5. Normal Approximation to the Binomial

Probabilities associated with binomial experiments are readily obtainable from the formula $b(x; n, p)$ of the binomial distribution or from Table A.1 when n is small.

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Probabilities associated with binomial experiments are readily obtainable from the formula $b(x; n, p)$ of the binomial distribution or from Table A.1 when n is small.

In addition, binomial probabilities are readily available in many software computer packages.

Normal Approximation to the Binomial

In Chapter 5 we illustrated how the **Poisson** distribution can be used to **approximate binomial** probabilities when n is quite large and p is very close to 0 .

Normal Approximation to the Binomial

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What if p is not close to 0?

Normal Approximation to the Binomial

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What if p is not close to 0?

We now state a theorem that allows us to use areas under the **normal** curve to **approximate binomial** properties when n is sufficiently large.

Normal Approximation to the Binomial

Theorem 6.2

If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $N(0, 1)$.

Normal Approximation to the Binomial

Example

We first draw the histogram for $b(x; 15, 0.4)$ and then superimpose the particular normal curve **having the same mean and variance** as the binomial variable X.

Normal Approximation to the Binomial

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We first draw the histogram for $b(x; 15, 0.4)$ and then superimpose the particular normal curve **having the same mean and variance** as the binomial variable X.Hence we draw a normal curve with

$$\mu = np = 6 \quad \text{and} \quad \sigma^2 = npq = 3.6$$

Normal Approximation to the Binomial

The histogram of $b(x; 15, 0.4)$ and the corresponding superimposed normal curve are illustrated by Figure 6.22

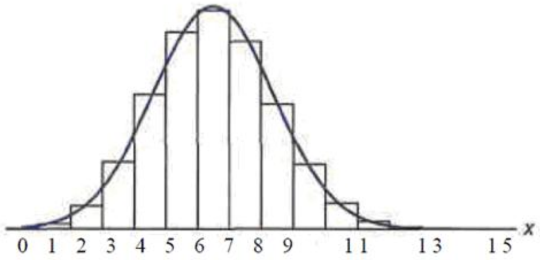

Figure 6.22: Normal approximation of $b(x; 15, 0.4)$.

Normal Approximation to the Binomial

For example, the exact probability that binomial random variable assumes the value 4 is equal to the area of the rectangle with base centered at $x = 4$. We find this area to be

$$P(X = 4) = b(4; 15, 0.4) = 0.1268$$

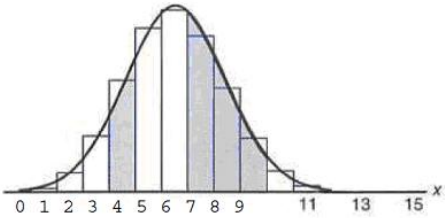

Figure 6.23: Normal approximation of $b(x; 15, 0.4)$ and $\sum_{x=7}^{9} b(x; 15, 0.4)$.

Normal Approximation to the Binomial

The area of the shaded region under the normal curve $N(6, 3.6)$ between the two ordinates $x_1 = 3.5$ and $x_2 = 4.5$ is

$$P(3.5 < Y < 4.5) = P(\frac{3.5-6}{\sqrt{3.6}} < \frac{Y-6}{\sqrt{3.6}} < \frac{4.5-6}{\sqrt{3.6}})$$

$= P(-1.32 < Z < -0.79) = P(Z < -0.79) - P(Z < -1.32)$
$= 0.2148 - 0.0934 = 0.1214$

Normal Approximation to the Binomial

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$= P(-1.32 < Z < -0.79) = P(Z < -0.79) - P(Z < -1.32)$
$= 0.2148 - 0.0934 = 0.1214$

This agrees very closely with the exact value of 0.1268.

Normal Approximation to the Binomial

$P(7 \leq X \leq 9) = \sum_{x=7}^{9} b(x; 15, 0.4)$
$= \sum_{x=0}^{9} b(x; 15, 0.4) - \sum_{x=0}^{7} b(x; 15, 0.4)$
$= 0.9662 - 0.6098 = 0.3564$

which is equal to the sum of the areas of the rectangles with bases centered at $x = 7, 8$, and 9 .

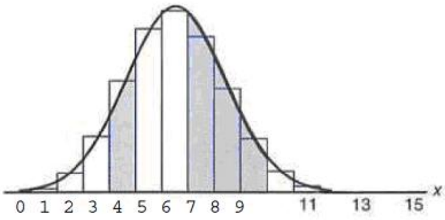

Figure 6.23: Normal approximation of $b(x; 15, 0.4)$ and $\sum_{x=7}^{9} b(x; 15, 0.4)$.

Normal Approximation to the Binomial

For the normal approximation we find that the area of the shaded region under the curve between the ordinates $x_1 = 6.5$ and $x_2 = 9.5$ in Figure 6.23.

