Chapter 3, Vector Spaces

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Outline

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- The operations of addition and scalar multiplication are used in many contexts in mathematics. Regardless of the context, however, these operations usually obey the same set of algebra rules. Thus a general theory of mathematical systems involving addition and scalar multiplication will have application to many areas in mathematics.
- Mathematical systems of this form are called vector spaces or linear spaces.

Outline

Euclidean Vector Spaces $\mathbb{R}^n$

In general, scalar multiplication and addition in $\mathbb{R}^n$ are defined by

$$\alpha x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} \quad \text{and} \quad x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

for any $x, y \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$.

$\mathbb{R}^{m \times n}$ denote the set of all $m \times n$ matrices with real entries.

Vector Space Axioms

Definition

Let V be a set on which the operations of addition and scalar multiplication are defined. By this we mean that, with each pair of elements x and y in V, we can associate a unique elements $x + y$ that is also in V, and with each element x in V and each scalar α, we can associate a unique element αx in V. The set V together with the operations of addition and scalar multiplication is said to form a **vector space** *if the following axioms are satisfied.*

A1 $x + y = y + x$ for any $x, y \in V$.

A2 $(x + y) + z = x + (y + z)$ for any $x, y, z \in V$.

A3 There exists an element 0 in V such that $x + 0 = x$ for each $x \in V$.

A4 For each $x \in V$, there exists an element $-x \in V$ such that $x + (-x) = 0$.

A5 $\alpha(x + y) = \alpha x + \alpha y$ for each scalar α and any $x, y \in V$.

A6 $(\alpha + \beta)x = \alpha x + \beta x$ for any scalars α and β and any $x \in V$.

A7 $(\alpha\beta)x = \alpha(x)$ for any scalars α and β and any $x \in V$.

A8 $1 \cdot x = x$ for all $x \in V$.

The closure properties of the two operations:

C1 If $x \in V$ and α is a scalar, then $\alpha x \in V$.

C2 If $x, y \in V$, then $x + y \in V$.

Example

Let

$$W = \{(a, 1) \mid a \text{ is real}\}$$

with addition and scalar Multiplication defined in the usual way.

Example

Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by

$$\begin{aligned} \alpha(x_1, x_2) &= (\alpha x_1, \alpha x_2) \\ (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, 0) \end{aligned}$$

We use the symbol $\oplus$ to denote the addition operation for this system avoid confusion with the usual addition $x + y$ of row vectors. Show that S, with the ordinary scalar multiplication and addition operation $\oplus$, is not a vector space. Which of the eight axioms fail to hold ?

The Vector Space $C[a, b]$

$C[a, b]$ denotes the set of all real-valued functions that are defined and continuous on the closed interval $[a, b]$.

$$(f + g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x)$$

The Vector Space P_n

P_n denotes the set of all polynomials of degree less than n.

$$(p + q)(x) = p(x) + q(x)$$
$$(\alpha p)(x) = \alpha p(x)$$

Theorem

If V is a vector space and $x \in V$ then

1. $0x = 0$.
2. $x + y = 0$ *implies that* $y = -x$.
3. $(-1)x = -x$.

Outline

Definition

If S is a nonempty subset of a vector space V, and S satisfies the following conditions:

1. $\alpha x \in S$ *whenever* $x \in S$ *for any scalar* α.
2. $x + y \in S$ *whenever* $x, y \in S$.

then S is said to be a **subspace** *of V.*

Example

Let

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_2 = 2x_1 \right\}.$$

Then S is a subset of $\mathbb{R}^2$.

Example

Let

$$S = \left\{ \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T \mid x_1 = x_2 \right\}.$$

Then S is a subset of $\mathbb{R}^3$.

- We refer $\{\mathbf{0}\}$ as the *zero subspace*.
- Every subspace of a vector space is a vector space in its own right.
- All the other subspace are referred to as *proper subspace*.

Example

Let

$$S = \left\{ \left(\begin{array}{c} x \\ 1 \end{array} \right) \mid x \in \mathbb{R} \right\}.$$

Then S is not a subset of $\mathbb{R}^2$.

Example

Let

$$S = \{A \in \mathbb{R}^{2\times 2} \mid a_{12} = -a_{21}\}$$

Then S is a subset of $\mathbb{R}^{2\times 2}$.

Example

Let S be the set of all polynomials of degree less than n with the property that $p(0) = 0$. The set S is nonempty since it contains the zero polynomial. We claim that S is a subspace of Pn.

Example

Let $C^n[a, b]$ be the set of all functions f that have a continuous nth derivative on $[a, b]$. We leave it to the reader to verify that $C^n[a, b]$ is a subspace of $C[a, b]$.

Example

The function $f(x) = |x|$ is in $C[-1, 1]$, but it is not differentiable at $x = 0$ and hence it is not in $C^1[-1, 1]$. This shows that $C^1[-1, 1]$ is a proper subspace of $C[-1, 1]$. The function $g(x) = x|x|$ is in $C^1[-1, 1]$, since it is differentiable at every point in $[-1, 1]$ and $g'(x) = 2|x|$ is continuous on $[-1, 1]$. However, $g \notin C^2[-1, 1]$, since $g''(x)$ is not defined when $x = 0$. Thus, the vector space $C^2[-1, 1]$ is a proper subspace of both $C[-1, 1]$ and $C^1[-1, 1]$.

