Chapter 12

Korteweg-de Vries Equation

12.1 Solitons

- solitons or solitary waves result from solution of the KdV equation
- KdV equation is a model for shallow water waves:

$$U_t + UU_x + U_{xxx} = 0 \quad \text{nonlinear PDE}$$

- analytical solutions exist
- solitons move in isolation and propagate without changing form. Velocity is amplitude dependent (linearly proportional to maximum amplitude).
- the nonlinear term causes waves to steepen (UU_x
- the dispersive term causes waves to disperse (U_{xxx})
- these effects are in exact balance for solitons → waveform maintains its size, shape and speed as it travels.
- solitons pass through each other without change of form except shifted.

12.2 Analytical solution

$$U_t + UU_x + U_{xxx} = 0$$

Let $U = f(\xi) = f(x - Vt)$ where $\xi = x - Vt$ then:

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} = -V\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi}$$

Let $f'(\xi) = \frac{df}{\partial \xi}$ then $U_t + UU_x + U_{xxx} = 0$ becomes: (using $U = f(\xi) = f(x - Vt)$)

$$-Vf' + ff' + f''' = 0$$

We integrate once: (use $ff' = \frac{d}{d\xi}(\frac{f^2}{2})$)

$$-V\int f'd\xi + \int \frac{d}{d\xi}(\frac{f^2}{2})d\xi + \int f'''d\xi = 0$$

$$\Rightarrow -Vf + \frac{f^2}{2} + f'' = C \tag{12.1}$$

Multiply by f' and integrate again:

$$-\int Vff'd\xi + \int f'\frac{f^2}{2}d\xi + \int f'f''d\xi = \int Cf'd\xi + c_0$$

$$\text{Term 1 } = \int Vff'd\xi = Vf^2 - \int Vff'd\xi \Rightarrow \int Vff'd\xi = \frac{V}{2}f^2$$

$$\text{Term 2 } = \int f'\frac{f^2}{2}d\xi = \frac{f^3}{2} - \int f^2f'd\xi \Rightarrow \int f'\frac{f^2}{2}d\xi = \frac{f^3}{6}$$

$$\text{Term 3 } = \int f'f''d\xi = (f')^2 - \int f''f'd\xi \Rightarrow \int f'f''d\xi = \frac{f'^2}{2}$$

So multiplying 12.1 by f' and integrating again gives:

$$\frac{-V}{2}f^2 + \frac{f^3}{6} + \frac{f'^2}{2} = Cf + C_0 \tag{12.2}$$

We assume boundary conditions $f(\xi) \to 0,\ f'(\xi) \to 0$ as $\xi \to \pm\infty$.

So Equation 12.2$\Rightarrow C_0 = 0$ and Equation 12.1$\Rightarrow C = 0$.

We assume initial conditions for soliton:

$$U(x, 0) = f(x) = A\text{sech}^2(\sqrt{\frac{A}{12}}x)$$

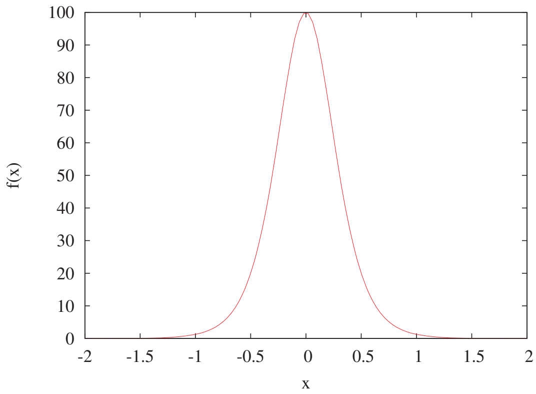

Figure 12.1: The initial conditions $U(x,0) = f(x) = A\text{sech}^2(\sqrt{\frac{A}{12}}x)$.

