Chapter 8

Numerical Solution of 1-D and 2-D Wave Equation

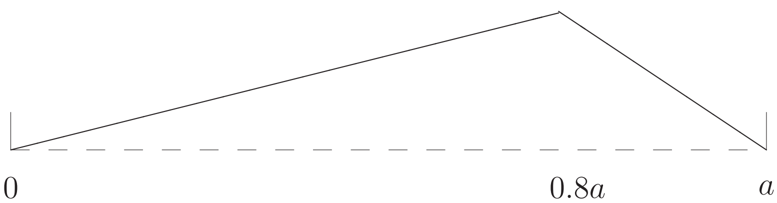
8.1 Explicit Central Difference for 1-D Wave Equation

$$U_{tt} = c^2 U_{xx}, \quad 0 \leq t \leq T,\ 0 \leq x \leq a$$

Discretise: $\Delta t = \frac{T}{m},\ \Delta x = \frac{a}{n+1}$,
$t_k = k\Delta t,\ 0 \leq k \leq m,\ x_j = j\Delta x$ and $0 \leq j \leq n+1$.

8.1.1 Example: plucking a string

The matlab code for this example is **Wave1D.m**.



A string is initially plucked or lifted from rest:
boundary conditions: $U(0,t) = 0,\ U(a,t) = 0$ or $U_0^k = 0,\ U_{n+1}^k = 0$
initial conditions: string is "plucked" or lifted 1mm at $x = 0.8a$:

$$U(x, t=0) = f(x) = \begin{cases} \frac{1.25x}{a}, & \text{for } x \leq 0.8a \\ 5(1 - \frac{x}{a}), & \text{for } x > 0.8a \end{cases}$$

Plucked string is released from rest:

$$\frac{\partial U}{\partial t}(x,0) = g(x) = 0$$

$$U(x,t=0) = f(x) \Rightarrow U_j^0 = f_j = f(x_j)$$

$$\frac{\partial U}{\partial t}(x,t=0) = g(x) \Rightarrow \underbrace{\frac{\partial U_j^0}{\partial t} \approx \frac{U_j^1 - U_j^{-1}}{2\Delta t}}_{\text{leap-frog in time}} = g_j = g(x_j)$$

We can solve for 'ghost' point U_j^{-1}:

$$U_j^{-}1 = U_j^1 - 2\Delta t g(x_j)$$

We approximate U_{tt} and U_{xx} using central differences:

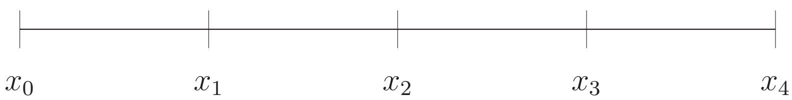
$$\begin{aligned} U_{tt} &= \frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{\Delta t^2} \\ U_{xx} &= \frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{\Delta x^2} \end{aligned}$$

Using $U_{tt} = c^2 U_{xx}$ and $s = \frac{c^2 \Delta t^2}{\Delta x^2}$, we solve for U_j^{k+1} at time step $k+1$:

$$U_j^{k+1} = \underbrace{-U_j^{k-1}}_{\text{solution at } t_{k-1}} + \underbrace{2U_j^k(1-s) + s(U_{j+1}^k + U_{j-1}^k)}_{\text{solution at } t_k}$$

In order to find U_j^2 we need to know U_j^0 and U_j^1.

We consider $n = 3$:



boundary conditions: $U_0^k = 0,\ U_4^k = 0$

initial conditions: $U_j^0 = f_j,\ U_j^{-1} = U_j^1 - 2\Delta t g(x_j) = U_j^1$, since $g(x_j) = 0$.

