Whitehead's problem

Hidetaka Noro

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Abstract

In this thesis we study the relation between free module and projective module. It is a folklore that every free module is projective (see. Theorem III.11), but in generally the converse does not hold. The Whitehead's problem is the question that whether $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z})$ implies A is free. In this thesis, we demonstrate models where Whitehead's problem holds true and where it does not.

To produce the not free Whitehead group we use the model MA $+\neg$ CH and construct the model that Whitehead group is free we use the Godël's constructive universe. I discusses references to the works of [Jec03] and [Kun11] in the section "Diamond Principle and Martin Axiom", [DF04], [Mac71], [Kaw76] and [Kaw77] in the sections "Derived Functor" and "Module" and [Ekl76] in the last section "Whitehead Problem".

Contents

I Diamond Principle and Martin Axiom

Definition I.1. For regular cardinal κ and stationary set $E \subset \kappa$. $\Diamond_E$ is the statement that there is a $\Diamond_E$-sequence $\langle D_\alpha \subset \kappa\,;\alpha \in E\rangle$ such that $\{\alpha \in E\,; E \cap \alpha = D_\alpha\}$ is a stationary subset. $\Diamond$ is the statement $\Diamond_{\omega_1}$.

Theorem I.2. $\Diamond_E \implies \mathfrak{c} \leq \cup E$. In particular $\Diamond \implies \mathfrak{c} = \omega_1$

Proof. Let $\langle D_\alpha\,;\alpha \in E\rangle$ be a $\Diamond_E$-sequence. We shall show that $\mathcal{P}(\omega) \subset \{D_\alpha\,;\alpha \in E\}$. Fix $A \in \mathcal{P}(\omega)$. Since $\{\gamma \in \cup E\,; \cup A \in \gamma\}$ is club, there is an $\alpha \in E$ such that $\cup A \in \alpha$ and $A \cap \alpha = D_\alpha$. Moreover, we have $A = D_\alpha$, now we complete the proof. $\square$

This result assert that $\neg\Diamond_E$ hold in the model ZFC $+2^{\aleph_0} > \kappa$ (where $\kappa = \cup E$). Thus, by using the Cohen forcing, it is easy (for example, use a forcing poset $\mathbb{P} = \mathrm{Fn}(\omega \times \aleph_{\kappa+1}, 2)$[i]) to construct the model for $\neg\Diamond_E$.

Theorem I.3 ([Jec03]). $V = L \implies \Diamond_E$

Proof. Fix a regular cardinal $\kappa = \cup E$. At the beginning of the proof,

we shall define a pair $\langle S_\alpha, C_\alpha\rangle$, $\alpha \in \kappa$, recursively in a uniform way, such that $S_\alpha \subset \kappa$ and C_α is club in κ:

1. $S_0 = C_0 = \emptyset$,
2. $S_{\alpha+1} = C_{\alpha+1} = \alpha + 1$,
3. For limit stages:

Case-I. (S_α, C_α) be a least $<_L$ least pair such that

a. $S_\alpha \subset \kappa$,

b. C_α is club in κ, and

c. $S_\alpha \cap \xi \neq S_\xi$ for any $\xi \in C_\alpha$.

Case-II. $S_\alpha = C_\alpha$ otherwise.

We shall show that the sequence $\langle S_\alpha\,; \alpha \in E\rangle$ is a $\Diamond_E$ sequence. Assume not, suppose that there is a subset $X \subset \kappa$ and a club C in κ which pair witnesses that

$$X \cap \alpha \neq C_\alpha \text{ for any } \alpha \in C$$

, we may assume that the pair (X, C) is a $<_L$-least pair.

Note that since $S_\alpha, X_\alpha, \kappa, X, C \subset \kappa$, we have $\langle(S_\alpha, C_\alpha)\,; \alpha \in E\rangle$, (X, C) contain in the model $(L(\kappa^+), \in)$ and moreover for a countable elementary submodel N with $E \in N$, since both sets are definable in the model $(L(\kappa^+), \in)$, we obtain that both contained in the model N and let $\delta = \kappa^+ \cap N$. For a transitive collapsing function $\pi\colon N \to L_\delta$. we have the following properties:

- $\pi(\gamma) = \kappa$,
- $\pi(X) = X \cap \delta$,
- $\pi(C) = C \cap \delta$, and
- $\pi(\langle(S_\alpha, C_\alpha)\,; \alpha \in E\rangle) = \langle(S_\alpha, C_\alpha)\,; \alpha \in \delta \cap E\rangle$.

Therefore, in $(L(\delta), \in)$, we have:

$$(X \cap \delta, C \cap \delta) \text{ is the } <_L\text{-least pair } (Z, D) \text{ such that } Z \subset \delta \text{ and } D \text{ is club in } \delta, \text{ and } Z \cap \xi \neq S_\xi \text{ for any } \xi \in D.$$

[i] See Theorem IV.7.17 in [Kun11]

Moreover, we have the property $(*)$ in L, so by the construction in Case I, we have that $(S_\delta, C_\delta) = (X\cap\delta, C\cap\delta)$. On the other hand since $C\cap\delta$ is unbounded in δ and C is closed in κ, we have $\delta \in C$. Therefore, $\delta \in C$ witnesses that $C\cap\delta = S_\delta$, a contradiction. $\square$

Theorem I.4. Let E be a stationary subset of regular cardinal κ and $\langle D_\alpha\,;\alpha\in E\rangle$ be a $\diamondsuit_E$-sequence. Let X, B be sets of size κ, $g\colon X\to C$ function, $\langle X_\alpha\,;\alpha\in\kappa\rangle$ be a strictly increasing sequence among $\subset$ of size $<\kappa$, such as $X = \cup\{X_\alpha\,;\alpha\in\kappa\}$. For a bijection $\nu\colon\kappa\to X\times C$, there is an $\alpha\in E$ such as D_α code $g|_{X_\alpha}$, i.e., $g|_{X_\alpha} = \nu[D_\alpha]$.

Proof. Let

$$X = \{\nu^{-1}(\langle a,c\rangle)\,;\langle a,c\rangle\in g\}$$
$$F = \{\alpha\in\kappa\,; g|_{X_\alpha}\subset\nu[\alpha]\subset X_\alpha\times C\}$$

First, we shall show that F is club. It is obvious that F is closed. To see that F is unbounded. Let $\xi_0\in\kappa$. Define ξ_α, $\alpha\in\kappa$, recursively:

1. Successor stage:

(1–a) $\xi_{\alpha+2n+1} = \cup\{\eta\in\kappa\,;\nu(\eta)\in g|_{X_{\xi_{\alpha+2n}}}\}+1$,

(1–b) $\nu[\xi_{\alpha+2n+1}]\subset X_{\xi_{\alpha+2n+2}}\times C$,

2. Limit stage:

(2–a) $\xi_\gamma = \cup\{\xi_\alpha\,;\alpha\in\gamma\}$ for any limit $\gamma\in\kappa$.

Then we have that $\xi = \cup\{\xi_\alpha\,;\alpha\in\kappa\}\in F$.

Since $\diamondsuit_E$ asserts that the set $\{\alpha\in E\,; X\cap\alpha = D_\alpha\}$ is stationary, there is an $\alpha\in F$ such that $X\cap\alpha = D_\alpha$. It is routine to obtain that $\nu[D_\alpha] = \nu[X\cap\alpha] = g|_{X_\alpha}$. $\square$

Definition I.5. MA is the statement that for any family $\mathcal{F}$ of dense subsets of ccc poset P of with $|\mathcal{F}|<\mathfrak{c}$ there is a generic filter G.

Fact I.6 ([Kun11])**.** Con(ZF) $\implies$ Con(ZF + MA + $\neg$ CH).

Corollary I.7. Assume MA + $\neg$ CH. Let A, B be sets of size $<2^\omega$ and $P\subset \mathrm{Fn}_{\mathfrak{c}}(A,B)$ be a non-empty family. If P satisfies the conditions

1. $\forall a\in A\forall f\in P\exists g\in P\ (f\subset g\wedge a\in\operatorname{dom} g)$, and
2. $\forall\mathcal{A}\in[P]^{\omega_1}\exists\{f_0,f_1\}\in[\mathcal{A}]^2\exists f_2\in P\ (f_0\cup f_1\subset f_3)$

then there is $g\in {}^AB$ such that for any finite set $F\in[A]^{<\omega}$ there is a $f\in P$ such that

$$g|_F = f|_F \text{ and } F\subset\operatorname{dom} f.$$

Proof. Since P is a ccc-forcing poset, with relation $\leq_P=\supset$, for a family $\mathcal{F} = \{D_F\,;F\in[A]^{<\omega}\}$ of size $<\mathfrak{c}$, where $D_F = \{f\in P\,;F\subset\operatorname{dom} f\}$, by applying MA there is the desired generic real $\cup G\in {}^AB$. $\square$

II Derived Functor

Definition II.1. Let $f\colon A \to B$ be a morphism. We define the followings:

1. The *kernel* for f is the morphism $\ker f\colon \operatorname{Ker} f \to A$ such that
   - $f \circ \ker f = 0$, and
   - For any $\varphi\colon C \to A$ with $f \circ \varphi = 0$ there is the unique $\psi\colon C \to \operatorname{Ker} f$ such that $\varphi = \ker f \circ \psi$, i.e.,

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r, "\ker f"] & A \arrow[r, "f"] & B \\ C \arrow[u, dotted, "\psi"] \arrow[ur, "\varphi"'] & & \end{tikzcd}$$

2. The *cokernel* for f is the morphism $\operatorname{coker} f\colon B \to \operatorname{Coker} f$ such that
   - $\operatorname{coker} f \circ f = 0$, and
   - For any $\varphi\colon B \to C$ with $\varphi \circ f = 0$ there is the unique $\psi\colon \operatorname{Coker} f \to C$ such that $\varphi = \psi \circ \operatorname{coker} f$, i.e.,

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "\operatorname{coker} f"] \arrow[dr, "\varphi"'] & \operatorname{Coker} f \arrow[d, dotted, "\psi"] \\ & & C \end{tikzcd}$$

3. The *image* of f is the kernel of the cokernel of f, $\operatorname{im} f = \ker \operatorname{coker} f$,
4. The *coimage* of f is the cokernel of the kernel of f, $\operatorname{coim} f = \operatorname{coker} \ker f$.

Definition II.2. An *Abelian category*, $\mathcal{A}$, is a category with following properties:

1. For any arrows $\cdot \xrightarrow{f_0, f_1} \cdot \xrightarrow{g_0, g_1} \cdot$ in $\mathcal{A}$, we have the additive operation $+$ such as
   (a) f_0, f_1 is an arrow in $\mathcal{A}$ with same domain and codomain,
   (b) $f_0 + f_1 = f_1 + f_0$,
   (c) The composition $(g_0 + g_1) \circ (f_0 + f_1) = g_0 f_0 \circ + g_0 f_1 \circ + g_1 \circ f_0 + g_1 \circ f_1$,
2. $\mathcal{A}$ possesses the null object, we denote 0,
3. Every arrow has the kernel and cokernel,
4. $\mathcal{A}$ has the binary biproduct, (we define at the following)
5. Every monic is the kernel for some arrow,
6. Every epi is the cokernel for some arrow.

