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Measuring potential gains from reallocation of resources

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Abstract In this work we consider the case when efficient operation of individual economic units does not necessarily imply efficiency for a group of these units. Merging theoretical findings of Li and Ng (Int Adv Econ Res, 1995, 1, 377.) and Färe and Zelenyuk (Eur J Oper Res, 2003, 146, 615), we develop new group-wise efficiency indexes that measure the extent to which the performance of a group of economic units can be enhanced, even if all these units are *individually* efficient. The existence of such potential improvement is attributed to non-optimal allocation of inputs across the individual economic units from the point of view of a group of these units.

Keywords Group decisions and negotiations · Group efficiency · Resource allocation

1 Introduction

For roughly half a century (Debreu 1951 and Farrell 1957 among others), scientists have been extensively developing the theory of efficiency measurement, followed by thousands of applications, mostly focusing on measuring efficiency of *individual* economic agents or, using conventional terminology, decision making units (hereafter DMUs): firms, departments or branches of a firm, countries, etc. Less attention was paid to measuring efficiency of a *group* of DMUs. The economic agents in the group context would be, for example, branches, departments or plants belonging to one firm. In this case, researchers might be interested in not only efficiency of each individual DMU but also in the efficiency of all the DMUs together, considered as one group. Another example of economic agents in the group context can be countries, where one might be interested in efficiency of a group of these countries (e.g. the group can be the European Union).

Perhaps the earliest development on group efficiency measures goes back to Farrell (1957), who introduced the notion of *structural efficiency of an industry*, measured as a weighted average of individual scores where the weights are the observed output shares (for the single output case). Another fundamental idea appeared in Førsund and Hjalmarsson (1979), who suggested the notion of the efficiency of an average DMU, elaborated by Li and Ng (1995). More recently, Färe and Zelenyuk (2003) have shown theoretical justification for Farrell's notion of structural efficiency of an industry and its multi-output generalization.

In our work, we take these ideas a step further: we merge the theoretical findings of Li and Ng (1995) and Färe and Zelenyuk (2003), creating a new group-wise efficiency index that measures the extent to which the performance of a group of DMUs can be enhanced, even if, *individually*, these DMUs are efficient. The source of such group inefficiency is attributed to non-optimal allocation of inputs (in the output oriented case) across the DMUs. We call this group efficiency indexes as *reallocative efficiency* measure

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and show the relationship between our measures and previously existing measures related to group efficiency.[1]

Our paper is organized as follows. In Sect. 2 and 3 we present the theoretical background for group efficiency measurement, while assuming that each DMU operates independently in the sense that inputs cannot be reallocated across the DMUs. Then, in Sect. 4, we introduce the possibility of input reallocation among firms and suggest a new group-wise efficiency measure in this case. Further, we address the issue of empirical implementation of our new measure in Sect. 5 and discuss limitations to our work in Sect. 6. Section 7 concludes and highlights areas for further research.

2 Individual technology and efficiency measures

Let us first consider *individual* efficiency measures. Assume there are K DMUs (indexed as $k = 1,\ldots,K$) within a group, which produce outputs $y^k \equiv (y_1^k,\ldots,y_M^k)' \in \Re_+^M$ from inputs $x^k \equiv (x_1^k,\ldots,x_N^k)' \in \Re_+^N$, according to the *individual technology* for each DMU k:

$$T^k \equiv \{(x^k, y^k) : \text{``}y^k \textit{ is producable from } x^k\text{''}\} \qquad (2.1)$$

Equivalently, technology of each DMU k ($k = 1,\ldots,K$) can be characterized by its *output set*:

$$P^k(x^k) \equiv \{y^k : (x^k, y^k) \in T^k\}, \quad x^k \in \Re_+^N. \qquad (2.2)$$

Hereafter, we assume that individual technologies satisfy standard regularity axioms of technology characterisation (see Färe and Primont (1995) for a detailed discussion), namely:

Axiom 1 ‘‘No free lunch’’: $y \notin P^k(0_N), \forall y \geq 0_M$.

Axiom 2 ‘‘Producing nothing is possible’’: $0_M \in P^k(x)$, $\forall x \in \Re_+^N$

Axiom 3 ‘‘‘Boundness’ of the Output set’’: $P^k(x)$ *is a bounded set*, $\forall\ x \in \Re_+^N$

Axiom 4 ‘‘‘Closedness’ of the Technology set T’’: *Technology set* T^k *is a closed set*

Axiom 5 ‘‘Free disposability of Outputs’’: $y^0 \in P^k(x) \Rightarrow y \in P^k(x)$, $\forall\ y \leqq y^0$, $x \in \Re_+^N$

[1] Conceptually similar, but analytically different ideas can also be found in Bogetoft and Wang (1999), Färe et al. (1994) and Soriano et al. (2003), as we will discuss latter in the paper.

The *individual revenue function* for a DMU k is defined as the maximum revenue achievable given individual technology, input endowment and output prices:

$$R^k(x^k, p) \equiv \max_y \{py : y \in P^k(x^k)\}, \qquad (2.3)$$

where $p \equiv (p_1,\ldots,p_M) \in \Re_+^M$ is a price vector, *common to all firms*. That is, prices are assumed to be the same for all DMUs—a necessary assumption to obtain revenue aggregation results, which we will use later on (see Färe and Zelenyuk (2003)).[2] Note that $R^k(x^k,p)$ represents a dual characterization of $P^k(x^k)$, as shown by Shephard (1970).

