11.1.1 Example 1: Using method of characteristics to solve the linear 1-D advection equation

$$\begin{aligned} U_t + cU_x &= 0 \\ \text{initial conditions: } x(s=0) = x_0, &\quad t(s=0) = 0 \\ U(x, t=0) = f(x) &\quad \text{or} \quad U(s=0) = f(x_0) \end{aligned}$$

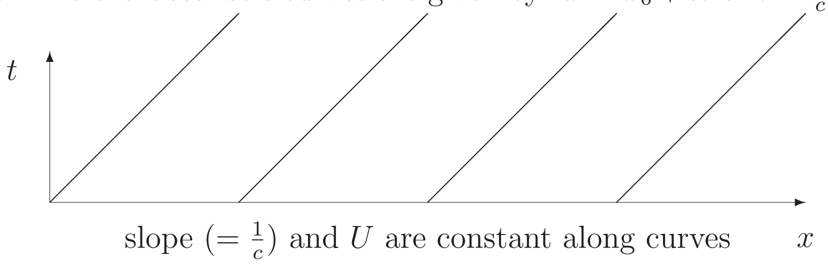
$$\begin{aligned} \frac{dx}{ds} = c \quad &\Rightarrow \quad x = cs + k_1, \ \text{use } x(0) = x_0 = k_1 \\ &\Rightarrow \quad x = cs + x_0 \end{aligned}$$

$$\begin{aligned} \frac{dt}{ds} = 1 \quad &\Rightarrow \quad t = s + k_2, \ \text{use } t(0) = 0 = k_2 \\ &\Rightarrow \quad t = s \end{aligned}$$

and since $t = s \Rightarrow x = ct + x_0$
Solve for U:

$$\frac{dU}{ds} = \frac{dt}{ds}U_t + \frac{dx}{ds}U_x = U_t + cU_x = 0$$

ie $\frac{dU}{ds} = 0 \Rightarrow U = k_3$ (U is constant along characteristic curves).
Use initial conditions $U(s = 0, x_0) = f(x_0) = k_3$
ie $U = f(x_0) = f(x - ct)$ since $x_0 = x - ct$.
This is the same as D'Alembert's solution for a wave moving to the right at speed c. The characteristic curves are given by: $x = x_0 + ct$ or $t = \frac{1}{c}(x - x_0)$.



slope ($= \frac{1}{c}$) and U are constant along curves

11.1.2 Example 2: Using method of characteristics to solve the *nonlinear* inviscid Burger's equation

Shock waves result when solving the nonlinear inviscid Burger's equation:

$$\begin{aligned} U_t + UU_x &= 0 \\ \text{initial conditions: } x(s=0) = x_0, &\quad t(s=0) = 0 \\ U(x, t=0) = f(x) &\quad \text{or} \quad U(s=0) = f(x_0) \end{aligned}$$

Now the wave speed is *not* constant but depends on the amplitude $U(x,t)$. The characteristic equations are:

$$\frac{dt}{ds} = 1 \Rightarrow t = s \quad (\text{using } t(0) = 0)$$
$$\frac{dx}{ds} = U \Rightarrow x = Ut + x_0 \quad (\text{using } x(0) = x_0 \text{ and } t = s)$$

Again:

$$\begin{aligned}
\frac{dU}{ds} &= \frac{dt}{ds}\frac{\partial U}{\partial t} + \frac{dx}{ds}\frac{\partial U}{\partial x} \\
&= U_t + UU_x = 0 \\
\Rightarrow U &= k_3 = f(x_0) = f(x - Ut)
\end{aligned}$$

So $U = f(x - Ut)$ is given *implicitly* since U is a function of itself. The characteristic curves given by

$$t = \frac{1}{U}(x - x_0) = \frac{1}{f(x_0)}(x - x_0)$$

The characteristic curves *no longer* have *constant* slope - they may cross (meaning U is multiply defined $\rightarrow$ *shock* waves) or be discontinuous (regions with no solution for $U \rightarrow$ *expansion* waves) as we will see in the next example.

