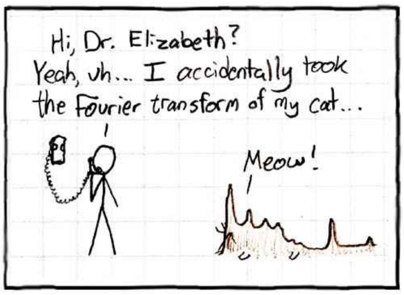
IV. Filtering in the Frequency Domain



IV. Filtering in the Frequency Domain

1. Background
2. Preliminary Concepts
3. Sampling and the Fourier Transform of Sampled Functions
4. The Discrete Fourier Transform (DFT) of One Variable
5. Extension to Functions of Two Variables
6. The Basics of Filtering in the Frequency Domain Filters
7. Image Sharpening Using Frequency Domain Filters
8. Selective Filtering

Background

- Fourier Series
  Any *periodic function* can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient
- Fourier Transform
  Even *nonperiodic function* can be expressed as the integral of sines and/or cosines multiplied by a weighting functions
  "*Nonperiodic*" → The period is infinite.
  Therefore every signal is periodic.


Jean Baptiste Joseph Fourier
(1768~1830)

Fourier Series

Periodic function: $f(t) = f(t + kT), \quad u_0 = \frac{1}{T}$ (Fundametal frequency)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi n u_0 t) + b_n \sin(2\pi n u_0 t)]$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(2\pi n u_0 t)dt, \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(2\pi n u_0 t)dt$$

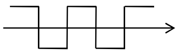
Using Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n u_0 t}$$

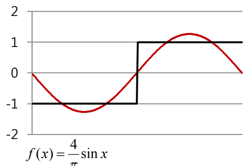
$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j2\pi n u_0 t} dt \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Fourier coefficients c_n → discrete spectrum (an integer multiple of the fundamental frequency; u_0, $2u_0$, $3u_0$,…)

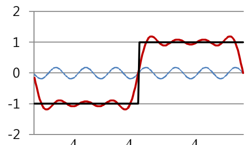
Example - Rectangular Wave



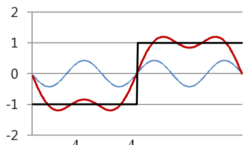
$$f(x)=\sum_{n=1}^{\infty}\frac{4}{(2n+1)\pi}\sin(2n+1)x=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin 3x+\frac{4}{5\pi}\sin 5x+\dots$$



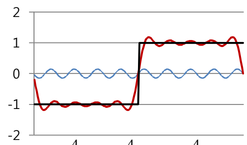
$f(x)=\frac{4}{\pi}\sin x$



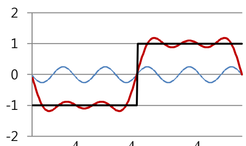
$f(x)=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin 3x+\frac{4}{5\pi}\sin 5x+\frac{4}{7\pi}\sin 7x$



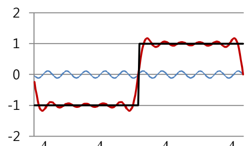
$f(x)=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin 3x$



$f(x)=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin 3x+\frac{4}{5\pi}\sin 5x+\frac{4}{7\pi}\sin 7x+\frac{4}{9\pi}\sin 9x$

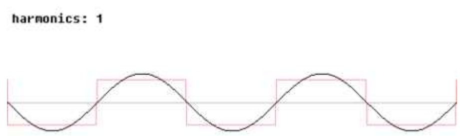


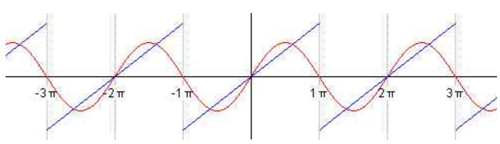
$f(x)=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin 3x+\frac{4}{5\pi}\sin 5x$



$f(x)=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin 3x+\frac{4}{5\pi}\sin 5x+\frac{4}{7\pi}\sin 7x+\frac{4}{9\pi}\sin 9x+\frac{4}{11\pi}\sin 11x$

Examples





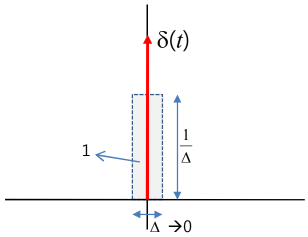
Animated plot of the first five successive partial Fourier series of a sawtooth wave.

Impulses and Their Sifting Property (Continuous Signal)

Unit Impulse

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

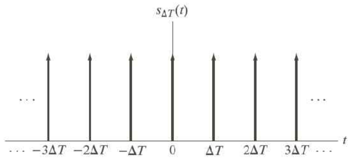
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



Sifting Property

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \int_{-\infty}^{\infty} f(0)\delta(t)dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = \int_{-\infty}^{\infty} f(t_0)\delta(t-t_0)dt = f(t_0)$$



Impulse Train : sum of infinitely many *periodic* impulses **Δ*T*** units apart

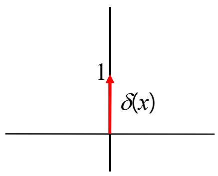
$$s_{\Delta t}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

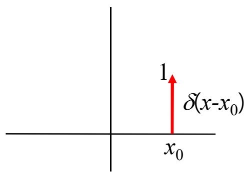
Impulses and Their Sifting Property (Discrete Signal)

Unit Impulse (x: discrete variable)

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$





Sifting Property

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

1D Fourier Transform

- Even *nonperiodic* function can be also expressed as the integral of sines and/or cosines multiplied by a weighting functions.
- "Nonperiodic" → It is also periodic, but its period is infinitely long.
- The fundamental frequency is infinitely small.
  - → Integral of all frequency components (∑→∫)
  - → The spectrum is a continuous function.

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n u_0 t} \qquad u_0 = \frac{1}{T}}$$

- Fourier transform

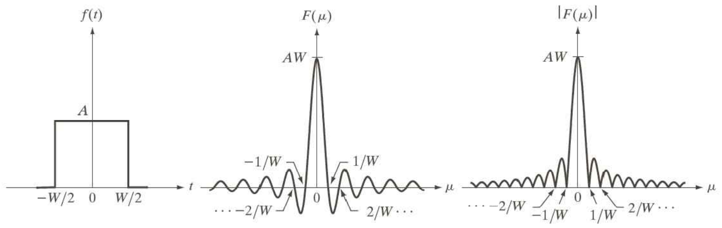
$$F(u) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} dt$$

- Inverse Fourier transform

$$f(t) = \Im^{-1}\{F(u)\} = \int_{-\infty}^{\infty} F(u) e^{+j2\pi ut} du$$

- Frequency (Fourier) domain : u
  - Unit : cycles/unit of t
    - - Hz (cycles/sec, 1/sec) for time [sec]
    - - cycles/meter (1/meter) for distance [meter]
- Fourier Spectrum (Frequency spectrum) : $|F(u)|$

Example



a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} dt = \int_{-W/2}^{W/2} A e^{-j2\pi ut} dt$$

$$= \frac{-A}{j2\pi u}[e^{-j\pi uW} - e^{-j\pi uW}]$$

$$= AW \frac{\sin(\pi Wu)}{\pi Wu} = AW\mathrm{sinc}(Wu) \qquad \mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{(\pi\theta)}$$

☞ Fourier transform of unit impulses, $\delta(t)$ and $\delta(t\text{-}t_0)$?

