Hereditarily Ordinal Definable Sets and the Independence of The Axiom of Choice

静岡大学大学院総合科学技術研究科
理学専攻数学コース

41230007　野呂秀貴

令和６年２月１日

Contents

Abstract

In the context of 20th-century developments, The Completeness and Soundness in the first order logic asserts that the proposition does not provable can shown by the existence the of suitable model. We draw connections to this fact, Gödel produces the inner model called the Gödel's constructive universe and Cohen produce the way to construct an extension model which called forcing.

The paper establishes the independence of AC (the axiom of choice) by constructing models for both $\mathrm{ZF} + \mathrm{AC}$ and $\mathrm{ZF} + \neg\,\mathrm{AC}$. Notably, the Gödel's construct universe serves as a desired model for $\mathrm{ZF} + \mathrm{AC}$, while forcing is employed to provide a model for $\mathrm{ZF} + \neg\,\mathrm{AC}$.

謝辞

本稿を完成させるにあたり, 数多の人々の支援および援助をいただき有難い限りでございます. とりわけ日本国及び陛下におかれましては, 小生に勉学と研究の機会を与えていただき深く感謝しております. 小生の指導教員である依岡輝幸先生には頗る多大なるご指導をして頂き, 同研究室の仲間にも多くの支援を頂き本稿を完成させることができました. 改めてこの世の生きとし生けるものに感謝の言葉を捧げたいと思います.

概要

選択公理 (AC) は 「無限個の集合から同時に元を取ってくることが出来る」という主張であり, 現代数学を学ぶ上で必要不可欠なものとして扱われているが, 1980 年代に確立された Zermelo-Fraenkel の公理系 (ZF) には含まれていない. しかし, もし ZF の公理系を満たすモデルが存在するならば, ZF の公理系に AC の肯定を加えた ZF+AC を満たすモデルが構成可能であり, さらに, AC の否定を加えた ZF+¬AC のモデルも構成可能である. 「AC が ZF から独立」とは, この 2 つのモデルが存在するということである. また, これらのモデルの存在より ZF の公理系から AC の肯定と否定のどちらも示されないことが直ちに分かる. この事実により選択公理を認めた数学も自然であることが分かることとなる. 本稿ではこれらの 2 つのモデルを ZF のモデルから具体的に構成する.

ZF+AC のモデルを構成する方法として, Gödel の構成可能宇宙がその証拠であることを用いた. Gödel の構成可能宇宙が選択公理を満たしているモデルであることは, クラス全体に整列な順序のようなものを定義することにより 任意の集合が整列可能という主張を確認した.

ZF+¬AC のモデルを構成する方法として, その存在が選択公理から導かれる「良い選択集合」を導入し, 良い選択集合が存在しない ZF のモデルを構成した. 具体的には, Cohen 強制法を用いて ZFC+GCH のモデルを拡大し, さらにその内部モデル $\mathrm{HOD}^{\mathbb{R}}$ が良い選択集合を持たないが ZF の公理系を満たす証拠となっていることを本稿で示した.

Chapter 1

Downward Löwenheim-Skolem-Tarski Theorem

In this section we prove the Downward Löwenheim-Skolem-Tarski Theorem and the Reflection theorem.

Before we start to show the theorems, we shall make some definitions and theorems.

Definition 1.1. Let B be a nonempty set.

1. An *α-ary function* on B is a function $f\colon {}^{\alpha}B \to B$.

2. A *$<\theta$-ary function* on B is an f such that $f\colon {}^{<\theta}B \to B$ for a cardinal θ.

3. A *finitary function* is a $<\omega$-ary function.

4. If $f\colon {}^{\alpha}B \to B$ then $A \subset B$ is *closed under* f provided that $f[{}^{\alpha}A] \subset A$, i.e., $f(x) \in A$ for any $x \in {}^{\alpha}A$.

5. Let $\mathcal{F}$ be a family of $<\theta$-ary functions on B. $A \subset B$ is *closed under* $\mathcal{F}$ provided that A is closed under every $f \in \mathcal{F}$, and the *closure of* $S \subset B$ *under* $\mathcal{F}$ is the least set A, according to the subset relation, such that $S \subset A \subset B$ and A is closed under $\mathcal{F}$.

Lemma 1.2. Let θ be a regular infinite cardinal, and let $\mathcal{F}$ be a family of $<\theta$-ary functions on B. Fix $S \subset B$, and define S_ξ, recursively,

1. $S_0 = S$,
2. $S_{\xi+1} = \cup\{S_\xi, \{f(x)\,;\, f \in \mathcal{F} \ \wedge \exists\alpha \in \xi + 1 \exists x\ (\text{ “}f \text{ is an } \alpha\text{-ary function”} \wedge x \in {}^{\alpha}S_\xi)\}\}$,
3. $S_\gamma = \cup\{S_\xi\,;\, \xi \in \gamma\}$ for any limit $\gamma \leq \theta$.

Then, S_θ is the closure of S under $\mathcal{F}$.

Proof. To see that S_θ is closed under $\mathcal{F}$. Let $f \in \mathcal{F}$ be an α-ary function and let $x \in {}^{\alpha}S_\theta$. Choose a $\beta \geq \alpha$ such as $x \in {}^{\alpha}S_\beta$, then we obtain that $f(x) \in S_{\beta+1} \subset S_\theta$.

To see the minimality. Let T be a superset of S which is closed under $\mathcal{F}$. We show that $S_\xi \subset T$ by induction on ξ. For the leading and limit stages are obvious. We check for the successor stages. Let $a \in S_{\xi+1}$ and assume that $a \notin S_\xi$ (if not we have $a \in T$ by induction hypothesis). Then there is $f \in \mathcal{F}$ and $x \in {}^{\theta}S_\xi \subset {}^{\theta}T$ such that $a = f(x)$. Then since T is closed under f we have $a \in T$. $\square$

Lemma 1.3. Let $\mathcal{F}$ be a family of finitary functions on B, κ an infinite cardinal, $S \subset B$, $|S| \leq \kappa$, and $|\mathcal{F}| \leq \kappa$. Then the cardinality of the closure of S under $\mathcal{F}$ is $\leq \kappa$.

Hereafter for a set A, $\vec{a} \in A$ is an abbreviation that $\vec{a}$ is a finite sequence of A, i.e., there is an $n \in \omega$ such that $\vec{a} \in {}^{n}A$.

Proof. Immediate by Lemma 1.2. $\square$

Lemma 1.4 (Tarski-Vaught criterion)**.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures with $\mathfrak{A} \subset \mathfrak{B}$. Then the following are equivalent.

1. $\mathfrak{A} \preccurlyeq \mathfrak{B}$.
2. For every existential formula $\varphi(\vec{x}) \equiv \exists y \psi(\vec{x}, y)$ and $\vec{a} \in A$, if $\mathfrak{B} \models \varphi[\vec{a}]$ then there is $b \in A$ such that $\mathfrak{B} \models \psi[\vec{a}, b]$.

Proof. To see ($\Rightarrow$). Let $\varphi(\vec{x}) \equiv \exists y \psi(\vec{x}, y)$ and $\vec{a} \in A$ with $\mathfrak{B} \models \varphi(\vec{a})$. Since, $\mathfrak{A} \preccurlyeq \mathfrak{B}$, we have $\mathfrak{A} \models \varphi(\vec{a})$ and so there is $b \in A$ such that $\mathfrak{A} \models \varphi(\vec{a}, b)$ moreover, $\mathfrak{B} \models \psi(\vec{a}, b)$.

To see ($\Leftarrow$). Let φ be a $\mathcal{L}$-formula. We distinguish some cases, according to the form of φ.

