Contents

I Basic Set Theory

I.1 Club Sets

Definition I.1.1 ([Jec03])**.** For a regular cardinal κ and a subset $C \subset \kappa$, C is *unbounded* provided that $\cup C = \kappa$ and C is *closed* provided that for any limit $\gamma \in \kappa$ if $\cup(C \cap \gamma) = \gamma$ then $\gamma \in C$. A subset C is a *club subset of* κ provided that C is closed and unbounded subset of κ. A susbet $S \subset \kappa$ is a *stationary subset of* κ provided that S meets every club subset of κ.

Note that a subset C is closed is equivalent to that for any subset $S \subset \kappa$ we have $\cup S \in C$ when $S \subset C$ and the size of stationary subset is κ since for a strictly increasing convegence sequence $\langle \alpha_\xi \in \kappa\,;\xi \in \kappa\rangle \nearrow \kappa$, $\langle \alpha_\xi\,;\eta \in \xi \wedge \xi \in \kappa\rangle$ is club for any $\eta \in \kappa$.

Theorem I.1.2 ([Kun11])**.** For a regular uncountable cardinal κ, we have the following:

1. The intersection of less than κ club subsets of κ is club,

2. The *diagonal intersection* of a κ-sequence of club subsets of κ

$$\Delta\langle C_\alpha\,;\alpha \in \kappa\rangle = \{\xi \in \kappa \setminus 1\,; \xi \in \cap\{C_\alpha\,;\alpha \in \xi\}\}$$

   is club.

Proof. 1. We shall prove the following by induction on $\beta_0 \in \kappa \setminus 1$:

For κ-club sets $\langle C_\xi\,;\xi \in \beta_0\rangle$, the intersection $C = \cap\{C_\xi\,;\xi \in \beta_0\}$ is club.

The case of the leading stage and the successor stages are clear. Let β_0 be limit. Define $D_\xi = \cap\{C_\eta\,;\eta \in \xi\}$ for $\xi \in \beta_0$ and note that by induction hypothesis every D_ξ is club and we obtain that the decreasing club sets $\langle D_\xi\,;\xi \in \beta_0\rangle$ such that the intersection, D, equals $\cap\{C_\xi\,;\xi \in \beta_0\}$. Note that manifestly D is closed. We shall show that D is unbounded in κ. Fix a $\gamma \in \kappa$ and define $\gamma_\xi \in D_\xi$, recursively such that:

- $\gamma_0 \in D_0$ such that $\gamma \in \gamma_0$,
- $\gamma_\xi \in D_\xi$ such that $\gamma_\eta \in \gamma_\xi$ for each $\eta \in \xi$.

Then we obtian an increasing sequence $x = \langle\gamma_\xi\,;\xi \in \beta_0\rangle$ in κ with limit $\beta \in \kappa$. Since for every $\xi \in \beta_0$ x is eventually in D_ξ so does β. Therefore, β witnesses that D is unbounded in κ.

2. To see that $\Delta\langle C_\alpha\,;\alpha \in \kappa\rangle$ is closed. Let $\langle\beta_\xi\,;\xi \in \beta_0\rangle \nearrow \beta$ be a sequence with limit point $\beta \in \kappa$. We shall show that $\forall\eta \in \beta\ (\beta \in C_\eta)$. For any $\eta \in \beta$ there is a $\xi_0 \in \beta_0$ such that $\eta \in \beta_\xi$ for any $\xi \ni \xi_0$. This shows that the sequence eventually contained in C_η so does the limit point β.

To see that $\Delta\langle C_\alpha\,;\alpha \in \kappa\rangle$ is unbounded. Fix $\delta_0 \in \kappa \setminus 1$ and define δ_n, $n \in \omega$, recursively such that:

$$\delta_{n+1} \ni \delta_n \text{ and } \delta_{n+1} \in \cap\{C_\alpha\,;\alpha \in \delta_n\} \text{ for any } n \in \omega.$$

Then for the limit point δ, since we have $\delta \in \cap\{C_\alpha\,;\alpha \in \delta\}$, $\delta \in \Delta\langle C_\alpha\,;\alpha \in \kappa\rangle$. This shows that $\Delta\langle C_\alpha\,;\alpha \in \kappa\rangle$ is unbounded. □

Definition I.1.3 ([Jec03])**.** For a set of ordinals A, a funcion $f\colon A \to A$ is *a regressive funcion* provided that $f(\alpha) \in \alpha$ for any $\alpha \in A$.

Theorem I.1.4 (**Hodor Theorem**, [Jec03])**.** Let f be a regressive funcion on a stationary set $S \subset \kappa$. Then there is a stationary set $T \subset S$ and $\gamma \in \kappa$ such that $f(\alpha) = \gamma$ for any $\alpha \in T$.

Proof. Assume that " $\{\alpha \in S\,; f(\alpha) = \gamma\}$ is not stationary." for all $\gamma \in \kappa$. For each $\gamma \in \kappa$ choose a club set C_γ which does not meet $\{\alpha \in S\,; f(\alpha) = \gamma\}$. Since $C = \Delta\{C_\gamma\,;\gamma \in \kappa\}$ is a club set, there is $\alpha \in S \cap C$ and for $\gamma \in \alpha$ we obtain that $\alpha \in C_\gamma$. This shows that $f(\alpha) \neq \gamma$ for any $\gamma \in \alpha$, this contrary to that $f(\alpha) \in \alpha$. □

Theorem I.1.5 (**Delta System Lemma**, [Kun11]). Let λ and κ be regular cardinals with $\omega \leq \lambda < \kappa$ and assume that $\forall \theta \in \kappa\ (\theta^{<\lambda} < \kappa)$. Then for a κ size family $\mathcal{A}$ with $|A| < \lambda$ for each $A \in \mathcal{A}$, there is a delta system $\mathcal{B} \in [\mathcal{A}]^\kappa$. Moreover,

Proof. III.6.15 □

II Forcing Notions

II.1 Definition for the Forcing

Definition II.1.1. A *forcing notion* $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{1})$ is a poset $(\mathbb{P}, \leq_\mathbb{P})$ with the maximal element $\mathbb{1}$. For $p \in \mathbb{P}$ an extension of p is $q \in \mathbb{P}$ satisfing $q \leq_\mathbb{P} p$.

1. A set $D \subset \mathbb{P}$ is a *dense subset below* $p \in \mathbb{P}$ provided that for every extension $q \leq_\mathbb{P} p$ there is an extension for p in D and we simply said D is *dense* in $\mathbb{P}$ if D is dense below $\mathbb{1}$.

2. A set $G \subset \mathbb{P}$ is a $(V, \mathbb{P})$*-generic filter* provided that:

   (a) $G \subset \mathbb{P}$, $G \in V$,

   (b) $\forall p, q \in G \exists r \in G\ r \leq_\mathbb{P} p, q$,

   (c) $\forall p \in G \forall q \in \mathbb{P}\ (p \leq_\mathbb{P} q \implies q \in G)$,

   (d) Every dense subset $D \subset \mathbb{P}$, D meets G if $D \in V$.

We will usually say that G is $\mathbb{P}$-generic and write $\leq$ for the $\leq_\mathbb{P}$ when there is no danger of confusing.

Definition II.1.2. Let $\mathbb{P}$ be a forcing notion and E be a subset of $\mathbb{P}$.

1. p and q are *compatible*, $p \not\perp q$, provided that there is a common extension $r \leq p$, q, for any conditions p, $q \in \mathbb{P}$,

2. p and q are *incompatible*, $p \perp q$, provided that $\neg(p \not\perp q)$,

3. E is *predense* provided that for every condition $p \in \mathbb{P}$ there is $q \in E$ such that $p \not\perp q$,

4. E is *open* provided that for $p \leq q$ if $p \in E$ then $q \in E$,

5. E is *dense open* provided that E is dense and open,

6. E is *an antichain* provided that there are no distinct $p, q \in E$ such that $p \not\perp q$,

7. E is *a maximal antichain* provided that E is an antichain and maxinum among the subset relation,

8. For a conditon $p \in \mathbb{P}$, $p \perp E$ is an abbreviation for $\forall q \in E\ p \perp q$.

Theorem II.1.3 ([Kun11])**.** III.3.60

Definition II.1.4. Let $\mathbb{P}$ be a forcing notion.

1. For names τ, σ, θ in $V^{\mathbb{P}}$, define, recursively,:

   (1-a) $p \Vdash \tau = \sigma$ provided that
   $\forall \theta \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\sigma) \forall q \leq p\ (q \Vdash \theta \in \tau \Leftrightarrow q \Vdash \theta \in \sigma)$,

   (1-b) $p \Vdash \tau \in \sigma$ provided that
   $\{q \leq p\,; \exists \langle \theta, r\rangle \in \sigma\ (q \leq r \wedge q \Vdash \tau = \theta)\}$ is dense below p.

2. For formulas $\varphi, \psi \in \mathcal{FL}_{\mathbb{P}}$,

   (2-a) $p \Vdash \varphi \wedge \psi$ provided that $p \Vdash \varphi$ and $p \Vdash \psi$,

   (2-b) $p \Vdash \neg\varphi$ provided that $\neg \exists q \leq p\ (q \Vdash \varphi)$,

   (2-c) $p \Vdash \forall x \varphi(x)$ provided that $p \Vdash \varphi(\tau)$ for any $\tau \in V^{\mathbb{P}}$,

   (2-d) $p \Vdash \exists x \varphi(x)$ provided that $\{q \leq p\,; \exists \tau \in V^{\mathbb{P}}\ q \Vdash \varphi(\tau)\}$ is dense below p.

Definition II.1.5. Let $\mathbb{P}$ be a forcing notion. A $\mathbb{P}$-*name*, τ, is a relation such that:

$$\forall \langle \sigma, p\rangle \in \tau\ (\text{“}\sigma \text{ is a } \mathbb{P}\text{-name.”} \wedge\ p \in \mathbb{P})$$

Let $V^{\mathbb{P}}$ be the class of all $\mathbb{P}$-names and for a transitive model M for ZF with $\mathbb{P} \in M$, define:

$$M^{\mathbb{P}} = M \cap V^{\mathbb{P}} = \{\tau \in M\,; M \models \text{“}\tau \text{ is } \mathbb{P}\text{-name.”}\}$$

Theorem II.1.6 ([Kun11])**.** For a transitive model M for ZF$-$P (resp. ZF,ZFC), $\mathbb{P}$ a forcing notion with $\mathbb{P} \in M$ and G a generic, $M[G]$ is a transitive model for ZF$-$P (resp. ZF,ZFC) with $M \subset M[G]$ and $G \in M[G]$. Conversely, if there is a transitive model N for ZF $-$P with $M \subset N$ and $G \in N$, we obtain that $M[G] \subset M$.

Proof. See IV2.15,2.19,2.26. □

II.2 General Properties for a Forcing

Definition II.2.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets and $i \colon \mathbb{P} \to \mathbb{Q}$ a function. A function i is a *complete embedding* provided that

1. $i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$,

2. $q \leq_{\mathbb{P}} p$ implies $i(q) \leq_{\mathbb{Q}} i(q)$ for any $p, q \in \mathbb{P}$,

3. $p \not\perp_{\mathbb{P}} q$ iff $i(p) \not\perp_{\mathbb{Q}} i(q)$ for any $p, q \in \mathbb{P}$, and

4. If A is a maximal antichain in $\mathbb{P}$, $i(A)$ is a maximal antichain in $\mathbb{Q}$, for any subset $A \subset \mathbb{P}$.

And a funciton i is a *dense embedding* provided that it is a complete embedding and

5. $i(\mathbb{P})$ is a dense subset of $\mathbb{Q}$.

Theorem II.2.2 ([Kun11])**.** Let M be a transitive model for ZFC, $\mathbb{P}, \mathbb{Q}$ forcing notions and a function $i\colon \mathbb{P} \to \mathbb{Q}$ with $\mathbb{Q}, \mathbb{P}, i \in M$ and i is a dense embedding. Define:

- $i_*\colon M^{\mathbb{P}} \to M^{\mathbb{Q}}$ be the function which assign to τ the

$$i_*(\tau) = \{\langle i_*(\sigma), i(p)\rangle \,; \langle \sigma, p\rangle \in \tau\}.$$

- $\tilde{i}(G) = \{q \in \mathbb{Q}\,; \exists p \in G\ i(p) \leq_{\mathbb{Q}} q\}$ for any subset G.

Then we have the followings:

1. For any $\mathbb{P}$-generic G, $\tilde{i}(G)$ is $\mathbb{Q}$-generic and $G = i^{-1}\tilde{i}(G)$, moreover we have $M[G] = M[\tilde{i}(G)]$,
2. For any $\mathbb{Q}$-generic H, $i^{-1}(H)$ is $\mathbb{P}$-generic and $H = \tilde{i}i^{-1}(H)$, moreover we have $M[H] = M[i^{-1}(H)]$,
3. For any formula $\varphi(x_1, \ldots, x_{n-1})$ of $\mathcal{L} = \{\in\}$, $p \in \mathbb{P}$ and $\tau_1, \ldots, \tau_{n-1} \in M^{\mathbb{P}}$, we have

$$p \Vdash_{\mathbb{P}} \varphi(\tau_1, \ldots, \tau_{n-1}) \text{ iff } i(p) \Vdash_{\mathbb{Q}} \varphi(i_*(\tau_1), \ldots, i_*(\tau_{n-1})).$$

Definition II.2.3 ([Jec03])**.** A foring poset $\mathbb{P}$ is *separative* provided that for any conditions $p, q \in \mathbb{P}$ if $p \not\leq q$ then there is $r \leq p$ such that $r \perp q$.

Theorem II.2.4 ([Jec03])**.** Let $\mathbb{P}$ be a forcing poset.

1. If $\mathbb{P}$ is separative, $p \leq q$ iff $p \Vdash q \in \mathring{G}$ for any $p, q \in \mathbb{P}$.
2. There is a separative forcing poset $\mathbb{Q}$ and dense embedding $i\colon \mathbb{P} \to \mathbb{Q}$.

Proof. Let us show the first assertion.

Let us show the second assertion. Define an equivalent relation $\sim \subset \mathbb{P} \times \mathbb{P}$ such that for any $p, q \in \mathbb{P}$:

$$p \sim q \Leftrightarrow \forall z \in \mathbb{P}\ (z \not\perp x \leftrightarrow z \not\perp y)$$

Then we have the following property:

$$x_0 \sim x_1,\ y_0 \sim y_1 \text{ and } \forall z \leq_{\mathbb{P}} x_0\ (z \not\perp y_0) \text{ implies } \forall z \leq_{\mathbb{P}} x_1\ (z \not\perp y_1).$$

Define a forcing poset $\mathbb{Q} = (\mathbb{P}/\sim, \leq_{\mathbb{Q}})$ where $[q] \leq_{\mathbb{Q}} [p]$ if and only if r is compatible with p for any extension $r \leq_{\mathbb{P}} q$. Then it is easy to see that $\mathbb{Q}$ is separative and to see that i is dense embedding since i is surjective, it suffcies to see the followings:

(a) $p \leq_{\mathbb{P}} q$ implies $i(p) \leq_{\mathbb{Q}} i(q)$ for any $p, q \in \mathbb{P}$,

(b) $p \not\perp_{\mathbb{P}} q$ iff $i(p) \not\perp_{\mathbb{Q}} i(q)$ for any $p, q \in \mathbb{P}$.

It is obvious that condition (a) and sufficient condition in (b). To see the necessary condition, let $r \in \mathbb{P}$ such that $[r] \leq_{\mathbb{Q}} [p], [q]$. $[r] \leq_{\mathbb{Q}} [p]$ asserts that there is a common extension $r' \leq_{\mathbb{P}} r, p$, and moreover $[r] \leq_{\mathbb{Q}} [q]$ asserts that there is a common extension $r'' \leq_{\mathbb{P}} r', q$. Therefore, r'' witnesses that $p \not\perp q$. □

Definition II.2.5. Let $\mathbb{P}$ be a forcing notion, θ be a cardinal and M be a ctm:

1. $\mathbb{P}$ has *θ-cc* provided that the size of antichain in $\mathbb{P}$ is less than θ,
2. $\mathbb{P}$ *preseves cofinalities* $\leq \theta$ provided that $M[G] \models \mathrm{cf}(\gamma) = \mathrm{cf}^M(\gamma)$ holds for every limit $\gamma \in o(M)$ with $\mathrm{cf}^M(\gamma) \geq \theta$,
3. $\mathbb{P}$ *preseves cardinals* $\leq \theta$ provided that $M \models$ " β is a cardinal." iff $M[G] \models$ " β is a cardinal." for $\theta \leq \beta \in o(M)$.

Lemma II.2.6. Let $\mathbb{P} \in M$ be a forcing notion with $M \models$ " θ is a regular cardinal." then we have:

1. $\mathbb{P}$ preseves cofinalities $\geq \theta$ iff for arbitrary limit β with $\theta \leq \beta \in o(M)$ we have that $M \models$ " β is regular." implies $M[G] \models$ " β is regular." ,
2. If $\mathbb{P}$ preseves cofinalities $\leq \theta$, so does cardinals $\leq \theta$.

Theorem II.2.7. Let $\mathbb{P} \in M$ and assume that $M \models$ " θ is a regular cardinal and $\mathbb{P}$ is θ-cc." . Then $\mathbb{P}$ preseves cofinalities $\geq \theta$.

Proof. By the previous lemma, it suffice to show that suppose that there is $\theta \leq \beta \in o(M)$ such that $M \models$ " β is regular." and $M[G] \models$ " β is singular." , a contradiction.

Let β be such an ordinal. Let X be a set in $M[G]$ and a function $f \colon \alpha \to X$ in $M[G]$ with $\alpha \in \beta$ and $p \Vdash_{\mathbb{P}}$ " $\mathring{f} \colon (\alpha, \in) \approx (X, \in)$ and $\cup X = \beta$." for some condition p. Define $F \colon \alpha \to \mathcal{P}(\beta)$ which assgin the ξ the $\{\eta \in \beta\,; \exists q \leq p\ (q \Vdash_{\mathbb{P}} \mathring{f}(\xi) = \eta)\}$ then $F \in M$. Now we assrt that $f(\xi) \in F(\xi)$ and $|F(\xi)| < \theta$ for arbitrary $\xi \in \alpha$. The former is immediate. For the later, for $\eta \in F(\xi)$ choose $p_\eta \leq p$ such that $p_\eta \Vdash_{\mathbb{P}} \mathring{f}(\xi) = \eta$. Then note that p_η, $\eta \in F(\xi)$, is pairwise incompatible, thus we obtain that $|F(\xi)| < \theta$. To conclude the proof, let $Y = \cup\{F(\xi)\,; \xi \in \alpha\}$, then we obtain that $Y \subset \beta$ and $\cup Y = \beta$. However since $|F(\xi)| < \beta$, $\alpha < \theta$ and β is regular, we have $|Y| < \beta$. This contrary to that $\beta = \mathrm{type}(Y)$. □

Definition II.2.8 ([Pal13])**.** A *dominating real* in a generic extension is a real $x \in {}^\omega\omega$ which eventually dominates all funcions $f \in {}^\omega\omega$ in V. Similarly a *unbounded real* in a generic extension is a real $x \in {}^\omega\omega$ which does not eventually dominated by any function $f \in {}^\omega\omega$ in V.

II.3 Iterated Forcing

Definition II.3.1 ([Kun11])**.** Let $\mathbb{P}$ be a forcing notion. The $\mathbb{P}$-name for a forcing notion is a triple pair of $\mathbb{P}$-names, $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$, such that $\mathring{\mathbb{1}}_{\mathbb{Q}} \in \mathrm{dom}(\mathring{\mathbb{Q}})$ and for any conditions $p \in \mathbb{P}$ force:

- $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \mathring{\mathbb{1}}_{\mathbb{Q}} \in \mathring{\mathbb{Q}}$, and
- $\mathring{\leq}_{\mathbb{Q}}$ is a pre-order of $\mathbb{Q}$ with the largest element $\mathring{\mathbb{1}}_{\mathbb{Q}}$.

We write $\mathring{\mathbb{Q}}$ for $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$, $\mathring{\leq}$ for $\mathring{\leq}_{\mathbb{Q}}$ and $\mathring{\mathbb{1}}$ for $\mathring{\mathbb{1}}_{\mathbb{Q}}$.

Definition II.3.2 ([Kun11])**.** Let $\mathbb{P}$ be a forcing notion and $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$ be a $\mathbb{P}$-name for a forcing notion, define the *product* $\mathbb{P} * \mathring{\mathbb{Q}}$ is the triple pair $(\mathbb{R}, \leq, \mathbb{1})$ such that;

1. $\mathbb{R} = \{(p, \mathring{q}) \in \mathbb{P} \times \mathrm{dom}\, \mathring{\mathbb{Q}}\, ; p \Vdash \mathring{q} \in \mathring{\mathbb{Q}}\}$,
2. $\leq = \{\langle (p_0, \mathring{q}_0), (p_0, \mathring{q}_0)\rangle\, ; p_1 \leq_{\mathbb{P}} p_2 \wedge p_1 \Vdash (\mathring{q}_1 \leq_{\mathring{\mathbb{Q}}} \mathring{q}_2)\}$,
3. $\mathbb{1} = (\mathbb{1}_{\mathbb{P}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$,

We simply write $(p, \mathring{q}) \in \mathbb{P} * \mathring{\mathbb{Q}}$ for $(p, \mathring{q}) \in \mathbb{R}$ and define $i \colon \mathbb{P} \to \mathbb{P} * \mathring{\mathbb{Q}}$ which assgin the p the $(p, \mathring{\mathbb{1}}_{\mathbb{Q}})$.

Theorem II.3.3. Using the notion of Definition II.3.2, with $p_0, p_1 \in \mathbb{P}$ and $\mathring{q}_0, \mathring{q}_1 \in \mathring{\mathbb{Q}}$. Then we have the following facts:

1. $\mathbb{P} * \mathring{\mathbb{Q}}$ is a forcing notion,
2. $p_0 \leq_{\mathbb{P}} p_1$ iff $i(p_0) \leq_{\mathbb{P} * \mathring{\mathbb{Q}}} i(p_1)$,
3. $i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{P} * \mathring{\mathbb{Q}}}$,
4. $p_0 \perp_{\mathbb{P}} p_1$ implies $(p_0, \mathring{q}_0) \perp_{\mathbb{P} * \mathring{\mathbb{Q}}} (p_1, \mathring{q}_1)$ if $(p_0, \mathring{q}_0), (p_1, \mathring{q}_1) \in \mathbb{P} * \mathbb{Q}$,
5. $p_0 \perp_{\mathbb{P}} p_1$ iff $(p_0, \mathbb{1}_{\mathbb{P} * \mathring{\mathbb{Q}}}) \perp_{\mathbb{P} * \mathring{\mathbb{Q}}} (p_1, \mathring{q}_1)$ whenever $(p_1, \mathring{q}_1) \in \mathbb{P} * \mathring{\mathbb{Q}}$,
6. $p_0 \perp_{\mathbb{P}} p_1$ iff $i(p_0) \perp_{\mathbb{P} * \mathring{\mathbb{Q}}} i(p_1)$,
7. i is a complete embedding.

