Countryman Line

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Abstract

In this paper, we shall demonstrate the existence of the countryman line within the framework of ZF set theory. The assertions and proofs presented herein are derived from our university seminar.

Definition 1. The linear order $(L, <_L)$ is a *countryman line* provided that there is countable many sublinear order $(C_n, <_{C_n})$ of $(L \times L, <_{L^2})$ such that $L \times L = \cup\{C_n\,; n \in \omega\}$

Example 2. $\mathbb{R}$ is not a countryman line.

Lemma 3. Every countryman line is an Aronszajn line.

Before we see that a Countryman line exists, we shall make definitions and lemmata.

Definition 4 (Shelah)**.** A ω_1-*laddersystem* is a ω_1-sequence $\langle C_\alpha\,; \alpha \in \omega_1\rangle$ which define recursively as follows:

1. $C_0 = \emptyset$,
2. $C_{\alpha+1} = \{\alpha\}$,
3. $C_\alpha = \{\beta_i \in \alpha\,; i \in \omega\}$ with $\beta_0 = 0$ and $\beta_i \nearrow \alpha$.

Now we note that for $\alpha \in \beta \in \omega_1$, we have $\alpha \in \cap(C_\beta \setminus \alpha) \in \beta$.

The *maximal weight*, $\rho_1 \colon \omega_1 \times \omega_1 \to \omega$, given by,

$$
\begin{aligned}
&\rho_1(\{\alpha, \beta\}) = \cup\{|\cap\{C_\beta, \alpha\}|, \rho_1(\alpha, \cap(C_\beta \backslash \alpha))\} && \text{if } \alpha \in \beta,\\
&\rho_1(\{\alpha, \beta\}) = 0 && \text{if } \alpha = \beta.
\end{aligned}
$$

Lemma 5. For $\alpha \in \omega_1$ define $\rho_1(\cdot, \alpha) \colon \alpha \to \omega_1\,; \beta \mapsto \rho(\{\alpha, \beta\})$. Then $\rho_1(\cdot, \alpha)$ is finite-to-one for arbitary $\alpha \in \omega_1$.

Proof. We prove by induction on $\alpha \in \omega_1$.

The case of the leading stage and the successor stages are clear.

If α is limit. Let $C_\alpha = \{\xi_i\,; i \in \omega\}$ and assume that there is an $n \in \omega$ and infinite $A \subset \alpha$ such that $\forall \xi \in A\ \rho_1(\xi, \alpha) = n$. We distinguish two cases, according to whether $\cup A = \alpha$. Let A be an infinite subset of α, we shall find an element $\xi \in A$ such that $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$. If $\cup A = \alpha$. There is a $\xi \in A$ such that $\xi \ni \xi_{n+1}$ and thus we have $\rho_1(\xi, \alpha) \geq |\cap\{C_\alpha, \xi\}| \geq |\cap\{C_\alpha, \xi_{n+1}\}| = n+1 \ni n$, a contradiction. If $\cup A \in \alpha$. Let $i \in \omega$ be the least ordinal such that $\cap\{A, \xi_i\}$ is infinite. Note that the induction hypothesis asserts that there are infinitely many $\xi \in A$ such that $\xi \in \xi_i$ and $\rho_1(\xi, \xi_i) \ni n$. Moreover, since there are only finitely many $\xi \in A$ such that $\xi \in \xi_{i-1}$, there is a $\xi \in A$ such that $\xi_{i-1} \in \xi \in \xi_i$ and $\rho_1(\xi, \xi_i) \ni n$. Then we obtian that $\rho_1(\xi, \alpha) \geq \rho_1(\xi, \cap(C_\alpha \setminus \xi)) = \rho_1(\xi, \xi_1) \ni n$, a contradiction. □

Lemma 6. For $\alpha \in \beta \in \omega_1$, we obtain that $\rho_1(\{\cdot, \alpha\}) =^* \rho_1(\{\cdot, \beta\})|_\alpha$.