Conventionally, the *individual revenue efficiency* of a DMU k is defined as the ratio of maximum revenue to its observed revenue for this DMU, i.e.,

$$RE^k \equiv RE^k(x^k, y^k, p) \equiv R^k(x^k, p)/py^k. \qquad (2.4)$$

The *individual technical efficiency* of the k-th DMU is defined, following Farrell (1957), as a scalar measure of maximum possible radial expansion of output vector within the individual output set:[3]

$$TE^k \equiv TE^k(x^k, y^k) \equiv \max_\theta \{\theta > 0 : \theta y^k \in P^k(x^k)\} \qquad (2.5)$$

Accordingly, the *individual technically efficient output* is equal to the actual observed output multiplied by the individual output technical efficiency score of the kth DMU, $y^{*k} \equiv y^k \cdot TE^k$. Finally, we define the *individual output allocative efficiency* of DMU k as the ratio of its individual maximal revenue to its revenue obtained from individual technically efficient output:

$$AE^k \equiv AE^k(x^k, y^k, p) \equiv R^k(x^k, p)/py^{*k}. \qquad (2.6)$$

The intuition behind this latter measure is that producing at the technically efficient output level does not guarantee the highest possible revenues because the firm may produce a sub-optimal output mix. AE^k reflects the discrepancy between individual maximal revenue and the revenue obtained from individual technically efficient output. Note that the inequality $R^k(x^k,p) \geq py^{*k}$ (also known as the Mahler inequality) must necessarily hold and hence $AE^k \geq 1$.

Using the fact that $y^{*k} \equiv y^k \cdot TE^k$, along with expressions (2.4) and (2.6), yields the famous multiplicative

[2] For a discussion of this type of ‘‘Law of One Price’’ assumption, see Kuosmanen et al. (2004).

[3] Hereafter, we do not explicitly include the word ‘‘output’’ in the terms used. Since only output-oriented case is considered in this work, thus distinction with input-oriented measures is unnecessary. The development of the input-oriented case would be similar and is omitted for the sake of brevity.

decomposition of revenue efficiency into technical and allocative efficiency components, which will be useful in further discussions:

$$RE^k = TE^k \cdot AE^k. \tag{2.7}$$

3 Group efficiency measurement when inputs can not be reallocated

In this section we turn to the construction of *group structural efficiency measures*, following Färe and Zelenyuk (2003).[4] These measures imply that the allocation of individual input endowments is taken as given. First, let us consider *group structural technology*, characterized through the *group structural output set*, defined as the sum of individual output sets:

$$\overline{P}(x^1, \ldots, x^K) \equiv \sum_{k=1}^{K} P^k(x^k). \tag{3.1}$$

The *group structural revenue* can be defined as in the individual case (2.3), but optimized relative to $\overline{P}(x^1, \ldots, x^K)$ under the common price p, when all DMUs produce from their input endowments,

$$\overline{R}(x^1, .., x^K, p) \equiv \max_y \{py : y \in \overline{P}(x^1, .., x^K)\}, \tag{3.2}$$

The following lemma gives a cornerstone of all further discussion of group efficiency measures.

Lemma 1 If each DMU faces the same output price p, the following equality holds:

$$\overline{R}(x^1, .., x^K, p) = \sum_{k=1}^{K} R^k(x^k, p). \tag{3.3}$$

This lemma says that the maximal revenue of a group of DMUs is the same as the sum of individually maximized revenues of each DMU—when the reallocation of inputs across these DMUs is not allowed and when all DMUs face the same prices for outputs. This result is an analogue to the theorem on aggregation of profit functions in Koopmans (1957), provided by Färe and Zelenyuk (2003), where the proof can also be found).

Parallel to the individual case (2.4), *group structural revenue efficiency* is defined as the ratio of the maximal revenue of the group to the observed total revenue of the group:

[4] The word ''structural'' will emphasize the fact that the structure—here, the allocation of inputs—is kept fixed. This notion is going back to Farrell's (1957) idea of structural efficiency of an industry.

$$\overline{RE} \equiv \overline{RE}(x^1, .., x^K, Y, p) \equiv \overline{R}(x^1, .., x^K, p)/pY. \tag{3.4}$$

Result (3.3) implies that the latter is equal to the weighted sum of individual revenue efficiencies, with the weights corresponding to the revenue shares of the individual firms:

$$\overline{RE} = \sum_{k=1}^{K} RE^k \cdot S^k, \tag{3.5}$$

where $S^k \equiv py^k/pY$ is the revenue share of the k[th] DMU, and $Y \equiv \sum_{k=1}^{K} y^k$. Moreover, Färe and Zelenyuk (2003) also multiplicatively decomposed group structural revenue efficiency into revenue weighted allocative technical efficiency terms:

$$\overline{RE} = \overline{AE} \cdot \overline{TE}, \tag{3.6}$$

where

$$\overline{TE} \equiv \overline{TE}(x^1, \ldots, x^K, Y) = \sum_{k=1}^{K} TE^k \cdot S^k, \tag{3.7}$$

and

$$\overline{AE} \equiv \sum_{k=1}^{K} AE^k \cdot S_a^k, \tag{3.8}$$

where $S_a^k = py^{k*}/pY^*$ is the revenue share of k-th DMU, which is now based on the sum of maximal outputs $Y^* \equiv \sum_{k=1}^{K} y^{*k}$. Note that the relationship (3.7) represents *group structural technical efficiency*, which is the multi-output generalization of the concept intuitively suggested by Farrell (1957) for the single output case, and $\overline{AE}$ in (3.8) is *group structural allocative efficiency*. Measures in (3.6) are very useful when one needs to infer the efficiency of a group of DMUs, when each economic unit can be considered as a separate entity, in the sense that reallocation of inputs across the DMUs is not of interest or merely impossible.

4 Measuring efficiency of a group if inputs reallocation is possible

In the preceding section, the group output set was assumed to be equivalent to the sum of individual output sets, according to definition (3.1). Since each DMU's output set is constructed given its individual input endowment, it follows that the allocation of inputs among DMUs is fixed under such an aggregation structure. One may suspect that if the DMUs operate at non-constant returns to scale, there may be unrealized output gains from input reallocation within the group, even when individual DMUs are efficient. In order to measure gains from such reallocations, we revise the classification of group technology, following the logic of Li and Ng (1995), and define the *group potential technology*. Subsequently, we introduce the *group potential efficiency measures* allowing for the possibility of resource reallocation. We go on to show the relationship between these measures and the past structural measures of group efficiency by introducing an additional component, termed *reallocative efficiency*.