Definition II.3.4. Let G be $\mathbb{P}$-generic over M and H be subset of $\mathring{\mathbb{Q}}_G$. Then $G * H = \{(p, \mathring{q}) \in \mathbb{P} * \mathring{\mathbb{Q}}\, ; p \in G \wedge \mathring{q}_G \in H\}$.

Theorem II.3.5. Let K be a $\mathbb{P} * \mathring{\mathbb{Q}}$-generic over M. Define $G = i^{-1}(K)$ and let $H = \{\mathring{q}_G\, ; \exists p \in \mathbb{P}\ (p, \mathring{q}) \in K\}$. Then we have the following:

1. G is $\mathbb{P}$-generic over M,
2. H is $\mathring{\mathbb{Q}}_G$-generic over $M[G]$,
3. $K = G * H$,

4. $M[K] = (M[G])[H]$.

Proof. For (1). To see that G is a filter. Let $p, q \in G$. Let $(r, \mathring{r}') \in K$ such that $(r, \mathring{r}') \leq_{\mathbb{P}*\mathring{\mathbb{Q}}} i^{-1}(p), i^{-1}(q)$. Thus we have $r \leq_{\mathbb{P}} p, q$ and $r \in G$. To see that meets every dense subset. Let D be a dense subset of $\mathbb{P}$ in M. By letting $D' = \{(p, \mathring{q}) ; p \in D \wedge \mathring{q} \in \mathring{\mathbb{Q}}_G\}$, D' is dense over $\mathbb{P} * \mathring{\mathbb{Q}}$. Therefore there is $(p, \mathring{q}) \in D' \cap K$, furthermore, $p \in D \cap G$.

For (2). To see that H is a filter. Let $(\mathring{q}_i)_G \in H$ with $p_i \in G$ such that $(p_i, \mathring{q}_i) \in K$ for $i \in 2$. Then there is a common extension $(r, \mathring{r}) \in K$. Then we have $r \Vdash_{\mathbb{P}}$ " $\mathring{r} \leq_{\mathring{\mathbb{Q}}} \mathring{q}_0$ and $\mathring{r} \leq_{\mathring{\mathbb{Q}}} \mathring{q}_1$." . Moreover since $r \in G$, $M[G] \models$ " $\mathring{r}_G \leq_{\mathring{\mathbb{Q}}_G} (\mathring{q}_0)_G$ and $\mathring{r}_G \leq_{\mathring{\mathbb{Q}}_G} (\mathring{q}_1)_G$." . To see that H meets every dense subset. Let D be dense subset of $\mathring{\mathbb{Q}}_G$ in $M[G]$. By letting $D' = \{(p, \mathring{q}) ; p \in G \wedge \mathring{q}_G \in H\}$, then since D' is dense over $\mathbb{P} * \mathring{\mathbb{Q}}$, there is $(p, \mathring{q}) \in D' \cap K$. Thus we obtain that $\mathring{q}_G \in D \cap H$.

For (3). To see that ($\subset$). Let $(p, \mathring{q}) \in K$. By the definition we have $p \in G$ and $\mathring{q}_G \in H$ thus $(p, \mathring{q}_G) \in G * H$. To see that ($\supset$). Let $p \in G$ and $\mathring{q}_G \in H$ with $(p', \mathring{q}) \in K$ for some $p' \in \mathbb{P}$. Since $(p, \mathring{\mathbb{1}}_{\mathbb{Q}}) \in K$, there is a common extension $(p'', \mathring{q}'') \leq (p, \mathring{\mathbb{1}}_{\mathbb{Q}}), (p', \mathring{q})$. Then we obtain that $p'' \leq_{\mathbb{P}} p$ and $p'' \Vdash_{\mathbb{P}} \mathring{q}'' \mathring{\leq}_{\mathbb{Q}} (p, \mathring{q})$. This shows that $(p, \mathring{q}) \in K$.

For (4). To see ($\subset$), by the minimality of extension, it suffices to show that $G * H \in (M[G])[H]$. By letting $\Gamma_{\mathbb{P}*\mathring{\mathbb{Q}}} = \{\langle (p, \mathring{q}), \mathring{q}_G \rangle ; \mathring{q}_G \in H\}$, we have $G * H = (\Gamma_{\mathbb{P}*\mathring{\mathbb{Q}}})_K \in (M[G])[H]$. To see ($\supset$), it suffices to show that $G \in M[K]$ and $H \in M[K]$. For the former, by letting $\Gamma_G = \{\langle p, (p, \mathring{\mathbb{1}}_{\mathbb{Q}}) \rangle ; p \in G\}$, we have $G = (\Gamma_G)_K \in M[K]$. For the later, by letting $\Gamma_H = \{(\mathring{q}_G, \langle \mathbb{1}_{\mathbb{P}}, \mathring{q}_G \rangle) ; \mathring{q}_G \in H\}$, we have $H = (\Gamma_H)_K \in M[K]$. $\square$

Definition II.3.6. For ordinal α, an *α-stage itetated forcing* is a pair $(\langle (\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) ; \xi \leq \alpha \rangle, \langle (\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_{\mathring{\mathbb{Q}}_\xi}, \mathbb{1}_{\mathring{\mathbb{Q}}_\xi}) ; \xi < \alpha \rangle)$ with following properties:

I–1 Every $(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi)$ is a forcing notion,

I–2 Every $(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_\xi, \mathbb{1}_\xi)$ is a $(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi)$-name for a forcing notion,

I–3 Every $p \in \mathbb{P}_\xi$ is a sequence of the form $\langle \mathring{q}_\mu ; \mu < \xi \rangle$ where each $\mathring{q}_\xi \in \mathrm{dom}(\mathring{\mathbb{Q}}_\mu)$,

I–4 $\xi < \eta$ and $p \in \mathbb{P}_\eta$ implies $p|_\xi \in \mathbb{P}_\xi$,

I–5 Let $\xi < \eta$, $p \in \mathbb{P}_\xi$ and p' be an η-sequence such that $p'|_\xi = p$ and $(p')|_\mu = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}}$, then $\xi \leq \mu < \eta$ implies $p' \in \mathbb{P}_\eta$ and we write $i^\eta_\xi(p)$ for p',

I–6 $\mathbb{1}_\xi$ is the sequence $\langle \mathring{q}_\mu ; \mu < \xi \rangle$, where each $\mathring{q}(\mu) = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}$,

I–7 For $p = \langle \mathring{q}_\mu ; \mu < \xi \rangle \in \mathbb{P}_\xi$ and $p' = \langle \mathring{q}'_\mu ; \mu < \xi \rangle \in \mathbb{P}_\xi$. $p \leq_\xi p'$ iff $p|_\mu \Vdash_{\mathbb{P}_\mu} \mathring{q}(\mu) \leq_{\mathring{\mathbb{Q}}_\mu} \mathring{q}'(\mu)$ for all $\mu < \xi$,

I–8 $\mathbb{P}_{\xi+1} = \{p^\frown \langle \mathring{q} \rangle ; p \in \mathbb{P}_\xi \wedge \mathring{q} \in \mathrm{dom}(\mathring{\mathbb{Q}}_\xi) \wedge p \Vdash_{\mathbb{P}_\xi} \mathring{q} \in \mathring{\mathbb{Q}}_\xi\}$ for every $\xi < \alpha$.

Definition II.3.7 ($<\kappa$-support iteretion)**.** For a sequence p with length ξ, the *support of* p is

$$\mathrm{supt}(p) = \{\mu < \xi\,;(p)_\mu \neq \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}\}.$$

For an infinite cardinal κ, the iteration is $<\kappa$*-support* provided that for all limit $\eta(\leq \alpha)$,

$$\mathbb{P}_\eta = \{p\,; \text{“}p \text{ is a sequence of length } \eta\text{”} \wedge |\,\mathrm{supt}(p)| < \kappa \wedge \forall \xi < \eta\ (p|_\xi \in \mathbb{P}_\xi)\}.$$

A finite support iteration is $<\aleph_0$-support and a countable support iteration is $<\aleph_1$-support iteration.

Definition II.3.8. For a limit ordinal α. $\mathbb{P}_\alpha$ is an *inverse limit* of $\langle \mathbb{P}_\beta\,; \beta \in \alpha\rangle$ provided that

$$\forall p (p \in \mathbb{P}_\alpha \leftrightarrow \forall \beta \in \alpha\ p|_\beta \in \mathbb{P}_\beta).$$

$\mathbb{P}_\alpha$ is a *direct limit* provided that

$$\forall p (p \in \mathbb{P}_\alpha \leftrightarrow \exists \beta \in \alpha\ (p|_\beta \in \mathbb{P}_\beta \wedge \forall \xi \in [\beta, \alpha)\ p(\xi) = \mathbb{1}_{\mathring{\mathbb{Q}}_\beta}).$$

Theorem II.3.9 ([Jec03])**.** Let $\mathbb{P}_\alpha$ be an iterated forcing and $\beta \in \alpha$ a limit ordinal. Then we have:

1. $\mathbb{P}_\beta$ is a set of finite support iff

   (1-a) $\mathbb{P}_\beta$ is a direct limit.

2. $\mathbb{P}_\beta$ is a set of countable support iff

   (2-a) If $\mathrm{cf}(\beta) = \omega$, $\mathbb{P}_\beta$ is an inverse limit,

   (2-b) If $\mathrm{cf}(\beta) \ni \omega$, $\mathbb{P}_\beta$ is a direct limit.

Proof. At the beginning of the proof, We write $\mathbb{P}'_\gamma$ for an inverse limit or a direct limit.

To see the statement 1, note that $\mathbb{P}_\gamma \subset \mathbb{P}'_\gamma$ and $\mathbb{P}'_\omega \subset \mathbb{P}_\omega$ hold. To see that $\mathbb{P}'_\gamma \subset \mathbb{P}_\gamma$ for $\gamma \ni \omega$, let $p \in \mathbb{P}_\gamma$ and let $\beta \in \gamma$ such that $p(\xi) = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi}$ for each $\xi \in [\beta, \gamma)$. Let δ be a limit ordinal and n a finite ordinal such that $\beta = \delta + n$. Then we obtain that $\mathrm{supt}(p) \subset \mathrm{supt}(p|_\delta) \cup [\delta, \delta + n)$ and this is a finite set by induction hypothesis.

To see the statement 2, we shall verify by the induction on γ and note that the case of $\gamma = \omega$ is obvious.

Case 2-a. $\omega \in \gamma$ with $\mathrm{cf}(\gamma) = \omega$. To see the $(\mathbb{P}_\gamma \subset \mathbb{P}'_\gamma)$. Let $p \in \mathbb{P}_\gamma$. Since for arbitrary $\beta \in \gamma$ we have $\mathrm{supt}(p|_\beta) \subset \mathrm{supt}(p)$ is countable. Thus by induction hypothesis, we have that $p|_\beta \in \mathbb{P}'_\beta = \mathbb{P}_\beta$. To see the $(\mathbb{P}_\gamma \supset \mathbb{P}'_\gamma)$. Let $p \in \mathbb{P}'_\gamma$ and $\langle \xi_j\,; j \in \omega\rangle$ with $\xi_j \nearrow \gamma$. Note that $i(p|_{\xi_j}) \to p$ where $i(p|_{\xi_j}) \in \mathbb{P}_\gamma$ which eventually assigns $\mathbb{1}$. Then we obtain that $\mathrm{supt}(p) = \cup\{\mathrm{supt}(p_{\xi_j})\,; j \in \omega\}$ and by induction hypothesis this is a countable set. Thus we have $p \in \mathbb{P}_\gamma$.

Case 2-b. γ with $\mathrm{cf}(\gamma) \ni \omega$. To see $(\mathbb{P}_\gamma \subset \mathbb{P}'_\gamma)$. Let $\langle \xi_\alpha\,; \alpha \in |\,\mathrm{cf}(\gamma)|\rangle$ with $\xi_\alpha \nearrow \gamma$. For $p \in \mathbb{P}_\gamma$ there is $\alpha \in |\,\mathrm{cf}(\gamma)|$ such that $\cup \mathrm{supt}(p) \in \xi_\alpha$ and this

α witnesses that $p \in \mathbb{P}'_\gamma$. To see $(\mathbb{P}_\gamma \supset \mathbb{P}'_\gamma)$. Let $p \in \mathbb{P}'_\gamma$ and $\beta \in \gamma$ such that $p|_\beta \in \mathbb{P}_\beta$ and $p(\xi) = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi}$ for $\xi \in [\beta, \gamma)$. Then by induction hypothesis, we obtain that $\mathrm{supt}(p)$ is countable. $\square$

Definition II.3.10 (Intermidiate stage,[Bau83])**.** Let $p \in \mathbb{P}_\alpha$. We define the followings:

1. $p^\beta = p|_{\{\gamma \in \alpha\,;\,\beta \leq \gamma\}}$ for $\beta \in \alpha$,
2. For $\beta \in \alpha$, $\mathbb{P}_{\beta\alpha} = \{p^\beta\,; p \in \mathbb{P}_\alpha\}$,
3. For $\mathbb{P}_\beta$-generic G_β, $f \leq g$ over $\mathbb{P}_{\beta\alpha}$ provided that there is $p \in G_\beta$ such that $p \cap f \leq_{\mathbb{P}_\alpha} p \cup g$.

$\mathbb{P}_{\beta\alpha}$ is a forcing notion with order in 3.

III Type of Forcings

III.1 ccc Forcing

Definition III.1.1. For a cardinal κ, a forcing notion is $< \kappa$*-cc forcing* provided that there are no antichains of size κ. A forcing notion is *ccc* provided that it is a $\aleph_1$-cc.

Example III.1.2.

1. Cohen forcing is a ccc forcing. (see Theorem IV.1.2),
2. σ-centerd forcing is ccc forcing.

Theorem III.1.3 ([Kun11])**.** ccc forcing preseves cofinalities.

Proof. IV.3.4 $\square$

Theorem III.1.4 ([Jec03])**.** Let κ be an uncountale regular cardinal and $\mathbb{P}_\alpha$ be a finite support iteration. If $\Vdash_\beta$ " $\mathring{\mathbb{Q}}_\beta$ is $< \kappa$-cc." , for each $\beta \in \alpha$, so does $\mathbb{P}_\alpha$.

Proof. We shall show by induction on α.

Case I. $\alpha = \gamma + 1$. Let $A = \{p_\xi^\frown \langle \mathring{q}_\xi \rangle\,; \xi \in \kappa\}$ be a subset of $\mathbb{P}_\alpha$ of size κ such that $p_\xi \in \mathbb{P}_\gamma$ and $\mathring{q}_\xi \in \mathring{\mathbb{Q}}_\gamma$, respectively for each $\xi \in \kappa$. Since κ is a regular and $\mathbb{P}_\gamma \models$ is $< \kappa$-cc, there is $B \in [A]^\kappa$ such that for any $\xi, \eta \in B$ $p_\xi = p_\eta$. Thus we obtian a κ size subset $\{\mathring{q}_\xi\,; \xi \in B\}$ of $\mathring{\mathbb{Q}}$. Moreover, since $\Vdash_\gamma$ " $\mathring{\mathbb{Q}}_\gamma$ is $< \kappa$-cc." , there are distinct ξ and η which is compatible. Let $\mathring{r} \in \mathring{\mathbb{Q}}_\gamma$ be a common extension. Then we obtain a common $p_\xi^\frown \langle \mathring{r} \rangle$ for $p_\xi^\frown \langle \mathring{q}_\xi \rangle$ and $p_\eta^\frown \langle \mathring{q}_\eta \rangle$. This shows that A is not an antichain, i.e., $\mathbb{P}_\alpha$ is $< \kappa$-cc.

Case II–i. α is limit and $\mathrm{cf}(\alpha) \ni \kappa$. Let $A = \{p_\xi \in \mathbb{P}_\alpha\,; \xi \in \kappa\}$ be a subset of size κ. Since $\mathbb{P}_\alpha$ is a finite support iteration, there is a $\beta \in \alpha$ such that

$$\forall \xi \in \kappa \forall \eta \in [\beta, \alpha)\ p_\xi(\eta) = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\eta}$$

moreover, we obtain that

$$\forall \xi \in \kappa \forall \eta \in [\beta, \alpha)\ p_\xi(\eta) = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\eta}$$

for a limit ordinal $\beta \in \alpha$. Now by induction hypothesis, there are distinct $\xi, \xi' \in \kappa$ and $p \in \mathbb{P}_\beta$ such that $p \leq_\beta p_\xi|_\beta, p_{\xi'}|_\beta$. Furthermore, $p^\frown \langle \mathring{\mathbb{1}} \rangle^\frown \cdots ^\frown \langle \mathring{\mathbb{1}} \rangle \leq_\alpha p_\xi, p_{\xi'}$. Therefore, A is not an antichain, i.e., $\mathbb{P}_\alpha$ is $< \kappa$-cc.

Case II–ii. α is limit and $\mathrm{cf}(\alpha) = \kappa$. Let $A = \{p_\xi\,; \xi \in \kappa\}$ be a subset of size κ and $\langle \alpha_\xi\,; \xi \in \kappa \rangle$ be a sequence with following properties:

- $\cup\{\mathrm{supt}(p_\eta)\,; \eta \in \xi\} \subset \alpha_\xi$ for all $\xi \in \kappa$,
- $\cup\{\alpha_\xi\,; \xi \in \kappa\} = \alpha$.

Then for each limit $\xi \in \kappa$, since $\mathrm{supt}(p_\xi)$ is finite and $\cup\{\mathrm{supt}(p_\eta)\,; \eta \in \xi\} \subset \alpha_\xi$, there is $\gamma_\xi \in \xi$ such that $\mathrm{supt}(p_\xi) \cap \alpha_\xi \subset \alpha_{\gamma_\xi}$.

Since the set $\kappa \cap \mathrm{Lim}$ is a stationary subset of κ, applying Theorem I.1.4, there is a stationary subset $S \subset C$ of size κ and $\gamma \in \kappa$ such that $\mathrm{supt}(p_\xi) \cap \alpha_\xi \subset \alpha_\gamma$ for any $\xi \in S$ and thus we obtain a subset $\{p_\xi|_\gamma\,; \xi \in S\} \subset \mathbb{P}_\gamma$. Now induction hypothesis and κ is regular assert that there are ξ and η such that $\gamma \in \xi \in \eta$ and $p_\xi|_\gamma$ and $p_\eta|_\gamma$ possesses a common extension, named $p \in \mathbb{P}_\gamma$.

Define an r by:

$$r(\beta) = \begin{cases} p(\beta) & \beta \in \alpha_\gamma \\ p_\xi(\beta) & \beta \in [\alpha_\gamma, \alpha_\eta) \\ p_\eta(\beta) & \beta \in [\alpha_\eta, \alpha) \end{cases}$$

then ovbiously we have $r \in \mathbb{P}_\alpha$. To see that r is an extension for both p_ξ and p_η, the former is derived by the fact $\eta \in \xi$ and the later is derived by the fact $\eta \in S$. Therefore, A possesses a compatible pair, i.e., $\mathbb{P}_\alpha$ is $< \kappa$-cc. □

III.2 σ-centerd Forcing

Definition III.2.1. Let $\mathbb{P}$ be a forcing notion and C a subset. C is *centered* provided that for any finite subset of C there is a common extension $p \in \mathbb{P}$. A forcing notion $\mathbb{P}$ is *σ-centered* provided that there are countably many centered subsets $C_i \subset \mathbb{P}$, $i \in \omega$, whose union is $\mathbb{P}$.

Example III.2.2.

1. Hechler forcing is σ-centered forcing. (see Theorem IV.2.4)

III.3 σ-closed Forcing

Definition III.3.1. A forcing notion $\mathbb{P}$ is *a σ-closed forcing* (or *ω_1-closed*, *countably closed*) provided that for every decreasing sequence $\langle p_n \in \mathbb{P}\,; n \in \omega \rangle$ with $p_{n+1} \leq p_n$ there is a condition $p \in \mathbb{P}$ such that $p \leq p_n$ for each $n \in \omega$.

III.4 Axiom A Forcing

Definition III.4.1 ([Bau83])**.** For a forcing notion $\mathbb{P}$, $\mathbb{P}$ satisfies Axiom A provided that there is an order sequence $\{\leq_n ; n \in \omega\}$ with following properties:

A–1 $\leq_{\mathbb{P}} = \leq_0$,

A–2 For every $n \in \omega$ and $p, q \in \mathbb{P}$, $q \leq_{n+1} q \implies p \leq_n q$,

A–3 For every sequence $\langle p_n \in \mathbb{P} ; n \in \omega \rangle$ with $p_{n+1} \leq_n p_n$ for each $n \in \omega$, there is $q \in \mathbb{P}$ such that $p \leq_n p_n$ for every $n \in \omega$. We call that the sequence is a *fusion sequence* and q is a *fusion*,

A–4 Let I be an antichain over $\mathbb{P}$, $p \in \mathbb{P}$ and $n \in \omega$, there is $q \in \mathbb{P}$ such that $q \leq_n p$ and $\{r \in I ; r \not\perp q\}$ is countable.

Example III.4.2.

1. A ccc forcing satisfies Axiom A.
2. A σ-closed forcing satisfies Axiom A.
3. A Sacks forcing satisfies Axiom A. (see Theorem IV.4.3)
4. A Mathias forcing satisfies Axiom A. (see Theorem IV.3.7)
5. A shooting a club by finite conditions forcing does not satisfy Axiom A. (see Theorem IV.5.4)

Theorem III.4.3. The following is equivalent to A–4:

A–4' Let $\mathring{a}$ be a $\mathbb{P}$-name, $p \in \mathbb{P}$ and $n \in \omega$. If $p \Vdash_{\mathbb{P}} \mathring{a} \in V$, equivalently $\exists b \in V\ (p \Vdash_{\mathbb{P}} \mathring{a} \in \check{b})$, then there is $X \in V$ and $q \in \mathbb{P}$ such that X is countable, $q \leq_n p$ and $q \Vdash_{\mathbb{P}} \mathring{a} \in \check{X}$.

