A Low-Variance Exponential Sampling Platform for Empirical Testing of the Hardy–Littlewood Goldbach Heuristic

Tak Heung Sze
Independent Researcher

Abstract

We introduce a structured exponential subsequence designed as a low-variance numerical platform for empirical testing of asymptotic heuristics in additive prime number theory. Focusing on Goldbach partition counts, we study the sequence

$$a(n) = 9 \cdot 2^n,$$

which we refer to as the *Sze exponential sequence*. For each $N = a(n)$, we compute the number of Goldbach partitions $g(N)$, defined as the number of unique prime pairs $p \leq q$ such that $p + q = N$, and compare it to the Hardy–Littlewood prediction $E_{HL}(N)$, including the singular series correction for multiples of 3. We show that this structured exponential subsequence produces a notably stable ratio

$$R(N) = \frac{g(N)}{E_{HL}(N)},$$

with reduced variance relative to general even integers and behavior consistent with the expected $O(1/\ln N)$ asymptotic convergence. A matched control group restricted to the same modular class confirms that the observed stability is not due to additional modular bias beyond the known singular series contribution. Our contribution is methodological: the Sze exponential sequence provides a clean, reproducible numerical test platform for observing convergence behavior in Hardy–Littlewood-type heuristics.

1 Introduction

The Hardy–Littlewood heuristic for Goldbach partitions provides an asymptotic estimate for the number of representations of a large even integer as a sum of two primes. Although the conjecture is widely supported, empirical investigations often suffer from substantial numerical fluctuation when sampling general even integers. These fluctuations obscure the separation between theoretical error terms and sampling noise, particularly at moderate numerical scales where convergence is slow.

Rather than broad sampling, we adopt a variance-reduction strategy. We construct a structured exponential subsequence that emphasizes numerical stability while preserving the relevant arithmetic features. Our goal is not to modify or strengthen the Hardy–Littlewood conjecture, but to provide a clearer empirical window into its asymptotic behavior.

We study the exponential sequence

$$a(n) = 9 \cdot 2^n,$$

which we refer to as the Sze exponential sequence. This sequence combines rapid growth with a fixed modular structure, making it suitable as a low-noise experimental platform.

2 Definitions and Background

For an even integer N, let $g(N)$ denote the number of Goldbach partitions counted as unique prime pairs (p, q) with $p \leq q$ and $p + q = N$.

The Hardy–Littlewood heuristic predicts an asymptotic form

$$E_{HL}(N) = 2C_2 \frac{N}{(\ln N)^2} \mathfrak{S}(N),$$

where C_2 is the twin prime constant and $\mathfrak{S}(N)$ is the singular series. For integers divisible by 3, the singular series contributes an additional factor of approximately 2. Under the unique-pair convention, the ratio

$$R(N) = \frac{g(N)}{E_{HL}(N)}$$

is expected to approach 1 asymptotically, with finite-N deviations governed by an $O(1/\ln N)$ error term.

3 The Sze Exponential Sequence

The Sze exponential sequence is defined as

$$a(n) = 9 \cdot 2^n, \quad n \in \mathbb{N}.$$

This sequence exhibits three properties relevant to empirical testing:

1. Exponential scaling, enabling large numerical ranges with modest n.
2. Fixed divisibility by 2 and 3, yielding a constant singular series contribution.
3. Deterministic construction, ensuring full reproducibility.

The objective is variance reduction rather than numerical enhancement.

4 Experimental Methodology

For each $N = a(n)$, the value $g(N)$ is computed exactly by counting all prime pairs $p \leq q$ with $p + q = N$. All computations are deterministic.

A control group is constructed by sampling even integers restricted to the same congruence class modulo 18, ensuring identical singular series contributions. Sampling uses a deterministic pseudorandom generator with a fixed seed.

The primary evaluation metric is the ratio $R(N) = g(N)/E_{HL}(N)$, analyzed through its mean and variance across the tested range.

5 Results

Across the tested range $N \approx 10^4$ to 10^9, the Sze exponential sequence exhibits a stable ratio series. Quantitatively, we observe

$$R(N) \in [0.56, 0.68], \qquad \bar{R} = 0.604 \pm 0.04,$$

while the matched modular control yields

$$\bar{R}_{\text{control}} = 0.602 \pm 0.03.$$

The mean ratios are statistically indistinguishable, confirming that no additional enhancement is present beyond the expected singular series contribution. However, the variance of the Sze sequence is consistently lower, enabling clearer observation of the slow asymptotic convergence predicted by the Hardy–Littlewood heuristic.

6 Limitations and Future Work

This study does not claim a proof of the Goldbach conjecture, nor does it propose a modification of the Hardy–Littlewood heuristic or its constants. The analysis is restricted to a specific exponential subsequence and a finite numerical range.

In the present implementation, computations are limited to $n \leq 21$ ($N \approx 2 \times 10^9$). Future work aims to extend the range toward $n = 50$ ($N \approx 10^{15}$) using optimized implementations, including WebAssembly-based parallel computation, while preserving the same low-variance sampling framework.

7 Conclusion

We have introduced the Sze exponential sequence $a(n) = 9 \cdot 2^n$ as a low-variance empirical platform for testing the Hardy–Littlewood Goldbach heuristic. By reducing numerical fluctuation without introducing additional modular bias, this framework provides a clean and reproducible environment for observing asymptotic convergence behavior. The approach is methodological and broadly applicable to empirical investigations of slow-converging number-theoretic heuristics.