Example

Let S be the set of all f in $C^2[a, b]$ such that

$$f''(x) + f(x) = 0$$

for all x in $[a, b]$. The set S is nonempty, since the zero function is in S.

The Nullspace of a Matrix

Definition

Let A be an $m \times n$ matrix. Let $N(A)$ denote the set of all solutions to the homogeneous system $Ax = 0$. Thus

$$N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$

Then the subspace $N(A)$ is called the **nullspace** *of A.*

Example

Determine $N(A)$ if

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}.$$

Using Gauss?Jordan reduction to solve $Ax = 0$, we have

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

The reduced row echelon form involves two free variables, x_3 and x_4:

$$\begin{aligned} x_1 &= x_3 - x_4 \\ x_2 &= -2x_3 + x_4. \end{aligned}$$

We set $x_3 = \alpha$ and $x_4 = \beta$, then

$$\mathbf{x} = \begin{pmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix}$$

Hence, the vector space $N(A)$ consists of all vectors of the form

$$\alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

The Span of a Set of Vectors

Definition

Let $v_1, v_2, \ldots, v_n$ be vectors in a vector space V. A sum of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n,$$

where α_j, $j = 1, \ldots, n$, are scalars, is called a **linear combination** *of $v_1, v_2, \ldots, v_n$. The set of all linear combinations of $v_1, v_2, \ldots, v_n$ is called the* **span***p of $v_1, v_2, \ldots, v_n$. The span of $v_1, v_2, \ldots, v_n$ will be denoted by*

$$Span(v_1, v_2, \ldots, v_n).$$

Spanning Set for a Vector Space

Theorem

If $v_1, v_2, \ldots, v_n$ *are elements of a vector space* V*, then* $Span(v_1, v_2, \ldots, v_n)$*. is a subspace of* V*.*

Definition

The set $v_1, v_2, \ldots, v_n$ is a **spanning set** *for V if and only if every vector in V can be written as a linear combination of $v_1, v_2, \ldots, v_n$.*

Example

- $\left\{e_1, e_2, e_3, (1,2,3)^T\right\}$.
- $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$.
- $\{(1,0,1)^T, (0,1,0)^T\}$.
- $\{(1,2,4)^T, (2,1,3)^T, (4,-1,1)^T\}$.

Outline

Consider the following vectors in $\mathbb{R}^3$

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad x_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad x_3 = \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix}$$

Conclusion

- If $v_1, v_2, \ldots, v_n$ span a vector space V and one of these vectors can be written as a linear combination of the other $n-1$ vectors, then those $n-1$ vectors span V.
- Given n vectors $v_1, v_2, \ldots, v_n$, it is possible to write one of the vectors as a linear combination of the other $n-1$ vectors if and only if there exist scalars $c_1, \ldots, c_n$ not all zero such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$$

Definition

The vectors $v_1, v_2, \ldots, v_n$ in a vector space V are said to be **linearly independent** *if*

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$$

implies that all the scalars $c_1, \ldots, c_n$ must equal 0.

Definition

The vectors $v_1, v_2, \ldots, v_n$ in a vector space V are said to be **linearly dependent** *if there exist scalars$c_1, \ldots, c_n$ not all zero such that*

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$$

Example

Which of the following collections of vectors are linearly independent in R^3?

- $(1,1,1)^T, (1,1,0)^T, (1,0,0)^T$.
- $(1,0,1)^T, (0,1,0)^T$.
- $(1,2,4)^T, (2,1,3)^T, (4,-1,1p)^T$.

Theorem

Let $x_1, x_2, \ldots, x_n$ be n vectors in $\mathbb{R}^n$ and $X = (x1, , xn)$. The vectors $x_1, x_2, \ldots, x_n$ will be linearly dependent if and only if pX is singular.

Example

Determine whether the vectors $(4, 2, 3)^T$, $(2, 3, 1)^T$, and $(2, -5, 3)^T$ are linearly dependent.

Example

Given

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ 0 \\ 7 \\ 7 \end{pmatrix},$$

determine if the vectors are linearly independent.

Theorem

Let $v_1, v_2, \ldots, v_n$ be vectors in a vector space V. A vector v in $Span(v_1, v_2, \ldots, v_n)$ can be written uniquely as a linear combination of $v_1, v_2, \ldots, v_n$ if and only if $v_1, v_2, \ldots, v_n$ are linearly independent.p

Outline

Definition

The vectors $v_1, v_2, \ldots, v_n$ form a basis for a vector space V if and only if

1. $v_1, v_2, \ldots, v_n$ *are linearly independent.*
2. $v_1, v_2, \ldots, v_n$ *span V.*

Example

p The standard basis for $\mathbb{R}^3$ is $\{e_1, e_2, e_3\}$, however, there are many bases that we could choose for $\mathbb{R}^3$.

Example

In $\mathbb{R}^{2\times 2}$, consider the set $\{E_11, E_12, E_21, E_22\}$, where

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem

If $v_1, v_2, \ldots, v_n$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$, is linearly dependent.