First find $\vec{U}^1 = \begin{pmatrix} U_1^1 \\ U_2^1 \\ U_3^1 \end{pmatrix}$

$$\vec{U}^1 = \begin{pmatrix} U_1^1 \\ U_2^1 \\ U_3^1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2(1-s) & s & 0 \\ s & 2(1-s) & s \\ 0 & s & 2(1-s) \end{pmatrix}}_{A} \begin{pmatrix} U_1^0 \\ U_2^0 \\ U_3^0 \end{pmatrix}$$

$$+ \quad \underbrace{s\begin{pmatrix} U_0^0 \\ 0 \\ U_4^0 \end{pmatrix}}_{b} - \begin{pmatrix} U_1^{-1} \\ U_2^{-1} \\ U_3^{-1} \end{pmatrix}$$

Use $U_j^0 = f_j$ and $U_j^{-1} = U_j^1 - 2\Delta t g_j$

$$\begin{aligned}
\vec{U}^1 &= \begin{pmatrix} U_1^1 \\ U_2^1 \\ U_3^1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2(1-s) & s & 0 \\ s & 2(1-s) & s \\ 0 & s & 2(1-s) \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \\
&+ \frac{s}{2}\begin{pmatrix} U_0^0 \\ 0 \\ U_4^0 \end{pmatrix} + \Delta t\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}
\end{aligned}$$

$$\vec{U}^1 = \frac{1}{2}A\vec{U}^0 + \frac{1}{2}\vec{b} + \vec{d}$$

For this example, $U_0^0 = 0,\ U_4^0 = 0$ and:

$$\frac{\partial U_j^0}{\partial t}(x, t = 0) = g(x_j) = 0 \Rightarrow \vec{d} = \vec{0}$$

for $\vec{U}^2, \ldots, \vec{U}^m$ we have:

$$U_j^{k+1} = 2U_j^k(1-s) + s(U_{j+1}^k + U_{j-1}^k) - U_j^{k-1}$$

for $1 \leq k \leq m$:

$$\begin{aligned}
\vec{U}^{k+1} &= \begin{pmatrix} U_1^{k+1} \\ U_2^{k+1} \\ U_3^{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 2(1-s) & s & 0 \\ s & 2(1-s) & s \\ 0 & s & 2(1-s) \end{pmatrix}}_{A}\begin{pmatrix} U_1^k \\ U_2^k \\ U_3^k \end{pmatrix} \\
&+ \underbrace{s\begin{pmatrix} U_0^k \\ 0 \\ U_4^k \end{pmatrix}}_{b} - \begin{pmatrix} U_1^{k-1} \\ U_2^{k-1} \\ U_3^{k-1} \end{pmatrix}
\end{aligned}$$

$$\vec{U}^{k+1} = A\vec{U}^k + \vec{b} - \vec{U}^{k-1}$$

The matlab code is **Wave1D.m**.

In our example $U_0^k = 0,\ U_4^k = 0$ and $\vec{b} = \vec{0}$, since $U_0^k = 0 = U_4^k$
At fixed boundaries $U(0,t) = 0 = U(a,t) \Rightarrow$ wave is reflected. We plot the numerical solution in figure 8.1.

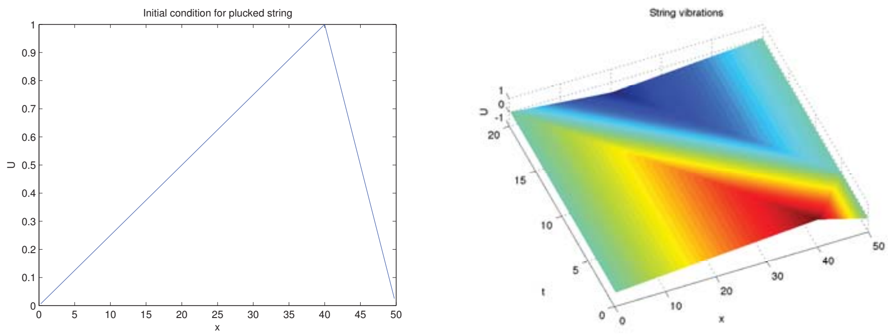


Figure 8.1: Initial conditions in (a) and matlab solution using explicit central difference method for 1D wave equation in (b)

We can compare with *D'Alembert's solution* which gives:

$$U(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] \text{ since } U_t(x,0) = 0$$
$$\text{where } U(x,0) = f(x) \text{ (initial conditions) for } -\infty < x < \infty$$

What if we want to solve the wave equation for $0 \le x \le a$, with fixed boundary condition $U(t,0) = 0 = U(t,a)$? We can extend D'Alembert's general solution for $U_{tt} = c^2 U_{xx}$ with initial conditions:$U(x,0) = f(x)$ $U_t(x,0) = g(x)$:

$$\begin{aligned} U(x,t) &= \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz \\ & \text{for } -\infty \le x \le \infty \end{aligned}$$