Proof. For A–4 $\implies$ A–4'. Assume $p \Vdash_{\mathbb{P}} \mathring{a} \in V$ and let $n \in \omega$. Since $\{p \in \mathbb{P} ; \exists b \in V\ (p \Vdash \mathring{a} = \check{b})\}$ is not empty let I be a maximal antichain of it and for $r \in I$ let $a_r \in V$ be the unique element which satisfies $r \Vdash_{\mathbb{P}} \mathring{a} = \check{a}_r$. Applying A–4, there is $q \leq_n p$ such that $\{r \in I ; r \not\perp_{\mathbb{P}} q\}$ is countable. By letting $X = \{a_r ; r \in I \wedge r \not\perp_{\mathbb{P}} q\}$, X is countable. Finally we check $q \Vdash_{\mathbb{P}} \mathring{a} \in \check{X}$. For arbitrary $q_1 \leq_{\mathbb{P}} q$ there is $r \in I$ such that $r \not\perp_{\mathbb{P}} q_1$, moreover there is $q_2 \in \mathbb{P}$ such that $q_2 \leq_{\mathbb{P}} q_1, r$. Then we obtain $q_2 \Vdash_{\mathbb{P}} \mathring{a} = \check{a}_r$.

For A–4' $\implies$ A–4. Let I be an antichain, $p \in \mathbb{P}$ and $n \in \omega$ and let I' be a maximal antichain finer than I. By letting $\mathring{a} = \{(\check{r}, r) ; r \in I'\}$, for any generic G we have $(\mathring{a})_G \in I'$, so we obtain $p \Vdash_{\mathbb{P}} \mathring{a} \in V$. By A–4' there is a coutable $X \in V$ and $q \in \mathbb{P}$ such that $q \leq_n p$ and $q \Vdash_{\mathbb{P}} \mathring{a} \in \check{X}$. Now we show that $\{r \in I' ; r \not\perp_{\mathbb{P}} q\} \subset X$. Let $r \in I'$ with $r \perp_{\mathbb{P}} q$. By $q \Vdash_{\mathbb{P}} \mathring{a} \in \check{X}$ there is q' and $x \in X$ such that $q' \Vdash_{\mathbb{P}} \mathring{a} = \check{x}$. Then for any generic G with $q' \in G$, since $r \in G$, we obtain that $r = (\mathring{a})_G = (\check{x})_G = x$. Thus, we have the subset relation and we complete the proof. $\square$

Theorem III.4.4. Let $\mathbb{P}_\alpha$ is an α-stage iteration over M and $\beta \in \alpha$. Let $i\colon \mathbb{P}_\alpha \to \mathbb{P}_\beta$ which assign the p the $p^\frown\langle\mathbb{1}\rangle^\frown\dots^\frown\langle\mathbb{1}\rangle \in \mathbb{P}_\alpha$. For $\mathbb{P}_\alpha$-generic K, define $G = i^{-1}(K)$ and $H = \{p^\beta\,;p \in K\}$. Then we have the followings:

1. G is $\mathbb{P}_\beta$-generic over M,
2. H is $\mathbb{P}_{\beta\alpha}$-generic over $M[G]$,
3. $K = G^\frown H = \{p^\frown q\,;p \in G \wedge q \in H\}$,
4. $M[K] = (M[G])[H]$.

Theorem III.4.5. Let $\mathbb{P}_\alpha$ is an α-stage iteration with countable support which satisfies $\mathbb{1} \Vdash_{\mathbb{P}_\beta}$ "$\mathring{\mathbb{Q}}_\beta$ satisfies Axiom A." for all $\beta \in \alpha$. Then $\mathbb{P}_\alpha$ does not collapse ω_1, i.e., preseves ω_1. And moreover if we have $\alpha < \omega_2$, $V \models \mathrm{ZFC} + \mathrm{CH}$ and $\Vdash_{\mathbb{P}_\beta} |\mathring{\mathbb{Q}}_\beta| \leq \aleph_1$ for all $\beta \in \alpha$, then $\mathbb{P}_\alpha$ has an $\aleph_2$-chain condition.

Before we show the theorem, we make some definition and lemmata. (See the proof.)

Definition III.4.6. For $\gamma \leq \alpha$, $F \in [\alpha]^{<\omega}$ and $n \in \omega$, we define the followings:

(i) For $p, q \in \mathbb{P}$, $p \leq_{F,n} q$ provided that $\forall \beta \in F\ (p|_\beta \Vdash_{\mathbb{P}_\beta} p(\beta) \mathring{\leq}^\beta_n q(p))$.

(ii) $D \subset \mathbb{P}_\gamma$ is (F,n)-*dense* provided that $\forall p \in \mathbb{P}_\gamma \exists q \in D\ (q \leq_{F,n} p)$.

(iii) A sequence $\langle (p_n, F_n)\,; n \in \omega\rangle$ is an (F,n)-*fusion sequence* of $\mathbb{P}_\alpha$ provided that

(iii–a) $p_n \in \mathbb{P}_\alpha$ and $F_n \in [\alpha]^{<\omega}$ for each $n \in \omega$,

(iii–b) $\forall n \in \omega\ (p_{n+1} \leq_{F_n,n} p_n)$,

(iii–c) $\forall n \in \omega\ (F_n \subset F_{n+1})$,

(iii–d) $\cup\{F_n\,; n \in \omega\} = \cup\{\mathrm{supt}(p_n)\,; n \in \omega\}$.

Lemma III.4.7. Let $\langle (p_n, F_n)\,; n \in \omega\rangle$ be an (F,n)-fusion sequence of $\mathbb{P}_\alpha$. There is $p \in \mathbb{P}_\alpha$ such that $\forall n \in \omega\ (p \leq_{F_n,n} p_n)$. Moreover we may assume that $\mathrm{supt}(p) = \cup\{F_n\,; n \in \omega\}$.

Proof. First we see the "moreover" part. Suppose there is the desired $q \in \mathbb{P}_\alpha$. By letting

$$q' = \{(\eta, q(\eta))\,; \eta \notin \cup\{\mathrm{supt}(p_n)\,; n \in \omega\}\} \cup \{(\eta, \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\eta})\,; \eta \in \cup\{\mathrm{supt}(p_n)\,; n \in \omega\}\}$$

we obtain that $\mathrm{supt}(q) = \cup\{\mathrm{supt}(p_n)\,; n \in \omega\}$.

To see the first statement, we show the following by the induction on β.

$$\forall \beta \leq \alpha \forall \langle (p_n, F_n)\,; n \in \omega\rangle \exists p \in \mathbb{P}_\beta \forall n \in \omega\ (p \leq_{F_n,n} p_n)$$

Where $\langle (p_n, F_n)\,; n \in \omega\rangle$ is an (F,n)-fusion sequence of $\mathbb{P}_\beta$.

(i) $\beta = 0$. For an (F, n)-fusion sequence $\langle (p_n, F_n)\,; n \in \omega\rangle$ of $\mathbb{P}_0$, by letting $p = \emptyset$, we have $p \in \mathbb{P}_0$ and for arbitrary $n \in \omega$, $p \leq_{0,n} p_n$ is trivial.

(ii) $\beta = \gamma + 1$. Let $\langle (p_n, F_n)\,; n \in \omega\rangle$ be an (F, n)-fusion sequence of $\mathbb{P}_{\gamma+1}$. By induction hypothesis, there is $p' \in \mathbb{P}_\gamma$ such that $\forall n \in \omega\ (p' \leq_{F_n,n} p_n)$ holds. We distinguish two cases, according to whether $\gamma \in \cup\{\mathrm{supt}(p_n)\,; n \in \omega\}$. If $\gamma \notin \cup\{\mathrm{supt}(p_n)\,; n \in \omega\}$, by letting $p = p' \cup \{(\gamma, \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\gamma})\}$, we have $p \in \mathbb{P}_{\gamma+1}$ and $\forall n \in \omega\ (p \leq_{F_n,n} p_n)$ holds. If $\gamma \in \cup\{\mathrm{supt}(p_n)\,; n \in \omega\}$. Let n_0 be an $\in$-minimal such that $\gamma \in F_{n_0}$. For $m \in \omega$, define $\mathring{q}_m$ for $p_{n_0}(\gamma)$ if $m \in n_0$ and $p_m(\gamma)$ if $m \notin n_0$. Then we note that $\mathbb{1} \Vdash_{\mathbb{P}_\gamma}$ " $\langle \mathring{q}_m\,; m \in \omega\rangle$ is a sequence of $\mathring{\mathbb{Q}}_\gamma$." and we assert that $p' \Vdash_{\mathbb{P}_\gamma} \forall n \in \omega\ (\mathring{q}_{m+1} \mathring{\leq}^{\gamma}_{m} \mathring{q}_m)$. To see the later, for $m \in \omega$, we distinguish two cases, according to whether $m \in n_0$. If $m \notin n_0$. Since we have $\gamma \in F_{n_0} \subset F_m$, $\mathring{q}_{m+1}(\gamma) = p_m(\gamma)$ and $\mathring{q}_{m+1}(\gamma) = p_{m+1}(\gamma)$, by the condition (a) in an (F, n)-fusion sequence, we obtain that $p_{m+1}|_\gamma \Vdash_{\mathbb{P}_\gamma} p_{m+1}(\gamma) \mathring{\leq}^{\gamma}_{m} p_m(\gamma)$. Since $p' \leq_{F_{m+1},m+1} p_{m+1}|_\gamma$ and $\gamma \in F_{n_0} \subset F_{m+1}$ hold, we have $\forall \delta < \gamma\ (p' \Vdash_{\mathbb{P}_\delta} p'(\delta) \mathring{\leq}^{\delta} p_{m+1}(\delta))$, moreover, $p' \leq_{\mathbb{P}_\gamma} p_{m+1}|_\gamma$. Therefore we obtain that $p' \Vdash_{\mathbb{P}_\gamma} p_{m+1}(\gamma) \mathring{\leq}^{\gamma}_{m} p_m(\gamma)$. If $m \in n_0$. Since $\mathring{q}_m(\gamma) = p_{n_0}(\gamma) = \mathring{q}_{m+1}(\gamma)$, we have $p' \Vdash_{\mathbb{P}_\gamma} p_{m+1}(\gamma) \mathring{\leq}^{\gamma}_{m} p_m(\gamma)$. Now since $p' \Vdash_{\mathbb{P}_\gamma}$ " $\mathring{\mathbb{Q}}_\gamma$ satisfies Axiom A." , by A–4', there is a $\mathring{q} \in V^{\mathbb{P}_\gamma}$ such that $p' \Vdash_{\mathbb{P}_\gamma} \mathring{q} \in \mathring{\mathbb{Q}}_\gamma$ and $\forall m \in \omega\ (\mathring{q} \mathring{\leq}^{\gamma}_{m} \mathring{q}_m)$ holds. Thus by letting $p = p'^\frown \mathring{q}$, we obtain that $p \in \mathbb{P}_\beta$ and $\forall n \in \omega\ (p \leq_{F_n,n} p_n)$.

(iii) β is limit. Let $\langle (p_n, F_n)\,; n \in \omega\rangle$ be an (F, n)-fusion sequence of $\mathbb{P}_\beta$. By induction hypothesis for every $\xi \in \beta$, choose $p'_\xi \in \mathbb{P}_\xi$ such that $\forall n \in \omega\ (p_\xi \leq_{F_n,n} p'_\xi)$. We distinguish two cases, according to whether $\omega \in \mathrm{cf}(\beta)$. If $\omega \in \mathrm{cf}(\beta)$. Since every F_n is finite there is $\eta \in \beta$ such that $\cup\{F_n\,; n \in \omega\} \subset \eta$. Since we have $\forall n \in \omega\ (p_\eta \leq_{F_n,n} p_\eta)$, we obtain that $\forall n \in \omega\ (p_\eta \leq_{F_n,n} p_n)$. If $\omega = \mathrm{cf}(\beta)$. Let $\xi_n \in \beta$ such that $\xi_n \nearrow \beta$ and $\cup\{F_m\,; m \leq n\} \subset \xi_n$. First, for an (F, n)-fusion sequence $\langle (p_n|_{\xi_0}, \cap\{F_n, \xi_0\})\,; n \in \omega\rangle$ of $\mathbb{P}_{\xi_0}$, there is $q_0 \in \mathbb{P}_{\xi_0}$ such that $q_0 \leq_{\cap\{F_n,\xi_0\},n} p_n|_{\xi_0}$ for every $n \in \omega$. Secondly, for $n \in \omega$, define $p^1_n = q_0 \cup f|_{[\xi_0,\xi_1)}$ then we obtain an (F, n)-fusion sequence $\langle (p^1_n, \cap\{F_n, \xi_1\})\,; i \in \omega\rangle$, there is $q_1 \in \mathbb{P}_{\xi_1}$ such that $q_1 \leq_{\cap\{F_n,\xi_n\},n} p^1_n$ for every $n \in \omega$. And by letting $q'_1 = q|_{\xi_0} \cup q_1|_{[\xi_0,\xi_1)}$, we have $q_0 \subset q'_1$. Continuing this process, we obtain $q_m \in \mathbb{P}_{\xi_m}$, $m \in \omega$, with following properties:

1. $q_0 \leq_{\cap\{F_n,\xi_0\},n} p_n|_{\xi_0}$ holds for each $n \ni 1$,
2. $q_m \leq_{\cap\{F_n,\xi_n\},i} p^m_n$ for every $n \in \omega$ holds for each $m \ni 0$, where $p^{m+1}_n = q_m \cup p_n|_{[\xi_m,\xi_{m+1})}$.

Let $q = \cup\{q_m\,; m \in \omega\}$. To conclude the proof we remain to show that $q \in \mathbb{P}_\beta$ and $q \leq_{F_n,n} p_n$ for every $n \in \omega$. The former is immediate by the fact $\mathbb{P}_\alpha$ is countable support and $q_m \subset q_{m+1}$ for every $m \in \omega$. To see the later statement. Let $n \in \omega$ and $\gamma \in F_n$. If $\gamma \in \xi_0$, we obtain $q \leq_{F_n,n} p_n$ immediately by $q_0 \subset q$. Thus we may assume that $\xi_0 \leq \gamma \in \beta$ and choose $m \in \omega$ such that $\gamma \in [\xi_{m-1}, \xi_m)$. Thus we obtain that $q_m|_\gamma \Vdash_{\mathbb{P}_\gamma} q_m(\gamma) \mathring{\leq}^{n}_{\gamma} p^m_n(\gamma)$, furthermore, by $q \supset q_m$ and $p^m_n(\gamma) = p_n(\gamma)$, we have that $q|_\gamma \Vdash_{\mathbb{P}_\gamma} q(\gamma) \mathring{\leq}^{n}_{\gamma} p_n(\gamma)$. Now we complete the proof of the limit case.

$\square$

Lemma III.4.8. For $\beta \leq \alpha$ we have the followings:

(a) If $\Vdash_{\mathbb{P}_\beta} \mathring{a} \in V$, then the following set is (F, n)-dense for all finite $F \subset \beta$ and $n \in \omega$;

$$\{q \in \mathbb{P}_\beta\,; \exists X\ (\text{“ } X \text{ is countable in } V.\text{” } \wedge q \Vdash_{\mathbb{P}_\beta} \mathring{a} \in \check{X})\},$$

(b) If $\Vdash_{\mathbb{P}_\beta}$ “ $\mathring{X}$ is a countable subset of V.” , then the following set is (F, n)-dense for all finite $F \subset \beta$ and $n \in \omega$;

$$\{q \in \mathbb{P}_\beta\,; \exists X\ (\text{“ } X \text{ is countable in } V.\text{” } \wedge q \Vdash_{\mathbb{P}_\beta} \mathring{X} \subset \check{X})\},$$

(c) If $\beta \in \gamma \leq \alpha$ and $\Vdash_{\mathbb{P}_\beta} \mathring{f} \in \mathbb{P}_{\beta\gamma}$, then the following set is (F, n)-dense for all finite $F \subset \beta$ and $n \in \omega$;

$$\{q \in \mathbb{P}_\beta\,; \exists f \in \mathbb{P}_{\beta\gamma}\ (q \Vdash_{\mathbb{P}_\beta} \mathring{f} = f)\}.$$

Proof. At the beginning of the proof, if $\beta \leq \alpha$ satisfies (a) (resp. (b), (c)), we say that (a) holds at a β-stage. To conclude the proof, thanks to induction, it suffices to show the followings:

1. (a) holds at a β-stage then so does (b),
2. (b) holds at a β-stage then so does (c),
3. (a) holds at the leading-stage, $\beta = 0$,
4. (b) holds at a β-stage then (a) holds at the $\beta + 1$-stage,
5. For any limit $\delta \leq \alpha$, if (b) and (c) hold at a $\beta \in \delta$ then (a) holds at a δ-stage.

We shall show 1. Assume that $\Vdash_{\mathbb{P}_\beta}$ “ $\mathring{X}$ is countable in V.” and $F \in [\beta]^{<\omega}$, $n \in \omega$ and $p \in \mathbb{P}_\beta$. Let $\mathring{a}_n$, $n \in \omega$, be $\mathbb{P}_\beta$-names such that $\Vdash_{\mathbb{P}_\beta} \mathring{X} = \{\mathring{a}_n\,; n \in \omega\}$. We define an (F, n)-fusion sequence $\langle (p_k, F_k)\,; k \in \omega\rangle$ of $\mathbb{P}_\beta$, recursively. Let $p_0 = p \in \mathbb{P}_\beta$ and $F_0 = F \cap \operatorname{supt}(p_0)$. Since $\mathbb{P}_\alpha$ is countable support, let $\mathring{\alpha}^0_i$, $i \in \omega$, such that $\{\mathring{\alpha}^0_i\,; i \in \omega\} = \operatorname{supt}(p_0)$. Assume that $p_k \in \mathbb{P}_\beta$ and F_k are defined. Applying (a) at a β-stage, there are $p_{k+1} \in \mathbb{P}_\beta$ and countable X_k in V such that $p_{k+1} \leq_{F_k, n+k} p_k$ and $p_{k+1} \Vdash_{\mathbb{P}_\beta} \mathring{a}_k \in \check{X}_k$ and let $\mathring{\alpha}^k_i$, $i \in \omega$, be $\mathbb{P}_\beta$-names such that $\{\mathring{\alpha}^k_i\,; i \in \omega\} = \operatorname{supt}(p_k)$. Then by letting $F_{k+1} = \cup\{F_k, \{\mathring{\alpha}^k_i\,; i \in \omega\}\}$. Therefore we obtain an (F, n)-fusion sequece $\langle (p_k, F_k)\,; k \in \omega\rangle$. Now by applying Lemma III.4.7, there is $q \in \mathbb{P}_\beta$ such that $q \leq_{F_k, n+k} p_k$ for every $k \in \omega$. Moreover, we have $q \leq_{F_0, n} p$. Thus we obtain that $q \leq_{F, n} p$ and we obtain that $q \Vdash_{\mathbb{P}_\beta} \mathring{X} \subset \check{\cup\{X_k\,; k \in \omega\}}$.

We shall show 2. Let γ be $\beta \in \gamma \leq \alpha$, $\Vdash_{\mathbb{P}_\beta} \mathring{f} \in \mathbb{P}_{\beta\gamma}$, $F \in [\beta]^{<\omega}$, $n \in \omega$ and $p \in \mathbb{P}_\beta$. Since $\Vdash_{\mathbb{P}_\beta}$ “ $\operatorname{supt}(p)$ is countable in V.” , by applying (b) at a β-stage, there is $q \in \mathbb{P}_\beta$ and countable X in V such that $q \leq_{F, n} p$ and $q \Vdash_{\mathbb{P}_\beta} \operatorname{supt}(p) \subset \check{X}$. To conclude the proof, we shall construct a function $f \in \mathbb{P}_{\beta\gamma}$ as follows:

- $f(\xi) = \mathbb{1}_{\mathring{\mathbb{Q}}_\xi}$ for all $\xi \notin X$

- $f(\xi)$ be a P_ξ-name such that $q \cup f|_\xi \Vdash_{\mathbb{P}_\xi} f(\xi) = \mathring{f}(\xi)$ for all $\xi \in X$

Then since $\mathrm{supt}(f)$ is countable we obtain $f \in \mathbb{P}_{\beta\gamma}$. We remain to show that $q \Vdash_\beta \mathring{f} = f$ and it suffcies to verify that $q \Vdash_\beta \forall\xi\ (\beta \leq \xi \wedge \xi \in \gamma \implies \mathring{f}(\xi) = f(\xi))$. (i) If $\xi = \beta$. Since $q \cup f|_\xi = q$, we obtain that $q \Vdash_{\mathbb{P}_\beta} f(\xi) = \mathring{f}(\xi)$. (ii) If ξ is successor or limit. We distinguish two cases, according to whether $\xi \in X$. If $\xi \notin X$. By the construction, we obtain that $q \Vdash_{\mathbb{P}_\beta} f(\xi) = \mathring{f}(\xi)$. If $\xi \in X$. By induction hypothesis, for arbitrary η with $\beta \leq \eta \in \xi$, we obtain $q \Vdash_{\mathbb{P}_\beta} (f|_\eta \Vdash_{\mathbb{P}_\eta} f(\eta) = \mathring{f}(\eta))$ and $q \Vdash_{\mathbb{P}_\beta} (\mathring{f}|_\eta \Vdash_{\mathbb{P}_\eta} f(\eta) = \mathring{f}(\eta))$. Thus we obtain that $q \Vdash_{\mathbb{P}_\beta} f(\xi) \leq_{\mathbb{P}_\xi} \mathring{f}(\xi)$ and $q \Vdash_{\mathbb{P}_\beta} f(\xi) \geq_{\mathbb{P}_\xi} \mathring{f}(\xi)$, furthermore, $q \Vdash_{\mathbb{P}_\beta} f(\xi) = \mathring{f}(\xi)$.

We shall show 3. First we note that $\mathbb{P}_\beta = \{\emptyset\}$ and for any condition there are no extensions without it self. By letting $X = \{\mathring{a}\} \subset V$ we obtain that $\emptyset \Vdash_{\mathbb{P}_0} \mathring{a} \in \check{X}$. This shows (a) holds at the 0-stage.