Theorem II.3 (Snake lemma,[Mac71])**.** For two exact sequences with morphism f, g, h as following diagram commutes. Then there is a long red exact sequence as in the diagram. Moreover, if we have the two blue we can extend the red sequence.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} f \arrow[r] \arrow[d] & \operatorname{Ker} g \arrow[r] \arrow[d] & \operatorname{Ker} h \arrow[d] \arrow[dddll, rounded corners, "s"] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
 & \operatorname{Coker} f \arrow[r] & \operatorname{Coker} g \arrow[r] & \operatorname{Coker} h \arrow[r] & 0
\end{tikzcd}$$

Definition II.4. For two objects a, b the *binary product object* $a \oplus b$ is the object with four morphisms and universal property such as:

- There are two inclusions, i, and two projections, p, such that
  1. $p_A \circ i_A = \mathrm{id}_A$,
  2. $p_B \circ i_B = \mathrm{id}_B$,
  3. $i_A \circ p_A + i_B \circ p_B = \mathrm{id}_{A \oplus B}$

$$\begin{tikzcd}
A \arrow[r, shift left, "i_A"] & A \oplus B \arrow[l, shift left, "p_A"] \arrow[r, shift left, "p_B"] & B \arrow[l, shift left, "i_B"]
\end{tikzcd}$$

In generally, since we have $p_A \circ i_B = p_A \circ (i_A \circ p_A + i_B \circ p_B) \circ i_B = p_A \circ i_B + p_A \circ i_B$, we obtain that $p_B \circ i_A = 0$ and $p_A \circ i_B = 0$. Hereafter, we use only for an Abelian category.

Definition II.5. Let f, g be composable pair. The sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is *exact* at B provided that any of following equivalent statements holds:

- $\operatorname{im} f = \ker g$,
- $\operatorname{coker} = f = \operatorname{coim} g$,
- $g \circ f = 0$ and for any $\varphi \in_m B$ if $g \circ \varphi \equiv 0$ then there is a morphism $\varphi' \in_m A$ such that $\varphi' \circ f \equiv \varphi$.

A sequence $A_0 \to A_1 \to \cdots \to A_n$ is exact provided that it is exact at any object.

A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *splits* provided that we have the one of the following equivalent statement:

1. there is a morphism $h\colon C \to B$ such that $g \circ h = \mathrm{id}_C$,

2. there is a morphism $h\colon A \to B$ such that $h \circ f = \mathrm{id}_A$,

3. $B = A \oplus C$.

Lemma II.6 (Splitting lemma)**.** For a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the following are equivalent.

1. f is right-cancellable, i.e., there is a morphism $f' \in \mathrm{Hom}(A, B)$ such that $f' \circ f = \mathrm{id}_A$,

2. g is left-cancellable, i.e., there is a morphism $g' \in \mathrm{Hom}(A, B)$ such that $g \circ g' = \mathrm{id}_C$,

3. There is $f' \in \mathrm{Hom}(A, B)$ and $g' \in \mathrm{Hom}(C, B)$ such that
   - $f' \circ f = \mathrm{id}_A$,
   - $g \circ g' = \mathrm{id}_C$, and
   - $f \circ f' + g' \circ g = \mathrm{id}_B$.

Proof. We shall show (1) $\implies$ (3) (the other cases are immediate by mimic this case or manifestly). Let $f'\colon B \to A$ be a morphism such that $f' \circ f = \mathrm{id}_B$. Since the short sequence exact at, we obtain that B, $\operatorname{coker} f = \operatorname{coim} g = g$ and $(1 - ff') \circ f = 0$, we have the canonical morphism $g'\colon C \to B$:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[dr, "\mathrm{id}_B - ff'"'] & C \arrow[d, dotted, "g'"] \\ & & B \end{tikzcd}$$

Moreover, since we have $g \circ g' \circ g = g \circ (\mathrm{id}_B - ff')$, $g \circ g' = \mathrm{id}_C$ holds (note that g is epi).

Therefore, we obtain

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "f'"] \arrow[r, shift left, "g"] & C \arrow[l, shift left, "g'"] \end{tikzcd}$$

with

1. $f' \circ f = \mathrm{id}_A$,

2. $g' \circ g = \mathrm{id}_C$, and

3. $f \circ f' + g \circ g' = \mathrm{id}_B$.

□

Definition II.7. For two Abelian categories $\mathcal{A}$ and $\mathcal{B}$ and a functor $T\colon \mathcal{A} \to \mathcal{B}$, we define the following:

1. A functor T is *additive* provided that for any parallel arrows f and g in $\mathcal{A}$, we have $T(f + g) = T(f) + T(g)$,

2. A covariant functor T is *left-exact* provided that for any short exact sequence $0 \to A \to B \to C$, a sequence $0 \to T(A) \to T(B)$ is exact.

3. A covariant functor T is *right-exact* provided that for any short exact sequence $A \to B \to C \to 0$, a sequence $T(A) \to T(B) \to 0$ is exact.

4. A covariant functor T is *exact* provided that it is right and left-exact,

5. A contravariant functor T is *left-exact* provided that for any short exact sequence $A \to B \to C \to 0$, a sequence $0 \to T(A) \to T(B)$ is exact.

6. A contravariant functor T is *right-exact* provided that for any short exact sequence $0 \to A \to B \to C$, a sequence $T(A) \to T(B) \to 0$ is exact.

7. A contravariant functor T is *exact* provided that it is right and left-exact.

Theorem II.8. Every additive functor preserves the binary product objects.

Theorem II.9. Every splits exact sequence $0 \to A \to B \to C \to 0$ preserved by any additive functor.

Definition II.10.

1. An object P is *projective* provided that for any φ and epi f whose codomain are the same, then there is ψ such that $\varphi = f \circ \psi$,

2. An object Q is *injective* provided that for any φ and monic f whose domain are the same, then there is ψ such that $\psi = \psi \circ f$.

Theorem II.11. For two projective objects P, Q, then the direct product object $P \oplus Q$ is projective.

Definition II.12. Let $\mathcal{C}$ be a countable sequence of pair of object and arrow, $\mathcal{C} = \langle (C_n, \delta_n)\, ; n \in \omega \rangle$

1. A sequence $\mathcal{C}$ is a *chain* provided that $\operatorname{dom} \delta_n = C_n = \operatorname{cod} \delta_{n+1}$ for any $n \in \omega$,

$$\cdots \to C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \to \cdots \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} \operatorname{cod} \delta_0$$

2. A sequence $\mathcal{C}$ is a *chain complex* provided that $\mathcal{C}$ is chain and $\delta_n \circ \delta_{n+1} = 0$ for any $n \in \omega$,

3. A sequence is a *projective resolution* for an object A provided that it is a chain complex such that every C_n is projective, $\operatorname{cod} \delta_0$ and the following sequence exact:

$$\cdots \to C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \to \cdots \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} A \to 0$$

In particular for an object A, a chain $\mathcal{C}$ with $\operatorname{cod} \delta_0 = A$ is called a chain for A.

Duality,

1. A sequence $\mathcal{C}$ is a *cochain* provided that $\operatorname{dom}\delta_{n+1} = C_n = \operatorname{cod}\delta_n$ for any $n \in \omega$,

$$\operatorname{dom}\delta_0 \xrightarrow{\delta_0} C_0 \xrightarrow{\delta_1} \cdots \to C_{n-1} \xrightarrow{\delta_n} C_n \xrightarrow{\delta_{n+1}} C_{n+1} \to \cdots,$$

2. A sequence $\mathcal{C}$ is a *cochain complex* provided that $\mathcal{C}$ is cochain and $\delta_{n+1} + \circ\delta_n = 0$ for any $n \in \omega$,

3. A sequence is a *injective resolution* for an object A provided that it is a cochain complex such that every C_n is injective, $\operatorname{dom}\delta_0 = A$ and the following sequence exact:

$$0 \to A \xrightarrow{\delta_0} C_0 \xrightarrow{\delta_1} \cdots \to C_{n-1} \xrightarrow{\delta_n} C_n \xrightarrow{\delta_{n+1}} C_{n+1} \to \cdots,$$

In particular for an object A, a chain $\mathcal{C}$ with $\operatorname{dom}\delta_0 = A$ is called a cochain for A.

Definition II.13. Let $\mathcal{A} = \langle (A_n, \delta_n)\,; n \in \omega\rangle$ be a chain complex. Define the *n-th homology object* as the quotient object, $H_n(\mathcal{A}) = H_n(\delta_n) = \operatorname{Ker}(\delta_n)/\operatorname{Im}(\delta_{n+1})$, i.e., the unique module as following universal property.

$$\begin{tikzcd}
 & \operatorname{Im}(\delta_{n+1}) \arrow[r, dotted, "u_n"] \arrow[d, "\operatorname{im}\delta_{n+1}"'] & \operatorname{Ker}(\delta_n) \arrow[r, "\operatorname{coker} u_n"] \arrow[dl, "\ker\delta_n"] & \operatorname{Coker} u_n \arrow[r, phantom, "="] & \operatorname{Ker}(\delta_n)/\operatorname{Im}(\delta_{n+1}) \\
A_{n+1} \arrow[r, "\delta_{n+1}"] & A_n \arrow[r, "\delta_n"] \arrow[d, "\operatorname{coker}\delta_{n+1}"'] & A_{n-1} & & \\
 & \operatorname{Coker}\delta_n \arrow[ur, dotted, "u"'] & & &
\end{tikzcd}$$

Definition II.14. Let $\mathcal{A} = \langle (A_n, \delta_n)\,; n \in \omega\rangle$ be a cochain complex. Define the *n-th cohomology object* as the quotient object, $H^*_n(\mathcal{A}) = H^*_n(\delta_n) = \operatorname{Ker}(\delta_n)/\operatorname{Im}(\delta_{n+1})$, i.e., the unique module as following universal property.

$$\begin{tikzcd}
 & \operatorname{Im}(\delta_n) \arrow[r, dotted, "u^*_n"] \arrow[d, "\operatorname{im}\delta_n"'] & \operatorname{Ker}(\delta_{n+1}) \arrow[r, "\operatorname{coker} u^*_n"] \arrow[dl, "\ker\delta_{n+1}"] & \operatorname{Coker} u^*_n \arrow[r, phantom, "="] & \operatorname{Ker}(\delta_{n+1})/\operatorname{Im}(\delta_n) \\
A_{n-1} \arrow[r, "\delta_n"] & A_n \arrow[r, "\delta_{n+1}"] \arrow[d, "\operatorname{coker}\delta_n"'] & A_{n+1} & & \\
 & \operatorname{Coker}\delta_n \arrow[ur, dotted, "u"'] & & &
\end{tikzcd}$$

Theorem II.15. For an object A and let $\mathcal{A} = \langle (A_n, \delta_n)\,; n \in \omega\rangle$ and $\mathcal{B} = \langle (B_n, \lambda_n)\,; n \in \omega\rangle$ cochains for A. If there is morphisms $\langle f_n \colon A_n \to B_n\,; n \in \omega\rangle \colon \mathcal{A} \to \mathcal{B}$ and $\langle g_n \colon B_n \to A_n\,; n \in \omega\rangle \colon \mathcal{B} \to \mathcal{A}$ with $g_n \circ f_n = \mathrm{id}_{A_n}$ for each $n \in \omega$. Then there is $\langle \varphi_n \colon H_n(\mathcal{A}) \to H_n(\mathcal{B})\,; n \in \omega\rangle$ and $\langle \psi_n\,; n \colon H_n(\mathcal{B}) \to H_n(\mathcal{A}) \in \omega\rangle$ such that $\psi_n \circ \varphi_n = \mathrm{id}_{H_n(\mathcal{A})}$. Moreover, if two chains are isomorphism, so does $H_n(\mathcal{A})$ and $H_n(\mathcal{B})$ for any $n \in \omega$.