In our example we have initial conditions:

$$U_t(x,0) = 0,\ U(x,0) = f(x) = \begin{cases} \frac{1.25x}{a}, & 0 \le x \le 0.8a \\ 5(1-\frac{x}{a}), & \text{for } x \ge 0.8a \end{cases}$$

$$0 \le x \le a$$

with *fixed* boundary conditions:
The boundary condition $U(0,t) = 0$ is equivalent to f and g being odd functions:

$$\begin{aligned} U(0,t) = 0 \Rightarrow f(-x) &= -f(x) \\ g(-x) &= -g(x) \end{aligned}$$
$$(f \text{ and } g \text{ are odd functions})$$

The boundary condition $U(a,t) = 0$ is equivalent to f and g being periodic with period $2a$:

$$\begin{array}{rcl} U(a,t) = 0 \Rightarrow f(x+2a) & = & f(x) \\ g(x+2a) & = & g(x) \\ (f \text{ and } g \text{ are periodic with period } 2a) & & \end{array}$$

Since $U_t(x,0) = g(x) = 0$ the analytical solution for our example:

$$U(t,x) = \frac{f(x+ct) + f(x-ct)}{2}$$

and we can compare the analytical solution with the numerical solution in figure 8.2.

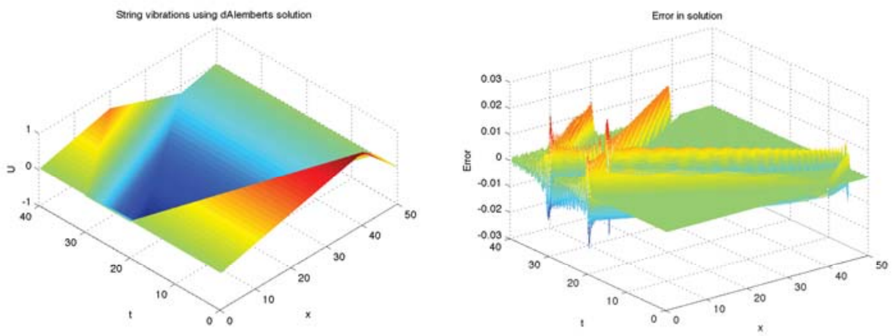


Figure 8.2: D'Alembert's solution in (a) and error using numerical matlab solution using explicit central difference method for 1D wave equation in (b)

8.1.2 1-D Wave Equation with Friction

The matlab code for this example is **Wave1DFriction.m**.

We consider friction due to viscosity of medium and density of string. Suppose we are solving:

$$\ddot{U} + 2\kappa\dot{U} = c^2 U_{xx}, \quad 0 \leq x \leq a = 50, \quad 0 \leq t \leq T = 20$$

The friction term κ opposes motion of string and means that eventually vibrations decay with time.

Suppose string is initially plucked in 2 places:

We have initial conditions:

$$
\begin{aligned}
U(x,0) &= \begin{cases} 0, & 0 \le x \le 0.1a \\ 5(10x-a), & 0.1a \le x \le 0.2a \\ 5(-10x+3a), & 0.2a \le x \le 0.3a \\ 0, & 0.3a \le x \le 0.7a \\ 5(10x-7a), & 0.7a \le x \le 0.8a \\ 5(-10x+9a), & 0.8a \le x \le 0.9a \\ 0, & x \ge 0.9a \end{cases} \\
U_t(x,0) &= 0
\end{aligned}
$$

and boundary conditions: $U(x,0)=0$, $U(x,a)=0$.
Again we use central difference for U_{xx} and U_{tt} as in section 8.1.1.
We use a leap-frog step for U_t

$$\frac{\partial U_j^k}{\partial t} = \frac{U_j^{k+1} - U_j^{k-1}}{2\Delta t}$$

Now we substitute difference approximations into $U_{tt} + 2\kappa U_t = c^2 U_{xx}$

$$\frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{\Delta t^2} + \kappa \frac{U_j^{k+1} - U_j^{k-1}}{\Delta t} = \frac{c^2(U_{j+1}^k - 2U_j^k + U_{j-1}^k)}{\Delta x^2}$$

let $s = \frac{c^2 \Delta t^2}{\Delta x^2}$
Rearranging for U_j^{k+1} gives:

$$U_j^{k+1} = \frac{1}{1+\kappa\Delta t}\left\{2(1-s)U_j^k - (1-\kappa\Delta t)U_j^{k-1} + s(U_{j+1}^k + U_{j-1}^k)\right\}$$