Example

Solving $U_t + UU_x = 0$ with the following initial conditions:

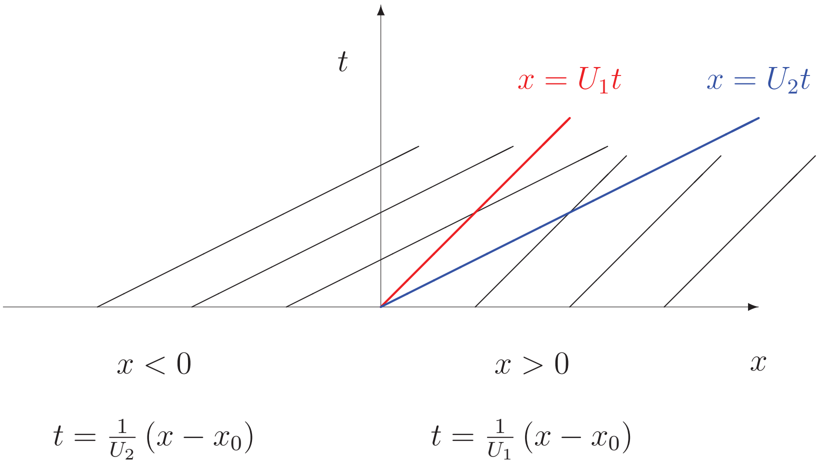
$$U(x, t = 0) = f(x) = \begin{cases} U_1, & x > 0 \\ U_2, & x < 0 \end{cases}$$

$$t = \begin{cases} \frac{1}{U_1}(x - x_0), & x > 0 \quad \text{or } x = U_1 t + x_0 \\ \frac{1}{U_2}(x - x_0), & x < 0 \quad \text{or } x = U_2 t + x_0 \end{cases}$$

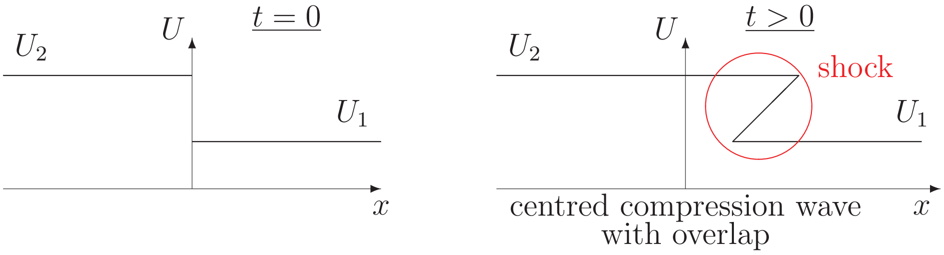
2 cases

$$\begin{aligned}
U_1 &< U_2 \text{ - compression wave} \rightarrow \textit{shock} \\
U_1 &> U_2 \text{ - expansion wave} \rightarrow \textit{rarefaction}
\end{aligned}$$

