Chapter 7

Flux conservative problems

7.1 Flux Conservative Equation

A large class of PDEs can be cast into the form of a flux conservative equation:

$$\frac{\partial \vec{U}}{\partial t} = \frac{-\partial f}{\partial x}(\vec{U}, \vec{U}_x, \vec{U}_{xx}, \ldots)$$

Example: flux conservative form for the wave equation

We consider the 1-D wave equation $U_{tt} = c^2 U_{xx}$. If we let:

$$\vec{w} = \begin{pmatrix} r \\ s \end{pmatrix}, \qquad \text{where } r = c\frac{\partial U}{\partial x}, \text{ and } s = \frac{\partial U}{\partial t}.$$

This means that:

$$\begin{aligned} \frac{\partial \vec{w}}{\partial t} &= \begin{pmatrix} \frac{\partial r}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} = \begin{pmatrix} c\frac{\partial s}{\partial x} \\ c\frac{\partial r}{\partial x} \end{pmatrix} \\ \text{or } \frac{\partial \vec{w}}{\partial t} &= -\frac{\partial}{\partial x}\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}\vec{w} = -\frac{\partial}{\partial x} f(\vec{w}) \end{aligned}$$

7.2 Stability analysis of numerical solutions of the first order flux conservative or 1-D advection equation

$$\frac{\partial U}{\partial t} = -c\frac{\partial U}{\partial x} \tag{7.1}$$

We introduce a change of variable $\xi = x - ct$ and:

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} = -c\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi}$$

We see that equation 7.1 holds: $-c\frac{\partial U}{\partial \xi} = -c\frac{\partial U}{\partial \xi}$. So, $U(x,t) = U(\xi) = f(x - ct)$ is the analytic general solution of equation 7.1, which is a wave propagating in the right (positive x) direction.

We study the stability of different finite difference schemes in solving the flux conservative or 1-D advection equation:

$$U_t = -cU_x, \ x_0 \leq x \leq x_1, \ t_0 \leq t \leq T$$

Again we discretise problem $\Delta x = \frac{x_1 - x_0}{n+1}, \ \Delta t = \frac{T - t_0}{m}$ and let $x_j = x_0 + j\Delta x, \ j = 0, \ldots, n+1, \ t_k = t_0 + k\Delta t, \ k = 0, \ldots, m$, and $U_j^k = U(x_j, t_k)$.

7.3 Forward Time Centred Space (FTCS)

Forward Euler method in time:

$$\frac{\partial U_j^k}{\partial t} = \frac{U_j^{k+1} - U_j^k}{\Delta t} + \mathrm{O}(\Delta t)$$

Leap-frog or centred difference in space:

$$\frac{\partial U_j^k}{\partial x} = \frac{U_{j+1}^k - U_{j-1}^k}{2\Delta x} + \mathrm{O}(\Delta x^2)$$

Using FTCS method: $U_t = -cU_x$ gives:

$$U_j^{k+1} = U_j^k - \frac{c\Delta t}{2\Delta x}[U_{j+1}^k - U_{j-1}^k] \tag{7.2}$$

7.3.1 von Neumann stability analysis of FTCS method

FTCS is unstable! Why?
We assume that independent solutions (eigenmodes) of equation 7.2 (or any difference equation) are of the form:

$$U_j^k = \xi^k e^{ipj\Delta x} \tag{7.3}$$

where p is a real spatial wavenumber and $\xi = \xi(p)$ is a complex number that depends on p.
Equation 7.3 shows that the time dependence of a single eigenmode U_j^k is

only through successive powers of $\xi(\xi^k)$. $\Rightarrow$ Difference equations are *unstable* if $|\xi(p)| > 1$ for some p. ξ is called the amplification factor.
To find $\xi(p)$ for FTCS method substitute $U_j^k = \xi^k e^{ipj\Delta x}$ into equation 7.2:

$$\begin{aligned}
\xi^{k+1} e^{ipj\Delta x} &= \xi^k e^{ipj\Delta x}\left(1 - \frac{c\Delta t}{2\Delta x}\underbrace{(e^{ip\Delta x} - e^{-ip\Delta x})}_{2i\sin(p\Delta x)}\right) \\
\Rightarrow \xi(p) &= 1 - \frac{ic\Delta t}{\Delta x}\sin(p\Delta x)
\end{aligned}$$

and $|\xi(p)| \geq 1 \; \forall \; p \Rightarrow$ FTCS scheme is unconditionally *unstable* for solving $U_t = -cU_x$.