Special care is again needed to solve for U_j^1 which needs U_j^0 and the ghost point, U_j^{-1}. To find U_j^{-1} we use initial condition:

$$\frac{\partial U}{\partial t}(x, t=0) = \frac{\partial U_j^0}{\partial t} = 0 = \frac{U_j^1 - U_j^{-1}}{2\Delta t}$$

$$\text{or } U_j^{-1} = U_j^1 \qquad (\text{since } U_t(x,0)=0)$$

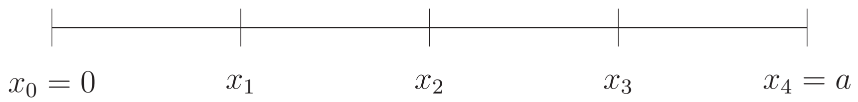
We evaluate U_j^1:

$$U_j^1 = \frac{1}{1+\kappa\Delta t}\left\{2(1-s)U_j^0 - (1-\kappa\Delta t)\underbrace{U_j^{-1}}_{=U_j^1} + s(U_{j+1}^0 - U_{j-1}^0)\right\}$$

$$\Rightarrow \frac{2}{1+\kappa\Delta t}U_j^1 = \frac{1}{1+\kappa\Delta t}\left\{2(1-s)U_j^0 + s(U_{j+1}^0 - U_{j-1}^0)\right\}$$

$$\Rightarrow U_j^1 = \frac{1}{2}\left\{2(1-s)U_j^0 + s(U_{j+1}^0 - U_{j-1}^0)\right\}$$

Example $n = 3$



$$U_0^k = 0 = U(0,t), \quad U_{n+1}^k = U_4^k = U(a,t)$$

Again we solve for time step $k = 1$, $\vec{U}^1$ first:

$$\vec{U}^1 = \begin{pmatrix} U_1^1 \\ U_2^1 \\ U_3^1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2(1-s) & s & 0 \\ s & 2(1-s) & s \\ 0 & s & 2(1-s) \end{pmatrix}\begin{pmatrix} U_1^0 \\ U_2^0 \\ U_3^0 \end{pmatrix} + \frac{s}{2}\begin{pmatrix} U_0^0 \\ 0 \\ U_4^0 \end{pmatrix}$$

and the solution for time steps, $k \geq 1$, $\vec{U}^{k+1}$ are given by:

$$\begin{aligned}\vec{U}^{k+1} &= \begin{pmatrix} U_1^{k+1} \\ U_2^{k+1} \\ U_3^{k+1} \end{pmatrix} = \underbrace{\frac{1}{1+\kappa\Delta t}\begin{pmatrix} 2(1-s) & s & 0 \\ s & 2(1-s) & s \\ 0 & s & 2(1-s) \end{pmatrix}}_{A}\begin{pmatrix} U_1^k \\ U_2^k \\ U_3^k \end{pmatrix} \\ &+ \underbrace{\frac{s}{1+\kappa\Delta t}\begin{pmatrix} U_0^k \\ 0 \\ U_4^k \end{pmatrix}}_{b} - \underbrace{\frac{1-\kappa\Delta t}{1+\kappa\Delta t}}_{e}\begin{pmatrix} U_1^{k-1} \\ U_2^{k-1} \\ U_3^{k-1} \end{pmatrix} \\ &= A\vec{U}^k + \vec{b} - e\vec{U}^{k-1}\end{aligned}$$

The numerical solution is plotted in figure 8.3 below.

