Chapter 1, Matrices and Systems of Equations

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Outline

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Systems of Linear Equations

A *linear equation* in n unknowns is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A *linear system* of m equations in n unknowns is then a system of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{n2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Here the a_{ij} and b_j are all real numbers, x_i are variables. The systems is called an $m \times n$ **linear systems**.

Example

$$\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 + 3x_2 = 8 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 + x_2 - x_3 = 4 \end{cases}$$

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 1 \\ x_1 = 4. \end{cases}$$

They are 2×2 system, 2×3 system and 3×2 system.

Definition

A linear system is **inconsistent** *if it has no solution.*

Definition

A linear system is **consistent** *if it has a solution*

Example

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 2 \end{cases}$$

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 + x_2 = 1 \end{cases}$$

$$\begin{cases} x_1 + x_2 = 2 \\ -x_1 - x_2 = -2 \end{cases}$$

Equivalent Systems

Definition

Two systems of equations involving the same variables are said to be **equivalent** *if they have the same solution set.*

Three Operations that can be used on a system to obtain an equivalent system:

- The order in which any two equations are written may be interchanged.
- Both sides of an equation may be multiplied by the same nonzero real number.
- A multiple of one equation may be added to (or subtracted from) another.

Example

$$\begin{cases} 3x_1 + 2x_2 - x_3 = -2 \\ \qquad\qquad\quad x_2 = 3 \\ \qquad\qquad\; 2x_3 = 4 \end{cases}$$

$$\begin{cases} 3x_1 + 2x_2 - x_3 = -2 \\ -3x_1 - x_2 + x_3 = 5 \\ 3x_1 + 2x_2 + x_3 = 2 \end{cases}$$

Definition

A linear system is said to be in **strict triangular form** *if in the k–th equation the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero $(k = 1, \ldots, n)$.*

Example

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 1 \\ \qquad\quad x_2 - x_3 = 2 \\ \qquad\qquad\quad 2x_3 = 4 \end{cases}$$

is in strict triangular form

Example

$$\begin{cases} 2x_1 - x_2 + 3x_3 - 2x_4 = 1 \\ \qquad\quad x_2 - 2x_3 + 3x_4 = 2 \\ \qquad\qquad\quad 4x_3 + 3x_4 = 3 \\ \qquad\qquad\qquad\quad 4x_4 = 4. \end{cases}$$

Elementary Row Operations:

- Interchange two rows.
- Multiply a row by a nonzero real number.
- Replace a row by its sum with a multiple of another row.

Example

Solve the system

$$\begin{cases} -x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 = -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 = 3 \end{cases}$$

Answer: $(2, -1, 3, 2)$.

Example

Solve the system

$$\begin{cases} -x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 = -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 = 3 \end{cases}$$

Answer: $(2, -1, 3, 2)$.

Example

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

Since there are no 5-tuples that could satisfy these equations, the system is inconsistent.

Example

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

Since there are no 5-tuples that could satisfy these equations, the system is inconsistent.

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right)$$

It is equal to

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Definition

A matrix is said to be in **row echelon form**

1. *If the first nonzero entry in each nonzero row is* 1.
2. *If row* k *does not consist entirely of zeros, the number of leading zero entries in row* $k+1$ *is greater than the number of leading zero entries in row k.*
3. *If there are rows whose entries are all zero, they are below the rows having nonzero entries.*

Outline

Determine whether the following matrices are in row echelon form or not.

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Overdetermined Systems

Definition

The process of using operations 1,2,3 to transform a linear system into one whose augmented matrix is in row echelon form is called **Gaussian elimination.**

Definition

A linear system is said to be **overdetermined** *if there are more equations than unknowns. A system of m linear equations in n unknowns is said to be* **underdetermined** *if there are fewer equations than unknowns $(m < n)$.*

Solve each of the following overdetermined systems

Example

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 3 \\ -x_1 + 2x_2 = -2 \end{cases}$$

Example

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 - x_2 + x_3 = 2 \\ 4x_1 + 3x_2 + 3x_3 = 4 \\ 2x_1 - x_2 + 3x_3 = 5 \end{cases}$$

Solve each of the following underdetermined systems

Example

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 4x_2 + 2x_3 = 3 \end{cases}$$

Example

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2 \end{cases}$$

Definition

A system of linear equations is said to be **homogeneous** *if the constants on the right-hand side are all zero.*

Theorem

An $m \times n$ homogeneous system of linear equations has a nontrivial solution if $n > m$.