7.4 Lax Method

Again we are solving the flux conservative equation: $U_t = -cU_x$. The instability in the FTCS method is removed in the Lax method by using the average for $U_j^k = \frac{U_{j+1}^k + U_{j-1}^k}{2}$ instead of U_j^k in approximating U_t:

$$\frac{\partial U_j^k}{\partial t} = \frac{U_j^{k+1} - \frac{1}{2}[U_{j+1}^k + U_{j-1}^k]}{\Delta t}$$

and centred difference again for U_x. Then $U_t = -cU_x$ becomes:

$$U_j^{k+1} = \frac{1}{2}[U_{j+1}^k + U_{j-1}^k] - \frac{c\Delta t}{2\Delta x}[U_{j+1}^k - U_{j-1}^k] \tag{7.4}$$

7.4.1 von Neumann Stability Analysis of Lax Method

The Lax method is conditionally stable. To see substitute $U_j^k = \xi^k e^{ipj\Delta x}$ into equation 7.4:

$$\xi^{k+1} e^{ipj\Delta x} = \xi^k e^{ipj\Delta x}\left(\underbrace{\frac{1}{2}[e^{ip\Delta x} + e^{-ip\Delta x}]}_{\cos(p\Delta x)} - \frac{c\Delta t}{2\Delta x}\underbrace{(e^{ip\Delta x} - e^{-ip\Delta x})}_{2i\sin(p\Delta x)}\right)$$

$$\Rightarrow \xi = \cos(p\Delta x) - i\frac{c\Delta t}{\Delta x}\sin(p\Delta x)$$

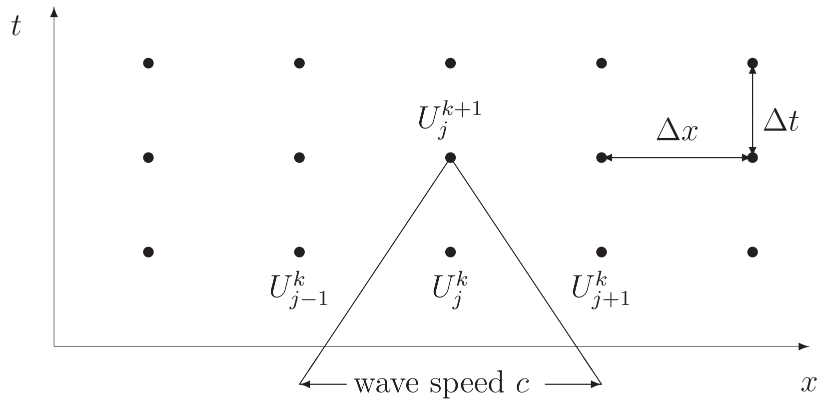
Lax method stable when $|\xi|^2 \leq 1$:

$$\Rightarrow |\cos^2(p\Delta x) + \frac{c^2\Delta t^2}{\Delta x^2}\sin^2(p\Delta x)| \leq 1$$

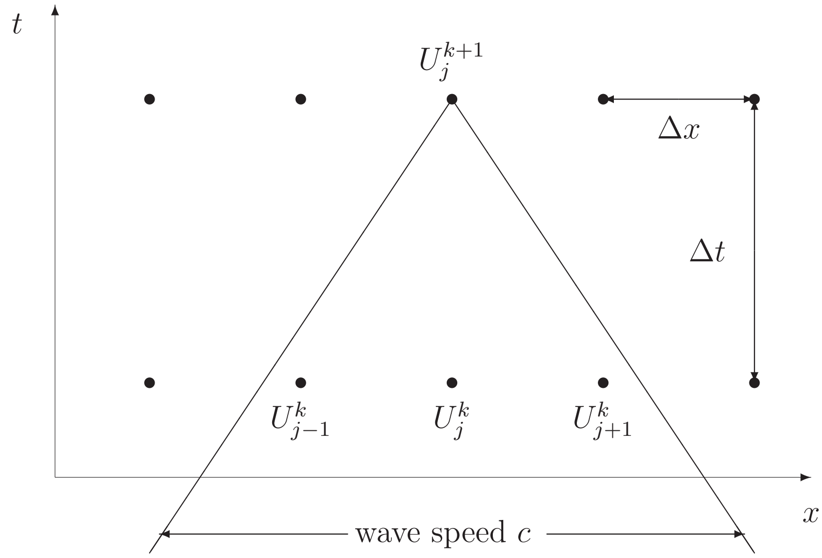
$$\text{or } |1-(1-\frac{c^2\Delta t^2}{\Delta x^2})\sin^2(p\Delta x)| \leq 1$$
$$\Rightarrow 1-\frac{c^2\Delta t^2}{\Delta x^2} \geq 0$$
$$\text{or } \frac{c^2\Delta t^2}{\Delta x^2} \leq 1$$
$$\text{or} \qquad \underbrace{\Delta t \leq \frac{\Delta x}{c}}_{\text{COURANT CONDITION}} \qquad (c > 0)$$