4.1 Potential production possibilities of a group of DMUs

Let us define the *group potential technology* that allows for resources reallocation within the group as

$$T^g \equiv \sum_{k=1}^{K} T^k, \tag{4.1}$$

where T^k is the individual technology defined in (2.1).[5] The definition (4.1) is equivalent to the definition of Li and Ng (1995) and implies that group technology represents the set of all combinations of *aggregate* inputs and outputs, regardless of their allocation across individual DMUs. An equivalent characterization of group potential technology is given by the *group potential output set*:

$$P^g \equiv P^g(X) \equiv \{y : (X, y) \in T^g\}, \tag{4.2}$$

where $X \equiv \sum_{k=1}^{K} x^k$. In contrast to the group structural output set defined in (3.1), the definition (4.2) allows for reallocation of resources across the DMUs, and what matters is the aggregate input and respective aggregate output of the group. It can be shown that $P^g(X)$ is less restrictive than $\overline{P}(x^1, \ldots, x^K)$, as is stated in the following lemma (see Appendix 1 for its proof).

Lemma 2

$$\overline{P}(x^1, \ldots, x^K) \subseteq P^g(X). \tag{4.3}$$

The above relationship says that the group potential output set (that allows for reallocation of inputs within the group) includes the group structural output set (where reallocation is not allowed).

[5] Hereafter, the superscript ‘‘g’’ indicates a group potential efficiency measure, i.e. when reallocation of inputs across the DMUs is possible.

4.2 Group potential efficiency measures

In order to define *group potential revenue efficiency*, we should first define *group potential revenue*, in a similar fashion as we did in (3.2). However, potential revenue is defined not on $\overline{P}(x^1, \ldots, x^K)$, as is group structural revenue, but on the group potential output set, $P^g(X)$, which allows for input reallocation between the DMUs,

$$R^g(X, p) \equiv \max_y \{py : y \in P^g(X)\}. \tag{4.4}$$

In words, group potential revenue is the maximal possible group revenue consistent with group potential production possibilities. It should be clear that $R^g(X, p) \geq \overline{R}(x^1, \ldots, x^K, p)$, which follows directly from Lemma 2. *Group potential revenue efficiency* is then defined as the ratio of group potential revenue to actual group revenue, i.e.,

$$RE^g(X, Y, p) \equiv R^g(X, p)/pY. \tag{4.5}$$

Similar to the case of the structural measure of group efficiency, (3.6), group potential revenue efficiency can be decomposed into two components: *group potential technical efficiency* and *group potential allocative efficiency*:

$$RE^g = TE^g \cdot AE^g, \tag{4.6}$$

where

$$TE^g \equiv TE^g(X, Y) \equiv \max_\theta \{\theta : \theta Y \in P^g(X)\}. \tag{4.7}$$

$$AE^g \equiv AE^g(X, Y, p) \equiv R^g(X, p)/pY^{**}, \tag{4.8}$$

and $Y^{**} \equiv Y \cdot TE^g$ is actual output of the group, adjusted for group potential technical inefficiency. Hence, if one desires to have the possibility of input reallocation within the group and thus turns to the group *potential* efficiency measures, then he or she is still able to decompose this group efficiency into technical (4.7) and allocative (4.8) components.

4.3 Reallocative efficiency of a group

To isolate the revenue gains from reallocation of inputs within the group, we now define the *group revenue reallocative efficiency* as the discrepancy between the group potential and the group structural revenues, i.e.,

$$RRE^g \equiv R^g(X, p)/\overline{R}(x^1, .., x^K, p). \tag{4.9}$$

A simple, but important result follows immediately from definitions (3.4), (4.5) and (4.9) and is stated formally in Proposition 1.

Proposition 1

$$RE^g = \overline{RE} \cdot RRE^g. \tag{4.10}$$

Intuitively, Proposition 1 says that RRE^g is the residual between *group potential revenue efficiency* and *group structural revenue efficiency*.[6] RRE^g shows the discrepancy between group revenue efficiency when inputs can be reallocated and group revenue efficiency when such reallocation is not possible. Thus, this measure shows by how much group revenues can be increased if inputs are allowed to be reallocated across the DMUs, ceteris paribus (i.e., given the current technologies, the technical and allocative efficiency level of each DMU and the total amount of inputs).

As was mentioned above, individual revenue maximization does not imply maximization of the group revenue if inputs can be reallocated within the group. In such a case RRE^g may be greater than unity. However, if individual optimizing goals are in accord with group ones, then RRE^g will be equal to 1.

The idea of group revenue reallocative efficiency is illustrated graphically in Fig. 1.

In Fig. 1, the line $\overline{R}(x^1, \ldots, x^K, p)$ corresponds to group maximal revenue function defined on the group output set and is equal to the sum of individual revenue functions. It presumes impossibility of input reallocation across DMUs. In contrast, $R^g(X,p)$ indicates the group *potential* revenue, i.e. maximum group revenue when inputs can be reallocated across DMUs. Clearly, even if $\overline{R}(\cdot)$ achieved at the output bundle Y^*, it may be increased further to $R^g(\cdot)$ which corresponds to group output Y^{**}. The trade-off between $\overline{R}(\cdot)$ and $R^g(\cdot)$ is attributed to the non-optimal input allocation *within* the group, from the point of view of group potential revenue maximization. This trade-off will be eliminated if inputs are reallocated in an optimal way in terms of the group revenue, in which case the group becomes revenue reallocatively efficient, $RRE^g = 1$.

