppose some the vertex set is partitioned as $A \cup B$. Take any $u \in A$ and $v \in B$. Then there exists a walk from u to v. Starting from u, show inductively that every vertex in this walk lies in A.

25. Characterise the connectedness of a graph in terms of (i) its incidence matrix, (ii) its adjacency matrix and (iii) the powers of its adjacency matrix.

Solution: Let G be a finite graph with incidence matrix A and adjacency matrix B. Then G is disconneced if and only with a permutation of rows and/or columns, A and B can be written as 2×2 partitioned matrices in which the non-diagonal entries are the zero matrices. Also G is connected if and only if B^r has no non-zero entries for some positive integer r, except in the trivial case where G has at most one vertex and no edges.

26. Express the condition in the Sudoku game in terms of the entries of a 9×9 matrix with 9 symbols as possible entries.

Solution: Let $A = (a_{ij}$ be a 9×9 matrix with 9 possible symbols as entries. Then the condition that no row has repeated entries is equivalent to saying that $a_{ij} = a_{ik}$ implies $j = k$. A similar translation holds for the column condition. For the condition on the nine 3×3 subsquares, for any real number x, denote by $\lceil x \rceil$ the **upper ceiling** of x, i.e. the smallest integer not less than x. Then the condition says that if $a_{ij} = a_{pq}$ and $\lceil i/3 \rceil = \lceil p/3 \rceil$ and $\lceil j/3 \rceil = \lceil q/3 \rceil$, then $i = j$ and $p = q$.

27. Construct magic sauares of odd orders.

Solution: There is a famous algorithm for doing this in which the n^2 terms of an $n \times n$ matrix $A = (a_{ij})$ are visited one-by-one and set equal to successive terms of an A.P., starting from the entry just below the entry at the centre. The rule to select the next entry is the 'south-east' rule, where you keep on moving one step forward and one downward except when you can't. In symbols, if a_{ij} is just filled, the next entry to visit is $a_{i+1,j+1}$. This choice is to be modified in the following cases:

(i) when the entry just visited is in the last row or column but not both i.e. when either $i = n$ or $j = n$ but not both. In this case take the next entry as $a_{1,j+1}$ or $a_{i+1,1}$ as the case may be

(ii) when the entry last visited is at the southeast corner, i.e. when $i = n = j$. In this case, the next entry is to be $a_{2,n}$.

(iii) when the entry to be visited next as given by the rule is already visited. In this case bounce back from that entry, that is take the next entry as $a_{i+2,j}$ (or as $a_{1,j}$ if $i = n - 1$).

It requires a proof that this algorithm really works, i.e. all the entries get visited exactly once and further that the resulting matrix is indeed a magic square. The squares resulting in the cases $n = 3$ and $n = 5$ are shown below.

4	9	2
3	5	7
8	1	6

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15