Symmetry Property

$$F(u) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ut}dt$$

$$f(t) = \Im^{-1}\{F(u)\} = \int_{-\infty}^{\infty} F(u)e^{+j2\pi ut}du$$

$$\Im\{F(t)\} = \int_{-\infty}^{\infty} F(t)e^{-j2\pi ut}dt = \int_{-\infty}^{\infty} F(t)e^{+j2\pi(-u)t}dt = f(-u)$$

Examples

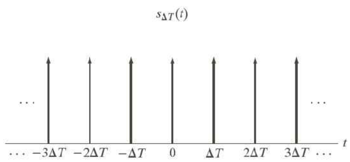
$\delta(t)$	$\Leftrightarrow$	1		1	$\Leftrightarrow$	$\delta(u)$
$\delta(t-t_0)$	$\Leftrightarrow$	$e^{-j2\pi ut_0}$	$\longleftrightarrow$	$e^{j2\pi u_0 t}$	$\Leftrightarrow$	$\delta(u-u_0)$
rectangular pulse	$\Leftrightarrow$	$\text{sinc}(u)$		sinc(t)	$\Leftrightarrow$	rectangular spectrum

Fourier Transform of an Impulse Train

$$s_{\Delta t}(t) = \sum_{n=-\infty}^{\infty}\delta(t-n\Delta T)$$
$$= \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta t}(t)e^{-j\frac{2\pi n}{\Delta T}t}dt = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2}\delta(t)e^{-j\frac{2\pi n}{\Delta T}t}dt = \frac{1}{\Delta T}$$

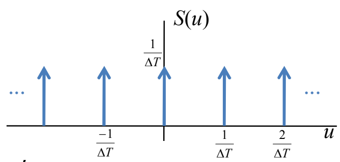
$$\therefore s_{\Delta t}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$



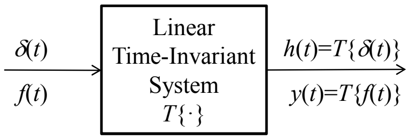
$$\Im\{e^{j\frac{2\pi n}{\Delta T}t}\} = \delta(u-\frac{n}{\Delta T})$$

$$S(u) = \Im\{s_{\Delta t}(t)\} = \Im\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\}$$
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta(u-\frac{n}{\Delta T})$$

→ *An impulse train, too!*



Convolution



Impulse response $h(t) = T\{\delta(t)\}$

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(\tau - t)d\tau$$ Sifting property

$$y(t) = T\{f(t)\} = T\{\int_{-\infty}^{\infty} f(\tau)\delta(\tau - t)d\tau\}$$

$$= \int_{-\infty}^{\infty} T\{f(\tau)\delta(\tau - t)\}d\tau$$ Additivity

$$= \int_{-\infty}^{\infty} f(\tau)T\{\delta(\tau - t)\}d\tau$$ Homogeneity

$$= \int_{-\infty}^{\infty} f(\tau)h(\tau - t)d\tau$$ Time-invariance

$$= f(t) * h(t)$$

Convolution Theorem

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau)d\tau$$

$$\Im\{f(t) * h(t)\} = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(\tau)h(t - \tau)d\tau]e^{-j2\pi ut}dt$$

$$= \int_{-\infty}^{\infty} f(\tau)[\int_{-\infty}^{\infty} h(t - \tau)e^{-j2\pi ut}dt]d\tau \qquad t - \tau = z$$

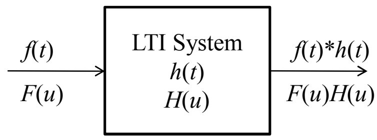
$$= \int_{-\infty}^{\infty} f(\tau)[\int_{-\infty}^{\infty} h(z)e^{-j2\pi u(z+\tau)}dz]d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)[e^{-j2\pi u\tau}\int_{-\infty}^{\infty} h(z)e^{-j2\pi uz}dz]d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)[e^{-j2\pi u\tau}H(u)]d\tau$$

$$= H(u)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi u\tau}d\tau$$

$$= H(u)F(u)$$



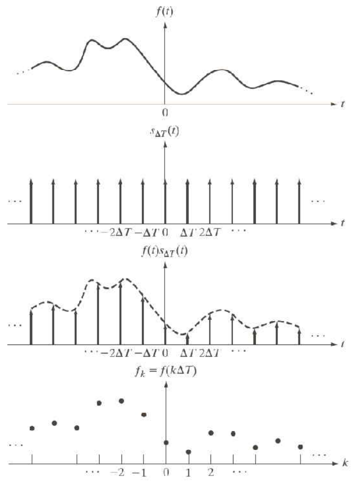
$$\boxed{\begin{array}{c} f(t)*h(t) \Leftrightarrow \mathrm{H(u)F(u)} \\ f(t)h(t) \Leftrightarrow \mathrm{H(u)*F(u)} \end{array}}$$

Sampling

Sample function $\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$

Arbitrary sample $f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt = f(k\Delta T)$

Sampling function

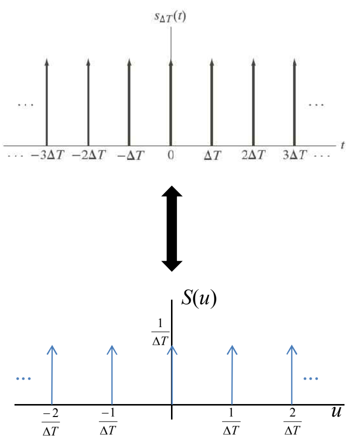


The Fourier Transform of Sampled Functions

$$\tilde{F}(u) = \Im\{\tilde{f}(t)\} = \Im\{f(t)s_{\Delta T}(t)\}$$
$$= F(u) * S(u)$$

$$S(u) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta(u - \frac{n}{\Delta T})$$

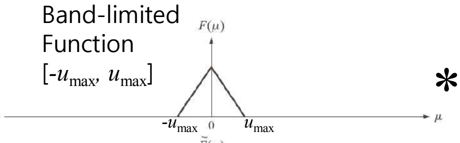
$$\tilde{F}(u) = F(u) * S(u) = \int_{-\infty}^{\infty} F(\upsilon)S(u-\upsilon)d\upsilon$$
$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\upsilon)\sum_{n=-\infty}^{\infty}\delta(u - \upsilon - \frac{n}{\Delta T})d\upsilon$$
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\upsilon)\delta(u - \upsilon - \frac{n}{\Delta T})d\upsilon$$
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F(u - \frac{n}{\Delta T})$$



The Fourier transform $\tilde{F}(u)$ of the sampled function $\tilde{f}(t)$ is an *infinite, periodic of copies* of the transform $F(u)$ of the original, continuous function $f(t)$

The Sampling Theorem

Band-limited Function
[$-u_{\max}$, $u_{\max}$]



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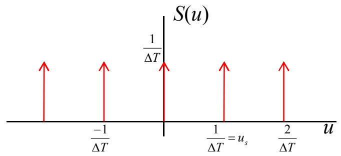
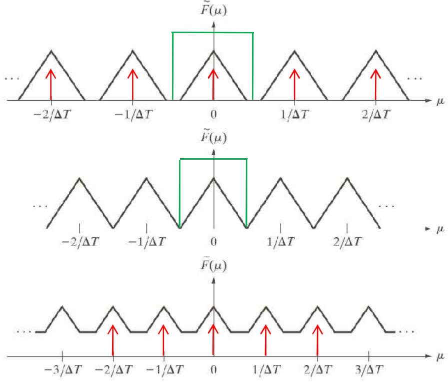


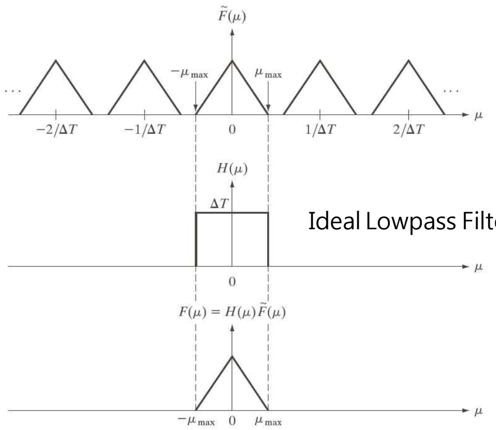


Over-sampling $u_s = \dfrac{1}{\Delta T} > 2u_{\max}$

Critically-sampling $u_s = \dfrac{1}{\Delta T} = 2u_{\max}$ *Nyquist rate*

Under-sampling $u_s = \dfrac{1}{\Delta T} < 2u_{\max}$

Reconstruction (Recovery) from Sampled Data



Ideal Lowpass Filter $H(u) = \begin{cases} \Delta T & -u_{\max} \le u \le u_{\max} \\ 0 & \text{otherwise} \end{cases}$

