Chapter 4. Mathematical Expectation

1. Mean of a Random Variable

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Suppose that the experiment yields:

no heads 4 times,

one head 7 times,

two heads 5 times.

Mean

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The numbers $4/16$, $7/16$, $5/16$ are the relative frequencies for the different values of X in our experiment.

Mean

Remark:

we can calculate the average number, by knowing the **distinct values** that occur and their **relative frequencies**, **without** any knowledge of the **total number** of observations.

Mean

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We shall refer to this average value as the **mean of the random variable** X and write it as μ_X.

It is also common to refer to this mean as the **mathematical expectation** or **expected value**, and denote it as $E(X)$.

Mean

Two fair coins were tossed, the sample space is

$$S = \{HH, HT, TH, TT\},$$

it follows that

$$\begin{aligned} P(X = 0) &= P(TT) = 1/4, \\ P(X = 1) &= P(TH) + P(HT) = 1/2, \\ P(X = 2) &= P(HH) = 1/4. \end{aligned}$$

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These probabilities are just the relative frequencies for the given events **in the long run**. Therefore,

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These probabilities are just the relative frequencies for the given events **in the long run**. Therefore,

$$\mu_X = E(X) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1.$$

Mean

the mean of any **discrete** random variable:

1. multiplying each of the values $x_1, x_2, \dots, x_n$ of r.v. X by its corresponding probability $p(x_1), p(x_2), \dots, p(x_n)$
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2. summing the products

in the case of **continuous** random variables: summations replaced by integrations

Mean

Definition 4.1

The **mean** or **expected value** of the random variable X is

$$\mu_X = E(X) = \sum_x x p(x)$$

if X is discrete, and

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

if X is continuous.

Mean

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The probability distribution of X is

$$p(x) = \frac{C_4^x C_3^{3-x}}{C_7^3}, \qquad x = 0, 1, 2, 3.$$

Mean

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$p(0) = 1/35, p(1) = 12/35, p(2) = 18/35, p(3) = 4/35.$

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$p(0) = 1/35, p(1) = 12/35, p(2) = 18/35, p(3) = 4/35.$

Therefore,

$$\mu_X = E(X) = 0 \times \frac{1}{35} + 1 \times \frac{12}{35} + 2 \times \frac{18}{35} + 3 \times \frac{4}{35} = \frac{12}{7}.$$

Mean

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Suppose Y is the amount that gambler can win; and the possible values of Y are \$5 and -\$3.

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Example 4.2 In a gambling game a man is paid \$5 if he gets all heads or all tails when three coins are tossed, and he will pay out \$3 if either one or two heads show. What is his expected gain?

Solution

Suppose Y is the amount that gambler can win; and the possible values of Y are \$5 and -\$3.

What are the corresponding probabilities?

$$P(Y = 5) =? \qquad P(Y = -3) =?$$

Mean

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$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

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Each of these possibilities is equally likely to occur ,with probability $1/8$. Therefore,

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$$P(Y = 5) = \frac{2}{8}, \qquad P(Y = -3) = \frac{6}{8}.$$

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The sample space for the possible outcomes when three coins are tossed simultaneously is
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Each of these possibilities is equally likely to occur ,with probability $1/8$. Therefore,

$$P(Y=5) = \frac{2}{8}, \qquad P(Y=-3) = \frac{6}{8}.$$

It follows that,

$$E(Y) = 5 \times \frac{2}{8} + (-3) \times \frac{6}{8} = -1.$$

Mean

Example 4.3 (continuous r.v.) Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \dfrac{20000}{x^3}, & x > 100 \\ 0, & \text{elsewhere}, \end{cases}$$

Find the expected life of this type of device.

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Solution

$$E(X) = \int_{100}^{\infty} x \frac{20000}{x^3} dx = \int_{100}^{\infty} \frac{20000}{x^2} dx = 200.$$

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Find the expected life of this type of device.

Solution

$$E(X) = \int_{100}^{\infty} x \frac{20000}{x^3} dx = \int_{100}^{\infty} \frac{20000}{x^2} dx = 200.$$

Therefore, we can expect this type of device to last 200 hours, on average.

Mean

Remark:

In example 4.3, an engineer is interested in the **mean life** of a certain type of electronic device.

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The expected values of the life of the device is an important parameter for its evaluation.

Mean

How to calculate the expect value of a new random variable $g(X)$, a function of X?

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Theorem 4.1 The mean or expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)p(x)$$

if X is discrete, and

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

if X is continuous.

Mean

Example 4.4 Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

4	5	6	7	8	9
$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular period.

Mean

4	5	6	7	8	9
$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Mean

4	5	6	7	8	9
$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Solution

By Theorem 4.1, the attendant can expect to receive

Mean

4	5	6	7	8	9
$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Solution

By Theorem 4.1, the attendant can expect to receive

$$E[g(X)] = E(2X-1) = \sum_{x=4}^{9}(2x-1)p(x)$$
$$= 7 \times \frac{1}{12} + 9 \times \frac{1}{12} + 11 \times \frac{1}{4} + 13 \times \frac{1}{4} + 15 \times \frac{1}{6} + 17 \times \frac{1}{6}$$
$$= \$12.67$$

Mean

Example 4.5 Let X be a random variables with density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere}, \end{cases}$$

Find the expect value of $g(X) = 4X + 3$.

Mean

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Solution......

$E(4X + 3) = 8$

Mean

extend the concept of mathematical expectation to the case of two random variables X and Y

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Definition 4.2 The mean or expected value of the random variable $g(X,Y)$ is

$$\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y)p(x,y)$$

if X and Y are discrete, and

$$\mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$

if X and Y are continuous.

Mean

Example 4.6 Let X and Y be random variables with joint probability distribution indicated in Table 3.1 of Example 3.8. Find the expected value of $g(X, Y) = XY$.

TABLE 3.1 Joint Probability Distribution for Example 3.8

$f(x, y)$		$x = 0$	$x = 1$	$x = 2$	Row totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$		$\frac{3}{7}$
	2	$\frac{1}{28}$			$\frac{1}{28}$
Column totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Mean

Example 4.7 Find $E(Y/X)$ for the density function

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0 < x < 2, 0 < y < 1 \\ 0, & \text{elsewhere,} \end{cases}$$

Solution......

Mean

Example 4.7 Find $E(Y/X)$ for the density function

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0 < x < 2, 0 < y < 1 \\ 0, & \text{elsewhere,} \end{cases}$$

Solution......

$$E(Y/X) = 5/8.$$

Mean

If $g(X, Y) = X$, we have

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If $g(X, Y) = X$, we have

$$E(X) = \sum_x \sum_y x p(x, y) = \sum_x x p_X(x) \qquad \text{(discrete case)}$$

$$E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) dx dy = \int_{-\infty}^{\infty} x f_X(x) dx$$

(continuous case)

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If $g(X, Y) = X$, we have

$$E(X) = \sum_x \sum_y x p(x, y) = \sum_x x p_X(x) \qquad \text{(discrete case)}$$

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(continuous case)

Therefore, in calculating $E(X)$ over a two-dimensional space, one may use either the joint probability distribution of X and Y or the marginal distribution of X.

Mean

Similarly, we have

$$E(Y) = \sum_x \sum_y yp(x, y) = \sum_y yp_Y(y) \qquad \text{(discrete case)}$$

$$E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x, y)dxdy = \int_{-\infty}^{\infty} yf_Y(y)dy$$

(continuous case)