(i) φ is an existential formula, $\exists y \psi(\vec{x}, y)$. Let $\vec{a} \in A$. First, we assume that $\mathfrak{A} \models \varphi(\vec{a})$. Then there is $b \in A$ with $\mathfrak{A} \models \psi(\vec{a}, b)$ moreover, $\mathfrak{B} \models \psi(\vec{a}, b)$. Second, we assume that $\mathfrak{B} \models \varphi(\vec{a})$. Then by assumption we have $b \in A$ with $\mathfrak{B} \models \psi(\vec{a}, b)$ and by induction hypothesis, $\mathfrak{A} \models \psi(\vec{a}, b)$. So, $\mathfrak{A} \models \varphi(\vec{a})$.

(ii) φ is implication, $\psi_1 \Rightarrow \psi_2$. First, we assume that $\mathfrak{A} \models \varphi$ and suppose that $\mathfrak{B} \models \psi_1$. Then by induction hypothesis, we have $\mathfrak{A} \models \psi_1$ moreover, $\mathfrak{A} \models \psi_2$ furthermore by induction hypothesis, $\mathcal{B} \models \psi_2$. Thus, we have $\mathfrak{B} \models \varphi$. Second, assume that $\mathcal{B} \models \varphi$. Mimic the proof of the first case.

(iii) φ is negation, $\neg\psi$. Mimic the proof of the case (ii). □

Theorem 1.5 (Downward Löwenheim-Skolem-Tarski Theorem)**.** Assume that $V \models \mathrm{ZFC}^-$. Let $\mathfrak{B}$ be an $\mathcal{L}$-structure and let κ be a cardinal with $\cup\{|\mathcal{L}|, \aleph_0\} \leq \kappa \leq |B|$ and fix $S \subset B$ with $|S| \leq \kappa$. Then there is an $\mathcal{L}$-structure $\mathfrak{A} \preccurlyeq \mathfrak{B}$ such that $S \subset A$ and $|A| = \kappa$.

Proof. For an existential formula $\varphi \equiv \exists y \psi(-, y)$ with n-free variables define a function $f_\varphi \colon {}^n B \to B\, ; \vec{x} \mapsto y$ if $\varphi(\vec{x}, y)$ and $\vec{x} \mapsto 0$ otherwise. Let $\mathcal{F}$ be the collection of such f_φ's then note that $|\mathcal{F}| \leq \kappa$ holds. For $S \subset B$, since $|S| \leq \kappa \leq |B|$, fix a set S' such that $S \subset S' \subset B$ and $|S'| = \kappa$ and let A the closure of S' under $\mathcal{F}$. Then by Lemma 1.3, $|A| \leq \kappa$, i.e., $|A| = \kappa$. Now we assert that for any n-ary function symbols $g \in \mathcal{L}$, A is closed under g. This is because that for $\vec{x} \in A$ there is an existential formula $\exists y (g(\vec{x}) = y)$, thus $g(\vec{x}) = y \in A$. Therefore, we obtain an $\mathcal{L}$-structure $\mathfrak{A}$ such that $|A| = \kappa$ and $S \subset A$. And $\mathfrak{A} \preccurlyeq \mathfrak{B}$ is derived from Lemma 1.4. □

Definition 1.6. A subformula of formula φ is defined as any followings:

- φ is a subformula of φ,
- if $\neg\psi$ is a subformula of φ, so is ψ,
- if $\varphi_0 \wedge \varphi_1$ is a subformula of φ, so are φ_0 and φ_1,
- if $\exists x\ \varphi(x)$ is a subformula of φ, so are $\varphi(t)$ for any t.

A list of formulas $\{\varphi_1, \cdots, \varphi_{n-1}\}$ is *subformula-closed* provided that every subformula of each φ_i is also in the list, and no formulas in the list use the universal quantifier.

Lemma 1.7. Let $\varphi_0, \cdots, \varphi_{n-1}$ be a subformula-closed list of formulas of $\mathcal{L} = \{\in\}$. Let A and B be classes with $\emptyset \neq A \subset B$. Then the following are equivalent:

1. $\bigwedge_{i\in n}(A \preccurlyeq_{\varphi_i} B)$, (which is an abbreviation for $(A \preccurlyeq_{\varphi_0} B) \wedge (A \preccurlyeq_{\varphi_1} B) \wedge (A \preccurlyeq_{\varphi_2} B) \cdots \wedge (A \preccurlyeq_{\varphi_{n-1}} B)$)

2. For all existential formulas $\exists y \psi_i(-, y)$ in the list,

$$\psi_i^B(\vec{a}, y) \Rightarrow \exists b \in A \psi_i^B(\vec{a}, b) \text{ for any } \vec{a} \in A.$$

Proof. Mimic the proof of Lemma 1.4 $\square$

This theorem asserts that to see weather A is an elementary submodel for B with respect to any formula φ_i, it suffices to find a witnesses for each extensional formulas.

Theorem 1.8 (**Reflection Theorem**). Let $\varphi_0, \cdots, \varphi_{n-1}$ be any formulas of $\mathcal{L} = \{\in\}$. Assume that B is a non-empty class and $A(\xi)$ is a set for each $\xi \in \mathrm{ON}$, such that

1. $\xi \in \eta \Rightarrow A(\xi) \subset A(\eta)$,
2. $A(\eta) = \cup\{A_\xi\,;\xi \in \eta\}$ for any limit η,
3. $B = \cup\{A_\xi\,;\xi \in \mathrm{ON}\}$.

Then $\forall \xi \in \mathrm{ON}\, \exists \eta \ni \xi (A(\eta) \neq \emptyset \wedge \bigwedge_{i<n}(A(\eta) \preccurlyeq_{\varphi_i} B) \wedge$ "η is limit").

Proof. At the beginning of the proof, we may assume that the list is subformula-closed. For a formula φ_i in the list, define $F_i\colon {}^{r}B \to \mathrm{ON}$ as follows:

$$\vec{a} \mapsto \begin{cases} \cap\{\xi\,; \exists b \in A(\xi)\ \varphi_i^B(\vec{a}, b)\} & \text{if } \varphi_i \text{ is an extensial fomula such that} \\ & \varphi_i^B(\vec{a}, b) \text{ holds for some } b \in B \\ 0 & \text{otherwise} \end{cases}$$

and, for $i \in n$ define $G_i\colon \mathrm{ON} \to \mathrm{ON}$ such as

$$\vec{\xi} \mapsto \begin{cases} \cup\{F_i(\vec{a})\,; \vec{a} \in A(\xi)\} & \text{if } \varphi^B(\vec{a}, b) \text{ holds for some } b \in B \\ 0 & \text{otherwise} \end{cases}$$

Now we define an assignment $K\colon \mathrm{ON} \to \mathrm{ON}$ such that $\xi \mapsto \cup\{\xi + 1, \cup\{G_i(\xi)\,; i \in n\}\}$.

Finally, we construct the desired η for any $\xi \in \mathrm{ON}$. We shall define η_n, $n \in \omega$, recursively, $\eta_0 = \cap\{\zeta \in \mathrm{ON}\,; A(\zeta) \neq \emptyset \wedge \zeta \ni \xi\}$ and $\eta_{n+1} = K(\eta_n)$.

Then we have an increasing sequence $\eta_0 \in \eta_1 \in \eta_2 \in \cdots$, thus by letting $\eta = \cup\{\eta_i\,; i \in \omega\}$ then we have that $\eta \ni \xi$, $A(\eta) \neq \emptyset$ and η is limit.