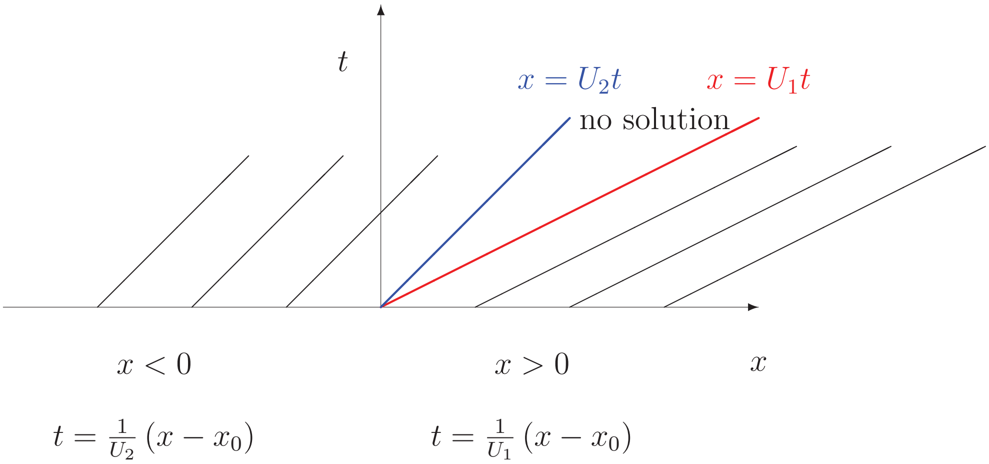
Case 1: Shock wave $U_1 < U_2$



In the fan bounded by $x = U_1 t$ and $x = U_2 t$ the characteristic curves are multi-valued leading to shocks (breaking waves). We illustrate this below:



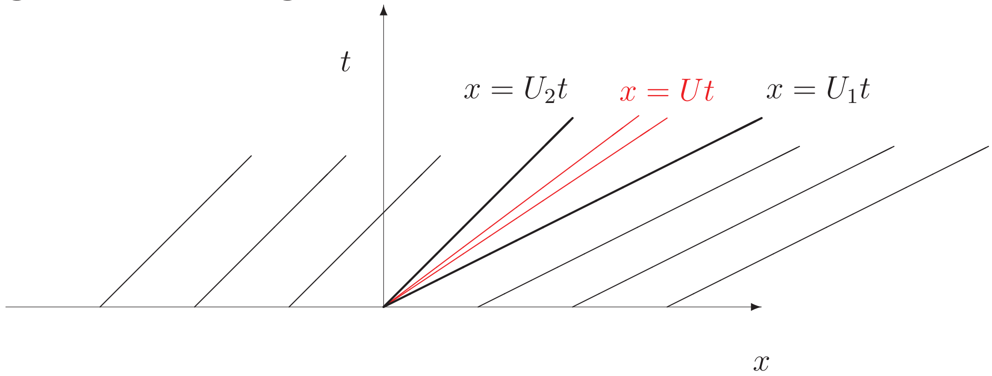
Case 2: Rarefaction or expansion wave $U_1 > U_2$



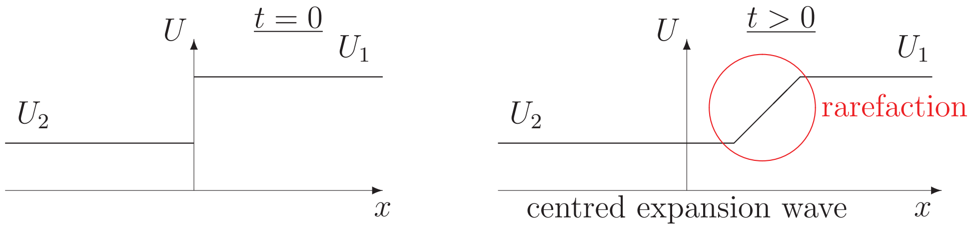
The solution is single-valued for $t > 0$ unlike the shock wave case. However in wedge between $x = U_2 t$ and $x = U_1 t$ there is *no* information. We assume $x = U_t$ in wedge since $U_2 t \leq x \leq U_1 t$ and speeds vary $U_2 \leq U \leq U_1$

and add solution to the wedge.

Adding solution to wedge:



$$\text{Thus } U = \begin{cases} U_2, & \frac{x}{t} < U_2 \\ \frac{x}{t}, & U_2 < \frac{x}{t} < U_1 \\ U_1, & \frac{x}{t} > U_1 \end{cases}$$



11.2 Numerical Solution for nonlinear Burger's Equation

$$U_t + UU_x = 0, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1$$

$$U(x, 0) = f(x) = \exp(-10(4x - 1)^2)$$

Solution given implicitly by $U(x,t) = f(x - Ut)$ so *speed* depends on *amplitude*, U.