So $f(0) = A$ and $f'(0) = 0$ and $f'(\xi) \leq 0$ for $\xi \geq 0$.
Rearranging Equation 12.2 gives

$$f'^2 = Vf^2 - \frac{f^3}{3} \tag{12.3}$$

Use

$$\begin{aligned} f(0) = A, \quad f'(0) = 0 \qquad &\Rightarrow \quad 0 = A^2(V - \frac{A}{3}) \\ &\Rightarrow \quad V = \frac{A}{3} \end{aligned}$$

Since $f'(\xi) \leq 0$ for $\xi \geq 0$ we take the negative square root of 12.3:

$$f' = \frac{-f}{\sqrt{3}}\sqrt{A-f}$$

This can be integrated analytically by making a change of variable if we let $f = A\text{sech}^2\theta$ then $df = -2A\text{sech}^2\theta \tanh\theta d\theta$ and integrating:

$$f' = \frac{-f}{\sqrt{3}}\sqrt{A-f}$$

Since we assumed $\xi \geq 0$ we integrate from $\xi = 0$ in integration limits:

$$\begin{aligned} \Rightarrow \int_0^\xi \frac{\frac{df}{d\xi}d\xi}{f\sqrt{A-f}} &= -\frac{1}{\sqrt{3}}\int_0^\xi d\xi \\ \Rightarrow -\frac{\xi}{\sqrt{3}} = \int_{f(0)}^{f(\xi)} \frac{df}{f\sqrt{A-f}} &= \int_A^f \frac{df}{f\sqrt{A-f}} \end{aligned}$$

now substitute $f = A\text{sech}^2\theta$

$$
\begin{aligned}
&= \int_0^\theta \frac{-2A\text{sech}^2\theta \tanh\theta d\theta}{A\text{sech}^2\theta\sqrt{A - A\text{sech}^2\theta}} \\
&= \int_0^\theta \frac{-2\tanh\theta d\theta}{\sqrt{A}\sqrt{1-\text{sech}^2\theta}}
\end{aligned}
$$

now use $1 - \text{sech}^2\theta = \tanh^2\theta$

$$
\begin{aligned}
\Rightarrow \int_0^\theta \frac{-2}{\sqrt{A}} d\theta &= \frac{-2\theta}{\sqrt{A}} = -\frac{\xi}{\sqrt{3}} \\
\Rightarrow \theta &= \sqrt{\frac{A}{12}}\xi \\
U(x,t) &= f(\xi) = A\text{sech}^2\theta \\
&= A\text{sech}^2\left(\sqrt{\frac{A}{12}}\xi\right) \\
\Rightarrow U(x,t) &= \underbrace{A\text{sech}^2\left(\sqrt{\frac{A}{12}}(x - \frac{A}{3}t)\right)}_{\text{soliton travelling to right}} \quad (\text{using } V = \frac{A}{3})
\end{aligned}
$$

where $\text{sech}x = \frac{1}{\cosh x}$ and $\cosh x = \frac{1}{2}(e^{-x} + e^{x})$.

12.3 Numerical solution of KdV Equation

$$
U_t + UU_x + U_{xxx} = 0, \quad 0 \leq x \leq 2\pi
$$

with periodic boundary conditions $U(0) = U(2\pi)$.
and initial conditions:

$$
U(x,0) = A\text{sech}^2(\sqrt{\frac{A}{12}}(x - \pi)), \quad A = 100
$$

The analytical solution is:

$$
U(x,t) = A\text{sech}^2\left(\sqrt{\frac{A}{12}}(x - \pi - \frac{A}{3}t)\right)
$$

If we apply the spectral method directly we find that the linear term, U_{xxx}, involves high frequencies making the numerical solution unstable as we will see in section 12.3.1. Section 12.3.2 shows how to modify this term to gain stability using a modified spectral method.

12.3.1 Solving directly with Spectral Method

$$U_t + UU_x + U_{xxx} = 0$$

The matlab code is **SpectralDirectSoliton.m**.
Take discrete Fourier transform of U:

$$\hat{U}_\nu = F(U) = \sum_{j=0}^{2n-1} U(x_j, t)\exp(-ix_j\nu), \quad \text{for } \nu = -n+1, \dots, n$$

and the inverse discrete Fourier transform of $\hat{U}$:

$$U_j = F^{-1}(\hat{U}) = \frac{1}{2n}\sum_{\nu=-n+1}^{n} \hat{U}_\nu \exp(ix_j\nu), \quad \text{for } j = 0, \dots, 2n-1$$