Definition

A matrix is said to be in **reduced row echelon form** *if:*

1. *The matrix is in row echelon form.*
2. *The first nonzero entry in each row is the only nonzero entry in its column.*

Example

Use Gauss-Jordan reduction to solve the system

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0 & 0 & 3\\0 & 1 & 0 & 2\\0 & 0 & 1 & 1\end{pmatrix} \quad \begin{pmatrix}0 & 1 & 2 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 0 & 1\\0 & 0 & 1 & 3\\0 & 0 & 0 & 0\end{pmatrix}$$

Example

$$\begin{cases} -x_1 + x_2 - x_3 + 3x_4 = 0 \\ 3x_1 + x_2 - x_3 - x_4 = 0 \\ 2x_1 - x_2 - 2x_3 - x_4 = 0 \end{cases}$$

Answer

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

Therefore, $(\alpha, -\alpha, \alpha, \alpha)$ are solutions of the system.

Example

$$\begin{cases} -x_1 + x_2 - x_3 + 3x_4 = 0 \\ 3x_1 + x_2 - x_3 - x_4 = 0 \\ 2x_1 - x_2 - 2x_3 - x_4 = 0 \end{cases}$$

Answer

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

Therefore, $(\alpha, -\alpha, \alpha, \alpha)$ are solutions of the system.

Outline

Matrix Notation

Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Vectors

- **Row vector** is a $1 \times n$ matrix

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

- **Column vector** is an $n \times 1$ matrix

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

If A is an $m \times n$ matrix, then the row vectors of A are given by

$$\vec{\mathbf{a}}_i = (a_{i1}, a_{i2}, \ldots, a_{in}), \quad i = 1, \ldots, m.$$

and the column vectors are given by

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

We have

$$A = \begin{pmatrix} \vec{\mathbf{a}}_1 \\ \vdots \\ \vec{\mathbf{a}}_m \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}.$$

Example

If

$$A = \begin{pmatrix} 3 & 2 & 5 \\ -1 & 8 & 4 \end{pmatrix}$$

then

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

and

$$\vec{\mathbf{a}}_2 = (-1, 8, 4).$$

Definition

Two $m \times n$ matrices A and B are said to be **equal** *if $a_{ij} = b_{ij}$ for each i and j.*

Scalar Multiplication: If A is a matrix and k is a scalar, then kA is the matrix formed by multiplying each of the entries of A by k.

Definition

If A is an $m \times n$ matrix and k is a scalar, then kA is the $m \times n$ matrix whose (i, j) entry is ka_{ij}.

Example

If

$$A = \begin{pmatrix} 4 & 8 & 2 \\ 6 & 8 & 10 \end{pmatrix}$$

then

$$\frac{1}{2}A \quad 3A.$$

Matrix Addition

Two matrices with the same dimensions can be added by adding their corresponding entries.

Definition

If $A = (a_{ij})$ *and* $B = (b_{ij})$ *are both mn matrices,then the sum* $A + B$ *is the* $m \times n$ *matrix whose* (i, j) *entry is* $a_{ij} + b_{ij}$ *for each ordered pair* (i, j)*.*

Example

Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 0 & 3 \\ 1 & 1 & -2 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then calculate $2A - 3I$.

Matrix Multiplication

Definition

If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix, then the product $AB = C = (c_{ij})$ is the $m \times r$ matrix whose entries are defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Example

If

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix},$$

then calculate AB.

Example

If

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

then calculate AB.

If O represents the matrix, with the same dimensions as A, where entries are all 0, then

$$A + O = O + A = A$$

We refer O as the zero matrix. Furthermore, each matrix A has an additive inverse. Indeed,

$$A + (-1)A = O = (-1)A + A.$$

Matrix Multiplication and Linear Systems

- case 1, One equation in Several unknowns.