We study the numerical solution using 3 methods but we will see in each case that the numerical solution fails to produce a shock wave because we are unable to produce multi-valued solutions.

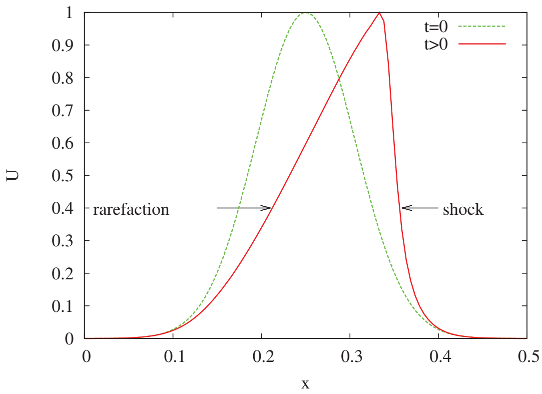


Figure 11.1: The analytical solution $U(x,t) = f(x - Ut)$ is plotted to show how shock and rarefaction develop for this example

11.2.1 Example I: Finite difference solution with *Lax Method*

The matlab code is **Shock_Lax.m**. We are solving:

$$U_t + UU_x = 0$$

$$\begin{aligned} \frac{\partial U_j^k}{\partial t} &= \frac{U_j^{k+1} - \frac{1}{2}(U_{j-1}^k + U_{j+1}^k)}{\Delta t} \quad \text{(Lax method for } U_t) \\ \frac{\partial U_j^k}{\partial x} &= \frac{U_{j+1}^k - U_{j-1}^k}{2\Delta x} \quad \text{(leap-frog for } U_x) \end{aligned}$$

The Courant condition only holds for *linear* wave equation. A good guess is $\Delta t << \frac{\Delta x}{\max(U)}$.(Waves travel at a maximum wave speed $U = 1$.)

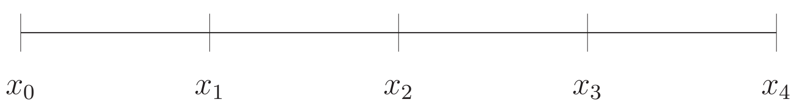
Put difference equations into PDE:

$$U_t + UU_x = 0 \quad \text{becomes:}$$

$$U_j^{k+1} = \frac{1}{2}\left\{U_{j+1}^k(1 - sU_j^k) + U_{j-1}^k(1 + sU_j^k)\right\} \tag{11.3}$$

Where $s = \frac{\Delta t}{\Delta x}$

Use boundary conditions $U(0,t) = U(1,t) = 0$ and for 4 elements:



So $U_0^k = 0 = U_4^k$ given by boundary conditions and we can rewrite Equation 11.3 as a matrix system of equations:

$$
\begin{aligned}
\vec{U}^{k+1} &= \begin{pmatrix} U_1^{k+1} \\ U_2^{k+1} \\ U_3^{k+1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 - sU_1^k & 0 \\ 1 + sU_2^k & 0 & 1 - sU_2^k \\ 0 & 1 + sU_3^k & 0 \end{pmatrix} \begin{pmatrix} U_1^k \\ U_2^k \\ U_3^k \end{pmatrix} \\
&+ \frac{1}{2} \begin{pmatrix} (1 + sU_1^k)U_0^k \\ 0 \\ (1 - sU_3^k)U_4^k \end{pmatrix} \\
&= \frac{1}{2} A\vec{U}^k + \frac{1}{2}\vec{b}
\end{aligned}
$$

A varies with time because of U_j^k term in matrix!

We can compare the difference between the matlab code for the linear 1D advective equation ($U_t + U_x = 0$), **Lax_Flux.m** in section 7.7.2 and the shock wave equation ($U_t + UU_x = 0$) above, **Shock_Lax.m**.