Proof. We shall prove by induction on β and the induction on $\alpha \in \beta$. It suffices to check the case of both α, β are limit ordinals. Since A possess an infinite subset whose order type is ω, we may assume that the order type of A is ω. Let $\gamma = \cup A$, $\delta = \cap(C_\beta \setminus \gamma)$, i be the maximum ordinal such that $\xi_i^\beta \in \gamma$ for $C_\beta = \{\xi_i^\beta\,; i \in \omega\}$ and $n = |\cap\{C_\beta, \gamma\}|$. Then by the previous theorem and the order type of A is ω, the set $B = \{\xi \in A\,; \xi \ni \xi_i^\beta \wedge \rho_1(\xi, \delta) \ni n\}$ is infinite.

Note that there are no ordinals $j \in \omega$ such that $\xi_i \in \xi_j \in \gamma$. For $\xi \in B$, since $\cap\{C_\beta, \xi\} = \cap\{C_\beta, \gamma\}$ and $C_\beta \setminus \xi = C_\beta \setminus \gamma$, we obtain that:

$$\begin{aligned}\rho_1(\xi, \beta) &= \cup\{|\cap\{C_\beta, \xi\}|, \rho_1(\xi, \cap(C_\beta \setminus \xi))\} \\ &= \cup\{|\cap\{C_\beta, \gamma\}|, \rho_1(\xi, \eta)\} \\ &= \cup\{n, \rho_1(\xi, \eta)\} \\ &= \rho_1(\xi, \eta).\ (\text{since } \rho_1(\xi, \eta) \ni n)\end{aligned}$$

If we have $\alpha = \eta$, $\eta \in B$ witnesses a contradiction. If we have $\eta \in \alpha$, since B is infinite, induction hypothesis shows that there is $\xi \in B$ such that $\rho_1(\xi, \eta) = \rho_1(\xi, \alpha)$. Thus we have $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$, a contradiction. □

Theorem 7. For $\alpha \in \beta \in \omega_1$, we define the follows:

$$\begin{aligned}&\Delta(\alpha, \beta) = \cap\{\cup\{\{\xi \in \omega_1\,; \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}, \{\alpha, \beta\}\}\}, \\ &\alpha <_{\rho_1} \beta \text{ provided that } \rho_1(\Delta(\alpha, \beta), \alpha)) \in \rho_1(\Delta(\alpha, \beta), \beta)), \\ &D_{\alpha\beta} = \{\xi \in \alpha + 1\,; \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}, \\ &n_{\alpha\beta} = \cup\{\cup\{\rho_1(\xi, \alpha), \rho_1(\xi, \beta)\}\,; \xi \in D_{\alpha\beta}\}, \\ &F_{\alpha\beta} = \{\xi \in \alpha + 1\,; \rho_1(\xi, \alpha) \leq n_{\alpha\beta} \wedge \rho_1(\xi, \beta) \leq n_{\alpha\beta}\}.\end{aligned}$$

Then we have the following properties:

1. every $D_{\alpha\beta}$ and $F_{\alpha\beta}$ are finite and $D_{\alpha\beta} \subset F_{\alpha\beta}$,
2. Let $\alpha, \beta, \gamma, \delta \in \omega_1$. If we assume the following:
   (a) $\Delta(\alpha, \beta) \in \alpha \in \beta$ and $\Delta(\gamma, \delta) \in \gamma \in \delta$,
   (b) $|F_{\alpha\beta}| = |F_{\gamma\delta}|$,
   (c) $n_{\alpha\beta} = n_{\gamma\delta}$
   (d) $\rho_1(\cdot, \alpha)|_{F_{\alpha\beta}} \approx \rho_1(\cdot, \gamma)|_{F_{\gamma\delta}}$ and $\rho_1(\cdot, \beta)|_{F_{\alpha\beta}} \approx \rho_1(\cdot, \delta)|_{F_{\gamma\delta}}$

   Then $\alpha <_{\rho_1} \gamma$ implies $\beta = \gamma$ or $\beta <_{\rho_1} \delta$.

Proof. We shall show that $\alpha <_{\rho_1} \gamma$ and $\beta \in \gamma$ implies $\beta <_{\rho_1} \delta$.