We calculate spatial derviatives using spectral method:

$$\begin{aligned} U_x &= \frac{\partial U_j^k}{\partial x} = \frac{1}{2n}\sum_{\nu=-n+1}^{n} \hat{U}_\nu(i\nu)\exp(ix_j\nu) \\ &= F^{-1}(i\nu\hat{U}) = F^{-1}(i\nu F(U)) \end{aligned}$$

and:

$$\begin{aligned} U_{xxx} &= \frac{\partial^3 U_j^k}{\partial x^3} = \frac{1}{2n}\sum_{\nu=-n+1}^{n} \hat{U}_\nu(-i\nu^3)\exp(ix_j\nu) \\ &= F^{-1}(-i\nu^3\hat{U}) = F^{-1}(-i\nu^3 F(U)) \end{aligned}$$

At high wavenumbers, ν, this term causes instabilities in solution.
We use a leap-frog approximation for U_t:

$$\frac{\partial U_j^k}{\partial t} = \frac{U_j^{k+1} - U_j^{k-1}}{2\Delta t}$$

Plug approximations into PDE: $U_t + UU_x + U_{xxx} = 0$,

$$U_j^{k+1} = U_j^{k-1} - 2\Delta t\left(U_j^k F^{-1}(i\nu F(U)) + F^{-1}(-i\nu^3 F(U))\right) \quad \Rightarrow \text{solution blows up!}$$

Figure 12.2 shows that the numerical solution for the KdV equation blows up using a direct spectral method. In the next section we modify the U_{xxx} term causing instabilities.

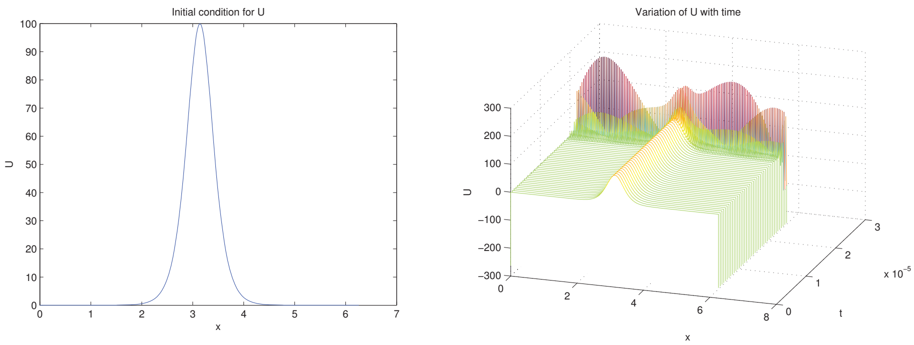

Figure 12.2: Initial conditions in (a) and solution for nonlinear KdV equation using the direct spectral method in (b)

12.3.2 Modifying U_{xxx} term causing instabilities in direct spectral method

The matlab code is **SpectralModifiedSoliton.m**.

The direct method solves:

$$U_j^{k+1} = U_j^{k-1} + 2\Delta t[U_j^k F^{-1}(ivF(U)) + F^{-1}(-iv^3F(U))]$$

The last term approximating U_{xxx} makes PDE very stiff at high wavenumbers.

To remove this instability for high wavenumbers we replace the last term with:

$$\sin(v^3\Delta t) \approx v^3\Delta t + 0(\Delta t^3) \quad \text{as } \Delta t \to 0 \text{ this is satisfied}$$

Using $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ and re-solve with this approximation:

$$U_j^{k+1} = U_j^{k-1} + 2\Delta t U_j^k F^{-1}(ivF(U)) + 2F^{-1}(-i\sin(v^3\Delta t)F(U))$$

$\Rightarrow$ This numerical solution is stable!
(See Fornberg and Whitham, Philos. Trans. Roy. Soc. London (1974))
Again use the same initial conditions:

$$\begin{aligned} U_j^{-1} &= U(x_j, -\Delta t) = U^0\left(x + \frac{A}{3}\Delta t\right) \\ &= f\left(x + \frac{A}{3}\Delta t\right) \\ &= A\text{sech}^2(\sqrt{\frac{A}{12}}\left(x + \frac{A}{3}\Delta t - \pi)\right) \end{aligned}$$