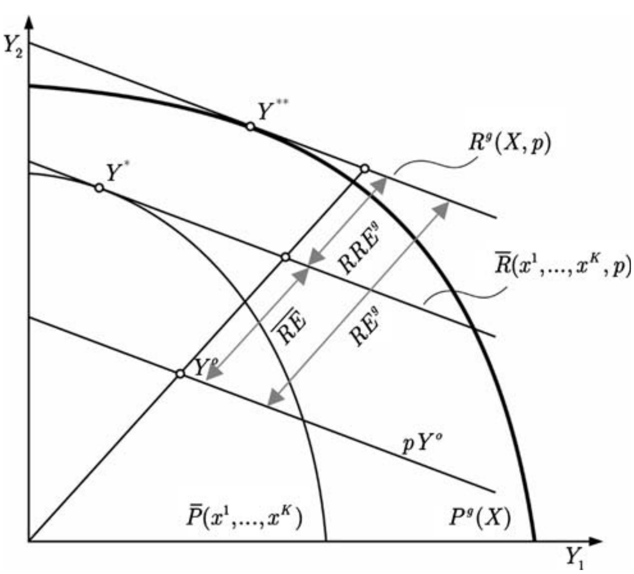

Fig. 1 Group revenue reallocative efficiency

[6] In fact, a similar relationship was defined by Soriano et al. (2003) in a somewhat different context, but the necessary proofs were not developed.

Before, we were interested in decomposing the revenue gains into technical and allocative components (e.g., see Eqs. (2.7), (3.6) and (4.6)). We can also do it now for the measure of gains from reallocation, which is stated formally in Proposition 2.

Proposition 2

$$RRE^g = TRE^g \cdot ARE^g, \tag{4.11}$$

where

$$TRE^g \equiv TE^g / \overline{TE}, \tag{4.12}$$

and

$$ARE^g \equiv AE^g / \overline{AE} \tag{4.13}$$

Proof The result follows from (3.6), (4.6) and (4.10).

Intuitively, this proposition says that group revenue reallocative inefficiency arises from two sources. The relative magnitude of these components reveals whether the goals of individual DMUs are more consistent with the maximization of group output or group revenue (i.e., respecting the output prices), or both. Output gains for the group (weighted by revenue shares of each member of the group) are reflected by group technical reallocative efficiency (TRE^g). It represents a factor by which the weighted group technically efficient output can be increased by reallocating resources across its constituents. The mismatch between the group and individually optimal output mixes is measured by group allocative reallocative efficiency (ARE^g). This, somewhat awkwardly named measure, represents the relative difference between group potential allocative efficiency (where input reallocation is possible) and structural allocative efficiency, proposed by Färe and Zelenyuk (2003) (which does not allow for input reallocation). If it is the case that there are no additional gains (or losses) by switching from individually optimal (individual revenue maximization) output mix to output mix that achieve group potential revenue, then ARE^g is equal to 1. However, it may be that $AE^g < \overline{AE}$, meaning that the current aggregate output mix is more consistent with the goal of achieving group potential revenue, rather than with maximization of individual revenues. Accordingly, such a situation will be reflected by $ARE^g < 1$. Alternatively, it may also happen that $ARE^g > 1$, which is the opposite case.

By merging (4.12) and (4.13) with (4.6), we obtain an additional useful decomposition:

$$RE^g = (\overline{TE} \cdot TRE^g) \cdot (\overline{AE} \cdot ARE^g). \tag{4.14}$$

This decomposition reveals all components of the revenue efficiency of a group. Structural efficiency components ($\overline{TE}$ and $\overline{AE}$) reflect gains from making individual DMUs efficient, while reallocative efficiency components (TRE^g and ARE^g) indicate unrealized gains due to non-optimal allocation of economic resources across the DMUs within the group. Taken together, they are absorbed by the measure of overall group revenue gains from reallocation of resources within the group, RE^g.

It is interesting to note that the group measures of reallocative efficiency can also be obtained from their individual analogues. Specifically, it can be easily shown that

$$RRE^g = \left(\sum_{k=1}^{K} (RRE^k)^{-1} \cdot S^k\right)^{-1}, \tag{4.15}$$

$$TRE^g = \left(\sum_{k=1}^{K} (TRE^k)^{-1} \cdot S^k\right)^{-1}, \tag{4.16}$$

$$ARE^g = \left(\sum_{k=1}^{K} (ARE^k)^{-1} \cdot S_a^k\right)^{-1}, \tag{4.17}$$

where $S^k \equiv py^k/pY$, $S_a^k = py^{*k}/pY^*$, $RRE^k \equiv RE^g/RE^k$, $TRE^k \equiv TE^g/TE^k$ and $ARE^k \equiv AE^g/AE^k$. That is, the group reallocative efficiency measures can be obtained by a weighted aggregation of the respective individual measures (RRE^k,TRE^k and ARE^k). Remarkably, the weights are the same as those of Färe and Zelenyuk (2003), i.e., the revenue shares as in (3.5) and (3.8), but the aggregation function is harmonic, rather than arithmetic.

4.4 Alternative measures of group structural and reallocative technical efficiencies

Färe and Zelenyuk (2003) proposed the measure of group structural technical efficiency defined in (3.7) and our definition of technical reallocative efficiency in (4.12) is closely related to that structural measure. However, the disadvantage of these measures may be that they depend on prices. Indeed, one may wonder why prices are incorporated into pure *technical* efficiency measures and why one should worry about prices when measuring the efficiency of the production process? It is worth noting, however, that price-dependent weights are not ad hoc but come directly from the objective criterion of firms: optimizing behaviour of economic agents (revenue maximization) and an additive aggregation structure. Since the essence of economic behaviour is tied to prices it is not surprising that the weights based on such a criterion also contain prices. If price dependency is not desirable or price information is unavailable, then additional assumptions can be made to get rid of prices in the weights (see Färe and Zelenyuk 2003, 2006). Another alternative, independent of prices, was suggested by Bogetoft and Wang (1999), and is discussed below.