The matlab code is **Wave1DFriction.m**

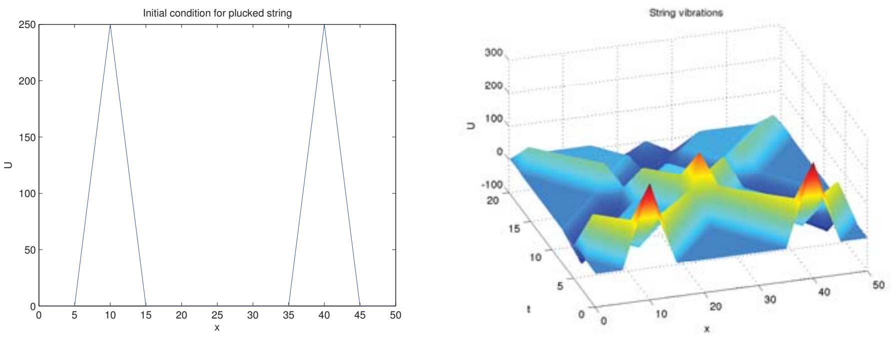


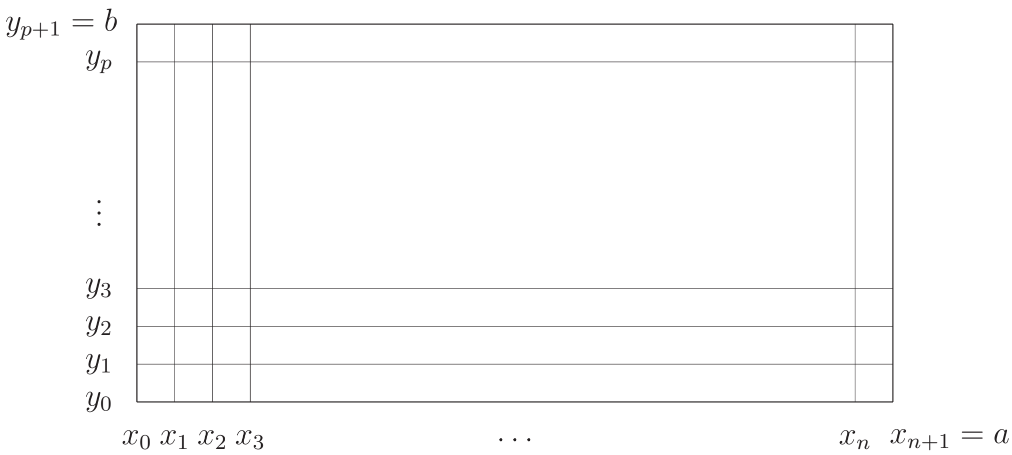
Figure 8.3: Initial conditions in (a) and matlab solution using explicit central difference method for 1D wave equation with friction in (b)

8.2 2-D Wave Equation

$$U_{tt} = \beta(U_{xx} + U_{yy}), \quad 0 \le x \le a,\ 0 \le y \le b,\ 0 \le t \le T$$

8.2.1 Example: vibrations of a thin elastic membrane fixed at its walls

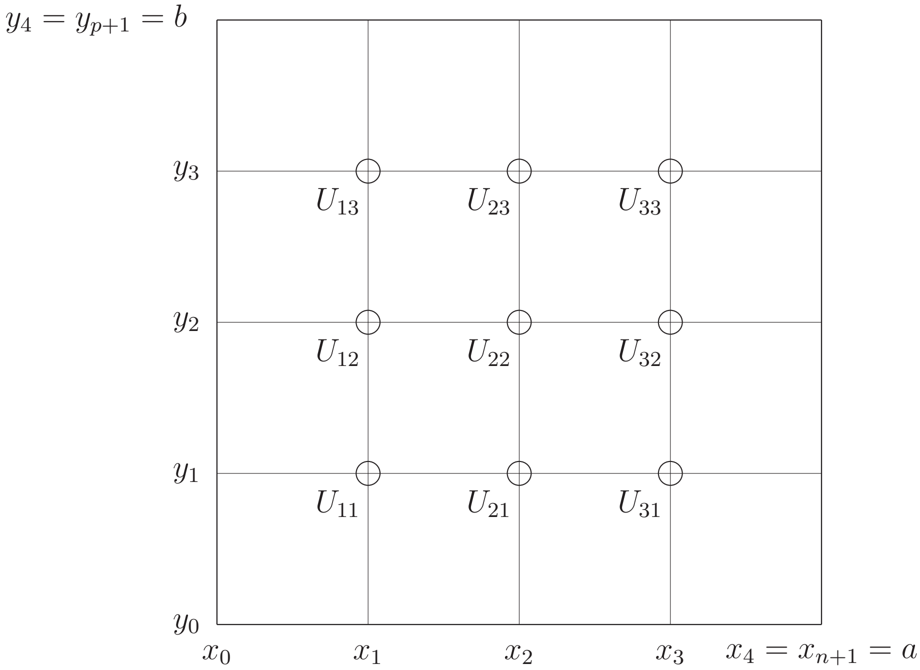
We discretise in x and y-directions:



We discretise: $\Delta t = \frac{T}{m},\ \Delta x = \frac{a}{n+1},\ \Delta y = \frac{b}{p+1},\ t_k = k\Delta t,\ x_i = i\Delta x,\ y_j = j\Delta y 0 \le k \le m,\ 0 \le i \le n+1,\ 0 \le j \le p+1$, and let $U_{ij}^k = U(t_k, x_i, y_j)$

Suppose we solve for $n = 3$ and $p = 3$ and have Dirichlet boundary conditions:
$U(0, y, t) = 0 = U^k_{oj}$, $U(a, y, t) = 0 = U^k_{n+1,j} = U^k_{4j}$, $U(x, 0, t) = 0 = U^k_{i0}$, $U(x, b, t) = 0 = U^k_{i,p+1} = U^k_{i4}$
and initial conditions:
$U(x, y, 0) = f(x, y) = f_{ij}$ $U_t(x, y, 0) = g(x, y) = g_{ij}$.

Since we have Dirichlet boundary conditions: the outer boundaries of the region we are solving for are known: $U^k_{0,j}, U^k_{n+1,j}, U^k_{i,0}, U^k_{i,p+1}$, and we need to find the interior values: $U^k_{i,j}$ for $1 \le i \le n$ and $1 \le j \le p$.



We will use the *2-D Central Difference Method*

$$
\begin{aligned}
U_{tt} &= \frac{U^{k+1}_{ij} - 2U^k_{ij} + U^{k-1}_{ij}}{\Delta t^2}, \\
U_{xx} &= \frac{U^k_{i+1,j} - 2U^k_{ij} + U^k_{i-1,j}}{\Delta x^2}, \\
U_{yy} &= \frac{U^k_{i,j+1} - 2U^k_{ij} + U^k_{i,j+1}}{\Delta y^2}
\end{aligned}
$$

We let $s_x = \frac{\beta \Delta t^2}{\Delta x^2}$, $s_y = \frac{\beta \Delta t^2}{\Delta y^2}$ and substitute the central difference approximations into our PDE, $U_{tt} = \beta(U_{xx} + U_{yy})$ we solve for U_{ij}^{k+1}:

$$U_{ij}^{k+1} = 2U_{ij}^{k}(1 - s_x - s_y) - U_{ij}^{k-1} + s_x(U_{i+1,j}^{k} + U_{i-1,j}^{k}) + s_y(U_{i,j+1}^{k} + U_{i,j-1}^{k})$$

computing $\vec{U}^{k+1}$ uses the solution at $\vec{U}^{k}$ and $\vec{U}^{k-1}$.
For first time step U_{ij}^{1} needs U_{ij}^{0} and U_{ij}^{-1}. Again we need to use the initial conditions to find the ghost point, U_{ij}^{-1}:

$$\frac{\partial U_{ij}^{0}}{\partial t} = U_t(x, y, 0) = \frac{U_{ij}^{1} - U_{ij}^{-1}}{2\Delta t} = g(x_i, y_j) = g_{ij} \Rightarrow U_{ij}^{-1} = U_{ij}^{1} - 2\Delta t g_{ij}$$

Solution at first time step $k = 1$:

$$U_{ij}^{1} = U_{ij}^{0}(1 - s_x - s_y) + \Delta t g_{ij} + \frac{s_x}{2}(U_{i+1,j}^{0} + U_{i-1,j}^{0}) + \frac{s_y}{2}(U_{i,j+1}^{0} + U_{i,j-1}^{0})$$