$\Downarrow$

Reconstruction Filter $H_R(u)$

$$\hat{F}(u) = H_R(u)\tilde{F}(u)$$

$$\hat{f}(t) = h_R(t) * \tilde{f}(t)$$

$\uparrow$

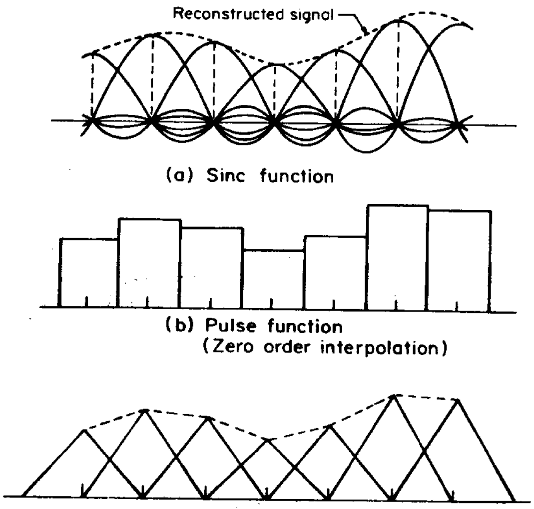
Interpolation with other functions
(nearest-neighbor, bilinear,...)

$$F(u) = H(u)\tilde{F}(u)$$

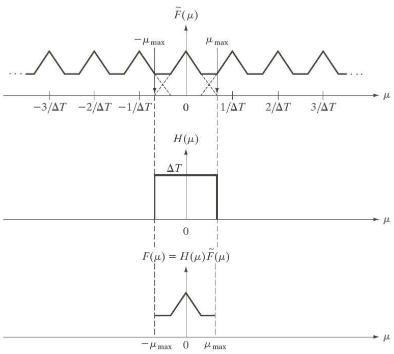
$$f(t) = h(t) * \tilde{f}(t) = \mathrm{sinc}(t) * \tilde{f}(t)$$

$$= \sum_{n=-\infty}^{\infty} f(n\Delta T)\,\mathrm{sinc}[(t - n\Delta T)/n\Delta T]$$ $\leftarrow$ Interpolation with sinc

Reconstruction Example



Aliasing



a
b
c

FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

Anti-aliasing filter before sampling
- smoothing the input function
- attenuating high frequency components

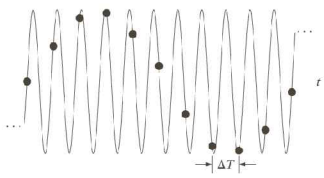
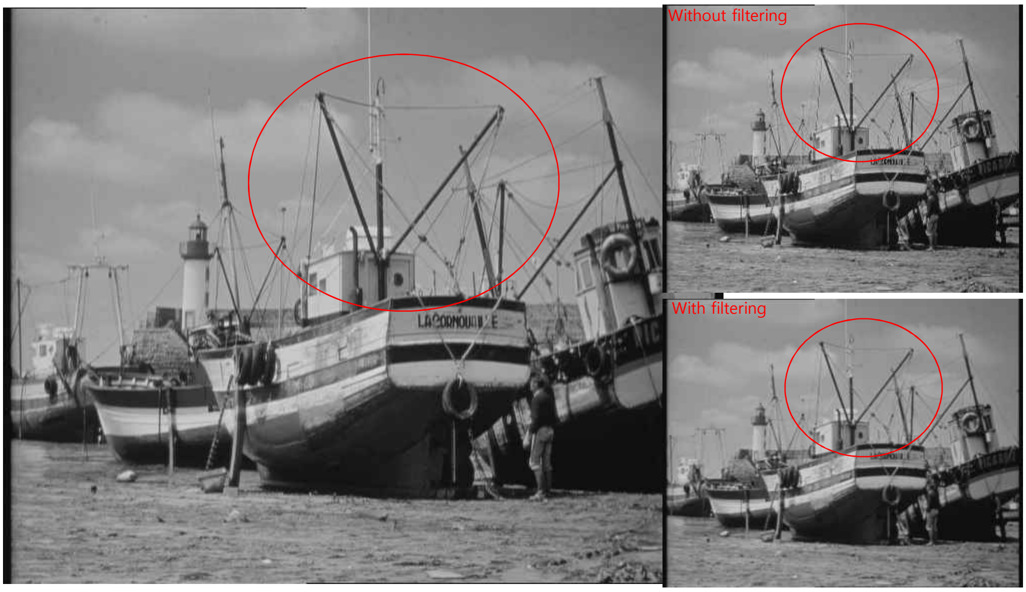


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

The wagon-wheel effect :
The "camera" constantly accelerates toward the right at the same rate with the objects sliding to the left. Halfway through the 24-second loop, the objects appear to suddenly shift and head backwards.

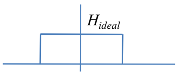
The wheel can appear to rotate more slowly than the true rotation, it can appear stationary, or it can appear to rotate in the opposite direction from the true rotation. This last form of the effect is sometimes called the reverse rotation effect.

Anti-aliasing Filter Before Subsampling



Band-Limited and Time-Limited

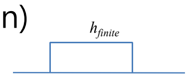
- Band-limited



$$F_{bl}(u) = H_{ideal}(u)F(u)$$

$$f_{bl}(t) = h_{ideal}(t) * f(t) = \mathrm{sinc}(t) * f(t) \qquad [-\infty, \infty]$$

- Time-limited (finite duration)



$$f_{finite}(t) = h_{finite}(t) f(t)$$

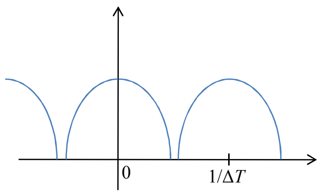
$$F_{finite}(u) = H_{finite}(u) * F(u) = \mathrm{sinc}(u) * F(u) \qquad [-\infty, \infty]$$

- No signal of finite duration can be band-limited.
  A band-limited signal must extend from -∞ to ∞ in time.

- Aliasing is inevitable for sampled data of finite length.

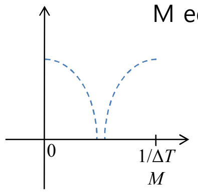
Discrete Fourier Transform (DFT)

- The Fourier transform of a *sampled, band-limited* function extending from -∞ to ∞ is a *continuous, periodic* function that also extends from -∞ to ∞.



$$
\begin{aligned}
\tilde{F}(u) &= \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi ut} dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t-n\Delta T) e^{-j2\pi ut} dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t-n\Delta T) e^{-j2\pi ut} dt \\
&= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi un\Delta T}
\end{aligned}
$$

→ Sampling one period $[0,\ 1/\Delta T]$ is the basis for the DFT.
M equally spaced samples of $\tilde{F}(u)$ taken over the period $[0,\ 1/\Delta T]$



$$u = \frac{1}{\Delta T}\frac{m}{M} \qquad m = 0,1,2,\ldots,M-1$$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0,1,2,\ldots,M-1$$

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{+j2\pi mn/M} \qquad n = 0,1,2,\ldots,M-1$$