If, on the contrary, prices do matter, e.g. a researcher wants to weight each firm efficiency by its market share (or, put another way, 'relative importance of each firm to the industry') price-dependent weights may be quite relevant. For the purpose of this subsection, we will denote the abovementioned measures as $\overline{TE}_{FZ1} \equiv \overline{TE}$ and $TRE^g_{FZ1} \equiv TRE^g$ and consider alternative measures of structural technical efficiency and technical reallocative efficiency of the group of DMUs. One such alternative is suggested in the work of Färe and Zelenyuk (2003), where group structural technical efficiency is defined as the maximal radial expansion of aggregate output subject to the frontier of $\overline{P}(x^1, \ldots, x^K)$, i.e.,

$$\overline{TE}_{FZ2} \equiv \max_{\theta}\{\theta > 0 : \theta Y \in \overline{P}(x^1, \ldots, x^K)\}. \tag{4.15}$$

This approach to the measurement of group structural technical efficiency gives rise to an alternative measure of group technical reallocative efficiency, defined as

$$TRE^g_{FZ2} \equiv TE^g/\overline{TE}_{FZ2}. \tag{4.16}$$

Such a measure may be considered as the ''first-best'' solution to the problem of measuring output gains from reallocation of inputs. It does not rely on prices and represents a pure trade-off between the frontier of $\overline{P}(x^1, \ldots, x^K)$ and that of $P^g(X)$. However, at this stage it is not clear how to estimate empirically the frontier of $\overline{P}(x^1, \ldots, x^K)$, as there is only one observation in a sample:(X,Y), where $X \equiv \sum_{k=1}^K x^k$ and $Y \equiv \sum_{k=1}^K y^k$.

One may also use another alternative measure, proposed by Bogetoft and Wang (1999). They defined $P^g(X)$ and TE^g in a similar manner as our (4.2) and (4.7) and then defined a measure of gains from reallocation of resources in the following way,

$$TRE^g_{BW} \equiv \max_{\theta}\{\theta > 0 : \theta \sum_{k=1}^{K} y^{*k} \in P^g(X)\}, \tag{4.17}$$

where $y^{*k} \equiv y^k \cdot TE^k$. That is, this reallocative efficiency index measures the maximal possible expansion of aggregate group output, given that all DMUs are individually

Table 1 Pro's and Con's of different measures of structural technical efficiency of groups

Measures	Pro's	Con's	Use in applications
FZ1	technically simple and intuitively appealing	incorporates price information in construction of weights	If price-dependent weights do not contradict the goals of research, or use additional assumption of Färe and Zelenyuk (2003, 2006) to obtain price-independent weights
FZ2	''first-best'' purely technical measure	Not applicable empirically (yet)	Good for theoretical purposes, practical use is still not clear
BW	Technically simple and intuitively appealing, does not depend on prices	It is not a radial measure. There is no certainty that $y\overline{TE}_{BW}$ will be on the boundary of $\overline{P}(x^1,\ldots,x^K)$.	May be a compromise between the two measures stated above

efficient. The authors also defined a variant of what we call structural efficiency of a group, as $\overline{TE}_{BW} \equiv TE^g/TRE^g_{BW}$, where $\overline{TE}_{BW}$ is our notation for their measure. Unfortunately, this measure of group output gains from reallocation (TRE^g_{BW}) does not, in principle, indicate the distance between the frontiers of $\overline{P}(x^1,\ldots,x^K)$ and $P^g(X)$. Still, it does not depend on prices, which may be an attractive feature in some cases.

A natural question which comes to mind from the above discussion is 'how are all these measures related and which of these, if any, is the best?' The second part of this question is an issue in and of itself and is left for further investigations. As to the first part, it can be shown that the following equalities hold for the *single-output case*:

$$\overline{TE}_{FZ1} = \overline{TE}_{FZ2} = \overline{TE}_{BW}, \quad \forall x \in \Re^N_+, y \in \Re^1_+ \tag{4.18}$$

$$TRE^g_{FZ1} = TRE^g_{FZ2} = TRE^g_{BW}, \quad \forall x \in \Re^N_+, y \in \Re^1_+ \tag{4.19}$$

That is, all these measures are equal to each other in the single output case. In a multi-output case, however, they may be different and, as a result, respective measures of structural allocative efficiency and their reallocative counterparts may be different as well. In Table 1, we provide some rationale for making the choice between these measures in a given application.

In addition, for the three measurement cases described above, we can define group structural allocative efficiencies as $\overline{AE}_{FZ1} \equiv \overline{RE}/\overline{TE}_{FZ1}$, $\overline{AE}_{FZ2} \equiv \overline{RE}/\overline{TE}_{FZ2}$ or $\overline{AE}_{BW} \equiv \overline{RE}/\overline{TE}_{BW}$. Respectively, group allocative reallocative efficiency may be redefined as $ARE^g_{FZ1} \equiv AE^g/\overline{AE}_{FZ1}$, $ARE^g_{FZ2} \equiv AE^g/\overline{AE}_{FZ2}$ or $ARE^g_{BW} \equiv AE^g/\overline{AE}_{BW}$.

5 Towards practical implementation

To apply the proposed measures of group-wise efficiency in practice, we place two additional assumptions on the individual technology. In particular, we assume that T^k is convex and identical across DMUs. These assumptions are usual for data envelopment analysis (DEA) (e.g., see Färe (1994)). These assumptions enable us to use the result of Li and Ng (1995), who showed that if such assumptions hold, then $T^g = KT$ (where $T = T^k$, $\forall$ $k = 1,\ldots,K$, is convex). Respectively, it can be shown that in this case $P^g(X) = KP(\tilde{x})$, where $\tilde{x} \equiv K^{-1}\sum_{k=1}^K x^k$ and $P(\tilde{x})$ is the output set of the average DMU in the group, i.e., the hypothetical DMU producing average group output from the average input amount. Furthermore, it can be shown that group potential efficiency measures are equal to the respective measures for the average DMU.[7] We state this result in the following lemma.