Normal Approximation to the Binomial

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$$P(6.5 < Y < 9.5) = P\left(\frac{6.5-6}{\sqrt{3.6}} < \frac{Y-6}{\sqrt{3.6}} < \frac{9.5-6}{\sqrt{3.6}}\right)$$

$$= P(0.26 < Z < 1.85) = P(Z < 1.85) - P(Z < 0.26)$$

$$= 0.9678 - 0.6026 = 0.3652$$

Normal Approximation to the Binomial

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$$P(6.5 < Y < 9.5) = P\left(\frac{6.5-6}{\sqrt{3.6}} < \frac{Y-6}{\sqrt{3.6}} < \frac{9.5-6}{\sqrt{3.6}}\right)$$

$$= P(0.26 < Z < 1.85) = P(Z < 1.85) - P(Z < 0.26)$$
$$= 0.9678 - 0.6026 = 0.3652$$

Once again, the normal-curve approximation provides a value that agrees very closely with the exact value of 0.3564.

Normal Approximation to the Binomial

The degree of accuracy, which depends on how well the curve fits the histogram, will increase as n increases.

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The degree of accuracy, which depends on how well the curve fits the histogram, will increase as n increases. This is particularly true when p is not very close to $1/2$ and the histogram is no longer symmetric.

Normal Approximation to the Binomial

Figure 6.24 and 6.25 show the histograms for $b(x; 6, 0.2)$ and $b(x; 15, 0.2)$, respectively.

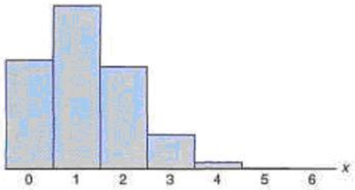

Figure 6.24: Histogram for $b(x; 6, 0.2)$.

Normal Approximation to the Binomial

It is evident that a normal curve would fit the histogram when $n = 15$ considerably better than when $n = 6$.

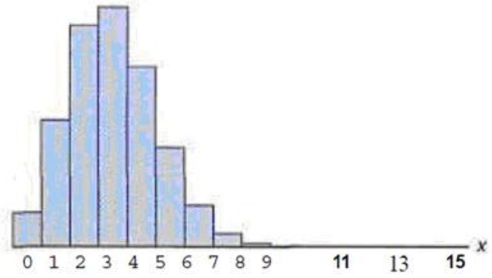

Figure 6.25: Histogram for $b(x; 15, 0.2)$.

Normal Approximation to the Binomial

Summary

We use the normal approximation to evaluate binomial probabilities whenever p is not close to 0.

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We use the normal approximation to evaluate binomial probabilities whenever p is not close to 0. The approximation is excellent when n is large and fairly good for small values of n if p is reasonably close to $1/2$.

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We use the normal approximation to evaluate binomial probabilities whenever p is not close to 0. The approximation is excellent when n is large and fairly good for small values of n if p is reasonably close to $1/2$.

One possible guide to determine when the normal approximation may be used is provided by calculating np and nq **are greater than or equal to** 5, the approximation will be good.

Normal Approximation to the Binomial

Example 6.15

The probability that a patient recovers from a rare blood disease is 0.4 . If 100 people are known to have contracted this disease, what is the probability that less than 30 survive?

Normal Approximation to the Binomial

Example 6.16

A multiple-choice quiz has 200 questions each with 4 possible answers of which only 1 is the correct answer. What is the probability that sheer guesswork yields from 25 to 30 correct answers for 80 of the 200 problems about which the student has no knowledge?

6. Gamma and Exponential Distributions

Although the normal distribution can be used to solve many problems in engineering and science, there are still numerous situations that require different types of density functions.

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Time between arrivals at service facilities,
time to failure of component parts and electronic systems,

6. Gamma and Exponential Distributions

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Time between arrivals at service facilities,
time to failure of component parts and electronic systems,
often are nicely modeled by the exponential distribution.

Gamma and Exponential Distributions

The gamma distribution derives its name from the well-known **gamma function**, studied in many areas of mathematics.

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Let us review this function and some of its important properties.

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Definition 6.2 The gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

for $\alpha > 0$.

Gamma and Exponential Distributions

For $\alpha > 1$, we have $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

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Gamma and Exponential Distributions

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Since $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, integrating by parts,we obtain

$$\Gamma(\alpha) = -e^{-x}x^{\alpha-1}|_0^\infty + \int_0^\infty e^{-x}(\alpha-1)x^{\alpha-2}dx$$
$$= (\alpha-1)\int_0^\infty e^{-x}x^{\alpha-2}dx$$

Gamma and Exponential Distributions

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$$= (\alpha - 1) \int_0^\infty e^{-x} x^{\alpha-2} dx$$

Repeated application of the recursion formula gives

$$\Gamma(\alpha) = (\alpha - 1)(\alpha - 2)\Gamma(\alpha - 2)$$
$$= (\alpha - 1)(\alpha - 2)(\alpha - 3)\Gamma(\alpha - 3) = \cdots$$