1D DFT

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0,1,2,\ldots,M-1$$

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u) e^{+j2\pi ux/M} \qquad x = 0,1,2,\ldots,M-1$$

$$F(u) = F(u+kM), \quad f(x) = f(x+kM)$$

- Both $F(u)$ and $f(x)$ are periodic and discrete.
- Convolution theorem of DFT → Circular Convolution

$$f(x) * h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

$$DFT\{f(x) * h(x)\} = F(u)H(u)$$

☞ Linear convolution

$$f(x) * h(x) = \sum_{m=-\infty}^{\infty} f(m)h(x-m)$$

$$\Im\{f(x) * h(x)\} = F(u)H(u)$$

Linear Convolution and Circular Convolution

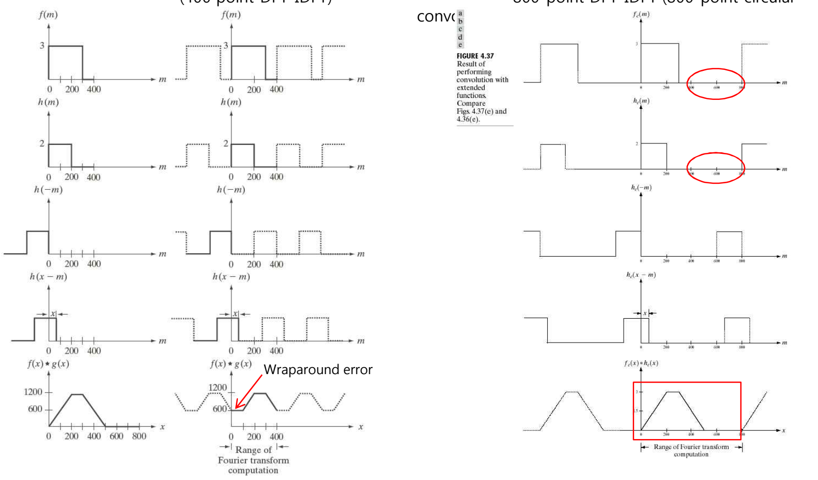
$$f(x)*h(x)=\sum_{m=-\infty}^{\infty} f(m)h(x-m)$$

$$f(x)*h(x)=\sum_{m=0}^{M-1} f(m)h(x-m)$$
$$=IDFT[DFT\{f(x)\}\cdot DFT\{h(t)\}]$$

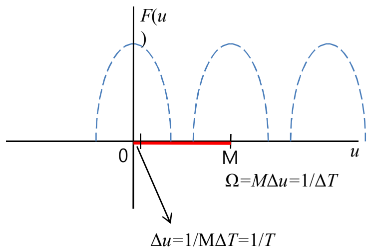
Linear convolution

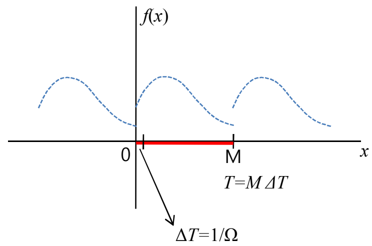
400-point circular convolution
(400 point DFT-IDFT)

→ Solution : Padding 400 zeros
800-point DFT-IDFT (800-point circular
conv



Relationship between the Sampling and Frequency Intervals





$$\Delta u=1/T$$
$$\Delta T=1/\Omega$$

ex) FOV=100 mm, 1000 pixels
→ Sampling interval (spatial resolution ΔT) =0.1 mm
1000-DFT → Frequency interval (Δu) =1/100 cycles/mm

Fast Fourier Transform (FFT)

- Fast algorithm of DFT
- Computations (multiplications and additions) : M^2 → $M\log_2 M$
- M-point DFT → 2 × (M/2-point DFT) → 4 × (M/4-point DFT)
  → 8 × (M/8-point DFT) → … → M/2 × (2-point DFT)

  ex) 1024-point DFT → ……… → 512 × (2-point DFT)

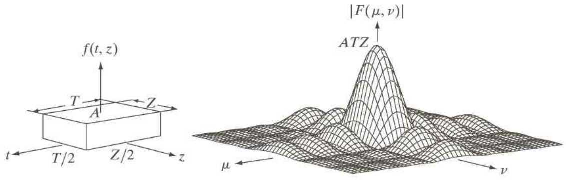
  total 10 steps ($\log_2 1024=10$)

  Computations : 1024×*1024* → 1024×*10* (100 times)
- Restriction : $M=2^n$

2D Fourier Transform

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{+j2\pi(ux+vy)}dudv$$

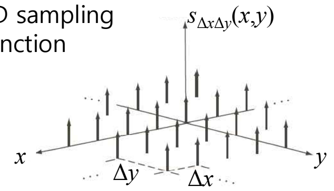


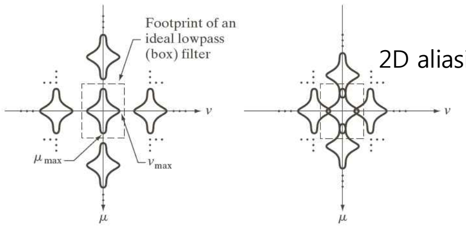
a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more “contracted” along the μ-axis. Compare with Fig. 4.4.

2D Aliasing

2D sampling function





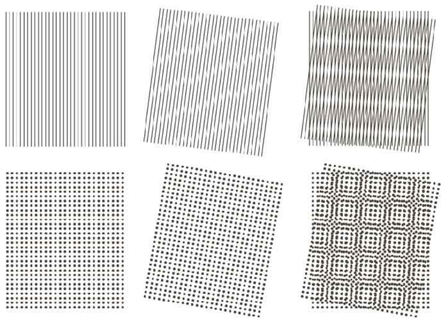
2D aliasing

2D Aliasing

a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Examples of Moiré Patterns



a b c
d e f

FIGURE 4.20 Examples of the moiré effect. These are ink drawings, not digitized patterns. Superimposing one pattern on the other is equivalent mathematically to multiplying the patterns.

2D DFT

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} \qquad u=0,1,2,\ldots,M-1 \qquad v=0,1,2,\ldots N-1$$

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{+j2\pi(ux/M+vy/N)} \qquad x=0,1,2,\ldots,M-1 \qquad y=0,1,2,\ldots N-1$$

2D DFT can be computed by applying 2 times a 1D DFT along orthogonal directions

$$F(u,v)=\sum_{y=0}^{N-1}\left[\sum_{x=0}^{M-1} f(x,y)e^{-j2\pi ux/M}\right] e^{-j2\pi vy/N}$$

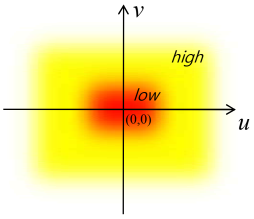
… $(M+N)$ 1D-DFTs

- Image $f(x,y)$: Real
  Spectrum $F(u,v)$: Complex, Conjugate symmetric
  $|F(u,v)|$: even, $\phi(u,v)$: odd
  Real$\{F(u,v)\}$: even, Imag$\{F(u,v)\}$: odd

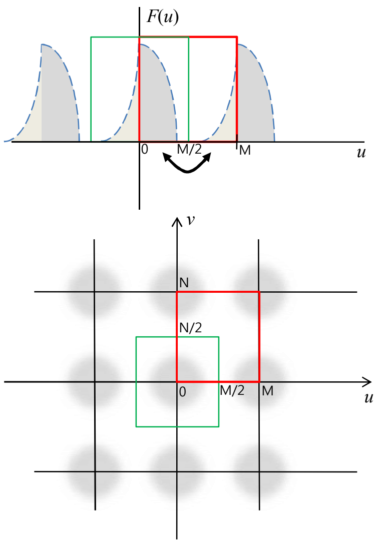
- $F(0,0)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$ → DC component $F(0,0)$ = sum of pixel values
  → Mean of pixel values $\bar{f}(x,y)=F(0,0)/MN$

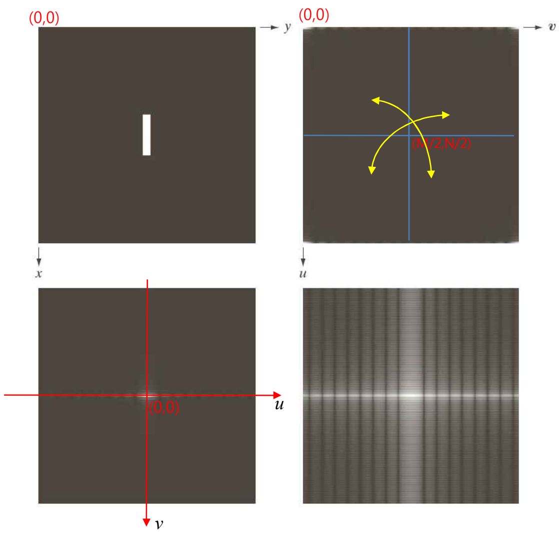
Remarks

- Each coefficient of the DFT depends on all pixel values of the image
- *DC component* $F(0,0)$ is the *average luminance* value of the image
- *Low frequency components* are related to slowly varying image features (i.e. smooth regions)
- *High frequency components* are related to quickly varying image features, i.e.
  - Textures
  - Edges
  - Noise