We shall show 4. Assume $\Vdash_{\mathbb{P}_{\beta+1}} \mathring{a} \in V$ and let $F \in [\beta+1]^{<\omega}$, $n \in \omega$ and $p \in \mathbb{P}_{\beta+1}$. Since we have $\mathbb{P}_{\beta+1}$ embbends $\mathbb{P}_\beta * \mathring{\mathbb{Q}}_{\beta+1}$ and $\Vdash_{\mathbb{P}_\beta} \mathring{a} \in V$, we obtain that $\mathbb{1}_{\mathring{\mathbb{Q}}_{\beta+1}} \Vdash_{\mathbb{P}_{\beta+1}} (\mathbb{1}_{\mathring{\mathbb{Q}}_\beta} \Vdash_{\mathbb{P}_\beta} \mathring{a} \in V)$ furthermore, $\mathbb{1}_{\mathring{\mathbb{Q}}_{\beta+1}} \Vdash_{\mathbb{P}_{\beta+1}} (p(\beta)_{\mathring{\mathbb{Q}}_\beta} \Vdash_{\mathbb{P}_\beta} \mathring{a} \in V)$. Since, $\Vdash_{\mathbb{P}_\beta}$ " $\mathring{\mathbb{Q}}$ satisfies Axiom A." , $\mathbb{1}_{\mathbb{Q}_{\mathring{\beta}+1}}$ forces " $\exists q \leq_n^\beta p(\beta) \exists X$ (" X is countable." $\wedge\ q \Vdash_{\mathbb{P}_\beta} \mathring{a} \in \check{X}$).". Moreover, by letting $X' = X \cap V$, since $\Vdash_{\mathbb{P}_{\beta+1}} \mathring{a} \in V$, we can forces " $\exists q \leq_n^\beta p(\beta) \exists X'$ (" X' is countable in V." $\wedge\ q \Vdash_{\mathbb{P}_\beta} \mathring{a} \in \check{X}'$)." . Now by applying (b), there is countable Y in V and $q' \leq_{F,n} p|_\beta$ such that $q'' \Vdash_{\mathbb{P}_\beta} (\exists q \leq_n^\beta p(\beta) \wedge q \Vdash_{\mathring{\mathbb{Q}}_\beta} \mathring{a} \in \check{Y})$. Then we obtain that $q''^\frown\langle q\rangle \leq_{F,n} p$ and $q''^\frown\langle q\rangle \Vdash_{\mathbb{P}_{\beta+1}} \mathring{a} \in \check{Y}$.

We shall show 5. Let $\delta \leq \alpha$ be a limit. Assume that $\Vdash_{\mathbb{P}_\gamma} \mathring{a} \in V$ and let $F \in [\delta]^{<\omega}$, $n \in \omega$ and $p \in \mathbb{P}_\delta$ and let γ be an ordinal with $F \subset \gamma \in \delta$. Since $\mathbb{P}_\delta$ embbends $\mathbb{P}_\gamma * \mathring{\mathbb{P}}_{\gamma\delta}$ and $\Vdash_{\mathbb{P}_\gamma} \mathring{a} \in V$, we have $\mathbb{1}_{\mathring{\mathbb{Q}}_\gamma} \Vdash_{\mathbb{P}_\gamma} (\mathbb{1}_{\mathbb{P}_{\gamma\delta}} \Vdash_{\mathbb{P}_{\gamma\delta}} \mathring{a} \in V)$, moreover, $\mathbb{1}_{\mathring{\mathbb{Q}}_\gamma} \Vdash_{\mathbb{P}_\gamma} (p^\gamma \Vdash_{\mathbb{P}_{\gamma\delta}} \mathring{a} \in V)$. Then we obtain that $\mathbb{1}_{\mathring{\mathbb{Q}}_\gamma}$ forces:

$$\text{there are } \mathbb{P}_{\gamma\delta}\text{-names } \mathring{f} \leq_n^\gamma p^\gamma \text{ and } b \in V \text{such that } \mathring{f} \Vdash_{\mathbb{P}_{\gamma\delta}} \mathring{a} = \check{b} \qquad (*)$$

For $(*)$, applying (c) at the γ-stage, there are $q \in \mathbb{P}_\gamma$ and $f \in \mathbb{P}_{\gamma\delta}$ such that $q \leq_{F,n} p|_\gamma$ and $q \Vdash_{\mathbb{P}_\gamma} \mathring{f} = f\ \cdots(*\text{–i})$. And for $(*)$, applying (a) at the γ-stage there are $q' \in \mathbb{P}_\gamma$ and countable X in V such that $q' \leq_{F,n} q$ and $q' \Vdash_{\mathbb{P}_\gamma} b \in \check{X}\ \cdots(*\text{–ii})$. Therefore from $(*)$, $(*\text{–i})$ and $(*\text{–ii})$, for $f \in \mathbb{P}_{\gamma\delta}$, $q' \in \mathbb{P}_\gamma$ and countable X, $q' \Vdash_{\mathbb{P}_\gamma} (f \leq_{F,n} p^\gamma \wedge f \Vdash_{\mathbb{P}_{\gamma\delta}} \mathring{a} \in \check{X})$. Recall that we have $q \leq_{F,n} p|_\gamma$, $f \leq_{\mathbb{P}_{\gamma\delta}} p^\gamma$ and $F \subset \gamma$, we obtain that $q \cup f \leq_{F,n} p|_\gamma \cup p^\gamma = p$.

□

Since we have already seen that the equivalent for A–4 and A–4', we have the following result:

Lemma III.4.9. The following is equivalent to (a) in Lemma III.4.8.

(a') If I is an antichain in $\mathbb{P}_\beta$, then the following set is (F,n)-dense for all finite $F \subset \beta$ and $n \in \omega$;

- $\{q \in \mathbb{P}_\beta\,;\,$ " $\{q \in I\,;\, p \not\perp q\}$ is countable in V." $\}$,

Lemma III.4.10. Assume that CH and let $\alpha \in \omega_2$ with $\Vdash_{\mathbb{P}_\beta} |\mathring{\mathbb{Q}}_\beta| \leq \aleph_1$. Then there is $D_\alpha \subset \mathbb{P}_\alpha$ such that $|D_\alpha| \leq \aleph_1$ and D_α is (F,n)-dense for arbitrary $F \in [\alpha]^{<\omega}$ and $n \in \omega$.

Proof. We show by induction on α.

For $\alpha = 0$. By letting $D_\alpha = \{\emptyset\}$, we are done.

The case of $\alpha = \beta + 1$. Let $D_\beta \subset \mathbb{P}_\beta$ with $\Vdash_\beta |\mathring{\mathbb{Q}}| \leq \aleph_1$ and D_β is (F,n)-dense for all $F \in [\beta]^{<\omega}$ and $n \in \omega$. Let $\{\mathring{d}_\xi\,;\,\xi \in \omega_1\}$ be a set such that $\Vdash_\beta \mathring{\mathbb{Q}}_\beta = \{\mathring{d}_\xi\,;\,\xi \in \omega_1\}$. Fix a $\mathbb{P}_\beta$-generic G_β. For antichain $I \subset D_\beta$ which is countable in V and a function $f \in {}^I\omega_1$ in V, define a $\mathbb{P}_\beta$-name $\mathring{q}(f)$ is; $\mathring{d}_{f(p)}$ if $I \cap G_\beta = \{p\}$ and $\mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\beta}$ if $I \cap G_\beta = \emptyset$. Since we have $V \models \mathrm{CH}$, the number of such a function f is at most $\aleph_1$, so the cardinality of $T = \{\mathring{q}(f)\,;\, f\}$ is $\leq \aleph_1$. Then by letting $D_\alpha = \{p \in \mathbb{P}_\alpha\,;\, p|_\beta \in D_\beta \wedge p(\beta) \in T\}$ we obtain that $|D_\alpha| \leq \aleph_1$. To see that D_α is (F,n)-dense for every $F \in [\alpha]^{<\omega}$ and $n \in \omega$. Let I be a maximal antichain of $\{d \in D_\beta\,;\, d \leq_{\mathbb{P}_\beta} p|_\beta \wedge \exists \xi \in \omega_1\ (d \Vdash_\beta p(\beta) = \mathring{d}_\xi)\}$. By Lemma III.4.9 and induction hypothesis, there is $d \in D_\beta$ such that $d \leq_{F,n} p$ and $J = \{q \in I\,;\, d \not\perp q\}$ is countable in V. Note that J is a dense antichain. So by letting $f\colon J \to \omega_1$ which assign the $q \in J$ the ξ if $q \Vdash_\beta p(\beta) = \mathring{d}_\xi$, the $\mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}}$ if otherwise. Note that since I is maximal, there is $d' \in J$ with $d \not\perp d'$ and $d' \Vdash_\beta p(\beta) = \mathring{d'}_\xi$ for some $\xi \in \omega_1$. Thus for a common extension d'' for d and d', we obtain that $d'' \Vdash_\beta p(\beta) = \mathring{q}(f)$. Then by letting $r = d''{}^\frown\langle \mathring{q}(f)\rangle \in D_\alpha$, since we have $d'' \leq_{F\cap\beta,n} p|_\beta$, we obtain that $r \leq_{F,n} p$.

The case of α is limit. For $\beta \in \alpha$ and $q \in D_\beta$, define $\overline{q} = q^\frown\langle\mathring{\mathbb{1}}\rangle^\frown \ldots {}^\frown\langle\mathring{\mathbb{1}}\rangle \in \mathbb{P}_\alpha$. Define $\overline{D}_\alpha = \{\overline{q}\,;\, \exists\beta \in \alpha\ q \in D_\beta\}$ and we shall find $D_\alpha \subset \mathbb{P}_\alpha$ with following conditions:

i. $|D_\alpha| \leq \aleph_1$,

ii. $\overline{D}_\alpha \subset D_\alpha$,

iii. For an (F,n)-fusion sequence $\{\langle p_n, F_n\rangle\,;\, n \in \omega\}$ in $\mathbb{P}_\alpha$ with $p_n \in D_\alpha$, there is a fusion p in D_α.

Fix $d \in D_\alpha$ and a fusion for (F,n)-fusion sequences in $\mathbb{P}_\alpha$. For $D \subset \mathbb{P}_\alpha$ let $f\colon {}^{<\omega_1}D \to D$ be a fuction which assings to x the:

$$\begin{cases} p & \text{if there is } F\colon \omega \to [\omega]^{<\omega} \text{ such that } \{\langle x(n), F(n)\rangle\,;\, n \in \omega\} \text{ is an} \\ & (F,n)\text{-fusion sequence and } p \text{ is the fusion for } \{\langle x(n), F(n)\rangle\,;\, n \in \omega\} \\ d & \text{if there is } n \in \omega\ x(n) \text{ is not a singleton or otherwise.} \end{cases}$$

Then there is a closure D_α such that $\overline{D}_\alpha \subset D_\alpha$ and D_α is closed under f and note that $|D_\alpha| \leq \omega_1^{<\omega} = \omega_1$ and triviality hold the conditions.

To see that D_α is (F,n)-dense, let $p \in \mathbb{P}_\alpha$ $F \in [\alpha]^{<\omega}$ and $n \in \omega$ and we distinguish two cases, according to whether $\mathrm{cf}(\alpha) = \omega$. If $\omega \in \mathrm{cf}(\alpha)$. Let $\beta \in \alpha$ with $\mathrm{supt}(p) \subset \beta$ and $F \subset \beta$. By induction hypothesis there is $q \in D_\beta$ such that $q \leq_{F,n} p|_\beta$ moreover, we obtain that $\overline{q} \leq_{F,n} p$. If $\mathrm{cf}(\alpha) = \omega$. Let $\xi_m \in \alpha$, $m \in \omega$, with $\xi_m \nearrow \alpha$ and $F \subset \xi_0$. Define $q_m \in D_{\xi_m}$, $m \geq n$, recursively, by induction hypothesis there is $q_n \in D_{\xi_0}$ such that $q_n \leq_{F \cap \beta, n} p|_{\xi_0}$ and for $q_m \in D_{\xi_m}$ there is $q_{m+1} \in D_{\xi_{m+1}}$ such that $q_{m+1} \leq_{F,n} q_m \cup p|_{[\xi_m, \xi_{m+1})}$ for $m \geq n$. Then we obtain an (F,n)-fusion sequence $\langle (p_m, F_m)\,;\, m \in \omega \rangle$ of $\mathbb{P}_\alpha$ such that $p_m = \overline{q_m}$ and $F_m = F$. Then by Lemma III.4.7, there is $q \in D_\alpha$ such that $q \leq_{F,m} p_m$ for all $m \in \omega$. Therefore we obtain that $q \leq_{F,n} p$. □

Proof of Theorem III.4.5. To see that $\mathbb{1} \Vdash \omega_1 = \omega_1^V$, we assume that if not, i.e., there is $p \in \mathbb{P}$ such that $p \Vdash (\exists f \colon \omega_1^V \to \omega$ " f is an injection."). Then we obtain that $p \Vdash \{f(n)\,;\, n \in \omega_1^V\} \subset \omega$. Now by applying (b) in Lemma III.4.8, we can show the general case in the similar way, there is an extension such that $q \in \mathbb{P}_\alpha$ and countable X in V such that $q \Vdash \{f(n)\,;\, n \in \omega_1^V\} \subset \check{X}$. This shows that $\omega_1 \subset X$ in V, a contradiction.

For the moreover part. We distinguish two cases, according whether $\alpha = \omega_2$. If $\alpha \in \omega_2$, Assume that there is an antichain $I \subset \mathbb{P}_\alpha$ of size $> \aleph_2$. We may assume that I is maximal. Lemma III.4.9 asserts that there is D_α such that for every $x \in I$, $F \in [\alpha]^{<\omega}$ and $n \in \omega$, there is $x_D \in D_\alpha$ so that $x_D \leq_{F,n} x$. Since $|D_\alpha| \leq \aleph_2$ and $|I| > \aleph_2$, Pigeonhole principle asserts that there is $\{x, y\} \in [I]^2$ such that $x_D = y_D$. Then for every $\eta \in \alpha$, for a finite set $\{\eta\}$ and $0 \in \omega$, we obtain that $x_D|_\eta \Vdash x_D(\eta) \leq_{\mathbb{P}_\eta} x(\eta), y(\eta)$. Thus we obtain that $x_D \leq_{\mathbb{P}_\alpha} x, y$. This contraly that $x \perp y$. If $\alpha = \omega_2$. Assume that there is an antichain I of size $\aleph_2$. Let $\mathcal{A} = \{\mathrm{supt}(r)\,;\, r \in I\}$. We assert that there is an ordinal $\beta \in \alpha$ and $\mathcal{I} \in [I]^{\aleph_2}$ such that $\forall \{p, q\} \in [\mathcal{I}]^2$ $\mathrm{supt}(p) \cap \mathrm{supt}(q) \in \beta$. If this holds, since $p|_\beta \not\perp q|_\beta$ iff $p \not\perp q$ for $p, q \in \mathcal{I}$, a contradiction. To see there is an ordinal and a family, we distinguish two cases, according to whether $|\cup \mathcal{A}| = \aleph_2$. If $|\cup \mathcal{A}| \in \aleph_2$, Pigeonhole principle shows that there is $\mathcal{I} \in [I]^{\aleph_2}$ which elemets are the same support and there is an ordinal β greater than the support them all. If $|\cup \mathcal{A}| = \aleph_2$. Since we have CH, we obtain that $\forall \theta \in \aleph_1$ $(\theta^{<\aleph_1} \in \aleph_2)$ holds. Therefore, by Delta System Lemma, there is $\mathcal{I} \in [I]^{\aleph_2}$ and a root R such that $\forall \{p, q\} \in [\mathcal{I}]^2$ $\mathrm{supt}(p) \cap \mathrm{supt}(q) = R \in \beta$ for some ordinal $\beta \in \alpha$. □

III.5 Proper Forcing

In this section we assume that a forcing notion is separative.

Definition III.5.1 ([Kun11]). For a regular cardinal κ, the collection of sets of *cardinality hereditarily* $< \kappa$ is $H_\kappa = \{x\,;\, |\mathrm{trcl}(x)| < \kappa\}$. Note that $H_\kappa \models \mathrm{ZFC} - P$ if κ is an uncountable reguler cardinal.

Definition III.5.2 ([Abr10]). Let $\mathbb{P}$ be a forcing notion, λ be an uncountable regular cardinal and $M \preccurlyeq H_\lambda$ with $\mathbb{P} \in M$. $q \in \mathbb{P}$ is $(M, \mathbb{P})$-*generic* provided that for any dense subset, D, of $\mathbb{P}$ if $D \in M$ $D \cap M$ is predense below q.

Lemma III.5.3 ([Abr10])**.** For $\mathbb{P}$ and M as per in previous definition, $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic iff for every dense subset D of $\mathbb{P}$ in M and $(V, \mathbb{P})$-generic G with $p \in G$ there is $\mathring{p} \in V^{\mathbb{P}}$ such that $q \Vdash \mathring{p} \in M \cap D \cap \mathring{G}$.

Proof. It suffices to show that for $q \in \mathbb{P}$ and $E \subset \mathbb{P}$, E is predense below q iff $q \Vdash E \cap \mathring{G} \neq \emptyset$ for arbitrary $(V, \mathbb{P})$-generic G with $p \in G$.

($\Rightarrow$) Let G be $(V, \mathbb{P})$-generic with $q \in G$. Since $F = \{p \in \mathbb{P} \,; p \perp q \vee \exists r \in E \ p \leq r\}$ is dense subset of $\mathbb{P}$, there is $p \in G \cap F$. Since $p \perp q$, we have $r \in G$. This shows that $q \Vdash E \cap \mathring{G} \neq \emptyset$.

($\Leftarrow$) Let $p \in q$ and G a $(V, \mathbb{P})$-generic with $p \in G$. Since $q \in G$ there is $r \in E \cap G$, which witnesses E is predense below q. $\square$

Definition III.5.4 ([Abr10])**.** A forcing notion $\mathbb{P}$ is *proper* provided that for an arbitrary uncountable regular cardinal $\lambda > 2^{|\mathbb{P}|}$, or simply says sufficiently large regular cardinal λ, and a contable elementary submodel $M \preccurlyeq H_\lambda$ with $\mathbb{P} \in M$, we have:

$$\forall p \in \mathbb{P} \cap M \exists q \leq p \text{ “} q \text{ is } (M, \mathbb{P})\text{-generic.”}$$

Example III.5.5 ([Abr10])**.**

1. A ccc forcing is proper.
2. A σ-closed forcing is proper.
3. An Axiom A forcing is proper.
4. A shooting a club by finite conditions forcing is proper. (see Theorem IV.5.2)

Proof. 1. Let λ be an uncountable regular cardinal with $\lambda > 2^{|\mathbb{P}|}$, $M \preccurlyeq H_\lambda$ with $\mathbb{P} \in M$ and $p \in \mathbb{P} \cap M$. We shall verify that p is $(M, \mathbb{P})$-generic. For a dense subset $D \subset \mathbb{P}$ with $D \in M$, let $A \subset D$ be a maximal antichain and by elementarity we may choose as $A \in M$. Since $\mathbb{P}$ is a ccc forcing, there is a surjection $f \colon \omega \to A$ in V. Furthermore, since $f \in H_\lambda$, we may assume that $f \in M$ and therefore, $A = f(\omega) \subset M$. Then A is predense below p and since $A \subset D \cap M$, so does D.

2. Let λ be an uncountable regular cardinal with $\lambda > 2^{|\mathbb{P}|}$, $M \preccurlyeq H_\lambda$ with $\mathbb{P} \in M$ and $p \in \mathbb{P} \cap M$. Let $\mathcal{D} = \{D \in M \,; \text{“ } D \subset \mathbb{P} \text{ is dense.” } \} \subset M$ and enumerate $\mathcal{D} = \{D_i \,; i \in \omega\}$ in V. Note that, by elementarity, for each $q \in \mathbb{P} \cap M$ there is an extension $r \in D \cap \ M$ for any $D \in \mathcal{D}$. Thus we obtain a decreasing sequence $\langle p_n \in \mathbb{P} \cap M \,; n \in \omega \rangle$ such that:

- $p_0 = p$,
- $p_{n+1} \leq p_n$ such that $p_{n+1} \in D_n \cap M$.

and moreover since $\mathbb{P}$ is σ-closed, there is $q \in \mathbb{P}$ such that $q \leq p_n$ for all $n \in \omega$. The statement that q is $(M, \mathbb{P})$-generic is immediate by the construction, thus we conclude the proof.

3. $\square$

Theorem III.5.6 ([Jec03])**.** A forcing notion $(\mathbb{P}, <)$ is proper iff for every uncountable cardinal λ every stationary subset of $[\lambda]^\omega$ remains stationary in the generic extension.

Before, we prove the theorem, we shall make some lemmata.

Lemma III.5.7. Let $\mathbb{P}$ be a ccc forcing. Every club subset $C \subset [\lambda]^\omega$ in $V[G]$ has a subset $D \in V$ which is club $D \in V$. Moreover, every stationary set $S \subset [\lambda]^\omega$ remains stationary in $V[G]$.

Lemma III.5.8. Let $\mathbb{P}$ be a σ-closed forcing. Every stationary set $S \subset [\lambda]^\omega$ remains stationary in $V[G]$.

Lemma III.5.9.

Definition III.5.10 ([Jec03])**.** Let $\mathbb{P}$ be a forcing notion and $p \in \mathbb{P}$. The *proper game* for $\mathbb{P}$, below p, is played as follows:

- I plays P-names $\mathring{\alpha}_n$ for ordinal numbers, and
- II plays ordinal numbers β_n.

Player II wins provided that there exists an extension $q \leq p$ which forces $\forall n \in \omega \exists k \in \omega\ \mathring{\alpha}_n = \beta_k$.

Theorem III.5.11 ([Jec03])**.** A forcing notion $\mathbb{P}$ is proper iff II has a winning strategy for the proper game for any $p \in \mathbb{P}$.

Theorem III.5.12 ([Abr10])**.** A proper forcing preseves ω_1.

Proof. Let $\mathring{f}$ be a $\mathbb{P}$-name and p a condition such that $p \Vdash \mathring{f} \colon \omega \to \omega_1^V$. We shall show that $p \Vdash$ " $\mathring{f}$ is not a surjection." . Fix an uncountable regular cardinal $\lambda > 2^{|\mathbb{P}|}$ such that $\mathring{f}, \mathbb{P}, \omega_1^V \in H_\lambda$ and let M be a countable elementary submodel of H_λ such that $\mathring{f}, \mathbb{P}, p, \omega_1^V \in M$. Since $\mathbb{P}$ is proper, there is an extension $q \leq p$ which is $(M, \mathbb{P})$-generic. Fix an $n \in \omega$. Define $D_n = \{r \in \mathbb{P}; (r \perp p) \vee (\exists \alpha \in \omega_1^V\ r \Vdash \mathring{f}(n) = \alpha)\}$ in H_λ. By elementary we obtain that $D_n \in M$, thus for an extension $q_1 \leq q$ there is $r \in D_n \cap M$ such that $q_1 \not\perp r$ and moreover since $q_1 \leq p$, there is $\alpha \in \omega_1$ such that $r \Vdash \mathring{f}(n) = \alpha$, furthermore, by elementary we may assume that $\alpha \in \omega_1^V \cap M$.

