Computation of heterogeneous agent models

Quantitative macroeconomics

Kazu Matsuda

Motivation

- What's the effect of government policy?
- We want to experiment government policy but can't…
- Let's build a model imitating your country on which we can simulate.

Model ingredients

- What should we include in the model?
  - Many households (you are one of them!)
  - Many firms
  - A government (introduce it later)
- And we have a notion of time $t = 1, 2, \cdots,$ in the model.

How introduce heterogeneity?

- Heterogeneity is necessary to study inequality in the economy.
- We assume that only households are heterogeneous.
    - Households face idiosyncratic labor income shock h.
    - Accordingly, households' savings a become heterogenous.
- There is going to be a distribution of households $\mu_t(a, h)$.

Stationary equilibrium

- Assume there is no aggregate shock (but there is idiosyncratic shock).
- Then the state of the economy is characterized only with μ_t.
- Focus on stationary equilibrium where…
  - Distribution μ_t constant but households move within the distribution.
  - Prices and aggregate variables are constant.
- Omit time subscripts t for the distribution, prices, and agg variables.

Households

- Continuum (measure 1) of households indexed by i.
- They work (supply 1 unit of labor), consume c_{it}, and save as assets a_{it+1}.
- Each household lives infinitely and maximizes the following utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}),\ u' > 0,\ u'' < 0,\ \beta \in (0,1)$$

Income shocks

- Ex-ante identical but faced with idiosyncratic income shocks.
- Stochastic idiosyncratic endowments of efficiency units $h_{it} \in \mathscr{H} = \{h_1, \cdots, h_{N_H}\}$
- The Markov process: $\pi(h_{it+1} \mid h_{it})$
  - π^* is the invariant distribution associated with π.
- Aggregate endowment of skills

$$H = \sum_{j=1}^{N_H} h_j \pi^*(h_j).$$

Household constraints

- Interest rates on assets r and wages w.
- Budget constraint

$$c_{it} + a_{it+1} = (1 + r)a_{it} + wh_{it}.$$

- Borrowing constraint

$$a_{it+1} \geq -\underline{B}$$

- For simplicity, we assume that households must choose asset levels from $\mathscr{A} = \{a_1, \cdots, a_{N_A}\}$.

Household problem

$$\max_{\{c_{it}\},\{a_{it+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}) \text{ s.t.}$$

$$c_{it} + a_{it+1} = (1+r)a_{it} + wh_{it}$$

$$a_{it+1} \geq -\underline{B},\ c_{it} \geq 0,\ a_{i0} \text{ given}$$

- How to solve this dynamic optimization problem?

Simplifying the problem (eliminate c_{it})

$$\max_{\{a_{it+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u((1+r)a_{it} + wh_{it} - a_{it+1}) \text{ s.t.}$$

$$a_{it+1} \geq -\underline{B}, (1+r)a_{it} + wh_{it} - a_{it+1} \geq 0, a_{i0} \text{ given}$$

- How to solve this dynamic optimization problem?

General dynamic problem

$$\max_{\{s_t\}} \sum_{t=0}^{T} \beta^t F(s_t, s_{t+1}) \text{ s.t. } s_{t+1} \in \Gamma(s_t),\ s_0 \text{ given.}$$

- For the earlier problem, set $s_t = a_{it}$ and
  - $T \to \infty$,
  - $F(s_t, s_{t+1}) = u(wh_{it} + (1+r)a_{it} - a_{it+1})$
  - $s_{t+1} \in \Gamma(s_t) = [-\underline{B}, wh_{it} + (1+r)a_{it}]$

Solving backward from the last period

- Jump ahead to period T. s_T will be given from earlier choices.

$$V_T(s_T) = \max_{s_{T+1}} F(s_T, s_{T+1}) \text{ s.t. } s_{T+1} \in \Gamma(s_T).$$

- $g_T(s_T)$ is the set of maximizers.
- Go to period $T-1$ and so on…

$$V_{T-1}(s_{T-1}) = \max_{s_T \in \Gamma(s_{T-1})} F(s_{T-1}, s_T) + \beta V_T(s_T) \text{ and } g_{T-1}(s_{T-1}) \text{ is maximizers…}$$

$$V_0(s_0) = \max_{s_1 \in \Gamma(s_0)} F(s_0, s_1) + \beta V_1(s_1) \text{ and } g_0(s_0) \text{ is maximizers.}$$

If $T \to \infty$? (Infinite periods)

- What if we let $T \to \infty$?
- Intuitively, there is always an infinite number of periods after the current period so we would think that all of Vs are the same.

$$V(s_t) = \max_{s_{t+1} \in \Gamma(s_t)} \{F(s_t, s_{t+1}) + \beta V(s_{t+1})\}.$$

This equation of functions is called Bellman equation.

- This form of the maximization problem is also called recursive form.