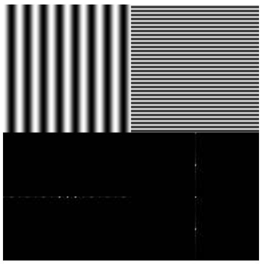


Centering the Fourier Transform

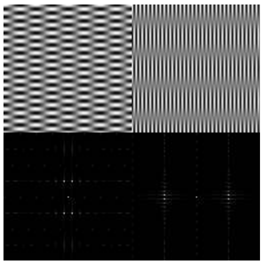




Example 1

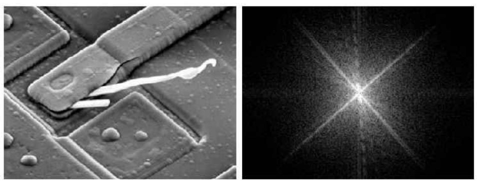


The images are a pure horizontal cosine of 8 cycles and a pure vertical cosine of 32 cycles. Notice that the FT for each just has a single component.



The left image has 4 cycles horizontally and 16 cycles vertically. The right image has 32 cycles horizontally and 2 cycles vertically.

Example 2



a b

FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

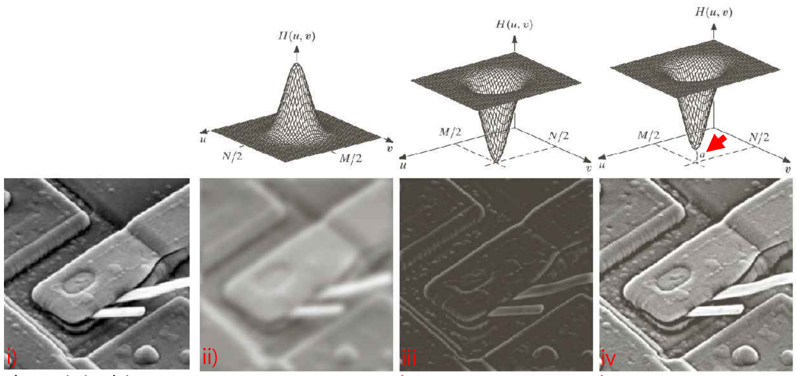
Note two principal features:

1) strong edges running at ±45˚

2) white oxide protrusions (→ off-axis slightly to the left)

Frequency Domain Filtering

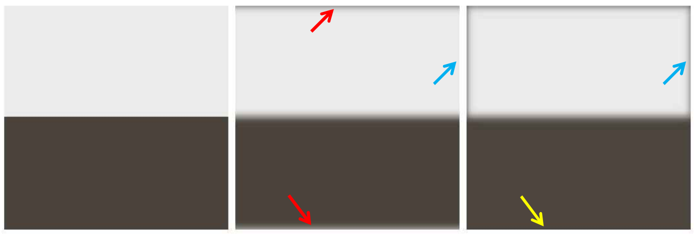
$g(x,y) = \textit{IDFT}\,[H(u,v) \cdot \textit{DFT}\{f(x,y)\}]$



the original image

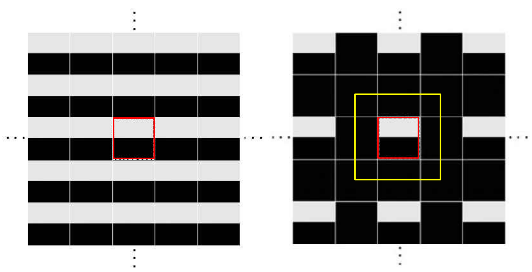
iii) + i)×a

Circular Convolution and Linear Convolution



a b c

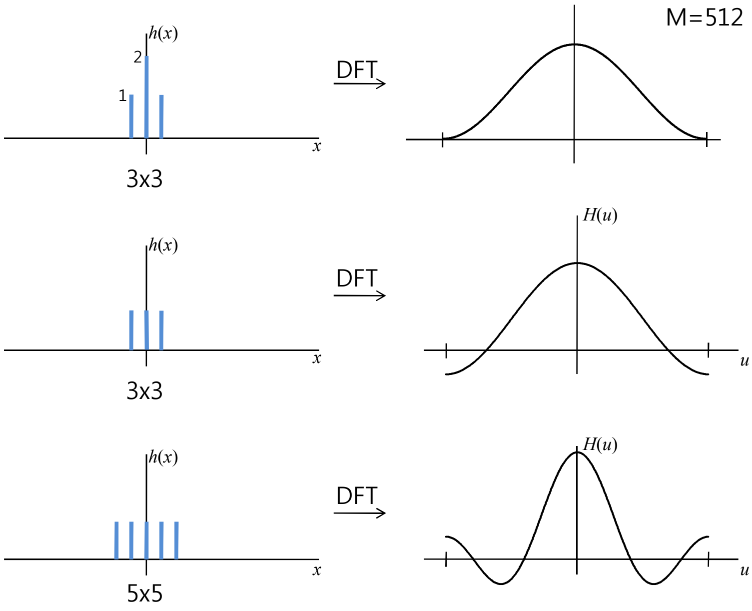
FIGURE 4.32 (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with padding. Compare the light area of the vertical edges in (b) and (c).



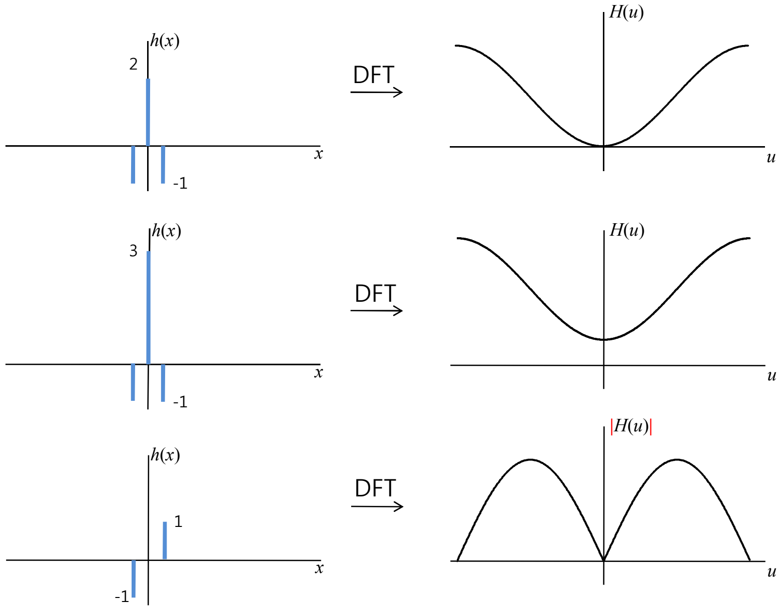
a b

FIGURE 4.33 2-D image periodicity inherent in using the DFT. (a) Periodicity without image padding. (b) Periodicity after padding with 0s (black). The dashed areas in the center correspond to the image in Fig. 4.32(a). (The thin white lines in both images are superimposed for clarity; they are not part of the data.)

