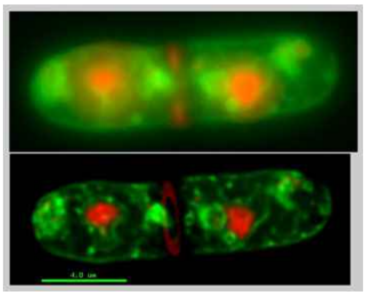
Medical Image Processing

V. Image Restoration and Reconstruction



"Things which we see are not by themselves what we see…" - I. Kant -

V. Image Restoration and Reconstruction

1. A model of the image **degradation**/restoration process
2. Noise models
3. Restoration in the presence of noise only – spatial filtering
4. Periodic noise reduction by frequency domain filtering – optimum notch filtering
5. Linear, position-invariant degradations
6. Inverse filtering
7. Minimum mean square error (Wiener) filtering
8. Constrained Least Squares Filtering
9. Image reconstruction from projections

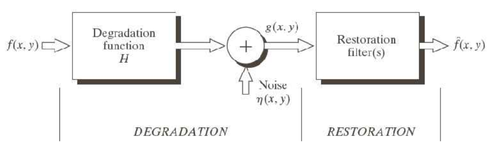
Preview

- Improve an image in some predefined sense
- Objective process

  (☞ Enhancement technique : subjective, heuristic)
- Reconstruct or recover an degraded image by using a priori knowledge of the degradation phenomenon
- Model the degradation and apply the inverse process in order to recover the original image

A model of the image degradation/restoration process

FIGURE 5.1 A model of the image degradation/restoration process.



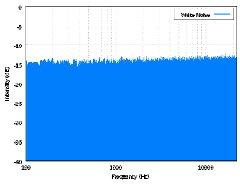
$$g(x,y) = h(x,y) * f(x,y) + \eta(x,y)$$
$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

Additive noise

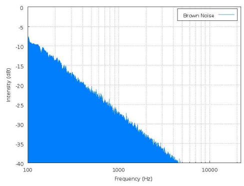
$$\hat{f}(x,y) = R\{g(x,y)\}$$

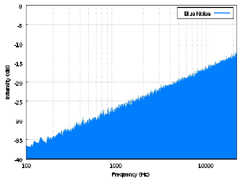
Noise Models

- Frequency properties
  - White noise : constant noise spectrum



  - Colored noise

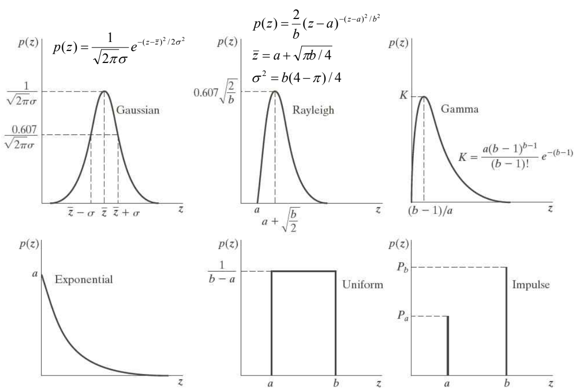




Noise Models

- Noise amplitude distribution : Probability density function

- Gaussian : electronic circuit noise, sensor noise
- Rayleigh : range imaging
- Exponential and Gamma : laser imaging
- Uniform : quantization
- Impulse : defective pixels, faulty switching



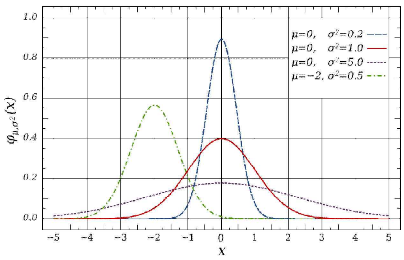
a b c
d e f

FIGURE 5.2 Some important probability density functions.

