Chapter 4, Linear Transformations

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Outline

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Definition

A mapping L from a vector space V into a vector space W is said to be a **linear transformation** *if*

$$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$$

for all $v_1, v_2 \in V$ and for all scalars α and β.

- A mapping L from a vector space V into a vector space W will be denoted

$$L : V \to W$$

  When the arrow notation is used, it will be assumed that V and W represent vector spaces.
- In the case that the vector spaces V and W are the same, we will refer to a linear transformation $L : V \to V$ as a linear operator on V. Thus, a linear operator is a linear transformation that maps a vector space V into itself

Linear Operators on $\mathbb{R}^2$

Example

Let L be the operator defined by

$$L(x) = 3x$$

for each $x \in \mathbb{R}^2$.

Example

Consider the mapping L *defined by*

$$L(x) = x_1 e_1$$

for each $x = (x_1, x_2)^T \in \mathbb{R}^2$. *Here* $e_1 = (1, 0)^T$.

Example

Let L *be the operator defined by*

$$L(x) = (x_1, -x_2)^T$$

for each $x = (x_1, x_2)^T \in \mathbb{R}^2$.

Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Example

The mapping $L : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$L(x) = x_1 + x_2 \quad x \in \mathbb{R}^2,$$

is a linear transformation.

Example

Consider the map $L : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$M(x) = \sqrt{x_1^2 + x_2^2}.$$

Example

The mapping L from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by

$$L(x) = (x_2, x_1, x_1 + x_2)^T$$

is a linear transformation.

Linear Transformations from V to W

If L is a linear transformation mapping a vector space V into a vector space W, then

1. Let 0_v and 0_w be the zero vectors in V and W, respectively, then

$$L(0_v) = 0_w$$

2. If $v_1, \ldots, v_n$ are elements of V and $\alpha_1, \ldots, \alpha_n$ are scalars, then

$$L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 L(v_1) + \alpha_2 L(v_2) + \cdots + \alpha_n L(v_n).$$

3.

$$L(-v) = -L(v) \quad \text{for all} \quad v \in V.$$

Example

If V is any vector space, then the identity operator I is defined by

$$I(v) = v$$

for all $v \in V$.

Example

Let L be the mapping from $C[a, b]$ to $\mathbb{R}$ defined by

$$L(f) = \int_a^b f(t)dt$$

Example

Let D be the linear transformation mapping $C^1[a,b]$ into $C[a,b]$ and defined by

$$D(f) = f' \quad \text{the derivative of } f$$

D is a linear transformation,

The Image and Kernel

Definition

Let $L : V \to W$ be a linear transformation. The **kernel** *of L, denoted $Ker(L)$, is defined by*

$$Ker(L) = \{v \in V \mid L(v) = 0_w\}$$

Definition

Let $L : V \to W$ be a linear transformation and let S be a subspace of V. The image of S, denoted $L(S)$, is defined by

$$L(S) = \{w \in W \mid w = L(v) \quad \text{for some} \quad v \in S\}.$$

The image of the entire vector space, $L(V)$, is called the range of L.

Theorem

If $L : V \to W$ is a linear transformation and S is a subspace of V, then

1. *$Ker(L)$ is a subspace of V.*
2. *$L(S)$ is a subspace of W.*

Example

Let L be the linear operator on $\mathbb{R}^2$ defined by

$$L(x) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \quad x \in \mathbb{R}^2.$$

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$L(x) = (x_1 + x_2, x_2 + x_3)^T$$

and let S be the subspace of $\mathbb{R}^3$ spanned by e_1 and e_3.

Outline

Theorem

If L is a linear transformation mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, there is an $m \times n$ matrix A such that

$$L(x) = Ax$$

for each $x \in \mathbb{R}^n$. In fact, the jth column vector of A is given by

$$a_j = L(e_j) \quad j = 1, 2, \ldots, n.p$$

Example

Define the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$L(x) = (x_1 + x_2, x_2 + x_3)^T.$$

Theorem

Matrix Representation Theorem If $E = [v_1, v_2, \ldots, v_n]$ *and* $F = [w_1, w_2, , w_p m]$ *are ordered bases for vector spaces* V *and* W*, respectively, then corresponding to each linear transformation*

$$L : V \to W$$

there is an $m \times n$ *matrix* A *such that*

$$[L(v)]_F = A[v]_E \quad \text{for each} \quad v \in V.$$

A *is the matrix presenting* L *relative to the ordered bases* E *and* F*. In fact,*

$$a_j = [L(v_j)]_F \quad j = 1, 2, \ldots, n.$$

If A is the matrix representing L with respect to the bases E and F and

- $x = [v]_E$ (the coordinate vector of v with respect to E)
- $y = [w]_F$ (the coordinate vector of w with respect to F)

then L maps v into w if and only if A maps x into y.