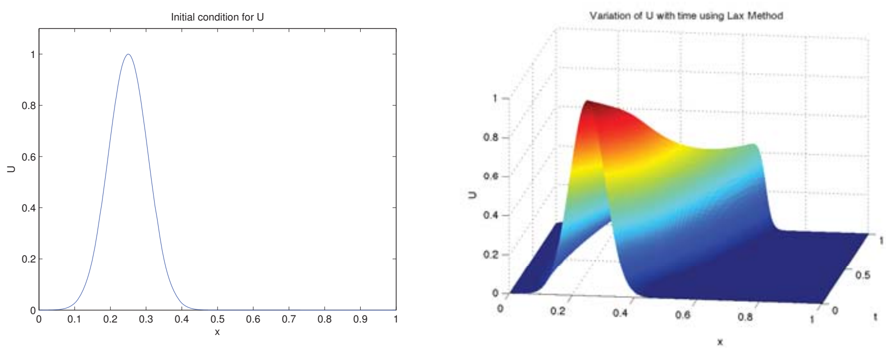


Figure 11.2: Initial conditions in (a) and solution for nonlinear Buger's equation using the Lax method in (b)

When we compare the analytical solution given by the method of characteristics to the numerical solution given by the Lax method we can see that the numerical solution is accurate for the linear 1D advection equation (see numerical solution in figure 7.1) but fails to give a shock wave for the nonlinear Burger's equation in figure 11.2. The Lax method introduces dispersion into the numerical solution and in the nonlinear case this "removes" the shock wave instability and flattens the wave front.

11.2.2 Example II: Solution using *Method of Lines*

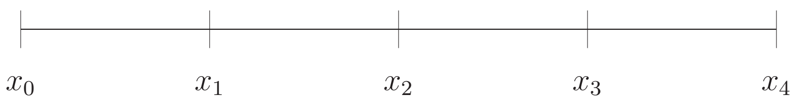
The matlab code is **Shock_mol.m** and **Uprime2.m**.

$$\frac{\partial U}{\partial t} = -UU_x$$

Using the method of lines solution (as demonstrated in section 2.3) we only replace spatial derivative U_x with *FD* approximation.

$$\frac{\partial U_j}{\partial x} = \frac{U_{j+1} - U_{j-1}}{2\Delta x}, \Rightarrow \frac{\partial U_j}{\partial t} = -U_j\left(\frac{U_{j+1} - U_{j-1}}{2\Delta x}\right)$$

and again we show the case with 4 elements:



$U_0 = 0 = U_4$ from boundary conditions.

$$\frac{\partial \vec{U}}{\partial t} = \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{pmatrix} = \frac{1}{2\Delta x}\begin{pmatrix} 0 & -U_1 & 0 \\ U_2 & 0 & -U_2 \\ 0 & U_3 & 0 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} + \frac{1}{2\Delta x}\begin{pmatrix} U_1U_0 \\ 0 \\ -U_3U_4 \end{pmatrix}$$

or

$$\dot{\vec{U}} = \frac{1}{2\Delta x}A(U)\vec{U} + \vec{b}$$

When the shock develops in figure 11.3(b) the numerical solution becomes *unstable* using the method of lines.

11.2.3 Example III: Solution using *Spectral Method*

The matlab code is **Shock_spectral.m**.

$$U_t = -UU_x, \quad 0 \leq x \leq 2\pi$$

(we can change variable: $\xi = \frac{x}{2\pi}$ later so that the range for ξ is $0 \leq \xi \leq 1$)