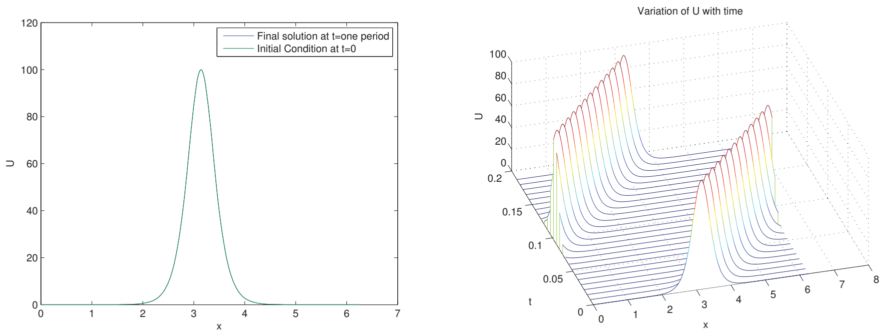

Figure 12.3: Initial conditions and final solution after one period in (a) and solution for nonlinear KdV equation using a modified spectral method in (b)

Figure 12.3 shows that the numerical solution for the KdV equation is stable using a spectral method where we have modified the U_{xxx} term causing instabilities.

The method of integrating factors can also be used to remove the instability due to U_{xxx} term (see Trefethen).

12.3.3 Interacting Solitons

The matlab code is **InteractingSoliton.m**.

When 2 solitons travelling at different speeds collide their waveform maintains same size, shape and speed but the smaller (and slower) soliton is backward shifted and the taller (and faster) soliton is forward shifted.

To show this feature of solitons we begin with intial conditions of two solitons with speeds of $V = 2A/3$ and $V = A/3$:

$$U(x,0) = f(x) = A\text{sech}^2\left(\sqrt{\frac{A}{12}}(x - \frac{3\pi}{2})\right) + 2A\text{sech}^2\left(\sqrt{\frac{2A}{12}}(x - \frac{\pi}{2})\right)$$

where $A = 100$.

Figure 12.5 shows the numerical solution for the KdV equation for 2 interacting solitons using the modified spectral method. In figure 12.5(a) we plot the initial conditions and final solution after one period. We see that after the interaction the smaller soliton is backward shifted and taller soliton forward shifted in time.

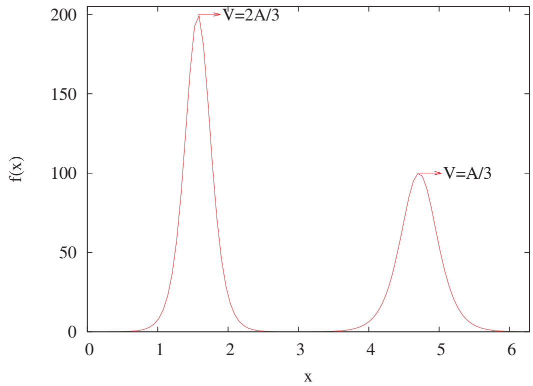

Figure 12.4: The initial conditions $U(x, 0) = f(x)$ is plotted to show the 2 solitions and their speeds

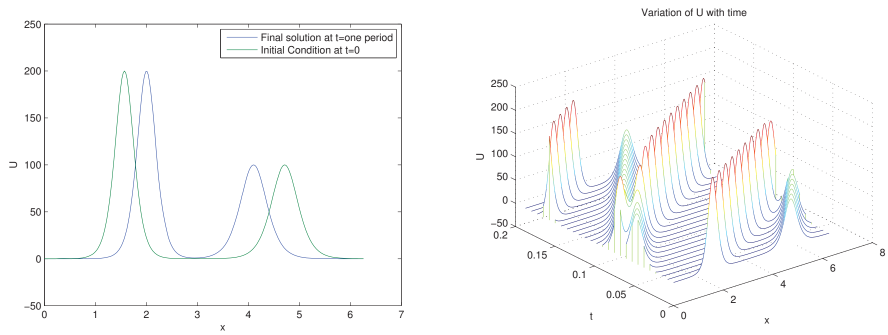

Figure 12.5: Initial conditions and final solution after one period in (a) and solution for nonlinear KdV equation for two interacting solitons in (b)