3. Counting Sample Points

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One of the problems that the statistician must consider and attempt to evaluate is the **probability of certain events** when an experiment is performed.

In many cases we shall be able to solve a probability problem by counting the number of points in the sample space without actually listing each element.

multiplication rule

the fundamental principle of counting:

multiplication rule

the fundamental principle of counting:

Theorem 2.1 If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

multiplication rule

Example 2.13 How many sample points are in the sample space when a pair of dice is thrown once?

multiplication rule

Example 2.13 How many sample points are in the sample space when a pair of dice is thrown once?

Solution The first dice can land in any one of $n_1 = 6$ ways. For each of these 6 ways the second die can also land in $n_2 = 6$ ways. Therefore, the pair of dice can land in

$$n_1 n_2 = 6 \times 6 = 36 \quad \text{possible ways.}$$

multiplication rule

The multiplication rule of Theorem 2.1 may be extended to cover any number of operations.

multiplication rule

The multiplication rule of Theorem 2.1 may be extended to cover any number of operations.

Theorem 2.2 If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \dots n_k$ ways.

multiplication rule

Example 2.15 Sam is going to assemble a computer by himself. He has the choice of ordering chips from two brands, a hard drive from four, memory from three and an accessory bundle from five local stores. How many different ways can Sam order the parts?

multiplication rule

Example 2.15 Sam is going to assemble a computer by himself. He has the choice of ordering chips from two brands, a hard drive from four, memory from three and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Solution Since $n_1 = 2$, $n_2 = 4$, $n_3 = 3$, and $n_4 = 5$, there are

$$n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$$

different ways to order the parts.

multiplication rule

Example 2.16 How many even four-digit numbers can be formed from the digits 0,1,2,5,6, and 9 if each digit can be used only once?

multiplication rule

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multiplication rule

Example 2.16 How many even four-digit numbers can be formed from the digits 0,1,2,5,6, and 9 if each digit can be used only once?

Solution We consider the units position by two parts, 0 or not 0. If the units position is 0, we have a total of

$$n_1 n_2 n_3 n_4 = 1 \times 5 \times 4 \times 3 = 60.$$

multiplication rule

Example 2.16 How many even four-digit numbers can be formed from the digits 0,1,2,5,6, and 9 if each digit can be used only once?

Solution We consider the units position by two parts, 0 or not 0. If the units position is 0, we have a total of

$$n_1 n_2 n_3 n_4 = 1 \times 5 \times 4 \times 3 = 60.$$

If the units position is not 0, there are a total of

$$n_1 n_2 n_3 n_4 = 2 \times 4 \times 4 \times 3 = 96.$$

even four-digit numbers.

multiplication rule

Example 2.16 How many even four-digit numbers can be formed from the digits 0,1,2,5,6, and 9 if each digit can be used only once?

Solution We consider the units position by two parts, 0 or not 0. If the units position is 0, we have a total of

$$n_1 n_2 n_3 n_4 = 1 \times 5 \times 4 \times 3 = 60.$$

If the units position is not 0, there are a total of

$$n_1 n_2 n_3 n_4 = 2 \times 4 \times 4 \times 3 = 96.$$

even four-digit numbers.

The total number of even four-digit numbers is $60 + 96 = 156$.

permutation

How many different arrangements are possible for sitting 6 people around a table?

permutation

How many different arrangements are possible for sitting 6 people around a table?
How many different orders are possible for drawing 2 lottery tickets from a total of 20?

permutation

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How many different orders are possible for drawing 2 lottery tickets from a total of 20?

The different arrangements are called **permutation**.

Definition 2.7 A **permutation** is an arrangement of all or part of a set of objects.

permutation

Consider the three letters a, b, and c.

permutation

Consider the three letters a, b, and c.

The possible permutations are abc, acb, bac, bca, cab, and cba. There are 6 different arrangement.

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Using Theorem 2.2 we could arrive at the answer 6 without listing the different orders.

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Consider the three letters a, b, and c.
The possible permutations are abc, acb, bac, bca, cab, and cba. There are 6 different arrangement.

Using Theorem 2.2 we could arrive at the answer 6 without listing the different orders.