We remain to show that $\bigwedge_{i\in n}(A(\eta) \preccurlyeq_{\varphi_i} B)$. It suffices to verify that, for any existential formula $\varphi_i(\vec{x}) \equiv \exists y \psi(\vec{x}, y)$ and $\vec{a} \in A(\eta)$, there is a witness $a \in A(\eta)$. Let $\eta_i \in \eta$ with $\vec{a} \in A(\eta_i)$. Then there is $b \in B$ such that $\psi^B(\vec{a}, b)$ and there is $\xi \in F_i(\vec{a})$ and $a \in A(\xi)$ with $\psi^B(\vec{a}, a)$. Now by construction we obtain $\xi \in F(\vec{a}) \leq G(\eta_i) \leq K(\eta_i) \in \eta$, furthermore $a \in A(\xi) \subset A(\eta)$, i.e., $\exists a \in A(\eta) \varphi_i^B(\vec{a}, a)$. $\square$

Lemma 1.9. If a model M have the following properties, it is a model for ZF.

1 M is a transitive model,

2 M satisfies the comprehension axiom,

3 For any collection $x \subset M$ there is $y \in M$ such that $x \subset y$.

Proof. Note that the extensionality and the foundation are immediate by the property 1.

To see the pairing. For $x, y \in M$, since there is $\{x, y\} \subset M$ property 3 assert that there is $A \in M$ such that $\{x, y\} \subset A$.

To see the infinity. Properties 1,3 asset that $\emptyset \in M$ and we have $\{x\} \in M$ for $x \in M$. Moreover since $M \models$ pairing+comprehension, $\{x, \{x\}\} \in M$ for any $x \in M$, we obtain that $\omega \in M$.

To see the replacement. Let $x \in M$ and φ be a formula such that x does not appear in its variables. Assume that $\forall z \in x \exists! y\ \varphi(z, y)$ over M. For a formula $\psi(z, y) \equiv \varphi(z, y) \wedge y \in M$, we have $\forall z \in x \exists! y\ \psi(z, y)$. Applying replacement over V, there is a set A such that

$$\forall y\ (y \in A \Leftrightarrow \exists z \in x\ \varphi(z, y) \wedge y \in M)$$

Then since $A \subset M$, property 3 asserts that there is $B \in M$ such that $A \subset B$ and we have:

$$M \models \forall z \in x \exists y \in B\ \varphi(z, y)$$

To see the power set. Let $x \in M$. Thanks to the property 3, it suffices to show that $\mathcal{P}(x) \subset M$. Let $y \in \mathcal{P}(x)$. Since M is transitive, we have $y \subset M$. Thus, by comprehension and property 3, we obtain that $y \in M$, i.e., $\mathcal{P}(x) \subset M$. $\square$

Chapter 2

Gödel's Constructive Universe

In this section, we produce the model for $\mathrm{ZFC} + \mathrm{AC} + \mathrm{GCH}$, called Gödel constructive universe.

Definition 2.1. Let R be a relation on set A such that R is well-founded and $\{b \in A\,; bRa\}$ is a set for any $a \in A$.

For $a \in A$, define the *rank*, recursively,

$$\mathrm{rank}_{R,A}(a) = \cup\{\mathrm{rank}_{R,A}(b) + 1\,; bRa\}.$$

Definition 2.2. For sets A and $P \subset A$, $\mathcal{D}(A, P)$ is the set of subsets of A which are *definable over* $\mathfrak{A} = (A, \in)$, i.e., $\mathcal{D}(A, P) = \{\{a \in A\,; \mathfrak{A} \models \varphi(\vec{b}, a)\}\,; \vec{b} \in P \wedge$ "φ is a formula" $\}$ and define $\mathcal{D}^+(A) = \mathcal{D}(A, A)$.

Definition 2.3. Define $L(\delta)$ by recursion on $\delta \in \mathrm{ON}$;

1. $L(0) = \emptyset$
2. $L(\beta + 1) = \mathcal{D}^+(L(\beta))$,
3. $L(\gamma) = \cup\{L(\alpha)\,; \alpha \in \gamma\}$ for any limit γ.

We call the class $L = \cup\{L(\alpha)\,; \alpha \in \mathrm{ON}\}$ *Gödel constructive universe.*

Lemma 2.4. For any ordinals α and β, we have

1. $L(\alpha) \subset R(\alpha)$, where $R(\alpha)$ is the Von Neumann universe,
2. $L(\beta)$ is transitive,

3. $\alpha \in \beta \Rightarrow L(\alpha) \subset L(\beta)$,
4. $\cap\{L(\beta), \mathrm{ON}\} = \beta$.

Proof. For 1. It suffices to show that $\mathcal{D}^+(A) \subset \mathcal{P}(A)$ for arbitrary set A and this is trivially by the definition of $\mathcal{D}^+(A)$.

For 2. We shall show by induction on β. The leading or limit stages are clear. Assume that $\beta = \alpha + 1$. Let $A \in L(\beta)$ and $x \in A$. Since $A \subset L(\alpha)$ and $L(\alpha)$ is transitive, $x \in L(\alpha)$. Therefore, we obtain $x = \{z \in L(\alpha)\,; z \in x\} \in L(\alpha+1)$. For 3. For arbitrary $\beta \in \mathrm{ON}$ we show the $\alpha \in \beta$ $L(\alpha) \subset L(\beta)$ by induction on β. The cases of β is limit is clear. When β is successor, $\beta = \gamma + 1$. It is enough to show that $L(\gamma) \subset L(\gamma + 1)$. For $x \in L(\gamma)$. Since $L(\gamma)$ is transitive, $x = \{y \in L(\gamma)\,; y \in x\}$ so $x \in L(\gamma + 1)$.

For 4. We show by induction on β. The cases that $\beta \in \mathrm{ON}$ is 0 and limits are clear. When β is a successor ordinal, $\beta = \gamma + 1$. To see ($\subset$). Let $\alpha \in \cap\{L(\gamma + 1), \mathrm{ON}\}$. Since $\alpha \subset L(\gamma)$, $\alpha \subset \cap\{L(\gamma), \mathrm{ON}\} = \gamma$, i.e., $\alpha \in \gamma+1$. To see ($\supset$), let $\alpha \in \gamma+1$. If $\alpha \in \gamma$, we have $\alpha \in \cap\{L(\gamma+1), \mathrm{ON}\}$. If $\alpha = \gamma$, since the assertion that "α is an ordinal" is a Δ_0-formula, we have $\gamma \in L(\gamma + 1)$. □

Definition 2.5. For $x \in L$ the *L-rank of* x, $\rho(x)$, is the least ordinal α such that $x \in L(\alpha + 1)$.

Lemma 2.6. Let α be an ordinal. We have $L(\alpha) = \{x \in L\,; \rho(x) \in \alpha\}$ and $L(\alpha + 1) \setminus L(\alpha) = \{x \in L\,; \rho(x) = \alpha\}$.

Proof. We proved by induction on α.

For $\alpha = 0$. Both of the equalities are clear.

For $\alpha = \beta + 1$. To see the first equality, $L(\beta + 1) = \cup\{L(\beta + 1) \setminus L(\beta), L(\beta)\} = \cup\{\{x \in L\,; \rho(x) = \beta\}, \{x \in L\,; \rho(x) \in \beta\}\} = \{x \in L\,; \rho(x) \in \beta+1\}$. To see the second equality, let $x \in L(\beta+2) \setminus L(\beta+1)$. $x \in L(\beta+2)$ shows that $\rho(x) \leq \beta + 1$, and $x \notin L(\beta + 1)$ implies that $\rho(x) \ni \beta$. Thus, $\rho(x) = \beta + 1$, i.e., $x \in \{x \in L\,; \rho(x) = \beta + 1\}$.

