Chapter 1

Overview of PDEs

1.1 Classification of PDEs

The classification of PDEs is important for the numerical solution you choose.

$$A(x,y)U_{xx} + 2B(x,y)U_{xy} + C(x,y)U_{yy} = F(x,y,U_x,U_y,U)$$

1.1.1 Elliptic

$$AC > B^2$$

For example, Laplace's equation:

$$U_{xx} + U_{yy} = 0$$

$A = C = 1$, $B = 0$

1.1.2 Hyperbolic

$$AC < B^2$$

For example the 1-D wave equation:

$$U_{xx} = \frac{1}{c^2} U_{tt}$$

$A = 1, C = -1/c^2, B = 0$

1.1.3 Parabolic

$$AC = B^2$$

For example, the heat or diffusion Equation

$$U_t = \beta U_{xx}$$

$A = 1, B = C = 0$

1.2 Implicit Vs Explicit Methods to Solve PDEs

Explicit Methods:

- possible to solve (at a point) directly for all unknown values in the finite difference scheme.

- stable only for certain time step sizes (or possibly never stable!). Stability can be checked using Fourier or von Neumann analysis. Time step size governed by *Courant condition* for wave equation.

Implicit Methods:

- there is no explicit formula at each point, only a set of simultaneous equations which must be solved over the whole grid.

- Implicit methods are stable for all step sizes.

1.3 Well-posed and ill-posed PDEs

The heat equation is *well-posed* $U_t = U_{xx}$. However the *backwards* heat equation is *ill-posed*: $U_t = -U_{xx} \Rightarrow$ at high frequencies this blows up!

In order to demonstrate this we let $U(x,t) = a_n(t)\sin(nx)$
then:

$$U_{xx} = -a_n(t)n^2\sin(nx), \qquad \text{and} \qquad U_t = \dot{a}_n(t)\sin(nx)$$

$$\underbrace{U_t = U_{xx}}_{\text{Heat Equation}} \quad \Rightarrow \dot{a}_n(t)\sin(nx) = -a_n(t)n^2\sin(nx)$$

$$\dot{a}_n = -a_n n^2 \Rightarrow a_n(t) = a_n(0)e^{-n^2 t}$$

For the heat equation the transient part of the solution decays and this has stable numerical solutions.

$$\underbrace{U_t = -U_{xx}}_{\text{Backwards Heat Equation}} \qquad \Rightarrow \dot{a}_n(t)\sin(nx) = a_n(t)n^2\sin(nx)$$

$$\dot{a}_n = a_n n^2 \Rightarrow a_n(t) = a_n(0)e^{n^2 t}$$

For the backwards heat equation the transient part of the solution blows up and the numerical solution would fail! In general it is difficult or impossible to obtain numerical solutions for ill-posed PDEs.

Part I

Numerical solution of parabolic equations