Definition 1. The linear order $(L, <_L)$ is a *countryman line* provided that there is coutable many sublinear order $(C_n, <_{C_n})$ of $(L \times L, <_{L^2})$ such that $L \times L = \cup\{C_n\,; n \in \omega\}$

Definition 2 (Shelah)**.** A ω_1-*laddersystem* is a ω_1-sequence $\langle C_\alpha\,; \alpha \in \omega_1\rangle$ which define recursively as follows:

1. $C_0 = \emptyset$,
2. $C_{\alpha+1} = \{\alpha\}$,
3. $C_\alpha = \{\beta_i \in \alpha\,; i \in \omega\}$ with $\beta_i \nearrow \alpha$.

Now we note that for $\alpha \in \beta \in \omega_1$, we have $\alpha \in \cap(C_\beta \setminus \alpha) \in \beta$.

4. The *maximal weight*, $\rho_1\colon [\omega_1]^2 \to \omega$, given by,
$$\rho_1(\{\alpha, \beta\}) = \cup\{|\cap\{C_\beta, \alpha\}|, \rho_1(\alpha, \cap(C_\beta \setminus \alpha)\}.$$

Lemma 3. For $\alpha \in \omega_1$ define $\rho_1(\cdot, \alpha)\colon \alpha \to \omega_1\,; \beta \mapsto \rho(\{\alpha, \beta\})$. Then $\rho_1(\cdot, \alpha)$ is finite to one for arbitary $\alpha \in \omega_1$.

Lemma 4. For $\alpha \in \beta \in \omega_1$, we obtain that $\rho_1(\{\cdot, \alpha\}) =^* \rho_1(\{\cdot, \beta\})$.

Theorem 5. For $\alpha \in \beta \in \omega_1$, we define the follows:

$$\begin{aligned}
&\Delta(\alpha, \beta) = \cap\{\cup\{\{\xi \in \omega_1\,; \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}, \{\alpha, \beta\}\}\},\\
&\alpha <_{\rho_1} \beta \text{ provided that } \rho_1(\Delta(\alpha, \beta), \alpha)) \in \rho_1(\Delta(\alpha, \beta), \beta)),\\
&D_{\alpha\beta} = \{\xi \in \alpha + 1\,; \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\},\\
&n_{\alpha\beta} = \cup\{\rho_1(\xi, \alpha), \rho_1(\xi, \beta)\,; \xi \in D_{\alpha\beta}\},\\
&F_{\alpha\beta} = \{\xi \in \alpha + 1\,; \rho_1(\xi, \alpha) \leq n_{\alpha\beta} \wedge \rho_1(\xi, \beta) \leq n_{\alpha\beta}\}.
\end{aligned}$$

Then we have the following properties:

1. every $D_{\alpha\beta}$ and $F_{\alpha\beta}$ are finite and $D_{\alpha\beta} \subset F_{\alpha\beta}$,
2. Let $\alpha, \beta, \gamma, \delta \in \omega_1$. If we assume the follows:
   (a) $\Delta(\alpha, \beta) \in \alpha \in \beta$ and $\Delta(\gamma, \delta) \in \gamma \in \delta$,
   (b) $|F_{\alpha\beta}| = |F_{\gamma\delta}|$,
   (c) $n_{\alpha\beta} = n_{\gamma\delta}$
   (d) $\rho_1(\cdot, \alpha)|_{F_{\alpha\beta}} \approx \rho_1(\cdot, \gamma)|_{F_{\gamma\delta}}$ and $\rho_1(\cdot, \beta)|_{F_{\alpha\beta}} \approx \rho_1(\cdot, \delta)|_{F_{\gamma\delta}}$

   Then $\alpha <_{\rho_1} \gamma$ implies $\beta = \gamma$ or $\beta <_{\rho_1} \delta$.

Proof. We shall show that $\alpha <_{\rho_1} \gamma$ and $\beta \in \gamma$ implies $\beta <_{\rho_1} \delta$.

