5. Additive Rules

Often it is easier to calculate the probability of some event from known probabilities of other events.

5. Additive Rules

Often it is easier to calculate the probability of some event from known probabilities of other events.

The event in question can be represented as the union of two other events, or as the complement of some event.

5. Additive Rules

Often it is easier to calculate the probability of some event from known probabilities of other events.

The event in question can be represented as the union of two other events, or as the complement of some event.

Several important laws frequently simplify the computation of probabilities.

Additive Rules

Theorem 2.10 If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

PROOF. Use the Venn diagram.

Additive Rules

Corollary 1 If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Additive Rules

Corollary 1 If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

PROOF. Since if A and B are mutually exclusive, $A \cap B = \emptyset$, corollary 1 is an immediate result of Theorem 2.10.

Additive Rules

Corollary 1 If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

PROOF. Since if A and B are mutually exclusive, $A \cap B = \emptyset$, corollary 1 is an immediate result of Theorem 2.10.

Corollary 2 If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Additive Rules

Example 2.27 John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8, and the probability he gets an offer from company B is 0.6. If, on the other hand, he believes that the probability that he will get offers from both companies is 0.5. What is the probability that he will get at least one offer from these two companies?

Additive Rules

Example 2.27 John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8, and the probability he gets an offer from company B is 0.6. If, on the other hand, he believes that the probability that he will get offers from both companies is 0.5. What is the probability that he will get at least one offer from these two companies?

Solution Using the additive rule we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.5 = 0.9.$$

Additive Rules

Example 2.28 What is the probability of getting a total of 7 or 11 when a pair of fair dice are tossed?

Additive Rules

Example 2.28 What is the probability of getting a total of 7 or 11 when a pair of fair dice are tossed?

Solution

A: the event that 7 occurs; B: the event that 11 occurs.

Additive Rules

Example 2.28 What is the probability of getting a total of 7 or 11 when a pair of fair dice are tossed?

Solution

A: the event that 7 occurs; B: the event that 11 occurs.

- a total of 7 occurs for 6 of the 36 sample points

Additive Rules

Example 2.28 What is the probability of getting a total of 7 or 11 when a pair of fair dice are tossed?

Solution

A: the event that 7 occurs; B: the event that 11 occurs.

- a total of 7 occurs for 6 of the 36 sample points

  a total of 11 occurs for only 2 of the sample points.

Additive Rules

Example 2.28 What is the probability of getting a total of 7 or 11 when a pair of fair dice are tossed?

Solution

A: the event that 7 occurs; B: the event that 11 occurs.

- a total of 7 occurs for 6 of the 36 sample points
  a total of 11 occurs for only 2 of the sample points.
- all sample points are equally likely, we have $P(A) = 1/6$ and $P(B) = 1/18$.

Additive Rules

Example 2.28 What is the probability of getting a total of 7 or 11 when a pair of fair dice are tossed?

Solution

A: the event that 7 occurs; B: the event that 11 occurs.

- a total of 7 occurs for 6 of the 36 sample points
  a total of 11 occurs for only 2 of the sample points.
- all sample points are equally likely, we have $P(A) = 1/6$ and $P(B) = 1/18$.

The events A and B are mutually exclusive, therefore

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}.$$

Additive Rules

How about the union of 3 events?

Additive Rules

How about the union of 3 events?

Theorem 2.11 For three events A, B, and C,
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$
$-P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

Additive Rules

How about the union of 3 events?

Theorem 2.11 For three events A, B, and C,
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$$
$$-P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

How to prove?

Additive Rules

about A and A'

Additive Rules

about A and A'

Theorem 2.12 If A and A' are complementary events, then

$$P(A) + P(A') = 1.$$

Additive Rules

about A and A'

Theorem 2.12 If A and A' are complementary events, then

$$P(A) + P(A') = 1.$$

Sometimes, it is more difficult to calculate the probability that an event occurs than it is to calculate the probability that the event does not occur.

Additive Rules

Example 2.30 If the probabilities that an automobile mechanic will service 3,4,5,6,7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10 and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Additive Rules

Example 2.30 If the probabilities that an automobile mechanic will service 3,4,5,6,7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10 and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Solution. Let E be the event that at least 5 cars are serviced.