  If we let $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, then we define the product AX by

$$AX = a_1x_1 + a_2x_2 + \cdots a_nx_n.$$

Example

If we set

$$A = (3, 2, 5), \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

then the equation

$$3x_1 + 2x_2 + 5x_3 = 4$$

can be written as the matrix equation

$$A\mathbf{x} = 4.$$

- Case 2, M equations in N Unknowns.

  If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, then we define the product AX by

$$AX = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A\mathbf{x} = ?$$

Example

$$A = \begin{pmatrix} -3 & 1 \\ 2 & 5 \\ 4 & 2 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$A\mathbf{x} = ?$$

Example

Write the following system of equations as a matrix equation of the form $A\mathbf{x} = \boldsymbol{b}$:

$$\begin{cases} -x_1 + x_2 - x_3 = 5 \\ 3x_1 + x_2 - x_3 = -2 \\ 2x_1 - x_2 - 2x_3 = 1 \end{cases}$$

Example

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 5 \\ 5x_1 - 4x_2 + 2x_3 = 6 \end{cases}$$

can be written as a matrix equation

$$x_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Example

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 5 \\ 5x_1 - 4x_2 + 2x_3 = 6 \end{cases}$$

can be written as a matrix equation

$$x_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Definition

If $a_1, a_2, \ldots, a_n$ are vectors in $\mathbb{R}^m$ and $c_1, c_2, \ldots, c_n$ are scalars, then a sum of the form

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n$$

is said to be a **linear combination** *of the vectors $a_1, a_2, \ldots, a_n$.*

Proposition

If A is an $m \times n$ matrix and $\mathbf{x}$ is a vector in $\mathbb{R}$, then

$$A\mathbf{x} = x_1 \mathbf{a}_1 + \cdots + x_n \mathbf{a}_n$$

Example

Since

$$x_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

If we choose $x_1 = 2$, $x_2 = 3$ and $x_3 = 4$. then

$$2 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ -4 \end{pmatrix} + 4 \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

It follows that the vector $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is a linear combination of the three column vectors of the coefficient matrix and the linear system is consistent and $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ is a solution of the system.

Theorem (Consistency Theorem for Linear Systems)

A linear system $A\mathbf{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ can be written as a linear combination of the column vectors of A.

Example

The linear system

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 1 \end{cases}$$

Matrix Multiplication

Definition

If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix, then the product $AB = C = (c_{ij})$ is the $m \times r$ matrix whose entries are defined by

$$c_{ij} = \vec{\mathbf{a}_i}\boldsymbol{b}_j = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Example

$$A = \begin{pmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{pmatrix}$$

then

$$AB = ? \quad BA = ?$$

Example

If

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{pmatrix}$$

then $BA =$ However, it is impossible to multiply A times B !

Example

If

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{pmatrix}$$

then $BA =$ However, it is impossible to multiply A times B !

Example

If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

then

$$AB = ? \quad BA = ?$$

Hence, $AB \neq BA$.

Multiplication of matrices is not commutative.

Example

If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

then

$$AB =? \quad BA =?$$

Hence, $AB \neq BA$.

Multiplication of matrices is not commutative.

Example

If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

then

$$AB = ? \quad BA = ?$$

Hence, $AB \neq BA$.

Multiplication of matrices is not commutative.

Example

If

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \quad C = \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}$$

then

$$A + BC =? \quad 3A + B =?$$

The Transpose of a Matrix

Definition

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix B defined by

$$b_{ji} = a_{ij}$$

for $j = 1, \ldots, n$ and $i = 1 \ldots, m$. The transpose of A is denoted by A^T.