Gamma and Exponential Distributions

Note that when $\alpha = n$, where n is a positive integer,

$$\Gamma(n) = (n-1)(n-2)\ldots\Gamma(1)$$

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$$\Gamma(n) = (n-1)(n-2)\ldots\Gamma(1)$$

However, by Definition 6.2

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$$

and hence

$$\Gamma(n) = (n-1)!$$

Gamma and Exponential Distributions

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and hence

$$\Gamma(n) = (n-1)!$$

Verify the important property of the gamma function

$$\Gamma(1/2) = \sqrt{\pi}$$

Gamma and Exponential Distributions

Gamma Distribution

The continuous random variable X has a gamma distribution, with parameters α and β, if its density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & elsewhere. \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

Gamma and Exponential Distributions

Graphs of several gamma distributions are shown in Figure 6.28 for certain specified values of the parameters α and β.

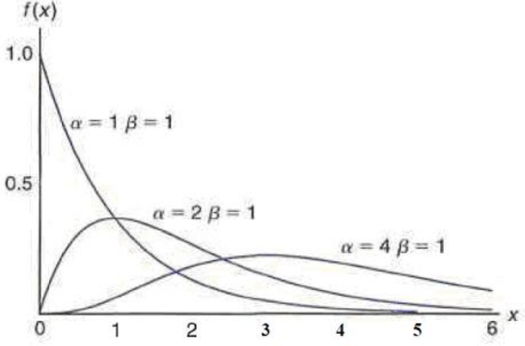

Figure 6.28: Gamma distributions.

Gamma and Exponential Distributions

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & elsewhere. \end{cases}$$

The special gamma distribution for which $\alpha = 1$ is called the **exponential distribution**.

Gamma and Exponential Distributions

Exponential Distribution The continuous random variable X has an exponential distribution, with parameter $\beta > 0$, if its density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta} e^{-x/\beta}, & x > 0 \\ 0, & elsewhere. \end{cases}$$

Gamma and Exponential Distributions

Exponential Distribution The continuous random variable X has an exponential distribution, with parameter $\beta > 0$, if its density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0 \\ 0, & elsewhere. \end{cases}$$

Let $\lambda = 1/\beta$, the density function turns to

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & elsewhere. \end{cases}$$

Gamma and Exponential Distributions

Theorem 6.3

The mean and variance of the gamma distribution are

$$\mu = \alpha\beta \quad \text{and} \quad \sigma^2 = \alpha\beta^2.$$

Gamma and Exponential Distributions

Theorem 6.3

The mean and variance of the gamma distribution are

$$\mu = \alpha\beta \quad \text{and} \quad \sigma^2 = \alpha\beta^2.$$

Corollary 1

The mean and variance of the exponential distribution are

$$\mu = \beta \quad \text{and} \quad \sigma^2 = \beta^2.$$

Gamma and Exponential Distributions

Corollary 2

The mean and variance of the exponential distribution are

$$\mu = \frac{1}{\lambda} \quad \text{and} \quad \sigma^2 = \frac{1}{\lambda^2}$$

Gamma and Exponential Distributions

Example 6.17

Suppose that a system contains type of component whose time in years to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta = 5$. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Gamma and Exponential Distributions

Solution

The probability that a given component is still functioning after 8 years is given by

$$P(T > 8) = \frac{1}{5}\int_8^\infty e^{-t/5}dt = e^{-8/5} \approx 0.2$$

Let X represent the number of components functioning after 8 years. Then using the binomial distribution

$$P(X \geq 2) = \sum_{x=2}^{5} b(x; 5, 0.2) = 1 - \sum_{x=0}^{1} b(x; 5, 0.2) = 0.2627$$

7. Other Continuous Distributions

Another very important special case of the gamma distribution is obtained by letting $\alpha = v/2$ and $\beta = 2$ in

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & elsewhere. \end{cases}$$

The result is called the **chi-squared distribution**.

Other Continuous Distributions

Chi-Squared Distribution

The continuous random variable X has a chi-squared distribution, with v degrees of freedom, if its density function is given by

$$f(x) = \begin{cases} \dfrac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Where v is a positive integer， called the **degrees of freedom**.

Other Continuous Distributions

Mean and Variance of Chi-squared The mean and variance of the chi-squared distribution are

$$\mu = v \quad \text{and} \quad \sigma^2 = 2v.$$

We can get the conclusion immediately from Theorem 6.3

Other Continuous Distributions

Lognormal Distribution

The continuous random variable X has a lognormal distribution if the random variable $Y = \ln(X)$ has a normal distribution with mean μ and standard deviation σ. The resulting density function of X is

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} e^{-[\ln(x)-\mu]^2/(2\sigma^2)}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Other Continuous Distributions

Weibull Distribution

The continuous random variable X has a Weibull distribution with parameter α and β if its density function is given by

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.