Proof.

$$
\begin{tikzcd}
 & A_n \arrow[rr, "\operatorname{coker}\delta_n"] \arrow[dd, "f^{n+1}"] \arrow[dddd, bend right=60, "\mathrm{id}_{A_n}"'] & & \operatorname{Coker}\delta_n \arrow[dl, dotted] & \\
\operatorname{Im}\delta_n \arrow[ur] \arrow[rr, "u_A^n"'] & & \operatorname{Ker}\delta_{n+1} \arrow[ul] \arrow[rr, "\operatorname{coker}u_A^n"] & & \operatorname{Coker}(u_A^n) \arrow[dd, dotted, "\varphi_n"] \arrow[dddd, dotted, bend left=60, "\mathrm{id}_{H_n(\mathcal{A})}"] \\
 & B_n \arrow[rr, "\operatorname{coker}\lambda_n"] \arrow[dd, "g^{n+1}"] & & \operatorname{Coker}\lambda_n \arrow[dl, dotted] & \\
\operatorname{Im}\lambda_n \arrow[ur] \arrow[rr, "u_B^n"'] & & \operatorname{Ker}\lambda_{n+1} \arrow[ul] \arrow[rr, "\operatorname{coker}u_B^n"] & & \operatorname{Coker}(u_B^n) \arrow[dd, dotted, "\psi_n"] \\
 & A_n \arrow[rr, "\operatorname{coker}\delta_n"] & & \operatorname{Coker}\delta_n \arrow[dl, dotted] & \\
\operatorname{Im}\delta_n \arrow[ur] \arrow[rr, "u_A^n"] & & \operatorname{Ker}\delta_{n+1} \arrow[ul] \arrow[rr, "\operatorname{coker}u_A^n"] & & \operatorname{Coker}(u_A^n)
\end{tikzcd}
$$

$\square$

Theorem II.16. For a cochain $\langle (X_n, \delta_n)\,; n \in \omega\rangle$, the n-th cohomology has the different form as in the following diagram:

$$
\begin{tikzcd}
\operatorname{Coker}u_{n-1} \arrow[d, equal] & \operatorname{Ker}\delta_n \arrow[l, "\operatorname{coker}u_{n-1}"'] \arrow[dr, "\ker\delta_n"'] & \operatorname{Im}\delta_{n-1} \arrow[l, "u_{n-1}"'] \arrow[d, "\operatorname{im}\delta_{n-1}"] & \operatorname{Im}\delta_n \arrow[r, "u_n"] \arrow[d, "\operatorname{im}\delta_n"'] & \operatorname{Ker}\delta_{n+1} \arrow[r, "\operatorname{coker}u_n"] \arrow[dl, "\ker\delta_{n+1}"] & \operatorname{Coker}u_n \arrow[d, equal] \\
H_{n-1}(X) \arrow[d, equal] & X_{n-2} \arrow[r, "\delta_{n-1}"] & X_{n-1} \arrow[r, "\delta_n"] \arrow[d, "\operatorname{coker}\delta_{n-1}"'] & X_n \arrow[r, "\delta_{n+1}"] & X_{n+1} & H_n(X) \arrow[d, equal] \\
\operatorname{Ker}\widetilde{\delta_n} \arrow[rr, "\ker\widetilde{\delta_n}"] & & \operatorname{Coker}\delta_{n-1} \arrow[r, dotted, "\widetilde{\delta_n}"] & \operatorname{Ker}\delta_{n+1} \arrow[u, "\ker\delta_{n+1}"'] \arrow[rr, "\operatorname{coker}\widetilde{\delta_n}"] & & \operatorname{Coker}\widetilde{\delta_n}
\end{tikzcd}
$$

Proof. To prove that $\operatorname{Coker}u_{n-1} = \operatorname{Ker}\widetilde{\delta_n}$, it is enough to show the following two equalities:

1. $\operatorname{coker}u_n = \operatorname{coim}(\operatorname{coker}\delta_{n-1} \circ \ker\delta_n)$,

   equivalently, $\operatorname{im}u_{n-1} = \ker(\operatorname{coker}\delta_{n-1} \circ \ker\delta_n)$,

2. $\ker\widetilde{\delta_n} = \operatorname{im}(\operatorname{coker}\delta_{n-1} \circ \ker\delta_n)$.

$$
\begin{tikzcd}
 & \operatorname{Im}\delta_{n-1} \arrow[d, "u_{n-1}"] & X_{n-2} \arrow[d, "\delta_{n-1}"] & X_n \arrow[r, "\delta_{n+1}"] & X_{n+1} \\
\operatorname{Im}u_{n-1} \arrow[r, "\operatorname{im}u_{n-1}"] & \operatorname{Ker}\delta_n \arrow[r, "\ker\delta_n"] \arrow[d, "\operatorname{coker}u_{n-1}"] & X_{n-1} \arrow[ur, "\delta_n"] \arrow[r, "\operatorname{coker}\delta_{n-1}"] & \operatorname{Coker}\delta_{n-1} \arrow[r, "\widetilde{\delta_n}"] & \operatorname{Ker}\delta_{n+1} \\
 & \operatorname{Coker}u_{n-1} & & \operatorname{Im}(\operatorname{coker}\delta_{n-1} \circ \ker\delta_n) \arrow[u, "\operatorname{im}(\operatorname{coker}\delta_{n-1}\circ\ker\delta_n)"] & \operatorname{Ker}\widetilde{\delta_n} \arrow[ul, "\operatorname{Ker}\widetilde{\delta_n}"']
\end{tikzcd}
$$

For 1, we shall verify

- $\operatorname{coker}\delta_{n-1} \circ \ker\delta_n \circ \operatorname{im}u_{n-1} = 0$,
- $\operatorname{coker}\delta_{n-1} \circ \ker\delta_n \circ \varphi = 0 \implies \exists!\psi\ \varphi = \operatorname{im}u_{n-1} \circ \psi$ for any composable φ.

The former is immediate by the facts that $u_{n-1} = \operatorname{im}u_{n-1} \circ \operatorname{coim}u_{n-1}$ and $\operatorname{coim}u_{n-1}$ is epi. To see the later, assume that for φ we have $\operatorname{coker}\delta_{n-1} \circ \ker\delta_n \circ \varphi = 0$. By the universal property for $\operatorname{im}\delta_{n-1}$ according to $\ker\delta_n \circ \varphi = 0$, there is ψ such that $\ker\delta_n \circ \varphi = \operatorname{im}\delta_{n-1} \circ \psi$. Thus, since $\ker\delta_n$ is monic, we obtain that $\varphi = u_{n-1} \circ \psi$. Moreover, by the universal property for $\operatorname{im} u_{n-1}$ according to $\operatorname{coker} u_{n-1} \circ \varphi = 0$, there is ψ' such that $\varphi = \operatorname{im} u_n \circ \psi'$. We complete the proof.

For 2, we shall verify

- $\operatorname{coker}(\operatorname{coker}\delta_{n-1} \circ \ker\delta_n) \circ \ker\widetilde{\delta_n} = 0$,
- $\operatorname{coker}(\operatorname{coker}\delta_{n-1} \circ \ker\delta_n) \circ \varphi = 0 \implies \exists^! \psi\ \varphi = \ker\widetilde{\delta_n} \circ \psi$ for any composable φ.

The former is immediate by the facts $\ker\delta_{n+1}$ is monic and in generally for composable f, g $g \circ f = 0$ implies $\operatorname{coker} f \circ \ker g = 0$.To see the later, assume for φ we have $\operatorname{coker}(\operatorname{coker}\delta_{n-1} \circ \ker\delta_n) \circ \varphi = 0$. The universal property for $\operatorname{coker}(\operatorname{coker}\delta_{n-1} \circ \ker\delta_n)$ according to $\widetilde{\delta_n} \circ \operatorname{coker}\delta_{n-1} \circ \ker\delta_n = 0$, there is ψ such that $\widetilde{\delta_n} = \psi \circ \operatorname{coker}(\operatorname{coker}\delta_{n-1} \circ \ker\delta_n)$, moreover by the universal property for $\ker\widetilde{\delta_n}$ according to $\ker\widetilde{\delta_n} \circ \psi = 0$, there is ψ' such as $\varphi = \ker\widetilde{\delta_n} \circ \varphi$. We complete the proof.

To prove that $\operatorname{Coker} u_{n+1} = \operatorname{Coker}\widetilde{\delta_n}$, we shall show the following two equalities:

At the beginning of the proof, by the universal property for $\operatorname{coker}\delta_{n-1}$ according to $\operatorname{coim}\delta_n \circ \delta_{n-1}$ there is φ such that $\operatorname{coim}\delta_n = \varphi \circ \operatorname{coker}\delta_{n-1}$. Moreover, the uniqueness property for $\widetilde{\delta_n}$ asserts that $\widetilde{\delta_n} = u_{n+1} \circ \varphi$.

$$
\begin{tikzcd}
X_{n-2} \arrow[r, "\delta_{n-1}"] & X_{n-1} \arrow[rr, "\delta_n"] \arrow[dl, "\operatorname{coker}\delta_{n-1}"'] \arrow[dr, "\operatorname{coim}\delta_n"] & & X_n \arrow[r, "\delta_{n+1}"] & X_{n+1} \\
\operatorname{Coker}\delta_{n-1} \arrow[rr, dotted, "\varphi"] \arrow[rrrr, dotted, bend right, "\widetilde{\delta_n}"'] & & \operatorname{Im}\delta_n \arrow[ur, "\operatorname{im}\delta_n"] \arrow[rr, "u_n"] & & \operatorname{Ker}\delta_{n+1} \arrow[ul, "\ker\delta_{n+1}"']
\end{tikzcd}
$$

To see that $\operatorname{coker} u_{n+1} = \operatorname{coker}\widetilde{\delta_n}$, we shall show that:

- $\operatorname{coker} u_{n+1} \circ \widetilde{\delta_n} = 0$,
- $\eta \circ \widetilde{\delta_n} = 0 \implies \exists^! \psi\ \eta = \psi \circ \operatorname{coker} u_{n+1}$ for any composable η.

Since former is immediate, we shall verify the later. Assume that $\eta \circ \widetilde{\delta_n} = 0$, we have $\eta \circ u_{n+1} \circ \operatorname{coim}\delta_n = 0$. Thus, by the universal property for $\operatorname{Coker} u_{n+1}$ according to $\eta \circ u_{n+1} = 0$, there is ψ such that $\eta = \psi \circ \operatorname{coker} u_{n+1}$.