First we verify $\Delta(\alpha, \gamma) = \Delta(\beta, \delta)$. We distingwish two cases, according to whether $F_{\alpha\beta} = F_{\gamma\delta}$.

If $F_{\alpha\beta} = F_{\gamma\delta}$. To see $(\leq)$, it suffices to show that $\forall\xi \in \Delta(\alpha, \gamma)$ $(\rho_1(\xi, \beta) = \rho_1(\xi, \delta) \wedge \xi \in \beta \wedge \xi \in \delta)$. To see $\forall\xi \in \Delta(\alpha, \gamma)$ $(\rho_1(\xi, \beta) = \rho_1(\xi, \delta) \wedge \xi \in \beta \wedge \xi \in \delta)$, let $\xi \in \Delta(\alpha, \gamma)$. Note that we have $\rho_1(\xi, \alpha) = \rho_1(\xi, \gamma)$. Case 1. If we have

$\rho_1(\xi,\alpha) = \rho_1(\xi,\beta)$ and $\rho_1(\xi,\delta) = \rho_1(\xi,\gamma)$, then we have $\rho_1(\xi,\beta) = \rho_1(\xi,\delta)$. Case 2. If we have $\rho_1(\xi,\alpha) \neq \rho_1(\xi,\beta)$ or $\rho_1(\xi,\delta) \neq \rho_1(\xi,\gamma)$, we have $\xi \in D_{\alpha\beta} \subset F_{\alpha\beta}$ or $\xi \in D_{\gamma\delta} \subset F_{\gamma\delta}$, so we obtain that $\xi \in F_{\alpha\beta} = F_{\gamma\delta}$. This shows that $\rho_1(\xi,\beta) = \rho_1(\xi,\delta)$. To see that $\Delta(\alpha,\gamma) \geq \Delta(\beta,\delta)$, it suffices to show that $\rho_1(\Delta(\alpha,\gamma),\beta) \neq \rho_1(\Delta(\alpha,\gamma),\delta)$. Note that we have $\rho_1(\Delta(\alpha,\gamma),\alpha) \neq \rho_1(\Delta(\alpha,\gamma),\gamma))$. Since we have the equality $\rho_1(\cdot,\alpha)|_{F_{\alpha\beta}} = \rho_1(\cdot,\gamma)|_{F_{\gamma\delta}}$, we have $\Delta(\alpha,\gamma) \notin F_{\alpha\beta} = F_{\gamma\delta}$, moreover, $\Delta(\alpha,\gamma) \notin D_{\alpha\beta} = D_{\gamma\delta}$. Hence we have $\rho_1(\Delta(\alpha,\gamma),\alpha) = \rho_1(\Delta(\alpha,\gamma),\beta)$ and $\rho_1(\Delta(\alpha,\gamma),\gamma) = \rho_1(\Delta(\alpha,\gamma),\delta)$, furthermore we have $\rho_1(\Delta(\alpha,\gamma),\beta) \neq \rho_1(\Delta(\alpha,\gamma),\delta)$.

If $F_{\alpha\beta} \neq F_{\gamma\delta}$. Let $\eta = \cap\{\cup\{F_{\alpha\beta}, F_{\gamma\delta}\}, \cap\{F_{\alpha\beta}, F_{\gamma\delta}\}\}$, then we have $\eta \leq \alpha,\beta,\gamma,\delta$. Assume that $\eta \in F_{\alpha\beta} \setminus F_{\gamma\delta}$ (the other case being handled in the obvious dual manner). $\eta \in F_{\alpha\beta}$ assert that $\rho_1(\eta,\alpha) \leq n_{\alpha\beta}$ and $\rho_1(\eta,\beta) \leq n_{\alpha\beta}$. And $\eta \notin F_{\gamma\delta}$ assert that $n_{\gamma\delta} \in \rho_1(\eta,\gamma) = \rho_1(\eta,\delta)$, we have $\rho_1(\eta,\alpha) \in \rho_1(\eta,\gamma)$ and $\rho_1(\eta,\beta) \in \rho_1(\eta,\delta)$, moreover, $\Delta(\alpha,\gamma) \leq \eta$ and $\Delta(\beta,\delta) \leq \eta$. This shows that, we suffice to verify that $\cap\{\Delta(\alpha,\gamma),\eta\} = \cap\{\Delta(\beta,\delta),\eta\}$. Since we have $\rho_1(\cdot,\alpha)|_{\cap\{F_{\alpha\beta},\eta\}} = \rho_1(\cdot,\gamma)|_{\cap\{F_{\gamma\delta},\eta\}}$ and $\rho_1(\cdot,\beta)|_{\cap\{F_{\alpha\beta},\eta\}} = \rho_1(\cdot,\delta)|_{\cap\{F_{\gamma\delta},\eta\}}$, by the mimik the case of $F_{\alpha\beta} = F_{\gamma\delta}$, we can verify the equality. Therefore, we have $\Delta(\alpha,\gamma) = \Delta(\beta,\delta)$ in generally.