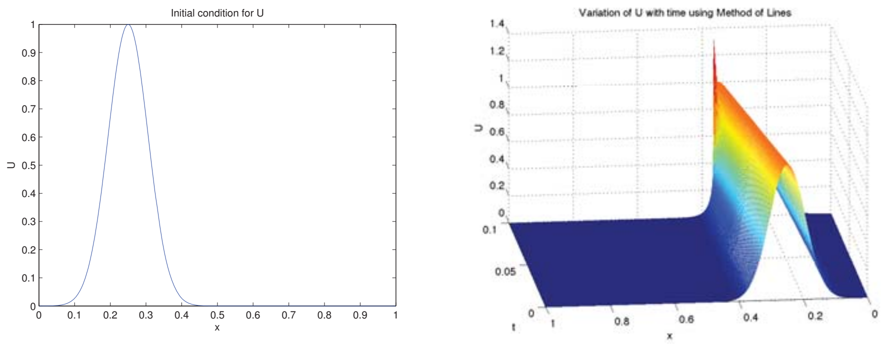


Figure 11.3: Initial conditions in (a) and solution for nonlinear Buger's equation using the method of lines in (b)

Spectral method

We let $U(x_j, t_k) = U_j^k,\ x_j = j\Delta x,\ j = 0, 1, \ldots, 2n-1,\ t_k = k\Delta t,\ k = 0, 1, \ldots, m,$ and $\Delta t = \frac{T}{m}$.

Take the discrete Fourier transform of U:

$$\begin{aligned}\hat{U}_\nu &= F(U) = \sum_{j=0}^{2n-1} U(x_j, t)\exp(-ix_j\nu) \text{ for } \nu = -n+1, \ldots, n \\ \text{where } x_j &= j\Delta x = \frac{j\pi}{n}\end{aligned}$$

then

$$\begin{aligned}U_j &= F^{-1}(\hat{U}) = \frac{1}{2n}\sum_{\nu=-n+1}^{n} \hat{U}_\nu \exp(ix_j\nu) \\ \text{for } j &= 0, 1, \ldots, 2n-1\end{aligned}$$

and

$$\begin{aligned}\frac{\partial U_j^k}{\partial x} &= \frac{1}{2n}\sum_{\nu=-n+1}^{n} i\nu\hat{U}_\nu \exp(ix_j\nu) \\ &= F^{-1}(i\nu\hat{U}) \\ &= F^{-1}(i\nu\, F(U))\end{aligned}$$

Use leap-frog for U_t:

$$U_t = \frac{U_j^{k+1} - U_j^{k-1}}{2\Delta t}$$

then $U_t = -UU_x$ becomes:

$$U_j^{k+1} = U_j^{k-1} - 2\Delta t U_j^k \; F^{-1}(i\nu \; F(U_j^k))$$

To find U_j^{-1} for spectral method we assume wave speed ≈ 1 and:

$$\begin{aligned} U_j^{-1} &= U(x, -\Delta t) = U^0(x - Ut) = f(x - Ut) \\ &\approx f(x + \Delta t) = \exp(-10(4(x + \Delta t) - 1)^2) \end{aligned}$$

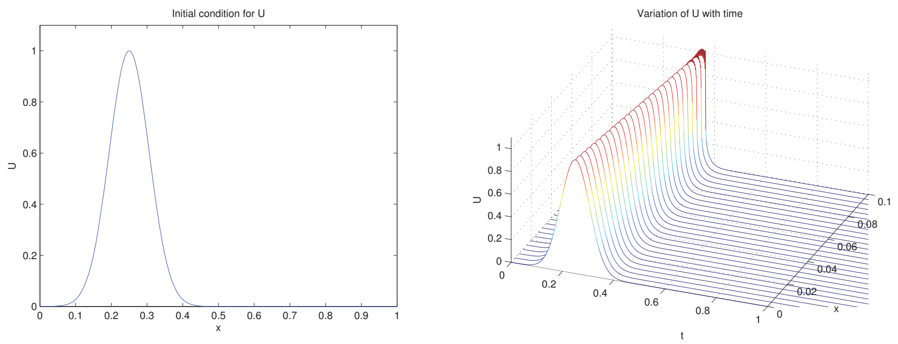


Figure 11.4: Initial conditions in (a) and solution for nonlinear Buger's equation using the spectral method in (b)

Again as in the method of lines the numerical solution becomes *unstable* as shock develops in figure 11.4 using the spectral method.