Gaussian Noise

- Amplitude distribution ← Gaussian function
- Normal distribution
- 70% of its values are in the range $-\sigma \leq \bar{z} \leq \sigma$
- Central limit theorem : The sum of a sufficiently large number of identically distributed independent random variables each with finite mean and variance will be approximately normally (Gaussian) distributed.

$$p(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-\bar{z})^2/2\sigma^2}$$



Examples 1

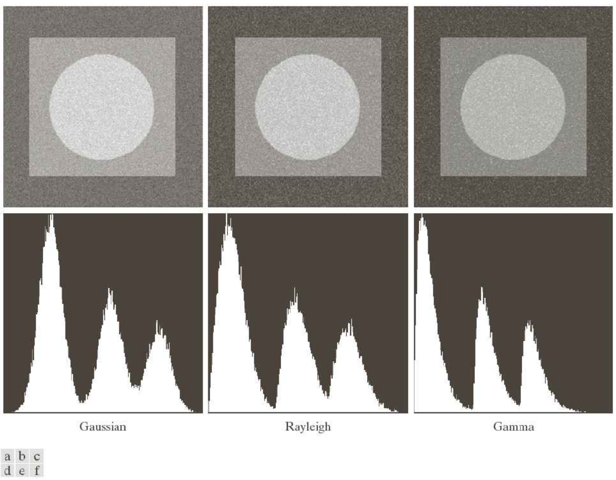


FIGURE 5.4 Images and histograms resulting from adding Gaussian, Rayleigh, and gamma noise to the image in Fig. 5.3.

Examples 2

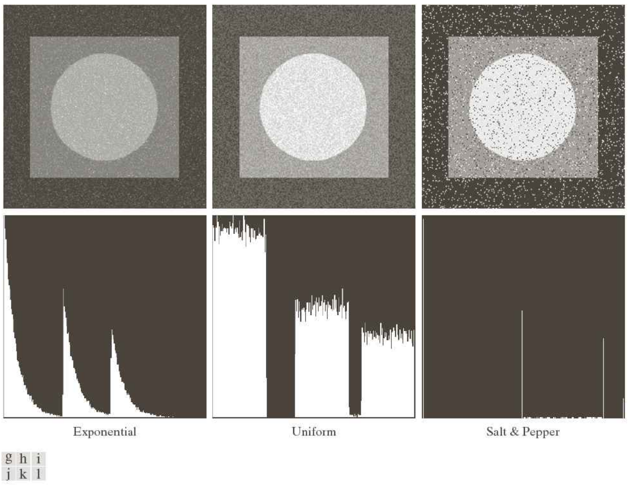
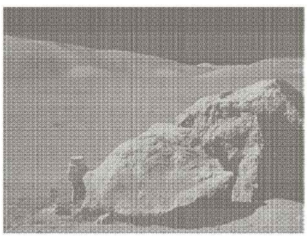
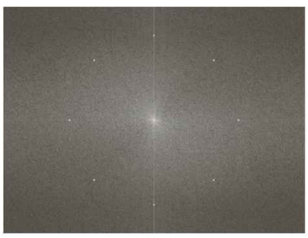


FIGURE 5.4 *(Continued)* Images and histograms resulting from adding exponential, uniform, and salt and pepper noise to the image in Fig. 5.3.

Periodic Noise

- Periodic noise in an image irises typically from electrical or electromechanical interference during image acquisition
- Can be reduced significantly via frequency domain filtering





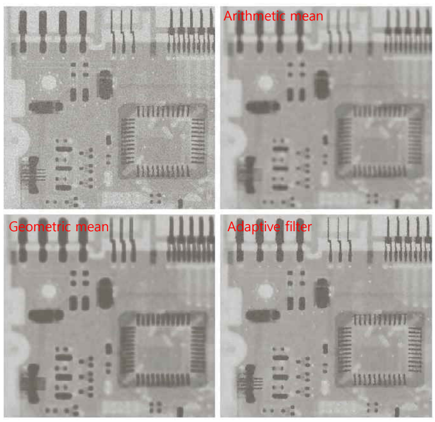
Restoration in the Presence of Noise Only - Spatial Filtering

