Note on dual energy formalism

Shinsuke Takasao

March 24, 2025

1 Introduction

The basic idea of dual energy formalism was first introduced by Bryan et al. (1995). Here any explicit non-ideal terms such as viscosity are ignored. The internal energy equation of an adiabatic gas with the effective adiabatic index of γ is written as

$$\frac{\partial e_{\mathrm{int}}}{\partial t} = -\nabla \cdot (\boldsymbol{v} e_{\mathrm{int}}) - (\gamma - 1) e_{\mathrm{int}} \nabla \cdot \boldsymbol{v} \equiv \left(\frac{\partial e_{\mathrm{int}}}{\partial t} \right)_{\mathrm{ad}} \tag{1}$$

Note that the dissipation heating coming from the numerical diffusion of the upwind scheme is ignored on the left-hand side. On the other hand, the energy conservation law gives the following equation.

$$\frac{\partial e_{\mathrm{int}}}{\partial t} = \frac{\partial e_{\mathrm{tot}}}{\partial t} - \frac{\partial}{\partial t} \left(e_{\mathrm{kin}} + e_{\mathrm{mag}} \right) \equiv \left(\frac{\partial e_{\mathrm{int}}}{\partial t} \right)_{\mathrm{tot}} \tag{2}$$

The left-hand side of this equation includes the heating due to numerical dissipation. Therefore, we generally get

$$\left(\frac{\partial e_{\mathrm{int}}}{\partial t} \right)_{\mathrm{tot}} \geq \left(\frac{\partial e_{\mathrm{int}}}{\partial t} \right)_{\mathrm{ad}} \tag{3}$$

If we integrate the two equations using the same e_{int}^{n}, the solution at n step, then at $n + 1$ step, we expect

$$e_{\mathrm{int,tot}}^{n+1} \geq e_{\mathrm{int,nc}}^{n+1}, \tag{4}$$

where $e_{\mathrm{int,tot}}$ indicates the internal energy density obtained from the total energy density e_{tot}, and $e_{\mathrm{int,nc}}$ denotes the solution of the internal energy equation for an adiabatic gas (nc denotes "non-conservative"). Considering this, we can use $e_{\mathrm{int,nc}}$ as a floor value. The floor value is spatially and temporally variable and has some physical meaning given by the internal energy equation for an adiabatic equation.

Negative pressure or internal energy density is a common issue in astrophysical simulations, particularly in highly magnetized regions and in shocks associated with high-Mach number flows. Adopting the dual energy formalism can serve as a reasonable compromise to address this problem (e.g. Takasao et al., 2022).

2 Method

2.1 A strategy

We explain the way we use $e_{\text{int,nc}}$ to improve the numerical stability. The requirements we propose are:

- Only modify energies in very low β regions

- Adopt a scheme that ensures the positivity of the internal energy density (at least for second-order schemes).

Positivity is essential because we want to calculate a floor value for the internal energy density, which must be positive.

The strategy you take should depend on the purpose and the property of your problems. Therefore, you need to test if your method is suitable for your problems.

2.2 Integration method

We first explain the integration method that ensures positivity. Equation 3 is integrated as follows. Just for simplicity, time integration is first-order in this explanation. e_{int}^n is the solution at step n. The advection term in the internal energy equation is solved using the Riemann solution so that the advection step is consistent with the density evolution. We define the solution at the advection step as e_{int}^*. The source term is integrated in an operator-splitting manner.

$$\frac{\partial e_{\text{int}}}{\partial t} = -(\gamma - 1)e_{\text{int}}\nabla \cdot \boldsymbol{v} \tag{5}$$

We analytically integrate this equation to get e_{int}^{n+1} as follows:

$$e_{\text{int}}^{n+1} = e_{\text{int}}^* \exp\left[-(\gamma - 1)\nabla \cdot \boldsymbol{v}\Delta t\right] \tag{6}$$

The velocity vectors at cell faces in $\nabla \cdot \boldsymbol{v}$ are obtained by the Riemann solver. As the solution of the Riemann problem is a time-averaged value between $[t, t + \Delta t]$, we can assume that the index of the exponential function is a constant during the timespan.

The above integration method ensures positivity (at least for second-order methods). e_{int}^* will not become negative in the advection step if the advection scheme is a robust upwind one. In addition, the source term integration guarantees positivity.

2.3 How to use the solution from the internal energy equation

Now we have $e_{\text{int,tot}}^{n+1}$ and $e_{\text{int,nc}}^{n+1}$. We define α as the ratio of the magnetic energy density to the internal energy density:

$$\alpha \equiv \frac{e_{\text{mag}}}{e_{\text{int,nc}}} \tag{7}$$

2

Note that we use $e_\text{int,nc}$ in the definition because $e_\text{int,tot}$ may be negative. α is similar to the inverse of the plasma β, but it can be quite different from β when γ is close to unity. For this reason, we use α. We have to make sure that α is calculated using the data at $n + 1$ step so that the time is the same among the quantities.

There should be different ways, but here we introduce one approach that combines the two e_int. We define a function $C(\alpha)$ which is close to unity and update the internal energy in the following way:

$$e_\text{int}^{n+1} = \begin{cases} e_\text{int,tot}^{n+1}, & e_\text{int,tot}^{n+1} > C(\alpha)e_\text{int,nc}^{n+1} \\ e_\text{int,nc}^{n+1}, & \text{otherwise} \end{cases} \tag{8}$$

A choice of $C(\alpha)$ is as follows (Iijima, 2016):

$$C(\alpha) = \min\left[C_\text{ub}, \max\left(C_\text{lb}, C_\text{var}(\alpha)\right)\right], \tag{9}$$

where

$$C_\text{var}(\alpha) = \frac{\alpha}{C_0 + \alpha}. \tag{10}$$

C_lb and C_ub are constants and set the lower and upper bounds for $C(\alpha)$, respectively. C_0 is another constant. We adopt $C_\text{lb} = 0.98, C_\text{ub} = 1.0, C_0 = 3.0$ as a fiducial set of parameters.

The current code allows users to change the switching function (i.e. methods corresponding to Equation 8).

2.4 Some cautions

When non-ideal source terms that can alter the internal energy density (e.g., thermal conduction, viscosity, ambipolar diffusion, etc.) are present in the energy equations, the code structure must be carefully considered to avoid incorrect calculations of the dissipation terms. When dissipation terms need to be included, the current code requires the use of the super time stepping (STS) method. Although STS is more computationally expensive, its code structure facilitates maintaining consistency in energy calculations.

References

Bryan, G. L., Norman, M. L., Stone, J. M., Cen, R., & Ostriker, J. P. 1995, Computer Physics Communications, 89, 149, doi: `10.1016/0010-4655(94)00191-4`

Iijima, H. 2016, PhD thesis, University of Tokyo, Department of Earth and Planetary Environmental Science

Takasao, S., Tomida, K., Iwasaki, K., & Suzuki, T. K. 2022, ApJ, 941, 73, doi: `10.3847/1538-4357/ac9eb1`