Basic idea of recursive form

- Time itself is not important. If the economy starts again from s_1^*, the optimal path doesn't change.
- Optimal solution has the property that the optimal choice for s tomorrow only depends upon s today, and not the actual period.
- Two ways to think about solving this problem
  1. Find $s_0^*, s_1^*, s_2^* \ldots$ (not in this lecture)
  2. $g(s)$ "optimal policy function" $s_0, g(s_0), g(g(s_0)), g(g(g(s_0))), \ldots$ (optimal way to respond)

Recursive form

$$V(a,h) = \max_{a'} u((1+r)a + wh - a') + \beta \sum_{h'} V(a',h')\pi(h' \,|\, h) \text{ s.t.}$$

$$-\underline{B} \leq a' \leq (1+r)a + wh.$$

- Time t doesn't matter. We only need the notion of today and tomorrow.
- Let $'$ denote the tomorrow variable (a' is the tomorrow assets).
- Solutions are policy functions $g_a(a,h)$.

Firms

- All the firms have production function

$$Y = F(K, H).$$

- Profit

$$\max_{K,H} F(K, H) - (r + \delta)K - wH$$

- Capital depreciates at δ and FOC:

$$r + \delta = F_1(K, H),\ w = F_2(K, H)$$

Markets

- How wage w and rent r are determined?
- Prices clear the 3 markets
  - Labor: w
  - Assets: r
  - Goods: normalize 1 in steady state

Evolution of measures

- The state of the economy is a distribution of households $\mu(a, h)$.
- How does μ evolve over time?

$$\mu(a', h') = \sum_a \sum_h \mathbf{1}\{a : g_a(a, h) \in a'\}\pi(h' | h)\mu(a, h)$$

  - $\mathbf{1}(x \in A)$ is an indicator function.
    - $\mathbf{1}(x \in A) = 1$ if $x \in A$ and $\mathbf{1}(x \in A) = 0$ if $x \notin A$

Stationary competitive equilibrium

What's the rigorous definition of an equilibrium of a dynamic problem?

- A *stationary competitive equilibrium* is a list of functions $V(a,h)$, $g_a(a,h)$, K, H, r, w, $\mu(a,h)$ s.t.

1. (Household optimization) Taking r and w as given, $V(a,h)$ solves

$$V(a,h) = \max_{a'} u((1+r)a + wh - a') + \beta \sum_{h'} V(a',h')\pi(h'|h) \text{ s.t.}$$

$$-\underline{B} \leq a' \leq (1+r)a + wh \text{ and } g_a(a,h) \text{ is an optimal decision rule.}$$

2. (Firm optimization) Taking r and w as given, K and H solve firms problem

$$\max_{k,h} F(k,h) - (r+\delta)k - wh \text{ such that } k \geq 0, h \geq 0.$$

3. (Market clearing)

(1) Labor $H = \sum_h h\pi^*(h)$,

(2) Assets $K = \sum_a \sum_h g_a(a,h)\mu(a,h)$,

(3) Goods $F(K,H) = \sum_a \sum_h ((1+r)a + wh - g_a(a,h))\mu(a,h) + \delta K$

4. (Aggregate law of motion) Distribution of agents over states μ is stationary

$$\mu(a',h') = \sum_a \sum_h \mathbf{1}\{a : g_a(a,h) \in a'\}\pi(h' \,|\, h)\mu(a,h)$$

Setting functions

- Utility $u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$.
- Production function $F(K, H) = K^{\alpha}H^{1-\alpha}$. Then firm's FOC

$$r + \delta = F_K(K, H) = \alpha\left(\frac{K}{H}\right)^{1-\alpha}$$

$$w = F_H(K, H) = (1-\alpha)\left(\frac{K}{H}\right)^{\alpha-1}$$

Setting the income process $\pi(h' \mid h)$

- Assume that efficiency units follow an AR1

$$\ln h' = \rho \ln h + \epsilon, \;\; \epsilon \sim N(0,\sigma_\epsilon^2)$$

- Discretize using Tauchen's method.

1. Make $\mathscr{H}$ evenly spaced from -1*$stdd(\ln h)$ to +1*$stdd(\ln h)$ with distance d.
2. Assume h_j goes to $h_{j'}$ if $\rho \ln h_j + \epsilon$ is in $[\ln h_{j'} - d/2, \ln h_{j'} + d/2]$.

$$\pi(h_{j'} \mid h_j) = N(\ln h_{j'} + d/2 - \rho \ln h_j) - N(\ln h_{j'} - d/2 - \rho \ln h_j)$$

Computation

1. Guess K^0 and calculate r^0 and w^0 using firm's FOC given H.
2. Given (r^0, w^0), solve household's problem to get $g_a^0(a, h)$.
3. Use policy function g_a^0 and transition $\pi(h' \mid h)$ to compute $\mu^0(a, h)$.
4. Use invariant distribution $\mu^0(a, h)$ to compute $\tilde{K}^0 = \sum_a \sum_h g_a(a, h)\mu^0(a, h)$.
5. Stop if $|\tilde{K}^0 - K^0| < tol$. Otherwise, update $K^{j+1} = \phi\tilde{K}^j + (1 - \phi)K^j$ and go to step 2.

