10.1.1 Example 1: Comparing the accuracy of solutions of a variable speed wave equation with either the spectral or finite difference method

Spectral method for variable speed wave equation

In this example we will compare the accuracy of either the spectral or finite difference method when solving the 1D advection equation with a variable wave speed, $c(x) = \frac{1}{5} + \sin^2(x-1)$. First we will derive the solution using the spectral method:

$$
\begin{aligned}
U_t + c(x)U(x) &= 0, \quad 0 \le x \le 2\pi \text{ and } 0 \le t \le 9 \\
U(x,0) &= \exp(-100(x-1)^2) \\
c(x) &= \frac{1}{5} + \sin^2(x-1) \\
U(0,t) &= U(2\pi,t) \quad \text{periodic boundary condition}
\end{aligned}
$$

The matlab code is **spectral_variable_wave_speed.m**

Again we discretise in space and time: $\Delta x = \frac{2\pi}{2n} = \frac{\pi}{n}$, $\Delta t = \frac{T}{m}$
$x_j = j\Delta x,\ j = 0,1,2,\ldots,2n-1$
$t_k = k\Delta t,\ k = 0,1,2,\ldots,m$
$U(x_j,t_k) = U_j^k$

The spectral method uses the discrete Fourier transform of $U(x_j,t)$:

$$
\begin{aligned}
\hat{U}_\nu &= F(U), \\
&= \sum_{j=0}^{2n-1} U(x_j,t)\exp(-ix_j\nu) \\
&= \sum_{j=0}^{2n-1} U(x_j,t)\exp(-i2\pi j\nu/(2n)), \text{ using } x_j = j\Delta x = j2\pi/2n \\
&= \sum_{j=0}^{2n-1} U(x_j,t)\exp(-i\pi j\nu/n)
\end{aligned}
$$

for $\nu = -n+1,\ldots,n$.

$U(x_j,t)$ is then defined as the inverse discrete Fourier transform of $\hat{U}_\nu$:

$$
\begin{aligned}
U(x_j,t) &= U_j = F^{-1}(\hat{U}), \\
&= \frac{1}{2n}\sum_{\nu=-n+1}^{n} \hat{U}_\nu \exp(ix_j\nu) \\
&= \frac{1}{2n}\sum_{\nu=-n+1}^{n} \hat{U}_\nu \exp(i2\pi j\nu/(2n))
\end{aligned}
$$

where $j = 0, \ldots, 2n-1$.

With this definition the spatial derivatives are:

$$\begin{aligned} \frac{\partial U(x_j,t)}{\partial x} &= \frac{1}{2n}\sum_{\nu=-n+1}^{n} i\nu \hat{U}_\nu \exp(ix_j\nu) \\ &= F^{-1}(i\nu\hat{U}) \\ &= F^{-1}(i\nu F(U)) \end{aligned}$$

We solve the advective equation with variable wave speeds and compare the solution with either FD or spectral method:

$$\begin{aligned} U_t + c(x)U_x &= 0 \\ \text{where } c(x) &= \frac{1}{5} + \sin^2(x-1) \\ \text{ic: } U(x,0) &= \exp(-100(x-1)^2) \\ \text{periodic bc: } U(0,t) &= U(\pi,t) \end{aligned}$$

The central difference approximation is used for U_t and spectral method for U_x:

$$\underbrace{\frac{U_j^{k+1} - U_j^{k-1}}{2\Delta t}}_{\text{leap-frog for } U_t} + c(x_j) \underbrace{F^{-1}(i\nu F(U_j^k))}_{\text{spectral method for } U_x} = 0$$

$$\text{or } U_j^{k+1} = U_j^{k-1} + 2\Delta t c(x_j) F^{-1}(i\nu F(U_j^k))$$

For U_j^1 need U_j^0 and U_j^{-1}

$$U(x,0) = U_j^0 = \exp(-100(x-1)^2)$$

since $c(x) \approx \frac{1}{5}$ we can assume a constant wave speed of $\approx 1/5$ to calculate U_j^{-1} at $t = -\Delta t$.

$$U_j^{-1} = U(x,-\Delta t) = U(x + c(x)\Delta t) \approx U^0(x + \frac{1}{5}\Delta t) = \exp(-100(x + \frac{\Delta t}{5} - 1)^2)$$

The matlab code is **spectral_variable_wave_speed.m**.

Comparing accuracy of solution with spectral method vs. finite difference method

Solve again using finite difference.

$$U_t + c(x)U_x = 0$$

This matlab code is **fd_variable_wave_speed.m**.