If we let $\vec{U}^{k} = \begin{pmatrix} U_{11}^{k} \\ U_{12}^{k} \\ U_{13}^{k} \\ U_{21}^{k} \\ U_{22}^{k} \\ U_{23}^{k} \\ U_{31}^{k} \\ U_{32}^{k} \\ U_{33}^{k} \end{pmatrix}$
then for time steps, $k > 1$, the solution is:

$$U_{ij}^{k+1} = 2U_{ij}^{k}(1 - s_x - s_y) - U_{ij}^{k-1} + s_x(U_{i+1,j}^{k} + U_{i-1,j}^{k}) + s_y(U_{i,j+1}^{k} + U_{i,j-1}^{k})$$

and we can write this in vector form:

$$\vec{U}^{k+1} = A\vec{U}^{k} + \vec{b} - \vec{U}^{k-1}$$

where:

$$\begin{aligned} A &= \begin{pmatrix} \lambda & s_y & 0 & s_x & 0 & 0 & 0 & 0 & 0 \\ s_y & \lambda & s_y & 0 & s_x & 0 & 0 & 0 & 0 \\ 0 & s_y & \lambda & 0 & 0 & s_x & 0 & 0 & 0 \\ s_x & 0 & 0 & \lambda & s_y & 0 & s_x & 0 & 0 \\ 0 & s_x & 0 & s_y & \lambda & s_y & 0 & s_x & 0 \\ 0 & 0 & s_x & 0 & s_y & \lambda & 0 & 0 & s_x \\ 0 & 0 & 0 & s_x & 0 & 0 & \lambda & s_y & 0 \\ 0 & 0 & 0 & 0 & s_x & 0 & s_y & \lambda & s_y \\ 0 & 0 & 0 & 0 & 0 & s_x & 0 & s_y & \lambda \end{pmatrix} \\ \text{and } \lambda &= 2(1 - s_x - s_y) \end{aligned}$$

$$
b = \begin{pmatrix} s_x U_{01}^k + s_y U_{10}^k \\ s_x U_{02}^k \\ s_x U_{03}^k + s_y U_{14}^k \\ s_y U_{20}^k \\ 0 \\ s_y U_{24}^k \\ s_x U_{41}^k + s_y U_{30}^k \\ s_x U_{42}^k \\ s_x U_{43}^k + s_y s_{34}^k \end{pmatrix}
$$

8.2.2 Examples of wave equation

1. **Elastic wave propagation through rocks** in *1-D*

$$
\sigma_{xx,x} = \rho U_{tt} \tag{8.1}
$$

where

$$
\begin{aligned}
\sigma_{xx} &= E\varepsilon_{xx},\ \sigma_{xx} = \text{stress},\ \varepsilon_{xx} = \text{strain} \\
&= E\frac{\partial U}{\partial x}
\end{aligned}
$$

$$
8.1 \Rightarrow EU_{xx} = \rho U_{tt} \quad \text{or } U_{tt} = \frac{E}{\rho} U_{xx}
$$

elastic waves propagate with speed $\sqrt{\frac{E}{\rho}}$

2. **Electromagnetic Wave Equation**

$$
c^2\nabla^2 E = \ddot{E} \text{ and } c^2\nabla^2 B = \ddot{B} \tag{8.2}
$$

From Maxwell's equations where E is electric field, B is magnetic field. Derived using:

$$
\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \times E = -\frac{\partial B}{\partial t} \tag{8.3}
$$

$$
\nabla \cdot B = 0, \quad \nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \tag{8.4}
$$

taking curl of 8.3 and 8.4 and using $\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2\vec{V}$ and $\nabla(\nabla \cdot E) = \nabla\left(\frac{\rho}{\epsilon_0}\right) = 0, \quad \nabla(\nabla \cdot B) = 0$ gives Equation 8.2 where $c = \sqrt{\frac{1}{\mu_0\varepsilon_0}} = 3 \times 10^8\text{m/s}$.

3. **Schrödinger's Wave Equation**

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

- for a wavefunction Ψ of a quantum system defined by Hamiltonian, H.
  eg. $H = KE + PE = -\frac{\hbar^2\nabla^2}{2m} + V(r)$
- numerical solutions also need to satisfy $\int_{-\infty}^{\infty}|\Psi(x)|^2_{dx} = 1$