$$\begin{tikzcd} v \in V \arrow[r, "L=L_A"] \arrow[d] & w = L(v) \in W \arrow[d] \\ x = [v]_E \in \mathbb{R}^n \arrow[r, "A"'] & Ax = [w]_F \in \mathbb{R}^m \end{tikzcd}$$

Example

Let L be a linear transformation mapping $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by

$$L(x) = x_1 b_1 + (x_2 + x_3) b_2 \quad \textit{for each} \quad x \in \mathbb{R}^3,$$

where

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad b_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Find the matrix A representing L with respect to the ordered bases $[e_1, e_2, e_3]$ and $[b_1, b_2]$.

Theorem

Let $E = [u_1, u_2, \ldots, u_n]$ *and* $F = [b_1, b_2, \ldots, b_m]$ *be ordered bases for* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively. If* $L : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear transformation and* A *is the matrix representing* L *with respect to* E *and* F*, then*

$$a_j = B^{-1}L(u_j) \quad \text{for} \quad j = 1, 2, \ldots, n,$$

where $B = (b_1, , b_m)$.

Corollary

If A *is the matrix representing the linear transformation* $L : \mathbb{R}^n \to \mathbb{R}^m$ *with respect to the bases* $E = [u_1, u_2, \ldots, u_n]$ *and* $F = [b_1, b_2, \ldots, b_m]$ *then the reduced row echelon form of* $(b_1, \ldots, b_m \mid L(u_1), , L(u_n))$ *is* $(I \mid A)$.

Example

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(x) = (x_2, x_1 + x_2, x_1 - x_2)^T.$$

Find the matrix representations of L with respect to the ordered bases $[u_1, u_2]$ and $[b_1, b_2, b_3]$, where

$$u_1 = (1, 2)^T, \quad u_2 = (3, 1)^T$$

and

$$b_1 = (1, 0, 0)^T, \quad b_2 = (1, 1, 0)^T, \quad b_3 = (1, 1, 1)^T.$$

Outline

If L is a linear operator on an n-dimensional vector space V, the matrix representation of L will depend on the ordered basis chosen for V. By using different bases, it is possible to represent L by different $n \times n$ matrices.

Example

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ *be the linear transformation defined by*

$$L(x) = (2x_1, x_1 + x_2)^T.$$

So the matrix representing L *with respect to* $[e_1, e_2]$ *is*

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

If there is another basis for $\mathbb{R}^2$*:*

$$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then

$$\begin{aligned} L(u_1) =& Au_1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ L(u_2) =& Au_2 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \end{aligned}$$

The transition matrix from $[u_1, u_2]$ to $[e_1, e_2]$ is

$$U = (u_1, u_2) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The transition matrix from $[e_1, e_2]$ to $[u_1, u_2]$ is

$$U^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Let matrix B representing L with respect to $[u_1, u_2]$,

$$b_1 = U^{-1}L(u_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad b_2 = U^{-1}L(u_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

Thus, if

1. B is the matrix representing L with respect to $[u1, u2]$
2. A is the matrix representing L with respect to $[e1, e2]$
3. U is the transition matrix corresponding to the change of basis from $[u1, u2]$ to $[e1, e2]$

then

$$B = U^{-1}AU.$$

Theorem

Let $E = [u_1, u_2, \ldots, u_n]$ and $F = [b_1, b_2, \ldots, b_m]$ be two ordered bases for a vector space V, and let L be a linear operator on V. Let S be the transition matrix representing the change from F to E. If A is the matrix representing L with respect to E, and B is the matrix representing L with respect to F, then

$$B = S^{-1} pAS.$$

Definition

Let A and B be $n \times n$ matrices. B is said to be **similar** *to A if there exists a nonsingular matrix S such that*

$$B = S^{-1}AS$$

.

Let L be the linear operator on $\mathbb{R}^2$ defined by $L(x) = Ax$, where

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

Thus the matrix A representing L with respect to $[e_1, e_2, e_3]$. Find the matrix representing L with respect to $[y_1, y_2, y_3]$, where

$$y_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad y_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Some Properties for Similar Matrices

1. Matrix A is similar to A itself.
2. If B is similar to A, then A is also similar to B.
3. If A is similar to B and B is similar to C, then A is similar to C.
4. If A and B are similar matrices, then $\det(A) = \det(B)$.
5. If A and B are similar matrices, then A^T is similar to B^T.
6. If A and B are similar matrices, then kA and kB are similar for each positive integer k.