Corollary

If $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_m$ are both bases for a vector space V, then $n = m$.

Definition

Let V be a vector space. If V has a basis consisting of n vectors, we say that V has **dimension** *n. The subspace $\{0\}$ of V is said to have dimension 0. V is said to be* **finite-dimensional** *if there is a finite set of vectors that spans V; otherwise, we say that V is* **infinite-dimensional**.

Theorem

If V is a vector space of dimension $n > 0$:

1. *Any set of n linearly independent vectors spans V.*
2. *Any n vectors that span V are linearly independent.*

Example

Show that

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis in $\mathbb{R}^3$.

Theorem

If V is a vector space of dimension $n > 0$, then:

1. *No set of less than n vectors can span V.*
2. *Any subset of less than n linearly independent vectors can be extended to form a basis for V.*
3. *Any spanning set containing more than n vectors can be pared down to form a basis for V.*

Outline

Changing Coordinates in $\mathbb{R}^2$.

- Let
$$x = x_1 e_1 + x_2 e_2$$
Then the coordinate of x is $(x_1, x_2)^T$.
- Let
$$x = \alpha y + \beta z$$
Then the coordinate of x is $(\alpha, \beta)^T$.

Let $[e_1, e_2]$ be the standard basis, $[u_1, u_2]$ is another basis.

$$u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Two problems:

1. Given a vector $x = (x1, x2)^T$, find its coordinates with respect to u_1 and u_2.
2. Given a vector $c_1 u_1 + c_2 u_2$, find its coordinates with respect to e_1 and e_2.

Definition

Let

$$x = Uc.$$

The matrix U is called the transition matrix from the ordered basis $[u1, u2]$ to the basis $[e1, e2]$.

Example

Let $u_1 = (3, 2)^T$, $u_2 = (1, 1)^T$, and $x = (7, 4)^T$. Find the coordinates of x with respect to u_1 and u_2.

Example

Let $b_1 = (1, -1)^T$, $b_2 = (-2, 3)^T$. Find the transition matrix from $[e1, e2]$ to $[b1, b2]$ and determine the coordinates of $x = (1, 2)^T$ with respect to $[b1, b2]$.

Example

Find the transition matrix corresponding to the change of basis from $[v1, v2]$ to $[u1, u2]$, where

$$v_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \quad \text{and} \quad u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Change of Basis for a General Vector Space

Definition

Let V be a vector space and let $E = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V. If v is any element of V, then v can be written in the form

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where $c_1, \ldots, c_n$ are scalars. Thus we can associate with each vector v a unique vector $c = (c_1, c_2, \ldots, c_n)^T$ in $\mathbb{R}^n$. The vector c defined in this way is called the **coordinate vector** *of v with respect to the ordered basis E and is denoted $[v]_E$. The c_i's are called the* **coordinates** *of v relative to E.*

Example

Let

$$E = [v1, v2, v3] = [(1,1,1)^T, (2,3,2)^T, (1,5,4)^T]$$
$$F = [u1, u2, u3] = [(1,1,0)^T, (1,2,0)^T, (1,2,1)^T].$$

Find the transition matrix from E to F. If

$$x = 3v_1 + 2v_2 - v_3 \quad \text{and} \quad y = v_1 - 3v_2 + 2v_3$$

find the coordinates of x and y with respect to the ordered basis F.p

Outline

Definition

If A is an $m \times n$ matrix, the subspace of $\mathbb{R}^{1\times n}$ spanned by the row vectors of A is called the **row space** *of A. The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the* **column space** *of A.*

Theorem

Two ropw equivalent matrices have the same row space.

Definition

The rank of a matrix A is the dimension of the row space of A.

Example

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{pmatrix}$$

Theorem (Consistency Theorem for Linear Systems)

A linear system $Ax = b$ is consistent if and only if b is in the column space of A.

Theorem

Let A be an $m \times n$ matrix. The linear system $Ax = b$ is consistent for every $b \in \mathbb{R}^m$ if and only if the column vectors of A span $\mathbb{R}^m$. The system $Ax = b$ has at most one solution for every $b \in \mathbb{R}^m$ if and only if the column vectors of A are linearly independent.

Corollary

An $n \times n$ matrix A is nonsingular if and only if the column vectors of A form a basis for $\mathbb{R}^n$.

Definition

The dimension of the nullspace of a matrix is called the nullity of the matrix.

Theorem

The Rank-Nullity Theorem If A is an $m \times n$ matrix, then the rank of A plus the nullity of A equals n.

Example

Let

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix}$$

Find a basis for the row space of A and a basis for $N(A)$. Verify that $\dim N(A) = n - r$.

Theorem

If A is an $m \times n$ matrix, the dimension of the row space of A equals the dimension of the column space of A.

Example

Let

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$$

Find a basis for the column space of A.

Example

Find the dimension of the subspace of $\mathbb{R}^4$ *spanned by*

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 2 \\ 5 \\ -3 \\ 2 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 0 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 3 \\ 8 \\ -5 \\ 4 \end{pmatrix}$$