- Mean filter
  - Arithmetic mean
  - Geometric mean
  - Harmonic mean
  - Contraharmonic mean $\hat{f}(x,y) = \dfrac{\sum g(s,t)^{Q+1}}{\sum g(s,t)^{Q}}$
- Order-statistics filter
  - Median
  - Max and min
  - Midpoint (min+max)/2
  - Alpha-trimmed mean : delete the d/2 lowest and the d/2 highest and average remaining pixels

Adaptive Filter

- Image characteristics vary from one point to another.
- Adaptive filter : Its behavior changes based on statistical characteristics inside the filter region.
- Improved performance ↔ Filter complexity
- Simple adaptive filter
  - If $\sigma_n^2=0$ → there is no noise → $\hat{f}(x,y) = g(x,y)$
  - If $\sigma_L^2 \gg \sigma_n^2$ → high contrast (edge) → $\hat{f}(x,y) \cong g(x,y)$
  - If $\sigma_L^2 \approx \sigma_n^2$ → noise in a smooth area → $\hat{f}(x,y) \cong m_L$

$$\Rightarrow \hat{f}(x,y) = g(x,y) - \frac{\sigma_n^2}{\sigma_L^2}[g(x,y) - m_L]$$

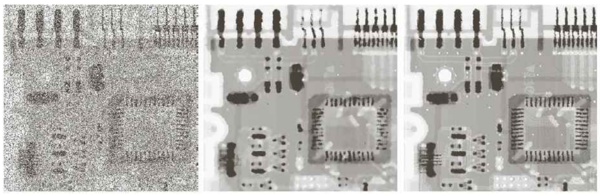


Adaptive Median Filter

- Main purposes
  - To remove salt-and-pepper noise
  - To provide smoothing of other noise (not impulsive)
  - To reduce distortion such as excessive thinning or thickness of object boundaries

- Stage A: (S_{max}= maximum allowed size of window)
  If $z_{min}<z_{med}<z_{max}$, go to stage B
  Else increase the window size
  If window size≤S_{max}, repeat stage A
  Else output z_{med}

- Stage B:
  If $z_{min}<z_{xy}<z_{max}$, output z_{xy}
  Else output z_{med}



a b c

FIGURE 5.14 (a) Image corrupted by salt-and-pepper noise with probabilities $P_a = P_b = 0.25$. (b) Result of filtering with a 7 × 7 median filter. (c) Result of adaptive median filtering with $S_{\max} = 7$.

Periodic Noise Reduction - Optimum Notch Filtering

- Optimum : Minimize local variances of the restored image
  1) Extract the principal frequency components $n(x, y)$ of the interference pattern
  2) Subtract a variable, weighted portion of the pattern from the corrupted image $g(x, y)$

$$1)\ N(u,v) = H_{NP}(u,v)G(u,v)$$

$$n(x,y) = \Im^{-1}\{N(u,v)\}$$

$$2)\ \hat{f}(x,y) = g(x,y) - w(x,y)n(x,y)$$

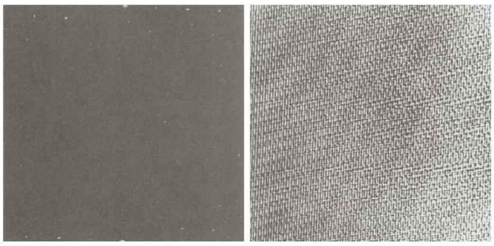
- $w(x, y)$: weighting (modulation) function
  → minimize the local variance of $\hat{f}(x,y)$

$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} [\hat{f}(x+s,y+t) - \bar{\hat{f}}(x,y)]^2$$

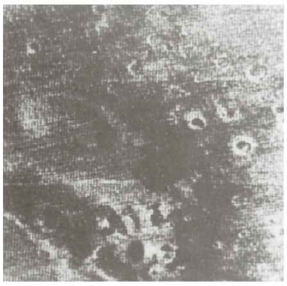
To minimize $\sigma^2(x,y)$, $\frac{\partial \sigma^2}{\partial w} = 0$ $\qquad (w(x,y) \cong w(x,y))$

