3.Areas Under the Normal Curve

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x;\mu,\sigma)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-(x-\mu)^2/(2\sigma^2)}dx$$

is represented by the area of the shaded region for the normal curve in Figure 6.6.

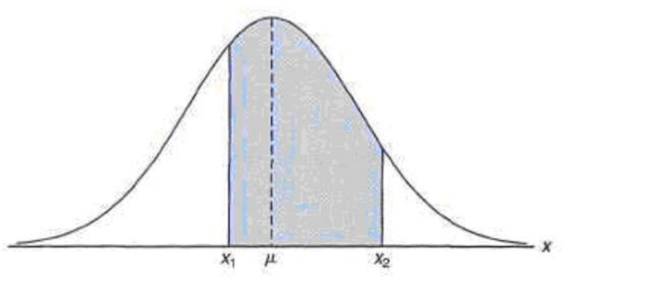

Figure 6.6: $P(x_1 < X < x_2)$ = area of the shaded region.

Areas Under the Normal Curve

The difficulty encountered in solving integrals of normal density functions necessitates the tabulation of normal curve areas for quick reference.

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However, it would be a hopeless task to attempt to set up separate tables for every conceivable value of μ and σ .

Areas Under the Normal Curve

Transform all the observations of any normal random variable X to a new set of observations of a normal random variable Z with mean 0 and variance 1.

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This can be done by the transformation

$$Z = \frac{X - \mu}{\sigma}.$$

Areas Under the Normal Curve

Therefore, if X falls between the values $x = x_1$ and $x = x_2$, the random variable Z will fall between the corresponding values $z_1 = (x_1 - \mu)/\sigma$ and $z_2 = (x_2 - \mu)/\sigma$.

Areas Under the Normal Curve

Therefore, if X falls between the values $x = x_1$ and $x = x_2$, the random variable Z will fall between the corresponding values $z_1 = (x_1 - \mu)/\sigma$ and $z_2 = (x_2 - \mu)/\sigma$. Consequently, we may write

$$P(x_1 < X < x_2) = P(\frac{x_1 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{x_2 - \mu}{\sigma})$$

$$= P(z_1 < Z < z_2)$$

Areas Under the Normal Curve

Definition 6.1

The distribution of a normal random variable with mean zero and variance 1 is called a **standard normal distribution.**

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Table A.3 indicates the area under the standard normal curve corresponding to $P(Z < z)$.

Areas Under the Normal Curve

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To find a z-value corresponding to a given probability, the process is reversed.For example, $P(Z < z) = 0.2148$, is seen to be -0.79.

Areas Under the Normal Curve

Example 6.2 Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $z = 1.84$,

(b) between $z = -1.97$ and $z = 0.86$.

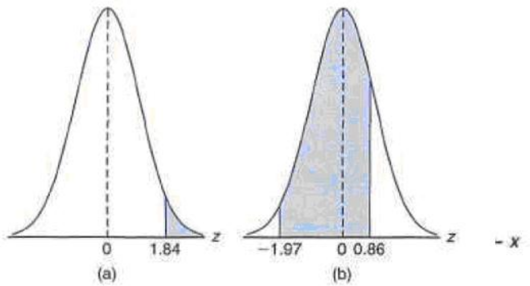


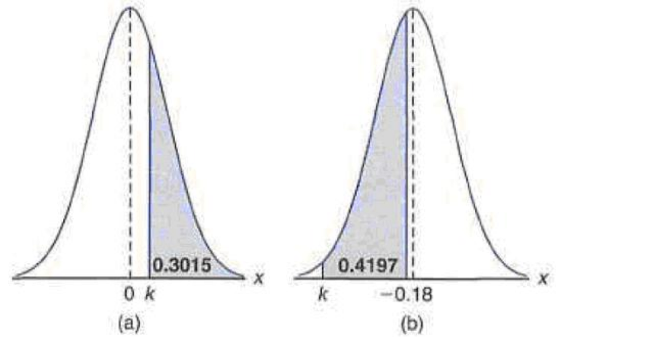
Figure 6.9: Areas for Example 6.2.

Areas Under the Normal Curve

Example 6.3 Given a standard normal distribution, find the value of k such that

(a) $P(Z > k) = 0.3015$,

(b) $P(k < Z < -0.18) = 0.4197$.



Areas Under the Normal Curve

Example 6.4 Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

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Solution

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$$= P(-0.5 < Z < 1.2)$$

$$= 0.8849 - 0.3085 = 0.5764$$

Areas Under the Normal Curve

Example 6.5 Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362 .

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Solution

$$P(X > 362) = P(\frac{X - 300}{50} > \frac{362 - 300}{50})$$

Areas Under the Normal Curve

Example 6.5 Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362 .

Solution

$$P(X > 362) = P(\frac{X - 300}{50} > \frac{362 - 300}{50})$$

$$= P(Z > 1.24) = 1 - P(Z < 1.24)$$

Areas Under the Normal Curve

Example 6.5 Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362 .

Solution

$$P(X > 362) = P(\frac{X-300}{50} > \frac{362-300}{50})$$

$$= P(Z > 1.24) = 1 - P(Z < 1.24)$$

$$= 1 - 0.8925 = 0.1075$$

Areas Under the Normal Curve

According to Chebyshev's theorem, the probability that a random variable assumes a value within 2 standard deviation of the mean is at least $3/4$. That is

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \geq 3/4.$$

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If the random variable has a **normal distribution**,

$$P(\mu - 2\sigma < X < \mu + 2\sigma)$$

$$= P\Big(\frac{(\mu - 2\sigma) - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{(\mu + 2\sigma) - \mu}{\sigma}\Big)$$

$$= P(-2 < Z < 2) = 0.9772 - 0.0228 = 0.9544$$

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If the random variable has a **normal distribution**,

$$\begin{aligned} &P(\mu - 2\sigma < X < \mu + 2\sigma) \\ &= P\Big(\frac{(\mu - 2\sigma) - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{(\mu + 2\sigma) - \mu}{\sigma}\Big) \\ &= P(-2 < Z < 2) = 0.9772 - 0.0228 = 0.9544 \end{aligned}$$

This is a much stronger statement than that stated by Chebyshev's theorem.

Areas Under the Normal Curve

Example 6.6

Given a normal distribution with $\mu = 40$ and $\sigma = 6$, find the value of x that has

(a) 45% of the area to the left,

(b) 14% of the area to the right.