Example

If

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 5 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

then

$$A^T = ? \quad B^T = ? \quad C^T = ?$$

Definition

An $n \times n$ matrix A is said to be **symmetric** *if $A^T = A$.*

Example

$$\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 5 \\ 4 & 5 & 1 \end{pmatrix}$$

Outline

Algebraic Rules

Theorem

Each of the following statements is valid for any scalars k and l and for any matrices A, B and C for which the indicated operations are defined.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B) + C = AC + BC$
6. $(kl)A = k(lA)$
7. $k(AB) = (kA)B = A(kB)$
8. $(k + l)A = kA + lA$
9. $k(A + B) = kA + kB$

Proof of Rule 4

Assume that A is an $m \times n$ matrix and B and C are both $n \times r$ matrices. Let $D = A(B + C)$ and $E = AB + AC$.

Example

If

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

verify that $A(BC) = (AB)C$ and $A(B+C) = AB + AC$.

Notation

If k is a positive integer, then

$$A^k = \underbrace{AA\cdots A}_{k}$$

Example

If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then

$$A^2 = ? \quad A^3 = ? \quad A^k = ?$$

Notation

If k is a positive integer, then

$$A^k = \underbrace{AA\cdots A}_{k}$$

Example

If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then

$$A^2 =? \quad A^3 =? \quad A^k =?$$

Notation

If k is a positive integer, then

$$A^k = \underbrace{AA\cdots A}_{k}$$

Example

If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then

$$A^2 =? \quad A^3 =? \quad A^k =?$$

The Identity Matrix

Definition

The $n \times n$ identity is the matrix $I = \delta_{i,j}$ where

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

The standard notation for the jth column vector of I is e_j. Thus, the $n \times n$ identity matrix can be written

$$I = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n).$$

Matrix Inversion

Definition

An $n \times n$ matrix A is said to be **nonsingular** *or* **invertible** *if there exists a matrix B such that $AB = BA = I$. Then matrix B is said to be a* **multiplicative inverse** *of A.*

Proposition

If B and C are both multiplicative inverse of A, then

$$B = C$$

Example

The matrices

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1/10 & 2/5 \\ 3/10 & -1/5 \end{pmatrix}$$

are inverses of each other.

Example

The matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

are inverses of each other.

Definition

An $n \times n$ matrix is said to be **singular** *if it does not have a multiplicative inverse.*

Remark

Only square matrices have multiplicative inverse. One should not use the terms singular and nonsigular when referring to nonsquare matrices.

Theorem

If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

Definition

The **transpose** *of an $m \times n$ matrix A is the $n \times m$ matrix B defined by*

$$b_{ji} = a_{ij}$$

for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. The transpose of A is denoted by A^T.

Algebra Rules for Transpose:

1. $(A^T)^T = A$.
2. $(kA)^T = kA^T$.
3. $(A+B)^T = A^T + B^T$.
4. $(AB)^T = B^T A^T$.

Proof of Rule 4

Note that

$$A = \begin{pmatrix} \vec{\mathbf{a}}_1 \\ \vdots \\ \vec{\mathbf{a}}_m \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}.$$

The (i, j) entry of $(AB)^T$ is the (j, i) entry of AB. It is

$$\vec{\mathbf{a}_j} \boldsymbol{b}_i$$

The (i, j) entry of $B^T A^T$ is given by

$$\boldsymbol{b}_i^T \vec{\mathbf{a}}_j^T.$$

Example

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix}$$

Verify $(AB)^T = B^T A^T$.

Homework 2

P43 6, 8.

P57 7, 12, 15, 29.

Outline

[Page 7] Elementary Row Operation

- Interchange two rows.
- Multiply a row by a nonzero real number
- Replace a row by it sum with a multiple of another row.

If we start with the identity matrix I and then perform exactly one elementary row operation, the resulting matrix is called an **elementary matrix**.

- Type I. An elementary matrix of type I is a matrix obtained by interchanging two rows of I.

Example

Let $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and A be 3×3 matrix then

$$E_1 A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$AE_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{pmatrix}$$

- Type II. An elementary matrix of type II is a matrix obtained by multiplying a row of I by a nonzero constant.

Example

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and A be 3×3 matrix. Then

$$E_2A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{pmatrix}$$

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- An elementary matrix of type III is a matrix obtained from I by adding a multiple of one row to another row.