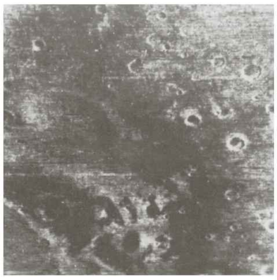
$$w(x,y) = \frac{\overline{g(x,y)n(x,y)} - \bar{g}(x,y)\bar{n}(x,y)}{\overline{n^2}(x,y) - \bar{n}^2(x,y)}$$

Result



a b
FIGURE 5.22 (a) Fourier spectrum of $N(u, v)$, and (b) corresponding noise interference pattern $\eta(x, y)$. (Courtesy of NASA.)





a=b=15

Linear, Position-Invariant Degradations

$$g(x, y) = h(x, y) * f(x, y) + n(x, y)$$
$$G(u, v) = H(u, v)F(u, v) + N(u, v)$$

- $h(x, y)$ is called point spread function (PSF)
- $H(u, v)$ is the frequency response of a linear space-invariant system
- The estimate of $f(x, y)$ is obtained by means of deconvolution methods (deconvolution filters applying the process in reverse)
- First, we need to estimate the degradation functions
  - estimation by image observation
  - estimation by experimentation
  - estimation by mathematical modeling
- $\hat{F}(u, v) = G(u, v) / H(u, v)$
- The process of restoring an image by using a degradation function sometimes is called ***blind deconvolution***, due to the fact that the true degradation function is seldom known completely.

Estimating the Degradation Function (1) Estimation by Image Observation

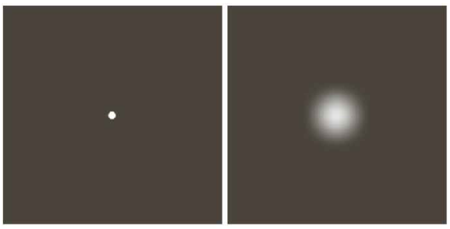
- Suppose that we are given a degraded image without any knowledge about the degradation function $H(u, v)$.
- Estimation ← Information from the image itself
  - Example: If the image is blurred, we can look at a small section of the image containing simple structures, like part of an object and the background
  1) Look for an area in which the signal content is strong (e.g., an area of high contrast)
  2) Process the subimage to arrive at a result is as unblurred as possible
- $H(u,v) \cong H_s(u,v) = G_s(u,v) / \hat{F}_s(u,v)$
- Laborious process → Not often applicable in practice
- Application example : Restoring an old photograph of historical value

(2) Estimation by Experimentation

- If equipment similar to the equipment used to acquire the degraded image is available, it is possible in principle to obtain an accurate estimate of the degradation.
- If the acquisition device is available, we can obtain an accurate estimate of the PSF.
- Acquire the image of a point (a small dot of light) under the same working conditions

a b

FIGURE 5.24 Degradation estimation by impulse characterization. (a) An impulse of light (shown magnified). (b) Imaged (degraded) impulse.



- $H(u, v) = G(u, v)/A$

(3) Estimation by Modeling

- Build a physical model of the degradation process
- Ex) Atmospheric turbulence $H(u, v)$=exp$\{-k(u^2+v^2)^{5/6}\}$

a b
c d

FIGURE 5.25 Illustration of the atmospheric turbulence model. (a) Negligible turbulence. (b) Severe turbulence, $k = 0.0025$. (c) Mild turbulence, $k = 0.001$. (d) Low turbulence, $k = 0.00025$. (Original image courtesy of NASA.)