$\square$

Theorem II.17. For chains $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and morphisms $\langle (f_n, g_n)\,;\, n \in \omega \rangle$ such that the sequence $0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$ exact and $f_{n+1} \circ \delta_n = \lambda_n \circ f_n$ and

$g_{n+1} \circ \lambda_n = \eta_n \circ g_n$ for each $n \in \omega$. Then we have two short exact sequences as in the following diagram draw by red and blue:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[dd, red] & 0 \arrow[d] & 0 \arrow[dd, blue] & 0 \arrow[d] & \\
A_{n-1} \arrow[rr, "\delta_{n-1}"] \arrow[dd, "f_{n-1}"] & & A_n \arrow[rr, "\delta_n"] \arrow[dl, "\operatorname{coker} \delta_{n-1}"] \arrow[dd, "f_n"] & & A_{n+1} \arrow[r, "\delta_{n-1}"] \arrow[dl, "\ker \delta_{n+1}"'] \arrow[dd, "f_{n+1}"] & A_{n+2} \\
& \operatorname{Coker} \delta_{n-1} \arrow[rr, "\widetilde{\delta_n}"'] \arrow[dd, red, "\color{red}\varphi_{n-1}"'] & & \operatorname{Ker} \delta_{n+1} \arrow[dd, blue, "\color{blue}\varphi_{n+1}"'] & & \\
B_{n-1} \arrow[rr, "\lambda_{n-1}"] \arrow[dd, "g_{n-1}"] & & B_n \arrow[rr, "\lambda_n"] \arrow[dl, "\operatorname{coker} \lambda_{n-1}"] \arrow[dd, "g_n"] & & B_{n+1} \arrow[r, "\lambda_{n-1}"] \arrow[dl, "\ker \lambda_{n+1}"] \arrow[dd, "g_{n+1}"] & B_{n+2} \\
& \operatorname{Coker} \lambda_{n-1} \arrow[rr, "\widetilde{\lambda_n}"'] \arrow[dd, red, "\color{red}\psi_{n-1}"'] & & \operatorname{Ker} \lambda_{n+1} \arrow[dd, blue, "\color{blue}\psi_{n+1}"'] & & \\
C_{n-1} \arrow[rr, "\eta_{n-1}"] \arrow[dd] & & C_n \arrow[rr, "\eta_n"] \arrow[dl, "\operatorname{coker} \eta_{n-1}"] \arrow[dd] & & C_{n+1} \arrow[r, "\eta_{n-1}"] \arrow[dl, "\ker \eta_{n+1}"] \arrow[dd] & C_{n+2} \\
& \operatorname{Coker} \eta_{n-1} \arrow[rr, "\widetilde{\eta_n}"'] \arrow[d, red] & & \operatorname{Ker} \eta_{n+1} \arrow[d, blue] & & \\
0 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Proof. Immediate by Snake Lemma. $\square$

Theorem II.18. Let $\mathcal{A} = \langle (A_n, \delta_n)\, ; n \in \omega \rangle$ and $\mathcal{C} = \langle (C_n, \eta_n)\, ; n \in \omega \rangle$ be projective resolutions. Then there is a projective resolutions $\mathcal{B} = \langle (B_n, \lambda_n)\, ; n \in \omega \rangle$ such that $0 \to A_n \to B_n \to C_n \to 0$ splits exact for each $n \in \omega \setminus 1$.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A_{n+2} \arrow[r, "\delta_{n+2}"] \arrow[d, "f_{n+1}"] & A_{n+1} \arrow[r, "\delta_n"] \arrow[d, "f_{n+1}"] & A_n \arrow[d] \\
B_{n+2} \arrow[r, "\lambda_{n+2}"] \arrow[d, "g_{n+1}"] & B_{n+1} \arrow[r, "\lambda_n"] \arrow[d, "g_{n+1}"] & B_n \arrow[d] \\
C_{n+2} \arrow[r, "\eta_{n+1}"] \arrow[d] & C_{n+1} \arrow[r, "\eta_n"] \arrow[d] & C_n \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

Proof. Manifestly by letting $B_n = A_n \oplus C_n$ and f_n and g_n be canonical inclusion and projection. $\square$

Definition II.19. A category $\mathcal{A}$ is *enough projectives* provided that for each object A there is an epimorphism $f\colon P \to A$ where P is projective. Duality, $\mathcal{A}$ is *enough injectives* provided that for each object A there is a monic $g\colon A \to Q$ where Q is injective.

Theorem II.20.

1. If $\mathcal{A}$ is enough projectives, every object has projective resolution,
2. If $\mathcal{A}$ is enough injectives, every object has injective resolution.

Proof. We prove the first statement. (Note that the second statement can being handled in an obvious dual manner.) Let X be an object in $\mathcal{A}$. There is projective P_0 and epimorphism $p_0\colon P_0 \to X$. Then we have the following sequence exact at P_0 and X,

$$\operatorname{Ker} p_0 \xrightarrow{\ker p_0} P_0 \xrightarrow{p_0} X \longrightarrow 0.$$

Continuing this processes, we have the projective objects P_n, $n \in \omega$,

$$P_{n+1} \xrightarrow{\ker p_n \circ p_{n+1}} P_n \xrightarrow{\ker p_{n-1} \circ p_n} \cdots \xrightarrow{\ker p_0 \circ p_1} P_0 \xrightarrow{p_0} X \longrightarrow 0$$

and the uniqueness for epi-monic factorization asserts that the sequence exact at any were. □

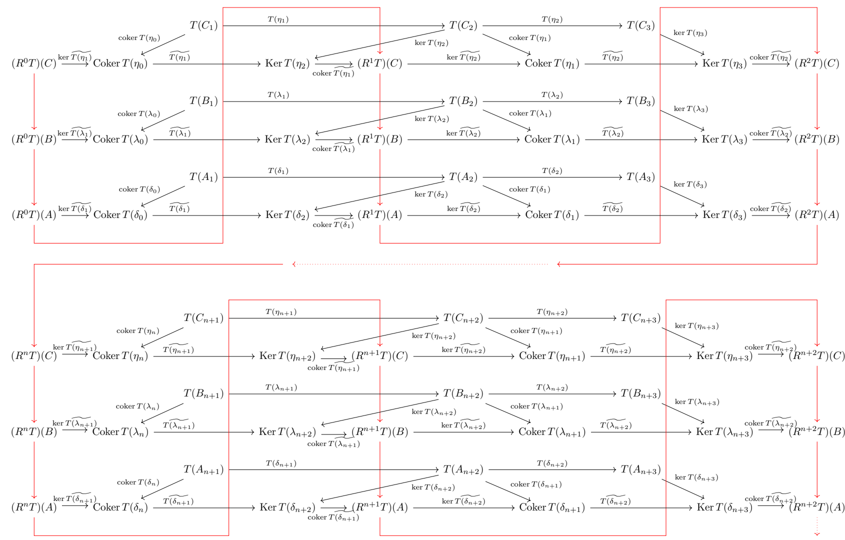
Definition II.21. Let $\mathcal{A}$ be enough projectives Abelian category. For an additive contravariant functor T, define a *right-derived functor*. For an object $A \in \mathcal{A}$, choose a projective resolution $\langle (A_n, \delta_n) \,;\, n \in \omega \rangle$ (we shall show that the definition is not depend on the choice of this resolution), we have the chain $\langle (T(A_n), T(\delta_n)) \,;\, n \in \omega \rangle$. The right-derived functor given by following data:

1. $(R^n T)(A) = H_n(T(A))$ or $(R^0 T)(A) = \operatorname{Ker} \delta_0$ for any object A,
2. $(R^n T)(f)\colon (R^n T)(A) \to (R^n T)(B)$ be the canonical morphism for any arrow $f\colon A \to B$.

Theorem II.22. For an object A and contravariant additive functor T, the derived functor does not depend on the choice of projective resolution.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be projective resolutions. The sequence $\mathcal{A} \oplus \mathcal{B} = \langle A_n \oplus B_n \,;\, n \in \omega \rangle$ is a projective resolution. It suffice to show that $(R^n T)(\mathcal{A}) = (R^n T)(\mathcal{A} \oplus \mathcal{B})$. There are morphism between two projective resolutions $\mathcal{A}$ and $f = \mathcal{A} \oplus \mathcal{B}$, $\langle i_1^n\colon A_n \to A_n \oplus B_n \,;\, n \in \omega \rangle$ and $\langle p_1^n\colon A_n \oplus B_n \to A_n \,;\, n \in \omega \rangle$ this morphism witness that two chains $T(\mathcal{A})$ and $T(\mathcal{A} \oplus \mathcal{B})$ are isomorphism. Thus by Theorem II.15 we have $(R^n T)(\mathcal{A}) = (R^n T)(\mathcal{A} \oplus \mathcal{B})$ for each $n \in \omega$. □

Theorem II.23. Let $\mathcal{A}$ be an enough projectives Abelian category and $T\colon \mathcal{A} \to \mathcal{B}$ be additive contravariant functor and let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. Then we have the following long long red exact sequence.



Lemma II.24. Let T be an additive contravariant functor and let R^nT be right-derived functor for T. If T is left-exact, $(R^0T) = T$. Moreover, for a projective object A, we have $(R^nT) = 0$ for any $n \in \omega \setminus 1$.

Proof. Immediate by that the sequence $0 \to T(A) \to T(P_0) \to T(P_1)$ exact for a projective resolution $\langle (P_0, \delta_0)\,; n \in \omega \rangle$. To see the moreover part, note that

$$\cdots \to 0 \to 0 \to A \to A \oplus A \to A \to 0$$

is a projective resolution for A, we obtain that $(R^nT) = 0$ for $n \in \omega \setminus 1$. $\square$

III Module

Definition III.1. Let R be a ring. M is a (*left*) R- *module* provided that M is an Abelian group and there is an action $\cdot\colon R \times M \to M$ such that:

- $\cdot(r +_R r', m) = \cdot(r, m) +_M \cdot(r', m)$, for any $r,' r \in R$ and $m \in M$,
- $\cdot(r, \cdot(r', m)) = \cdot(rr', m)$, for any $r,' r \in R$ and $m \in M$,
- $\cdot(r, m +_M m') = \cdot(r, m) +_M \cdot(r', m)$, for any $r \in R$ and $m, m' \in M$,
- If R processes identity id_R, $\cdot(1_R, m) = m$ for any $m \in M$

We simply write $r \cdot m$ for $\cdot(r, m)$, or more simply rm when there is no danger of confusing.

For an Abelian group A, we can regard A as the $\mathbb{Z}$-module with following natural action:

$$(n + 1)a = na + a$$

for any $n \in \mathbb{Z}$ and any $a \in A$.

Definition III.2. An R-module homomorphism φ form R-module $(M, +M)$ into R-module $(N, +_N)$ is a function such that

- $\varphi(x +_M y) = \varphi(x) +_N \varphi(y)$ for all $x, y \in M$ and
- $\varphi(rm) = r\varphi(m)$ for all $r \in R$ and $m \in M$.