Computing aggregate labor H

1. Start with initial $\pi^{*0}(h_j) = 1/N_H$. Solve forward.

2. First, set initially $\pi^{*1}(h_j) = 0$ for each h_j. Then for each h_j on grid $\mathscr{H}$,

$$\pi^{*1}(h_{j'}) \leftarrow \pi^{*1}(h_{j'}) + \pi(h_{j'} \mid h_j)\pi^{*0}(h_j) \text{ for each } h_{j'} \text{ on grid } \mathscr{H}.$$

(Note that they are not equal signs. “Accumulate” in the code).

3. If $d(\pi^{*1}, \pi^{*0}) < tol$, done. Otherwise, update $\pi^{*0} = \pi^{*1}$ and go to step 2. After finishing this, get $H = \sum_{j=1}^{N_H} h_j \pi^*(h_j)$

Solving the household problem

1. Given V_0 and g_{a0}, for each $(a_i, h_j) \in \mathscr{A} \times \mathscr{H}$,

   (1) Find $a' \in \mathscr{A}$ on the grid such that

   $$g_{a1}(a_i, h_j) = a' \in \arg\max_{a' \in \mathscr{A}} u(wh_j + (1+r)a_i - a') + \beta \sum_{h' \in \mathscr{H}} V_0(a', h')\pi(h' \mid h_j)$$

   (2) Update
   $$V_1(a_i, h_j) = u(wh_j + (1+r)a_i - g_{a1}(a_i, h_j)) + \beta \sum_{h' \in \mathscr{H}} V_0(g_{a1}(a_i, h_j), h')\pi(h' \mid h_j)$$

2. If $d(V_0, V_1) < tol$ or $d(g_{a0}, g_{a1}) < tol$, done. Otherwise return to 3 with new guess V_1 and g_{a1}.

Computing distribution μ

- Start with initial $\mu^0(a_i, h_j) = \dfrac{1}{N_A N_H}$.
- Now update μ^0 to μ^1. First, set initially $\mu^1(a_i, h_j) = 0$ for each (a_i, h_j).
- Then for each (a_i, h_j) on grid $\mathscr{A} \times \mathscr{H}$, for each j'

$$\mu^1(g_a(a_i, h_j), h_{j'}) \leftarrow \mu^1(g_a(a_i, h_j), h_{j'}) + \pi(h_{j'} \mid h_j)\mu^0(a_i, h_j)$$

for each j. (Note that they are not equal signs. Accumulate in the code)

- Repeat this until $d(\mu^1, \mu^0) < tol$.

Equilibrium with labor income tax

- Now we assume that the government introduces labor income tax with rate τ (exogenous) and rebate it as lump-sum transfer T (endogenous).

$$\max_{\{c_{it}\},\{a_{it+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}) \text{ s.t.}$$

$$c_{it} + a_{it+1} = (1+r)a_{it} + (1-\tau)wh_{it} + T$$

$$a_{it+1} \geq -\underline{B},\ a_{i0} \text{ given}$$

Stationary competitive equilibrium

- A stationary CE with policy is a list of functions $V(a,h)$, $g_a(a,h)$, K, H, r, w, $\mu(a,h)$, T s.t.

1. (Household optimization) Taking r and w as given, $V(a,h)$ solves

$$V(a,h) = \max_{a'} u((1+r)a + (1-\tau)wh + T - a') + \beta \sum_{h'} V(a',h')\pi(h' \mid h) \text{ s.t.}$$

$$-\underline{B} \leq a' \leq (1+r)a + (1-\tau)wh + T \text{ and } g_a(a,h) \text{ is an optimal decision rule.}$$

2. (Firm optimization) Taking r and w as given, K and H solve firms problem

$$\max_{k,h} F(k,h) - (r+\delta)k - wh \text{ such that } k \geq 0,\ h \geq 0.$$

3. (Government) $\tau w H = T$

4. (Market clearing)

   (1) Labor $H = \sum_h h\pi^*(h)$,

   (2) Assets $K = \sum_a \sum_h g_a(a, h)\mu(a, h)$,

   (3) Goods $F(K, H) = \sum_a \sum_h ((1 + r)a + (1 - \tau)wh + T - g_a(a, h))\mu(a, h) + \delta K$

5. (Aggregate law of motion) Distribution of agents over states μ is stationary

$$\mu(a', h') = \sum_a \sum_h \mathbf{1}\{a : g_a(a, h) \in a'\}\pi(h' | h)\mu(a, h)$$

Computation of the tax model

1. Guess K^0 Calculate r^0 and w^0 using firm's FOC. Calculate $T^0 = \tau w^0 H$.
2. Given (r^0, w^0, T^0), solve household's problem to get $g_a^0(a, h)$.
3. Use policy function g_a^0 and transition $\pi(h' \mid h)$ to compute $\mu^0(a, h)$.
4. Use invariant distribution $\mu^0(a, h)$ to compute $\tilde{K}^0 = \sum_a \sum_h g_a(a, h)\mu^0(a, h)$.
5. Stop if $|\tilde{K}^0 - K^0| < tol$. Otherwise, update $K^{j+1} = \phi K^j + (1-\phi)\tilde{K}^j$ and go to step 2.