For α is limit. $L(\alpha) = \cup\{x \in L\,; \rho(x) \in \gamma \wedge \gamma \in \alpha\} = \{x \in L\,; \rho(x) \in \alpha\}$ and $L(\alpha{+}1)\setminus L(\alpha) = \{x \in L\,; \rho(x) \leq \alpha \wedge \rho(x) \geq \alpha\} = \{x \in L\,; \rho(x) = \alpha\}$. □

Lemma 2.7. For an ordinal α, we have $L(\alpha) \in L$ and $\rho(L(\alpha)) = \rho(\alpha) = \alpha$.

Proof. The first statement is immediate by a fact $L(\alpha) = \{x \in L(\alpha)\,; x = x\} \in L(\alpha + 1)$. We shall show that $\rho(L(\alpha)) = \alpha$ by induction on α.

For $\alpha = 0$. $\rho(0) = 0$.

For $\alpha = \beta + 1$. $\rho(L(\alpha)) = \{\rho(x)\,;x \in L(\alpha)\} = \cup\{\rho(x)\,;x \in L(\gamma+1) \setminus L(\gamma)\,;\gamma \leq \beta\} = \cup\{\rho(x)\,;\rho(x) = \gamma \wedge \gamma \leq \beta\} = \beta + 1 = \alpha$.

For α is limit. $\rho(L(\alpha)) = \cup\{\{\rho(x)\,;x \in L(\gamma+1) \setminus L(\gamma)\}\,;\gamma \in \alpha\} = \alpha$. □

Lemma 2.8. *$L(\alpha + 1)$ has all finite subsets of $L(\alpha)$.*

Proof. For a finite subset A of $L(\alpha)$, by concatenate the finitely many "$\vee$(or)", we obtain $A \in \mathcal{D}^+(L(\alpha)) = L(\alpha + 1)$. □

Lemma 2.9. *$L(\alpha) = R(\alpha)$ for all $\alpha \leq \omega$ and $L(\omega + 1) \subsetneq R(\omega + 1)$.*

Proof. $L(\alpha) = R(\alpha)$ for $\alpha \in \omega$ is by previous lemma we have $L(\omega) = R(\omega) =$ HF is countable, $L(\omega + 1) = \mathcal{D}^+(\mathrm{HF})$ is countable. However, $R(\omega + 1) = \mathcal{P}(\mathrm{HF})$ does not countable. Thus we have the proper subset relation $L(\omega + 1) \subsetneq R(\omega + 1)$. □

Lemma 2.10. *Assuming AC, $|\mathcal{D}^+(A)| = |A|$ for an infinite set A.*

Proof. For ($\geq$). For every $a \in A$, we have $\{a\} \subset \mathcal{D}^+(A)$, so $|A| \leq |\mathcal{D}^+(A)|$.

For ($\leq$). For every element x in $\mathcal{D}^+(A)$ there is a sequence $\vec{a} \in A$ and a formula φ such that the pair is a witness. Thus there is a surjection from $[A]^{<\omega} \times \omega$ to $\mathcal{D}^+(A)$. So by assuming AC, we obtain that $|A| = [A]^{<\omega} \geq |\mathcal{D}^+(A)|$. □

Lemma 2.11. *Assuming AC, $|L(\alpha)| = |\alpha|$ for an infinite ordinal α.*

Proof. Since for arbitrary $\alpha \geq \omega$, $\alpha \subset L(\alpha)$ assert that $|\alpha| \leq |L(\alpha)|$. So we remain to show that $|L(\alpha)| \leq |\alpha|$ for $\alpha \geq \omega$ and we show by induction on α.

For $\alpha = \omega$. This is immediate by $L(\omega) = \mathrm{HF}$.

For $\alpha = \beta + 1$. Since $|L(\beta)| = |\beta|$ and Lemma 2.10 assert that $|L(\beta+1)| = |L(\beta)| = |\beta + 1|$.

For α is limit. For $\beta < \alpha$, $|L(\beta)| = |\beta| \leq |\alpha|$ asserts that $|\cup\{L(\beta)\,;\beta < \alpha\}| \leq |\alpha|$ holds by AC. □

Theorem 2.12. *$V \models \mathrm{ZF} \Rightarrow L \models \mathrm{ZF}$.*

Proof. Thanks to Lemma 1.9, it is enough to verify that:

(ii) $L \models$ "comprehension" ,

(iii) $\forall x \subset L \exists y \in L\ (x \subset y)$.

Let us show that (ii). Let $x \in L$ and ψ be a formula such that x does not appear in its variables. Reflection theorem asserts that there is an ordinal $\beta \in \mathrm{ON}$ such that $L(\beta) \prec_\psi L$ and $x \in L(\beta)$. Let φ be a formula and $\vec{b}$ variables which pair witnesses $x \in L(\beta)$. Then we obtain that $\{z \in x\,; L \models \psi(z)\} = \{z \in L(\beta)\,; L(\beta) \models \varphi(\vec{b}, z) \wedge \psi(z)\}$.

Let us show that (iii). Fix a $x \subset L$ and define an ordinal $\beta_0 = \cup\{\rho(z) + 1\,; z \in x\}$. Then we obtain that $L(\beta_0) \in L$ and $x \subset L(\beta_0)$. □

Lemma 2.13. For $x, y \in L$, we have the following:

1. $\{x, y\} \in L$ and $\rho(\{x, y\}) = \cup\{\rho(x), \rho(y)\} + 1$,
2. $\langle x, y\rangle \in L$ and $\rho(\langle x, y\rangle) = \cup\{\rho(x), \rho(y)\} + 2$,
3. $\cup x \in L$ and $\rho(\cup x) \leq \rho(x)$,
4. $\cup\{x, y\} \in L$ and $\rho(\cup\{x, y\}) \leq \cup\{\rho(x), \rho(y)\}$.

Proof. The assertion that $\{x, y\}$, $\langle x, y\rangle$, $\cup x$ and $\cup\{x, y\}$ are in L is immediate by $L \models \mathrm{ZF}$, so we remain to show that the equations.

To see that 1. Let $\alpha = \cup\{\rho(x), \rho(y)\}$ and we may assume that $\rho(x) \leq \rho(y)$. We distinguish two cases, according to $\rho(x) = \rho(y)$. Case I. $\rho(x) = \rho(y)$. Since $\rho(\{x, y\}) \leq \alpha + 1$ and if $\rho(\{x, y\}) \in \alpha + 1$, contrary to that $\rho(x) = \alpha$. We have $\rho(\{x, y\}) = \alpha + 1$ Case II. $\rho(x) \in \rho(y)$. Note that we have $x \notin L(\alpha)$. $\{x, y\} \not\subset L(\alpha)$ shows that $\{x, y\} \notin L(\alpha + 1)$ and $x, y \in L(\alpha + 1)$ shows that $\{x, y\} \in L(\alpha + 2)$. This asserts that $\rho(\{x, y\}) \leq \alpha + 1$

To see that 2. $\rho(\langle x, y\rangle) = \cup\{\rho(x), \cup\{\rho(x), \rho(y)\} + 1\} = \cup\{\rho(x), \rho(y)\} + 2$.

To see that 3. Let $\alpha = \rho(x)$. Since $x \in L(\alpha + 1)$, for some formula $\varphi(y)$, x has the form $\{y \in L(\alpha)\,; \varphi(y)\}$ and $L(\alpha)$ is transitive. Therefore, $\cup x = \{z \in L(\alpha)\,; \exists y(z \in y \wedge \varphi(y))\}$. So, $\rho(\cup x) \leq \alpha = \rho(x)$.