First we verify $\Delta(\alpha,\gamma) = \Delta(\beta,\delta)$. We distinguish two cases, according to whether $F_{\alpha\beta} = F_{\gamma\delta}$.

If $F_{\alpha\beta} = F_{\gamma\delta}$. To see $(\leq)$, it suffices to show that $\forall\xi \in \Delta(\alpha,\gamma)$ $(\rho_1(\xi,\beta) = \rho_1(\xi,\delta) \wedge \xi \in \beta \wedge \xi \in \delta)$. To see $\forall\xi \in \Delta(\alpha,\gamma)$ $(\rho_1(\xi,\beta) = \rho_1(\xi,\delta) \wedge \xi \in \beta \wedge \xi \in \delta)$, let $\xi \in \Delta(\alpha,\gamma)$. Note that we have $\rho_1(\xi,\alpha) = \rho_1(\xi,\gamma)$. Case 1. If we have $\rho_1(\xi,\alpha) = \rho_1(\xi,\beta)$ and $\rho_1(\xi,\delta) = \rho_1(\xi,\gamma)$, then we have $\rho_1(\xi,\beta) = \rho_1(\xi,\delta)$. Case 2. If we have $\rho_1(\xi,\alpha) \neq \rho_1(\xi,\beta)$ or $\rho_1(\xi,\delta) \neq \rho_1(\xi,\gamma)$, we have $\xi \in D_{\alpha\beta} \subset F_{\alpha\beta}$ or $\xi \in D_{\gamma\delta} \subset F_{\gamma\delta}$, so we obtain that $\xi \in F_{\alpha\beta} = F_{\gamma\delta}$. This shows that $\rho_1(\xi,\beta) = \rho_1(\xi,\delta)$. To see that $\Delta(\alpha,\gamma) \geq \Delta(\beta,\delta)$, it suffices to show that $\rho_1(\Delta(\alpha,\gamma),\beta) \neq \rho_1(\Delta(\alpha,\gamma),\delta)$. Note that we have $\rho_1(\Delta(\alpha,\gamma),\alpha) \neq \rho_1(\Delta(\alpha,\gamma),\gamma))$. Since we have the equality $\rho_1(\cdot,\alpha)|_{F_{\alpha\beta}} = \rho_1(\cdot,\gamma)|_{F_{\gamma\delta}}$, we have $\Delta(\alpha,\gamma) \notin F_{\alpha\beta} = F_{\gamma\delta}$, moreover, $\Delta(\alpha,\gamma) \notin D_{\alpha\beta} = D_{\gamma\delta}$. Hence we have $\rho_1(\Delta(\alpha,\gamma),\alpha) = \rho_1(\Delta(\alpha,\gamma),\beta)$ and $\rho_1(\Delta(\alpha,\gamma),\gamma) = \rho_1(\Delta(\alpha,\gamma),\delta)$, furthermore we have $\rho_1(\Delta(\alpha,\gamma),\beta) \neq \rho_1(\Delta(\alpha,\gamma),\delta)$.