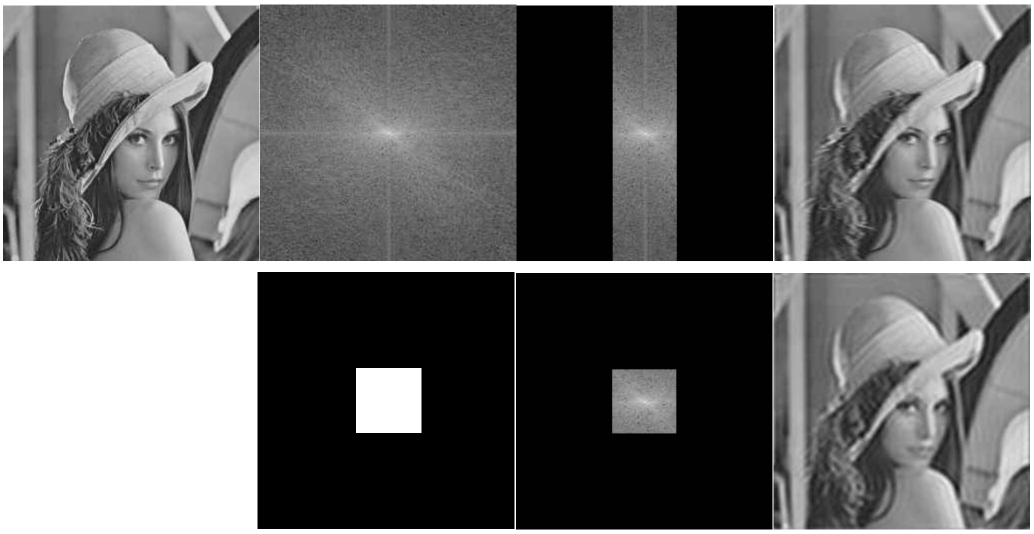
Smoothing Spatial Filter



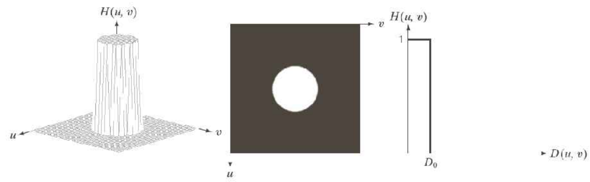
Laplacian and Sobel Operators



Ideal Lowpass Filtering

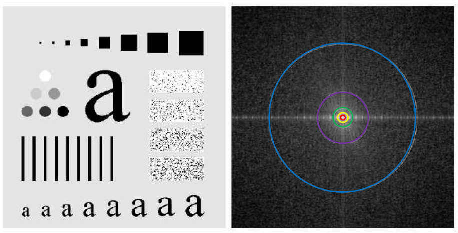


Cutoff Frequency



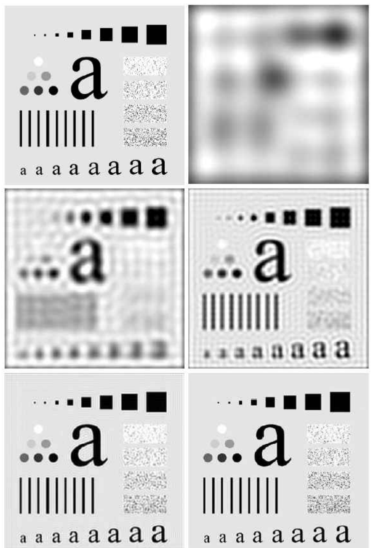
a b c

FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.



a b

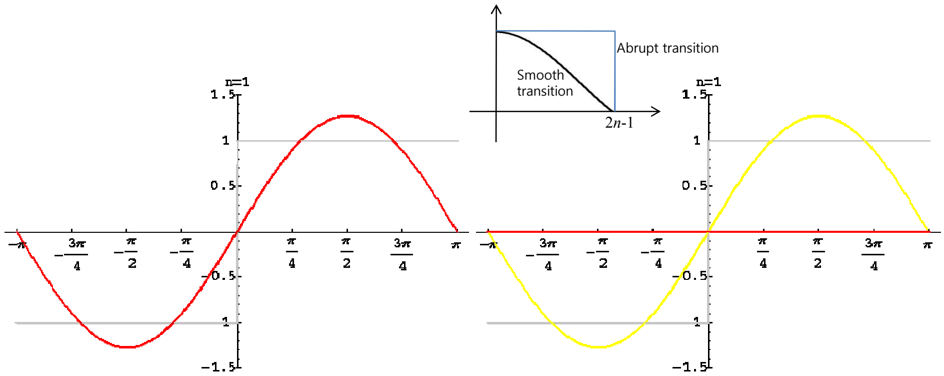
FIGURE 4.41 (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



a b
c d
e f

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

Gibbs Phenomenon – Ringing Artifacts



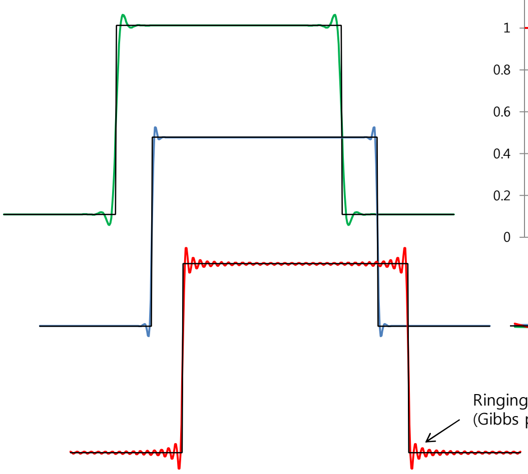
$$f(x)=\frac{4}{\pi}\{\sin(x)+\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)+...+\frac{1}{(2n-1)}\sin((2n-1)x)\}$$

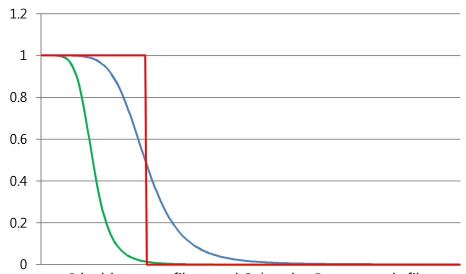
$$f(x)=\frac{4}{\pi}\{\mathrm{sinc}(\frac{1}{2n-1})\sin(x)+\mathrm{sinc}(\frac{3}{2n-1})\frac{1}{3}\sin(3x)+\mathrm{sinc}(\frac{5}{2n-1})\frac{1}{5}\sin(5x)+$$

$$...+\mathrm{sinc}(\frac{(2n-1)}{2n-1})\frac{1}{2n-1}\sin((2n-1)x)\}$$

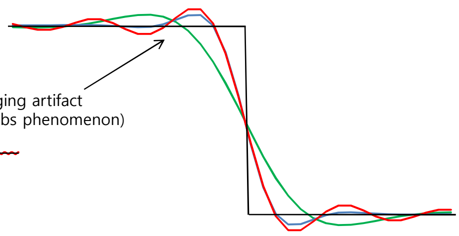
$$\mathrm{sinc}(a)=\frac{\sin(\pi a)}{\pi a}$$

Filter and Ringing Artifact





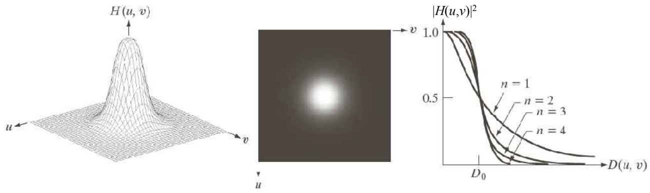
Ideal lowpass filter and 3^{rd}-order Butterworth filters with 2 cutoff frequencies



Ringing artifact
(Gibbs phenomenon)