To see that 4. Let $\alpha = \cup\{\rho(x), \rho(y)\}$ and φ_x, φ_y such that $x = \{z \in L(\alpha)\,; \varphi_x(z)\}$, $y = \{z \in L(\alpha)\,; \varphi_y(z)\}$, respectively. Then $\cup\{x, y\} = \{x \in L(\alpha)\,; \varphi_x(z) \vee \varphi_y(z)\} \in L(\alpha + 1)$. Thus, $\cup\{x, y\} \in L$ and $\rho(\cup\{x, y\}) \leq \max\{\rho(x), \rho(y)\}$ hold. □

Lemma 2.14. The function $L(\delta)$, $\delta \in \mathrm{ON}$, is absolute for transitive model $M \models \mathrm{ZF} - \mathrm{P}$.

Proof. Recall that the ordinals are absolute for transitive model M and $L(\delta)$ is defined by the theory over $\mathrm{ZF} - \mathrm{P}$. Thus we have $L(\delta)^M = L(\delta)$ for every $\delta \in \mathrm{ON}$ by induction on δ. □

Lemma 2.15. $L \models V = L$.

Proof. Recall that L is a transitive model for $\mathrm{ZF} - \mathrm{P}$, it suffices to show that $\forall x \in L \exists \delta \in \mathrm{ON}^L (x \in L(\delta)^L)$. Since "$\delta$ is an ordinal" and "function $L(\cdot)$" are absolute for L, it is enough to show that $\forall x \in L \exists \delta \in \mathrm{ON}(x \in L(\delta))$ and this is obviously. $\square$

Definition 2.16. For a transitive set M, $o(M) = \cap\{M, \mathrm{ON}\}$.

Lemma 2.17. Let M be a transitive set such that $M \models \mathrm{ZF} - \mathrm{P}$. Then $M \models V = L$ iff $M = L(o(M))$.

Proof. ($\Rightarrow$). Let $\gamma = o(M)$. Since $M \models \mathrm{ZF} - \mathrm{P}$, $L(\delta) \in M$ for some $\delta \in \gamma$ and since M is transitive, $M \supset \cup\{L(\delta)\,; \delta \in \gamma\} = L(o(M))$. Let $x \in M$. Since $M \models V = L(o(M))$ holds, $x \in L(\delta)$ for all $\delta \in o(M)$. This shows that $x \in L(o(M))$, i.e, $M \subset L(o(M))$.

($\Leftarrow$). It suffices to show that $\forall x \in M \exists \delta \in o(M)\ x \in L(\delta)$ and this is immediate by assumption. $\square$

Definition 2.18. Let $n \in \omega$. For a set A_i and ordered relation R_i on A_i for $i \in n$, the *lexicographic order*, $\lhd$, on $\Pi\{A_i\,; i \in \omega\}$ with $\langle R_i\,; i \in n\rangle$ is an ordered relation such that $\langle a_0, \ldots, a_{n-1}\rangle \lhd \langle b_0, \ldots, b_{n-1}$ iff

- $a_j \neq b_j$ for some $j \in n$,
- For $k = \cap\{j \in n\,; a_j \neq b_j\}$, $a_k R_k b_k$ holds.

We will write the lexicographic order with $\langle R_i\,; i \in n\rangle$ for $\lhd\langle R_i\,; i \in n\rangle$ when there is no danger of confusing and we simply write R_0^{n-1} for $\lhd\langle R_i\,; i \in n\rangle$ when $R_i = R_j$ for any $i, j \in n$.

Note that if every R_i is totally-ordered, then $\lhd\langle R_i\,; i \in n\rangle$ is also totally-ordered. Similarly, if every R_i is well-ordered, then $\lhd\langle R_i\,; i \in n\rangle$ is also well-ordered.

Before we shall show that the choice holds in V, we recall that we assume that every formula and variable contained in $\mathrm{HF} = R(\omega)$.

Theorem 2.19. There is a class $<_L$ which well-orders L, that is for every set $A \subset L$, $\cap\{<_L, A \times A\}$ well-orders A. Therefore, $V = L \models \mathrm{ZF} + \mathrm{AC}$

Proof. Fix a bijection $\Gamma\colon \mathrm{HF} \to \omega$ and let $<_\Gamma$ be the order on WF such as Γ preserves order. At the beginning of the proof, we shall define a well-order $\lhd_\gamma$ in $L(\gamma)$ for $\gamma \in \mathrm{ON}$. Let $\lhd_0 = \emptyset$.

When γ is a successor, $\gamma = \alpha + 1$, For any $S \in L(\alpha+1) \setminus L(\alpha)$, there is a formula φ and variables $\vec{b} \in L(\alpha)$ which witnesses that $S = \{a \in L(\alpha) \,;\, L(\alpha) \models \varphi(\vec{b}, a)\}$. Since a pair of formula and variables which witnesses $S \in L(\alpha+1)$ is not uniquely determine, we said a pair $\langle \varphi, \vec{b} \rangle$ is a *good-witnesser* for S provided that a pair is a $\lhd$-least element which witnesses S where $\lhd$ is the lexicographic order $\langle <_\Gamma, \cup\{\lhd_\alpha^n \,;\, n \in \omega\}\rangle$. For $S_0, S_1 \in L(\alpha+1)$, define $S_0 \lhd_{\alpha+1} S_1$ iff

1. $S_0, S_1 \in L(\alpha)$ and $S_0 \lhd_\alpha S_1$, or
2. $S_0 \in L(\alpha)$ and $S_1 \in L(\alpha+1) \setminus L(\alpha)$, or
3. $S_0, S_1 \in L(\alpha+1) \setminus L(\alpha)$ and

$$\langle \varphi_0, \vec{b}_0 \rangle \lhd \langle \varphi_1, \vec{b}_1 \rangle.$$

where $\langle \varphi_i, \vec{b}_i \rangle$ be a good-witnesser pair for $S_i \in L(\alpha+1)$ for each $i \in 2$.

Then $\lhd_{\alpha+1}$ well-orders $L(\alpha+1)$ and $\lhd_\alpha \subset \lhd_{\alpha+1}$ holds.

When γ is a limit, let $\lhd_\gamma = \cup\{\lhd_\beta \,;\, \beta \in \gamma\}$.

Then the class $<_L = \cup\{\lhd_\gamma \,;\, \gamma \in \mathrm{ON}\}$ can well-orders L. □

Definition 2.20. For any cardinal κ, define $H(\kappa) = \{x \in \mathrm{WF} \,;\, |\operatorname{trcl}(x)| \in \kappa\}$. $\mathrm{HC} = H(\omega_1)$ is called the set of hereditarily countable sets.

Lemma 2.21. For any infinite cardinal κ, $H(\kappa) \subset R(\kappa)$ and $H(\kappa)$ is a set of size $2^{<\kappa}$.

Proof. For $H(\kappa) \subset R(\kappa)$, let $x \in \mathrm{WF}$ with $|\operatorname{trcl}(x)| < \kappa$. For arbitrary $\xi < \operatorname{rank}(x)$, by lemma 9.16 in [Kun11] there is $z \in \operatorname{trcl}(x)$ such that $\xi = \operatorname{rank}(z)$. Thus, we have $\operatorname{rank}(x) = \{\operatorname{rank} z \,;\, z \in \operatorname{trcl}(x)\}$. So we obtain $\operatorname{rank}(x) < \kappa$.