7.5 Courant Condition

- The Courant condition means Lax method is stable when $\Delta t \leq \Delta x / c$
- The physical meaning is that value U_j^{k+1} is computed from information at points $j-1$ and $j+1$ at time k in a stable scheme, where the wave speed is less that the mesh spacing divided by time. ie. in a continuum wave equation information propagates at maximum speed c, so Lax method is *stable* when $\frac{\Delta x}{\Delta t} \geq c$. This is shown in the plot below:



- Unstable schemes arise when $\frac{\Delta x}{\Delta t} \leq c$ ie. when the time step Δt becomes too large because U_j^{k+1} requires information from points outside $[U_{j-1}^k, U_{j+1}^k]$ as shown in the plot below. (see Press et al, Numerical Recipes, p. 825-830)



7.6 von Neumann Stability Analysis For Wave Equation

$$
\begin{aligned}
U_{tt} &= c^2 U_{xx} \\
\text{let} \quad \vec{w} &= \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} cU_x \\ U_t \end{pmatrix}
\end{aligned}
$$

We saw earlier that:

$$
\frac{\partial \vec{w}}{\partial t} = \begin{pmatrix} \frac{\partial r}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} = \begin{pmatrix} c\frac{\partial s}{\partial x} \\ c\frac{\partial r}{\partial x} \end{pmatrix} = c\frac{\partial}{\partial x} \begin{pmatrix} s \\ r \end{pmatrix}
$$

7.6.1 Lax method

We will solve the wave equation using the Lax method:
$r_t = cs_x$ becomes:

$$
r_j^{k+1} = \frac{1}{2}[r_{j-1}^k + r_{j+1}^k] + \frac{c\Delta t}{2\Delta x}(s_{j+1}^k - s_{j-1}^k) \tag{7.5}
$$

$s_t = cr_x$ becomes:

$$s_j^{k+1} = \frac{1}{2}[s_{j-1}^k + s_{j+1}^k] + \frac{c\Delta t}{2\Delta x}(r_{j+1}^k - r_{j-1}^k) \tag{7.6}$$

where $r_j^k = r(x_j, t_k)$ and $s_j^k = s(x_j, t_k)$

For von Neumann stability analysis assume eigen-modes for r_j^k and s_j^k are of the form:

$$\begin{pmatrix} r_j^k \\ s_j^k \end{pmatrix} = \xi^k e^{ipj\Delta x} \begin{pmatrix} r_j^0 \\ s_j^0 \end{pmatrix} \Rightarrow \text{solutions stable if } |\xi| \leq 1$$

Equation 7.5 and 7.6 give:

$$\begin{pmatrix} \xi - \cos(p\Delta x) & -\frac{ic\Delta t}{\Delta x}\sin(p\Delta x) \\ -\frac{ic\Delta t}{\Delta x}\sin(p\Delta x) & \xi - \cos(p\Delta x) \end{pmatrix} \begin{pmatrix} r^0 \\ s^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- This has a solution only if determinant = 0.
- This gives $\xi = \cos(p\Delta x) \pm i\frac{c\Delta t}{\Delta x}\sin(p\Delta x)$.
- This is stable if $|\xi|^2 \leq 1$ which gives same *Courant condition* $\Delta t \leq \frac{\Delta x}{c}$.