This shows that $q \Vdash \forall n \in \omega\ \mathring{f}(n) \in M$, i.e., $p \Vdash$ " $\mathring{f}$ is not a surjective." . Therefore, $\Vdash \omega_1 = \omega_1^V$. $\square$

Hereafter, let λ be a sufficiently large regular cardinal such that $\lambda > 2^{|\mathbb{P}|}$.

Theorem III.5.13 ([Abr10])**.** Let $\mathbb{P}$ be a forcing notion and $M \preccurlyeq H_\lambda$ be countable with $\mathbb{P} \in M$ and suppose that $p \in \mathbb{P}$ is a $(M, \mathbb{P})$-generic condition. For $x, y \in M \cap V^{\mathbb{P}}$, if we have $p \Vdash x = y$, then there is an extension $r \leq p$ and $(z, q) \in y \cap M$ such that $z \in \mathbb{P}$, $q \in V^{\mathbb{P}}$, $r \leq a$ and $r \Vdash x = z$.

Proof. Define $E = \{r \in \mathbb{P}\,; \exists (z,q) \in x\ (r \leq q \wedge r \Vdash x = z)\}$ and $E' = \{r \in \mathbb{P}\,; r \in E \vee r \perp E\}$. Then E' is a dense set in $\mathbb{P}$ with $E' \in M$. Then there is $r \in M \cap E'$ which is compatible with p and there is $(z,q) \in x$ such that $r \leq q$ and $r \Vdash x = z$. Furthermore, since $M \preccurlyeq H_\lambda$, we may assume that $(z,q) \in M$, we are done. $\square$

Lemma III.5.14 ([Abr10])**.** Let $\mathbb{P}$ be a forcing notion and $\Vdash_{\mathbb{P}}$ " $\mathbb{Q}$ is a forcing notion." , $M \preccurlyeq H_\lambda$ countable with $\mathbb{P}, \mathring{\mathbb{Q}}_G \in M$ and $p \in \mathbb{P} \cap M$ and $\mathring{q} \in \mathring{\mathbb{Q}} \cap M$ where G is $(V, \mathbb{P})$-generic. Then

$$(p, \mathring{q}) \text{ is } (M, \mathbb{P} * \mathbb{Q})\text{-generic iff}$$

$$p \text{ is } (M, \mathbb{P})\text{-generic and } p \Vdash \text{“ } \mathring{q} \text{ is } (M[\mathring{G}], \mathring{\mathbb{Q}})\text{-generic.” }.$$

Proof. $(\Rightarrow)$. To see that p is $(M, \mathbb{P})$-generic. Let $D \subset \mathbb{P}$ be a dense subset with $D \in M$. Then $D' = \{(d, \mathring{q})\,; d \in D\}$ be a dense subset of $\mathbb{P} * \mathring{\mathbb{Q}}$ with $D' \in M$. Therefore there is $(d, \mathring{q}) \in D' \cap M$ such that $(d, \mathring{q}') \leq_{\mathbb{P} * \mathring{\mathbb{Q}}} (p, \mathring{q})$. Moreover, by the elementary there is $d \in D \cap M$ such that $d \leq_{\mathbb{P}} p$.

To see that $p \Vdash_{\mathbb{P}}$ " $\mathring{q}$ is $(M[\mathring{G}], \mathring{\mathbb{Q}})$-generic." , it suffices show that for any dense open subset $\mathring{D} \in M[\mathring{G}]$ in $\mathring{\mathbb{Q}}$ and $\mathring{q}' \leq_{\mathring{\mathbb{Q}}} \mathring{q}$, p forces:

$$\exists \mathring{q}'' \in \mathring{D} \cap M[\mathring{G}]\ \mathring{q}'' \not\perp_{\mathring{\mathbb{Q}}} \mathring{q}.$$

Let $p' \leq_{\mathbb{P}} p$. Define $E = \{(s, \mathring{t}) \in \mathbb{P} * \mathring{\mathbb{Q}}\,; s \Vdash \mathring{t} \in \mathring{D}\}$ and $F = \{(s, \mathring{t}) \in \mathbb{P} * \mathring{\mathbb{Q}}\,;(s, \mathring{t}) \in E \vee (s, \mathring{t}) \perp_{\mathbb{P} * \mathring{\mathbb{Q}}} E\}$. Then $F \in M$ and F is dense in $\mathbb{P} * \mathring{\mathbb{Q}}$. Then there is $(s, \mathring{t}) \in F \cap M$ which compatible with $(p', \mathring{q})$ and let $(p_2, \mathring{q}_2)$ be a common extension and $\mathring{q}'' \in \mathring{D}$ with $\mathring{q}'' \leq_{\mathring{\mathbb{Q}}} \mathring{q}_2$. Then we obtain that $p_2 \Vdash \mathring{q}'' \leq_{\mathring{\mathbb{Q}}} \mathring{t}$. So, $(s, \mathring{t}) \in E$, moreover, $p_2 \Vdash \mathring{t} \in \mathring{D}$ and note that $\mathring{t} \in M$.

This argument shows that there is $(p_2, \mathring{t}) \in \mathbb{P} * \mathring{\mathbb{Q}}$ such that $p_2 \leq p'$, $p_2 \Vdash \mathring{q}' \not\perp_{\mathring{\mathbb{Q}}} \mathring{t} \wedge \mathring{t} \in \mathring{D} \wedge \mathring{t} \in M[\mathring{G}]$. Furthermore, since there is densely many extension, we have:

$$p \Vdash \exists \mathring{q}'' \in \mathring{D} \cap M[\mathring{G}]\ \mathring{q} \not\perp_{\mathring{\mathbb{Q}}} q''.$$

$(\Leftarrow)$. Let $D \subset \mathbb{P} * \mathring{\mathbb{Q}}$ be a dense open set with $D \in M$ and let (p_1, q_1) be an extension for (p, q). We shall find a $(p', q') \in D \cap M$ such that $(p_1, q_1) \not\perp_{\mathbb{P} * \mathring{\mathbb{Q}}} (p', q')$. Let $\mathring{E} = \{\mathring{q} \in \mathring{\mathbb{Q}}\,; \exists p' \in \mathring{G}\ (p', \mathring{q}) \in D\}$. To see that $\Vdash_{\mathbb{P}}$ " E is dense." , it suffices to verify that:

$$\forall (s, \mathring{t}) \in \mathbb{P} * \mathring{\mathbb{Q}} \exists s' \leq_{\mathbb{P}} s\ s' \Vdash_{\mathbb{P}} (\mathring{q} \in \mathring{E} \wedge \mathring{q} \leq_{\mathring{\mathbb{Q}}} \mathring{t}).$$

For $(s, \mathring{t})$, since D is dense, there is an extension $(s', \mathring{t}')$ and $s' \Vdash_{\mathbb{P}} (\mathring{t}' \leq_{\mathbb{Q}} \mathring{t} \wedge s \in \mathring{G})$ holds. Thus, $s \Vdash_{\mathbb{P}} \mathring{t} \in \mathring{E}$, and we obtain that $\Vdash_{\mathbb{P}}$ " $\mathring{E}$ is dense." and note that $\Vdash_{\mathbb{P}} \mathring{E} \in M[\mathring{G}]$, so by assumption we have $p_1 \Vdash_{\mathbb{P}} \exists \mathring{r} \in \mathring{E} \cap M[\mathring{G}]\ \mathring{r} \not\perp_{\mathring{\mathbb{Q}}} \mathring{q}_1$, furthermore there is $\mathring{r} \in V^{\mathbb{P}} \cap M$ and $\mathring{t}' \in \mathring{\mathbb{Q}}$ such that $p_1 \Vdash_{\mathbb{P}} \mathring{r} \in \mathring{E} \wedge \mathring{t}' \leq_{\mathring{\mathbb{Q}}} \mathring{r}, \mathring{q}_1$.

Define $F = \{s \in \mathbb{P}\,;(s, \mathring{r}) \in D\}$ and $F' = \{s \in \mathbb{P}\,; s \in F \vee s \perp_{\mathbb{P}} F\}$ and note that F' is predense in $\mathbb{P}$ and $F' \in M$. Then there is $s \in F' \cap M$ such that $s \not\perp_{\mathbb{P}} p_1$ and by letting s' be a common extension, we have $s' \Vdash_{\mathbb{P}} \mathring{r} \in \mathring{E}$. Hence, there is a t such that $s' \Vdash_{\mathbb{P}} (t \in \mathring{G} \wedge (t, \mathring{r}) \in D)$ and note that we obtain $s' \leq_{\mathbb{P}} t$ and $t \in F$ witnesses $s \in F$.

Therefore, $(s, \mathring{r}) \in D \cap M$ and $(s', \mathring{t})$ witnesses $(s, \mathring{r}) \not\perp_{\mathbb{P} * \mathring{\mathbb{Q}}} (p_1, \mathring{q}_1)$ $\square$

Theorem III.5.15 ([Abr10])**.** Let $\mathbb{P}_0$ be a proper forcing notion, $\mathring{\mathbb{P}}_1 \in V^{\mathbb{P}_0}$ with $\Vdash_{\mathbb{P}_0}$ " $\mathring{\mathbb{P}}_1$ is proper." , $\mathbb{Q} = \mathbb{P}_0 * \mathring{\mathbb{P}}_1$ and $\pi \colon \mathbb{Q} \to \mathbb{P}_0 ; (p, \mathring{q}) \mapsto p$. Then for a countable model $M \preccurlyeq H_\lambda$ with $\mathbb{Q} \in M$ and $(M, \mathbb{P}_0)$-generic $p_0 \in \mathbb{P}_0$ we have the following.

For any $\mathring{r} \in V^{\mathbb{P}_0} \cap M$ if $p_0 \Vdash_{\mathbb{P}_0} (\mathring{r} \in \mathbb{Q} \wedge \pi(\mathring{r}) \in \mathring{G}_0)$, then there is $\mathring{p}_1 \in V^{\mathbb{P}_0}$ such that $(p_0, \mathring{p}_1) \in \mathbb{Q}$ is $(M, \mathbb{Q})$-generic and $(p_0, \mathring{p}_1) \Vdash_{\mathbb{Q}} \mathring{r} \in \mathring{G}$. Where G_0 and G is $\mathbb{P}_0$ and $\mathbb{Q}$ generic, respectively.

Moreover, $\mathbb{Q}$ is proper.

Proof. Let $p_0 \in \mathbb{P}_0$ be $(M, \mathbb{P}_0)$-generic and $(r_0, \mathring{r}_1) \in R \cap M$ with $p_0 \Vdash_{\mathbb{P}_0} r_0 \in \mathring{G}_0$. Since $p_0 \Vdash_{\mathbb{P}_0}$ " $\mathring{\mathbb{P}}_1$ is proper." and $M[\mathring{G}] \preccurlyeq H_\lambda[\mathring{G}]$, there is $\mathring{p}_1 \in V^{\mathbb{P}_0}$ such that $p_0 \Vdash_{\mathbb{P}_0} (\mathring{p}_1 \Vdash_{\mathbb{P}_1} \mathring{r}_1) \wedge$ " $\mathring{p}_1$ is $(M[\mathring{G}], \mathring{\mathbb{P}}_1)$-generic." and since $p_0 \leq_{\mathbb{P}_0} r_0$, we obtain that $(p_0, \mathring{p}_1) \leq_{\mathbb{Q}} (r_0, \mathring{r}_1)$ and $p_0 \Vdash_{\mathbb{P}_0}$ " $\mathring{p}_1$ is $(M[\mathring{G}], \mathring{\mathbb{P}}_1)$-generic." . Therefore, by the previous lemma, we obtain that $(p_0, \mathring{p}_1)$ is $(M, \mathbb{Q})$-generic and $(p_0, \mathring{p}_1) \Vdash_{\mathbb{Q}} (r_0, \mathring{r}_1) \in \mathring{G}$.

The moreover part is immediate by the first assertion. $\square$

Lemma III.5.16 (**The Properness Extension Lemma**, [Abr10])**.** Let γ be a limit ordinal, $\mathbb{P}_\gamma = \langle \mathbb{P}_\alpha ; \alpha \in \gamma \rangle$ a countable support iteration of proper forcing and $M \preccurlyeq H_\lambda$ a countable submodel with sufficiently large cardinal λ with $\mathbb{P}_\gamma, \gamma \in M$.

For any $\gamma_0 \in \gamma \cap M$, $q_0 \in \mathbb{P}_{\gamma_0}$ and name $\mathring{p}_0 \in V^{\mathbb{P}_{\gamma_0}}$.

If $q_0 \in \mathbb{P}_{\gamma_0}$ forces at $\mathbb{P}_{\gamma_0}$ stage:

$$\mathring{p}_0 \in \mathbb{P}_\gamma \cap M \text{ and } \mathring{p}_0|_{\gamma_0} \in \mathring{G}_0$$

then there is an $(M, \mathbb{P}_\gamma)$-generic condition q such that $q|_{\gamma_0} = q_0$ and $q \Vdash_{\mathbb{P}_\gamma} i_*(\mathring{p}_0) \in \mathring{G}_\gamma$

Proof. For $\gamma_0 \in \gamma \cap M$, $\mathring{p}_{\gamma_0} \in V^{\mathbb{P}_{\gamma_0}}$ and $(M, \mathbb{P}_{\gamma_0})$-generic q_{γ_0} we assume that

$$q_{\gamma_0} \Vdash_{\mathbb{P}_{\gamma_0}} (\mathring{p}_{\gamma_0} \in \mathbb{P}_\gamma \cap M \wedge \mathring{p}_{\gamma_0}|_{\gamma_0} \in \mathring{G}_{\gamma_0})$$

We shall found the desired $(M, \mathbb{P}_\gamma)$-generic condition by induction on γ.

(i) $\gamma = 0$. It is immediate that there are no any elements in $\gamma \cap M$.

(ii) $\gamma = \gamma' + 1$. We have seen the case of $\gamma' = \gamma_0$ in Theorem III.5.15, so we may assume that $\gamma_0 \in \gamma'$.

Note that we have $\gamma', \mathbb{P}_{\gamma'} \in M$ and $\lambda > 2^{|\mathbb{P}_{\gamma'}|}$ and since q_{γ_0} forces

$$\mathring{p}_{\gamma_0}|_{\gamma'} \in \mathbb{P}_{\gamma'} \cap M \text{ and } (p_{\gamma_0}|_{\gamma'})|_{\gamma_0} \in \mathring{G}_{\gamma_0},$$

by indution hypothesis, there is an $(M, \mathbb{P}_{\gamma'})$-generic $q_{\gamma'}$ such that

$$q_{\gamma'}|_{\gamma_0} = q_{\gamma_0} \text{ and } q_{\gamma'} \Vdash_{\mathbb{P}_{\gamma'}} i_*(\mathring{p}_{\gamma_0})|_{\gamma'} = \mathring{p}_{\gamma_0}|_{\gamma'} \in \mathring{G}_{\gamma'}.$$

Theorem III.5.15 assets that there is $(M, \mathbb{P}_\gamma)$-generic $q \in \mathbb{P}_\gamma$ such that

$$q|_{\gamma'} = q_{\gamma'} \text{ and } q \Vdash_{\mathbb{P}_\gamma} i_*(\mathring{p}_{\gamma_0}) \in \mathring{G}_\gamma$$

and moreover, we have $q|_{\gamma_0} = q_{\gamma_0}$.

(ii) γ is limit. Since $\gamma_0 \in \gamma \cap M$ and $\mathrm{cf}(\gamma \cap M) = \omega$, there is $\langle \gamma_n \,; n \in \omega \rangle \nearrow \gamma \cap M$ and enumerate the dense sets in $\mathbb{P}_\gamma \cap M$, $\{D_n \,; n \in \omega\}$. Now we define $q_n \in \mathbb{P}_{\gamma_n}$ and $p_n \in V^{\mathbb{P}_{\gamma_n}}$ recursively such as:

1. $q_0 = q_{\gamma_0}$,
2. q_n is $(M, \mathbb{P}_{\gamma_n})$-generic such as $q_{n+1}|_{\gamma_n} = q_n$,
3. $\mathring{p}_0 = \mathring{p}_{\gamma_0}$,
4. q_{n+1} forces, over $\mathbb{P}_{\gamma_{n+1}}$, and

(4–a.) $p_{n+1} \in \mathbb{P}_\gamma \cap M$,

(4–b.) $\mathring{p}_{n+1}|_{\gamma_{n+1}} \in \mathring{G}_{\gamma_{n+1}}$,

(4–c.) $\mathring{p}_{n+1} \leq_{\mathbb{P}_{\gamma_n}} i_*(p_n)$, and

(4–d.) $\mathring{p}_{n+1} \in D_n$.

Suppose we have p_n and q_n, we shall found p_{n+1} and q_{n+1}.

Fix a $q' \leq_{\mathbb{P}_{\gamma_n}} q_n$. There is $q'' \leq_{\mathbb{P}_{\gamma_n}} q'$ and $p \in \mathbb{P}_\gamma \cap M$ such that $q' \Vdash \mathring{p}_n = p$. By letting $E = \{r|_{\gamma_n} \in \mathbb{P}_{\gamma_n} \,; r \in D_n \wedge r \leq_{\mathbb{P}_{\gamma_n}} p\}$, we have $E \in M$ and is dense below $p|_{\gamma_n}$. Since $q'' \Vdash_{\mathbb{P}_{\gamma_n}} p|_{\gamma_n} \in \mathring{G}_{\gamma_n}$ and q'' is $(M, \mathbb{P}_{\gamma_n})$-generic, by Lemma III.5.3 there is $\mathring{r} \in V^{\mathbb{P}_{\gamma_n}}$ such that $q'' \Vdash_{\mathbb{P}_{\gamma_n}} \mathring{r} \in E \cap M \cap \mathring{G}_{\gamma_n}$ and moreover, there is $\mathring{p} \in V^{\mathbb{P}_{\gamma_n}}$ such that q'' forces on $\mathbb{P}_{\gamma_n}$

$$\mathring{p} \in D_n \cap M,\ \mathring{p}|_{\gamma_n} \in \mathring{G}_{\gamma_n} \text{ and } \mathring{p} \leq_{\mathbb{P}_{\gamma_n}} \mathring{p}_n.$$

Therefore we obtain that q_n forces on $\mathbb{P}_{\gamma_n}$:

$$\mathring{p}|_{\gamma+n+1} \in \mathbb{P}_{\gamma_{n+1}} \cap M \text{ and } (p|_{\gamma_{n+1}})|_{\gamma_n} \in \mathring{G}_{\gamma_n}$$

and hence applying induction hypothesis there is $(M, \mathbb{P}_{\gamma_{n+1}})$-generic $q_{n+1} \in \mathbb{P}_{\gamma_{n+1}}$ such that

$$q_{n+1}|_{\gamma_n} = q_n \text{ and } q_{\gamma_{n+1}} \Vdash_{\mathbb{P}_{\gamma_{n+1}}} i_*(p|_{\gamma_{n+1}}) \in \mathring{G}_{\gamma_{n+1}}.$$

By letting $\mathring{p}_{n+1} = i_* \mathring{p}$, we have that $q_{n+1} \leq_{\mathbb{P}_{\gamma_{n+1}}} i(q_n)$ and $i(q_n)$ forces on $\mathbb{P}_{\gamma_{n+1}}$

$$p_{n+1} \in \mathbb{P}_\gamma \cap M,\ \mathring{p}_{n+1} \leq_{\mathbb{P}_{\gamma_{n+1}}} i_*(\mathring{p}_n),\ \mathring{p}_{n+1}|_{\gamma_{n+1}} \in \mathring{G}_{\gamma+1} \text{ and } \mathring{p}_{n+1} \in D_n.$$

Now we complete the construction of p_n and q_n for $n \in \omega$.

Before we conclude the proof, we shall show that for $q = \cup\{q_n \in \mathbb{P}_{\gamma_n}\,; n \in \omega\} \in \mathbb{P}_\gamma$, we have

$$\forall n \in \omega \; q \Vdash_{\mathbb{P}_\gamma} i_*(p_n) \in \mathring{G}_{\gamma_n} \tag{$*$}$$

Fix an $n \in \omega$. For $m \ni n$, since we have $q_m \Vdash_{\mathbb{P}_{\gamma_m}} \mathring{p_m} \leq_{\mathbb{P}_\gamma} i_*(\mathring{p_n})$ and $q_m \Vdash_{\mathbb{P}_{\gamma_m}} \mathring{p_m}|_{\gamma_m} \in \mathring{G}_{\gamma_m}$, we have

$$q \Vdash_{\mathbb{P}_\gamma} i_*(\mathring{p_n}) = i_*(i_*(\mathring{p_n})|_{\gamma_m}) \in i_*(\mathring{G}_{\gamma_m})$$

For any extension $q' \leq_{\mathbb{P}_\gamma} q$ there is an extension $q'' \leq_{\mathbb{P}_\gamma} q'$ and $p \in \mathbb{P}_\gamma \cap M$ such that $q'' \Vdash_{\mathbb{P}_\gamma} i_*(\mathring{p_n}) = p$ and thus we have

$$q'' \Vdash_{\mathbb{P}_\gamma} p|_{\gamma_n} \in i_*(\mathring{G}_{\gamma_m}), \text{ moreover } q''|_{\gamma_m} \leq_{\mathbb{P}_{\gamma_m}} p|_{\gamma_m}.$$

Therefore we obtain $q'' \leq_{\mathbb{P}_\gamma} p$, i.e, $q'' \Vdash_{\mathbb{P}_\gamma} p \in \mathring{G}_\gamma$. Hence, we have $q'' \Vdash_{\mathbb{P}_\gamma} i_*(p_n) \in \mathring{G}_\gamma$ and $(*)$ hold.

