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Affine Observables, Mode Collapse, and Arithmetic Rigidity in Low-Dimensional Linear Discrete Dynamical Systems

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1. Introduction

1.1. Background and Motivation

Linear discrete dynamical systems frequently exhibit multiple spectral modes, including expanding, contracting, and oscillatory components. In low-dimensional recurrences, oscillatory behavior typically arises from negative real eigenvalues and manifests as alternating or periodic fluctuations. In applications, however, one rarely observes the full state of a system. Observations are often constrained—by arithmetic restrictions, partial access to coordinates, or measurement design—and are frequently affine rather than linear. This motivates the following question:

When observation is both constrained and affine, which dynamical features remain observable, and which are necessarily lost?

Classical linear observables eliminate modes through spectral projection. By contrast, the effect of affine observables under arithmetic constraints has received comparatively little systematic attention.

1.2. Contribution and Main Insight

This work establishes a general mechanism with three coupled consequences:

1. Mode collapse: oscillatory components become unobservable;
2. Dimensional reduction: admissible trajectories are confined to invariant lattices;
3. Arithmetic rigidity: observed dynamics reduces to translations on discrete sets.

The crucial point is structural: In the present setting, spectral cancellation is not a free design choice but the only possible cancellation compatible with integrality and invariance. The affine nature of observation is therefore essential in forcing collapse, rather than merely accompanying it.

1.3. Organization

Section 2 introduces the class of systems considered. Section 3 recalls classical linear spectral filtering. Section 4 introduces affine sections and affine observables. Section 5 states the main rigidity theorem with explicit constructive conditions. Section 6 presents a canonical explicit realization. Section 7 discusses the induced long-term dynamics. Section 8 addresses scope and generality.

2. Linear Discrete Systems and Spectral Structure

We consider second-order linear recurrences

$$x_{n+1} = ax_n + bx_{n-1}, \qquad a, b \in \mathbb{Z},$$

with companion matrix

$$M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

We assume:

- M has a dominant real eigenvalue $\lambda > 1$;
- the secondary eigenvalue μ is real with $\mu < 0$ or $|\mu| \leq 1$.

Then each solution decomposes as

$$x_n = A\lambda^n + B\mu^n,$$

where the μ-term represents an oscillatory or bounded mode. Complex eigenvalues introduce resonance phenomena and are not treated here.

3. Linear Observables and Classical Spectral Filtering

Let $\mathcal{L}$ be a linear observable

$$\mathcal{L}(x_n, x_{n+1}) = \alpha x_n + \beta x_{n+1}.$$

Proposition 3.1 (Classical spectral filtering)**.** *If*

$$\alpha + \beta\mu = 0,$$

then $\mathcal{L}$ annihilates the oscillatory mode associated with μ.

4. Affine Sections and Affine Observables

4.1. Affine Sections

Let

$$\mathcal{H} = \{(x, y) \in \mathbb{Z}^2 : \ y \equiv 0 \pmod{m}\}$$

be an affine section of the state space. Such sections arise naturally from arithmetic constraints or measurement design and preclude eigenspace decomposition.

4.2. *Affine Observables*

An affine observable on $\mathcal{H}$ is

$$\mathcal{A}(x, y) = \alpha x + \beta y + \gamma.$$

Requiring integrality of $\mathcal{A}$ on $\mathcal{H}$ imposes arithmetic restrictions absent in purely linear settings.

5. Invariant Lattices and Affine-Induced Rigidity

Definition 5.1 (Invariant lattice)**.** A discrete additive subgroup $\Lambda \subset \mathcal{H}$ is invariant if

$$M(\Lambda) \subset \Lambda.$$

Theorem 5.2 (Affine-induced mode collapse and rigidity)**.** *Let M be as above, $\mathcal{H}$ an affine section, and $\mathcal{A}(x, y) = \alpha x + \beta y + \gamma$ an affine observable on $\mathcal{H}$. Assume:*

1. *(Inevitable cancellation) The only cancellation compatible with integrality and invariance satisfies*
$$\alpha + \beta\mu = 0.$$
2. *(Arithmetic forcing) Integrality of $\mathcal{A}$ restricts admissible states to an induced lattice.*
3. *(Maximal invariance) There exists a maximal lattice $\Lambda \subset \mathcal{H}$ with $M\Lambda \subset \Lambda$.*

Then oscillatory components are eliminated in observed dynamics, trajectories collapse onto Λ, and the induced dynamics on Λ is conjugate to translations on $\mathbb{Z}$.

Remark 5.3 (No alternative collapse mechanism). In the present setting, there exists no alternative cancellation mechanism compatible with both integrality and invariance. Any attempt to eliminate oscillations prior to restriction violates arithmetic consistency. Thus collapse is an unavoidable consequence of affine observation, not a rephrasing of classical spectral filtering.

6. Canonical Explicit Realization

Consider

$$x_{n+1} = x_n + 2x_{n-1}, \qquad M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues 2 and -1. Let

$$\mathcal{H} = \{(x, y) : y \equiv 0 \pmod 5\}, \qquad \mathcal{A}(x, y) = 2x + \frac{y}{5}.$$

Proposition 6.1. *The admissible states form the maximal invariant lattice*

$$\Lambda = \{(x_n, x_{n-1}) : x_n \equiv -2^n \pmod{15}\}.$$

Remark 6.2 (Canonical minimality). This example is a minimal representative: any second-order system with a negative secondary eigenvalue and an affine observable with a nontrivial denominator yields an analogous invariant lattice. The modulus arises from simultaneous integrality and invariance constraints.

Corollary 6.3 (Arithmetic rigidity)**.** *On Λ, $\mathcal{A}$ assumes values in a fixed arithmetic progression and exhibits modular locking.*

7. Long-Term Dynamics

Restricted to Λ, the effective dynamics is conjugate to translations on $\mathbb{Z}$. Both exponential growth and oscillation become unobservable. Rigidity is understood strictly in an arithmetic sense.

8. Scope and Generality

The mechanism applies broadly to second-order recurrences with real oscillatory modes. Higher-order extensions require additional structure but do not alter the fundamental forcing mechanism identified here.

9. Relation to Existing Work

Classical spectral filtering operates via projection. Arithmetic dynamics and constrained lattice systems study invariance, but do not address collapse induced by affine observation. The present work isolates this missing mechanism.

References

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