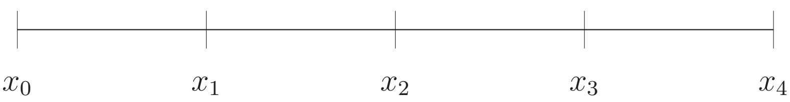
The Lax method is used for U_t and central difference method for U_x:

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{U_j^{k+1} - \frac{1}{2}(U_{j-1}^k + U_{j+1}^k)}{\Delta t} \\
\frac{\partial U}{\partial x} &= \frac{U_{j+1}^k - U_{j-1}^k}{2\Delta x} \\
c(x_j) &= c_j.
\end{aligned}$$

Plug the formulas into the PDE:

$$\begin{aligned}
\underbrace{\frac{U_j^{k+1} - \frac{1}{2}(U_{j-1}^k + U_{j+1}^k)}{\Delta t}}_{U_t} + c(x_j)\underbrace{\frac{U_{j+1}^k - U_{j-1}^k}{2\Delta x}}_{U_x} &= 0 \\
\text{or } U_j^{k+1} &= \frac{1}{2}(1 + sc_j)U_{j-1}^k + \frac{1}{2}(1 - sc_j)U_{j+1}^k
\end{aligned}$$

where $s = \Delta t/\Delta x$. Using 4 elements:



$U_0^k = 0 = U_4^k$ given by boundary conditions
$U_j^0 = U(x, 0)$ given by initial conditions

$$\vec{U}^{k+1} = \begin{pmatrix} U_1^{k+1} \\ U_2^{k+1} \\ U_3^{k+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 - sc_1 & 0 \\ 1 + sc_2 & 0 & 1 - sc_2 \\ 0 & 1 + sc_3 & 0 \end{pmatrix}\begin{pmatrix} U_1^k \\ U_2^k \\ U_3^k \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (1 + sc_1)U_0^k \\ 0 \\ (1 - sc_3)U_4^k \end{pmatrix}$$

or $\vec{U}^{k+1} = A\vec{U}^k + \vec{b}$

Figure 10.1(b) shows that the solution using finite differences is *much* worse than the spectral method because dispersion is introduced in FD method when a variable wave speed is applied. However figure 10.1(a) shows the spectral method performs well when smooth initial conditions of a Gaussian pulse are used and there is very little dispersion present.

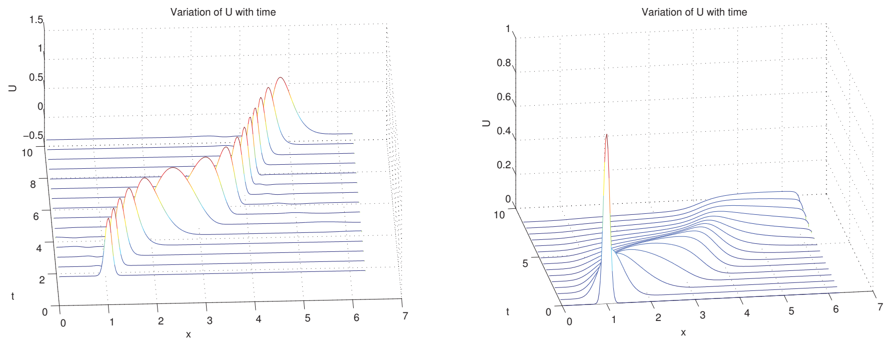


Figure 10.1: Numerical solution for 1D advection equation with initial conditions of a smooth Gaussian pulse with variable wave speed using the spectral method in (a) and finite difference method in (b)

10.1.2 Example 2 Comparing spectral and finite difference methods with constant wave speed conditions and initial conditions of a non-smooth pulse

We solve the advective equation with constant wave speed ($c = 1$) with initial conditions of a box pulse and compare the solution with either FD or spectral method:

$$\begin{aligned} U_t + U_x &= 0 \\ \text{ic: } U(x,0) &= \begin{cases} 1, & 0.5 \le x \le 1. \\ 0, & \text{otherwise} \end{cases} \\ \text{periodic bc: } U(0,t) &= U(\pi,t) \end{aligned}$$

- The matlab code for the *spectral* solution is **spectral_c_1_box.m**.
- The matlab code for the *FD* solution is **fd_c_1_box.m**. In this code we have chosen the time step carefully so that no dispersion is present for a constant wave speed of $c = 1$. Please see section 7.7.2 for a discussion on dispersion in finite difference methods.

Figure 10.2(a) shows that the solution with an initial condition which is not smooth like the box pulse we used here using the spectral method is *much* worse than the finite difference method. This is because the spectral method uses Fourier series to approximate the initial conditions and is unable to

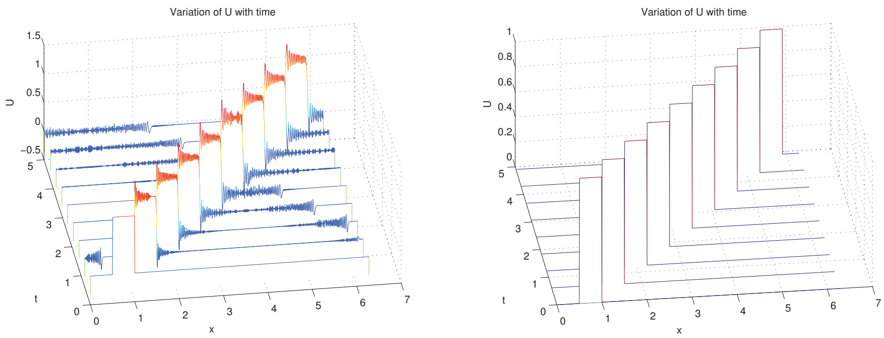


Figure 10.2: Numerical solution for 1D advection equation with initial conditions of a box pulse with a constant wave speed using the spectral method in (a) and finite difference method in (b)

approximate non-smooth initial conditions accurately. It is important to note that for the numerical solution using the FD method in figure 10.2(b) that we were able to remove dispersion in this example by carefully choosing $\Delta t = \Delta x/c$ (see section 7.7.2).