Additive Rules

Example 2.30 If the probabilities that an automobile mechanic will service 3,4,5,6,7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10 and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Solution. Let E be the event that at least 5 cars are serviced. Now, $P(E) = 1 - P(E')$, where E' is the event that fewer than 5 cars are serviced.

Additive Rules

Example 2.30 If the probabilities that an automobile mechanic will service 3,4,5,6,7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10 and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Solution. Let E be the event that at least 5 cars are serviced. Now, $P(E) = 1 - P(E')$, where E' is the event that fewer than 5 cars are serviced.

$P(E') = 0.12 + 0.19 = 0.31,$

Additive Rules

Example 2.30 If the probabilities that an automobile mechanic will service 3,4,5,6,7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10 and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Solution. Let E be the event that at least 5 cars are serviced. Now, $P(E) = 1 - P(E')$, where E' is the event that fewer than 5 cars are serviced.

$P(E') = 0.12 + 0.19 = 0.31$,

it follows that $P(E) = 1 - 0.31 = 0.69$.

Additive Rules

Additive Rules

A collection of events $\{A_1, A_2, \ldots, A_n\}$ of a sample space S is called **a partition of** S

Additive Rules

A collection of events $\{A_1, A_2, \ldots, A_n\}$ of a sample space S is called **a partition of** S

1. $A_1, A_2, \ldots, A_n$ are mutually exclusive

2. $A_1 \cup A_2 \cup \cdots \cup A_n = S$.

Additive Rules

A collection of events $\{A_1, A_2, \ldots, A_n\}$ of a sample space S is called **a partition of** S

1. $A_1, A_2, \ldots, A_n$ are mutually exclusive

2. $A_1 \cup A_2 \cup \cdots \cup A_n = S$.

Corollary 3 If $\{A_1, A_2, \ldots, A_n\}$ is a partition of S, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1$$

6. Conditional Probability

The probability of an event B occurring when it is known that some event A has occurred.

6. Conditional Probability

The probability of an event B occurring when it is known that some event A has occurred.

denoted by $P(B|A)$, read "the probability of B, given A".

Conditional Probability

Example

A die is constructed such that the even numbers are twice as likely to occur as the odd numbers.

Conditional Probability

Example

A die is constructed such that the even numbers are twice as likely to occur as the odd numbers.

i.e. the sample space is $S = \{1, 2, 3, 4, 5, 6\}$, $1/9$ and $2/9$ assigned to the odd and the even numbers respectively.

Conditional Probability

Example

A die is constructed such that the even numbers are twice as likely to occur as the odd numbers.

i.e. the sample space is $S = \{1, 2, 3, 4, 5, 6\}$, $1/9$ and $2/9$ assigned to the odd and the even numbers respectively.

event B: get a perfect square

Conditional Probability

Example

A die is constructed such that the even numbers are twice as likely to occur as the odd numbers.

i.e. the sample space is $S = \{1, 2, 3, 4, 5, 6\}$, $1/9$ and $2/9$ assigned to the odd and the even numbers respectively.

event B: get a perfect square
$B = \{1, 4\}$, $P(B) = 1/3$

Conditional Probability

If it is known that the toss of the die resulted in a number greater than 3.

Conditional Probability

If it is known that the toss of the die resulted in a number greater than 3. What's the probability that B occurs under this condition?

Conditional Probability

If it is known that the toss of the die resulted in a number greater than 3. What's the probability that B occurs under this condition?

dealing with a **reduced sample space**, i.e. $A = \{4, 5, 6\}$, a subset of S.