7.7 Other sources of error

7.7.1 Phase Errors (through dispersion)

- Fourier analysis of the Lax method shows how phase errors arise.
- The Fourier mode $U(x,t) = e^{i(px+\omega t)}$ is an exact solution of $U_t = -cU_x$ if ω and p satisfy the dispersion relation $\omega = -cp$, then $U(x,t) = e^{ip(x-ct)} = f(x-ct)$ gives the exact solution of $U_t = -cU_x$.
- ie. this mode is completely undamped and the amplitude is constant (no dispersion) for the numerical solution using a time step which satisifes this dispersion relation.
- We will show the effects of phase errors by studying the numerical solution of the 1-D advection equation using different time steps which lead to dispersion being absent or present in section 7.7.2.

Dispersion relation for the Lax Method

The dispersion relation is only satisfied if: $\Delta t = \frac{\Delta x}{c}$.
Why? Consider $U_j^k = \xi^k e^{ipj\Delta x}$
In section 7.4.1 we found:

$$\begin{aligned} \xi &= \cos(p\Delta x) - i\frac{c\Delta t}{\Delta x}\sin(p\Delta x) \\ &= e^{-ip\Delta x} + i(1 - \frac{c\Delta t}{\Delta x})\sin(p\Delta x) \end{aligned}$$

If we let $\Delta t = \frac{\Delta x}{c} \Rightarrow \xi = e^{-ip\Delta x}$ and $U_j^k = \xi^k e^{ipj\Delta x} = e^{ip(-k\Delta x + j\Delta x)}$
When we substitute $x_j = j\Delta x,\ t_k = k\Delta t$ and the dipsersion relation $\Delta x = c\Delta t$ then: $U_j^k = e^{ip(-ck\Delta t + j\Delta x)} = e^{ip(x_j - ct_k)} = \underbrace{f(x_j - ct_k)}_{\text{exact solution}}$.

Thus the Lax method has no dispersion present when the time step satisfies the dispersion relation exactly: $\Delta t = \frac{\Delta x}{c}$. We will show this in the next section.

x

7.7.2 Dispersion in the numerical solution of the 1-D advection equation using the Lax method

The matlab code for this section is **Lax_Flux.m**.

Use Lax Method to solve:

$$U_t + U_x = 0, \quad 0 \le x \le 2 \approx \infty, \quad 0 \le t \le 1$$

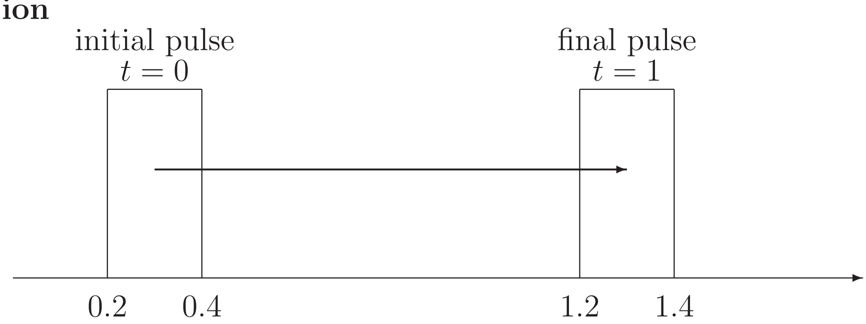
initial conditions:

$$U(x,0) = \begin{cases} 1, & 0.2 \le x \le 0.4 \\ 0, & \text{otherwise} \end{cases} = U^0(x)$$

boundary conditions:

$$U(0,t) = U(2,t) = 0$$

Exact solution



$$U(x,t) = U^0(x - ct)$$

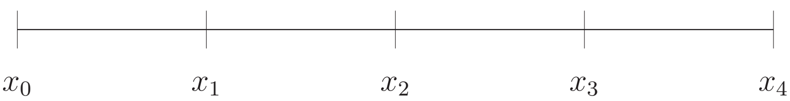
$$= \quad U^0(x-t) \qquad (c=1)$$

In the next section we compare the above exact solution with the numerical solution using the Lax method with different time steps:

- $\Delta t = \frac{\Delta x}{c} \Rightarrow$ *no dispersion* matches analytic solution.
- $\Delta t = \frac{\Delta x}{2c} \Rightarrow$ *dispersion* present but pulse matches speed of wave.
- $\Delta t = \frac{1.001\Delta x}{c} \Rightarrow$ courant condition not met $\rightarrow$ unstable!