Lemma 3 If individual technology sets are identical and convex, then the following is true:

$$RE^g(X,Y,p) = RE(\tilde{x},\tilde{y},p) \tag{5.1}$$

$$TE^g(X,Y) = TE(\tilde{x},\tilde{y}) \tag{5.2}$$

$$AE^g(X,Y,p) = AE(\tilde{x},\tilde{y},p) = RE/TE \tag{5.3}$$

where $\tilde{x} \equiv K^{-1}\sum_{k=1}^K x^k$ and $\tilde{y} \equiv K^{-1}\sum_{k=1}^K y^k$.

Intuitively, Lemma 3 gives us a simple way to estimate group potential efficiency measures. This way avoids estimating a group-wise frontier, which may be especially hard if the data is scarce. (For the proof of this lemma, see Appendix 2.) Further, once the group potential efficiency measures are estimated and the structural measures are calculated as the weighted averages of individual scores, as in (3.5–3.8), then the reallocative efficiency scores can be estimated as well by applying (4.11–4.13) or (4.15–4.17).

In the following table we use simulated data borrowed from Färe and Zelenyuk (2003) to illustrate our new measures, and compare it to the results from their measure of group efficiency. Column 1 in Table 2 lists the id of 20 imaginary firms, while columns 2–3 and 4–5 list inputs and outputs of these firms. Columns 6–8 list the canonical

[7] Noteworthy, this is the idea put forward by Førsund and Hjalmarsson (1979), which was elaborated on later by Li and Ng (1995). For further discussion refer to Ylvinger(2000).

Table 2 Illustrative example of various group efficiency measures

D M U	x_1^k	x_2^k	y_1^k	y_2^k	Rev. Eff. (RE^k)	Tech. Eff. (TE^k)	Alloc. Eff. (AE^k)	Rev. Realloc. Eff. (RRE^k)	Tech. Realloc. Eff. (TRE^k)	Alloc. Realloc. Eff. (ARE^k)
1	39.00	49.00	12.00	17.53	2.200	1.772	1.242	0.650	0.773	0.841
2	37.00	45.00	19.00	22.00	1.000	1.000	1.000	1.430	1.370	1.044
3	35.00	55.00	17.29	17.00	1.929	1.891	1.020	0.741	0.724	1.024
4	34.00	63.97	25.00	12.97	1.957	1.884	1.039	0.731	0.727	1.005
5	33.00	53.00	28.00	18.72	1.000	1.000	1.000	1.430	1.370	1.044
6	70.00	50.00	35.00	43.00	1.516	1.476	1.027	0.943	0.928	1.017
7	45.00	55.56	25.00	0.00	1.915	1.712	1.119	0.747	0.800	0.933
8	60.00	62.38	45.00	37.42	1.540	1.477	1.043	0.929	0.928	1.001
9	30.00	83.33	75.00	64.03	1.000	1.000	1.000	1.430	1.370	1.044
10	40.00	90.00	34.00	59.27	2.279	1.397	1.631	0.627	0.981	0.640
11	75.00	75.00	82.00	75.00	1.107	1.072	1.033	1.292	1.278	1.011
12	45.00	125.00	78.00	101.70	1.086	1.000	1.086	1.317	1.370	0.961
13	60.00	93.75	35.00	93.54	2.417	1.256	1.924	0.592	1.091	0.543
14	87.00	53.57	75.00	120.00	1.000	1.000	1.000	1.430	1.370	1.044
15	85.00	66.18	85.00	111.00	1.009	1.000	1.009	1.417	1.370	1.035
16	91.00	99.00	100.00	171.00	1.102	1.000	1.102	1.298	1.370	0.947
17	115.20	169.00	115.00	212.00	1.198	1.000	1.198	1.194	1.370	0.871
18	86.40	240.00	80.00	151.00	1.425	1.104	1.291	1.004	1.241	0.809
19	247.00	189.00	230.00	347.00	1.069	1.035	1.033	1.338	1.324	1.011
20	240.00	180.00	247.00	359.00	1.000	1.000	1.000	1.430	1.370	1.044
Group structural efficiency					**1.218**	**1.099**	**1.108**	**1.174**	**1.247**	**0.943**
Group potential efficiency					**1.430**	**1.370**	**1.044**			
Group reallocative efficiency					**1.174**	**1.247**	**0.943**			

Notes: The data is from Färe and Zelenyuk (2003). Output prices are 1 and 0.1 for the first and second outputs, respectively. Estimates are obtained using DEA, under assumption of variable returns to scale, using OnFront software (but any other linear programming solvers can be used).

output oriented revenue, technical and allocative efficiency scores of these firms, estimated using DEA. In the first line after these 20 estimates, of columns 6–8, we present the estimates of *group structural efficiency measures* derived by Färe and Zelenyuk (2003), which were obtained as weighted sums of the individual analogues, using (3.5), (3.7) and (3.8). The line below these estimates lists the estimated *group potential revenue, technical and allocative efficiency measures* described in Sect. 4.2, but obtained using (5.1), (5.2) and (5.3).