To see the equality $|H(\kappa)| = 2^{<\kappa}$. $(\geq)$. For $\lambda \in \kappa$ and $\mu \subset \lambda$. We have $\mu \in H(\kappa)$. This shows that $\forall \lambda \in \kappa (\mathcal{P}(\lambda) \subset H(\kappa))$, moreover, $2^{<\kappa} \leq |H(\kappa)|$. For $(\leq)$. For $x \in H(\kappa)$, define $\lambda = |\operatorname{trcl}(x) \cup \{x\}|$. Then since $\lambda \in \kappa$, there is a relation $F(x)$ on λ such that $(\lambda, F(x)) \approx (\operatorname{trcl}(x) \cup \{x\}, \in)$. This endowed a function $F\colon H(\kappa) \to \cup\{\mathcal{P}(\lambda \times \lambda) \,;\, \lambda \in \kappa\}$, $x \mapsto F(x)$. Clearly, this F is injective. So we have $|H(\kappa)| \leq 2^{<\kappa}$. □

Lemma 2.22 (AC)**.** If κ is any regular uncountable cardinals, then $L(\kappa) \models \mathrm{ZF} - \mathrm{P} + V = L$.

Proof. For comprehension. Let $x \in L(x)$ and $\varphi(y)$ be a formula over $L(\kappa)$. Reflection theorem assert that there is $\alpha < \kappa$ such that $x \in L(\alpha)$ and $L(\alpha) \preccurlyeq_\varphi L(\kappa)$. Since $\{y \in L(\alpha)\,; y \in x \wedge \varphi(y)\} \in L(\alpha+1)$ and $L(\alpha)$ is transitive. Thus, $\{y \in x\,; \psi(y)\} \in L(x)$.

For Replacement. Let $A \in L(\kappa)$ and $\varphi(x,y)$ be a formula over $L(\kappa)$ such that $L(\kappa) \models \forall x \in A \exists ! y\ \varphi(x,y)$. Let f be a function from A into $L(\kappa)$ which assign to $x \in A$ the unique y such that $L(\kappa) \models \varphi(x,y)$. Let $\beta = \sup\{\rho(f(x)) + 1\,; x \in A\}$. Then since $|A| < \kappa$, $\forall x \in A(|\rho(f(x))| < \kappa)$, $\beta < \kappa + 1$. Thus, we have $\rho(L(\beta)) < \kappa$ and $L(\beta)$ is desired one. □

Theorem 2.23. If $V = L$, then $L(\kappa) = H(\kappa)$ holds for a cardinal $\kappa \geq \omega$. Moreover, $V = L$ implies GCH.

Proof. First, we show that the second assertion by the fact of the first assertion. Let $\kappa \geq \omega$. Then by Lemma 2.21, we have $\mathcal{P}(\kappa) \subset H(\kappa^+) = L(\kappa^+)$ and by Lemma 2.11 $|L(\kappa^+)| = \kappa^+$. Therefore, $2^\kappa \leq \kappa^+$, i.e., $2^\kappa = \kappa^+$ holds for $\kappa \geq \omega$.

From now on, we verify the first assertion. The case of $\omega = \kappa$ is by Lemma 2.10 and the definition Definition 2.20, and the limit case, γ, is clear if it holds for all $\lambda \in \gamma$.

From now on we verify the equality $L(\kappa) = V(\kappa)$ for $\kappa \geq \omega$.

For ($\subset$). Let $x \in L(\kappa)$. Since, κ is a limit cardinal, there is an ordinal $\alpha < \kappa$ such that $x \in L(\alpha)$. Thus we obtain $\mathrm{trcl}(x) \subset L(\alpha)$, so $|\,\mathrm{trcl}(x)| \leq |L(\alpha)|$ holds and by Lemma 2.11, $|L(\alpha)| = |\alpha| = \alpha < \kappa$. Therefore, $x \in H(\kappa)$.

For ($\supset$). It suffices to show that $L(\kappa) = H(\kappa) \Rightarrow L(\kappa^+) \supset H(\kappa^+)$. Let $b \in H(\kappa^+)$ and let $T = \mathrm{trcl}(\{b\})$ then note that $b \in T$ and $|T| \leq \kappa$. Since $V = L$ there is a regular cardinal $\theta > \sigma(T)$ by the definition of σ. Then by Lemma 2.22 we have $T \subset L(\theta)$ and $L(\theta) \models \mathrm{ZF} - \mathrm{P} + V = L$. For κ, Downward Löwenheim Skolem-Tarski Theorem assert that there is A such that $A \prec L(\theta)$, $T \subset A$ and $|A| \leq \lambda$. Then for the Mostoski Collapsing function, mos, over A and $B = \mathrm{ran}(mos)$ we obtain $mos\colon (A, \in) \approx (B, \in)$ and B is transitive and $mos(x) = x$ for all $x \in T$. Moreover, $B \models \mathrm{ZF} - \mathrm{P} + V = L$ and Lemma 2.17 assert that $B = L(\beta)$ for $\beta = o(M)$. Then by Lemma 2.11, $|\beta| = |L(\beta)| = |B| = |A| \leq \lambda$. So, $\beta < \lambda^+$ furthermore, $b \in L(\beta) \subset L(\lambda^+)$. □

Definition 2.24. A *nice choice set* is an $\mathcal{E} \subset \mathcal{P}(\omega)$ such that;
$\forall x \in \mathcal{P}(\omega)\ (x \in \mathcal{E} \Leftrightarrow (\omega \backslash x \notin \mathcal{E})) \wedge \forall x, y \in \mathcal{P}(\omega)\ (x =^* y \Rightarrow (x \in \mathcal{E} \Leftrightarrow y \in \mathcal{E}))$

Since a non-principle ultrafilter exists over the ZFC, we have the following fact.

Fact 2.25. In ZFC. There is a nice choice set.

Definition 2.26. For forcing posets $\mathbb{P}$, $\mathbb{Q}$ and $i\colon \mathbb{Q} \to \mathbb{P}$, define $i_*\colon V^{\mathbb{Q}} \to V^{\mathbb{P}}$ recursively:

$$i_*(\tau) = \{\langle i_*(\sigma), i(q)\rangle \,;\langle \sigma, q\rangle \in \tau\}$$

Theorem 2.27. Let M be a ctm for ZFC and let $\mathbb{P} = \mathrm{Fn}((\omega_1)^M, 2)$. Let $\varphi(x)$ be a formula with one free variable in $\mathcal{FL}_{\mathbb{P}} \cap M$ such that all names which mentioned are either nice names for subsets of ω, or names of the form $\check{b}$, where $b \in M$. Let G be a $\mathbb{P}$-generic over M. Then $M[G]$ satisfies the sentence asserting that $\{x \in \mathcal{P}(\omega)\,; \varphi(x)\}$ is not a nice choice set.

Proof. At the beginning of the proof, for $S \subset \omega_1^M$, we define $i^S\colon \mathbb{P} \to \mathbb{P}$ which assigns to $r \in \mathbb{P}$ the $i^S(r) \in \mathbb{P}$ where $i^S(r)$ is $\xi \mapsto r(\xi)$ if $\xi \in \mathrm{dom}(r)$ and $\xi \in S$, $\xi \mapsto 1 - r(\xi)$ if $x \in \mathrm{dom}(r)$ and $x \notin S$. And for $\alpha < \omega_1^M$, define $\theta_\alpha = \{\langle n, \{(\alpha + n, 1)\}\rangle\,; n \in \omega\}$. We shall show the following:

1. i^S is dense embedding for each $S \subset \omega_1^M$,
2. $\Vdash(\theta_\alpha, i_*^S(\theta_\alpha) \in \mathcal{P}(\omega))$ for each $S \subset \omega_1^M$,
3. $\Vdash(i_*^S(\theta) =^* \omega \setminus \theta_\alpha)$ if $\forall^\infty n \in \omega\ (\alpha + n \notin S)$.