Notice that, for fixed ring R, the collection of R-modules and the collection of R-module homomorphisms be an (enough projectives) Abelian category.

The finite dimensional vector space (linear space) over a field F is a (free-) F module and the F-module homomorphism is the linear transformation or the associated matrix.

Definition III.3. For a set A, the *free R-module* on the set A, $F(A)$ is an R-module with following universal property:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "\varphi"'] & F(A) \arrow[d, dotted, "\Phi"] \\ & M \end{tikzcd}$$

For set morphism $f\colon A \to B$ there is a unique R-module morphism $F(f)$ such that TFDC:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"] & F(A) \arrow[d, "F(f)"] \\ B \arrow[r, hook] & F(B) \end{tikzcd}$$

In this sense, we can define the free functor $F\colon \mathbb{S}et \to R\text{-}\operatorname{Mod}$ and note that F is an additive functor.

An R-module M is free, provided that there is a set A such that $M = F(A)$. In particular, we said that the free $\mathbb{Z}$-module as *free Abelian group*.

Theorem III.4 ([DF04])**.** Every free module is projective.

Definition III.5. For an ordinal α, a sequence $\langle A_\xi\,; \xi \in \alpha\rangle$ is a *smooth chain* provided that

1. $A_\xi \subset A_{\xi+1}$ for any $\xi \in \alpha$,
2. $A_\xi = \cup\{A_\eta\,; \eta \in \xi\}$ for any limit $\xi \in \alpha$

We said that a smooth chain is *smooth chain of modules* if A_ξ is module and A_ξ is a submodules of $A_{\xi+1}$ for any $\xi \in \alpha$.

Theorem III.6. For a smooth chain of R-modules $\langle M_\xi\,; \xi \in \alpha\rangle$. If M_0 and $M_{\eta+1}/M_\eta$ are free, for any $\eta \in \alpha$, the R-module $M = \cup\{M_\xi\,; \xi \in \alpha\}$ is free. Moreover, M/M_ξ is free for any $\xi \in \alpha$.

Proof. We shall show by induction on α. For the leading stage is immediate.

To see the successor stages. Let A and B be sets such that $F(A) = M_\xi$ and $F(B) = M_\xi / M_\xi$. Note that since the free functor $F \colon \mathbb{S}et \to \mathbb{Z}\text{-}\operatorname{Mod}$ is additive, we have $F(A \oplus B) = F(A) \oplus F(B)$. Since we may assume that A and B are disjoint, we obtain that $F(A \cup B) = F(A) \oplus F(B) = M_\xi \oplus M_{\xi+1}/M_\xi = M_{\xi+1}$. We complete the proof.

We prove the limit stages. Induction hypothesis asserts that there are sets X_ξ, $\xi \in \alpha$, such that $X_\xi \subset X_{\xi+1}$ and $F(X_\xi) = M_\xi$ for any $\xi \in \alpha$. To complete the proof, it is routine to see that $F(\cup\{X_\xi\,; \xi \in \alpha\}) = \cup\{M_\xi\,; \xi \in \alpha\}$ by check the universal property.

To see the moreover part, let X and Y be basis for M_ξ and M respectively such as $X \subset Y$ we note there is a unique φ by the universal property for cokernel and ψ by the universal property for free module:

$$
\begin{tikzcd}
 & & & M/M_\xi \arrow[r] \arrow[d, dotted, shift right, "\varphi"'] & 0 \\
0 \arrow[r] & M_\xi \arrow[r, "f"] & M \arrow[ur, "\operatorname{coker} f"] \arrow[r, shift left, "p"] & F(Y \setminus X) \arrow[l, shift left, "i"] \arrow[u, shift right, "\psi"'] \arrow[r] & 0 \\
 & X \arrow[u, hook] \arrow[r, hook] & Y \arrow[u, hook] & Y \setminus X \arrow[u, hook] \arrow[l, hook']
\end{tikzcd}
$$

Here, p and i is the canonical inclusion and projection since $M = M_\xi \oplus F(Y \setminus X)$.

The equality $\varphi \circ \psi = \mathrm{id}$ is immediate by the fact that φ is epi and $\varphi = \varphi \circ \psi \circ \psi$. The equality $\psi \circ \varphi = \mathrm{id}$ is immediate by following steps:

$$
\begin{gathered}
\varphi = \varphi \circ \operatorname{coker} f \circ i \circ \varphi \\
\mathrm{id} = \operatorname{coker} f \circ i \circ \varphi \\
\psi \circ \varphi = \operatorname{coker} f \circ i \circ \varphi \circ \psi \circ \varphi = \mathrm{id}
\end{gathered}
$$

$\square$

Definition III.7. An R-module is *projective* provided that it is projective in the sense in an Abelian category $R\text{-}\operatorname{Mod}$.

Theorem III.8. $R\text{-}\operatorname{Mod}$ is enough projectives.

Lemma III.9. For any $R\text{-}\operatorname{Mod}$ M, there is an projective resolution.

Definition III.10. Let A, B, C be R-modules. An exact sequence $0 \to A \to B \xrightarrow{f} C \to 0$ *splits* provided that there is a R-module homomorphism $g \colon C \to B$ such that $g \circ f = \mathrm{id}_C$.

Theorem III.11 ([DF04])**.** Let R be a ring with identity and P an R- module. TFAE.

1. $\operatorname{Hom}_R(-, P)$ is exact,
2. $\operatorname{Ext}^1_{\mathbb{Z}}(-, P) := (R^1 T) = 0$, here T is an additive right-exact contravariant hom-functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, P)$ from $\mathbb{Z}\text{-}\operatorname{Mod}$,

3. If P is a quotient of the R-module M then P is isomorphic to a direct summand of M, i.e., every short exact sequence $0 \to L \to M \to P \to 0$ splits,

4. P is a direct summand of a free R-module.

Definition III.12. For an integral domain R and R-module M, we define a *torsion submodule of* M and an *annihilator* of module M.

1. $\mathrm{Tor}(M) = \{x \in M\,; \exists r \in R \setminus \{0\}\ rx = 0\}$,

2. $\mathrm{Ann}(M) = \{r \in R\,; \forall m \in M\ rm = 0\}$.

We said that M is *torsion free* provided that $\mathrm{Tor}(M) = 0$.

Theorem III.13 (Fundamental Theorem (invariant factor form), [DF04])**.** Let R be an P.I.D., and M a finitely generated R-module. Then we have:

1. There is a finite sequence of R, $\langle a_i\,; i \in n\rangle$, and $r \in \omega$ such as
   - $(a_0) \subset (a_1) \subset \cdots \subset (a_{n-1})$,
   - $M \cong R^r \oplus R/(a_0) \oplus R/(a_1) \oplus \ldots R/(a_{n-1})$.

2. M is torsion free if and only if M is free,

3. In the decomposition in (1),

$$\mathrm{Tor}(M) \cong R/(a_0) \oplus R/(a_1) \oplus \ldots R/(a_{n-1})$$

   Moreover, M is torsion module if and only if $r = 0$ and in this case the annihilator of M is the ideal (a_0).

Definition III.14. Let A be a torsion free group and let B be a normal subgroup of A. B is a *pure subgroup of* A provided the quotient A/B is torsion free. For a subset B the *pure closure for* B is the subset such that $B' = \{a \in A\,; \exists n \in \mathbb{Z} \setminus \{0\}\ n \cdot a \in B\}$.

Note that the pure closure B' is a pure subgroup for torsion free group A.

Theorem III.15 (Pontryagin's Criterion)**.** Let A be a countable torsion free group such as every finitely generated subgroup of A is contained in some pure finitely generated subgroup. A is free.

Proof. Enumerate $A = \{a_n \in n \in \omega\}$. Thanks to the assumption we can easy to define an increasing sequence, among $\subset$, of finitely generated pure subgroup of A $\{B_n \in n \in \omega\}$ such that

- $a_n \in B_n$, and
- B_{n+1}/B_n is free. (See Theorem III.13)

The proof of the theorem is completed by an application of Theorem III.6. □

IV Whitehead Problem

In this section, since we treat a $\mathbb{Z}$-module, we simply say Abelian group.

Definition IV.1. An Abelian group A is a *Whitehead group* provided that $\mathrm{Ext}^1_{\mathbb{Z}}(A) = 0$

Definition IV.2. Let A be an Abelian group. C is an $(A, \mathbb{Z})$-*group* provided that it is an Abelian group whose underlying set is $B \times \mathbb{Z}$ (i.e., for the forgetful functor $U \colon \mathbb{Z}\text{-}\mathrm{Mod} \to \mathbb{S}et$, $U(C) = A \times \mathbb{Z}$), $(0, n) +_C (0, m) = (0, n+m)$ for each n, m, and a map $\pi \colon C \to A$ given by $(a, n) \mapsto a$ is a group morphism.

Note that since the kernel for π is $\mathbb{Z}$, if π is left cancellable by splitting lemma we have $C = \mathbb{Z} \oplus A$.

Theorem IV.3. Let B_1 be a Whitehead group and B_0 be a subgroup of B_1 such that the quotient B_0/B_1 is not Whitehead. Then there is $\varphi \in \mathrm{Hom}(B_0, \mathbb{Z})$ such that there are no $\psi \in \mathrm{Hom}(B_1, \mathbb{Z})$ such as TFDC:

$$\begin{tikzcd}
B_0 \arrow[d, hook, "\iota"'] \arrow[dr, "\varphi"] & \\
B_1 \arrow[r, "\psi"'] & \mathbb{Z}
\end{tikzcd}$$

Proof. For a short exact sequence

$$0 \to B_0 \xrightarrow{\iota} B_1 \xrightarrow{\mathrm{coker}\,\iota} B_1/B_0 \to 0,$$

we have an exact sequence

$$\mathrm{Hom}(B_1, \mathbb{Z}) \xrightarrow{\iota^*} \mathrm{Hom}(B_0, \mathbb{Z}) \xrightarrow{\varphi} \mathrm{Ext}(B_1/B_0, \mathbb{Z}) \to \mathrm{Ext}(B_1, \mathbb{Z}) = 0.$$

To conclude the proof, we remain to see that the ι^* is not epi. Suppose not, we have $\mathrm{Hom}(B_0, \mathbb{Z}) = \mathrm{Im}\,\iota^* = \mathrm{Ker}\,\varphi$. This asserts that $\varphi = 0$ and moreover, since φ is monic, we obtain that $\mathrm{Ext}(B_1/B_0, \mathbb{Z}) = 0$, a contradiction. $\square$

Lemma IV.4.

1. Every free Abelian group is a Whitehead group,
2. Every subgroup of a Whitehead group is also a Whitehead group,
3. Every Whitehead group is torsion free,
4. For a Whitehead group A and a subgroup B such that A/B is not a Whitehead group. Let C_0 be a $(B_0, \mathbb{Z})$-group with the left cancellable canonical morphism $\pi_0 \colon C_0 \to B_0$. There is a $(B_1, \mathbb{Z})$-group C_1 such which is an extension for C_0 and there are no left inverse for π_1 which is an extension for a left inverse for π_0.

5. Every countable Whitehead group is free,

Proof. The statement (1) is manifestly.