Finally we shall show that $\rho_1(\Delta(\beta,\delta),\beta) \in \rho_1(\Delta(\beta,\delta),\delta)$. Let $\xi = \Delta(\alpha,\gamma) = \Delta(\beta,\delta)$. Note that since $\alpha <_L \gamma$, $\rho_1(\xi,\alpha) \in \rho_1(\xi,\gamma)$ holds. We distinguish two cases, according to whether $F_{\alpha\beta} = F_{\gamma\delta}$, again. If $F_{\alpha\beta} = F_{\gamma\delta}$ holds. Since $\rho_1(\xi,\alpha) = \rho_1(\xi,\gamma)$ and $\rho_1(\xi,\beta) = \rho_1(\xi,\delta)$, we obtain the desired equality. If $F_{\alpha\beta} \neq F_{\gamma\delta}$. By letting η as we have seen above, we obtian $\xi \in \eta$ and $\cap\{F_{\alpha\beta},\eta\} = \cap\{F_{\gamma\delta},\eta\}$, we obtain that $\rho_1(\xi,\alpha) = \rho_1(\xi,\gamma)$ and $\rho_1(\xi,\beta) = \rho_1(\xi,\delta)$. Now we comple the proof.

□

This theorem assert that if there is an uncountable set A such that for every $\{\alpha,\beta\} \in [A]^2$, $\Delta(\alpha,\beta) \in \cap\{\alpha,\beta\}$. Then for $n \in \omega$ and p, $q \in {}^{<\omega}\omega$ by letting:

$$S^n_{p,q} = \left\{ \langle\alpha,\beta\rangle \in A \times A\,;\ \begin{array}{c} \alpha \in \beta \wedge n_{\alpha\beta} = n \\ \wedge\, \rho_1(\cdot,\alpha)|_{F_{\alpha\beta}} \approx p \wedge \rho_1(\cdot,\beta)|_{F_{\alpha\beta}} \approx q \end{array} \right\}$$

be a liner set. Moreover, by letting $Q^n_{p,q} = \{\langle\beta,\alpha\rangle\,;\langle\alpha,\beta\rangle \in S^n_{p,q}\}$ and $P = \{\langle\alpha,\alpha\rangle\,;\alpha \in A\}$, then we obtain that

$$A \times A = \cup\{\cup\{S^n_{p,q}, Q^n_{p,q}\,; p,\, q \in {}^{<\omega}\omega \wedge n \in \omega\}, P\}$$

Therefore, A is a Countryman line. To see that a Countryman line exists, we remain to show that an A exists.

Lemma 6 (Continued fraction)**.** Define funcion $\mathcal{C}\colon {}^{<\omega}\omega \to \mathbb{R}$ which assgins to f the real:

$$-\cfrac{1}{f(0)+1+\cfrac{1}{f(1)+1+\cfrac{1}{f(2)+1+\cfrac{1}{\ddots + \cfrac{1}{f(|f|-1)+1}}}}}$$

and define $\mathcal{C}\colon {}^{\omega}\omega \to \mathbb{R}$ which assgins to f the limit point for $\langle \mathcal{C}(f|_n)\,; n \in \omega\rangle$. Then, $\mathcal{C}\colon {}^{\leq\omega}\omega \to \mathbb{R}$ preseve the order.