Mathematical Model

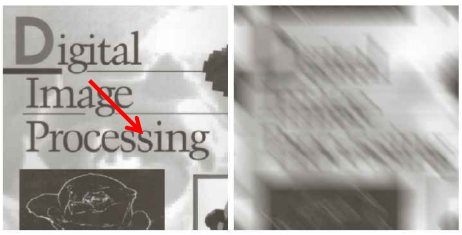
- Ex) Blur by uniform linear motion

$$g(x,y)=\int_0^T f[x-x_0(t),y-y_0]dt$$

$$G(u,v)=\Im\{g(x,y)\}=F(u,v)\int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

$$H(u,v)=\frac{G(u,v)}{F(u,v)}=\int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

$$=\frac{T}{\pi(ua+vb)}\sin[\pi(ua+vb)]e^{-j\pi(ua+vb)} \qquad x_0(t)=\frac{at}{T},\, y_0(t)=\frac{bt}{T}$$



a b

FIGURE 5.26 (a) Original image. (b) Result of blurring using the function in Eq. (5.6-11) with $a = b = 0.1$ and $T = 1$.

Inverse Filtering

- H : degradation function, G : degraded image

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)}$$

- Take noise into account

$$\hat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

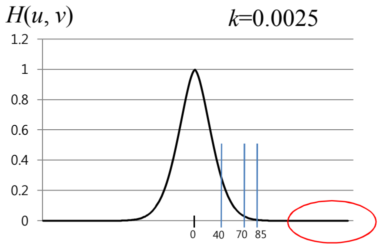
- If $H(u,v)$ has zero or very small values, $N(u,v)$ / $H(u,v)$ could easily dominate the estimate $\hat{F}(u,v)$

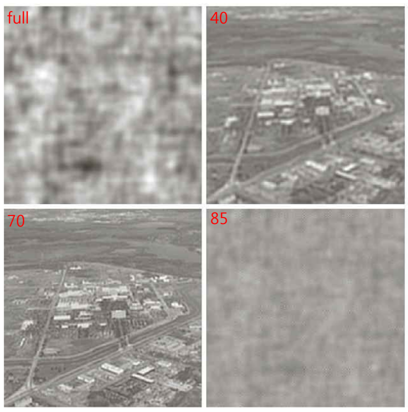
  → *This is frequently the case!*

  → Limit the filter frequencies to values near the origin

Example

- Turbulence degradation $H(u, v)=\exp\{-k(u^2+v^2)^{5/6}\}$





a b
c d

FIGURE 5.27 Restoring Fig. 5.25(b) with Eq. (5.7-1). (a) Result of using the full filter. (b) Result with H cut off outside a radius of 40; (c) outside a radius of 70; and (d) outside a radius of 85.

Minimum Mean Square Error (Wiener) Filtering

- Incorporate both the degradation function and statistical characteristics of noise into the restoration process.
- Assume
  - Consider images and noise as uncorrelated stationary random variables.
  - One or the other has zero mean.
  - The intensity levels in the estimate are a linear function of the levels in the degraded image.
- Find an estimate $\hat{f}$ of the uncorrupted image f such that the **MSE** between them is minimized.

  MSE $e^2 = E\{(f-\hat{f})^2\}$
- Wiener filter, optimal filter, least square error (LSE) filter

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)}\frac{|H(u,v)|^2}{|H(u,v)|^2 + S_n(u,v)/S_f(u,v)}\right]G(u,v)$$

H : degradation function

$S_n = E\{|N(u,v)|^2\}$: power spectrum of the noise

$S_f = E\{|F(u,v)|^2\}$: power spectrum of the undegraded image

Derivation (Spatial Domain)