To prove (2). Let A be an Abelian group and B a subgroup of A. Let $i\colon B \to A$ be the natural inclusion. Let $f\colon C \to B$ be a module homomorphism such that

$$0 \to \mathbb{Z} \to C \xrightarrow{f} B \xrightarrow{\operatorname{coker} f} 0 \text{ is exact.}$$

We shall find a right-inverse $g\colon B \to C$ for f. Define a function $\varphi\colon A \to B$ which assigns x to x if $x \in B$ and 0 if $A \setminus B$. Then we obtain an exact sequence such that

$$0 \to \mathbb{Z} \to C \xrightarrow{i \circ f} A \xrightarrow{\operatorname{coker} f \circ \varphi} 0.$$

Since A is Whitehead, there is a module morphism $g'\colon A \to C$ such that $i \circ f \circ g' = \mathrm{id}_A$. Moreover, since $\varphi \circ i = \mathrm{id}_B$, we have that $f \circ (g \circ i) = \mathrm{id}_B$.

To prove (3). Let A be a Whitehead group and assume that there is $a \in \mathrm{Tor}(A) \setminus \{0\}$ and let $n \in \mathbb{Z}$ with $n \cdot a = 0$. Let $\pi\colon A \to A/\langle a\rangle$ be the canonical projection. Then the kernel is $\ker \pi\colon A \to A$ with canonical assignments. Then since previous fact asserts that the finite subgroup $\langle a\rangle$ is a Whitehead group, we obtain that $A = A \oplus A/\langle a\rangle$, a contradiction.

To prove (4). Note that $C_0 = \mathbb{Z} \oplus B_0$. Fix a morphism $\psi \in \mathrm{Hom}(B_0, \mathbb{Z})$ as in Theorem IV.3 and define $\gamma \in \mathrm{Hom}(C_0, C_1)$ which assigns (b, n) the $(b, n + \varphi(b))$. Assume that there is a left inverse for π_1 and assume that $\gamma \circ \rho_0 = \rho_1 \circ \iota_B$, i.e., the right square in the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, shift left] \arrow[d, hook, "\mathrm{id}_{\mathbb{Z}}"] & C_0 \arrow[l, shift left, "p_0"] \arrow[r, shift left, "\pi_0"] \arrow[d, "\gamma"] & B_0 \arrow[l, shift left, "\rho_0"] \arrow[d, hook, "\iota_B"] \arrow[ll, bend right, "\varphi"'] \\
\mathbb{Z} \arrow[r, shift left] & C_1 \arrow[l, shift left, "p_1"] \arrow[r, shift left, "\pi_1"] & B_1 \arrow[l, shift left, "\rho_1"] \arrow[ll, bend left, "\psi = p_1 \circ \rho_1"]
\end{tikzcd}$$

Now, for arbitrary $b \in B_0$ we obtain that

$$\begin{aligned}
\mathrm{id}_{\mathbb{Z}} \circ \varphi(b) &= (p_1 \circ \gamma)(b, 0) \\
&= (p_1 \circ \gamma \circ \rho_0)(b) \\
&= \psi \circ \iota_B(b)
\end{aligned}$$

This contradicts that the choice for φ.

To prove (5). Thanks to the results (3), if for every finitely generated subgroup there is a finitely generated pure subgroup which contains it, Pontryagin's Criterion, we obtain the result. We remain to see the case of that there is a finitely generated subgroup B_0 such that there are no finitely generated pure subgroup which contains B_0. Fix a finite set S which generates B_0 and pure closure B of B_0. First we note that for any morphism $\rho\colon B \to C$ (C is any Abelian group), since

- $\rho|_{B_0}$ can be recovered from ρ_S, and
- ρ can be recovered from ρ_B, this is because that B is torsion free,

ρ can be recovered from ρ_S. For a canonical projection $\pi_S \colon S \times \mathbb{Z} \to S$, enumerate all the right-inverse of π, $\{g_n \colon S \to S \times \mathbb{Z} \,; n \in \omega\}$. Now we construct a $(B_n, \mathbb{Z})$-group C_n and a right-inverse $\rho_n \colon B_n \to C_n$ for a canonical morphism $\pi_n \colon C_n \to B_n$, as following steps, by using the result (4):

1. Leading stage:

   1–a $C_0 = B_0 \oplus \mathbb{Z}$.

   1–b If there is a right-inverse ρ_0 for π_0 such that $g_0 \subset \rho_0$, choose such a morphism and if otherwise choose arbitrary right-inverse.

2. Successor stages:

   2–a Choose a $(B_{n+1}, \mathbb{Z})$-group C_{n+1} which is an extension for C_n, such that there are no right-inverse of $\pi_{n+1} \colon B_{n+1} \to C_{n+1}$ which is an extension for ρ_n.

   2–b If there is a right-inverse ρ_{n+1} for π_{n+1} such that $g_{n+1} \subset \rho_{n+1}$, choose such a morphism and if otherwise choose arbitrary right-inverse.

To complete the proof, we shall show that that for $(B, \mathbb{Z})$-group C the canonical morphism $\pi \colon C \to B$ does not possess a right-inverse, if we done we have a contradiction that B is a Whitehead group. Suppose that there is a right-inverse $\rho \colon B \to C$. Then there is an $n \in \omega$ such that $g_n \subset \rho|_S$, and moreover since $\rho|_{B_n}$ is a right-inverse of π_n, we obtain that $\rho_n = \rho|_{B_n}$. However $\rho|_{B_{n+1}}$ is a right-inverse of π_{n+1} and is an extension of a right-inverse $\rho|_{B_n}$ of π_n, a contradiction.

$\square$

Definition IV.5. Let A be an Abelian group. A is $\aleph_1$*-free* provided that every countable subgroup is free. A subgroup B is $\aleph_1$*-pure* provided that the quotient A/B is $\aleph_1$-free. An $\aleph_1$-free group A has a *Chase's condition* provided that

(Chase) every countable subgroup of A is contained in a countable $\aleph_1$-pure subgroup of A.

Note that the previous lemma shows that every Whitehead group is $\aleph_1$-free.

Lemma IV.6. A group of size $\aleph_1$ has the chase's condition if and only if A is a union of a smooth chain $\langle A_\alpha \,; \alpha \in \omega_1 \rangle$ such that $A_0 = 0$ and A_α is free and $A_{\alpha+1}$ is an $\aleph_1$-pure subgroup of A.

Proof. First, we show the sufficiently condition. Assume that A satisfies the chase's condition. Enumerate $\{a_\alpha \,; \alpha \in \omega_1\}$ and define a A_α, $\alpha \in \omega_1$, recursively:

1. Leading stage.

   $A_0 = 0$,

2. Successor stages.

   $A_{\alpha+1}$ be an $\aleph_1$-pure subgroup which containing A_α and a_α,

3. limit stages.

   $A_\gamma = \cup\{A_\alpha\,;\alpha \in \gamma\}$.

Note that the existence of an $\aleph_1$-pure subgroup is by the chase's condition.

Second, we show the enough condition but this is immediate that since for any countable subset B of A there is $\alpha \in \omega_1$ such that $B \subset \{a_\beta\,;\beta \in \alpha\}$. $\square$

From now on, we will demonstrate that the statement "every Whitehead group of size $\aleph_1$ is free" is independent from ZF. First, we will establish the existence of a model such that every Whitehead group of size $\aleph_1$ is free. Specifically, the truth of the statement that the Whitehead group of size $\aleph_1$ is free holds in a model that satisfies the $\diamondsuit_E$ axiom.

Lemma IV.7. Let $\langle A_\alpha\,;\alpha \in \omega_1\rangle$ be a smooth chain for countable Abelian groups. Define $A = \cup\{A_\alpha\,;\alpha \in \omega_1\}$ and $E = \{\alpha \in \omega_1\,;A_{\alpha+1}/A_\alpha \text{ is not free}\}$. Then if E is stationary and $\diamondsuit_E$ holds, A is not a Whitehead group.

Proof. Fix a $\diamondsuit_E$-sequence $\langle D_\alpha\,;\alpha \in E\rangle$ and bijection $\nu\colon \omega_1 \to \mathbb{Z} \times A$. Define a $(\mathbb{Z}, A_\alpha)$-group C_α and the canonical epimorphism $f_\alpha\colon C_\alpha \to A_\alpha$ with $\mathbb{Z} = \operatorname{Ker} f_\alpha$, $\alpha \le \omega_1$, recursively:

1. Leading stage, 0,

   (1–a) $C_0 = \mathbb{Z} \times A_0$ and $f_\alpha = p_0$: projection.

2. Successor stages, $\alpha + 1$,

   (2–a) If $\alpha \in E$ and $g_\alpha = \nu[D_\alpha]$ is a function with $\operatorname{dom}(g_\alpha) = A_\alpha$ such that $f_\alpha \circ g_\alpha = \mathrm{id}_{A_\alpha}$.
   $f_{\alpha+1}\colon C_{\alpha+1} \to A_{\alpha+1}$ is an epimorphism such that $\operatorname{Ker}(f_{\alpha+1}) = \mathbb{Z}$, $(\mathbb{Z}, A_\alpha)$- group C_α and there are no $g\colon A_{\alpha+1} \to C_{\alpha+1}$ such as $g|_{A_\alpha} = g_\alpha$ and $f_{\alpha+1} \circ g = \mathrm{id}_{A_{\alpha+1}}$.

   (2–b) Otherwise.
   Choose an epimorphism $f_{\alpha+1}\colon C_{\alpha+1} \to A_{\alpha+1}$ such with TFDC and exact:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & C_\alpha \arrow[r, "f_\alpha"] \arrow[d] & A_\alpha \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & C_{\alpha+1} \arrow[r, "f_{\alpha+1}"] & A_{\alpha+1} \arrow[r] & 0 \end{tikzcd}$$

3. Limit stages, α,

   (3–a) $f_\alpha = \cup\{f_\beta\,;\beta \in \alpha\}$, $C_\alpha = \cup\{C_\beta\,;\beta \in \alpha\}$.

For (2–a) we note that since $\alpha \in E$ shows that $\mathrm{Ext}^1_{\mathbb{Z}}(A_{\alpha+1}/A_\alpha, \mathbb{Z}) \neq 0$, so we can apply 4 in Lemma IV.4.

Now we show that $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \neq 0$. If there is an epimorphism $g\colon A \to C_{\omega_1}$ such that $f_{\omega_1} \circ g = \mathrm{id}_A$. Since $\diamondsuit_E$ asserts that there is an $\alpha \in E$ such that $g_\alpha = g|_{A_\alpha}$. Then we obtain that $f_\alpha \circ g_\alpha = \mathrm{id}_{A_\alpha}$, $(g|_{A_{\alpha+1}})|_{A_\alpha} = g|_{A_\alpha}$ and $f_{\alpha+1} \circ g|_{A_{\alpha+1}} = \mathrm{id}_{A_{\alpha+1}}$. This contrary to that the construction in (2–a) step. □

Theorem IV.8 (Shelah)**.** Assume that $\diamondsuit_E$ for any stationary subsets $E \subset \omega_1$. Every Whitehead group of size $\aleph_1$ is free.