Conditional Probability

If it is known that the toss of the die resulted in a number greater than 3. What's the probability that B occurs under this condition?

dealing with a **reduced sample space**, i.e. $A = \{4, 5, 6\}$, a subset of S.

find the probability that B occurs, relative to the space A

Conditional Probability

If it is known that the toss of the die resulted in a number greater than 3. What's the probability that B occurs under this condition?

dealing with a **reduced sample space**, i.e. $A = \{4, 5, 6\}$, a subset of S.

find the probability that B occurs, relative to the space A denote this event by $B|A$,

Conditional Probability

If it is known that the toss of the die resulted in a number greater than 3. What's the probability that B occurs under this condition?

dealing with a **reduced sample space**, i.e. $A = \{4, 5, 6\}$, a subset of S.

find the probability that B occurs, relative to the space A denote this event by $B|A$, $B|A = \{4\}$,

Conditional Probability

If it is known that the toss of the die resulted in a number greater than 3. What's the probability that B occurs under this condition?

dealing with a **reduced sample space**, i.e. $A = \{4, 5, 6\}$, a subset of S.

find the probability that B occurs, relative to the space A denote this event by $B|A$, $B|A = \{4\}$,

$P(B|A) = 2/5$.

Conditional Probability

Remark: Events may have different probabilities when considered relative to different sample spaces.

Conditional Probability

Remark: Events may have different probabilities when considered relative to different sample spaces.

We can also write

$$P(B|A) = \frac{2}{5} = \frac{2/9}{5/9} = \frac{P(A \cap B)}{P(A)}.$$

Conditional Probability

Remark: Events may have different probabilities when considered relative to different sample spaces.

We can also write

$$P(B|A) = \frac{2}{5} = \frac{2/9}{5/9} = \frac{P(A \cap B)}{P(A)}.$$

$P(A \cap B)$ and $P(A)$: from the original sample space S.

Conditional Probability

Definition 2.9 The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{if} \quad P(A) > 0.$$

Conditional Probability

Example 2.31 The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane (a) arrives on time given that it departed on time, (b) departed on time given that it has arrived on time.

Conditional Probability

Example 2.31 The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane (a) arrives on time given that it departed on time, (b) departed on time given that it has arrived on time.

Solution

a. $$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94.$$

Conditional Probability

Example 2.31 The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane (a) arrives on time given that it departed on time, (b) departed on time given that it has arrived on time.

Solution

a. $$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94.$$

b. $$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95.$$

Conditional Probability

It is important to know the probability that the flight arrives on time.

Conditional Probability

It is important to know the probability that the flight arrives on time.

One is given the information that the flight did not depart on time. **Armed with this additional information,** the more pertinent probability is $P(A|D')$.

Conditional Probability

It is important to know the probability that the flight arrives on time.

One is given the information that the flight did not depart on time. **Armed with this additional information,** the more pertinent probability is $P(A|D')$.

$$P(A|D') = \frac{P(A \cap D')}{P(D')} = \frac{0.82 - 0.78}{0.17} = 0.24$$

Conditional Probability

It is important to know the probability that the flight arrives on time.

One is given the information that the flight did not depart on time. **Armed with this additional information,** the more pertinent probability is $P(A|D')$.

$$P(A|D') = \frac{P(A \cap D')}{P(D')} = \frac{0.82 - 0.78}{0.17} = 0.24$$

The probability of an on-time arrival is diminished severely in the presence of the additional information.

Conditional Probability

It is important to know the probability that the flight arrives on time.

One is given the information that the flight did not depart on time. **Armed with this additional information,** the more pertinent probability is $P(A|D')$.

$$P(A|D') = \frac{P(A \cap D')}{P(D')} = \frac{0.82 - 0.78}{0.17} = 0.24$$

The probability of an on-time arrival is diminished severely in the presence of the additional information.

$P(A|D')$ is an '**updating**' of $P(A)$

Independent Events

previous example: the die-tossing experiment

Independent Events

previous example: the die-tossing experiment

$P(B|A) = 2/5$, $P(B) = 1/3$

Independent Events

previous example: the die-tossing experiment

$P(B|A) = 2/5$, $P(B) = 1/3$

$P(B|A) \neq P(B)$, indicating that B **depends** on A

Independent Events

Example 2 cards are drawn in succession from an ordinary deck, **with replacement**.

Independent Events

Example 2 cards are drawn in succession from an ordinary deck, **with replacement**.

A: the first card is an ace,

B: the second card is a spade.