For (1). Immediate by $i^S \circ i^S = \mathrm{id}_{\mathbb{P}}$.

For (2). For $M[G] \models \theta_\alpha \in \mathcal{P}(\omega)$. Immediate from $(\theta_\alpha)_G = \{\check{n}_G\,; \exists p \in G\ (p = \{(\alpha + n, 1)\})\} \subset \omega$. For $M[G] \models i_*^S(\theta_\alpha) \in \mathcal{P}(\omega)$. Immediate from $i_*^S(\theta_\alpha)_G = \{i_*^S(\check{n})_G\,; \exists p \in G\ p = i^S(\{(\alpha + n, 1)\})\} \subset \omega$.

For (3). For $(\subset^*)$. For $n \in i_*^S(\theta_\alpha)_G$, note that we have $i(\{(\alpha+n, 1)\}) \in G$. If $\alpha+n \notin S$, we have $\{(\alpha+n, 0)\} \in G$, furthermore since $D_\beta = \{\{\beta, m\}\,; m \in 2\}$ is dense in $\mathbb{P}$ for arbitrary $\beta \in \omega_1^M$, $\{(\alpha + n, 1)\} \notin G$ furthermore, $n \notin (\theta_\alpha)_G$. Therefore since we have $\forall^\infty n \in \omega\ \alpha + n \notin S$, we have $i_*^S(\theta_\alpha)_G \subset^* \omega \setminus (\theta_\alpha)_G$. For $(\supset^*)$. Let $n \notin (\theta_\alpha)_G$. Note that if $\alpha + n \notin S$, we obtain $i^S(\{(\alpha + n, 0)\}) \notin G$. Since $D_\alpha = \{p \in \mathbb{P}\,; \alpha \in \mathrm{dom}(p)\}$ is dense in $\mathbb{P}$, we obtain that $i^S(\{(\alpha + n, 1)\}) \in G$ if $\alpha + n \notin S$. Therefore, $\forall^\infty n \in \omega\ (n \in \omega \setminus (\theta_\alpha)_G \Rightarrow n \in i^S(\theta_\alpha)_G)$.

Assume that $M[G] \models$ "$\{x\,;\varphi(x)\}$ is a nice choice set" and choose $p \in G$ such that p forces that:

$$\forall x \in \mathcal{P}(\omega)\ (\varphi(x) \Leftrightarrow \neg\varphi(\omega \setminus x)), \mathit{and}$$
$$\forall x, y \in \mathcal{P}(\omega)\ (x =^* y \Rightarrow (\varphi(x) \leftrightarrow \varphi(y))).$$

Working in M, write out $\varphi(x)$ as $\psi(x, \tau_0, \ldots, \tau_{m-1}, \check{b}_0, \ldots, \check{b}_{n-1})$ where τ_i are nice names for a subset of ω and $b_j \in M$. Recall that by Lemma.IV.4.7 in [Kun11] for $q \in \mathbb{P}$, $\sigma \in M^{\mathbb{P}}$ and $i \in \cap\{\mathrm{Aut}\mathbb{P}, M\}$ which is a dense embedding, we have:

$q \models \psi(\sigma, \tau_0, \cdots \tau_{m-1}, \check{b}_0, \cdots, \check{b}_{n-1})$

iff $i(q) \models \psi(i_*(\sigma), i_*(\tau)_0, \cdots, i_*(\tau)_{m-1}, i_*(\check{b}_0), \cdots, i_*(\check{b}_{n-1}))$

and note that $i_*(\check{b}_l) = \check{b}_l$ holds.

Since, τ_i are nice names for a subset ω, the number of antichains which mentioned in some τ_i is countable, we can choose $\alpha \in \omega_1^M$ such that $\mathbb{P}_\alpha = \mathrm{Fn}(\alpha, 2)$ contains them all. Furthermore, if we have $i|_{\mathbb{P}_\alpha} = \mathrm{id}$, $i_*(\tau_i) = \tau_i$ holds. In this situation $q \Vdash \varphi(\sigma) \Leftrightarrow i(q) \Vdash \varphi(i_*(\sigma))$.

Now we conclude the proof. Letting $\sigma = \theta_\alpha$, $q \in G$ with $q \leq p$ and $q \Vdash \varphi(\sigma) \vee q \Vdash \neg\varphi(\sigma)$ and $S = \cup\{\alpha, \mathrm{dom}(q)\}$. Note that the following hold:

- $i(q) = q$,
- $q \Vdash \varphi(\sigma) \Leftrightarrow i(q) \Vdash \varphi(i_*(\sigma))$,
- $q \Vdash i_*^S(\sigma) =^* \omega \setminus \sigma$.

Therefore, we obtain that:

$$q \Vdash \varphi(\sigma) \Leftrightarrow q \Vdash \varphi(i_*(\sigma)) \Leftrightarrow q \Vdash \varphi(\omega \setminus \sigma) \Leftrightarrow q \Vdash \neg\varphi(\sigma),$$

a contradiction. $\square$

Chapter 3

Hereditarily Ordinal Definable Sets

Definition 3.1. For transitive set N, define:

$\mathrm{OD}_N^{\mathbb{R}} = \{a \in N\,; \exists\varphi\exists\vec{b} \in \cap\{N, \cup\{\mathcal{P}(\omega), \mathrm{ON}\}\ (N \models \varphi(\vec{b}, a) \wedge N \models \exists! y\varphi(\vec{b}, y))\}$.

where φ is a formula.

Definition 3.2. $\mathrm{OD}^{\mathbb{R}} = \cup\{\mathrm{OD}_{R(\eta)}^{\mathbb{R}}\,; \eta \in \mathrm{ON}\}$.

Definition 3.3. $\mathrm{HOD}^{\mathbb{R}} = \{x \in \mathrm{OD}^{\mathbb{R}}\,; \mathrm{trcl}(x) \subset \mathrm{OD}^{\mathbb{R}}\}$.

Theorem 3.4.

1. $\mathrm{ON} \subset \mathrm{HOD}^{\mathbb{R}} \subset \mathrm{OD}^{\mathbb{R}}$ and $\mathrm{HOD}^{\mathbb{R}}$ is transitive,
2. For any set a, $a \in \mathrm{HOD}^{\mathbb{R}}$ iff $a \in \mathrm{OD}^{\mathbb{R}} \wedge a \subset \mathrm{HOD}^{\mathbb{R}}$,
3. For an ordinal α, $\cap\{R(\alpha), \mathrm{HOD}^{\mathbb{R}}\} \in \mathrm{HOD}^{\mathbb{R}}$,
4. $\mathrm{HOD}^{\mathbb{R}} \models \mathrm{ZF}$.

Proof. 1. For $\mathrm{ON} \subset \mathrm{HOD}^{\mathbb{R}}$ is immediate by recursion on $\alpha \in \mathrm{ON}$. $\mathrm{HOD}^{\mathbb{R}}$ is transitive is immediate by the facts $y \in \mathrm{trcl}(x)$ and $\mathrm{trcl}(y) \subset \mathrm{trcl}(x)$ for any sets x, y such that $y \in x \in \mathrm{HOD}^{\mathbb{R}}$.