There are $n_1 = 3$ choices for the first position, then $n_2 = 2$ for the second, leaving only $n_3 = 1$ choice for the last position, giving a total of

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$$n_1 n_2 n_3 = 3 \times 2 \times 1 = 6 \text{ permutations.}$$

permutation

Theorem 2.3 The number of permutations of n distinct objects is $n!$.

$n!$ is read 'n factorial'

$n! = n(n-1)(n-2)\cdots(3)(2)(1)$

permutation

The number of permutations of the four letters a, b, c, and d will be $4! = 24$.

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What's the number of permutations that are possible by taking the four letters two at a time?

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permutation

The number of permutations of the four letters a, b, c, and d will be $4! = 24$.

What's the number of permutations that are possible by taking the four letters two at a time?

Using Theorem 2.1, we have $n_1 = 4$ choices for the first position and $n_2 = 3$ for the second.
A total of $n_1 n_2 = 12$ permutations.

permutation

In general, n distinct objects taken r at a time can be arranged in $n(n-1)(n-2)\cdots(n-r+1)$ ways.

permutation

In general, n distinct objects taken r at a time can be arranged in $n(n-1)(n-2)\cdots(n-r+1)$ ways.

Theorem 2.4 The number of permutations of n distinct objects taken r at a time is

$$P_{r,n} = \frac{n!}{(n-r)!}.$$

permutation

Example 2.17 Three awards (research, teaching and service) will be given one year for a class of 25 graduate students in a statistic department. If each student can receive at most one award, how many possible selections are there?

permutation

Example 2.17 Three awards (research, teaching and service) will be given one year for a class of 25 graduate students in a statistic department. If each student can receive at most one award, how many possible selections are there?

Solution Since the awards are distinguishable, it is a permutation problem. The total number of sample points is

$$P_{3,25} = \frac{25!}{(25-3)!} = 13800.$$

permutation

Example 2.18 A president and a treasure are to be chosen from a student club consisting of 50 people. How many different choices of offers are possible if

permutation

Example 2.18 A president and a treasure are to be chosen from a student club consisting of 50 people. How many different choices of offers are possible if

a. there are no restrictions;

b. A will serve only if he is president;

c. B and C will serve together or not at all;

d. D and E will not serve together?

permutation

circular permutation: permutation that occur by arranging objects in a circle

permutation

circular permutation: permutation that occur by arranging objects in a circle

Example 4 people are playing bridge, we do not have a new permutation if they all move one position in a clockwise direction. How many different arrangements are there for the bridge game?

permutation

circular permutation: permutation that occur by arranging objects in a circle

Example 4 people are playing bridge, we do not have a new permutation if they all move one position in a clockwise direction. How many different arrangements are there for the bridge game?

By considering one person in a fixed position and arranging the other three in $3!$ ways.

permutation

Theorem 2.5 The number of permutations of n distinct objects arranged in a circle is $(n-1)!$

combinations

combinations: the number of ways of selecting r objects from n without regard to order.

The number of such combinations is denoted by $C_{r,n}$.

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The number of such combinations is denoted by $C_{r,n}$.

Consider the four letters a, b, c, and d.

- the ways of selecting one letter from four?

  a, b, c, and d that is

$$C_{1,4} = \frac{4!}{1!(4-1)!} = 4 \text{ ways.}$$

combinations

- the ways of selecting two letters from four?

  ab, ac, ad, bc, bd, cd; that is

$$C_{2,4} = \frac{4!}{2!(4-2)!} = 6 \text{ ways.}$$

combinations

- the ways of selecting two letters from four?

  ab, ac, ad, bc, bd, cd; that is

$$C_{2,4} = \frac{4!}{2!(4-2)!} = 6 \text{ ways.}$$

- the ways of selecting three letters from four?

  abc, abd, acd, bcd; that is

$$C_{3,4} = \frac{4!}{3!(4-3)!} = 4 \text{ ways.}$$

combinations

Theorem 2.8 The number of combinations of n distinct objects taken r at a time is

$$C_{r,n} = \frac{n!}{r!(n-r)!}.$$

combinations

Question 1: What's the number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind,..., n_k of a kth kind?