$\hat{f}(m) = w(m) * g(m)$ $\quad w$: Wiener filter (linear filter)

error $e(m) = f(m) - \hat{f}(m)$

MSE
$$\begin{aligned}
E\{e^2\} &= E\{(f-\hat{f})^2\} \\
&= E\{f^2\} - 2E\{f\cdot\hat{f}\} + E\{\hat{f}^2\} \\
&= E\{f^2\} - 2E\{f\cdot(w*g)\} + E\{(w*g)^2\} \\
&= E\{f^2\} - 2E\{f\cdot\sum_{i=1}^{L} w(i)g(m-i)\} + E\{\left[\sum(w(i)g(m-i)\right]^2\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial w(j)}E\{e^2\} &= 0 - 2E\{g(m-i)f\} + 2E\{\left[\sum w(i)g(m-i)\right]g(m-j)\} \\
&= -2E\{\sum g(m-i)f\} + 2\sum w(i)E\{g(m-i)g(m-j)\}
\end{aligned}$$

cross-correlation $R_{gg}(m) = E\{g(n)g(n+m)\}$

autocorrelation $R_{fg}(m) = E\{f(n)g(n+m)\}$

$$\frac{\partial}{\partial w(j)}E\{e^2\} = -2R_{gf}(j) + 2\sum w(i)R_{gg}(i-j) = 0$$

$\sum w(i)R_{gg}(i-j) = R_{gf}(j)$

$w * R_{gg} = R_{gf}$

$S_g = \Im\{R_{gg}\},\ S_{gf} = \Im\{R_{gf}\}$

$W \cdot S_g = S_{gf}$

$g = f + n$

$R_{gf} = R_{(f+n)f} = R_{ff},\quad R_{gg} = R_{ff} + R_{nn}$

$$\therefore\ W = \frac{S_{gf}}{S_g} = \frac{S_f}{S_f + S_n} = \frac{1}{1 + S_n/S_f}$$

If we assume white noise (constant power spectral density), the Wiener filter is a low-pass filter.
It introduces significant blurring.

$g(m) = h(m) * f(m) + n(m)$

$R_{gf} = R_{(h*f+n)f} = h(-m) * R_{ff},\quad R_{gg} = h(-m)*h(m)*R_{ff} + R_{nn}$

$S_{gf} = H^* \cdot S_f,\quad S_g = S_f H^* H = S_f |H|^2$

$$\therefore\ W = \frac{S_{gf}}{S_g} = \frac{H^*\cdot S_f}{S_f|H|^2 + S_n} = \frac{1}{H}\frac{|H|^2\cdot S_f}{S_f|H|^2+S_n} = \frac{1}{H}\frac{|H|^2}{|H|^2 + S_n/S_f}$$

If noise is zero (S_n=0), the Wiener filter W is the inverse filter.

Derivation (Frequency Domain)

$g = h * f + n$ h: degradation function

$\hat{f} = w * g$ w: Wiener filter (linear filter)

$\hat{F} = W \cdot G = W(HF + N)$

error $e = F - \hat{F}$

MSE $E\{e^2\} = E\{|F - \hat{F}|^2\}$

$$
\begin{aligned}
&= E\{|F - W(HF+N)|^2\} = E\{|(1-WH)F - WN|^2\} \\
&= E\{[(1-WH)F - WN][(1-WH)^* F^* - W^* N^*]\} \\
&= (1-WH)(1-WH)^* E\{FF^*\} + (1-WH)W^* E\{FN^*\} + W(1-WH)^* E\{F^* N\} + WW^* E\{NN^*\} \\
&= (1-WH)(1-WH)^* E\{|F|^2\} + 0 + 0 + WW^* E\{|N|^2\} \\
&= (1-WH)(1-WH)^* S_f + WW^* S_n
\end{aligned}
$$

$$\frac{\partial}{\partial W} E\{e^2\} = -H(1 - W^* H^*) S_f + W^* S_n = 0$$

$$W^*(HH^* S_f + S_n) = HS_f$$

$$W = \frac{H^* S_f}{|H|^2 S_f + S_n} = \frac{1}{H} \frac{|H|^2 S_f}{|H|^2 S_f + S_n}$$

Note that if noise is zero (S_n=0), the Wiener filter W is the inverse filter $1/H$.

$g = f + n$ (only noise degradation)

$H = 1$

$$W = \frac{S_f}{S_f + S_n} = \frac{S_f}{S_f + c}$$

If we assume white noise (constant power spectral density), the Wiener filter is a low-pass filter. It introduces significant blurring.

