Chapter 7(1). Some Probability Limit Theorems

1. Law of Large Numbers

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n	m	m/n
2048	1061	0.518
4040	2048	0.5069
12000	6019	0.5016
24000	12012	0.5005
30000	14994	0.4998

Law of Large Numbers

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as $n \rightarrow \infty$.Actually, this is a special case of 'law of large numbers'.

Law of Large Numbers

The general case: Consider n repeated Bernoulli trials

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$i = 1, \ldots, n$.

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The frequency of success $\frac{\sum_{i=1}^{n} X_i}{n} \to p$, as $n \to \infty$.

Law of Large Numbers

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$$\lim_{n\to\infty} P\left(\left|\frac{\sum_{i=1}^{n} X_i}{n} - p\right| < \varepsilon\right) = 1$$

for any $\varepsilon > 0$.

Law of Large Numbers

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We will introduce **Chebyshev's law of large numbers**, a more general one. First, introduce another form of **Chebyshev's inequality.**

Theorem 1 Let X be a random variable with mean $E(X) = \mu$, variance $V(X) = \sigma^2$, then

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

for any $\varepsilon > 0$.

Law of Large Numbers

Proof

Suppose X is a continuous random variable, with density function $f(x)$, then

$$P(|X-\mu| \geq \varepsilon) = \int_{|x-\mu|\geq\varepsilon} f(x)dx \leq \int_{|x-\mu|\geq\varepsilon} \frac{(x-\mu)^2}{\varepsilon^2} f(x)dx$$
$$\leq \frac{1}{\varepsilon^2}\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \frac{\sigma^2}{\varepsilon^2}$$

Law of Large Numbers

Theorem 2

Suppose the sequence of random variables X_1, X_2, ..., X_n is **independent and identically distributed**, with $E(X_i) = \mu$, $V(X_i) = \sigma^2$ $(i = 1, 2, \ldots, n)$, then

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| < \varepsilon\right) = 1$$

for any $\varepsilon > 0$.

Law of Large Numbers

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According to the Chebyshev's inequality, we have

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2/n}{\varepsilon^2} \to 0,$$

as $n \to \infty$.

2.Central Limit Theorem

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Answer: Under some conditions, $\sum_{i=1}^{n} X_i$ will follow **the normal distribution** as $n \to \infty$.

Central Limit Theorem

We have

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \to N(0,1),$$

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This actually is the **central limit theorem**.

Central Limit Theorem

Theorem 3

Suppose that random variables X_1, X_2, ..., X_n is **independent and identically distributed**, with $E(X_i) = \mu$, $V(X_i) = \sigma^2 \neq 0$ $(i = 1, 2, \ldots, n)$, for any x, it has

$$\lim_{n\to\infty} P\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \leq x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \Phi(x)$$

where $\Phi(x)$ is the **standard normal distribution function.**

Central Limit Theorem

Special case

X_1, X_2, ..., X_n is independent and identically distributed, with probability mass function

0	1
$1-p$	p

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$$\begin{array}{cc} 0 & 1 \\ 1-p & p \end{array}$$

then $\sum_{i=1}^n X_i \sim b(x; n, p)$,

Central Limit Theorem

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X_1, X_2, ..., X_n is independent and identically distributed, with probability mass function

0	1
$1-p$	p

then $\sum_{i=1}^{n} X_i \sim b(x; n, p)$,by using the **central limit theorem**, it has

$$\lim_{n \to n} P\left(\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{np(1-p)}} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

Central Limit Theorem

Example

X_1, X_2, ..., X_n is independent and identically distributed, X_i is Poisson distributed with parameter $\mu = 2$. Calculate $P(190 < \sum_{i=1}^{100} X_i < 210)$.

Central Limit Theorem

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X_1, X_2, ..., X_n is independent and identically distributed, X_i is Poisson distributed with parameter $\mu = 2$. Calculate $P(190 < \sum_{i=1}^{100} X_i < 210)$.

It is hard to find the precise distribution for $\sum_{i=1}^{100} X_i$, so we use central limit theorem.

Central Limit Theorem

general case:

$$P(a_1 < \sum_{i=1}^{n} X_i < a_2) = P(\frac{a_1 - n\mu}{\sqrt{n\sigma^2}} < \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} < \frac{a_2 - n\mu}{\sqrt{n\sigma^2}})$$

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$$\approx \Phi(\frac{a_2 - n\mu}{\sqrt{n\sigma^2}}) - \Phi(\frac{a_1 - n\mu}{\sqrt{n\sigma^2}})$$

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Solution Since $E(X_i) = 2$, $V(X_i) = 2$, $i = 1, 2, \ldots, 100$

$$P(190 < \textstyle\sum_{i=1}^{100} X_i < 210)$$

$$= P(\frac{190-200}{\sqrt{200}} < \frac{\sum_{i=1}^{n} X_i - 200}{\sqrt{200}} < \frac{210-200}{\sqrt{200}})$$

$$\approx \Phi(\frac{210-200}{\sqrt{200}}) - \Phi(\frac{190-200}{\sqrt{200}})$$