Independent Events

Example 2 cards are drawn in succession from an ordinary deck, **with replacement**.

A: the first card is an ace,

B: the second card is a spade.

Since the first card is replaced, our sample spaces for both the first and second draws are the same.

Independent Events

Example 2 cards are drawn in succession from an ordinary deck, **with replacement**.

A: the first card is an ace,

B: the second card is a spade.

Since the first card is replaced, our sample spaces for both the first and second draws are the same.

$$P(B|A) = 13/52 \quad \text{and} \quad P(B) = 13/52.$$

Independent Events

Example 2 cards are drawn in succession from an ordinary deck, **with replacement**.

A: the first card is an ace,
B: the second card is a spade.

Since the first card is replaced, our sample spaces for both the first and second draws are the same.

$$P(B|A) = 13/52 \quad \text{and} \quad P(B) = 13/52.$$

We have $P(B|A) = P(B)$, the events A and B are said to be **independent**.

Independent Events

Definition 2.10 Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A).$$

Otherwise, A and B are **dependent**.

7. Multiplicative Rules

Multiplying the formula

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

by $P(A)$, we obtain the **multiplicative rule**.

7. Multiplicative Rules

Multiplying the formula

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

by $P(A)$, we obtain the **multiplicative rule**.

Theorem 2.13 If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A).$$

7. Multiplicative Rules

Multiplying the formula

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

by $P(A)$, we obtain the **multiplicative rule**.

Theorem 2.13 If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A).$$

We can also write $P(A \cap B) = P(B)P(A|B)$.

Multiplicative Rules

Example 2.32 Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Multiplicative Rules

Example 2.32 Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution

event A: the first fuse is defective

Multiplicative Rules

Example 2.32 Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution

event A: the first fuse is defective

event B: the second fuse is defective

Multiplicative Rules

Example 2.32 Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution

event A: the first fuse is defective

event B: the second fuse is defective

$A \cap B$: A occurs, and then B occurs after A has occurred

Multiplicative Rules

Example 2.32 Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution

event A: the first fuse is defective

event B: the second fuse is defective

$A \cap B$: A occurs, and then B occurs after A has occurred

$$P(A \cap B) = P(A)P(B|A) = \frac{5}{20} \times \frac{4}{19} = \frac{1}{19}.$$

Multiplicative Rules

Example 2.33 One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Multiplicative Rules

Example 2.33 One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Solution B_1: a black ball from bag 1;
B_2: a black ball from 2; W_1: a white ball from bag 1.

Multiplicative Rules

Example 2.33 One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Solution B_1: a black ball from bag 1;
B_2: a black ball from 2; W_1: a white ball from bag 1.

$$P[(B_1 \cap B_2) \text{ or } (W_1 \cap B_2)] = P(B_1 \cap B_2) + P(W_1 \cap B_2)$$
$$= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1) = (\tfrac{3}{7})(\tfrac{6}{9}) + (\tfrac{4}{7})(\tfrac{5}{9})$$

Multiplicative Rules

Theorem 2.14 Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Multiplicative Rules

Theorem 2.14 Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

How to prove?

Multiplicative Rules

Example 2.34 A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance ia available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available.

Multiplicative Rules

Example 2.34 A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance ia available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available.

Solution A: the fire engine is available

B: the ambulance is available

Multiplicative Rules

Example 2.34 A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance ia available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available.

Solution A: the fire engine is available

B: the ambulance is available

$$P(A \cap B) = P(A)P(B) = 0.98 \times 0.92 = 0.9016$$

Multiplicative Rules

Theorem 2.15

If the events $A_1, A_2, A_3, \cdots A_k$ can occur, then $P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{k-1})$.

Multiplicative Rules

Theorem 2.15

If the events $A_1, A_2, A_3, \cdots A_k$ can occur, then $P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{k-1})$.

If the events $A_1, A_2, A_3, \cdots A_k$ are independent, then $P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) = P(A_1)P(A_2)P(A_3) \cdots P(A_k)$.