Example

$E_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and A be 3×3 matrix. Then

$$\begin{aligned} E_3 A &= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + 3a_{31} & a_{12} + 3a_{32} & a_{13} + 3a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

$$AE_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 3a_{11} + a_{13} \\ a_{21} & a_{22} & 3a_{21} + a_{23} \\ a_{31} & a_{32} & 3a_{31} + a_{33} \end{pmatrix}$$

- In general, suppose that E is an $n \times n$ elementary matrix. E is obtained by either a row operation or a column operation.
- If A is an $n \times r$ matrix, **premultiplying** A by E has the effect of performing that same row operation on A.
- If B is an $m \times n$ matrix, **postmultiplying** B by E is equivalent to performing that same column operation on B.

Example

Let

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Evaluate E_iA and AE_i, $i = 1, 2$.

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Evaluate E_iA and AE_i, $i = 1, 2$.

Example

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_{31} & a_{32} + 3a_{33} & a_{33} \\ a_{21} & a_{22} + 3a_{23} & a_{23} \\ a_{11} & a_{12} + 3a_{13} & a_{13} \end{pmatrix}$$

Find the elementary matrices P_1 and P_2, such that

$$B = P_1 A P_2.$$

Solution

$$P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

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Theorem

If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

Example

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Definition

A matrix B is **row equivalent** *to A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that*

$$B = E_k E_{k-1} \cdots E_1 A$$

A method for finding the inverse of a matrix

If A is nonsingular, then A is row equivalent to I and hence there exist elementary matrices $E_1, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_1 A = I$$

Multiplying both sides of above equation on the right by A^{-1}, we have

$$E_k E_{k-1} \cdots E_1 I = A^{-1}$$

Thus

$$(A \mid I) \xrightarrow{\text{row operations}} (I \mid A^{-1}).$$

Example

Compute A^{-1} if

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$$

Example

Solve the system

$$\begin{cases} x_1 + 4x_2 + 3x_3 = 12 \\ -x_1 - 2x_2 = -12 \\ 2x_1 + 2x_2 + 3x_3 = 8 \end{cases}$$

Theorem (Equivalent Conditions for Nonsingularity)

Let A be an $n \times n$ matrix. The following are equivalent:

- *A is nonsingular.*
- *$Ax = 0$ has only the trivial solution 0.*
- *A is row equivalent to I.*

Theorem

The system of n linear equations in n unknowns $Ax = b$ has a unique solution if and only if A is nonsingular.

Example

Page 67, Problem 17, 18.

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Page 67, Problem 17, 18.

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Example

Page 67, Problem 17, 18.

Diagonal and Triangular Matrices

- An $n \times n$ matrix A is said to be **upper triangular** if $a_{ij} = 0$ for $i > j$ and **lower triangular** if $a_{ij} = 0$ for $i < j$.
- An $n \times n$ matrix A is said to be **triangular** if it is either upper triangular or lower triangular.
- An $n \times n$ matrix A is said to be **diagonal** if $a_{ij} = 0$ whenever $i \neq j$.

Example

$$\begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 4 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Triangular Factorization

Let

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix}$$

We have

$$E_3E_2E_1A = U = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

Triangular Factorization

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$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix}$$

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Let

$$L = E_1^{-1}E_2^{-2}E_3^{-3} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

We have

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix} = A$$

It is called the *LU factorization.*

Outline

Let

$$C = \begin{pmatrix} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{pmatrix}.$$

Then $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ where

$$C_{11} = \begin{pmatrix} 1 & -2 & 4 \\ 2 & 1 & 1 \end{pmatrix} \quad C_{12} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

$$C_{21} = \begin{pmatrix} 3 & 3 & 2 \\ 4 & 6 & 2 \end{pmatrix} \quad \text{and} \quad C_{22} = \begin{pmatrix} -1 & 2 \\ 2 & 4 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -1 & 2 & -1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{pmatrix}$$