combinations

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hint: n objects, n positions

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hint: n objects, n positions

$$C_{n_1,n}C_{n_2,n-n_1}\cdots C_{n_k,n-n_1-\cdots-n_{k-1}} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$

combinations

Example 2.19 In a college football training session, the defensive coordinator needs to have 10 player standing in a row. Among these 10 players, there are 1 freshman, 2 sophomore, 4 juniors and 3 seniors, respectively. How many different ways can they be arranged in a row if only their class level will be distinguished?

combinations

Example 2.19 In a college football training session, the defensive coordinator needs to have 10 player standing in a row. Among these 10 players, there are 1 freshman, 2 sophomore, 4 juniors and 3 seniors, respectively. How many different ways can they be arranged in a row if only their class level will be distinguished?

Solution The total number of arrangements is

$$\frac{10!}{1!2!4!3!} = 12600.$$

combinations

Question 2: What's the number of arrangements of a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth?

combinations

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$$C_{n_1,n}C_{n_2,n-n_1}\cdots C_{n_r,n-n_1-\cdots-n_{r-1}} = \frac{n!}{n_1!n_2!\cdots n_r!}.$$

combinations

Example 2.20 In how many ways can 7 scientists be assigned to one triple and two double hotel rooms?

combinations

Example 2.20 In how many ways can 7 scientists be assigned to one triple and two double hotel rooms?

Solution The total number of possible partitions would be

$$\frac{7!}{3!2!2!} = 210.$$

4. Probability of an Event

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In an effort to increase their winnings, gamblers called upon **mathematicians** to provide optimum strategies for various games of chance.

4. Probability of an Event

Perhaps it was man's unquenchable thirst for **gambling** that led to the early development of **probability theory**.

In an effort to increase their winnings, gamblers called upon **mathematicians** to provide optimum strategies for various games of chance.

Some of the mathematicians providing these strategies were Pascal, Leibniz, Fermat, and James Bernoulli.

Probability of an Event

Probability theory has branched out far beyond games of chance to encompass many other fields associated with chance occurrences, such as politics, business, weather forecasting, and scientific research.

Probability of an Event

Given an experiment and a sample space S, the objective of probability is to assign to each event A a number $P(A)$.

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$P(A)$ is the probability of the event A, which will give a precise measure of the chance that A will occur.

Probability

To ensure that the probability assignment will be consistent with our intuitive notions of probability, all assignments should satisfying the following axioms (basic properties) of probability.

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AXIOM 2 $P(S) = 1, P(\emptyset) = 0$.

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AXIOM 1 For any event A, $P(A) \geq 0$.
AXIOM 2 $P(S) = 1, P(\emptyset) = 0$.

AXIOM 3
a. If $A_1, A_2, \cdots A_k$ is a finite collection of mutually exclusive events, then $P(A_1 \cup A_2 \cup \cdots \cup A_k) = \sum_{i=1}^{k} P(A_i)$.
b. If $A_1, A_2, A_3, \cdots$ is an infinite collection of mutually exclusive events, then $P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$.

Probability

Example 2.22 A coin is tossed twice. What is the probability that at least one head occurs?

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The coin is balanced, each of these outcomes would be equally likely to occur. The probability of each sample point is $1/4$.

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The coin is balanced, each of these outcomes would be equally likely to occur. The probability of each sample point is $1/4$.

event A: at least one head occurring. $A = \{HH, HT, TH\}$

$$P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Probability

Example 2.23 A dice is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

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Solution

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$, $E = \{1, 2, 3\}$.

Probability

Example 2.23 A dice is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

Solution

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$, $E = \{1, 2, 3\}$.

assign a probability of ω to each odd number, and 2ω to each even number, we have $9\omega = 1$.

Probability

$1/9$, and $2/9$ are assigned to each odd and even number, respectively.