Proof. Since for positive reals a and c, we have $\frac{1}{a+c} < \frac{1}{a}$, $\mathcal{C}\colon {}^{<\omega}\omega \to \mathbb{R}$ preseves order. To see the $\mathcal{C}\colon {}^{\omega}\omega \to \mathbb{R}$ is a function. For $f \in {}^{\omega}\omega$, since $\mathcal{C}(f|_n) \leq \mathcal{C}(f|_{n+1}) < 0$ for each $n \in \omega$, $\mathcal{C}(f) \in \mathbb{R}$ and preseves order. $\square$

Lemma 7. There is an uncountable set $A \subset \omega_1$ such that for α, $\beta \in A$, $\alpha \in \beta$ implies $\rho_1(\cdot, \alpha) \not\subset \rho_1(\cdot, \beta)$. Equivalently, $\forall\{\alpha, \beta\} \in [A]^2\ \Delta(\alpha, \beta) \in \cap\{\alpha, \beta\}$.

Proof. Define $\varphi\colon \omega_1 \to {}^{\omega}\omega\,; \alpha \mapsto (f_\alpha\colon n \mapsto |\{\xi \in \alpha\,; \rho_1(\xi, \alpha) = n\}|)$. Then for α, $\beta \in \omega_1$ if we have $\rho_1(\cdot, \alpha) \subset \rho_1(\cdot, \beta)$, we have $f_\alpha \leq f_\beta$ for pointwise and we have $f_\alpha < f_\beta$ if $\alpha \in \beta$.

For $\alpha \in \omega_1$ if there is an ordinal δ such that $\rho_1(\cdot, \alpha) \subset \rho_1(\cdot, \delta)$ and $\alpha \in \delta$ then, choose the δ_α such that the funtion $\rho_1(\cdot, \delta)$ is minimal among the subset relation. If such an ordinal δ_α exists choose a rational $q_\alpha \in \mathbb{Q}$ such that $(\mathcal{C} \circ \varphi)(\alpha) <_{\mathbb{R}} q_\alpha <_{\mathbb{R}} (\mathcal{C} \circ \varphi)(\delta_\alpha)$. We distinguish two cases, according wherther such an α exists for uncounbtably many.

If such an α exists for uncountable many, Pigeonhole principle assert that there is a rational $q \in \mathbb{Q}$ such that $q = q_\alpha$ for uncountable many $\alpha \in \omega_1$ and let A be the set of such ordinals. To see that A is the desired set, suppose that there are $\langle \alpha, \beta \rangle \in \cap\{A \times A, \in\}$ such that $\rho_1(\cdot, \alpha) \subset \rho_1(\cdot, \beta)$ holds. By the minimality of δ_α asserts that $\rho_1(\cdot, \alpha) \subset \rho_1(\cdot, \delta_\alpha) \subset \rho_1(\cdot, \beta)$ and since $\alpha \in \delta_\alpha$. We obtain that $q_\alpha <_{\mathbb{R}} (\mathcal{C} \circ \varphi)(\delta_\alpha) \leq_{\mathbb{R}} (\mathcal{C} \circ \varphi)(\beta) <_{\mathbb{R}} q_\beta$, a contradiction.

If such an α exists for only countably many, let $\beta \in \omega_1$ such that for arbitrary $\alpha \in \omega_1$ with $\alpha \ni \beta$, α does not possess a δ_α. Define $A = \{\alpha \in \omega_1\,; \alpha \ni \beta\}$ and to conclude that A is the desired set, let $\langle \alpha, \beta \rangle \in \cap\{A \times A, \in\}$ and we shall show that $\rho_1(\cdot, \alpha) \not\subset \rho_1(\cdot, \beta)$. This relation is immediate by that δ_α does not exist. $\square$