2. Immediate by that $\mathrm{HOD}^{\mathbb{R}}$ is transitive.

3. For an ordinal α, it suffices to show that $\cap\{R(\alpha), \mathrm{HOD}^{\mathbb{R}}\} \in \mathrm{OD}^{\mathbb{R}}$. Since we have $\cap\{R(\alpha), \mathrm{HOD}^{\mathbb{R}}\} \in \mathrm{OD}^{\mathbb{R}}$, we remain to verify that $\mathrm{OD}_V^{\mathbb{R}} \subset$

$\mathrm{OD}^{\mathbb{R}}$. Let $x \in \mathrm{OD}^{\mathbb{R}}_V$ and choose a formula φ and variables $\vec{b} \in \cup\{\mathcal{P}(\omega), \mathrm{ON}\}$ which witness $x \in \mathrm{HOD}^{\mathbb{R}}_V$. By Reflection theorem there is an ordinal $\alpha \in \mathrm{ON}$ such that $R(\alpha) \prec_{\varphi} V$ and $\vec{b} \in R(\alpha)$. Therefore a pair α, φ and $\vec{b}$ witness $x \in \mathrm{OD}^{\mathbb{R}}_{R(\alpha)} \subset \mathrm{OD}^{\mathbb{R}}$.

4. Thanks to Lemma 1.9 we suffices to show that:

(4-a) $\mathrm{HOD}^{\mathbb{R}} \models$ "Comprehension" ,

(4-b) $\forall x \subset \mathrm{HOD}^{\mathbb{R}}\, \exists y \in \mathrm{HOD}^{\mathbb{R}}\ (x \subset y)$.

Let us show the comprehension axiom. Let $a \in \mathrm{HOD}^{\mathbb{R}}$ and φ be a formula. As in the previous, choose α, φ and $\vec{b}$. Then a formula $\psi(\vec{b}, \alpha, w)$:

$$\forall v\ (v \in w \leftrightarrow (\exists z\ ((R(\alpha) \models \varphi(\vec{b}, z)) \wedge v \in z \wedge \varphi(v))))$$

witnesses $\{v \in x\,;\varphi(v)\} \in \mathrm{OD}^{\mathbb{R}}$ and $\{v \in x\,;\varphi(v)\} \subset \mathrm{HOD}^{\mathbb{R}}$, we obtain that $\{v \in x\,;\varphi(v)\} \in \mathrm{HOD}^{\mathbb{R}}$.

Let us show the condition $(4–a)$. For a $x \subset \mathrm{HOD}^{\mathbb{R}}$, it is clear that $\cap\{R(\alpha+1), \mathrm{HOD}^{\mathbb{R}}\}$ is a witnesses that $x \subset \cap\{R(\alpha+1), \mathrm{HOD}^{\mathbb{R}}\}$ for an ordinal α with $x \in R(\alpha)$. □

Theorem 3.5. $\mathrm{HOD}^{\mathbb{R}}$ satisfies AC iff there is a well-order of $\mathcal{P}(\omega)$ that is in $\mathrm{HOD}^{\mathbb{R}}$.

Proof. Since the sufficient condition is manifestly since $\mathcal{P}(\omega) \in \mathrm{HOD}^{\mathbb{R}}$, we remain to show the necessary condition.

Fix a well-ordered set R in $\mathcal{P}(\omega)$. For $x \in \mathrm{HOD}^{\mathbb{R}}$ there is an ordinal α, formula φ and variables $\vec{b} \in \cap\{R(\alpha), \mathrm{ON}\}$ and $\vec{c} \in \cap\{R(\alpha), \mathcal{P}(\omega)\}$ which witness x. We write $\lhd$ for the lexicographic order $\rangle \lhd \langle \in, <_{\Gamma}, \in, \cup\{\in^i\,;i \in \omega\}, \cup\{R^i\,;i \in \omega\}\rangle$. A pair $\langle \alpha, \varphi, \vec{b}, \vec{c}\rangle$ is a good-witnesser if the pair $\langle \alpha, \varphi, |\vec{b}|, \vec{b}, \vec{c}\rangle$ is a $\lhd$-least which witness x. (since the number of variables equals the sum for the size $\vec{b}$ and $\vec{c}$.)

For $x_i \in \mathrm{HOD}^{\mathbb{R}}$, let $\langle \alpha_i, \varphi_i, |\vec{b}_i|, \vec{b}_i, \vec{c}_i\rangle$ be a good-witnesser for x_i for each $i \in 2$. Define an order $x_0 \lhd x_1$ iff

$$\langle \alpha_0, \varphi_0, |\vec{b}_0|, \vec{b}_0, \vec{c}_0\rangle \lhd \langle \alpha_1, \varphi_1, |\vec{b}_1|, \vec{b}_1, \vec{c}_1\rangle$$

Then $\lhd$ well-orders $\mathrm{HOD}^{\mathbb{R}}$. □

Theorem 3.6. Assume there is a ctm for ZFC. There are ctms N_1 and N_2 such that N_1 is a model for ZFC + GCH + "there are no nice choice sets in $\mathrm{HOD}^{\mathbb{R}}$" , and N_2 is a model for ZF + GCH + "there are no nice choice sets" .

Proof. Fix a ctm, M, for ZFC + GCH. Let $\mathbb{P} = \mathrm{Fn}(\omega_1^{N_1}, 2)$ and G a $\mathbb{P}$-generic over M. Then we have $M[G] \models \mathrm{ZFC} + \mathrm{GCH}$. Let $N_1 = M[G]$ and $N_2 = (\mathrm{HOD}^{\mathbb{R}})^{N_1}$, to conclude the proof, it suffices to show that $N_1 \models$ "there are no nice choice set in $\mathrm{HOD}^{\mathbb{R}}$" . Suppose there is a nice choice set $\mathcal{E} \in \mathcal{P}(\omega^{N_1})$. Since $\mathcal{E} \in (\mathrm{HOD}^{\mathbb{R}})^{N_1} \subset (\mathrm{OD}^{\mathbb{R}})^{N_1}$, there are $\alpha \in \mathrm{ON}$, formula φ and a sequence $\vec{b} \in \cap\{R^{N_1}(\alpha), \cup\{\mathrm{ON}, \mathcal{P}^{N_1}(\omega)\}\}$ such that $R^{N_1}(\alpha) \models \varphi(\vec{b}, \mathcal{E})$ and $R^{N_1}(\alpha) \models \exists! y\ \varphi(\vec{b}, y)$. Let ψ be a formula such as:

$$\forall x\ (N_1 \models \psi(x) \Leftrightarrow R^{N_1}(\alpha) \models \exists! y\ \varphi(\vec{b}, y) \wedge y \in x).$$

Then we obtain that $N_1 \models \mathcal{E} = \{x \in \mathcal{P}^{N_1}(\omega)\,;\psi(x)\}$ and we may assume that every name that appeared in $\vec{b}$ are nice names, this contrary to Theorem 2.27. $\square$

Theorem 3.7. If there are no nice choice sets, there are no total orders of $\mathcal{P}(\mathcal{P}(\omega))$.

Proof. We prove by contraposition. Assume that R is a total order of $\mathcal{P}(\mathcal{P}(\omega))$. For $x \in \mathcal{P}(\omega)$ define $x^* = \{y \in \mathcal{P}(\omega)\,; y \subset^* x\} \in \mathcal{P}(\mathcal{P}(\omega))$ and define $\mathcal{E} = \{x \in \mathcal{P}(\omega)\,;\langle x^*, (\omega \setminus x)^*\rangle \in R\}$ and we assert that $\mathcal{E}$ is a nice choice set. To see that $\mathcal{E}$ is a nice choice set. It suffices to check $x^* \neq (\omega \setminus x)^*$ and this is immediate by that x or $\omega \setminus x$ are infinite. $\square$

Bibliography

[Kun11] Kenneth Kunen. *Set theory*, volume 34 of *Studies in Logic (London)*. College Publications, London, 2011.