Lax Method for $U_t + U_x = 0$

Equation 7.4 gives: $U_j^{k+1} = \frac{1}{2}[U_{j+1}^k + U_{j-1}^k] - \frac{c\Delta t}{2\Delta x}[U_{j+1}^k - U_{j-1}^k]$
let $s = \frac{c\Delta t}{\Delta x}$
$\Rightarrow U_j^{k+1} = \frac{1}{2}(1-s)U_{j+1}^k + U_{j-1}^k + \frac{1}{2}(1+s)U_{j-1}^k$
Again for simplicity we only consider 4 elements in x:



Solve for U_j^{k+1} for $0 \le k \le m$, $1 \le j \le 3$ with *boundary conditions*: $U_0^k = 0$, $U_4^k = 0$. We have:

$$\begin{pmatrix} U_1^{k+1} \\ U_2^{k+1} \\ U_3^{k+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}(1-s) & 0 \\ \frac{1}{2}(1+s) & 0 & \frac{1}{2}(1-s) \\ 0 & \frac{1}{2}(1+s) & 0 \end{pmatrix} \begin{pmatrix} U_1^k \\ U_2^k \\ U_3^k \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(1+s)U_0^k \\ 0 \\ \frac{1}{2}(1-s)U_4^k \end{pmatrix}$$

$$\text{or} \quad \vec{U}^{k+1} = A\vec{U}^k + \vec{b}$$

Dispersion means the initial pulse changes shape (unlike analytical solution) because wave components with different frequencies travel at different speeds. The matlab code is **Lax_Flux.m**.

The numerical solution changes for different time steps depending on whether or not the scheme is stable or dispersion is present. Figure 2.1 shows how the solution changes for different time steps depending on whether or not the scheme is stable or dispersion is present.

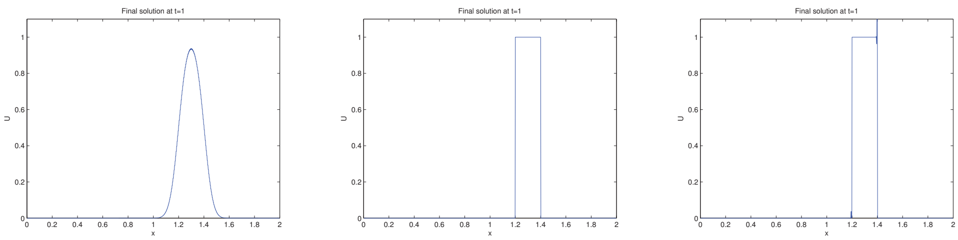


Figure 7.1: Solution at $t = 1$ using the Lax method with different time steps, (a) $\Delta t = \Delta x/2c$ where dispersion is present but the pulse matches the analytical solution for the speed of the wave, (b) $\Delta t = \Delta x/c$ where no dispersion is present and numerical solution matches analytical solution exactly, and (c) $\Delta t = 1.001\Delta x/c$ where the Courant condition is not met and solution is becoming unstable.

7.7.3 Error due to nonlinear terms

Example

Shock wave equation:

$$U_t + \underbrace{UU_x}_{\text{nonlinear term}} = 0$$

- nonlinear term causes wave profile to steepen resulting in a shock.
- schemes stable for linear problems can become unstable.
- this will be discussed later in chapter 11

7.7.4 Aliasing error

Example

Alising error occurs when a short wavelength (λ_1) is not represented well by the mesh-spacing (Δx), and may be misinterpreted as a longer wavelength oscillation (λ_2).

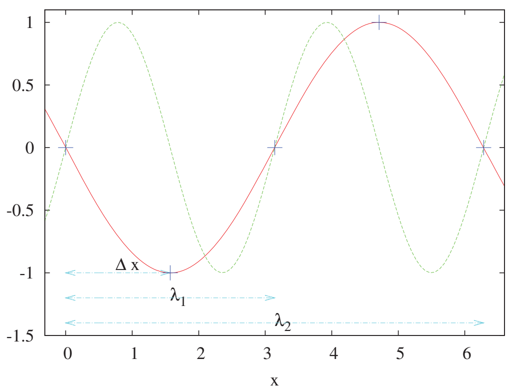


Figure 7.2: Aliasing error occurs when the mesh spacing Δx is too large to represent the smallest wavelength λ_1 and misinterprets it as a longer wavelength oscillation λ_2