Proof. Let A be a Whitehead group of size $\aleph_1$. At the beginning of the proof, we shall show that A satisfies the chase condition. Suppose not, since A is an $\aleph_1$-free, there is a countable $A_0 \leq A$ such that for any B' if $A_0 \leq B$ there is $A_1 \leq A$ such that A_1/A_0 is not free. By continuing this processes we can define a $\subset$-increasing sequence $\langle A_\alpha \,; \alpha \in \omega_1 \rangle$ such that $A_{\alpha+1}/A_\alpha$ is not free and $A = \cup\{A_\alpha \,; \alpha \in \omega_1\}$. Since $\diamondsuit_{\omega_1}$ shows that A is not a Whitehead group, a contradiction.

Fix a smooth chain sequence of countable subgroups $\langle B_\alpha \subset A \,; \alpha \in \omega_1 \rangle$ such that $B_{\alpha+1}$ is α_1-pure subgroup of B_α for any $\alpha \in \omega_1$ and $A = \cup\{B_\alpha \,; \alpha \in \omega_1\}$. Define $E = \{\alpha \in \omega_1 \,; B_{\alpha+1}/B_\alpha \text{ is not free}\}$. Note that the previous lemma shows E is not stationary set. We shall show the equality:

$$E = \{\alpha \in \omega_1 \,; B_\alpha \text{ is not an } \aleph_1\text{-pure subgroup of } A\}$$

To prove $(\subset)$. If B_α is an α_1-pure subgroup of A, A/B_α is α_1-free. So $B_{\alpha+1}/B_\alpha$ is free. To prove $(\supset)$. If $B_{\alpha+1}/B_\alpha$ is free. For any $\beta \ni \alpha$, since we have

$$B_\beta/B_{\alpha+1} \approx (B_\beta/B_\alpha)/(B_{\alpha+1}/B_\alpha)$$

and $A/B_{\alpha+1}$ is $\aleph_1$-free asserts that $B_\beta/B_{\alpha+1}$ is free, so does B_β/B_α is free. Furthermore, B_α is not α_1-pure subgroup of A. Thus, we obtain the equality.

Since, E is not a stationary set, there is a club set $C \subset \omega_1 \backslash E$. We asserts that for $\alpha, \beta \in C$ with $\alpha \in \beta$, B_β/B_α is free. We distinguish two cases, according to whether $\beta = \alpha + 1$. If $\beta = \alpha + 1$. The statement that B_β/B_α is free is manifestly. If $\beta \ni \alpha + 1$. As we have seen in before, B_β/B_α is free. Therefore, $A = \cup\{B_\alpha \,; \alpha \in \omega_1\}$ is free. □

Second, we shall produce the model that there is a model such that there is a Whitehead group of size $\aleph_1$ but free. To establish a Whitehead group of size $\aleph_1$ but free, we shall prove the two statement as following:

IV.9 There is an Abelian group of size $\aleph_1$ with Chase's condition but free,

IV.10 Every Abelian group with Chase's condition is Whitehead.

Theorem IV.9. There is an Abelian group of size $\aleph_1$ with Chase's condition but free,

Proof. We shall construct the desired A as the union for a chain $\langle A_\alpha \,; \alpha \in \omega_1 \rangle$ with following properties:

1. $A_\alpha \subsetneq A_{\alpha+1}$, for each $\alpha \in \omega_1$,
2. A_α is free, for each $\alpha \in \omega_1$,
3. $A_\beta / A_{\alpha+1}$ is free, for each $\alpha \in \beta \in \omega_1$, and
4. $A_{\gamma+1}/A_\gamma$ is non-free, for each limit $\gamma \in \omega_1$.

Before to verify that the sooth chain exists, we shall show that the union A has a chase's condition but free. To see the chase's condition, for $\xi \in \omega_1$ and countable subset B of $A/A_{\xi+1}$ there is an A_ν such that $B \leq A_\mu / A_\xi$ thus, $A_{\xi+1}$ is free. Therefore, Lemma IV.6 $A = \{A_{\xi+1}\,; \xi \in \omega_1\}$ has a chase's condition. To see that A is not free, since set $\omega_1 \cap \mathrm{Lim}$ is not stationary so by Lemma IV.11 A is not free.

From now on, we shall establish the smooth chain. Fix $A_0 = 0$

Successor stages, $\beta = \alpha + 1$. We distinguish two cases, according to whether α is a limit ordinal. (Case A.) If α is not a limit ordinal. Let $A_\beta = A_\alpha \oplus \mathbb{Z}$. Then it is clearly the chain $\langle A_\xi\,; \xi \in \beta+1\rangle$ satisfies the all of these conditions.

(Case B.) If α is a limit ordinal. Fix a strictly increasing sequence $\langle \xi_n\,; n \in \omega\rangle \nearrow \alpha$ and let X_n be a base for A_{ξ_n} such as $X_n \subset X_{n+1}$. Choose $x_n \in X_{n+1} \setminus X_n$ for every $n \in \omega$ and let $Y_k = X_k \setminus \{x_m\,; m \in k\}$ and $B = F(\cup\{Y_k\,; k \in \omega\}$ be a subgroup of A_α and let $P = \Pi\{\langle x_n\rangle n \in \omega\}$ be a product group.

For set morphism $\varphi\colon \{x_n\,; n \in \omega\} \to P$,

$$x_n \mapsto \left(m \mapsto \begin{cases} \frac{(m+1)!}{(n+1)!} x_m & \text{if } m+1 \ni n \\ 0 & \text{if } m \in n \end{cases} \right)$$

we have the following morphism:

$$\begin{tikzcd} \{x_n\,; n \in \omega\} \arrow[r, hook] \arrow[dr, "\varphi"'] & A_\alpha \setminus B \arrow[d, dotted, "\psi"] \\ & P \end{tikzcd}$$

Define $P' := \psi(A_\alpha \setminus B) = F(\{\psi(x_n)\,; n \in \omega\})$ and note that the set $\{\psi(x_n)\,; n \in \omega\}$ is linearly independent in P and define $A_{\alpha+1} = B \oplus P'$.

To check the four conditions, 2 is clear and 1 is immediate by the fact, we can identify x_n with $(n+1)! \cdot \varphi(x_{n+1}) - n! \cdot \varphi(x_n)$. To see the condition 3, let $\gamma \in \alpha$. Fix an $n \in \omega$ such as $\gamma + 1 \in \xi_n$. Then since we have

$$A/A_{\xi_n} \cong F(\cup\{Y_k\,; k+1 \ni n\}) \oplus F(\{\varphi(x_k\,; k+1 \ni n\})$$

and induction hypothesis assets that $A_{\xi_n}/A_{\gamma+1}$ is free, so is $A/A_{\gamma+1}$.

To see the condition 4. First we assert that for any $n \in \omega$ we have $(n+1)! \cdot \varphi(x_n) - \varphi(x_0)$ is a function which assigns only finitely many non-zero value, i.e., it can embeds into the direct sum, free module, A_α, i.e., $(n+1)! \cdot \varphi(x_n) - \varphi(x_0) = 0 + A_\alpha$ in $A_{\alpha+1}/A_\alpha$ for every $n \in \omega$. Note that $\varphi(x_0) \notin A_\alpha$ and we shall there are no finitely many pair of $\varphi(x_m) + A_\alpha$'s (without $\varphi(x_0) + A_\alpha$) which can presents $\varphi(x_0) + A_\alpha$.

Suppose there are finitely many pair $z_i = \varphi(x_-)$, $i \in n$ and $\eta_n \in \mathbb{Z}$, such that:

$$(\eta_n \cdot z_0 + A_\alpha) + \cdots + (\eta_{n-1} \cdot z_{n-1} + A_\alpha) = \varphi(x_0) + A_\alpha$$

by multiplying $(\eta_0)! \cdot (\eta_1)! \cdots (\eta_{n-1})!$, we have

$$m \cdot \varphi(x_0) + A_\alpha = \varphi(x_0) + A_\alpha$$

for some integer m, a contradiction.

Limit stages, γ. Let $A_\gamma = \{A_\alpha \,; \alpha \in \gamma\}$. Then the conditions (1) and (4) are immediate. Fix a sequence $\langle \xi_n \,; n \in \omega \rangle \nearrow \gamma$. Then we obtain $A_\gamma = \cup\{A_{\xi_n+1} \,; n \in \omega\}$ and $A_{\xi(n+1)+1}/A_{\xi(n)+1}$ is free for any $n \in \omega$, then by Theorem III.6, so is A_γ ((2) hold!) and note that $A_\gamma/A_{\xi(n)+1}$ is free for each $n \in \omega$. To see the remaining condition (3). It suffice to see that for any $\alpha \in \gamma$, $A_\gamma/A_{\alpha+1}$ is free. Fix an $n \in \omega$ such that $\alpha \in \xi(n)+1$, since we have

$$A_\gamma/A_{\xi(n)+1} \cong (A_\gamma/A_{\alpha+1})/(A_{\xi(n)+1}/A_{\alpha+1})$$

we obtain that $A_\gamma/A_{\alpha+1}$ is free. $\square$

After completing the mathematical proof, it is essential to recognize that the crux of this proof is the case B of the successor step. In this step, to collapse the property free for the quotient A_β/A_α, we use the fact that the direct sum can embeds to the direct product. Since we can identify x_n with $(n+1)! \cdot \varphi(x_{n+1}) - n! \cdot \varphi(x_n)$, the direct sum, free module, $F(\{x_n \,; n \in \omega\})$ can embeds into P but every $\varphi(x_n)$ can be.

Theorem IV.10. Assume MA $+\neg$ CH. Every Abelian group with Chase's condition is Whitehead.

Proof. Let A be an Abelian group with Chase's condition. For any B and epimorphism $f \in \mathrm{Hom}(B, A)$ with $\mathrm{Ker}\, f = \mathbb{Z}$. We shall find a right-inverse g for f.

Define a set

$$P = \{\varphi \in \mathrm{Hom}(S, B) \,; \exists S\colon \text{finitely generated pure subgroup of } A \ f \circ \varphi = \mathrm{id}_S\}$$

If P satisfies the conditions in Theorem I.7, there is $g \in {}^{A}B$ such that for any finite set $F \subset A$ there is a map $\varphi \in P$ such that $F \subset \mathrm{dom}\, \varphi$ and $g|_F = \varphi|_F$, this shows that

- $(f \circ g)(a) = a$ for all $a \in A$, and
- for any $a \in A$ and $n, m \in \mathbb{Z}$ there is $\varphi \in P$ which witnesses

$$g((n+m) \cdot (a+a')) = \varphi((n+m) \cdot (a+a')) = (n+m) \cdot (g(a) + g(a'))$$

Thus, g is a module morphism and clearly that g is a right-inverse for f.