To see that q is $(M, \mathbb{P}_\gamma)$-generic thanks to Lemma III.5.3, we shall verify that

$$\forall n \in \omega \; q \Vdash_{\mathbb{P}_\gamma} i_*(\mathring{p_{n+1}}) \in D_n \cap M \cap \mathring{G}_\gamma.$$

For an $n \in \omega$, we have that $i(q_{n+1}) \Vdash_{\mathbb{P}_\gamma} i_*(p_{n+1}) \in D_n \cap M$ and $q \leq_{\mathbb{P}_\gamma} i_*(q_{n+1})$, we obtain that

$$q \Vdash_{\mathbb{P}_\gamma} i_*(\mathring{p_{n+1}}) \in D_n \cap M \in \mathring{G}_\gamma.$$

Thus, q is $(M, \mathbb{P}_\gamma)$-generic and since we have $q|_{\gamma_0} = q_{\gamma_0}$, $(*)$ asserts that we have

$$q \Vdash_{\mathbb{P}_\gamma} i_*(\mathring{p_{\gamma_0}}) \in \mathring{G}_\gamma.$$

□

Theorem III.5.17 ([Abr10]). For a limit ordinal δ and a countable support iteration of proper forcings $\langle \mathbb{P}_\alpha\,; \alpha \in \delta\rangle$, $\mathbb{P}_\delta$ is proper.

Proof. Let $\lambda > 2^{|\mathbb{P}_\delta|}$ be a regular uncountable cardinal and $M \preccurlyeq H_\lambda$ be an elementary substrucrure with $\mathbb{P}_\delta \in M$ and we may assume that $\mathbb{P}_\delta \cap M \neq \emptyset$.

Let $p \in \mathbb{P}_\delta \cap M$. Since $\mathbb{P}_0$ is proper, there is an $(M, \mathbb{P}_0)$-generic $q_0 \in \mathbb{P}_0$ with $q_0 \leq_{\mathbb{P}_0} p|_0$. Thus by applying The Properness Extension Lemma, there is an $(M, \mathbb{P}_\delta)$-generic $q \in \mathbb{P}_\delta$ such that $q|_1 = q_0$ and $q \Vdash_{\mathbb{P}_\delta} p \in \mathring{G}_\delta$. Thus we obtain that $q \leq_{\mathbb{P}_\delta} p$.

□

Theorem III.5.18 ([Abr10]). Assume CH and δ be a limit ordinal. For a countable support iteration $\mathbb{P}_\delta = \langle \mathbb{P}_i\,; i \in \delta\rangle$ with:

$$\Vdash_\gamma(\text{ “ } \mathring{\mathbb{Q}}_\gamma \text{ is proper.” } \wedge |\mathring{\mathbb{Q}}_\gamma| = \aleph_1) \text{ for any } \gamma \in \delta.$$

Then, $\mathbb{P}_\delta$ satisfies an $\aleph_2$-cc.

Proof. Fix $\{r_\xi\,; \xi \in \aleph_2\}$, regular $\lambda \geq \max\{2^{|\mathbb{P}_\delta|}, \aleph_3\}$ and well-order $\lhd$ over H_λ. For each $\xi \in \aleph_2$ let $M_\xi = \langle M_\xi, \in|_{M_\xi}, \lhd|_{M_\xi}, \mathbb{P}_\delta, r_\xi\rangle$ be a countable submodel of H_λ.

First of all, we shall found an $I \in [\omega_2]^{\omega_2}$ and cuntable $C \in [\omega_2]^\omega$ such that:

1. $\forall\{\xi_0, \xi_1\} \in [I]^2 \exists f\colon (M_{\xi_0}, \in) \to (M_{\xi_1}, \in)$

   f is order-isomorphism and $f(r_{\xi_0}) = r_{\xi_1}$,

2. $\forall\{\xi_0, \xi_1\} \in [I]^2\ M_{\xi_0} \cap M_{\xi_1} \cap \omega_2 = C$,

3. $\forall \xi_0, \xi_1 \in I\ \xi_0 \in \xi_1 \implies (\cup C \in \mu_{\xi_0} \wedge \cup(M_{\xi_0} \cap \omega_2) \in \mu_{\xi_1})$,

   where $\mu_\xi = \cap((M_\xi \cap \omega_2) \setminus C)$.

4. C is an initial segment of $M_\xi \cap \omega_1$ for any $\xi \in I$, that is:

   $\forall y \in C \forall x \in M_\xi \cap \omega_1\ x \in y \implies x \in C$.

To see the condition 4. Fix a distinct $\eta \in I$. Let $y \in C$ and $x \in y \cap M_3 \cap \omega_1$. We shall show that $x \in M_\eta$. $y \in M_\xi \cap \omega_2$ asserts that there is a function $f_\xi \in M_\xi$ such that $M_\xi \models($ " $f_\xi\colon \omega_1 \to y$ is surjective." $\wedge$ " f_ξ is $\lhd$-minimal." $)$. So does over H_λ. Similarly we found a funciton $f_\eta \in M_\eta$. Since the minimality assert that $f_\xi = f_\eta$ in H_λ and an $\alpha \in M_\xi \cap \omega_1$ such that $f_\xi(\alpha) = x$ witnesses $f_\eta(\alpha) = x \in M_\eta$.

Let ξ, $\eta \in I$ such that $\mu := \mu(\xi) \leq \mu(\eta)$ and define $r_1 = r_\xi|_{\cup C}$ and $r_2 = r_\eta|_{\cup C}$. Note that $r_\xi = i(r_\xi|_{\cup C})$ for any $\xi \in I$. Since $r_1|_\mu \in M_\xi \cap \mathbb{P}_\mu$, there is an $(M_\xi, \mathbb{P}_\mu)$-generic extension $p \in \mathbb{P}_\mu$ and thanks to the result of the following lemma, we have:

$$\text{If } p \leq_{\mathbb{P}_\mu} r_1|_\mu \text{ is } (M_\xi, \mathbb{P}_\mu)\text{-generic, then } i(p) \leq_{\mathbb{P}_{\mu_\eta}} r_2|_{\mu_\eta}.$$

Therefore, we obtain a common extension $r' \in \mathbb{P}_\mu$ such that $r' \leq_{\mathbb{P}_{\cup C}} r_1|_{\cup C}$ and $r' \leq_{\mathbb{P}_{\cup C}} r_2|_{\cup C}$. Define a fucntion r which assigns:

$$\alpha \mapsto \begin{cases} r'(\alpha) & \alpha \in \cup C \\ r_\xi(\alpha) & \alpha \in \operatorname{supt}(r_\xi) \setminus \cup C \\ r_\eta(\alpha) & \alpha \in \operatorname{supt}(r_\eta) \setminus \cup C \\ \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\alpha} & \textit{otherwise} \end{cases}$$

We remain to see that $r \in \mathbb{P}_\delta$ and r is a common extension for r_ξ and r_η in $\mathbb{P}_\delta$. □

Lemma III.5.19. Let M_1, M_2 be two isomorphic countable elementary submodels of H_λ. Let $h\colon M_1 \to M_2$ be an isomorphism and $\mu \in M_1 \cap \omega_2$ be such that $\mu \leq h(\mu)$ and identity on $\mu \cap M_1$. Then if $p \in \mathbb{P}_\mu$ is any $(M_1, \mathbb{P}_\mu)$-generic condition then for any condition $r \in \mathbb{P}_\mu \cap M_1$, $p \leq_{\mathbb{P}_\mu} r$ implies $p \leq_{\mathbb{P}_{h(\mu)}} h(r)$.

IV List of Forcing Notions

IV.1 Cohen Forcing

Definition IV.1.1. Let $\mathrm{Fn}(I, J) = \cup\{{}^A J \,;\, A \in [I]^{<\omega}\}$ be a forcing notion with order $\leq_{\mathbb{P}} = \supset$.

Theorem IV.1.2. $\mathrm{Fn}(I, J)$ has ccc iff I is empty or J is countable.

Proof. ($\Rightarrow$). We shall verify that if I is not empty, J is countable. Fix an $n \in I$ then since $\{\{\langle n, j\rangle\} \,;\, j \in J\}$ is an antichain, we obtain that J is countable.

($\Leftarrow$). Since the case of I is an empty set is clear, we may assume that $I \neq \emptyset$ and J is countable. Suppose that there is an uncountable anticahin A, the Delta system lemma asserts that there is $B \in [A]^{\aleph_1}$ such that $\forall\{f, g\} \in [B]^2\ f \cap g = h$ for some $h \in {}^I J$. This shows that there are distinct $f, g \in B$ such that $f \not\perp g$, a contradiction. $\square$

IV.2 Hechler Forcing

Definition IV.2.1 ([Bla10])**.** Let $\mathbb{P}$ be a forcing notion such that:

1. $(s, f) \in \mathbb{P}$ provided that $s \in {}^{<\omega}\omega$ and $f \in {}^{\omega}\omega$,
2. $(s', f') \leq (s, f)$ provided that
   (a) $s' \supset s$,
   (b) $f' \geq f$,
   (c) $\forall n \in \mathrm{dom}(s') \setminus \mathrm{dom}(s)\ s'(n) \ni f(n)$.

Theorem IV.2.2 ([Pal13])**.** Let G be a generic in $\mathbb{P}$, as par Definition IV.2.1, a *Hechler real* x is a function $x = \cup\{s \,;\, \exists f\ (s, f) \in G\} \colon \omega \to \omega$. Then, G can be recovered form x.

Proof. First, we show that x is a function with domain ω. Since G is a filter, manifestly, x is a fucntion and since $D_n = \{(s, f) \in \mathbb{P} \,;\, n \in |s| \wedge f \in {}^{\omega}\omega\}$ is dense for every $n \in \omega$, $\mathrm{dom}(x) = \omega$.

Secondly, let $G' = \{(s, f) \in \mathbb{P} \,;\, s \subset x \wedge \forall n \in \omega \setminus \mathrm{dom}(s)\ f(n) \leq x(n)\}$ and we assert that $G = G'$ holds. To see $G \subset G'$, let $(s, f) \in G$ and we show that $f(n) \leq x(n)$ for all $n \in \omega \setminus \mathrm{dom}(s)$. Since $\{(t, g) \in \mathbb{P} \,;\, n \in \mathrm{dom}(t)\}$ is dense, there is $(t, g) \in G$ with $n \in \mathrm{dom}(t)$ and let (u, h) be a common extension. Then we obtain that $f(n) \leq t(n) = x(n)$. To see $G \supset G'$, let $(s, f) \in G'$. Since $\{(t, g) \,;\, t \supset s \wedge g \geq f\}$ is dense, there is $(t, g) \in G \cap D$. Then we have $(t, g) \leq (s, f)$, i.e., $(s, f) \in G$. $\square$

Theorem IV.2.3 ([Bla10])**.** A Hechler real x is a dominating real.

Proof. Let $f \in {}^{\omega}\omega \cap V$. Since $D = \{(s, g) \in \mathbb{P} \,;\, f \leq g\}$ is dense, there is $(s, g) \in G$ with $f \leq g$. Thanks to the previous theorem we have that $M[G] \models G = G'$, furthermore, $M[G] \models g \leq x$. Thus we obtain that $M[G] \models \forall f \in {}^{\omega}\omega \cap V\ f \leq x$. $\square$

Theorem IV.2.4 ([Bla10])**.** A Hechler forcing notion is σ-centered, in paticular ccc poset.

Proof. Manifestly, by letting $\mathbb{P} = \cup\{\{(s,f) \in \mathbb{P}\}\,; s \in {}^{<\omega}\omega\}$. $\square$

Definition IV.2.5 ([Jec03])**.** Fix a non-empty family $\mathcal{G} \subset {}^{\omega}\omega$ in the ground model. Let $\mathbb{P}$ be a forcing notion such that:

1. $(s, E) \in \mathbb{P}$ provided that $s \in {}^{<\omega}\omega$ and $F \in [\mathcal{G}]^{<\omega}$,
2. $(s', E') \leq (s, E)$ provided that
   (a) $s' \supset s$,
   (b) $E' \supset E$,
   (c) $\forall n \in \mathrm{dom}(s') \setminus \mathrm{dom}(s) \forall h \in E\ h(n) \in s'(n)$.

Theorem IV.2.6. Let G be a generic in $\mathbb{P}$, as par Definition IV.2.5, a *Hechler real* x is a function $x = \cup\{s\,; \exists E\ (s, E) \in G\} \colon \omega \to \omega$. Then, V can be recoverd form x.

Proof. At the beginning of the proof, we note that since G is a filter and $\{(s, E) \in \mathbb{P}\,; \exists n \ni m\ n \in \mathrm{dom}(s)\}$ is dense for every n, x is a function with domain ω.

Let $G' = \{(s, E) \in \mathbb{P}\,; s \subset x \wedge \forall f \in E \forall n \in \omega \setminus \mathrm{dom}(s)\ (f(n) \leq x(n))\}$ and we verify that $G = G'$. To see $G \subset G'$. Let $(s, E) \in G$. We shall show that $h(n) \leq x(n)$ for any $h \in E$ and $n \in \omega \setminus \mathrm{dom}(s)$. Let $n \in \omega \setminus \mathrm{dom}(s)$. Since $D_n = \{(t, F) \in \mathbb{P}\,; t \supset s \wedge n \in \mathrm{dom}(t) \wedge F \supset E\}$ is dense, there is $(t, F) \in D \cap G$ and let (u, H) be a common extension in G. Then $(u, H) \leq (s, E)$ shows that $h(n) \leq u(n) = x(n)$ for any $h \in E$. To see $G' \subset G$. Let $(s, E) \in G'$. Since $\{(u, H) \in \mathbb{P} \supset u \supset t \wedge H \supset F\}$ is dense, there is $(u, H) \in G \cap D$. Then since for $n \in \mathrm{dom}(u) \setminus \mathrm{dom}(t)$ we have $u(n) = x(n) \geq h(n)$, we obtain that $(u, H) \leq (t, F)$, i.e., $(t, F) \in G$. $\square$

Similarly to Theorem IV.2.3, we have the following theorem:

Theorem IV.2.7 ([Jec03])**.** A Hechler real x is a dominating real.

Theorem IV.2.8. A Hechler forcing notion is σ-centered, in paticular ccc poset.

Proof. Manifestly, by letting $\mathbb{P} = \cup\{\{(s,f) \in \mathbb{P}\}\,; s \in {}^{<\omega}\omega\}$. $\square$

IV.3 Mathias Forcing

Definition IV.3.1. Let $\mathbb{P}$ be a forcing notion such that $(s, A) \in \mathbb{P}$ provided that $s \in [\omega]^{<\omega}$, $A \in [\omega]^{\omega}$ and $\cup s \in \cap A$ and the order $\leq_{\mathbb{P}}$ is given by $(s, A) \leq_{\mathbb{P}} (t, B)$ provided that $s \supset t$, $A \subset B$ and $s \setminus t \subset\ B$.

Theorem IV.3.2. For a generic G the *Mathias real* x is an infinite subset of ω defined by $\cup\{s \in [\omega]^{<\omega}\,; \exists A \in [\omega]^{\omega}\ (s, A) \in G\}$. G can be recoverd from x.

Proof. At the beginning of the proof, since $D_n = \{(s, A) \in \mathbb{P}\,; \exists m \ni n\ m \subset s\}$ is dense for every $n \in \omega$, x is an infinite set.

We shall show that $G = G'$ for $G' = \{(s, A) \in \mathbb{P}\,; s \subset x \wedge x \subset s \cup A\}$. To see $G \subset G'$. Let $(s, A) \in G$. It suffices to verify that $x \subset s \cup A$. For any $a \in x$, there is $(t, B) \in G$ such that $a \in t$ and there is a common extension (u, C). Since $a \in u$, we must have $a \in s$ or $a \in A$. Therefore, we have the subset relation. To see $G' \subset G$. Let $(s, A) \in G'$. Since a set

$$D_A = \{(t, B) \in \mathbb{P}\,; |A \cap B\| < \aleph_0 \vee B \subset A\}$$

is dense, there is $(t, B) \in G \cap D$. Since $(t, B) \in G'$, elements in x are eventually contained in both A and B. This shows $B \subset A$. On the other hand, since a set

$$D_s = \{(t, B) \in \mathbb{P}\,; t \supset \vee s \not\subset \cup\{t, B\}\}$$

is dense, there is $(t', B') \in D_s \cap G$. Since we have $(t', B') \in G'$ we have $t' \supset s$.

Let (u, C) be a common extension for (t, B) and (t', B') in G. To see that (u, C) is an extension of (s, A), we remain to verify that $u \setminus s \subset A$ but this is immediate from since $(u, c) \in G'$ and $(s, A) \in G'$ implies $u \subset x \subset s \cup A$. $\square$

Theorem IV.3.3. A Mathias real x is a dominating real, i.e., a function $x\colon n \mapsto x_n$ where $\mathrm{rank}_{\in, x}(x_n) = n$ is a dominating real.

Proof. Let $f \in {}^{\omega}\omega$ in V. We may assume that f is a strictly increasing fucntion, $f(0) = 0$ and identify f with an infiite set $\{f(i)\,; i \in \omega\}$. Since a set, D_f,

$$\{(t, B) \in \mathbb{P}\,;\ \text{“every interval in } \Pi_B \text{ possesses more than two elements in } f.\text{”}\}$$

is a dense set, where Π_f is an iterval partition endowed with f, let $(t, B) \in G \cap D_f$. Thanks to the previous theorem, since we have $M[G] \models G = G'$, we obtain that $M[G] \models x \subset s \cup f$. This shows that, in $M[G]$, by construction there is some $n \in \omega$ such that the n-th least element in x is grater than in f, i.e., $M[G] \models \forall^\infty n \in \omega f(n) \leq x(n)$. $\square$

Theorem IV.3.4. Let $\mathbb{P}_{\omega_2}$ a ω_2-countable support iteration with

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\mathring{\mathbb{Q}}_\alpha \text{ is a Mathias ordering.”}$$

Then we have $\Vdash_{\mathbb{P}_{\omega_2}} \mathfrak{c} = \aleph_2$.

Proof. To see $(\geq)$. Since the iteration adjoining ω_2 many Mathias reals, dominating reals, we have $\Vdash_{\omega_2} \mathfrak{c} \geq \aleph_2$. To see $(\leq)$. Note that by Theorem III.4.5 $\mathbb{P}_{\omega_2}$ preserves all cardinals. $\square$

Theorem IV.3.5 ([Bla10])**.** For the Mathias real X and a dense open family $\mathcal{D}$, there is $D \in \mathcal{D}$ such that $\Vdash X \subset \check{D}$.

Proof. Let G be generic, X the Mathias real and $\mathcal{D}$ a dense open family. Let (s, A) be any condition. Since $\mathcal{D}$ is dense, there is an extension $(s, A') \leq (s, A)$ with $A' \in \mathcal{D}$. We assert that $(s, A') \Vdash \mathring{X} \setminus s \subset A'$. Let $n \in \omega \setminus s$. If $n \in \mathring{X}$. There is $(\mathring{t}, \mathring{B}) \in \mathring{G}$ such that $n \in \mathring{t}$ and $(\mathring{t}, \mathring{B}) \leq (s, A')$, then we have $n \in \mathring{t} \setminus s \subset A'$. This shows that $\{(t, B) \in \mathbb{P}\,;(t, B) \Vdash (n \in \mathring{X} \implies n \in A')\}$ is dense below (s, A'). Therefore we obtain that $(s, A') \Vdash (n \in \mathring{X} \implies n \in A')$ for each $n \in \omega \setminus s$. Moreover, $(s, A') \Vdash \mathring{X} \subset \cup\{A', s\}$ and we have $\cup\{A', s\} \in \mathcal{D}$ since $\mathcal{D}$ is dense open. Therefore, there is $D \in \mathcal{D}$ such that $\Vdash X \subset \check{D}$. $\square$

Theorem IV.3.6 ([Bla10])**.** Let $\mathbb{P}_{\omega_2}$ be a forcing notion as per Theorem IV.3.4. We have $\mathfrak{h} = \aleph_2 = \mathfrak{c}$. Moreover, $\mathfrak{b} = \mathfrak{g} = \mathfrak{s} = \mathfrak{r} = \mathfrak{d} = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} = \mathbf{non}(\mathcal{B}) = \mathbf{cof}(\mathcal{B}) = \mathbf{non}(\mathcal{L}) = \mathbf{cof}(\mathcal{L}) = \mathfrak{c}$.

Proof. For the first statement, it suffices to show that $\Vdash_{\omega_2} \aleph_1 < \mathfrak{h}$. Let $\{\mathcal{D}_\xi\,; \xi \in \omega_1\}$ be a family of dense open sets. Let $\alpha \in \omega_2$ be an ordinal such that D_ξ in $V[G_\alpha]$ and let X be a Mathias real adjoined in $\alpha + 1$ step. Previous theorem asserts that there is $D_\xi \in \mathcal{D}_\xi$ such that $X \subset D$ for each $\xi \in \omega_1$. Then we obtain that $\Vdash_{\alpha+1} \forall \xi \in \omega_1 \exists D \in \mathcal{D}_\xi\ X \subset D$, this shows that $\Vdash_{\alpha+1} \forall \xi \in \omega_1\ X \in \mathcal{D}_\xi$. Therefore, $\Vdash_{\omega_2} \aleph_1 < \mathfrak{h}$ holds.

For the moreover part is immediate by the general consequence ($\mathfrak{h} \leq \mathfrak{s}, \mathfrak{b}, \mathfrak{g}$), ($\mathfrak{s} \leq \mathbf{non}(L)$), ($\mathfrak{b} \leq \mathfrak{a}$) and ($\mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{u}$). (see 3.8, 5.19, 6.9, 6.27, 8.4 and 9.7 in [Bla10].) $\square$

Theorem IV.3.7. $\mathbb{P}$ satisfies Axiom A.

Proof. Define $\leq_n$ as

1. $\leq_0 = \leq_{\mathbb{P}}$
2. $(s, A) \leq_n (t, B)$ provided that

   (2–a) $(s, A) \leq_{\mathbb{P}} (t, B)$,

   (2–b) $s = t$,

   (2–c) $\{x \in B\,; \mathrm{rank}_{B,\in}(x) \in n\} \subset A$ for each $n \ni 0$.

For the conditions A–1, A–2 are clear. We remain to check the conditions A–3 and A–4'.

For A–3. Let $\langle (s_i, A_i)\,; i \in \omega\rangle$ be a fusion suquence. Define $s = s_0$ and $A = \cap\{A_i\,; i \in \omega\}$. To show $(s, A) \in \mathbb{P}$ and $\forall i \in \omega\ ((s, A) \leq_n (s_n, A_n))$, it remain to check that $A \in [\omega]^\omega$. For $n \in \omega$ there is $m \in A_{n+1}$ such that $\mathrm{rank}_{A_{n+1},\in}(m) = n$. Then we have $n \in m$ and $\forall i \in \omega\ (m \in A_i)$. Therefore we obtain that $\exists^\infty n \in \omega\ (n \in A)$, i.e., $A \in [\omega]^\omega$.