As one would expect, the estimate of *group potential revenue efficiency* is larger than the estimate of *group structural revenue efficiency*, suggesting that higher revenues can be gained when DMUs are considered not independently but as a group with possibility of reallocation of inputs between these DMUs. (One could also present the reciprocals of these estimates to get their percentage meaning.) The same pattern is observed for group potential technical efficiency vs. group structural efficiency. Note, however, that the estimate of group *potential* allocative efficiency is smaller than the estimate of group *structural* allocative efficiency. This means that if we consider all those firms as a group, allowing reallocation of inputs across the DMUs, then there will still be allocative inefficiency (due to non-optimal allocation of outputs), but it will be smaller than if we were to consider the DMUs individually. In other words, in this particular example, from the point of view of revenue optimization, the current output mix of all DMUs (e.g., industry) is closer to the 'group optimum' than it is to the 'individual optimum'. That is, there is an immediate improvement in allocative efficiency (of outputs) once the firms are considered as a group with possibilities to reallocate inputs. As a consequence, the estimate of *group allocative reallocative efficiency*, presented in the next line, is less than unity (0.943), indicating this improvement. Another consequence of it is that the estimate of *group revenue reallocative efficiency* is smaller than the estimate of the *group structural revenue efficiency* (1.174 vs. 1.218), while *group technical reallocative efficiency* is larger than the estimate of *group structural technical efficiency* (1.247 vs. 1.099). The economic meaning of this is that the most of the gain due to reallocation of inputs across firms, when they are considered as one group, would be due to technological (and not price-related) reasons.

Finally, columns 9–11 list the estimates of the individual reallocative (for revenue, technical and allocative) efficiencies for the 20 imaginary DMUs. The line below these estimates gives the weighted sums of these estimates, according to (4.15), (4.16) and (4.17), which are alternative ways to obtain estimates of *group reallocative efficiencies* and are identical to those just discussed.

6 Limitations and extensions

One problem which may arise is the possibility of fixed or individual specific inputs, i.e. those which cannot be reallocated between the DMUs. For example, there may exist fixed costs, or DMU-specific inputs, which cannot be reallocated. Moreover, if the DMUs are geographically disperse, it may be impossible or very costly to reallocate inputs across them. As a result, inputs may be reallocated, but only across some DMUs and not all. Often, these specifics are minor so our framework can still be applied as a reasonable approximation to more complicated scenarios. Whenever these complications are critical (and this might vary from one application to another), then the present framework can be used as a stepping-stone for a corresponding extension, accounting for these complications.

Another issue is the possibility of altering the number of DMUs within an enterprise. Specifically, it may be very useful, from the practical point of view, to compare gains from the reallocation of inputs to those from merging/separating the DMUs, or/and from establishing new ones. (Although it might not be possible if DMUs are not owned by the group, e.g., when DMUs are countries, etc.). Some foundation was laid out with regards to mergers by Bogetoft and Wang (1999), but the question of measuring efficiency gains from separating existing DMUs and establishing new DMUs is still open to debate and is another interesting extension to the present work.

In our study we argued that, to achieve higher group revenue, even if all DMUs are individually efficient, one should take into account the possibility of reallocating the inputs across the DMUs. However, a practical problem might appear with the decision about *how* such reallocation is to be made. As one possibility, the manager could perform a cost-benefit analysis of input reallocation. Although the benefits are more or less clear (higher revenues), the cost side of this process may not always be transparent. The additional expenditures may include transaction and transportation costs, miscellaneous labor costs, including unemployment compensations and hiring costs, restructuring costs, and social costs to name a few. Clearly, decision makers should take these costs into account when making a decision about input reallocation across DMUs. A simple rule of thumb, for example, would be: do not bother with reallocation of inputs within the group if the reallocative inefficiencies are small (near 1) but think about possible reallocation scenarios if these measures are above some threshold level (maybe 1.1 or 1.2).

Yet another natural extension would be consideration of the intertemporal framework: for example, extending the Malmquist productivity indexes (MPIs) to the case when the reallocation of inputs (for output oriented framework) is possible. In this respect, results of this work can be merged with the aggregation results for MPIs in Zelenyuk (2006).

It is also worth noting here that the bootstrap technique may also be used to obtain more accurate estimates of the various individual and group efficiency scores, their bias corrected analogues, confidence intervals, standard errors, etc. One way to do it would be to follow the approach of Simar and Zelenyuk (2006). This is, however, beyond the focus of current research, but it would certainly make a natural extension to it in the future.

7 Conclusions

In this paper, we have merged prior work on group-wise efficiency measurement with the necessary extensions to provide a theoretical foundation for the measurement of group efficiency when *reallocation* of inputs within a group of DMUs (e.g., firms) is possible. More succinctly, in addition to the measure of group structural efficiency that does not allow for input reallocation (the one proposed by Färe and Zelenyuk 2003), we have introduced a measure that allows for this possibility. We have shown that this measure can be decomposed into the standard technical and allocative components, allowing the source of overall inefficiency to be more clearly identified. Further, we have established the link between the two measures of group efficiency, potential and structural efficiencies. This link itself represents an efficiency measure, which we call reallocative efficiency. This measure shows how much group revenues may be increased, even if all its DMUs are individually efficient, and represents the revenue gains due to the reallocation of inputs (in the output oriented context) across the firms in a group. We further decomposed this measure to uncover the sources of such gains, which may appear either from the unrealized group output (technical side), or from gains due to a better match between individual and group-wise output mixes (price or allocative side). Finally, we suggest a simple procedure for an empirical implementation of the proposed measures and compared our results with those of Färe and Zelenyuk

(2003). Overall, we believe that the group measures of efficiency we have introduced here will be invaluable for the analysis of the efficiency of a firm (horizontally) composed of many units (e.g., branches of a bank) or for analysis of efficiency of unions of countries (e.g., European Union), where reallocation of inputs across individual units within the group is possible.