Example

Practical Issues :

- $H(u,v)$ must be estimated
- The noise power spectrum $S_n(u,v)$ is unknown → we often assume white noise → $S_n(u,v)$ is constant.
- The power spectrum $S_f(u,v)$ of the undegraded image is unknown → can be estimated from a collection of similar images.

$$W(u,v) = \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_n(u,v)/S_f(u,v)} \approx \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + K}$$

a b c

FIGURE 5.28 Comparison of inverse and Wiener filtering. (a) Result of full inverse filtering of Fig. 5.25(b). (b) Radially limited inverse filter result. (c) Wiener filter result.

Example 2

FIGURE 5.29 (a) 8-bit image corrupted by motion blur and additive noise. (b) Result of inverse filtering. (c) Result of Wiener filtering. (d)–(f) Same sequence, but with noise variance one order of magnitude less. (g)–(i) Same sequence, but noise variance reduced by five orders of magnitude from (a). Note in (h) how the deblurred image is quite visible through a "curtain" of noise.

Constrained Least Squares Filtering

- Wiener filtering requires knowledge of
  - Degradation function, $H(u,v)$
  - Power spectrum of the undegraded image, $S_f(u,v)$
  - Noise power spectrum, $S_n(u,v)$
- A constant estimate of the ratio $S_n(u,v)/S_f(u,v)$ of the power spectra is not always a suitable solution.

$$W(u,v) = \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_n(u,v)/S_f(u,v)} \approx \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + K}$$

- Constrained Least Square Filtering requires
  - $H(u,v)$
  - the mean and the variance of the noise ← degraded image

Constrained Least Squares Filtering

g=h*f+n $$h(x,y)*f(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$

Vector matrix form $\mathbf{g}=\mathbf{Hf}+\mathbf{n}$

- If f, g, and n are of size M×N, vectors $\mathbf{g}$, $\mathbf{f}$, and $\mathbf{n}$ have dimensions (MN)×1.
- The matrix $\mathbf{H}$ has dimensions (MN) × (MN).
- The matrix $\mathbf{H}$ is highly sensitive to noise.
- → The problem cannot be solved by a simple matrix manipulation.

Constrained Least Squares Filtering

- One way to alleviate the noise sensitivity problem is to base optimality of restoration on a measure of smoothness, such as the second derivative (the Laplacian) of an image.
- To be meaningful, the restoration must be constrained by the parameters of the problems at hand. → Find the minimum of a criterion function C, defined as

$$C=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}[\nabla^2 f(x,y)]^2$$

subject to the constraint $\left\|\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}\right\|^2=\left\|\mathbf{n}\right\|^2$

Euclidean vector norm
$\|\mathbf{w}\|^2=\mathbf{w}^T\mathbf{w}=\sum_{k=1}^{N}w_k^2$

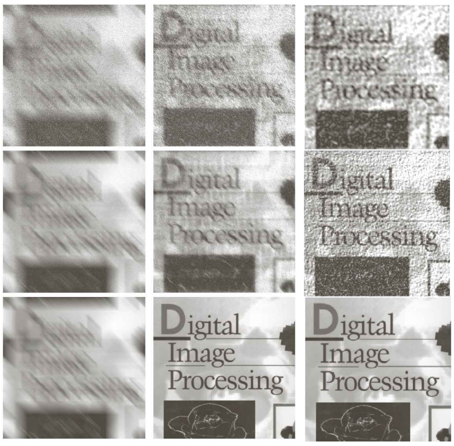
- The frequency domain solution to this optimization problem

$$\hat{F}(u,v)=\left[\frac{H^*(u,v)}{|H(u,v)|^2+\gamma\,|P(u,v)|^2}\right]G(u,v) \qquad p(x,y)=\begin{bmatrix}0&-1&0\\-1&4&-1\\0&-1&0\end{bmatrix}$$

(if γ=0, it reduces to inverse filtering)

Example

- γ is manually selected.
- The CLS filtering yield better results.
- The result based on manually selecting γ would be a more accurate estimate.
  - γ is a scalar, while K is an approximation to the ratio of two unknown functions.
  - The ratio of two functions seldom is constant.