For A–4'. Let $(s, A) \in \mathbb{P}$ with $(s, A) \Vdash \mathring{a} \in V$ and let $n \in \omega$. Define $t = \{a \in A\,; \mathrm{rank}_{A,\in}(a) \in n\}$ and enumerate $\mathcal{P}(t)$ such that $\{t_i\,; i \in k\} = \mathcal{P}(t)$. By the following lemma, there are B_i, $i \in k$ and X_{i+1}, $i \in k-1$, recursively, satisfing:

1. $A \supset B_i \supset B_{i+1}$ for $i \in k - 1$,

2. $(s \cup t_i, B_{i+1}) \Vdash \mathring{a} \in X_{i+1}\check{}$.

Define $B = t \cup B_k$ and $X = \cup\{X_{i+1}\,; i \in k-1\}$ and we verify that $(s, B) \leq_n (s, A)$ and $(s, B) \Vdash \mathring{a} \in \check{X}$. The former is clear by the definition of t. To see that $(s, B) \Vdash \mathring{a} \in \check{X}$. Let $(u, C) \leq_{\mathbb{P}} (s, B)$ since $u \setminus s \subset B$ there are $t_i \subset t$ and $u' \subset B_k \subset B$ such that $u \setminus s = t_i \cup u'$. We shall show that $(t, C \setminus t) \in \mathbb{P}$, $(u, c \setminus t) \leq (t, C)$ and $(u, C \setminus t) \Vdash \mathring{a} \in X$. The first two properties are immediate. To see the last property, we shall show $(u, C \setminus t) \leq (s \cup t_i, B_{i+1})$ and this is immediate by the properties $B = t \cup B_k$ and $B_k \subset B_{i+1}$. □

Lemma IV.3.8. For $(s, A) \in \mathbb{P}$, suppose that $(s, A) \Vdash \mathring{a} \in V$. Then there are $B \subset A$ and countable set X in V such that $(s, B) \Vdash \mathring{a} \in \check{X}$. Moreover, B may be chosen such that if $(t, C) \leq_{\mathbb{P}} (s, B)$, $a \in V$ and $(t, C) \Vdash \mathring{a} = a$, then $(t, B \setminus (\cup t + 1)) \Vdash \mathring{a} = a$.

Proof. First we construct $b_j \in A$ and $B_j \subset A$, recursively, which satisfies:

1. $B_0 \supset B_1 \supset \cdots$,
2. $\forall b \in B_{j+1}\ (b_j \in b)$.

For the leading stage, let $B_0 = A$. Suppose that b_j and B_j, $j \in n$, are defined. Enumerate the subsets of $\{b_j\,; j \in n\}$, $\{s_{n,0}, \ldots, s_{n,k-1}\} = \mathcal{P}(\{b_j\,; j \in n\})$. We construct a sequence $B_0^n \supset B_1^n \supset \cdots \supset B_k^n$ as $B_0^n = B_{n-1}$ and

$$B_{i+1}^n = \begin{cases} C \subset B_i^n & \text{such that } (s \cup s_{n,i}, C) \Vdash \mathring{a} \in V, \\ B_i^n & \text{“ if such a } C \text{ does not exist.”} \end{cases}$$

Let $b_n = \cap B_k^n$, $B_{n+1} = B_k^n \setminus \{b_n\}$. Then by letting, $B = \{b_j\,; j \in \omega\}$ and $X = \{a \in V\,; \exists t \subset B\ (\cup\{s, t\}, B \setminus (\cup t + 1)) \Vdash \mathring{a} = a\}$, we assert that X is countable and $(s, B) \Vdash \mathring{a} \in X$.

We assert that $(s \cup s_{n,i}, B_{i+1}^n) \Vdash \mathring{a} \in V$ for every $n \in \omega$ and $i \in k_n$. Let $(t, C) \leq (s, B)$. Since $t \setminus s \subset B \subset \{b_i\,; i \in \omega\}$, there is $n \in \omega$ and $i \in k_n$ such that $t \setminus s = s_{n,i}$. Then we have $t = s \cup s_{n,i}$. Note that $(t, C) \leq (t, B_{i+1}^n)$ and $(t, B \setminus (\cup t + 1)) \leq (t, B_{i+1}^n)$ (we check at the end of the proof). The first inequality asserts that there is an $a \in V$ such that $(t, C) \Vdash \mathring{a} = a$ and second inequality asserts that the a in X.

For the moreover part, by letting t as in above, let $a' \in V$ such that $(t, B_{n+1}^i) \Vdash \mathring{a} = a'$. The first inequality asserts that $a = a'$ and the second inequality asserts that $a' \in X$.

Now we show the inequalities. To see $(t, C) \leq (t, B_{i+1}^n)$. Immediate by $t \subset \{b_i\,; i \in n\}$, $B^{n+1} \subset B_{i+1}^n$ and $B^{n+1} \subset B \setminus \{b_i \setminus i \in n\}$. To see $(t, B \setminus (\cup t + 1)) \leq (t, B_{i+1}^n)$. Immediate by $B \setminus (\cup t + 1) \subset B^{n+1}$. □

Theorem IV.3.9 (Pre-decision property, [Bau84]). Let φ be a sentence of the language of forcing. For any $(s, A) \in P$ there is $B \subset A$ such that either $(s, B) \Vdash \varphi$ or $(s, B) \Vdash \neg\varphi$.

Proof. Let A_0 and A_1 be maximal antichains in $\{p \in \mathbb{P}\,; p \Vdash \varphi\}$ and $\{p \in \mathbb{P}\,; p \Vdash \neg\varphi\}$, respectively and let $\mathring{a}$ be a nice name induced by antichain $\cup\{A_i\,; i \in 2\}$, then note that we have $(s, A) \Vdash \mathring{a} \in V$.

Lemma IV.3.8 asserts that there is countable $B' \subset A$ with follows:

(a) For $(t, C) \leq (s, B')$, $(t, C) \Vdash \varphi$ implies $(t, B' \setminus (\cup t + 1)) \Vdash \varphi$, and

(b) for $(t, C) \leq (s, B')$, $(t, C) \Vdash \neg\varphi$ implies $(t, B' \setminus (\cup t + 1)) \Vdash \neg\varphi$.

Now, we define a sequence $\langle b_n \in B'\,; n \in \omega\rangle$ and $\langle B_n \in [B']^\omega\,; n \in \omega\rangle$, recursively, with following properties:

- $b_n \in b_{n+1}$ for each $n \in \omega$,
- $B_n \supset B_{n+1}$ for each $n \in \omega$.

Let $B_0 = B'$ and assume that $B_n \subset B'$ and b_n are defined for $n \in \omega$. We shall find an infinite B'_{n+1} such that for arbitrary $s' \subset \{b_i\,; i \in n\}$ we have the one of the following properties:

(i) $\forall b \in B'_{n+1}\ (s \cup s' \cup \{b\}, B' \setminus (\cup \cup \{s', \{b\}\} + 1)) \Vdash \varphi$,

(ii) $\forall b \in B'_{n+1}\ (s \cup s' \cup \{b\}, B' \setminus (\cup \cup \{s', \{b\}\} + 1)) \Vdash \neg\varphi$,

(iii) both of (i) and (ii) does not hold.

Let $\{s_{n,i}\,; i \in k_n\} = \mathcal{P}(\{b_i\,; i \in n\})$ and define a decreasing sequence of infinite sets $B^n = B = B^n_0 \supset B^n_1 \supset \cdots \supset B^n_{k_n}$ such that B^n_{i+1} and $s_{n,i}$ satisfies one of the condition in (i), (ii), (iii) by following processes:

Assume B^n_i is defined, for $i \in k_n$, since the one of the following sets are infinte, let B^{n+1}_i be such an infinite set:

- $T = \{b \in B^n_i\,; (\cup\{s, s_{n,i}, \{b\}\}, B' \setminus (\cup \cup \{s_{n,i}, \{b\}\} + 1)) \Vdash \varphi\}$,
- $F = \{b \in B^n_i\,; (\cup\{s, s_{n,i}, \{b\}\}, B' \setminus (\cup \cup \{s_{n,i}, \{b\}\} + 1)) \Vdash \neg\varphi\}$,
- $B^n_i \setminus \cup\{T, F\}$.

And let $B'_{n+1} = B^n_{k_n}$ and $b_n = \cap B'_{n+1}$ and $B_n = B'_{n+1} \setminus \{b_n\}$. Define $B = \{b_n\,; n \in \omega\}$.

To conclude the proof, we shall show that (s, B) forces φ or its negation. Assume that it does not force $\neg\varphi$. Then there is an extension $(t, C) \leq (s, B)$ which forces φ and we may assume that t has the least cardinality among them. We distinguish two case, according to whether $|s| = |t|$. If $|s| = |t|$. Since we have $s = t$, (s, C) forces φ, so does (s, B), since $(t, B' \setminus (\cup t + 1))$ is an extension of (t, B) and by (a). If $|s| \in |t|$. Since $t \setminus s \subset B$ there is an $n \in \omega$ such that $\cup(t \setminus s) = b_n$. Let $s' = t \setminus (s \cup \{b_n\})$. If we have $(s \cup s', B'_{n+1})$ forces φ, Since $(s \cup s', B'_{n+1}) \leq (s, B)$, the minimality for the cardinality asserts that $|t| \leq |s \cup s'| = |t| - 1$, a contradiction and we complete the proof.

Thus, we remain to show that $(s \cup s', B'_{n+1})$ forces φ. Since (t, C) forces φ, by (a), so does $(t, B' \setminus (\cup t + 1))$. Furthermore, $s' = t \setminus (s \cup \{b_n\})$ and $b_n \in B'_{n+1}$ witnesses that B'_{n+1} holds the condition (i). To see that $(s \cup s', B'_{n+1})$ forces φ, let (u, D) be an extension. We distinguish two cases, according to whether $u \setminus (s \cup s') = \emptyset$. Case I. $u \setminus \cup\{s, s'\} = \emptyset$. Let $b = \cap D$. Then we have $(u \cup \{b\}, D \setminus \{b\})$ is an extension for (u, D) and $(s \cup s' \cup \{d\}, B'_{n+1} \setminus (\cup(s' \cup \{d\} + 1))$. And the later conditon forces φ, we obtain that $(\cup\{s, s'\}, B'_{n+1}) \Vdash \varphi$. Case II. $u \setminus (s \cup s') \neq \emptyset$. Let $b = u \setminus \cup\{s, s'\}$ and $d = u \setminus (s \cup s' \cup \{b\})$. Since $(s \cup s' \cup \{b\}, B' \setminus \cup(s \cup \{b\} + 1))$ forces φ, we shall show that this condition possesses an extension (u, D). To see that $D \subset B' \setminus (\cup(s' \cup \{b\} + 1)$, we note that $D \subset B'_{n+1} \subset B'$ and for arbitrary $c \in D$, $\cup u \in x$, $\cup s' \leq \cup u \in x$ and $b \leq \cup u \in x$ imply $x \notin \cup(s' \cup \{b\} + 1)$. To see that $u \setminus (s \cup s' \cup \{b\}) \subset B' \setminus (\cup(s' \cup \{b\} + 1))$. It suffices to see that $d \subset B' \setminus (\cup(s' \cup \{b\} + 1))$. For $x \in d$, $x \in u \setminus \{s, s'\}$ shows that $x \in B'_{n+1} \subset B'$ and $x \notin s' \cup \{b\}$ shows that $x \notin \cup(s' \cup \{b\} + 1)$. ☐

Corollary IV.3.10. Let X be a finite set in V, $(s, A) \Vdash \mathring{a} \in X$ then there is $B \subset A$ and $a \in X$ such that $(s, B) \Vdash \mathring{a} = a$. Moreover, for $n \in \omega$ there is $(t, B) \leq_n (s, A)$ and $Y \subset X$ of size $\leq 2^n$ such that $(t, B) \Vdash \mathring{a} \in Y$.

Proof. Let $X = \{a_i\,; i \in k\}$ in V and we define the squence $A \supset B_0 \supset B_1 \supset \cdots \supset B_k$, recursively, such that $(s, B_i) \Vdash \mathring{a} \in X \setminus \{a_j\,; j \in i\}$ or $\exists j \leq i\ (s, B_i) \Vdash \mathring{a} = a_j$. For the leading stage. Theorem IV.3.9 asserts that there is finite $B_0 \subset A$ such that either

(i) $(s, B_0) \Vdash \mathring{a} = a_0$, or

(ii) $(s, B_0) \Vdash \mathring{a} \neq a_0$.

Note that $(s, B_0) \Vdash a \neq a_0$ implies $(s, B_0) \Vdash \mathring{a} \in X \setminus \{a_0\}$. For the successor stages. If B_i have the condition (i), let $B_{i+1} = B_i$. If B_i have condition (ii), by Theorem IV.3.9 there is $B_{i+1} \subset B_i$ such that either $(s, B_{i+1}) \Vdash \mathring{a} = a_{i+1}$ or $(s, B_{i+1}) \Vdash \mathring{a} \in \{a_j\,; j \in n+1\}$. Since we have $(s, B_n) \nVdash \mathring{a} \in \emptyset$. There is $j \leq k$ such that $(s, B_n) \Vdash \mathring{a} = a_j$. To see the moreover part, fix an n. For the leading stage, since $(s, A) \Vdash \mathring{a} \in X$ there is an extension (t, B) and $x \in X$ such that $(t, B) \Vdash \mathring{a} = x$. So by letting $Y = \{x\}$ we have the $|Y| \leq 2^0 = 1$ and $(s, B) \Vdash \mathring{a} \in Y$. For the successor stages. Let $U_n = \{a \in A\,; \mathrm{rank}_{\in, A}(a) \in n\}$ and enumerate $\{s_i^n\,; i \in 2^n\} = \mathcal{P}(U_n)$. Define B_0^n, $B_{i+1}^n \subset A$ and $a_i \in X$ for $i \in 2^n$, recursively:

- $B_0^n = A$,
- B_{i+1}^n and a_i be sets such that $(\cup\{s, s_i^n\}, B_{i+1}^n) \Vdash \mathring{a} = a_i$.

note that for the successor stage, since $(\cup\{s, s_i^n\}, B_i^n \setminus (\cup s_i^n + 1)$ is an extension for (s, A), the first statement asserts that there are desired sets B_{i+1}^n and a_i. Let $B = \cup\{B_{2^n}^n, U_n\}$ and $Y = \{a_i\,; i \in 2^n\}$. To conclude the proof, we shall verify that $|Y| \leq 2^n$, $(s, B) \leq_n (s, A)$ and $(s, B) \Vdash \mathring{a} \in Y$. The first two statements are obvious, we see the last condition. Let (t, C) be an extension for (s, B).

By letting $W = (t \setminus s) \cap B^n_{2^n}$ and $s^n_j = (t \setminus s) \cap U_n$ for some $j \in 2^n$, we have $t \setminus s = \cup\{W, s^n_j\}$. To conclude the proof, we shall show that $(t, C \cap B^n_{j+1})$ is an extension for (t, C) and $(\cup\{s, s^n_j\}, B^n_{j+1})$. The former is immediate and the later is immediate by the fact that $W \subset B^n_{2^n} \subset B^n_{j+1}$. Then, since the later condition forces $\mathring{a} \in Y$, we obtain that $(s, B) \Vdash \mathring{a} \in Y$. $\square$

IV.4 Sacks Forcing (Perfect Set Forcing)

Definition IV.4.1. Let $\mathbb{S}$ be a poset with order $\leq_{\mathbb{S}} = \subset$ such that $p \in \mathbb{S}$ provided that

(a) $\emptyset \neq p \subset \cup\{{}^n 2\,; n \in \omega\}$,

(b) $\forall s \in p \forall n \in \omega\ (s|_n \in p)$,

(c) $\forall q \in p \exists s, t \in p\ (q \subset s \wedge q \subset t \wedge \exists n \in \omega\ s(n) \neq t(n))$.

Note that the condition (c) asserts that every $p \in \mathbb{S}$ is a parfect set in $2^{<\omega}$.

For $p \in \mathbb{S}$ and $s \in p$ define *the degree of s in p*, $\deg(s,p)$, by the cardinality of $\{n \in \omega\,; \exists t \in p\ t(n) \neq s(n)\}$. $s \in p$ is a *splitting node of p* provided that $s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in p$ in addtion if we have $\deg(s,p) = n$ then s is an *n-splitting node of p*. $p|_s$ denote the codition $\{t \in p\,; t \subset s \vee s \subset t\}$.

Theorem IV.4.2 ([Jec03])**.** For generic G *the Sacks real, or generic branch,* is a function $f \colon \omega \to 2$ given by $f = \cap\{\cup p\,; p \in G\}$. Then G can be recovered form x.

Proof. First, we assert that f is a function with $\omega \to 2$. Since G is a filter, for any $n \in \omega$ and for distinct $p, q \in G$ we obtain that $p \cap q \cap {}^n 2 \neq \emptyset$. Moreover since ${}^n 2$ is finite, we obtain that $S_n = (\cap G) \cap {}^n 2 \neq \emptyset$. Furthermore, since $D_n = \{p \in \mathbb{S}\,;$ " $\cup p$ does not possess either $\langle n-1, 0\rangle$ or $\langle n-1, 1\rangle$." $\}$ is dense in $\mathbb{S}$ for $n \ni 0$, S_n must be a singleton. Therefore, $f \colon \omega \to 2$ is a function.

Second, we assert that $M[f] = M[G]$. Since $f \in M[G]$ manifestly holds, it suffices to verify that $G = G'$ for $G' = \{p \in \mathbb{S}\,; f \subset \cup p\}$. The relation $(\subset)$ is obvious. To see the $(\supset)$. Since $D = \{q \in G\,; q \subset p \vee \neg(\exists f \in (\cup p) \cap (\cup q)\ f \in {}^\omega 2)\}$ is dense, there is $q \in G \cap D$. Since $q \in G'$, we have $f \in (\cup p) \cap (\cup q)$. Thus we obtain that $q \subset p$, i.e., $p \in G$. $\square$

Theorem IV.4.3.

(i) For $p \in \mathbb{S}$, p possesses a binary subtree,

(ii) Define $\leq_n$ as $p \leq_n q$ provided that $p \leq_{\mathbb{S}} q$ and $\forall s \in q\ (\deg(s,q) \leq n \implies s \in p)$. Note that $\leq_0 = \leq_{\mathbb{S}}$ holds. Then,

- $\mathbb{S}$ satisfies Axiom A. Moreover we can say that $|\{r \in I\,; r \not\leq q\}| \leq 2^n$ in the condition A–4.

Proof. For (i). Let $p \in \mathbb{S}$. Define a binary subtree $T_p = \cup\{T^n_p = {}^{({}^n 2)}p\,; n \in \omega\}$, recursively on a height $n \in \omega$,

1. For a leading stage. $T_p^0 = \{\emptyset\}$.

2. For a successor stages. For $s^\frown\langle i\rangle \in {}^{(n+1)}2$, let $N = \min\{m \in \omega\,; \exists t, u \in p\ (s \subset t_0, t_1 \wedge \mathrm{dom}(t_0) = \mathrm{dom}(t_1) = m + 1 \wedge t_0(m) = 0 \wedge t_1(m) = 1)\}$ and $t_0, t_1 \in p$ which endow N. Define $T_p^{n+1}(s^\frown\langle i\rangle) = t_i$.

Then we obtain a subtree T_p. Note that the heigh of s in T_p and the degree of s in p are the same.

For (ii). The conditions A–1 and A–2 are manifestly. We check the conditions A–3 and A–4.

For condition A–3. Let $\langle p_n \in \mathbb{S}\,; n \in \omega\rangle$ with $p_{n+1} \leq_n p_n$ for each $n \in \omega$. Let $q = \cap\{p_n\,; n \in \omega\}$. First we check that $q \in \mathbb{S}$. Since the condition (a) and (b) are obvious, we remain to see the condition (c). Let $s \in q$ and $n = \deg(s, p_0)$. Since we have $\deg(s, p_{n+1}) \leq n$, $p_{n+1} \in \mathbb{S}$ and by the result of (i), there are $t, u \in p_{n+1}$ such that $\deg(t, p_{n+1}) = \deg(u, p_{n+1}) = n + 1$. To see that $t, u \in q$, let $m \in \omega$ and we shall distinguish two cases, according to whether $m \leq n + 1$. If $m \leq n + 1$. Immediate by the fact $p_{n+1} \subset p_m$. If $n + 1 \in m$. Note that for $s, t \in p_{n+1}$ with $\deg(s, p_{n+1}) \leq n + 1$, $p_{n+2} \leq_{n+1} p_{n+1}$ implies $s, t \in p_{n+2}$. Thus, by inductively, we obtain that $s, t \in p_m$. Note that the conditions $q \subset s, t$, $s \not\subset t$ and $t \not\subset s$ are clear, thus $q \in \mathbb{S}$. To conclude the proof for condition A–3, we show that $q \subset p_n$, for every $n \in \omega$. This can prove by the mimik the proof of $t, u \in q$.

For condition A–4. Let I be an antichain over $\mathbb{S}$, $p \in \mathbb{S}$ and $n \in \omega$. Let $\Gamma(p, n) = \{s \in p\,; \deg(s, p) = n \wedge s^\frown\langle 0\rangle \in p \wedge s^\frown\langle 1\rangle \in p\}$ and $p|_s = \{t \in p\,; t \subset s \vee s \subset t\} \in \mathbb{S}$. For $s \in \Gamma(p, n)$ since I is an antichain there is $q_s \leq_{\mathbb{S}} p|_s$ such that either $\exists r_s \in I\ (q_s \leq_{\mathbb{S}} r_s)$ or $\forall r \in I\ (q_s \perp_{\mathbb{S}} r)$. We choose q_s and r_s (if exists and note that uniquely exists) for each $s \in \Gamma(p, n)$ and define $q = \cup\{q_s\,; s \in \Gamma(p, n)\}$ and we check the q is the desired one. For $q \leq_n p$. Let $t \in p$ with $\deg(t, p) \leq n$, by the result of (i), there is $s \in \Gamma(p, n)$ such that $t \subset s$. Since we have $t \in p|_s$, $\deg(t, p|_s) \leq \deg(s, p|_s) = 0$ and $q_s \leq_0 p|_s$, we obtain $t \in q_s \subset q$. Thus $q \leq_n p$. For $|\{r \in I\,; r \not\perp q\}| \leq 2^n$. First we note that $|\Gamma(p, n)| \leq 2^n$ since the number of nodes in a binary tree with height n is 2^n. If we obtain the equation $\{r \in I\,; r \not\perp_{\mathbb{S}} q\} = \{r \in I\,; \exists s \in \Gamma(p, n)\ (q_s \not\perp_{\mathbb{S}} r)\}$, we have $|\{r \in I\,; r \not\perp_{\mathbb{S}} q\}| \leq 2^n$.

