Chapter 6

Analytical solutions to the 1-D Wave equation

6.1 1-D Wave equation

$$\begin{aligned} U_{tt} - c^2 U_{xx} &= 0 \\ \text{or } (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})U &= 0 \end{aligned} \tag{6.1}$$

This is a hyperbolic equation since $A = 1, C = -c^2, B = 0$ so that $AC < B^2$

6.2 d'Alembert's solution

We introduce a change of variables:
$\xi = x + ct$
$\eta = x - ct$
Then:

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = \frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}$$
$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \eta}\frac{\partial}{\partial t} = \frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}$$

So equation 6.1 becomes:

$$\begin{aligned} \Rightarrow -c\frac{\partial}{\partial \eta}(c\frac{\partial}{\partial \xi})U &= -c^2 U_{\xi\eta} = 0 \\ \Rightarrow U(\xi, \eta) &= g(\xi) + f(\eta) \\ &= g(x + ct) + f(x - ct) \end{aligned}$$

- $g(x + ct)$ defines waves that travel in left direction with speed c
- $f(x - ct)$ defines waves that travel in right direction with speed c.
- The pulses move without dispersion and the initial pulse breaks into a left and right pulse.

6.3 Separation of variables

This time we will derive the analytical solution using the separation of variables technique as we did for the 1-D heat equation in section 2.1. We want to solve the 1-D heat equation:

$$\begin{aligned} U_{tt} &= c^2 U_{xx} \qquad (6.2) \\ \text{with periodic boundary conditions } U(0,t) &= 0 = U(L,t) \end{aligned}$$

Again we assume $U(x,t) = X(x)T(t)$ then substitute into equation 6.2:

$$\begin{aligned} X(x)T''(t) &= c^2 X''(x)T(t) \\ &\text{then divide by } XT \quad \Rightarrow \\ \underbrace{\frac{T''}{T}}_{\text{function of } t \text{ only}} &= \underbrace{c^2\frac{X''}{X}}_{\text{function of } x \text{ only}} = -\omega^2 \text{ (constant)} \end{aligned}$$

Solving $X'' = -k^2 X$, where $k = \omega/c$ for $X(x)$ gives:

$$X = A\sin(kx) + B\cos(kx)$$

The boundary conditions give $X(0) = B = 0$ and $X(L) = A\sin kL = 0 \quad \Rightarrow$ $k = k_n = n\pi/L, \quad n = 0, 1, \ldots$. Thus the general solution for $X(x)$ is:

$$X(x) = \sum_n a_n \sin(\frac{n\pi x}{L})$$

Similarly if we solve $T'' = -\omega_n^2 T$ (where $\omega_n = ck_n$) we find the general solution:

$$T(t) = C\sin(\omega_n t) + D\cos(\omega_n t)$$

. So the solution for $U(x,t)$ is:

$$\begin{aligned} U(x,t) &= X(x)T(t) \\ &= \sum_n [a_n \sin(\omega_n t) + b_n \cos(\omega_n t)] \sin(k_n x) \end{aligned}$$

where a_n, b_n are given by initial conditions:

$$\begin{aligned} U(x,0) = U_0(x), \qquad & \frac{\partial U}{\partial t}(x,0) = V_0(x) \\ \Rightarrow U_0 = \sum_n b_n \sin(k_n x) \qquad & V_0 = \sum_n a_n \omega_n \sin(k_n x) \end{aligned}$$

using orthogonality of sine functions: $\int_0^L \sin(k_m x)\ \sin(k_n x) dx = \delta_{nm} \Rightarrow$

$$\begin{aligned} b_m &= \frac{2}{L}\int_0^L U_0(x)\sin(k_m x)dx \\ a_m &= \frac{2}{w_m L}\int_0^L V_0(x)\sin(k_m x)dx \end{aligned}$$

where $k_n = n\pi/L$ and $k_m = m\pi/L$.

Chapter 7

Flux conservative problems

7.1 Flux Conservative Equation

A large class of PDEs can be cast into the form of a flux conservative equation:

$$\frac{\partial \vec{U}}{\partial t} = \frac{-\partial f}{\partial x}(\vec{U}, \vec{U}_x, \vec{U}_{xx}, ...)$$

Example: flux conservative form for the wave equation

We consider the 1-D wave equation $U_{tt} = c^2 U_{xx}$. If we let:

$$\vec{w} = \begin{pmatrix} r \\ s \end{pmatrix}, \qquad \text{where } r = c\frac{\partial U}{\partial x}, \text{ and } s = \frac{\partial U}{\partial t}.$$

This means that:

$$\begin{aligned} \frac{\partial \vec{w}}{\partial t} &= \begin{pmatrix} \frac{\partial r}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} = \begin{pmatrix} c\frac{\partial s}{\partial x} \\ c\frac{\partial r}{\partial x} \end{pmatrix} \\ \text{or } \frac{\partial \vec{w}}{\partial t} &= -\frac{\partial}{\partial x}\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}\vec{w} = -\frac{\partial}{\partial x} f(\vec{w}) \end{aligned}$$

7.2 Stability analysis of numerical solutions of the first order flux conservative or 1-D advection equation

$$\frac{\partial U}{\partial t} = -c\frac{\partial U}{\partial x} \tag{7.1}$$