To prove the former condition. Let $\varphi \in P$ with $\mathrm{dom}(P) = S$ and $F \in [A]^{<\omega}$. Since A is $\aleph_1$-free and by the third isomorphism theorem, there is a finitely generated pure subgroup $S' = \langle S \cup F \rangle$ of A such that $S \cup F \subset S'$. Moreover since S'/S is finitely generated torsion free, so is free. Let X be a set such that and $F(X) = S/S'$. Fix a right inverse set morphism $f^{-1}\colon A \to B$ for f and note that every free group is Whitehead, we obtain a canonical morphism $\psi\colon S' \to B$ as following diagram:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, shift left, "i=f\circ\varphi"] \arrow[dr, "\varphi"'] & S' \arrow[l, shift left, "\pi'"] \arrow[r, shift left, "\operatorname{coker} i"] \arrow[d, dotted, "\psi"] & S'/S \arrow[l, shift left, "\pi"] \arrow[dl, "\varphi'"] \arrow[r] & 0 \\
& & B & X \arrow[l, "f^{-1}|_{S'}\circ\pi\circ\iota"] \arrow[u, hook, "\iota"'] &
\end{tikzcd}$$

We remain to check that $f \circ \psi = \mathrm{id}_{S'}$.

Note that the uniqueness property for free group, we have $\pi = f \circ \psi$:

$$\begin{tikzcd}
& & S'/S \arrow[dll, bend right, "\pi"'] \arrow[dl, "\varphi'"'] \\
S' & B \arrow[l, "f|_{S'}"] & X \arrow[l, "f^{-1}|_{S'}\circ\pi\circ\iota"] \arrow[u, hook, "\iota"']
\end{tikzcd}$$

Therefore, we have

$$\begin{aligned}
f \circ \varphi &= f(\varphi \circ \pi' + \varphi' \circ \operatorname{coker} i) \\
&= i \circ \pi' + \pi \circ \operatorname{coker} i \\
&= \mathrm{id}_{S'}
\end{aligned}$$

To prove the latter condition, we shall verify it through the following two steps:

Step–I. For any uncountable $P' \subset P$, if there is a pure subgroup A' which is free and $\operatorname{dom}\varphi \subset A$ for any $\varphi \in P'$ then

$$\exists\{\varphi_0, \varphi_1\} \in [P']^2 \exists \psi \in P' \ (\varphi_0 \cup \psi_1 \subset \psi).$$

Step–II. For any uncountable $P' \subset P$ there is uncountable $P'' \subset P'$ and a free pure subgroup A' such that

$$\operatorname{dom}(\varphi) \subset A' \text{ for any } \varphi \in P''.$$

Step–I. Let $P' \in [P]^{\omega_1}$ and assume that there is a pure subgroup A' of A such that A' is free and $\operatorname{dom}(f) \subset A'$ for any $f \in P'$. Enumerate $X = \{x_\alpha\,; \alpha \in \omega_1\}$ be a base of A'. Since for every $\varphi \in P'$ there is $m \in \omega$ such that there is a function $\xi \in {}^m\omega_1$ such that $\operatorname{dom}(\varphi) = F(x_{\xi(n)}\,; n \in m\})$, Pigeonhole Principle asserts that there is $P'' \in [P']^{\omega_1}$ such that

$$\forall \in P'' \exists \xi \in {}^m\omega_1 \ \operatorname{dom}(\varphi) = F(\{x_{\xi(n)}\,; n \in m\})$$

Enumerate $P'' = \{\varphi_\alpha\,; \alpha \in \omega_1\}$ and let Y_α be a base of $\operatorname{dom}(\varphi)$ for each $\alpha \in \omega_1$.

Fix a maximal subset $T \subset X$, among the subset relation, respect to the property

$$\exists \mathcal{A} \in [\omega_1]^{\omega_1} \forall \alpha \in \mathcal{A} \ (T \subset Y_\alpha)$$

Note that there are at most countable many morphisms from $F(T)$ to $\mathrm{Ker}\, f = \mathbb{Z}$, (recall that the morphism $\varphi\colon F(T) \to \mathbb{Z}$ can be recovered from $\varphi \circ \iota$), so there are at most countable many morphisms from $F(T)$ to A. Thus, there is $\mathcal{A} \in [\omega_1]^{\omega_1}$ such that

- $T \subset Y_\alpha$ for all $\alpha \in \mathcal{A}$, and
- $\varphi_\alpha|_{F(T)} = \varphi_\beta|_{F(T)}$ for all $\alpha, \beta \in \mathcal{A}$.

Fix $\alpha \in \mathcal{A}$ and let $y \in Y_\alpha \setminus T$ (assume not empty). The maximality assert that there are only countably many $\beta \in \mathcal{A}$ such that $y \in Y_\beta$. Hence for $\{y_0, \cdots, y_{n-1}\} = Y_\alpha \setminus T$, there are $\mathcal{A} \supset \mathcal{A}_0 \supset \cdots \supset \mathcal{A}_{n-1}$ such as each of them are uncountable and $y_m \notin A_m$ for $m \in n$.

Therefore, there is $\beta \in \mathcal{A}$ such that $\alpha \neq \beta$ and $Y_\alpha \cap Y_\beta = T$ and $\psi\colon F(Y_\alpha \cup Y_\beta) \to B$ be the canonical extension. Furthermore, since we have

$$(A/F(Y_\alpha \cup Y_\beta))/(A'/F(Y_\alpha \cup Y_\beta)) = A/A'$$

and $A'/F(Y_\alpha \cup Y_\beta)$ and A/A' are torsion free, $A/F(Y_\alpha \cup Y_\beta)$ is torsion free, i.e., $F(Y_\alpha \cup Y_\beta)$ is a pure subgroup of A. Thus, $\psi \in P$ is an extension for φ_α and φ_β.

Step–II. As we have seen in step–I, let P'' be uncountable subset such that $|S| = |S'|$ for any $S, S' \in P''$, and moreover by the Delta system lemma, there is uncountable $P''' \subset P''$ and a root T such that

$$S \cap S' = T \text{ for any distinct } S, S' \in P'''$$

and we may assume that T be a $\subset$-maximal among a root and T be a pure subgroup of A. (if necessary take the pure closure.)

Enumerate $P''' = \{S_\alpha\,;\alpha \in \omega_1\}$ and define and Y_α be a base for $S_\alpha \setminus T$ for any $\alpha \in \omega_1$. To conclude the proof, we shall construct the desired A' as the union for a smooth chain $\langle A_\alpha \in \omega_1\rangle$ such as

- A_α is a countable pure subgroup of A,
- $A_{\alpha+1}/A_\alpha$ is free.

and it is easy to see that A' is free (see Theorem III.6) and A' is a pure subgroup of A.

Let $A_0 = T$ and for limit stages $A_\gamma = \cup\{A_\alpha\,;\alpha \in \gamma\}$. We shall see the successor stages, $\beta = \alpha + 1$. Thanks to the chase's condition fix a countable $\aleph_1$-pure subgroup C_α of A such that $A_\alpha \subset C_\alpha$. We assert that there is $\xi_\alpha \in \omega_1$ such that $\langle Y_{\xi_\alpha}\rangle \cap C_\alpha = 0$. If otherwise, there is a single element $t \in C_\alpha$ which meets uncountably many $\langle Y_\xi\rangle$ with ξ. Then the pure closure for $\langle T \cup \{t\}\rangle$ witness a contradiction and note that we can choose such as $\xi_\alpha \ni \eta$ for any fixed $\eta \in \omega_1$. Let $A_\beta \supset S_{\xi_\alpha}$ be the pure closure of $\langle A_\alpha \cup \{Y_{\xi_\alpha}\}\rangle$. Since $\langle Y_{\xi_\alpha}\rangle \cap C_\alpha = 0$ follows that $A_\beta \cap C_\alpha = A_\alpha$, so by second isomorphism theorem, we obtain:

$$A_\beta/A_\alpha \cong (A_\alpha \cdot C_\alpha)/C_\alpha \leq A/C_\alpha$$

and sine C_α is an $\aleph_1$-pure subgroup, A_β/A_α is free.

Therefore, we obtain a free pure subgroup A and uncountable many $\{\varphi_{\xi_\alpha}\,;\alpha\in\omega_1\}$ such that $\operatorname{dom}\varphi_{\xi_\alpha}\subset A$. □

Lemma IV.11. Let $\langle A_\alpha\,;\alpha\in\omega_1\rangle$ be a smooth chain of countable Abelian groups and $A=\cup\{A_\alpha\,;\alpha\in\omega_1\}$. For a set $E=\{\gamma\in\omega_1\cap\mathrm{Lim}\,;$ " A_γ is not an $\aleph_1$-pure subgroup.." $\}$. The following are equivalent:

1. A is free,
2. E is not a stationary subset of ω_1.

Proof. At the beginning of the proof, we note that we may assume that $A_0=0$.

(1) $\Longrightarrow$ (2). Let X be a base for A. We shall define a $X_\alpha\subset X$ and $\xi_\alpha\in\omega_1$, $\alpha\in\omega_1$, such as:

- $X_\alpha\subsetneq X_\beta$ for any $\alpha\in\beta\in\omega_1$,
- $\xi_\alpha\in\xi_\beta$ for any $\alpha\in\beta\in\omega$, and
- $F(X_\alpha)=A_{\xi_\alpha}$ for any $\alpha\in\omega_1$.

If we are done, we have a club set $\{\xi_\alpha\,;\alpha\in\omega_1\}$ such that every A_{ξ_α} is an $\aleph_1$-pure subgroup (note that $A/A_{\xi_\alpha}=F(X\setminus X_\alpha)$ holds). Therefore, to conclude the proof, we shall define X_α and ξ_α, $\alpha\in\omega_1$, recursively.

1. Leading Stage:
   - $X_0=0$,
   - $\xi_0=0$.
2. Successor Stages:

   We shall define X_α^n and ξ_α^n, $n\in\omega$, iteratively:

   (a) Leading Stage:

   a–i. X_α^0 be a countable set with $X_\alpha\subsetneq X_\alpha^0$, and

   a–ii. ξ_α^0 be an ordinal such that $F(X_\alpha^0)\subset A_{\xi_\alpha^0}$.

   (b) Successor Stages:

   b–i. $A_{\xi_\alpha^n}\subset F(X_\alpha^{n+1})$, and

   b–ii. $F(X_\alpha^{n+1})\subsetneq A_{\xi_\alpha^{n+1}})$.

   Let $X_{\alpha+1}=\cup\{X_\alpha^n\,;n\in\omega\}$ and $A_{\alpha+1}=\cup\{A_{\xi_\alpha^n}\,;n\in\omega\}$.
3. Limit Stages:
   - $X_\gamma=\cup\{X_\alpha\,;\alpha\in\gamma\}$,
   - $\xi_\gamma=\cup\{\xi_\alpha\,;\alpha\in\gamma\}$.

(2) $\Longrightarrow$ (1). Let $\langle\xi_\alpha\,;\alpha\in\omega_1\rangle\nearrow\omega_1$ be a sequence such that $\xi_\gamma=\cup\{\xi_\alpha\,;\alpha\in\gamma\}$ for any limit γ and who does not meet E. For every $\alpha\in\omega$, since A_{ξ_α} is $\aleph_1$-pure, we have $A_{\xi_{\alpha+1}}/A_{\xi_\alpha}$ is free. Therefore, $A=\cup\{A_{\xi_\alpha}\,;\alpha\in\omega_1\}$ is free by Theorem III.6. □

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