Therefore we remain to show the equality. The relation, $(\supset)$, is immediate by $q_s \subset q$. Before we show the $(\subset)$, we assert that the following trivial facts.

1. For $\{s_0, s_1\} \in [\Gamma(p, n)]^2$, $q_{s_0} \perp_{\mathbb{S}} q_{s_1}$.

2. For $\mathcal{D}, \mathcal{F} \in [\Gamma(p, n)]^{<\omega}$, if $\mathcal{D} \cap \mathcal{F} = \emptyset$ then $\cup\{q_s\,; s \in \mathcal{D}\} \perp_{\mathbb{S}} \cup\{q_s\,; s \in \mathcal{F}\}$.

To see $(\subset)$. Let $r \in I$ and let $q' \leq_{\mathbb{S}} p$ with $q' \leq_{\mathbb{S}} q, r$. Suppose that we have $\forall s \in \Gamma(p, n)\ (q_s \perp_{\mathbb{S}} q')$, inductively and by the above facts, we obtain that $\cup\{q_s\,; s \in \Gamma(p, n)\} \perp_{\mathbb{S}} q'$, i.e., $q \perp_{\mathbb{S}} q'$, a contradiction. Thus for some $s \in \Gamma(p, n)$ we have $q_s \not\perp_{\mathbb{S}} q'$ moreover, $q_s \not\perp_{\mathbb{S}} r$. □

Definition IV.4.4 ([Jec03])**.** A generic filter G is *minimal over the groud model* M provided that every set of ordinals in $M[G]$, either $X \in M$ or $G \in M[X]$. Here $M[X]$ is the least extension with $M \subset M[X]$ and $X \in M[X]$.

Theorem IV.4.5 ([Jec03])**.** A generic filter G of a Sacks forcing notion is minimal over the ground model.

Proof. Let $X \in V[G]$ be a set of ordinals and assume that $\not\Vdash \mathring{X} \in V$ and let p be a condition which forces $\mathring{X} \notin V$.

Define a fusion sequence $\langle p_n\,; n \in \omega\rangle$, and ordinals γ_s in $V[G]$, recursively. Let $p_0 = p$. Let S_n be the set of an n-splitting nodes of p_{n-1}. For each $s \in S_n$ let γ_s be an ordinal which does not decide $\gamma_s \in \mathring{X}$ by $p_{n-1}|_s$. Then for $i \in 2$, since $p_{n-1}|_{s^\frown\langle i\rangle}$ is an extension of $p_{n-1}|_s$, there is an extension q_s^i such that $q_s^0 \Vdash \gamma_s \notin \mathring{X}$ and $q_s^1 \Vdash \gamma_s \in \mathring{X}$. Then by letting p be *the amalgamation* $\{q_s^i\,; s \in S_n \wedge i \in 2\}$ *into* p_{n-1}:

$$p_n = \cup\{\{t \in p_{n-1}\,;(t \subset s) \vee (s \subset t \wedge t \in q_s^i)\}\,; s \in S_n \wedge i \in 2\} \in \mathbb{P}$$

we have $p_n \leq_n p_{n-1}$ and $q_s^i \leq p_n$ for each $s \in S_n$ and $i \in 2$. And moreover, there is a fusion q such that $q \leq_n p_n$ for each $n \in \omega$. □

Theorem IV.4.6 (**Sacks property**, [Bla10])**.** Let $s \in \mathbb{S}$ and $Y \in V$ satisfying $s \Vdash \mathring{f}\colon \omega \to \check{Y}$ there is $g\colon \omega \to X$ in V with $X \in V$ such that $f(n) \in g(n)$ and $|g(n)| \leq 2^n$ for each $n \in \omega$.

Proof. For every $n \in \omega$ we have $s \Vdash (\mathring{f})(n) \in \check{Y}$, so by applying A–4', there is q_n and $X_n \in V$ with $|X_n| \leq 2^n$ such that $q_n \leq_n s$ and $q_n \Vdash (\mathring{f})(n) \in \check{X}_n$. By letting the function g with assign the n the X_n, we obtain the desired one. □

IV.5 Shooting A Club by Finite Conditions

Definition IV.5.1 ([Bau84])**.** Let $\mathbb{P}$ be a forcing notion with an order $\leq_{\mathbb{P}}$ such that:

1. $p \in \mathbb{P}$ provided that p is a fucntion $p\colon 2n \to \omega_1$ for $n \in \omega \setminus 1$ such that $p(2i+1) \in p(2i+2)$ for any $i \in n-1$, we simply write $p = \{\langle \xi_{2i}, \xi_{2i+1}\rangle\,; i \in n\}$,
2. For p, $q \in \mathbb{P}$, $q \leq_{\mathbb{P}} p$ provided that $q \supset p$.

Theorem IV.5.2. $\mathbb{P}$ is proper.

Proof. Let $\lambda > 2^{|\mathbb{P}|}$ be regular uncountable, $M \preccurlyeq H_\lambda$ with $\mathbb{P} \in M$ and $p \in M \cap \mathbb{P}$. We shall find an extension $q \leq p$ which is $(M, \mathbb{P})$-generic. Fix a $\beta_0 \in \omega_1$ such that $\omega_1 \cap M \in \beta_0$ and let $p' = p \cup \{\langle \omega_1 \cap M, \beta_0\rangle\}$. We assert that p' is a witness. Let $D \subset \mathbb{P}$ be a dense set with $D \in M$ and let $I \subset D$ be a maximal antichain. Let $q \leq p'$ be an extension. Since there is $r \in I$ such that $r \not\perp q$. Now we note that the following properties (we shall proved at the end of the proof):

(a.) $s \cap M \in M$ for any condition $s \in \mathbb{P}$,

(b.) $s \in M$ implies $\cup s \subset M$ for any conditon $s \in \mathbb{P}$.

Thus, there is an $r' \in I \cap M$ such that $r \cap M = r' \cap M$ and $r' \subset M$. Then $r' \cup q = (r \cap M) \cup q \in \mathbb{P}$, i.e., $r' \not\perp q$.

To see that (a). For $s \in \mathbb{P}$. Since $s \cap M$ is finite, there is a bijection $f\colon |s \cap M| \to s \cap M$ and thus $s \cap M \in M$. To see that (b). For $s \in M \cap \mathbb{P}$, there is a surjection $f\colon \omega \to s$ in M. Thus we obtain that $\cup s \subset M$. $\square$

Theorem IV.5.3. For any generic G, $\mathbb{1} \Vdash_{\mathbb{P}}$ " $\mathring{C} = \{\alpha \in \omega_1\,; \exists \beta \in \omega_1\ \langle \alpha, \beta \rangle \in \cup \mathring{G}\}$ is club in ω_1." .

Proof. To see $\mathring{C}$ is unbounded. Let $\gamma \in \omega_1$. Since $D_\gamma = \{p \in \mathbb{P}\,; \exists \langle \alpha, \beta \rangle \in \cup p\ \gamma \in [\alpha, \beta]\}$ is dense in $\mathbb{P}$, there is $p \in \mathring{G}$ and $\langle \alpha, \beta \rangle \in p$ such that $\gamma \in [\alpha, \beta]$. And, similarly, for $\beta + 1$ there is $q \in \mathring{G}$ such that $\beta + 1 \in [\alpha', \beta']$ for some $\langle \alpha', \beta' \rangle \in q$. Then since $\beta + 1 \leq \alpha'$, $\beta + 1$ witnesses that $\mathring{C}$ possesses an element $\geq \gamma$. Thus $\models \cup \mathring{C} = \omega_1$.

To see that $\mathring{C}$ is closed set, it suffices show that $\omega_1 \setminus \mathring{C} = \cup\{[\alpha+1, \beta]\,; \langle \alpha, \beta \rangle \in \cup \mathring{G}\}$ since $[\alpha, \beta]$ is an open interval $(\alpha, \beta + 1)$. For $(\subset)$. Let $\gamma \in \omega_1 \setminus \mathring{C}$. By genericity there is $\langle \alpha, \beta \rangle \in \cup \mathring{G}$ such that $\gamma \in [\alpha, \beta]$, moreover since $\gamma \notin \mathring{C}$, $\gamma \in [\alpha + 1, \beta]$. For $(\supset)$. Let $\gamma \in [\alpha + 1, \beta]$ and $\langle \alpha, \beta \rangle \in p \in G$. If $\gamma \in \mathring{C}$, there is $\delta \in \omega_1$ such that $\langle \gamma, \delta \rangle \in q$ for some $q \in G$. However, this contradicts that $\alpha = \gamma$ since p and q are compatible. $\square$

Theorem IV.5.4. There are no sequence of order relations $\langle \leq_n\,; n \in \omega \rangle$ which satisfies Axiom A.

Proof. Assume that there is a sequence of order relations $\langle \leq_n\,; n \in \omega \rangle$ which satisfies Axiom A. At the beginning of the proof, we assert that for $p \in \mathbb{P}$ there is an uncountable antichain I_p such that every $q \in I_p$ is compatible with p. Since p is a finite set there is $\alpha \in \omega_1$ such that $\cup \cup p \in \alpha$. Then by letting $I_p = \{\cup\{p, \{\langle \alpha, \beta \rangle\}\}\,; \alpha \in \beta \in \omega_1\}$ we obtain the desired uncountable antichain.

To heve a contradiction, we define a fusion sequence p_n, $n \in \omega$ with $|p_n| \geq n$, recursively,

1. For a leading stage. Fix arbitrary $p_0 \in \mathbb{P}$,

2. For a successor stages. For p_n and I_{p_n}, by applying the condition A–4, we obtain a $q \in \mathbb{P}$ such that $q \leq_{n+1} p_n$ and $\{r \in I_{p_n}\,; r \perp_{\mathbb{P}} q\}$ is countable. If $|q| \leq n$, by $q \leq_{n+1} p_n$, $|p_n| \geq n$ and the conditon A–2 shows that $q = p_n$. However this contrary to that $\{r \in I_{p_n}\,; r \perp p_n\}$ is countable. Thus we have $q \in \mathbb{P}$ such that $q \leq_{n+1} p_n$ and $|q| \geq n + 1$.

Therefore, we obtain a specific fusion sequence. However the condition A–3 asserts that there is a fusion $p \in \mathbb{P}$ such that $q \leq_n p_n$ for every n, moreover $|q| \geq n$ for every n, a contradiction. $\square$

V The Concrete Forcing Examples

V.1 Axiom of Choice

V.2 Continnum Hypothsis

V.3 Martin Axiom

V.4 Borel Conjecture

Definition V.4.1. A subset of reals A is *strong measure zero* provided that for each sequence of positive reals $\langle \varepsilon_n ; n \in \omega \rangle$ there is a sequence of intervals $\langle I_n \subset \mathbb{R} ; n \in \omega \rangle$ such that $|I_n| \leq \varepsilon_n$ for each $n \in \omega$ and $A \subset \cup \{ I_n ; n \in \omega \}$.

Theorem V.4.2. *X is strongly measure zero over ${}^{\omega}2$ iff $\forall f \in {}^{\omega}\omega \exists \langle g_n \colon f(n) \to 2 ; n \in \omega \rangle$ $(\forall h \in X \exists n \in \omega\ g_n \subset h)$.*

Lemma V.4.3 ([JSW90])**.** *$\mathfrak{b} = \aleph_1$ implies there is an uncountable strongly measure zero set.*

Proof. At the beginning of the proof, we shall define a set $X \subset \mathbb{R}$ is *concentrated on* $\mathbb{Q}$ provided that for arbitrary open set $\mathcal{U}$ with $\mathcal{U} \supset \mathbb{Q}$ we have $X \setminus \mathcal{U}$ is countable and we assert that X is concentrated on $\mathbb{Q}$ implies that X is a strongly measure zero set.

Emumerate $\mathbb{Q} = \{ q_i ; i \in \omega \}$ and let $\langle \varepsilon_i ; i \in \omega \rangle$ be a sequence of positive reals. Since $\mathcal{U} = \cup \{ (q_i - \varepsilon_i, q_i + \varepsilon_i) \subset \mathbb{R} ; i \in \omega \}$ is an open set with $\mathcal{U} \supset \mathbb{Q}$, $X \setminus \mathcal{U}$ is countable. Moreover since X have the form of the union of $\mathcal{U}$ and countable $X \setminus \mathcal{U}$, so X can covered by countable intervals.

To conclude the proof, let $\mathcal{F}$ be an unbounded family of size ω_1 and assume that $f_i <^* f_j$ holds for $i \in j \in \omega$. Identify ${}^{\omega}\omega$ with $P = [0,1] \setminus Q$. To see that $\mathcal{F}$ is councentrated on $\mathbb{Q}$, let $\mathcal{U}$ be an open set with $\mathcal{U} \supset \mathbb{Q}$. Since $K = [0,1] \setminus \mathcal{U}$ is compact and K is covered by $\cup \{ \{ f \in {}^{\omega}\omega ; f < g \} ; g \in {}^{\omega}\omega \}$, there is $g \in {}^{\omega}\omega$ such that K is covered by $\{ f \in {}^{\omega}\omega ; f <^* g \}$ for some $g \in {}^{\omega}\omega$. Furthermore, we have $\mathcal{F} \setminus \mathcal{U} \subset \{ f \in \mathcal{F} ; f <^* g \}$. Therefore, since $\mathcal{F}$ is unbounded family, $\mathcal{F} \setminus \mathcal{U}$ is countable of size $\aleph_1$, $\mathcal{F}$ is an uncountable strongly measure zero set. $\square$

Hereafter, let $\mathbb{P}_\alpha$ be an α-stage iteration with countable support such that $\beta \in \alpha$:

$$\mathbb{1} \Vdash_{\mathbb{P}_\beta} \text{“}\mathring{\mathbb{Q}}_\beta \text{ is the Mathias ordering.”}$$

Lemma V.4.4. *Let X be finite in V and $p \Vdash_\alpha \mathring{a} \in X$. For any finite $F \subset \alpha$ and $n \in \omega$ there is $q \leq_{F,n} p$ and $Y \subset X$ such that $|Y| \leq 2^{2 \cdot |F|}$ and $q \Vdash_\alpha \mathring{a} \in Y$.*

Proof. We prove by induction on α. The leading stage is immediate by Theorem IV.3.10. For the successor stages, $\alpha = \beta + 1$. We distinguish two cases, according to whether $\beta \notin F$. If $\beta \notin F$, note that we have $F \subset \beta$. Since we have $p \Vdash_\alpha \mathring{a} \in X$, there is $\mathring{f}$, $\mathring{b} \in V^{\mathbb{P}_\beta}$ such that $p|_\beta$ forces, at the β-stage,

$$\mathring{f} \mathring{\leq}_{\mathring{\mathbb{Q}}_\beta} p(\beta), \mathring{b} \in X \text{ and } \mathring{f} \Vdash_{\mathring{\mathbb{Q}}_\beta} \mathring{a} = \mathring{b}.$$

Then, by induction hypothesis, there is an extension $q' \leq_{F,n} p|_\beta$ and $Y \subset X$ such that $|Y| \leq 2^{2\cdot|F|}$ and q' forces, at the β-stage,

$$\mathring{f} \mathring{\leq}_{\mathring{\mathbb{Q}}_\beta} p(\beta) \text{ and } \exists \mathring{b} \in Y \ \mathring{f} \Vdash_{\mathring{\mathbb{Q}}_\beta} \mathring{a} = \mathring{b}.$$

letting $q = \cup\{q', \langle\beta, \mathring{f}\rangle\} \in \mathbb{P}_\alpha$, we have $q \leq_{F,n} p$ and $q \Vdash \mathring{a} \in Y$. If $\beta \in F$. Applying Theorem IV.3.10 to $p|_\beta \Vdash_{\mathbb{P}_\beta} (p(\beta) \Vdash_{\mathring{\mathbb{Q}}_\beta} \mathring{a} \in X)$, there is $\mathring{q}$, $\mathring{f}$, $\mathring{Y} \in V^{\mathbb{P}_\beta}$ such that $p|_\beta$ forces, at β-stage,

$$\mathring{q} \leq^\beta_n p(\beta),\ \mathring{Y} \subset X,\ \exists \mathring{f} \colon \mathring{Y} \to 2^n \text{ (“}\mathring{f}\text{ is injective” and } \mathring{q} \Vdash_{\mathring{\mathbb{Q}}_\beta} \mathring{a} \in \mathring{Y}).$$

Since for each $i \in k_n$ we have $p|_\beta \Vdash_{\mathbb{P}_\beta} \mathring{f}(i) \in X$ by induction hypothesis we obtain the sequence q_i and Y_i with

- $p|_\beta = q_0 \geq_{F\cap\beta,n} q_1 \geq_{F\cap\beta,n} q_2 \geq_{F\cap\beta,n} \cdots \geq_{F\cap\beta,n} q_{k_n}$,
- $|Y_i| \leq 2^{n\cdot(|F|-1)}$,
- $q_{i+1} \Vdash_{\mathbb{P}_\beta} \mathring{f}(\beta) \in Y_{i+1}$.

Then we have $q_{k_n} \geq_{F\cap\beta,n} p|_\beta$ and $q_{k_n} \Vdash_{\mathbb{P}_\beta} \mathring{f}(i) \in Y_{i+1}$ for each $i \in k_n$. Then by letting $Y = \cup\{Y_{i+1}\,;i \in 2^n\}$ and $q = \cup\{q_{k_n}, \langle\beta, \mathring{q}\rangle\}$, we obtain that $q \in \mathbb{P}_\alpha$, $q \geq_{F\cap\beta,n} p$ and $|Y| \leq 2^{n\cdot(|F|-1)} \cdot 2^n = 2^{n\cdot|F|}$. For the limit stages. Since there is β such that $F \subset \beta \subset \alpha$. We can prove by mimik the case of $\beta \notin F$ in the successor stages. □

Lemma V.4.5. Let $\langle X_n\,; n \in \omega\rangle$ be a sequence of finite sets and assume that $p \Vdash_\alpha \forall n \in \omega\ f(n) \in X_n$. Then there is $q \leq p$ and a sequence $\langle Y_n\,; n \in \omega\rangle$ in V such that $Y_n \subset X_n$ and $|Y_n| \leq 2^{(n^2)}$ for each $n \in \omega$ and $q \Vdash_\alpha \forall n \in \omega\ \mathring{f}(n) \in Y_n$.

Proof. We shall define a sequence of sets $\langle Y_n \in n \in \omega\rangle$ and an (F,n)-fusion sequence, $\langle(p_n, F_n)\,; n \in \omega\rangle$ such that $|F_n| \leq n$ and $p_{n+1} \Vdash_{\mathbb{P}_\alpha} \mathring{f}(n) \in Y_n$.

Let $p_0 = p$ and $F_0 = \emptyset$. Suppose we have p_n and F_n, we shall find a p_{n+1} and F_{n+1}. Since $p_n \Vdash \mathring{f}(n) \in X_n$ by Lemma V.4.4 there is $p_{n+1} \leq_{F_n,n} p_n$ and $Y_n \subset X_n$ such that $|Y_n| \leq 2^{n\cdot|F_n|}$ and $p_{n+1} \Vdash \mathring{f}(n) \in Y_n$. Fix funcitons h_1 and h_2 which assigns to $\Sigma\{j\,; j \in n\} + i$ to n and to j, respectively, for $i \in n$ and enumerate $\mathrm{supt}(p_m) = \{\xi^m_i\,; i \in \omega\}$. $Y_{n+1} = \cup\{F_n, \{\xi^{h_1(n)}_{h_2(n)}\}\}$. Then we obtain that $\cup\{F_n\,; n \in \omega\} = \cup\{\mathrm{supt}(p_n)\,; n \in \omega\}$. Thus we obtain the (F,n)-fusion sequence $\langle(p_n, F_n)\,; n \in \omega\rangle$.

Lemma III.4.7 asserts that there is a fusion $q \in \mathbb{P}_\alpha$. Then we have $q \Vdash_{\mathbb{P}_\alpha} \forall n \in \omega\ \mathring{f}(n) \in Y_n$ and moreover we have $q \leq_{\mathbb{P}_\alpha} p$, $Y_n \subset X_n$ and $|Y_n| \leq 2^{(n^2)}$ for any $n \in \omega$. □

Theorem V.4.6. Assume that CH. $\Vdash_{\omega_2}$ “ Borel conjecture.” $+\ \mathfrak{c} = \aleph_2$.

Proof. We have seen that $\Vdash_{\omega_2} \mathfrak{c} = \aleph_2$ in Theorem IV.3.4.

At the beginning of the proof, we show that there is a $\beta \in \omega_2$ such that $X \in [G_\beta]$. To see this statement, enumerate $X = \{x_\alpha\,; \alpha \in \omega\}$ in $V[G_{\omega_2}]$. For arbitrary α and $n \in \omega$, since $\{p \in \mathbb{P}_{\omega_2}\,; \exists i \in 2\ p \Vdash_{\omega_2} x_\alpha(n) = i\}$ is dense in $\mathbb{P}_{\omega_2}$, let A_n^α be a maximal antichain in it dense set and $f_n^\alpha \colon A_n^\alpha \to 2$ a function in V which assgin to condition p the witness $i \in 2$. Then we obtain that $p \Vdash_{\omega_2} \mathring{x_\alpha}(n) = \check{f_\alpha^n}(p)$. Note that since $\mathbb{P}_{\omega_1}$ has an $\aleph_2$-cc, there is a $\beta \in \omega_2$ such that

$$\bigcup_{\alpha \in \omega_1} \bigcup_{n \in \omega} \bigcup_{p \in A_n^\alpha} \operatorname{supt}(p) \in \beta.$$

Therefore, for a name $\mathring{\check{Y}} = \{\{\langle n, \check{f_\alpha^n}(p)\rangle\,; p \in A_n \wedge n \in \omega\}\,; \alpha \in \omega\}$, we obtain that $Y \in V[G_\beta]$ and $\Vdash_{\omega_2} \mathring{\check{Y}} = X$. $\square$

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