Degraded Wiener CLS

CLS filter

$$U(u,v) = \frac{H^*(u,v)}{|H(u,v)|^2 + \gamma\,|P(u,v)|^2}$$

Wiener filter

$$W(u,v) = \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_n(u,v)/S_f(u,v)}$$

$$\approx \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + K}$$

More Improvement

- Adjusting the parameter γ interactively → automatic selection
- Procedure for computing γ by iteration
  - Residual vector $\mathbf{r} = \mathbf{g} - \mathbf{H}\hat{\mathbf{f}}$
  - Adjust γ so that $\|\mathbf{r}\|^2 = \|\mathbf{n}\|^2 \pm a$ (a : accuracy factor)

    1) Specify an initial value of γ

    2) Compute $\|\mathbf{r}\|$

    3) Stop if $\|\mathbf{r}\|^2 = \|\mathbf{n}\|^2 \pm a$;

    Otherwise return to step 2

    after increasing γ if $\|\mathbf{r}\|^2 < \|\mathbf{n}\|^2 - a$

    or decreasing γ if $\|\mathbf{r}\|^2 = \|\mathbf{n}\|^2 + a$

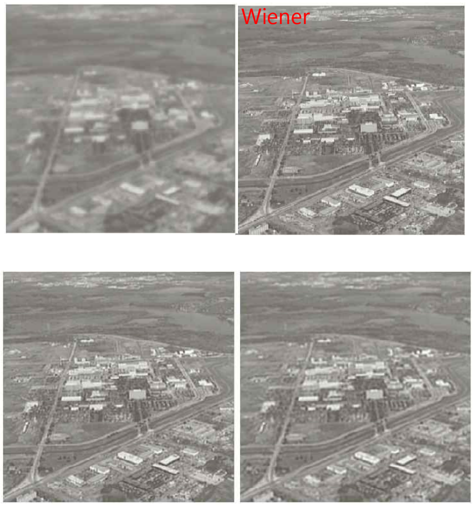
$$R(u,v) = G(u,v) - H(u,v)\hat{F}(u,v)$$

$$\|\mathbf{r}\|^2 = \sum\sum r^2(x,y) \qquad r(x,y) = \Im^{-1}\{R(u,v)\}$$

$$\sigma_n^2 = \frac{1}{MN}\sum\sum n^2(x,y) - m_n^{\ 2}$$

$$\|\mathbf{n}\|^2 = MN[\sigma_n^2 + m_n^2]$$ ☜

Example



a b

FIGURE 5.31 (a) Iteratively determined constrained least squares restoration of Fig. 5.16(b), using correct noise parameters. (b) Result obtained with wrong noise parameters.

Geometric Mean Filter

- Generalization of the Wiener filter

$$\hat{F}(u,v)=\left[\frac{H^*(u,v)}{|H(u,v)|^2}\right]^{\alpha}\left[\frac{H^*(u,v)}{|H(u,v)|^2+\beta\dfrac{S_n(u,v)}{S_f(u,v)}}\right]^{1-\alpha}G(u,v)$$

- α=1 → inverse filter
- α=0 → parametric Wiener filter (β=1 → standard Wiener filter)
- β=1
  - As α decreases below ½, the filter behaves more like the Wiener filter.
  - As α increases above ½, the filter behaves more like the inverse filter.

- Useful when implementing restoration filters

  Because it represents a family of filters combined into a single expression.

Homework #5

- The image 2) is the result of 15x15 averaging of 1). It looks much blurred.
- We cannot recover the undegraded image by using a simple sharpening filter.
- Here, we know the exact degradation function.
- Try to restore the undegraded image from 2) and 3) using inverse filtering and Wiener filtering.
- Discuss the results.


1) Original


2) 15x15 averaging


3) 2) + noise


4) simple sharpening