Multiplicative Rules

Example 2.36 Three cards are draw in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

Multiplicative Rules

Example 2.36 Three cards are draw in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

Solution We have

$$P(A_1) = 2/52, \; P(A_2|A_1) = 8/51, \; P(A_3|A_1 \cap A_2) = 12/50,$$

Multiplicative Rules

Example 2.36 Three cards are draw in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

Solution We have

$$P(A_1) = 2/52, \ P(A_2|A_1) = 8/51, \ P(A_3|A_1 \cap A_2) = 12/50,$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) = 8/5525.$$

Multiplicative Rules

Example 2.37 A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails and 1 head?

Multiplicative Rules

Example 2.37 A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails and 1 head?

Solution the sample space

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$

Multiplicative Rules

Example 2.37 A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails and 1 head?

Solution the sample space
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
$P(H) = 2/3, \ P(T) = 1/3$

Multiplicative Rules

Example 2.37 A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails and 1 head?

Solution the sample space
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
$P(H) = 2/3,\ P(T) = 1/3$

event A: get 2 tails and 1 head; $A = \{TTH, THT, HTT\}$
$P(TTH) = P(T)P(T)P(H) = (\frac{1}{3})(\frac{1}{3})(\frac{2}{3}) = 2/27$.

Multiplicative Rules

Example 2.37 A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails and 1 head?

Solution the sample space
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
$P(H) = 2/3,\ P(T) = 1/3$

event A: get 2 tails and 1 head; $A = \{TTH, THT, HTT\}$
$P(TTH) = P(T)P(T)P(H) = (\frac{1}{3})(\frac{1}{3})(\frac{2}{3}) = 2/27.$

Similarly, $P(THT) = P(HTT) = 2/27,$
$P(A) = 2/27 + 2/27 + 2/27 = 2/9.$

8. Bayes' Rule

Theorem of Total Probability

Example 2.38 In a certain assembly plant, three machines, B_1, B_2 and B_3 make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3% and 2% of the products made by each machine, respectively, are defective. Suppose that a finished products is randomly selected. What is the probability that it is defective?

Theorem of Total Probability

Example 2.38 In a certain assembly plant, three machines, B_1, B_2 and B_3 make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3% and 2% of the products made by each machine, respectively, are defective. Suppose that a finished products is randomly selected. What is the probability that it is defective?

Solution

Theorem of Total Probability

Example 2.38 In a certain assembly plant, three machines, B_1, B_2 and B_3 make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3% and 2% of the products made by each machine, respectively, are defective. Suppose that a finished products is randomly selected. What is the probability that it is defective?

Solution

$$\begin{aligned} P(A) &= P(B_1 \cap A) + P(B_2 \cap A) + P(B_3 \cap A) \\ &= P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3) \end{aligned}$$

Theorem of Total Probability

On the other hand,

$P(B_1)P(A|B_1) = 0.3 \times 0.02 = 0.006,$

$P(B_2)P(A|B_2) = 0.45 \times 0.03 = 0.0135,$

$P(B_3)P(A|B_3) = 0.25 \times 0.02 = 0.005.$

Theorem of Total Probability

On the other hand,

$P(B_1)P(A|B_1) = 0.3 \times 0.02 = 0.006,$

$P(B_2)P(A|B_2) = 0.45 \times 0.03 = 0.0135,$

$P(B_3)P(A|B_3) = 0.25 \times 0.02 = 0.005.$

Therefore,

$$P(A) = 0.006 + 0.0135 + 0.005 = 0.0245$$

Theorem of Total Probability

Theorem 2.16 If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$

Bayes' Rule

Example 2.39 With reference to Example 2.38, if a product were chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

Bayes' Rule

Example 2.39 With reference to Example 2.38, if a product were chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

Solution

$P(B_3|A) = P(B_3 \cap A)/P(A)$

$$= \frac{P(B_3)P(A|B_3)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$= \frac{0.005}{0.006 + 0.0135 + 0.005} = \frac{0.005}{0.0245} = \frac{10}{49}.$$

Bayes' Rule

Theorem 2.17 (Bayes' Rule) If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S, where $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}, \text{ for } r = 1, 2, \ldots, k$$