If $F_{\alpha\beta} \neq F_{\gamma\delta}$. Let $\eta = \cap\{\cup\{F_{\alpha\beta}, F_{\gamma\delta}\}, \cap\{F_{\alpha\beta}, F_{\gamma\delta}\}\}$, then we have $\eta \leq \alpha,\beta,\gamma,\delta$. Assume that $\eta \in F_{\alpha\beta} \setminus F_{\gamma\delta}$ (the other case being handled in the obvious dual manner). $\eta \in F_{\alpha\beta}$ assert that $\rho_1(\eta,\alpha) \leq n_{\alpha\beta}$ and $\rho_1(\eta,\beta) \leq n_{\alpha\beta}$. And $\eta \notin F_{\gamma\delta}$ assert that $n_{\gamma\delta} \in \rho_1(\eta,\gamma) = \rho_1(\eta,\delta)$, we have $\rho_1(\eta,\alpha) \in \rho_1(\eta,\gamma)$ and $\rho_1(\eta,\beta) \in \rho_1(\eta,\delta)$, moreover, $\Delta(\alpha,\gamma) \leq \eta$ and $\Delta(\beta,\delta) \leq \eta$. This shows that, we suffice to verify that $\cap\{\Delta(\alpha,\gamma),\eta\} = \cap\{\Delta(\beta,\delta),\eta\}$. Since we have $\rho_1(\cdot,\alpha)|_{\cap\{F_{\alpha\beta},\eta\}} = \rho_1(\cdot,\gamma)|_{\cap\{F_{\gamma\delta},\eta\}}$ and $\rho_1(\cdot,\beta)|_{\cap\{F_{\alpha\beta},\eta\}} = \rho_1(\cdot,\delta)|_{\cap\{F_{\gamma\delta},\eta\}}$, by the mimik the case of $F_{\alpha\beta} = F_{\gamma\delta}$, we can verify the equality. Therefore, we have $\Delta(\alpha,\gamma) = \Delta(\beta,\delta)$ in generally.

Finally we shall show that $\rho_1(\Delta(\beta,\delta),\beta) \in \rho_1(\Delta(\beta,\delta),\delta)$. Let $\xi = \Delta(\alpha,\gamma) = \Delta(\beta,\delta)$. Note that since $\alpha <_L \gamma$, $\rho_1(\xi,\alpha) \in \rho_1(\xi,\gamma)$ holds. We distinguish two cases, according to whether $F_{\alpha\beta} = F_{\gamma\delta}$, again. If $F_{\alpha\beta} = F_{\gamma\delta}$ holds. Since $\rho_1(\xi,\alpha) = \rho_1(\xi,\gamma)$ and $\rho_1(\xi,\beta) = \rho_1(\xi,\delta)$, we obtain the desired equality. If $F_{\alpha\beta} \neq F_{\gamma\delta}$. By letting η as we have seen above, we obtian $\xi \in \eta$ and $\cap\{F_{\alpha\beta},\eta\} = \cap\{F_{\gamma\delta},\eta\}$, we obtain that $\rho_1(\xi,\alpha) = \rho_1(\xi,\gamma)$ and $\rho_1(\xi,\beta) = \rho_1(\xi,\delta)$. Now we complete the proof.

□

This theorem assert that if there is an uncountable set A such that for every $\{\alpha,\beta\} \in [A]^2$, $\Delta(\alpha,\beta) \in \cap\{\alpha,\beta\}$. Then for $n \in \omega$ and p, $q \in {}^{<\omega}\omega$ by letting:

$$S^n_{p,q} = \left\{ \langle\alpha,\beta\rangle \in A \times A\,; \begin{array}{c} \alpha \in \beta \wedge n_{\alpha\beta} = n \\ \wedge\, \rho_1(\cdot,\alpha)|_{F_{\alpha\beta}} \approx p \wedge \rho_1(\cdot,\beta)|_{F_{\alpha\beta}} \approx q \end{array} \right\}$$

$S^n_{p,q}$ be a linear set. Moreover, by letting $Q^n_{p,q} = \{\langle\beta,\alpha\rangle\,;\langle\alpha,\beta\rangle \in S^n_{p,q}\}$ and $P = \{\langle\alpha,\alpha\rangle\,; \alpha \in A\}$, then we obtain that

$$A \times A = \cup\{\cup\{S^n_{p,q}, Q^n_{p,q}\,; p,\, q \in {}^{<\omega}\omega \wedge n \in \omega\}, P\}$$

Therefore, A is a Countryman line. To see that a Countryman line exists, we remain to show that an A exists.

Lemma 8 (Continued fraction)**.** Define funcion $\mathcal{C}\colon {}^{<\omega}\omega \to \mathbb{R}$ which assigns to f the real:

$$-\cfrac{1}{f(0)+1+\cfrac{1}{f(1)+1+\cfrac{1}{f(2)+1+\cfrac{1}{\ddots+\cfrac{1}{f(|f|-1)+1}}}}}$$

and define $\mathcal{C}\colon {}^{\omega}\omega \to \mathbb{R}$ which assigns to f the limit point for $\langle \mathcal{C}(f|_n)\,;n\in\omega\rangle$. Then, $\mathcal{C}\colon {}^{\leq\omega}\omega \to \mathbb{R}$ preseve the order.