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Appendix 1

Proof of Lemma 2 We want to prove that $\overline{P}(x^1,\ldots,x^K) \subseteq P^g\left(\sum_{k=1}^K x^k\right)$. Let

$$Y^0 \in \overline{P}(x^{0,1},\ldots,x^{0,K}). \qquad (*)$$

Then, $Y^0 \in \overline{P}(x^{0,1},\ldots,x^{0,K}) \Rightarrow Y^0 \in \sum_{k=1}^K P^k(x^{0,k})$ (by definition (3.1)), and therefore $\exists y^{0,k} : y^{0,k} \in P^k(x^{0,k})$, $k = 1,\ldots,K, \sum_{k=1}^K y^{0,k} = Y^0$. Moreover, under standard regularity conditions, $P^k(x^k)$ $(\forall\ x^k \in \Re_+^N)$ and T^k are equivalent characterizations of technology (see Färe and Primont (1995)), i.e., $y^{0,k} \in P^k(x^{0,k}) \Leftrightarrow (x^{0,k}, y^{0,k}) \in T^k, \forall k = 1,\ldots,K$. Furthermore, $\sum_{k=1}^K (x^{0,k}, y^{0,k}) \in \sum_{k=1}^K T^k \Leftrightarrow \left(\sum_{k=1}^K x^{0,k}, \sum_{k=1}^K y^{0,k}\right) \in \sum_{k=1}^K T^k \Leftrightarrow (X^0, Y^0) \in T^g$, where $X^0 = \sum_{k=1}^K x^{0,k}$. Now, similar to the individual case, under the same regularity conditions, $P^g(X^0)$ $(X^0 \in \Re_+^N)$ and T^g are equivalent characterizations of group potential technology, i.e. $(X^0, Y^0) \in T^g \Leftrightarrow Y^0 \in P^g\left(\sum_{k=1}^K x^{0,k}\right)$. Combining this with (*) we get the desired result. □

Appendix 2

Proof of Lemma 3 A set S is convex if and only if $\alpha_1 S + \alpha_2 S = (\alpha_1 + \alpha_2)S$ for all $\alpha_1, \alpha_2 > 0$ (Li and Ng (1995)). This implies that for $T^k = T$, $\forall$ $k = 1,\ldots,K$, $T^g \equiv \sum_{k=1}^K T^k = KT$. Using this result, Li and Ng (1995) showed that $TE^g(X, Y) \equiv \max\{\theta : (X, \theta Y) \in T^g\} \equiv TE(\tilde{x}, \tilde{y})$, where $\tilde{x} \equiv K^{-1} \sum_{k=1}^K x^k{}^{\theta}$ and $\tilde{y} \equiv K^{-1} \sum_{k=1}^K y^k$, $x^k \in \Re_+^N$, $y^k \in \Re_+^M$. Using the same logic, we obtain

$$\begin{aligned} P^g(X) &= \{y : (X, y) \in T^g\} = \{y : (X, y) \in KT\} \\ &= \{y : (X, y)/K \in T\} = K \cdot \{y/K : (\tilde{x}, y/K) \in T\} \\ &= K \cdot P(\tilde{x}). \end{aligned}$$

And, therefore:

$$\begin{aligned} R^g(X,p) &= \max_Y\{pY : Y \in P^g(X)\} = \max_Y\{pY : Y/K \in P(\tilde{x})\} \\ &= K \cdot \max_{Y/K}\{pY/K : Y/K \in P(\tilde{x})\} \\ &= K \cdot \max_{\tilde{y}}\{p\tilde{y} : \tilde{y} \in P(\tilde{x})\} \equiv K \cdot R(\tilde{x}, p). \end{aligned}$$

Hence,

$$\begin{aligned} RE^g(X, Y, p) &\equiv R^g(X, p)/pY = K \cdot R(\tilde{x}, p)/pY \\ &= R(\tilde{x}, p)/(pY/K) \\ &= R(\tilde{x}, p)/p\tilde{y} \equiv RE(\tilde{x}, \tilde{y}, p), \end{aligned}$$

which proves (5.1). In its turn,

$$\begin{aligned} AE^g(X, Y, p) &\equiv R^g(X, p)/(pY \cdot TE^g) \\ &= K \cdot R(\tilde{x}, p)/(pK \cdot \tilde{y} \cdot TE) \\ &= R(\tilde{x}, p)/(p\tilde{y} \cdot TE) \equiv AE(\tilde{x}, \tilde{y}, p). \end{aligned}$$

□

References

Bogetoft P, Wang D (1999) Estimating the potential gains from mergers. J Prod Anal (under revision)

Debreu G (1951) The coefficient of resource utilization. Econometrica 19:273–292

Färe R, Grosskopf S, Lovell CAK (1994) Production Frontiers, Cambridge University Press

Färe R, Primont D (1995) Multi-output production and duality: theory and applications. Kluwer Academic Publishers, Boston

Färe R, Zelenyuk V (2003) On aggregate farrell efficiencies. Eur J Oper Res 146:615–620

Färe R, Zelenyuk V (2006) Extending Färe and Zelenyuk (2003). Eur J Oper Res, forthcoming

Farrell MJ (1957) The measurement of productive efficiency. J Roy Stat Soc 120:11–290

Førsund F, Hjalmarsson L (1979) Generalized farrell measures of efficiency: an application to milk processing in Swedish diary plants. Econ J 89:294–315

Koopmans TC (1957) Three essays on the state of economic analysis. McGraw-Hill, New York

Kuosmanen T, Cherchye L, Sipilainen T (2004) The law of one price in data envelopment analysis: restricting weight flexibility across firms, Eur J Oper Res, forthcoming

Li S-K, Ng Y-C, (1995) Measuring the productive efficiency of a group of firms. Int Adv Econ Res 1(4):377–390

Shephard RW (1970) Theory of cost and production function. Princeton University Press, Princeton

Simar L, Zelenyuk V (2006a) Statistical inference for aggregates of farrell-type efficiencies. J Appl Econom, forthcoming

Soriano F, Rao P, Coelli T (2003) Aggregation of Farrell Efficiencies With Possible Reallocation of Inputs, paper presented at the 8th European Workshop of Efficiency and Productivity Analysis, Oviedo, Spain, 2003

Ylvinger S (2000) Industry performance and structural efficiency measures: solutions to problems in firm models. Eur J Oper Res 121(1):164

Zelenyuk V (2006) On aggregation of malmquist productivity indexes. Eur J Oper Res, forthcoming