Then

$$AB = A \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{pmatrix} = \begin{pmatrix} A\boldsymbol{b}_1 & A\boldsymbol{b}_2 & A\boldsymbol{b}_3 \end{pmatrix}$$

In general, if A is an $m \times n$ matrix and B is an $n \times r$ that has been partitioned into columns $\begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{pmatrix}$, then the block multiplication of A times B is given by

$$\begin{pmatrix} A\boldsymbol{b}_1 & A\boldsymbol{b}_2 & \dots & A\boldsymbol{b}_r \end{pmatrix}$$

If we partition A into rows, then $A = \begin{pmatrix} \vec{\boldsymbol{a}}_1 \\ \vec{\boldsymbol{a}}_2 \\ \vdots \\ \vec{\boldsymbol{a}}_m \end{pmatrix}$. Therefore, the product AB can be partitioned into rows as follows:

$$AB = \begin{pmatrix} \vec{\boldsymbol{a}}_1 B \\ \vec{\boldsymbol{a}}_2 B \\ \vdots \\ \vec{\boldsymbol{a}}_m B \end{pmatrix}$$

Block Multiplication

- Let A be an $m \times n$ matrix and B an $n \times r$ matrix.

Case 1 $B = (B_1 B_2)$, where B_1 is an $n \times t$ matrix and B_2 is an $n \times (r-t)$ matrix.

$$\begin{aligned} AB &= A\begin{pmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_t & \boldsymbol{b}_{t+1} & \cdots & \boldsymbol{b}_r \end{pmatrix} \\ &= \begin{pmatrix} A\boldsymbol{b}_1 & \cdots & A\boldsymbol{b}_t & A\boldsymbol{b}_{t+1} & \cdots & A\boldsymbol{b}_r \end{pmatrix} \\ &= \begin{pmatrix} AB_1 & AB_2 \end{pmatrix} \end{aligned}$$

Case 2 $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$,where A_1 is a $k \times n$ matrix and A_2 is an $(m-k) \times n$ matrix. Thus

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} B = \begin{pmatrix} A_1 B \\ A_2 B \end{pmatrix}$$

case 3 $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, where A_1 is an $m \times s$ matrix and A_2 is an $m \times (n-s)$ matrix, B_1 is an $s \times r$ matrix and B_2 is an $(n-s) \times r$ matrix. Thus

$$\begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = A_1 B_1 + A_2 B_2.$$

Case 4 Let A and B both be partitioned as follows

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where

$$A_{11} \in \mathbb{R}^{k\times s},\ A_{12} \in \mathbb{R}^{k\times(n-s)},\ A_{21} \in \mathbb{R}^{(n-k)\times s},\ \text{and } A_{22} \in \mathbb{R}^{(m-k)\times(n-s)}.$$

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where

$$B_{11} \in \mathbb{R}^{s\times t},\ B_{12} \in \mathbb{R}^{s\times(r-t)},\ B_{21} \in \mathbb{R}^{(n-s)\times t},\ \text{and } B_{22} \in \mathbb{R}^{(n-s)\times(r-t)}.$$

Then

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

In general, if the blocks have the proper dimensions, the block multiplication can be carried out in the same manner as ordinary matrix multiplication.

Example

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then calculate AB.

Example

Let A be an $n \times n$ matrix of the form

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

where A_{11} is a $k \times k$ matrix $(k < n)$. Show that A is nonsingular if and only if A_{11} and A_{22} are nonsingular.

Outer Product Expansions

This representation is referred to as an **outer product expansion**.

$$XY^T = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) \begin{pmatrix} \boldsymbol{y}_1^T \\ \boldsymbol{y}_2^T \\ \vdots \\ \boldsymbol{y}_n^T \end{pmatrix} = \boldsymbol{x}_1 \boldsymbol{y}_1^T + \cdots + \boldsymbol{x}_n \boldsymbol{y}_n^T$$

Example

Given

$$X = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix} \quad Y = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{pmatrix}$$

computer the outer product expansion of XY^T

$$XY^T = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 \end{pmatrix}$$

Homework 3

P66 6, 10(g)(h), 12(a), 17, 18.

P76 13 15.