Proof. Since for positive reals a and c, we have $\frac{1}{a+c} < \frac{1}{a}$, $\mathcal{C}\colon {}^{<\omega}\omega \to \mathbb{R}$ preseves order. To see the $\mathcal{C}\colon {}^{\omega}\omega \to \mathbb{R}$ is a function. For $f \in {}^{\omega}\omega$, since $\mathcal{C}(f|_n) \leq \mathcal{C}(f|_{n+1}) < 0$ for each $n\in\omega$, $\mathcal{C}(f)\in\mathbb{R}$ and preseves order. $\square$

Lemma 9. There is an uncountable set $A \subset \omega_1$ such that for $\alpha,\beta\in A$, $\alpha\in\beta$ implies $\rho_1(\cdot,\alpha)\not\subset\rho_1(\cdot,\beta)$. Equivalently, $\forall\{\alpha,\beta\}\in[A]^2\ \Delta(\alpha,\beta)\in\cap\{\alpha,\beta\}$.

Proof. Define $\varphi\colon \omega_1 \to {}^{\omega}\omega\,;\alpha\mapsto(f_\alpha\colon n\mapsto|\{\xi\in\alpha\,;\rho_1(\xi,\alpha)=n\}|)$. Then for $\alpha,\beta\in\omega_1$ if we have $\rho_1(\cdot,\alpha)\subset\rho_1(\cdot,\beta)$, we have $f_\alpha\leq f_\beta$ for pointwise and we have $f_\alpha<f_\beta$ if $\alpha\in\beta$.

For $\alpha\in\omega_1$ if there is an ordinal δ such that $\rho_1(\cdot,\alpha)\subset\rho_1(\cdot,\delta)$ and $\alpha\in\delta$ then, choose the δ_α such that the function $\rho_1(\cdot,\delta)$ is minimal among the subset relation. If such an ordinal δ_α exists choose a rational $q_\alpha\in\mathbb{Q}$ such that $(\mathcal{C}\circ\varphi)(\alpha) <_{\mathbb{R}} q_\alpha <_{\mathbb{R}} (\mathcal{C}\circ\varphi)(\delta_\alpha)$. We distinguish two cases, according to whether such an α exists for uncountably many.

If such an α exists for uncountable many, Pigeonhole principle asserts that there is a rational $q\in\mathbb{Q}$ such that $q=q_\alpha$ for uncountable many $\alpha\in\omega_1$ and let A be the set of such ordinals. To see that A is the desired set, suppose that there are $\langle\alpha,\beta\rangle\in\cap\{A\times A,\in\}$ such that $\rho_1(\cdot,\alpha)\subset\rho_1(\cdot,\beta)$ holds. The minimality of δ_α asserts that $\rho_1(\cdot,\alpha)\subset\rho_1(\cdot,\delta_\alpha)\subset\rho_1(\cdot,\beta)$ and since $\alpha\in\delta_\alpha$. We obtain that $q_\alpha <_{\mathbb{R}} (\mathcal{C}\circ\varphi)(\delta_\alpha) \leq_{\mathbb{R}} (\mathcal{C}\circ\varphi)(\beta) <_{\mathbb{R}} q_\beta$, a contradiction.

If such an α exists for only countably many, let $\beta\in\omega_1$ such that for arbitrary $\alpha\in\omega_1$ with $\alpha\ni\beta$, α does not possess a δ_α. Define $A=\{\alpha\in\omega_1\,;\alpha\ni\beta\}$ and to conclude that A is the desired set, let $\langle\alpha,\beta\rangle\in\cap\{A\times A,\in\}$ and we shall show that $\rho_1(\cdot,\alpha)\not\subset\rho_1(\cdot,\beta)$. This relation is immediate from that δ_α does not exist. $\square$