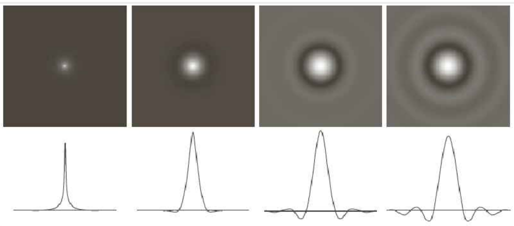
Butterworth Lowpass Filter

$$|H(u,v)|^2 = \frac{1}{1+[D(u,v)/D_0]^{2n}} \qquad D(u,v) = \sqrt{u^2+v^2}$$



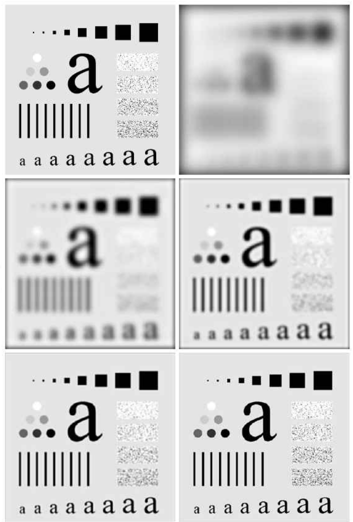
a b c

FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.



a b c d

FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

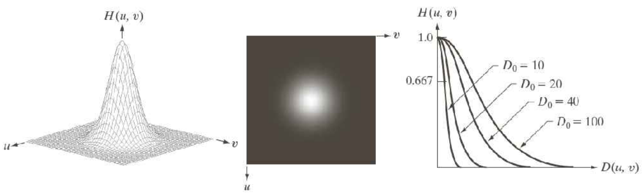


a b
c d
e f

FIGURE 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

Gaussian Lowpass Filter

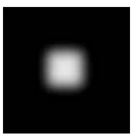
$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$
$$= e^{-D^2(u,v)/2D_0^2} \quad \text{for } \sigma = D_0$$

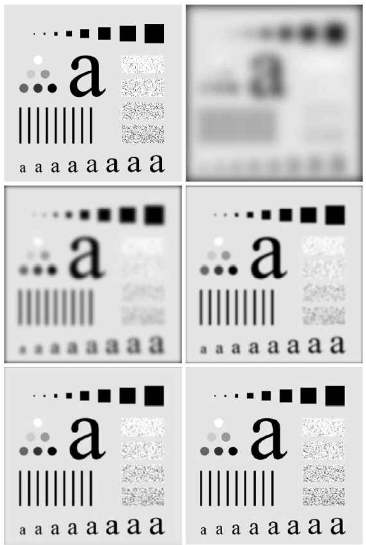


a b c

FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

$$H(u,v) = e^{-u^2/2u_{c1}^2} \cdot e^{-v^2/2u_{c2}^2}$$





a b
c d
e f

FIGURE 4.48 (a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.

Image Sharpening Using Frequency Domain Filters

$$H_{\mathrm{HP}}(u,v)=1-H_{\mathrm{LP}}(u,v)$$

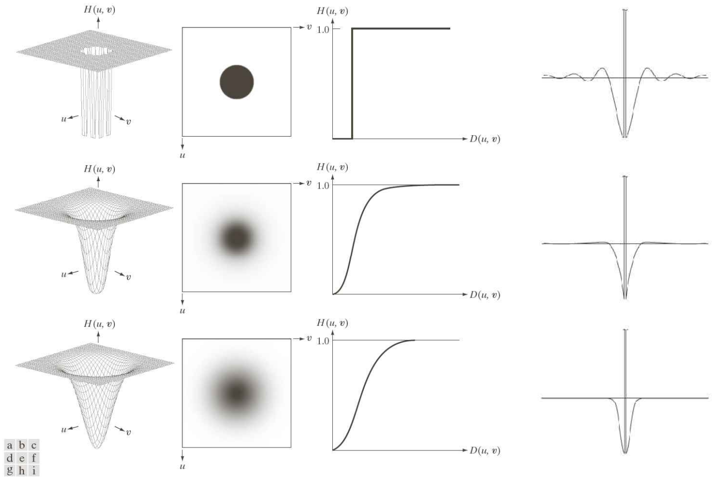


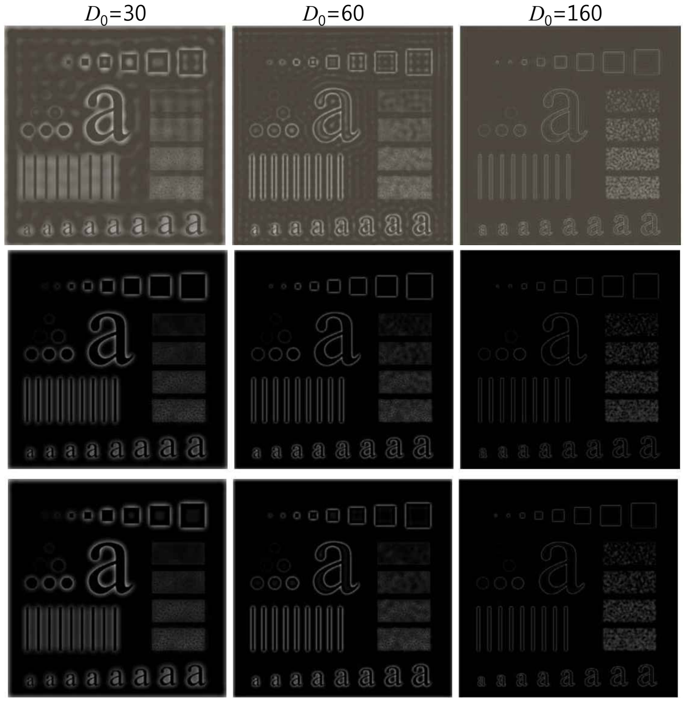
FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

D_0=30 D_0=60 D_0=160

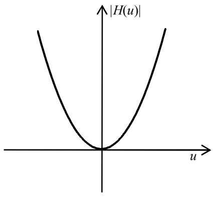
Ideal HPF

Butterworth HPF

Gaussian HPF



The Laplacian in the Frequency Domain



$$\nabla^2 f(x,y) = \frac{\partial^2}{\partial x^2} f(x,y) + \frac{\partial^2}{\partial y^2} f(x,y)$$

$$\Im\{\nabla^2 f(x,y)\} = \Im\{\frac{\partial^2 f}{\partial x^2}\} + \Im\{\frac{\partial^2 f}{\partial y^2}\}$$

$$= (j2\pi u)^2 F(u,v) + (j2\pi v)^2 F(u,v) \qquad \because \Im\{\frac{\partial f}{\partial x}\} = j2\pi u\, F(u,v)$$

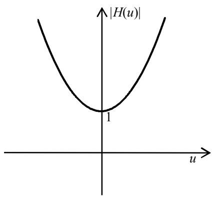
$$= -4\pi^2 (u^2+v^2) F(u,v)$$

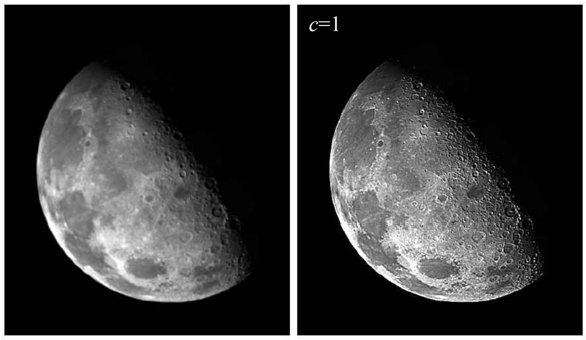
$$H(u,v) = -4\pi^2 (u^2+v^2)$$

$$g(x,y) = f(x,y) + c\nabla^2 f(x,y)$$

$$= \Im^{-1}\{F(u,v) - cH(u,v)F(u,v)\}$$

$$= \Im^{-1}\{[1+4\pi^2 c(u^2+v^2)]F(u,v)\}$$





Unsharp Masking and Highboost Filtering

$$g_{mask}(x,y) = f(x,y) - f_{\mathrm{LP}}(x,y)$$

$$g(x,y) = f(x,y) + k \cdot g_{mask}(x,y)$$

$$= \Im^{-1}\{[1 + k \cdot [1 - H_{\mathrm{LP}}(u,v)]]F(u,v)\}$$

$$= \Im^{-1}\{[1 + k \cdot H_{\mathrm{HP}}(u,v)]F(u,v)\}$$

High-frequency-emphasis filter

$$g(x,y) = \Im^{-1}\{[k_1 + k_2 \cdot H_{\mathrm{HP}}(u,v)]F(u,v)\}$$

$$H(u,v) = k_1 + k_2 \cdot H_{\mathrm{HP}}(u,v)$$

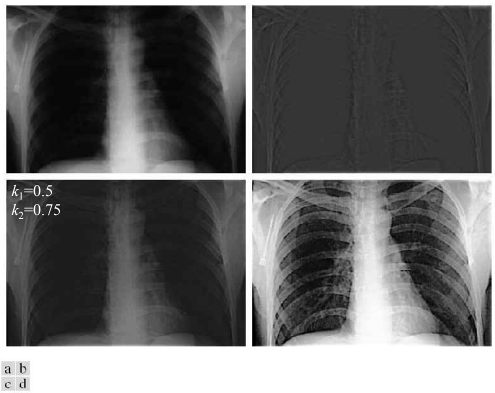
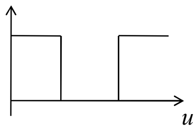
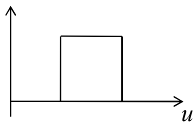


FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

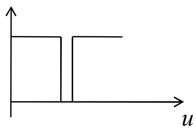
Selective Filtering

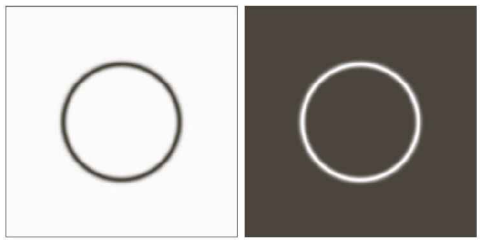
- Band-reject and band-pass filters

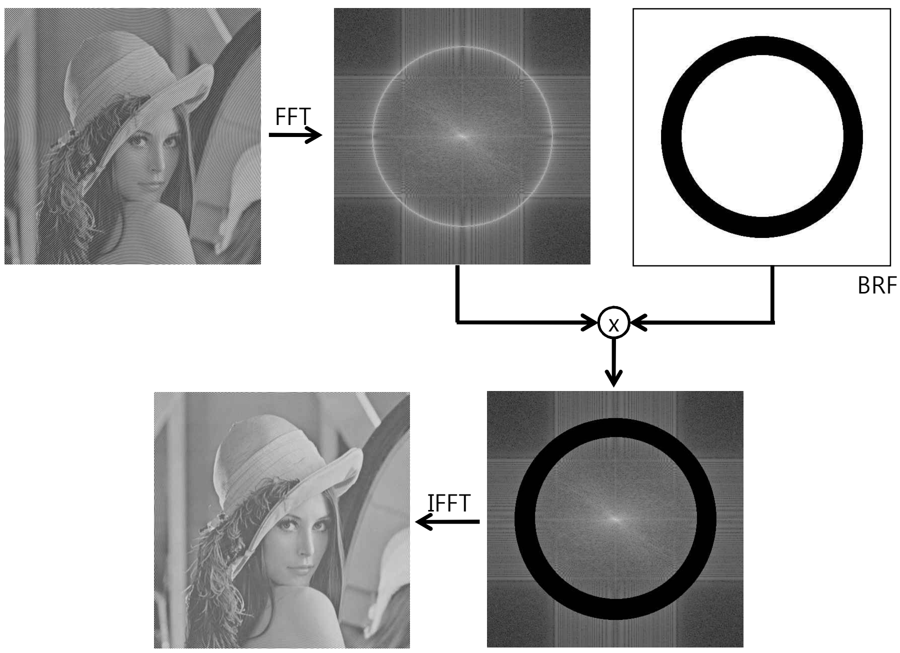


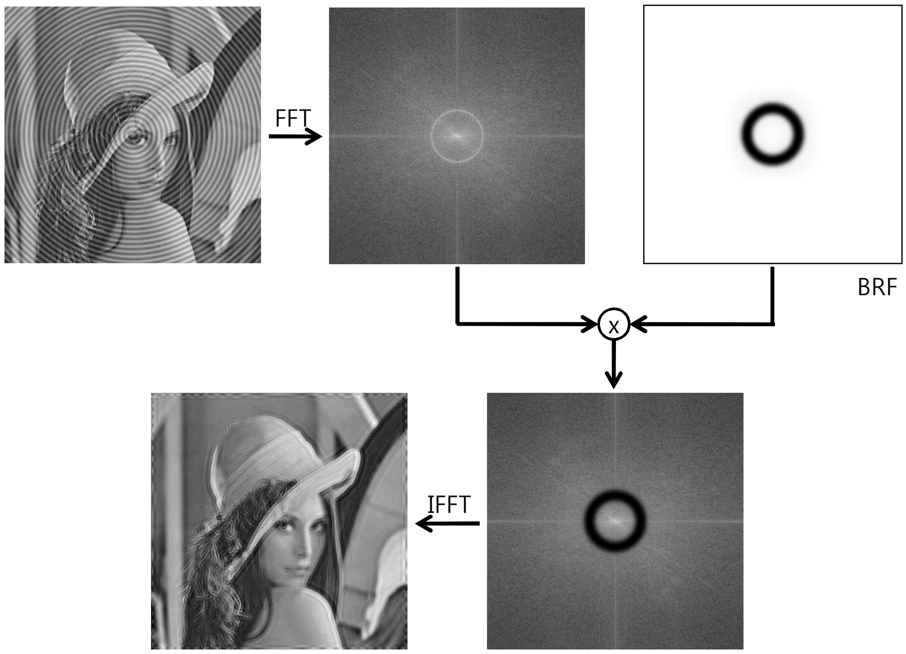


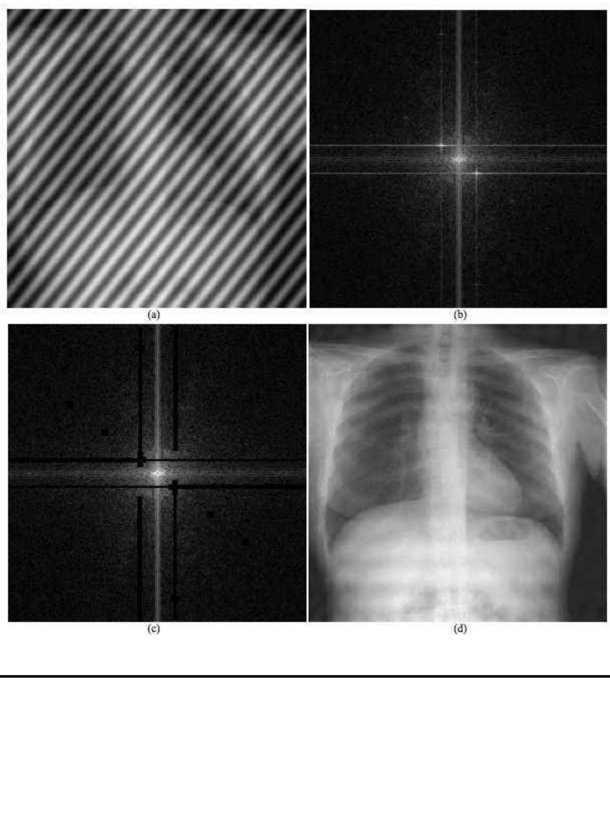
- Notch filter








FFT
BRF
x
IFFT


(a)
(b)
(c)
(d)

Discussion

- Why filtering in the frequency domain?
  - Weigh pros and cons of filtering in the frequency domain.

Homework #4

- Find the cutoff frequency u_c of the rectangular Gaussian lowpass filter which improves SNR as much as a 5x5 averaging filter.
- Compare the edges of both images. Which edge looks more blurred?

- The SNR improvement of filters depends on the characteristics of images. (Why?)
  Use an uniform phantom image (512x512) below

$$H(u,v) = e^{-u^2/2u_c^2} \cdot e^{-v^2/2u_c^2}$$