Chapter 2, Determinants

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Outline

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With each square matrix it is possible to associate a real number called the determinant of the matrix. The value of this number will tell us whether the matrix is singular.

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case 1 : 1×1 Matrices

If $A = (a)$ is a 1×1 matrix, then A will have a multiplicative inverse if and only if $a \neq 0$. Thus, if we define

$$\det(A) = a$$

Then A will be nonsingular if and only if $\det(A) \neq 0$.

case 2 : 2×2 Matrices

- Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Notation

If

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

then

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

represents the determinant of A

case 3 : 3×3 Matrices

If

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Example

Let

$$A = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$$

then calculate $\det(A)$.

Minor, Cofactor

Definition

Let $A = (a_{ij})$ be an $n \times n$ matrix and let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of M_{ij} is called the **minor** *of a_{ij}. We define the* **cofactor** *A_{ij} of a_{ij} by*

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

Example

If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then

$$M_{11} = (a_{22}), \quad M_{12} = (a_{21}).$$

If

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then

$$M_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \quad M_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$A_{13} = (-1)^{1+3} \det(M_{13}) = a_{21}a_{32} - a_{22}a_{31}.$$

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Examples

Since

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

we have

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

and

$$\det(A) = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

Example

If

$$A = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$$

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Examples

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$$A = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$$

then $\det(A)$

Definition

The determinant of an $n \times n$ matrix A, denoted $\det(A)$, is a scalar associated with the matrix A that is defined inductively as follows:

$$\det(A) = \begin{cases} a_{11} & \textit{if} \quad n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \textit{if} \quad n \neq 1 \end{cases}$$

where

$$A_{1j} = (-1)^{1+j}\det(M_{1j}) \quad j = 1, \cdots, n$$

are the cofactors associated with the entries in the first row of A.

Theorem

If A is an $n \times n$ matrix with $n \neq 2$, then $\det(A)$ can be expressed as a cofactor expansion using any row or column of A.

$$\begin{aligned} \det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \end{aligned}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

Example

Evaluate

$$\det(A) = \begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix}.$$

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for $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

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Theorem

If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.

Theorem

If A is an $n \times n$ triangular matrix, the determinant of A equals the product of the diagonal elements of A.

Theorem

Let A be an $n \times n$ matrix.

1. *If A has a row or column consisting entirely of zeros, then $\det(A) = 0$.*
2. *If A has two identical rows or two identical columns, then $\det(A) = 0$.*

Outline

Lemma

Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, \ldots, n$, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots a_{in}A_{jn} = \begin{cases} \det(A) & i = j \\ 0 & i \neq j \end{cases}$$

Effects of row operation on the the value of a determinant

- Row Operation I: *Two rows are interchanged*

  Suppose that E is an elementary matrix of type I, then

$$\det(EA) = -\det(A) = \det(E)\det(A)$$

- Row Operation II: *A row of A is multiplied by a nonzero constant*

  Let E denote the elementary matrix of type II formed from I by multiplying the i–th row by the nonzero constant.

$$\det(EA) = \alpha \det(A) = \det(E) \det(A).$$

- Row Operation III: *A multiple of one row is added to another row.*

  Let E be the elementary matrix of type III formed from I by adding c times the ith row to the jth row.

$$\det(EA) = \det(A) = \det(E)\det(A)$$

- Interchanging two rows (or columns) of a matrix changes the sign of the determinant.
- Multiplying a single row or column of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar.
- Adding a multiple of one row (or column) to another does not change the value of the determinant.

Theorem

An $n \times n$ matrix A is singular if and only if

$$\det(A) = 0.$$

Proof.

The matrix A can be reduced to row *echelon form* with a finite number of row operation. Thus

$$U = E_k E_{k-1} \cdots E_1 A$$

where U is in row echelon and the E_i's are all elementary matrices. □

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Example

Evaluate

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix}.$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix} = 60$$

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Theorem

If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

Outline

Definition (The Adjoint of a Matrix)

Let A be an $n \times n$ matrix. We define a new matrix called the **adjoint** *of A by*

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

Since

$$A(\operatorname{adj} A) = \det(A)I,$$

we have

$$A\left(\frac{1}{\det(A)}\operatorname{adj} A\right) = I.$$

Then we can see that

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj} A \quad \text{when} \quad \det(A) \neq 0.$$

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Example

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Compute $\operatorname{adj} A$ *and* A^{-1}.

Theorem (Cramer's Rule)

Let A be an $n \times n$ nonsingular matrix, and let $b \in \mathbb{R}^n$. Let A_i be the matrix obtained by replacing the i–th column of A by b. If x is the unique solution to $Ax = b$, then

$$x_i = \frac{\det(A_i)}{\det(A)} \quad \textit{for} \quad i = 1, 2, \cdots, n$$

Example

Use Cramer's rule to solve

$$\begin{cases} x_1 + 2x_2 + x_3 & = 5 \\ 2x_1 + 2x_2 + x_3 & = 6 \\ x_1 + 2x_2 + 3x_3 & = 9 \end{cases}$$

Homework

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