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Since $E = \{1, 2, 3\}$, we have

$$P(E) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}.$$

Probability

Example 2.24 In Example 2.23 let A be the event that an even number turns up and let B be the event that a number divisible by 3 occurs. Find $P(A \cup B)$ and $P(A \cap B)$

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Solution We can get $A = \{2, 4, 6\}$ and $B = \{3, 6\}$, therefore,

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$$A \cup B = \{2, 3, 4, 6\} \qquad \text{and} \qquad A \cap B = \{6\}.$$

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assign: $1/9 \rightarrow$ each odd number, $2/9 \rightarrow$ each even number, we have

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Solution We can get $A = \{2, 4, 6\}$ and $B = \{3, 6\}$, therefore,

$$A \cup B = \{2, 3, 4, 6\} \qquad \text{and} \qquad A \cap B = \{6\}.$$

assign: $1/9 \to$ each odd number, $2/9 \to$ each even number, we have

$$P(A \cup B) = \frac{2}{9} + \frac{1}{9} + \frac{2}{9} + \frac{2}{9} \qquad P(A \cap B) = \frac{2}{9}.$$

Probability

Theorem 2.9 If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}.$$

Probability

Example 2.25 A statistic class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major, (b) a civil engineering or an electrical engineering major.

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Solution

Denote by $I, M, E,$ and C the students majoring in industrial, mechanical, electrical and civil engineering, respectively.

Probability

Example 2.25 A statistic class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major, (b) a civil engineering or an electrical engineering major.

Solution

Denote by $I, M, E,$ and C the students majoring in industrial, mechanical, electrical and civil engineering, respectively.
students in the class: equally to be selected;
the total number: 53.

Probability

(a) Since 25 of the 53 students are majoring in industrial engineering, the probability of the event I is

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Probability

(a) Since 25 of the 53 students are majoring in industrial engineering, the probability of the event I is

$$P(I) = \frac{25}{53}.$$

(b) Since 18 of the 53 students are civil and electrical engineering majors, it follows that

$$P(C \cup E) = \frac{18}{53}.$$

Probability

Example 2.26 In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

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Solution

The number of ways of being dealt 2 aces from 4 is

$$C_{2,4} = \frac{4!}{2!2!} = 6.$$

Probability

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Solution

The number of ways of being dealt 2 aces from 4 is

$$C_{2,4} = \frac{4!}{2!2!} = 6.$$

The number of ways of being dealt 3 jacks from 4 is

$$C_{3,4} = \frac{4!}{3!1!} = 4.$$

Probability

Example 2.26 In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

Solution

The number of ways of being dealt 2 aces from 4 is

$$C_{2,4} = \frac{4!}{2!2!} = 6.$$

The number of ways of being dealt 3 jacks from 4 is

$$C_{3,4} = \frac{4!}{3!1!} = 4.$$

There are $n = 6 \times 4 = 24$ hands with 2 aces and 3 jacks.

Probability

The total number of 5-card poker hands is

$$N = C_{5,52} = \frac{52!}{5!47!} = 2,598,960.$$

Probability

The total number of 5-card poker hands is

$$N = C_{5,52} = \frac{52!}{5!47!} = 2,598,960.$$

Therefore, the probability of event C of getting 2 aces and 3 jacks in a 5-card poker hand is

$$P(C) = \frac{24}{2,598,960} = 0.9 \times 10^{-5}.$$

frequency and probability

If event A occurs n_A times in N repeated experiments under a certain conditions, then **frequency** of A occurring in N experiments is defined as:

$$F_N(A) = \frac{n_A}{N}.$$

frequency and probability

When N is large enough, the frequency turns out to have a kind of **stability**.

frequency and probability

When N is large enough, the frequency turns out to have a kind of **stability**.

i.e., the values of $F_N(A)$ show fluctuations which become progressively weaker as N increases, until ultimately $F_N(A)$ stabilizes to a constant.

frequency and probability

Tossing a fair coin, let $A = \{\text{head comes up}\}$.

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Experimenter	Number of tosses	Times of head	Frequency
Buffon	4040	2048	0.5069
Pearson	12000	6019	0.5016
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Buffon	4040	2048	0.5069
Pearson	12000	6019	0.5016
Pearson	24000	12012	0.5005

Frequency stabilizes to $1/2$.

frequency and probability

The statistical definition of probability:

frequency and probability

The statistical definition of probability:

The constant to which the frequency of the event A stabilizes is called the **probability** of the occurrence of event A ($P(A)$).